

CHAPTER 2: POWER SERIES

OBJECTIVES: At the end of this chapter students will be able to

1. Use polynomials to approximate the values of some functions which have no exact values.
2. Determine the Power series representation of a given function.
3. Derive Taylor series representation of a given function.
4. Determine the Taylor series representation of a given function.
5. Determine the intervals of convergence of a power series.
6. Differentiate and integrate a power series.
7. Determine the power series representation of a function using the properties of derivatives and integrals of power series.
8. Determine the Fourier series representations of functions.
9. Use Fourier series expansion to obtain the sum of an infinite series.

POLYNOMIAL APPROXIMATION

Suppose f is defined at $x = a$ and in the neighborhood of a that is for values of x near $x = a$,

$$f'(a) \approx \frac{f(x) - f(a)}{x - a} \Rightarrow f(x) \approx f(a) + f'(a)(x - a) = c_0 + c_1(x - a) \dots (1)$$

Where $c_0 = f(a)$ and $c_1 = f'(a)$

Eq. 1 is the linear approximation of the function f near $x = a$.

Let, $p_1(a) = f(a) + f'(a)(x - a)$, then we will have the following relations hold.

$$p_1(a) = f(a) \text{ and } p_1'(a) = f'(a) \quad \dots (2)$$

The above relation can be generalized on the basic assumption that the approximation becomes better and better as we proceed to higher degree polynomials such as quadratic, cubic, and etc.

Table 1: The approximated values of $f(x) = \ln x$ near $x = 1$ using polynomial approximations

Taylor Polynomial Function	Value at $x = 1.1$
$p_1 = 1 - x$	-0.1
$p_2 = -1.5 + 2x - 0.5x^2$	0.0950
$p_3 = -1.8333 + 3x - 1.5x^2 + 0.3333x^3$	0.0953223
$p_4 = -2.0833 + 4x - 3x^2 + 1.3333x^3 - 0.25x^4$	0.0952973

Now, we may generalize Eq. 2 for the n^{th} degree polynomial

$$p_n(a) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n \dots (3)$$

$$\left\{ \begin{array}{l} p_1(a) = f(a) \\ p_1'(a) = f'(a) \\ p_1''(a) = f''(a) \\ \vdots \\ p_1^{(n)}(a) = f^{(n)}(a) \end{array} \right. \dots (4)$$

Since we have $n + 1$ coefficients $c_0, c_1, c_2, c_3, \dots, c_n$ and $n + 1$ equations (Eq. 4), we can obtain each coefficients. Finally we obtain the Taylor polynomial

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x - a)^k}{k!}$$

First some examples

Consider the following example series:

$$f(x) = \sum_{k=0}^n \frac{2^k}{k!}$$

- *What does our intuition tell us about the convergence or divergence of this series?*
- *What test should we use to confirm our intuition?*

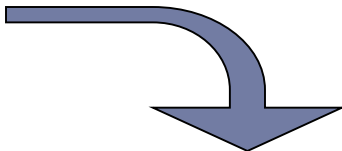
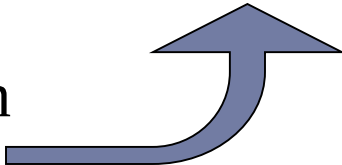
Power Series

Now we consider a whole family of similar series:

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \quad \sum_{k=0}^{\infty} \frac{(-4)^k}{k!} \quad \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{k!} \quad \sum_{k=0}^{\infty} \frac{214^k}{k!}$$

- What about the convergence or divergence of these series?
- What test should we use to confirm our intuition?

We should use the ratio test; furthermore, we can use the similarity between the series to test them all at once.


$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$


How does it go? We start by setting up the appropriate limit.

$$\text{Let } a_k = \frac{x^k}{k!}, \text{ then } \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{|x|}{k+1} = 0$$

By the Ratio Test, the interval of convergence is $(-\infty, \infty)$ and the radius of convergence

is $R = \infty$. That is the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely for all values of x .

What are Power Series?

