

Chapter 4

Advanced counting methods

4.1 The principle of Inclusion – Exclusion

Let A, B and C be sets. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Where $|X|$ is cardinality X or the number of elements of set X .

Theorem:- (Inclusion – Exclusion)

Let $A_1, A_2, \dots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$$

Note:- 1. For 'n' number of sets, we have $2^n - 1$ number of terms.

2. The coefficient of a term is -1 if the number of sets in the intersection is even and 1 otherwise.

Exercise:- Give a formula for the number of elements in the union of four sets.

Note :- The inclusion – exclusion principle is used to solve problems that ask for the number of elements in a set that have none of 'n' properties $p_1, p_2, \dots, p_n$, denoted by $N(p_1' p_2' \dots p_n')$ and defined by

$N(p_1' p_2' \dots p_n') = N - |A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n|$, where N is the no of elements which belong to the universal set (U) and $A_i \in U$ and A_i has property p_i

Example:- 12 balls are painted in the following ways: 2 are painted red, 1 is painted blue, 1 is painted white, 2 are painted red & blue, 1 is painted red & white and 3 are painted red, blue & white. How many balls are not painted?

Solution:- Let p_1, p_2, p_3 denotes the properties that a ball is painted red, blue, white respectively; then

$$N(p_1) = 8, \quad N(p_2) = 6, \quad N(p_3) = 5, \quad N(p_1 p_2) = 5, \quad N(p_1 p_3) = 4 \\ N(p_2 p_3) = 3, \quad N(p_1 p_2 p_3) = 3$$

$$\begin{aligned} \text{Then } N(p_1' p_2' p_3') &= N - |A_1 \cup A_2 \cup A_3| \\ &= N - (N(p_1) + N(p_2) + N(p_3) - N(p_1 p_2) \\ &\quad - N(p_1 p_3) - N(p_2 p_3) + N(p_1 p_2 p_3)) = 12 - [8 + 6 + 5 - 4 - 3 + 3] = 2 \end{aligned}$$

Example:- 1. Determine the number of positive integer 'n' where $1 \leq n \leq 100$ and n is not divisible by 2, 3, 5.

Solution:- Here $U = \{1, 2, 3, \dots, 100\}$, $N = 100$,

for $n \in U$, n satisfies: p_2 if n is divisible by 2.

p_3 if n is divisible by 3.

p_5 if n is divisible by 5.

Note:- $[x]$ is the greatest integer l such that $l \leq x$.

$$\text{Now, } N(p_2) = \left\lfloor \frac{100}{2} \right\rfloor = 50, A_2 = \{2, 4, 6, \dots\}, N(p_2 p_3) = \left\lfloor \frac{100}{6} \right\rfloor = 16$$

$$N(p_3) = \left\lfloor \frac{100}{3} \right\rfloor = 33, A_3 = \{3, 6, 9, \dots\}, N(p_2 p_5) = \left\lfloor \frac{100}{10} \right\rfloor = 10$$

$$N(p_5) = \left\lfloor \frac{100}{5} \right\rfloor = 20, A_5 = \{5, 10, 15, \dots\}, N(p_3 p_5) = \left\lfloor \frac{100}{15} \right\rfloor = 6$$

$$N(p_2 p_3 p_5) = \left\lfloor \frac{100}{30} \right\rfloor = 3$$

$$\begin{aligned} \text{Then } N(p_2' p_3' p_5') &= N - |A_2 \cup A_3 \cup A_5| \\ &= N - (N(p_2) + N(p_3) + N(p_5) - N(p_2 p_3) \\ &\quad - N(p_2 p_5) - N(p_3 p_5) + N(p_2 p_3 p_5)) \\ &= 100 - (50 + 33 + 20 - 10 - 16 - 6 + 3) \\ &= 26 \text{ number are not divisible by 2, 3, 5.} \end{aligned}$$

2. Using question (1);

i) find the number of positive integers that are not divisible by 2 nor by 5 but are divisible by 3?

