Chapter 7

Polynomial Functions

What You'll Learn

- **Lessons 7-1 and 7-3** Evaluate polynomial functions and solve polynomial equations.
- **Lessons 7-2 and 7-9** Graph polynomial and square root functions.
- **Lessons 7-4, 7-5, and 7-6** Find factors and zeros of polynomial functions.
- **Lesson 7-7** Find the composition of functions.
- **Lesson 7-8** Determine the inverses of functions or relations.

Key Vocabulary

- polynomial function (p. 347)
- synthetic substitution (p. 365)
- Fundamental Theorem of Algebra (p. 371)
- composition of functions (p. 384)
- inverse function (p. 391)

Why It's Important

According to the Fundamental Theorem of Algebra, every polynomial equation has at least one root. Sometimes the roots have real-world meaning. Many real-world situations that cannot be modeled using a linear function can be approximated using a polynomial function.

You will learn how the power generated by a windmill can be modeled by a polynomial function in Lesson 7-1.

Getting Started

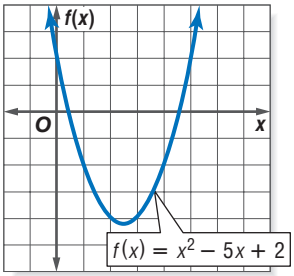
Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 7.

For Lesson 7-2 — Solve Equations by Graphing

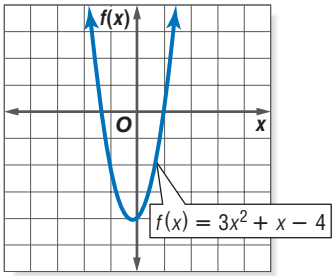
Use the related graph of each equation to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.

(For review, see Lesson 6-2.)

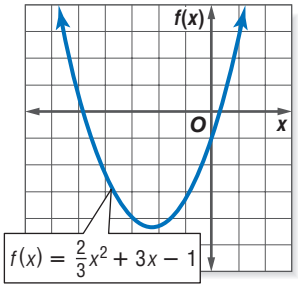
1. $x^2 - 5x + 2 = 0$



2. $3x^2 + x - 4 = 0$



3. $\frac{2}{3}x^2 + 3x - 1 = 0$



For Lesson 7-3 — Quadratic Formula

Solve each equation. *(For review, see Lesson 6-5.)*

4. $x^2 - 17x + 60 = 0$ **5.** $14x^2 + 23x + 3 = 0$ **6.** $2x^2 + 5x + 1 = 0$

For Lessons 7-4 through 7-6 — Synthetic Division

Simplify each expression using synthetic division. *(For review, see Lesson 5-3.)*

7. $(3x^2 - 14x - 24) \div (x - 6)$ **8.** $(a^2 - 2a - 30) \div (a + 7)$

For Lessons 7-1 and 7-7 — Evaluating Functions

Find each value if $f(x) = 4x - 7$ and $g(x) = 2x^2 - 3x + 1$. *(For review, see Lesson 2-1.)*

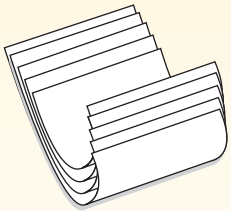
9. $f(-3)$ **10.** $g(2a)$ **11.** $f(4b^2) + g(b)$

FOLDABLES™ Study Organizer

Polynomial Functions Make this Foldable to help you organize your notes. Begin with five sheets of plain $8\frac{1}{2}$" by 11" paper.

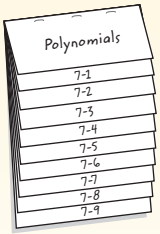
Step 1 Stack and Fold

Stack sheets of paper with edges $\frac{3}{4}$-inch apart. Fold up the the bottom edges to create equal tabs.



Step 2 Staple and Label

Staple along the fold. Label the tabs with lesson numbers.



Reading and Writing As you read and study the chapter, use each page to write notes and examples.

7-1 Polynomial Functions

What You'll Learn

- Evaluate polynomial functions.
- Identify general shapes of graphs of polynomial functions.