It's convenient to think of a power series as an infinite polynomial:

Polynomials:

$$2 - x + 3x^2 - 12x^3$$

$$1 + (x - 1) + 3(x - 1)^2 - \frac{1}{4}(x - 1)^3$$

Power Series:

$$1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{k=0}^{\infty} (k + 1) x^k$$

$$1 - \frac{(x + 3)}{3!} + \frac{(x + 3)^2}{5!} - \frac{(x + 3)^3}{7!} + \frac{(x + 3)^4}{9!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{(x + 3)^k}{(2k + 1)!}$$

Definition: Power Series

An expression of the form

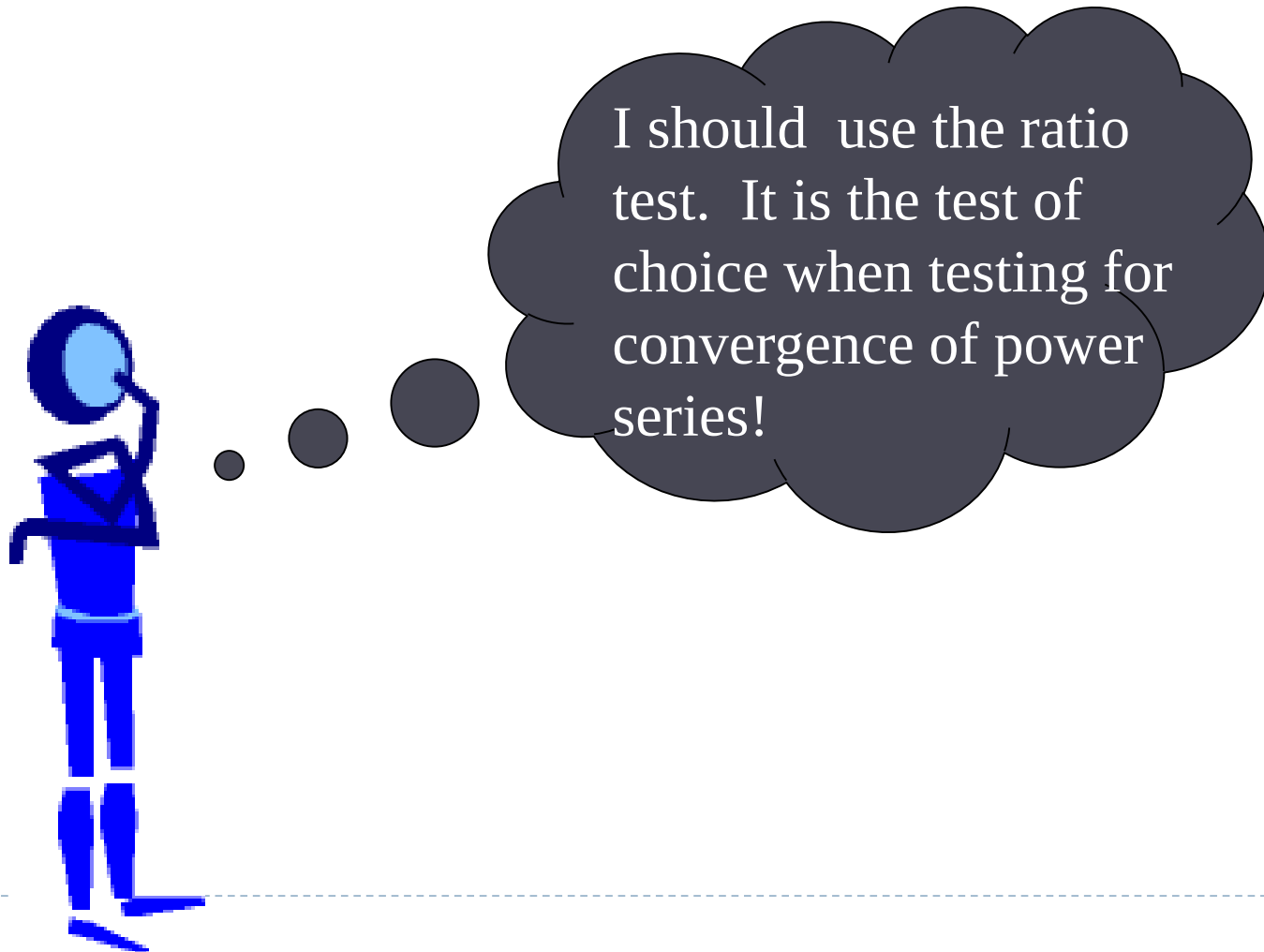
$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

is a power series centered at $x = 0$ and an expression of the form

$$c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x - a)^n$$

is a power series centered at $x = a$.

Checking for Convergence



Checking for Convergence

Example 1: Checking on the convergence of

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{k=0}^{\infty} (k+1)x^k$$

How does it go? We start by setting up the appropriate limit.