ii) Find the number of positive integers that are not divisible by 5 but are divisible by 2 & 3

$$\text{Solution:- i) } N(p_2' p_3 p_5') = N(p_3) - N(p_2 p_3) - N(p_3 p_5) + N(p_2 p_3 p_5) = 33 - 16 - 6 + 3$$

$$\text{ii) } N(p_2 p_3 p_5') = N(p_2 p_3) - N(p_2 p_3 p_5) = 16 - 3 = 13$$

4.2. Recurrence Relation

Consider the sequence 2, 4, 8, 16, ... can be defined recursively like

$$a_{k+1} = 2a_k, \quad a_1 = 2, k \geq 1 \quad \dots (*)$$

The equation $a_{k+1} = 2a_k$ in (*), which defines one member of the sequence in terms of a previous one is called a recurrence relation.

The equation $a_1 = 2$ is called an initial (boundary) condition.

Example:- 1. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} - a_{n-2}, \text{ for } n = 2, 3, 4, \dots, a_0 = 3 \text{ and } a_1 = 5. \text{ Then find } a_2, a_3, a_4, a_5$$

2. Suppose that f is defined recursively by $f(0) = 3, f(n+1) = 2f(n) + 3$. Then find $f(1), f(2), f(3), f(4)$

3. The Fibonacci numbers $f_0, f_1, f_2, \dots$ are defined by the equations $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, for $n = 2, 3, 4, \dots$. Then find $f_2, f_3, f_4, f_5, f_6, f_7$

Definition:- The general linear recurrence relation of degree k with constant coefficients has the form $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$, where

$c_0, c_1, c_2, \dots, c_k$ are real numbers, $c_0, c_k \neq 0$ and $k \in \mathbb{N}$

Note:- When $f(n) = 0, \forall n \in \mathbb{N}$, the relation is called homogeneous, otherwise non-homogeneous.

Example:- The recurrence relation

a) $p_n = 5p_{n-1}$ is linear, homogeneous with degree 1.

b) $f_n = f_{n-1} + f_{n-2}$ is linear, homogeneous with degree 2.

- c) $a_n = a_{n-5}$ is linear, homogeneous with degree 5.
 d) $f_n = f_{n-1} + f_{n-2}^3$ is not linear, it is homogeneous with constant coefficient.
 e) $f_n = 2f_{n-1}f_{n-2} + 1$ is not linear, not homogeneous but it has constant coeffi.
 f) $B_n = nB_{n-1}$ is linear, homogeneous but the coefficient is not constant.

4.3 Solution of recurrence relation

A. First order recurrence relation

➤ For the homogeneous case can be written as:

$$a_n = ra_{n-1}, n \geq 1, a_0 = A, r \text{ is constant. Its general solution is } a_n = Ar^n$$

1. Solve the recurrence relation

- a) $a_n = 3a_{n-1}, n \geq 1, a_0 = 5$
 b) $a_n = 7a_{n-1}, n \geq 1, a_2 = 98$
 c) $a_n = na_{n-1}, n \geq 1, a_0 = 1$

Solution:- a) Since $r = 3, A = 5$, then $a_n = 5(3^n)$ is the solution

b) Since $r = 7, A = a_0 = 2$ because $a_2 = 7a_1$ and $a_1 = 7a_0 \Rightarrow a_0 = 2$
 Then $a_n = 2(7^n)$ is the solution

c) Since $r = n$, which is not constant, we can not use the above formula.

$$a_1 = 1, a_0 = 1, a_2 = 2a_1 = 2.1, a_3 = 3a_2 = 3.2, a_4 = 4a_3 = 4.3.2 = 4!
 a_5 = 5! . \text{ Thus } a_n = n!$$

B. Second order recurrence relation

➤ For the homogeneous case can be written as:

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} = 0, n \geq 2, \text{ where } c_0, c_1, c_2 \in R \text{ (real no)}, c_0, c_2 \neq 0 \dots (*)$$

First we assume for solutions of the form $a_n = Ar^n$, where $A, r \neq 0$ and substituting in to the above equation (*), we have

$$c_0Ar^n + c_1Ar^{n-1} + c_2Ar^{n-2} = 0 \\ \Rightarrow c_0r^2 + c_1r + c_2 = 0 \dots \dots \dots (**)$$

which is called the characteristic equation of the recurrence relation.