Vocabulary

- polynomial in one variable
- degree of a polynomial
- leading coefficients
- polynomial function
- end behavior

Where are polynomial functions found in nature?

If you look at a cross section of a honeycomb, you see a pattern of hexagons. This pattern has one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function $f(r) = 3r^2 - 3r + 1$, where r is the number of rings and $f(r)$ is the number of hexagons.

POLYNOMIAL FUNCTIONS Recall that a polynomial is a monomial or a sum of monomials. The expression $3r^2 - 3r + 1$ is a **polynomial in one variable** since it only contains one variable, r.

Key Concept — Polynomial in One Variable

- **Words** A polynomial of degree n in one variable x is an expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$, represent real numbers, a_0 is not zero, and n represents a nonnegative integer.
- **Examples** $3x^5 + 2x^4 - 5x^3 + x^2 + 1$

  $n = 5$, $a_0 = 3$, $a_1 = 2$, $a_2 = -5$, $a_3 = 1$, $a_4 = 0$, and $a_5 = 1$

The **degree of a polynomial** in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial	Expression	Degree	Leading Coefficient
Constant	9	0	9
Linear	$x - 2$	1	1
Quadratic	$3x^2 + 4x - 5$	2	3
Cubic	$4x^3 - 6$	3	4
General	$a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$	n	a_0

Example 1 Find Degrees and Leading Coefficients

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $7x^4 + 5x^2 + x - 9$

This is a polynomial in one variable.

The degree is 4, and the leading coefficient is 7.

b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, x and y.

c. $7x^6 - 4x^3 + \frac{1}{x}$

This is not a polynomial. The term $\frac{1}{x}$ cannot be written in the form x^n, where n is a nonnegative integer.

d. $\frac{1}{2}x^2 + 2x^3 - x^5$

Rewrite the expression so the powers of x are in decreasing order.

$$-x^5 + 2x^3 + \frac{1}{2}x^2$$

This is a polynomial in one variable with degree of 5 and leading coefficient of -1.

Study Tip

Power Function
A common type of function is a **power function**, which has an equation in the form $f(x) = ax^b$, where a and b are real numbers. When b is a positive integer, $f(x) = ax^b$ is a polynomial function.

A polynomial equation used to represent a function is called a **polynomial function**. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

Key Concept — *Definition of a Polynomial Function*

- **Words** A polynomial function of degree n can be described by an equation of the form $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, a_2, \ldots, a_n$, represent real numbers, a_0 is not zero, and n represents a nonnegative integer.
- **Examples** $f(x) = 4x^2 - 3x + 2$
  $n = 2, a_0 = 4, a_1 = -3, a_2 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

Example 2 Evaluate a Polynomial Function

NATURE **Refer to the application at the beginning of the lesson.**

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1$, 2, and 3.

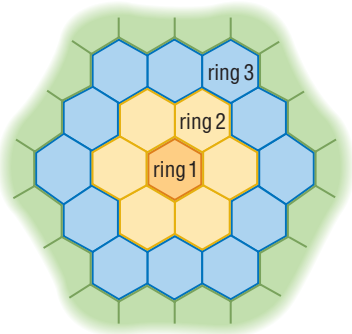
Find the values of $f(1)$, $f(2)$, and $f(3)$.

$f(r) = 3r^2 - 3r + 1$
$f(1) = 3(1)^2 - 3(1) + 1$
$= 3 - 3 + 1$ or 1

$f(r) = 3r^2 - 3r + 1$
$f(2) = 3(2)^2 - 3(2) + 1$
$= 12 - 6 + 1$ or 7

$f(r) = 3r^2 - 3r + 1$
$f(3) = 3(3)^2 - 3(3) + 1$
$= 27 - 9 + 1$ or 19

From the information given, you know the number of hexagons in the first ring is 1, the number of hexagons in the second ring is 6, and the number of hexagons in the third ring is 12. So, the total number of hexagons with one ring is 1, two rings is $6 + 1$ or 7, and three rings is $12 + 6 + 1$ or 19. These match the functional values for $r = 1$, 2, and 3, respectively.



Rings of a Honeycomb

b. Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$ — Original function
$f(12) = 3(12)^2 - 3(12) + 1$ — Replace r with 12.
$= 432 - 36 + 1$ or 397 — Simplify.