Let $a_k = (k+1)x^k$, then $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left(\frac{k+2}{k+1} \right) |x| = |x|$

By the Ratio Test, the series converges when $|x| < 1 \Rightarrow -1 < x < 1$.

At the end points we get the series $\sum_{k=0}^{\infty} (k+1)(-1)^k$ and $\sum_{k=0}^{\infty} (k+1)$ which are both divergent.

Hence, the interval of convergence is $(-1, 1)$ and the radius of convergence is 1.

Example 2: What about the convergence of

$$1 - \frac{(x+3)}{3!} + \frac{(x+3)^2}{5!} - \frac{(x+3)^3}{7!} + \frac{(x+3)^4}{9!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{(x+3)^k}{(2k+1)!}$$

How does it go? We start by setting up the appropriate limit.

$$\text{Let } a_k = (-1)^k \frac{(x+3)^k}{(2k+1)!}, \text{ then } \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{|x+3|}{(2k+2)(2k+3)} = 0$$

By the Ratio Test, the interval of convergence is $(-\infty, \infty)$ and the radius of convergence

is $R = \infty$. That is the series $\sum_{k=0}^{\infty} (-1)^k \frac{(x+3)^k}{(2k+1)!}$ converges absolutely for all values of x .

Now you work out the convergence of

$$1 - \frac{(x+3)}{3} + \frac{(x+3)^2}{5} - \frac{(x+3)^3}{7} + \frac{(x+3)^4}{9} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{(x+3)^k}{(2k+1)}$$



Don't forget those
absolute values!

How does it go? We start by setting up the appropriate limit.

$$\text{Let } a_k = (-1)^k \frac{(x+3)^k}{(2k+1)}, \text{ then } \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left(\frac{2k+1}{2k+3} \right) |x+3| = |x+3|$$

By the Root Test, the series converges when $|x+3| < 1 \Rightarrow -4 < x < -2$.

In addition to this, we tested for convergence at the end points as follows:

At $x = -4$, we get $\sum_{k=0}^{\infty} \frac{1}{(2k+1)}$ which is divergent by the Integral Test.

At $x = -2$, we get $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}$ which is convergent by Alternating Series Test.

Therefore, the interval of convergence is $(-4, -2]$ and the radius of convergence is $\frac{-2 - (-4)}{2} = 1$.

Interval of Convergence of a power series

Definition: The set of all real numbers x for which a power series converges is said to be the interval of convergence.

A power series can have the following intervals of convergence.

- 1) $(-r, r)$, $[-r, r)$, $(-r, r]$, or $[-r, r]$. Here the radius of convergence is r .
- 2) $(-\infty, \infty)$. Here the radius of convergence is ∞ .
- 3) At a single point $x = a$. Here the radius of convergence is 0 .

Example 3: Find the interval and radius of convergence of the following power series.

$$1. \sum_{n=0}^{\infty} n! (x-2)^n$$

Solution: Let $a_n = n! (x-2)^n$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} (n+1)|x-2|$

By the ratio test, the series converges when $\lim_{n \rightarrow \infty} (n+1)|x-2| < 1 \Rightarrow |x-2| = 0 \Rightarrow x = 2$.

Hence the series converges at only $x = 2$ and the radius of convergence is $R = 0$

$$2. \sum_{n=0}^{\infty} \frac{(x-3)^n}{n2^n}$$

Solution: Let $a_n = \frac{(x-3)^n}{n2^n}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} \left| \frac{x-3}{2} \right|$

By the Ratio Test the series converges when $\left| \frac{x-3}{2} \right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$.

In addition to this, we tested for convergence at the end points as follows:

At $x = 1$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is convergent by Alternating Series Test.

At $x = 5$, we get $\sum_{n=0}^{\infty} \frac{1}{n}$ which is divergent by P – Series Test.

Therefore, the interval of convergence is $(1, 5]$ and the radius of convergence is $\frac{5-1}{2} = 2$.



$$3. \sum_{n=0}^{\infty} \frac{(x-4)^n}{n^2 2^n}$$

Solution: Let $a_n = \frac{(x-4)^n}{n^2 2^n}$, then $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{x-4}{2} \right| = \left| \frac{x-4}{2} \right|$

By the Root Test, the series converges when $\left| \frac{x-4}{2} \right| < 1 \Rightarrow |x-4| < 2 \Rightarrow 2 < x < 6$.

At the end points we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=0}^{\infty} \frac{1}{n^2}$ which are both divergent.