Then we may have two roots, r_1 and r_2 are called the characteristic roots.

For these roots we have 3 cases:

Case1:- Distinct real roots

In this case the general solution of the recurrence relation is

$$a_n = b_1r_1^n + b_2r_2^n, \text{ where } b_1, b_2 \text{ are arbitrary constants.}$$

Example:- Solve the recurrence relations

- a) $a_n + a_{n-1} - 6a_{n-2} = 0, n \geq 2, a_0 = -1, a_1 = 8$
 b) $a_n = a_{n-1} + 2a_{n-2}, n \geq 2, a_0 = 2, a_1 = 7$
 c) $a_{n+2} = a_{n+1} + a_n, n \geq 0, a_0 = 0, a_1 = 1$

Solution:- a) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on

$$a_n + a_{n-1} - 6a_{n-2} = 0, \text{ we have } cr^n + cr^{n-1} - 6cr^{n-2} = 0 \\ \Rightarrow r^2 + r - 6 = 0 \\ \Rightarrow r = 2 \text{ or } r = -3$$

Here we have two distinct real roots, so, we can $a_n = 2^n$ and $a_n = (-3)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $(-3)^n = k(2)^n$ for all $n \in \mathbb{N}$. Therefore its general solution is the form $a_n = b_1 2^n + b_2 (-3)^n$, where b_1, b_2 are arbitrary constants.

Now to determine b_1 and b_2 , use $a_0 = -1, a_1 = 8$.

$$\text{If } a_0 = -1 = b_1 2^0 + b_2 (-3)^0 \Rightarrow -1 = b_1 + b_2$$

$$\text{If } a_1 = 8 = b_1 2^1 + b_2 (-3)^1 \Rightarrow 8 = 2b_1 - 3b_2$$

Solve this system of equations, we have $b_1 = 1, b_2 = -2$

$\therefore a_n = 2^n - 2(-3)^n$ for $n \geq 0$ is its general solution.

b) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on $a_n = a_{n-1} + 2a_{n-2}$, we have $cr^n = cr^{n-1} + 2cr^{n-2}$

$$\Rightarrow r^2 - r - 2 = 0$$

$$\Rightarrow r = -1 \text{ or } r = 2$$

Here we have two distinct real roots, so, we can $a_n = 2^n$ and $a_n = (-1)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $(-1)^n = k(2)^n$ for all $n \in \mathbb{N}$.

Therefore its general solution is the form $a_n = b_1 2^n + b_2 (-1)^n$, where b_1, b_2 are arbitrary constants.

Now to determine b_1 and b_2 , use $a_0 = 2, a_1 = 7$.

$$\text{If } a_0 = 2 = b_1 2^0 + b_2 (-1)^0 \Rightarrow 2 = b_1 + b_2$$

$$\text{If } a_1 = 7 = b_1 2^1 + b_2 (-1)^1 \Rightarrow 7 = 2b_1 - b_2$$

Solve this system of equations, we have $b_1 = 3, b_2 = -1$

$\therefore a_n = 3(2^n) - 1(-1)^n$ for $n \geq 0$ is its general solution.

c) Let $a_n = cr^n$ with $c, r \neq 0$. Then substitute this on $a_{n+2} = a_{n+1} + a_n$, we have $cr^{n+2} = cr^{n+1} + cr^n$

$$\Rightarrow r^2 - r - 1 = 0$$

$$\Rightarrow r = \frac{1 - \sqrt{5}}{2} \text{ or } r = \frac{1 + \sqrt{5}}{2}$$

Here we have two distinct real roots, so, we can $a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ and $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ which are both solutions. They are linearly independent solutions because one is not a multiple of the other, i.e, there is no real constant k such that $\left(\frac{1-\sqrt{5}}{2}\right)^n = k \left(\frac{1+\sqrt{5}}{2}\right)^n$ for all $n \in \mathbb{N}$.

Therefore its general solution is the form $a_n = b_1 \left(\frac{1-\sqrt{5}}{2}\right)^n + b_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$, where b_1, b_2 are arbitrary constants.