You can also evaluate functions for variables and algebraic expressions.

Example 3 Functional Values of Variables

a. Find $p(a^2)$ if $p(x) = x^3 + 4x^2 - 5x$.

$p(x) = x^3 + 4x^2 - 5x$ Original function

$p(a^2) = (a^2)^3 + 4(a^2)^2 - 5(a^2)$ Replace x with a^2.

$= a^6 + 4a^4 - 5a^2$ Property of powers

b. Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

To evaluate $q(a + 1)$, replace x in $q(x)$ with $a + 1$.

$q(x) = x^2 + 3x + 4$ Original function

$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$ Replace x with $a + 1$.

$= a^2 + 2a + 1 + 3a + 3 + 4$ Evaluate $(a + 1)^2$ and $3(a + 1)$.

$= a^2 + 5a + 8$ Simplify.

To evaluate $2q(a)$, replace x with a in $q(x)$, then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$ Original function

$2q(a) = 2(a^2 + 3a + 4)$ Replace x with a.

$= 2a^2 + 6a + 8$ Distributive Property

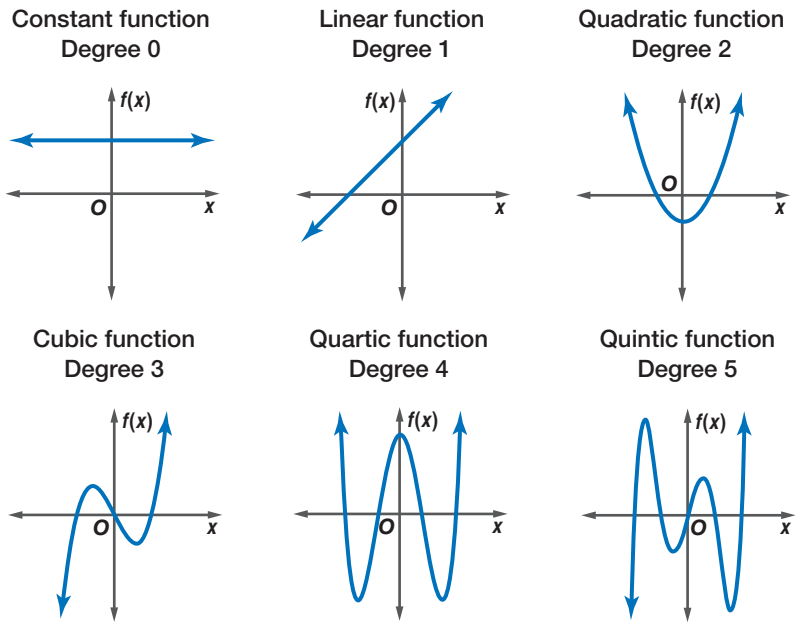
Now evaluate $q(a + 1) - 2q(a)$.

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$ Replace $q(a + 1)$ and $2q(a)$ with evaluated expressions.

$= a^2 + 5a + 8 - 2a^2 - 6a - 8$

$= -a^2 - a$ Simplify.

GRAPHS OF POLYNOMIAL FUNCTIONS The general shapes of the graphs of several polynomial functions are shown below. These graphs show the *maximum* number of times the graph of each type of polynomial may intersect the x-axis. Recall that the x-coordinate of the point at which the graph intersects the x-axis is called a *zero* of a function. How does the degree compare to the maximum number of real zeros?



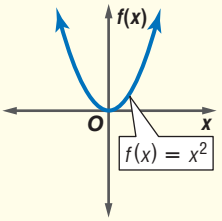
Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph's end behavior.