Hence, the interval of convergence is $(2, 6)$ and the radius of convergence is 2.

$$4. \sum_{n=0}^{\infty} \frac{x^n}{(\ln n)^n}$$

Solution: Let $a_n = \frac{x^n}{(\ln n)^n}$, then $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{x}{\ln n} \right| = 0$

By the Root Test, the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

Exercises: Find the interval and radius of convergence of the following power series.

$$1. \sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{n 2^{n+3}} \quad 2) \sum_{n=0}^{\infty} n! (3x + 1)^n \quad 3) \sum_{n=0}^{\infty} \frac{5^n (x - 1)^{3n}}{(3n + 1)!}$$



Additional Exercises

The interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{3n+1}$$

(a) $(-1, 1)$

(b) $(1, 3)$

(c) $[1, 3]$

(d) $(1, 3]$

(e) $(-1, 3]$

The interval of convergence I and the radius of convergence R of the power series

$$\sum_{n=1}^{+\infty} \frac{(x-1)^n}{n \cdot 2^n}$$

- (a) $I = [-1, 3]$ and $R = 2$
- (b) $I = (-1, 3)$ and $R = 2$
- (c) $I = (-1, 1)$ and $R = 1$
- (d) $I = [-2, 2]$ and $R = 2$
- (e) $I = [-1, 3)$ and $R = 2$

The interval of convergence I and the radius of convergence R of the power series $\sum_{n=1}^{+\infty} (-1)^n \frac{x^n}{4^n \cdot n^3}$ are

(a) $I = [-3, 3)$, $R = 3$

(b) $I = (-4, 4]$, $R = 4$

(c) $I = (3, 4)$, $R = \frac{1}{2}$

(d) $I = [-4, 4]$, $R = 4$

(e) $I = (-4, 4)$, $R = 4$

If the power series $\sum_{n=0}^{+\infty} c_n(x+2)^n$ has a radius of convergence $R = 3$, then which one of the following is **TRUE**?

- (a) $\sum_{n=0}^{+\infty} \frac{c_n}{2^n}$ is divergent
- (b) $\sum_{n=0}^{+\infty} (-1)^n c_n 5^n$ is convergent
- (c) $\sum_{n=0}^{+\infty} c_n 2^n$ is divergent
- (d) $\sum_{n=0}^{+\infty} c_n 4^n$ is convergent
- (e) $\sum_{n=0}^{+\infty} c_n$ is convergent

The interval of convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n4^n}$ is

(a) $[-6, 2]$

(b) $(-6, 2]$

(c) $[-2, 2]$

(d) $[-6, 2)$

(e) $(-6, 2)$

The radius of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n n^p (x+1)^n, \text{ where } p \text{ is a constant,}$$

is

(a) $\frac{1}{2}$

(b) 2

(c) 1

(d) 4

(e) $\frac{1}{4}$

The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$ is

(a) $\left[-\frac{1}{2}, 0\right]$

(b) $\left[\frac{1}{4}, 1\right]$

(c) $\left(-\frac{1}{2}, 0\right)$

(d) $\left(\frac{1}{4}, 1\right]$

(e) $\left[-\frac{1}{2}, 0\right)$

The interval of convergence of the power series $\sum_{n=1}^{+\infty} \frac{(2x-3)^n}{n4^{2n}}$ is

(a) $\left[-\frac{13}{2}, \frac{19}{2}\right)$

(b) $\left(-\frac{13}{2}, \frac{19}{2}\right)$

(c) $\left(-\frac{13}{2}, \frac{19}{2}\right]$

(d) $\left[-\frac{13}{2}, \frac{19}{2}\right]$

(e) $\left[\frac{13}{2}, \frac{19}{2}\right)$

Theorem: Differentiation and Integration of Power Series

If the power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, the f is both differentiable and integrable on the interval $(a-R, a+R)$, and

$$1) f'(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n(x-a)^n \right) = \sum_{n=1}^{\infty} \frac{d}{dx} [c_n(x-a)^n] = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$$

$$2) \int f(x) dx = \int \left(\sum_{n=0}^{\infty} c_n(x-a)^n \right) dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

Example 1: If $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, show that $f'(x) = f(x)$.