Now to determine b_1 and b_2 , use $a_0 = 0, a_1 = 1$.

$$\text{If } a_0 = 0 = b_1 \left(\frac{1-\sqrt{5}}{2} \right)^0 + b_2 \left(\frac{1+\sqrt{5}}{2} \right)^0 \Rightarrow 0 = b_1 + b_2 \Rightarrow b_1 = -b_2$$

$$\text{If } a_1 = 1 = b_1 \left(\frac{1-\sqrt{5}}{2} \right)^1 + b_2 \left(\frac{1+\sqrt{5}}{2} \right)^1 \Rightarrow 1 = b_1 \left(\frac{1-\sqrt{5}}{2} \right) + b_2 \left(\frac{1+\sqrt{5}}{2} \right)$$

Solve this system of equations, we have $b_1 = -\frac{1}{\sqrt{5}}, b_2 = \frac{1}{\sqrt{5}}$

$$\therefore a_n = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \text{ for } n \geq 0 \text{ is its general solution.}$$

Case 2:- Repeated real roots

If $r_1 = r_2 = r$, then the general solution of the recurrence relation is

$a_n = b_1 r_1^n + b_2 n r_2^n$ or $a_n = b_1 n r_1^n + b_2 r_2^n$, where b_1, b_2 are arbitrary constants.

Example:- Solve the recurrence relation

a) $a_{n+2} = 4a_{n+1} - 4a_n, n \geq 0$ and $a_0 = 1, a_1 = 3$

b) $a_n = 6a_{n-1} - 9a_{n-2}, n \geq 2$ and $a_0 = 1, a_1 = 12$

c) $a_n = 6a_{n-1} - 9a_{n-2}, n \geq 2$ and $a_0 = 1, a_1 = 6$

Solution:- a) Let $a_n = cr^n$ with $c, r \neq 0$ and $n \geq 0$ Then substitute this on

$$a_{n+2} = 4a_{n+1} - 4a_n, \text{ we have } cr^{n+2} = 4cr^{n+1} - 4cr^n \\ \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow r = 4$$

Thus its general solution is the form $a_n = b_1 2^n + b_2 n 2^n$, where b_1, b_2 are arbitrary constants.

Now to determine b_1 and b_2 , use $a_0 = 1, a_1 = 3$.

$$\text{If } a_0 = 1 = b_1 2^0 + b_2(0)(2)^0 \Rightarrow 1 = b_1$$

$$\text{If } a_1 = 3 = b_1 2^1 + b_2(1)(2)^1 \Rightarrow 3 = 2b_1 + b_2 \Rightarrow b_2 = \frac{1}{2}$$

$$\therefore \text{General solution } a_n = 2^n + \frac{1}{2} n 2^n, n \geq 0$$

Case3:- Complex roots

In this case the solution could be expressed in the same way as in case of distinct real roots. Before getting into the case of complex roots, we recall **Demoiver's theorem**:

$$\diamond (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta; n \geq 0$$

$\diamond$ If $z = x + iy \in \mathbb{C}; z \neq 0$, where $\mathbb{C}$ is complex number, then can write

i) For $x \neq 0, z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$

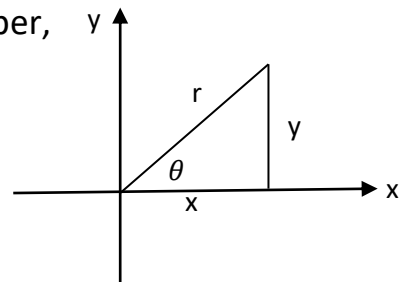
ii) for $x = 0$ and $y > 0$,

$$z = iy = r(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = y i \sin\frac{\pi}{2}$$

iii) for $x = 0$ and $y < 0$,

$$z = iy = r(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}) = r(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})$$

$\diamond$ In all case $z^n = r^n(\cos n\theta + i\sin n\theta)$, for $n \geq 0$ by **Demoiver's theorem**