The **end behavior** is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). This is represented as $x \to +\infty$ and $x \to -\infty$, respectively. $x \to +\infty$ is read *x approaches positive infinity.*

Concept Summary — *End Behavior of a Polynomial Function*

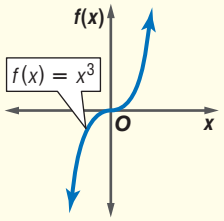
Degree: even
Leading Coefficient: positive
End Behavior:

$f(x) \to +\infty$ as $x \to -\infty$; $f(x) \to +\infty$ as $x \to +\infty$



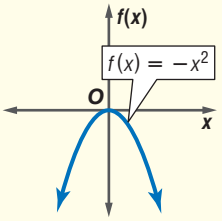
Degree: odd
Leading Coefficient: positive
End Behavior:

$f(x) \to +\infty$ as $x \to +\infty$



$f(x) \to -\infty$ as $x \to -\infty$

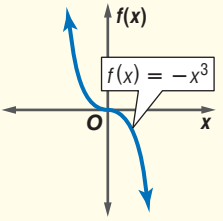
Degree: even
Leading Coefficient: negative
End Behavior:



$f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to -\infty$ as $x \to +\infty$

Degree: odd
Leading Coefficient: negative
End Behavior:

$f(x) \to +\infty$ as $x \to -\infty$

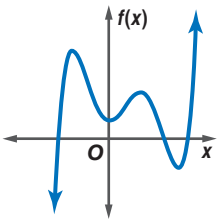


$f(x) \to -\infty$ as $x \to +\infty$

The graph of an even-degree function may or may not intersect the x-axis, depending on its location in the coordinate plane. If it intersects the x-axis in two places, the function has two real zeros. If it does not intersect the x-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the x-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the x-axis at least once, and thus the function always has at least one real zero.

Study Tip

Number of Zeros
The number of zeros of an odd-degree function may be less than the maximum by a multiple of 2. For example, the graph of a quintic function may only cross the x-axis 3 times.



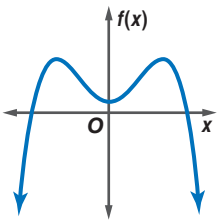
The same is true for an even-degree function. One exception is when the graph of $f(x)$ touches the x-axis.

Example 4 Graphs of Polynomial Functions

For each graph,

- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

a.



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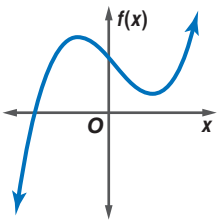
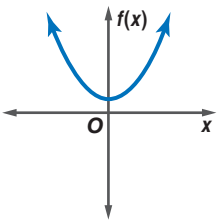


c.



a.
- $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The graph intersects the x-axis at two points, so the function has two real zeros.

b.
- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an odd-degree polynomial function.
- The graph has one real zero.

c.
- $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- This graph does not intersect the x-axis, so the function has no real zeros.

Check for Understanding

Concept Check

1. **Explain** why a constant polynomial such as $f(x) = 4$ has degree 0 and a linear polynomial such as $f(x) = x + 5$ has degree 1.
2. **Describe** the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive.
3. **OPEN ENDED** Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots.
4. **Tell** whether the following statement is *always*, *sometimes* or *never* true. Explain.
   A polynomial function that has four real roots is a fourth-degree polynomial.

Guided Practice

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

5. $5x^6 - 8x^2$
6. $2b + 4b^3 - 3b^5 - 7$

Find $p(3)$ and $p(-1)$ for each function.

7. $p(x) = -x^3 + x^2 - x$
8. $p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$

If $p(x) = 2x^3 + 6x - 12$ and $q(x) = 5x^2 + 4$, find each value.

9. $p(a^3)$
10. $5[q(2a)]$
11. $3p(a) - q(a + 1)$

For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

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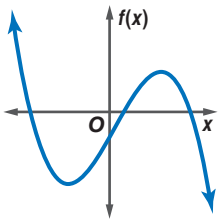

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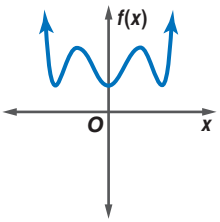

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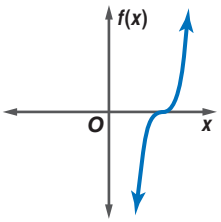

Application

15. **BIOLOGY** The intensity of light emitted by a firefly can be determined by $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is temperature in degrees Celsius and $L(t)$ is light intensity in lumens. If the temperature is 30°C, find the light intensity.

Practice and Apply

Homework Help

For Exercises	See Examples
16–21	1
22–29, 45	2
30–38	3
39–44, 46–48	4

Extra Practice
See page 842.