Solution:
$$f'(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

Example: Find the power series representation of the following functions.

$$a) \ln(1+x^3) \quad b) \frac{x}{(2-x)^2} \quad c) \ln\left(\frac{1+x}{1-x}\right)$$

$$\text{Solution a): } \ln(1+x^3) = \int_0^x \frac{3x^2}{1+x^3} dx = \int_0^x 3x^2 \frac{1}{1-(-x^3)} dx = \int_0^x 3x^2 \sum_{n=0}^{\infty} (-x)^{3n} dx$$

$$= \int_0^x 3x^2 \sum_{n=0}^{\infty} (-1)^n (x)^{3n} dx = 3 \int_0^x \sum_{n=0}^{\infty} (-1)^n (x)^{3n+2} dx$$

$$= 3 \sum_{n=0}^{\infty} \int_0^x (-1)^n (x)^{3n+2} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{n+1}$$

$$\begin{aligned}
 \text{Solution b): } \frac{x}{(2-x)^2} &= x \frac{d}{dx} \left(\frac{1}{2-x} \right) = \frac{x}{2} \frac{d}{dx} \left(\frac{1}{1-\frac{x}{2}} \right) = \frac{x}{2} \frac{d}{dx} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n \text{ for } |x| < 2 = \frac{x}{2} \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{x}{2} \right)^n \\
 &= \frac{x}{2} \sum_{n=1}^{\infty} \frac{n}{2} \left(\frac{x}{2} \right)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2} \frac{x}{2} \left(\frac{x}{2} \right)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{2} \left(\frac{x}{2} \right)^n \text{ for } |x| < 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Solution c): } \ln\left(\frac{1+x}{1-x}\right) &= \int_0^x \frac{1}{1+x} dx + \int_0^x \frac{1}{1-x} dx = \int_0^x \sum_{n=0}^{\infty} (-x)^n dx + \int_0^x \sum_{n=0}^{\infty} x^n dx \text{ for } |x| < 1 \\
 &= \sum_{n=0}^{\infty} \int_0^x (-x)^n dx + \sum_{n=0}^{\infty} \int_0^x x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
 \end{aligned}$$

Problems: Find the power series representation of the following functions.

$$a) \frac{\tan^{-1}x}{x}$$

$$b) \frac{x+1}{x^2-5x+6}$$

Exercise: Evaluate the indefinite integral as a power series.

$$a) \int_0^x \frac{x}{1-x^2} dx$$

$$b) \int_0^x \frac{\ln(1-x)}{x} dx$$

Theorem (Taylor Series): Suppose that f has derivatives of all orders at a , then the Taylor Series

generated by f is the power series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

Note that: When $a = 0$, we obtain a special type of series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!}$$

which is known as Maclaurine series.

Example 1: Find the Taylor Series for the $f(x)$ centered at the given point.

a) $f(x) = 2 - 3x + 5x^2, a = 2.$ b) $f(x) = \ln x, a = 1$

Solution a):
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)(x-2)^n}{n!}$$

$$= \frac{f^{(0)}(2)(x-2)^0}{0!} + \frac{f^{(1)}(2)(x-2)^1}{1!} + \frac{f^{(2)}(2)(x-2)^2}{2!} + \frac{f^{(3)}(2)(x-2)^3}{3!}$$

$$+ \frac{f^{(4)}(2)(x-2)^4}{4!} + \dots = 16 + 17(x-2) + 10(x-2)^2$$

Solution b): $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, f^{(4)}(x) = -\frac{6}{x^4}, f^{(5)}(x) = \frac{24}{x^5}, \dots$

$$\Rightarrow f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \Rightarrow f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n}$$

Hence
$$f(x) = \sum_{n=1}^{\infty} \frac{\frac{(-1)^{n+1}(n-1)!}{2^n} f^{(n)}(2)(x-2)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$$

Example 2: Find the Maclaurine series for $f(x)$.

a) $f(x) = e^{kx}$ where k is any real number different from 0.

Solution: $f^{(n)}(x) = k^n e^{kx} \Rightarrow f^{(n)}(0) = k^n \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!}$

Note that: a) When $k = 1$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ b) for $m \in \mathbb{R}$, $x^m e^x = x^m \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+m}}{n!}$

$$b) f(x) = \cos x$$

$$\text{Solution: } f^{(n)}(0) = \begin{cases} 1, & \text{when } n = 0, 4, 8, \dots \\ 0, & \text{when } n = 1, 3, 5, \dots \\ -1, & \text{when } n = 2, 6, 10, \dots \end{cases}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\text{Note that: } \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

Exercises

1. Find the Taylor Series representations for the following functions centered at the indicated points.

$$a) f(x) = 5 + 2x - 3x^2, a = 1. \quad b) f(x) = x \ln x, a = 2 \quad c) f(x) = x^2 e^x$$