Example1:- Determine $(1 + i\sqrt{3})^{10}$

Solution:- $(1 + i\sqrt{3})^{10} = (r(\cos\theta + i\sin\theta))^{10}$ but $r = 2, x = 1, y = \sqrt{3}$

$$= \left(2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^{10} = 2^{10} \left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) = 2^{10} \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)$$

Example2:- Solve the recurrence relations

- a) $a_{n+2} + 4a_n = 0, n \geq 0$ and $a_0 = -1, a_1 = 4$
- b) $a_n = 2(a_{n-1} - a_{n-2}), n \geq 2$ and $a_0 = 1, a_1 = 2$
- c) $a_{n+2} + 4a_{n+1} + 16a_n = 0, n \geq 0$ and $a_0 = 2, a_1 = 10$

Solution:- a) Let $a_n = br^n$, where $b, r \neq 0$ and $n \geq 0$.

Then $a_{n+2} + 4a_n = 0$ becomes $br^{n+2} + 4br^n = 0$

$$\Rightarrow r^2 + 4 = 0 \Rightarrow a_1 = 2i, a_2 = -2i$$

It follows that the general solution of the recurrence relation is given by

$$a_n = b_1(2i)^n + b_2(-2i)^n$$

$$\begin{aligned} \Rightarrow a_n &= 2^n \left[b_1 \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \right)^n + b_2 \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} \right)^n \right] \\ &= 2^n \left[b_1 \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} \right) + b_2 \left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2} \right) \right] \\ &= 2^n \left[(b_1 + b_2)\cos\frac{n\pi}{2} + i(b_1 - b_2)\sin\frac{n\pi}{2} \right] \\ &= 2^n \left[c_1\cos\frac{n\pi}{2} + c_2\sin\frac{n\pi}{2} \right], \text{ where } c_1 = b_1 + b_2, c_2 = i(b_1 - b_2) \end{aligned}$$

$$\text{But } a_0 = -1 = 2^0[c_1\cos 0 + c_2\sin 0] = c_1 \Rightarrow c_1 = -1$$

$$a_1 = 4 = 2^1 \left[c_1\cos\frac{\pi}{2} + c_2\sin\frac{\pi}{2} \right] = 2c_2 \Rightarrow c_2 = 2$$

Therefore, $a_n = 2^n \left[-\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2} \right]$ is the solution.

b) Let $a_n = br^n$, where $b, r \neq 0$ and $n \geq 0$. Then $a_n = 2(a_{n-1} - a_{n-2})$ becomes

$$br^n = 2(br^{n-1} - br^{n-2})$$

$$\Rightarrow r^n = 2(r^{n-1} - r^{n-2}), \text{ since divide both side by } b.$$

$$\Rightarrow r^2 - 2r + 2 = 0$$

$$\Rightarrow r_1 = 1 + i, r_2 = 1 - i, \text{ Now since } x = 1, y = 1, \text{ then } r = \sqrt{2}$$

Then the general solution is $a_n = b_1(1+i)^n + b_2(1-i)^n$

$$\begin{aligned} \Rightarrow a_n &= (\sqrt{2})^n \left[b_1 \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \right)^n + b_2 \left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4} \right)^n \right] \\ &= (\sqrt{2})^n \left[b_1 \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} \right) + b_2 \left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4} \right) \right] \\ &= (\sqrt{2})^n \left[(b_1 + b_2)\cos\frac{n\pi}{4} + (b_1 - b_2)i\sin\frac{n\pi}{4} \right] \\ &= (\sqrt{2})^n \left[c_1\cos\frac{n\pi}{4} + c_2\sin\frac{n\pi}{4} \right], \text{ where } c_1 = b_1 + b_2, c_2 = (b_1 - b_2)i \end{aligned}$$

To find c_1 and c_2 , use initial condition, i.e.

$$a_0 = 1 = (\sqrt{2})^0 [c_1\cos 0 + c_2\sin 0] \Rightarrow c_1 = 1$$

$$\begin{aligned} a_1 = 2 &= (\sqrt{2})^1 \left[c_1\cos\frac{\pi}{4} + c_2\sin\frac{\pi}{4} \right] \Rightarrow 2 = \sqrt{2} \left(c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2} \right) \\ &\Rightarrow 2 = c_1 + c_2 \Rightarrow c_2 = 1 \end{aligned}$$