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

16. $7 - x$
17. $(a + 1)(a^2 - 4)$
18. $a^2 + 2ab + b^2$
19. $6x^4 + 3x^2 + 4x - 8$
20. $7 + 3x^2 - 5x^3 + 6x^2 - 2x$
21. $c^2 + c - \frac{1}{c}$

Find $p(4)$ and $p(-2)$ for each function.

22. $p(x) = 2 - x$
23. $p(x) = x^2 - 3x + 8$
24. $p(x) = 2x^3 - x^2 + 5x - 7$
25. $p(x) = x^5 - x^2$
26. $p(x) = x^4 - 7x^3 + 8x - 6$
27. $p(x) = 7x^2 - 9x + 10$
28. $p(x) = \frac{1}{2}x^4 - 2x^2 + 4$
29. $p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

30. $r(3a)$ **31.** $4p(a)$ **32.** $p(a^2)$

33. $p(2a^3)$ **34.** $r(x + 1)$ **35.** $p(x^2 + 3)$

36. $2[p(x + 4)]$ **37.** $r(x + 1) - r(x^2)$ **38.** $3[p(x^2 - 1)] + 4p(x)$

For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

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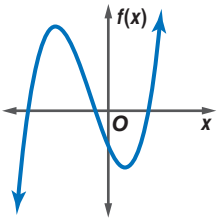

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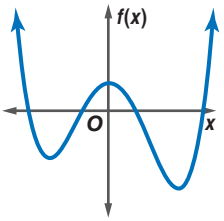

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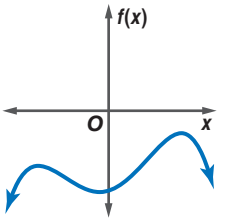

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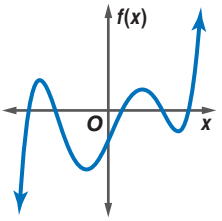

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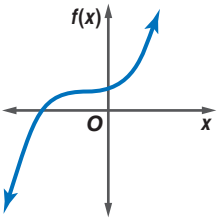

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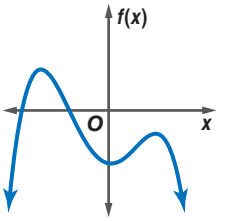
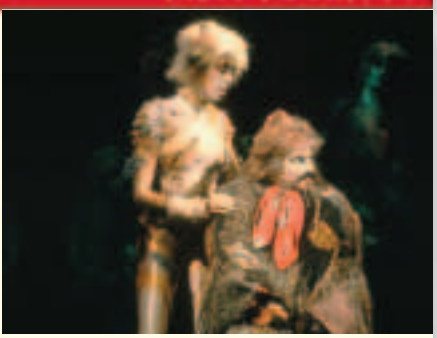


More About...



Theater

In 1997, *Cats* surpassed *A Chorus Line* as the longest-running Broadway show.

Source: www.newsherald.com

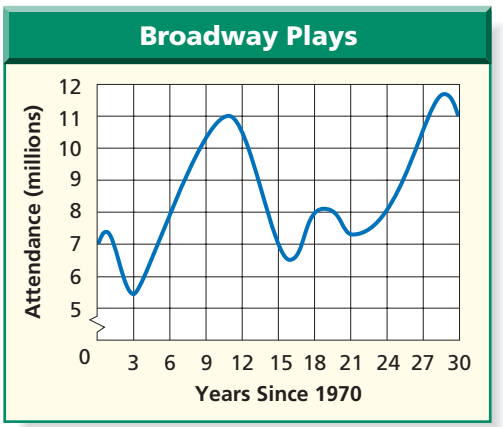
45. ENERGY The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour.

THEATER For Exercises 46–48, use the graph that models the attendance to Broadway plays (in millions) from 1970–2000.

46. Is the graph an odd-degree or even-degree function?

47. Discuss the end behavior of the graph.

48. Do you think attendance at Broadway plays will increase or decrease after 2000? Explain your reasoning.



CRITICAL THINKING For Exercises 49–52, use the following information.
The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at (5, 15).