2. Find the Maclaurine Series for $f(x)$.

$$a) f(x) = \frac{e^x}{x^3} \quad b) f(x) = \sin 3x \quad c) f(x) = 1 + 2\sin^2 x$$

The Fourier Series of a function

Let $f(x)$ is periodic function of period $2T$ on the interval $-T \leq x \leq T$, then $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right) \rightarrow 1$$

where the fourier coefficients a_0, a_n and b_n are

$$a_0 = \frac{1}{T} \int_{-T}^T f(x) dx \quad \rightarrow 2$$

$$a_n = \frac{1}{T} \int_{-T}^T f(x) \cos \frac{n\pi x}{T} dx \quad \rightarrow 3$$

$$b_n = \frac{1}{T} \int_{-T}^T f(x) \sin \frac{n\pi x}{T} dx \quad \rightarrow 4 \quad n = 1, 2, 3, \dots$$

Note.1. The Fourier series (1) converges to $f(x)$ at all points where $f(x)$ is continuous. ---

Note.2. The Fourier series (1) converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at all points of jump discontinuity of $f(x)$

Note.3. Periodic Function: Any function which repeats itself after a certain period is known as periodic function.

Examples :

$$\sin(x + 2\pi) = \sin x \quad , \text{Period } 2\pi$$

$$\cos(x + 2\pi) = \cos x \quad , \text{Period } 2\pi$$

$$f(x + 2T) = f(x) \quad , \text{Period } 2T$$

$$\sin(x + 2n\pi) = \sin x \quad , \text{Period } 2\pi$$

$$f(x) = f(x + P) = f(x + 2P) = f(x + 3P) = \dots, \quad \text{Period } P.$$

Special Case

If $f(x)$ is a periodic function of period 2π on interval $[-\pi, \pi]$, the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow 5$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \rightarrow 6$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \rightarrow 7$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \rightarrow 8 \quad n = 1, 2, 3, \dots$$

Example: 1.

Find the Fourier series of the function

$$f(x) = x \quad , \quad -\pi < x < \pi \quad , \quad f(x + 2\pi) = f(x)$$

Solution:

Fouries series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow 9$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dn = \frac{1}{\pi} \cdot \frac{x^2}{2} \Big|_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^2}{2} \right] = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx = \frac{1}{\pi} \left[x \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right] \Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\pi \frac{\sin n\pi}{n} - \pi \frac{\sin(-n\pi)}{n} \right] \\ &\quad + \frac{1}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{\cos(-n\pi)}{n^2} \right] \\ &= \frac{1}{\pi} [0 - 0] + \frac{1}{\pi} [0] = 0 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{1}{\pi} \left[-x \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[-x \frac{\cos nx}{n} \right]_{-\pi}^{\pi} + 0
 \end{aligned}$$

$$b_n = \frac{1}{\pi} \left[-2\pi \frac{(-1)^n}{n} \right] = \frac{-2}{n} (-1)^n$$

Substituting values of a_0 , a_n and b_n in equation 9

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \text{ is Fourier series of } f(x) = x, -\pi \leq x \leq \pi$$

Example:2

Find the Fourier senie of the function

$$f(x) = x^2 \quad , \quad -1 \leq x \leq 1 \quad , \quad f(x+2) = f(x)$$

Solution:

$$2T = 2 \Rightarrow T = 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

$$\text{where } a_0 = \int_{-1}^1 x^2 dx$$

$$a_n = \int_{-1}^1 x^2 \cos n\pi x dx$$

$$b_n = \int_{-1}^1 x^2 \sin n\pi x dx$$

$$a_0 = \frac{x^3}{3} \Big|_{-1}^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\begin{aligned} a_n &= \int_{-1}^1 x^2 \cos n\pi x dx = \left[x^2 \frac{\sin n\pi x}{n\pi} + 2x \frac{\cos n\pi x}{n^2 \pi^2} - 2 \frac{\sin n\pi x}{n^3 \pi^3} \right]_{-1}^1 \\ &= \frac{4 \cos n\pi}{n^2 \pi^2} = \frac{4(-1)^n}{n^2 \pi^2}, \quad n = 1, 2, 3 \end{aligned}$$

$$b_n = \int_{-1}^1 x^2 \sin n\pi x dx = \left[-x^2 \frac{\cos n\pi x}{n\pi} + 2x \frac{\sin n\pi x}{n^2 \pi^2} + 2 \frac{\cos n\pi x}{n^3 \pi^3} \right]_{-1}^1 = 0$$

The Fouries series of $f(x) = x^2$, $-1 < x < 1$ is

$$\begin{aligned} f(x) &\approx \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2 \pi^2} \cos n\pi x \\ &\approx \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x \end{aligned}$$

Example:3.