Thus the general solution is $a_n = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4} \right]$.

c) Let $a_n = br^n$, where $b, r \neq 0$ and $n \geq 0$. Then $a_{n+2} + 4a_{n+1} + 16a_n = 0$ becomes
 $br^{n+2} + 4br^{n+1} + 16br^n = 0$

$$\Rightarrow r^2 + 4r + 16 = 0$$

$$\Rightarrow r_1 = -2 + 2\sqrt{3}i, r_2 = -2 - 2\sqrt{3}i$$

Since $x = -2$ and $y = 2\sqrt{3}$, $r = 4$

Then the general solution is $a_n = b_1(-2 + 2\sqrt{3}i)^n + b_2(-2 - 2\sqrt{3}i)^n$

$$\Rightarrow a_n = 4^n \left[b_1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^n + b_2 \left(\cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3} \right)^n \right]$$

$$= 4^n \left[b_1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^n + b_2 \left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right)^n \right]$$

$$= 4^n \left[b_1 \left(\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right) + b_2 \left(\cos \frac{2n\pi}{3} - i \sin \frac{2n\pi}{3} \right) \right]$$

$$= 4^n \left[(b_1 + b_2) \cos \frac{2n\pi}{3} + (b_1 - b_2) i \sin \frac{2n\pi}{3} \right]$$

$$= 4^n \left[c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} \right], \text{ where } c_1 = b_1 + b_2, c_2 = (b_1 - b_2)i$$

To find the value of c_1 and c_2 , use initial condition, i.e.

$$a_0 = 2 = 4^0 [c_1 \cos 0 + c_2 \sin 0] \Rightarrow c_1 = 2$$

$$a_1 = 10 = 4^1 \left[c_1 \cos \frac{2\pi}{3} + c_2 \sin \frac{2\pi}{3} \right] \Rightarrow \frac{10}{4} = c_1 \left(\frac{-1}{2} \right) + c_2 \left(\frac{2\sqrt{3}}{4} \right)$$

$$\Rightarrow \frac{5}{2} = \frac{-1}{2} c_1 + \frac{\sqrt{3}}{2} c_2$$

$$\Rightarrow c_2 = \frac{7\sqrt{3}}{3}$$

Hence the general solution is $a_n = 4^n \left[2 \cos \frac{2n\pi}{3} + \frac{7\sqrt{3}}{3} \sin \frac{2n\pi}{3} \right]$.

C. First and second order recurrence relation for non-homogeneous:-

The general solution for non-homogeneous recurrence relation is

$a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(h)}$ is homogeneous solution,

$a_n^{(p)}$ is particular solution.

i) Consider a non-homogeneous first order recurrence relation $a_n - ka_{n-1} = cr^n$, where c is a constant and $n \in \mathbb{Z}^+$

➤ If r^n is not a solution of a homogeneous relation $a_n - ka_{n-1} = 0$, then

$a_n^{(p)} = Ar^n$, where A is constant.

➤ If r^n is a solution of a homogeneous relation $a_n - ka_{n-1} = 0$, then

$a_n^{(p)} = Bnr^n$, where B is constant.

Example:- Solve a) $a_n - 3a_{n-1} = 5(7^n), n \geq 1, a_0 = 2$ ————(*)

b) $a_n - 3a_{n-1} = 5(3^n), n \geq 1, a_0 = 2$ ————(**)

c) $a_n - 2a_{n-1} = 5(3^n), n \geq 1, a_0 = 2$ ————(***)

Solution:- a) Let $a_n^{(h)} = Cr^n$, where $C, r \neq 0$. Then substitute on $a_n - 3a_{n-1} = 0$.