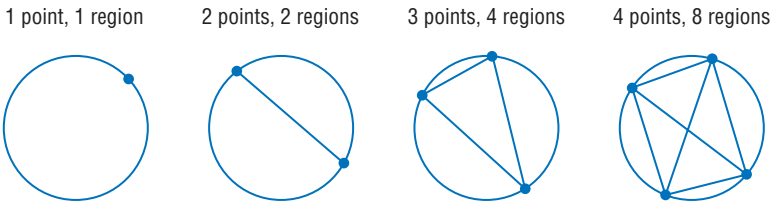
49. Find the value of a.

50. For what value(s) of x will $f(x) = 0$?

51. Rewrite the function as a cubic function.

52. Sketch the graph of the function.

PATTERNS **For Exercises 53–55, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.**



53. The maximum number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$. What is the degree of this polynomial function?

54. Find the maximum number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.

55. How many points would you have to connect to form 99 regions?

56. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Where are polynomial functions found in nature?

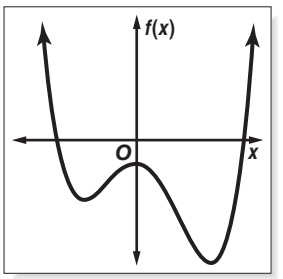
Include the following in your answer:

- an explanation of how you could use the equation to find the number of hexagons in the tenth ring, and
- any other examples of patterns found in nature that might be modeled by a polynomial equation.

Standardized Test Practice

57. The figure at the right shows the graph of the polynomial function $f(x)$. Which of the following could be the degree of $f(x)$?

Ⓐ 2 Ⓑ 3 Ⓒ 4 Ⓓ 5



58. If $\frac{1}{2}x^2 - 6x + 2 = 0$, then x could equal which of the following?

Ⓐ -1.84 Ⓑ -0.81 Ⓒ 0.34 Ⓓ 2.37

Maintain Your Skills

Mixed Review

Solve each inequality algebraically. *(Lesson 6-7)*

59. $x^2 - 8x + 12 < 0$ **60.** $x^2 + 2x - 86 \geq -23$ **61.** $15x^2 + 3x - 12 \leq 0$

Graph each function. *(Lesson 6-6)*

62. $y = -2(x - 2)^2 + 3$ **63.** $y = \frac{1}{3}(x + 5)^2 - 1$ **64.** $y = \frac{1}{2}x^2 + x + \frac{3}{2}$

Solve each equation by completing the square. *(Lesson 6-4)*

65. $x^2 - 8x - 2 = 0$ **66.** $x^2 + \frac{1}{3}x - \frac{35}{36} = 0$

67. **BUSINESS** Becca is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by p. Marty's salary is \$23,450 now. Write a polynomial to represent Marty's salary after one year and another to represent Marty's salary after three years. Assume that the rate of increase will be the same for each of the three years. *(Lesson 5-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Graph each equation by making a table of values.**

*(To review **graphing quadratic functions**, see Lesson 6-1.)*

68. $y = x^2 + 4$ **69.** $y = -x^2 + 6x - 5$ **70.** $y = \frac{1}{2}x^2 + 2x - 6$

7-2 Graphing Polynomial Functions

What You'll Learn

- Graph polynomial functions and locate their real zeros.
- Find the maxima and minima of polynomial functions.

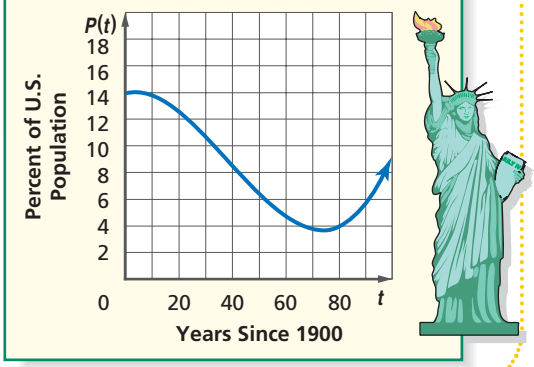
Vocabulary

- Location Principle
- relative maximum
- relative minimum

How can graphs of polynomial functions show trends in data?