Find the Fourier series for the square wave

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

Solution:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cdot dx + \int_0^{\pi} 1 \cdot dx \right] = \frac{1}{\pi} \int_0^{\pi} dx = \frac{1}{\pi} x \Big|_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = \frac{1}{\pi} \frac{\sin nx}{n} \Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = -\frac{1}{\pi} \frac{\cos nx}{n} \Big|_0^{\pi} \\ = \frac{-[(-1)^n - 1]}{n\pi}$$

The Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases} \quad n = 1, 2, 3, \dots$

is

$$f(x) \approx \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-[(-1)^n - 1]}{n\pi} \sin nx \\ \approx \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n\pi} \sin nx$$

Example: 4.

Find the Fourier series of

$$f(x) = e^{-x} \quad , \quad -\pi < x < \pi$$

Solution:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} dx = \frac{1}{\pi} (-e^{-x}) \Big|_{-\pi}^{\pi} = \frac{1}{\pi} [-e^{-\pi} + e^{\pi}]$$

$$a_0 = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[\frac{e^{-\pi}}{1+n^2} (-\cos n\pi + n \sin n\pi) - \frac{e^{\pi}}{1+n^2} (-\cos(-n\pi) + n \sin(-n\pi)) \right] \\
&= \frac{1}{\pi(1+n^2)} \left[e^{-\pi} \cdot [-(-1)^n] - e^{\pi} (-(-1)^n) \right] \\
&= \frac{(-1)^n}{\pi(1+n^2)} [e^{\pi} - e^{-\pi}] \quad n=1,2,3,\dots
\end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\sin nx + n \cos nx) \right]_{-\pi}^{\pi} \\
 &= \frac{-n}{\pi(1+n^2)} \left[e^{-x} \cos nx \right]_{-\pi}^{\pi} \\
 &= \frac{-n}{\pi(1+n^2)} \left[e^{-\pi} \cos n\pi - e^{\pi} \cos(-n\pi) \right] \\
 &= \frac{-n}{\pi(1+n^2)} \left[e^{-\pi} (-1)^n - e^{\pi} (-1)^n \right] \\
 &= \frac{n(-1)^n}{\pi(1+n^2)} \left[e^{\pi} - e^{-\pi} \right] \quad n = 1, 2, 3, \dots
 \end{aligned}$$

$$\begin{aligned}
 f(x) &\approx \frac{1}{2} \left[\frac{e^{\pi} - e^{-\pi}}{\pi} \right] \\
 &+ \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\pi(1+n^2)} \left[e^{\pi} - e^{-\pi} \right] \cos nx + \frac{n(-1)^n}{\pi(1+n^2)} \left[e^{\pi} - e^{-\pi} \right] \sin nx \right] \dots
 \end{aligned}$$



Example:5.

Find the Fourier series of $f(x) = |x|$, $-\pi < x < \pi$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = |x| = \begin{cases} x & \text{if } 0 \leq x < \pi \\ -x & \text{if } -\pi < x < 0 \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{-\pi}^0 (-x) dx \right] \\ &= \frac{1}{\pi} \left[\frac{x^2}{2} \Big|_0^{\pi} + \left(\frac{-x^2}{2} \right) \Big|_{-\pi}^0 \right] \\ &= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^2}{2} \right] = \frac{1}{\pi} \times \pi^2 = \pi \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \left[\frac{1}{\pi} \int_0^{\pi} x \cos nx dx - \frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx \right]$$

$$= \frac{2}{\pi n^2} [\cos n\pi - 1] = \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx dx = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx - \frac{1}{\pi} \int_{-\pi}^0 x \sin nx dx = 0$$

The Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ is

$$f(x) \approx \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2} \cos nx.$$

Note (i) $f(x) = x + |x|$ on $[-\pi, \pi]$

$$\begin{aligned} f(x) &= \begin{cases} x - x & \text{if } -\pi < x \leq 0 \\ x - (-x) & \text{if } 0 < x < \pi \end{cases} \\ &= \begin{cases} 0 & \text{if } -\pi < x \leq 0 \\ 2x & \text{if } 0 < x < \pi \end{cases} \end{aligned}$$

(ii) $f(x) = 1 - |x|$, $-2 < x < 2$

$$f(x) = \begin{cases} 1 + x & \text{if } -2 < x \leq 0 \\ 1 - x & \text{if } 0 < x < 2 \end{cases}$$

Even and Odd functions

Even function

A function $f(x)$ is an even function, if $f(-x) = f(x)$.