$$\text{We have } Cr^n - 3Cr^{n-1} = 0$$

$$\Rightarrow r - 3 = 0 \text{ (by dividing both side with } Cr^{n-1})$$

$$\Rightarrow r = 3$$

$$\text{Thus } a_n^{(h)} = C3^n$$

Since 7^n is not a solution for homogeneous, $a_n^{(p)} = A7^n$ and substitute on $a_n - 3a_{n-1} = 5(7^n)$. We have $A7^n - 3A7^{n-1} = 5(7^n)$

$$\Rightarrow A - \frac{3}{7}A = 5 \Rightarrow A = \frac{35}{4}$$

$$\text{Thus } a_n^{(h)} = \frac{35}{4} 7^n$$

Hence the general solution of (*) is the form $a_n = a_n^{(h)} + a_n^{(p)}$

$$\Rightarrow a_n = C3^n + \frac{35}{4} 7^n$$

$$\text{To find } C, \text{ use initial condition } a_0 = 2 = C(3^0) + \frac{35}{4} (7^0) \Rightarrow C = \frac{-27}{4}$$

Therefore $a_n = \frac{-27}{4} (3^n) + \frac{35}{4} (7^n)$ is the general solution of (*)

b) Let $a_n^{(h)} = Cr^n$, where $C, r \neq 0$. Then substitute on $a_n - 3a_{n-1} = 0$.

$$\text{We have } Cr^n - 3Cr^{n-1} = 0$$

$$\Rightarrow r - 3 = 0 \text{ (by dividing both side with } Cr^{n-1})$$

$$\Rightarrow r = 3$$

$$\text{Thus } a_n^{(h)} = C3^n$$

Since 3^n is a solution for homogeneous, $a_n^{(p)} = An3^n$ and substitute on $a_n - 3a_{n-1} = 5(3^n)$. We have $An3^n - 3A(n-1)3^{n-1} = 5(3^n)$

$$\Rightarrow An - \frac{3}{3}A(n-1) = 5 \Rightarrow A = 5$$

$$\text{Thus } a_n^{(p)} = 5n3^n$$

Hence the general solution of (**) is the form $a_n = a_n^{(h)} + a_n^{(p)}$

$$\Rightarrow a_n = C3^n + 5n3^n$$

$$\text{To find } C, \text{ use initial condition } a_0 = 2 = C(3^0) + 5 \cdot 0 \cdot (3^0) \Rightarrow C = 2$$

Therefore $a_n = 2(3^n) + 5n3^n$ is general solution of (**)

c) Exercise

ii) Consider a non-homogeneous 2^{nd} order recurrence relation,

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = f(n), n \geq 0, \text{ where } c_0, c_1, c_2 \text{ are real numbers}$$

$$\& c_0, c_2 \neq 0 \text{ ----- (*)}$$

Suppose $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$, where c_1, c_2 are real numbers, $c_2 \neq 0$ and $f(n) = p(n)r^n$, say $p(n)$ is polynomial.

✚ When r is not a root of characteristic equation of homogeneous.

There is $a_n^{(p)} = q(n)r^n$, where $q(n)$ is polynomial.

✚ When r is a root of characteristic equation of homogeneous and its multiplicity is 'm', there is $a_n^{(p)} = n^m q(n)r^n$, where $q(n)$ is polynomial.

Example:- What form $a_n^{(p)}$ for $a_n = 6a_{n-1} - 9a_{n-2} + f(n)$ have:
when $f(n) = 3^n$, $f(n) = n3^n$, $f(n) = n^2 2^n$ and $f(n) = (n+1)^2 3^n$?

Solution:- 1st find $a_n^{(h)}$?

Let $a_n = cr^n$, where $c, r \neq 0$ and substitute on homogeneous. We have

$$cr^n = 6cr^{n-1} - 9cr^{n-2}$$

$$\Rightarrow r^n - 6r + 9 = 0$$

$$\Rightarrow (r - 3)^n = 0 \text{ has a single root with multiplicity 2. } \therefore a_n^{(h)} = b_1 3^n + b_2 n 3^n$$

2nd find $a_n^{(p)}$?