Odd function

A function $f(x)$ is an odd function, if $f(-x) = -f(x)$.

Example: 1.

Test the nature of function

(i) $f(x) = x^2$

(ii) $f(x) = x^3$

(iii) $f(x) = \cos x$

Solution:

(i) $f(x) = x^2$

$$f(-x) = (-x)^2 = x^2 \Rightarrow f(x) = x^2 \quad \text{is even.}$$

(ii) $f(x) = x^3$

$$f(-x) = (-x)^3 = -x^3 \Rightarrow f(x) = x^3 \quad \text{is odd.}$$

(iii) $f(x) = \cos x$

$$f(-x) = \cos(-x) = \cos x \Rightarrow f(x) = \cos x \quad \text{is even.}$$

<i>Function</i>	<i>Classification</i>
x^2	even
x^3	odd
$\cos x$	even
$\sin x$	odd
$ x $	even
$3x+2$	neither
$x + x^2 \sin x$	odd
$\sin x + \cos x$	neither

Properties of even and odd function

i) $(\text{even}) \times (\text{even}) = \text{even}$

ii) $(\text{odd}) \times (\text{odd}) = \text{even}$

iii) $(\text{even}) \times (\text{odd}) = \text{odd}$

(iv) If $f(x)$ is even , then $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

(v) If $f(x)$ is odd , then $\int_{-a}^a f(x)dx = 0$

Properties of even and odd function

- (vi) The Fourier series of an even function consists only of cosine terms with *coefficients*

$$a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{n\pi x}{T} dx, \quad n = 0, 1, 2, \dots$$

- (vii) The Fourier series of an odd function consists only of sine terms with coefficients:

$$b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T} dx, \quad n = 1, 2, 3, \dots$$

Properties of even and odd function

(viii) Fourier Series : If $f(x)$ is even function on $[-T, T]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{T} \quad \rightarrow 1$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{n\pi x}{T} dx, \quad n = 0, 1, 2, \dots \rightarrow 2$$

(ix) Fourier Series: If $f(x)$ is an odd function on $[-T, T]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{T} \quad \rightarrow 3$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T} dx \quad n = 1, 2, 3, \dots \rightarrow 4$$

Solution:

$$f(x) = |x|$$

$$f(-x) = |-x| = |x| \Rightarrow f(x) \text{ is even function} \Rightarrow b_n = 0.$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \cdot \frac{x^2}{2} \Big|_0^{\pi} = \pi$$

Example:1

Find the Fourier series of

$$f(x) = |x| \text{ on } (-\pi, \pi)$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{\cos n\pi - \cos 0}{n^2} \right] \\ &= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right] \end{aligned}$$

$$n = 1, 2, 3, \dots$$

The Fourier series of $|x|$ on $-\pi < x < \pi$ is

$$f(x) \approx \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right] \cos nx$$

Example:2

Find the Fourier series of the function

$$f(x) = 2 - x^2, \quad -2 < x < 2$$

Solution:

$$f(-x) = 2 - (-x)^2 = 2 - x^2 = f(x) \Rightarrow f(x) \text{ is even function}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} x$$

$$a_0 = \frac{2}{2} \int_0^2 (2 - x^2) dx = 2x - \frac{x^3}{3} \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 (2 - x^2) \cos \frac{n\pi}{2} x dx \\ &= \left[(2 - x^2) \frac{2}{n\pi} \sin \frac{n\pi}{2} x - \frac{4}{n^2 \pi^2} \cdot 2x \left(\cos \frac{n\pi}{2} \right) + \frac{16}{n^3 \pi^3} \sin \frac{4\pi}{2} x \right]_{-2}^2 \\ &= -\frac{16}{n^2 \pi^2} (\cos \pi) = \frac{-16}{n^2 \pi^2} (-1)^n, \quad n = 1, 2, 3, \dots \end{aligned}$$

Fourier series of $f(x) = 2 - x^2$ on $[-2, 2]$ is

$$\begin{aligned} f(x) &\approx \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-16}{n^2 \pi^2} (-1)^n \cos \frac{n\pi}{2} x \\ &= \frac{2}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos \frac{n\pi}{2} x \end{aligned}$$