Since $r = 3$ is a root with multiplicity $m = 2$, but $r = 2$ is not a root of characteristic equation for homogeneous. Then

$a_n^{(p)}$ has the form $n^2 A 3^n$ if $f(n) = 3^n$

$a_n^{(p)}$ has the form $n^2 (p_1 n + p_2) 3^n$ if $f(n) = n 3^n$

$a_n^{(p)}$ has the form $(p_2 n^2 + p_1 n + p_0) 2^n$ if $f(n) = n^2 2^n$

$a_n^{(p)}$ has the form $n^2 (p_2 n^2 + p_1 n + p_0) 3^n$ if $f(n) = (n+1)^2 3^n$

iii) To find $a_n^{(p)}$ let us use the method of undetermined coefficients, that is, assume $a_n^{(p)}$ for the following $f(n)$ on the table: (where c, A, A_0, A_1, A_2 are constants.)

$f(n)$	$a_n^{(p)}$
C	A
C n	$A_0 + A_1 n$
C n ²	$A_0 + A_1 n + A_2 n^2$
$A \sin n\theta$ or $A \cos n\theta$	$A_1 \sin n\theta + A_2 \cos n\theta$
$A r^n \sin n\theta$ or $A r^n \cos n\theta$	$A_1 r^n \sin n\theta + A_2 r^n \cos n\theta$
$A n^m r^n$	$r^n (A_0 + A_1 n + \dots + A_m n^m)$

Example:- Find the general solution of a) $a_n = 3a_{n-1} + 2n$, $a_1 = 3$

b) $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Solution:- a) Let $a_n^{(h)} = Ar^n$, where A is constant, $r \neq 0$. and substitute this on the homogeneous $a_n = 3a_{n-1}$, we have $a_n^{(h)} = A 3^n$

since $f(n) = 2n$ is a polynomial with degree 1, $a_n^{(p)}$ is a linear function in n ,

i.e, $a_n^{(p)} = c n + d$, where c, d are constants. Then

$$a_n = 3a_{n-1} + 2n \text{ becomes } c n + d = 3(c(n-1) + d) + 2n$$

$$\Rightarrow c n + d = 3c n - 3c + 3d + 2n$$

$$\Rightarrow (3c + 2 - c)n - 3c = d - 3d$$

$$\Rightarrow (2c + 2)n + (2d - 3c) = 0$$

$$\Rightarrow (2c + 2) = 1 \text{ and } (2d - 3c) = 0$$

$$\Rightarrow c = -1, \quad d = \frac{-3}{2}$$

Thus $a_n^{(p)} = -n - \frac{3}{2}$ is a particular solution.

All solutions are the form $a_n = a_n^{(h)} + a_n^{(p)} = b_1 3^n - n - \frac{3}{2}$

To find b_1 :- use initial condition $a_1 = 3$

$$a_1 = 3 = b_1 3^1 - 1 - \frac{3}{2} \Rightarrow 3b_1 = 3 + 1 + \frac{3}{2} \Rightarrow b_1 = \frac{11}{6}$$

$\therefore a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$ is a general solution.

b) To find $a_n^{(h)}$

Let $a_n = cr^n$, where $c, r \neq 0$ and substitute on $a_n = 5a_{n-1} - 6a_{n-2}$, we have

$$cr^n = 5cr^{n-1} - 6cr^{n-2}$$

$$\Rightarrow r^2 - 5r + 6 = 0$$

$$\Rightarrow r = \frac{5 \pm \sqrt{25-24}}{2} \text{ Thus the roots are } r_1 = 3, r_2 = 2$$

$$a_n^{(h)} = b_1(3^n) + b_2(2^n)$$

Since $f(x) = 7^n$, $a_n^{(h)} = A(7^n)$, where A is constant. Then substitute on

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n, \text{ we have } A7^n = 5A7^{n-1} - 6A7^{n-2} + 7^n$$

$$\Rightarrow A7^2 = 5A7 - 6A + 7^2$$

$$\Rightarrow (49 - 35 + 6)A = 49$$

$$\Rightarrow A = \frac{49}{20}$$

$$\text{Thus } a_n^{(p)} = \frac{49}{20}(7^n)$$

$$a_n = a_n^{(h)} + a_n^{(p)} = b_1 3^n + b_2 2^n + \frac{49}{20}(7^n)$$