The percent of the United States population that was foreign-born since 1900 can be modeled by $P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14$, where $t = 0$ in 1900. Notice that the graph is decreasing from $t = 5$ to $t = 75$ and then it begins to increase. The points at $t = 5$ and $t = 75$ are turning points in the graph.

Foreign-Born Population



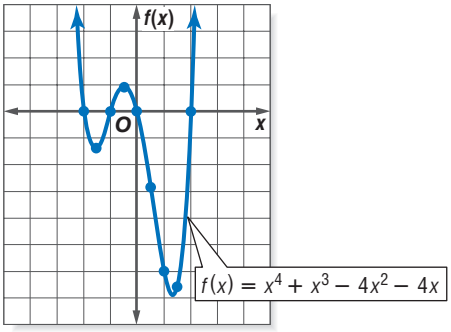
GRAPH POLYNOMIAL FUNCTIONS To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

Example 1 Graph a Polynomial Function

Graph $f(x) = x^4 + x^3 - 4x^2 - 4x$ by making a table of values.

x	$f(x)$
−2.5	≈ 8.4
−2.0	0.0
−1.5	≈ −1.3
−1.0	0.0
−0.5	≈ 0.9

x	$f(x)$
0.0	0.0
0.5	≈ −2.8
1.0	−6.0
1.5	≈ −6.6
2.0	0.0



This is an even-degree polynomial with a positive leading coefficient, so $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$. Notice that the graph intersects the x-axis at four points, indicating there are four real zeros of this function.

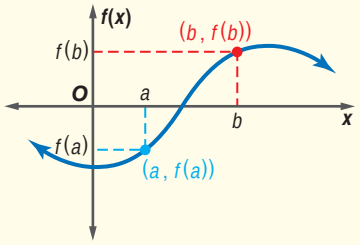
Study Tip

Graphing Polynomial Functions
To graph polynomial functions it will often be necessary to include x values that are not integers.

In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, the graph of a polynomial function will cross the x-axis somewhere between pairs of x values at which the corresponding $f(x)$ values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x values. This property for locating zeros is called the **Location Principle**.

Key Concept — Location Principle

- **Words** Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.
- **Model**

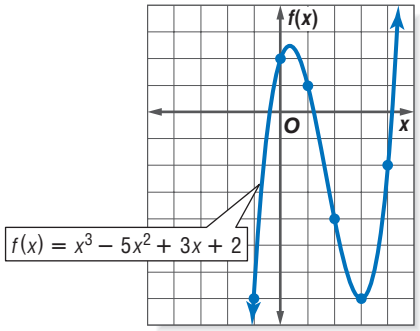


Example 2 Locate Zeros of a Function

Determine consecutive values of x between which each real zero of the function $f(x) = x^3 - 5x^2 + 3x + 2$ is located. Then draw the graph.

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 1, 2, or 3 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$	
−2	−32	
−1	−7	change in signs
0	2	
1	1	change in signs
2	−4	
3	−7	
4	−2	change in signs
5	17	

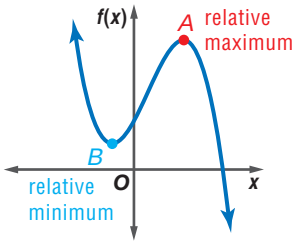


The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$.

MAXIMUM AND MINIMUM POINTS

The graph at the right shows the shape of a general third-degree polynomial function.

Point A on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. These points are often referred to as *turning points*. The graph of a polynomial function of degree n has at most $n - 1$ turning points.



Study Tip

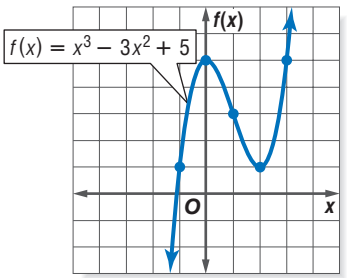
Reading Math
The plurals of maximum and minimum are *maxima* and *minima*.

Example 3 Maximum and Minimum Points

Graph $f(x) = x^3 - 3x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

Make a table of values and graph the equation.

x	$f(x)$	
−2	−15	zero between $x = -2$ and $x = -1$
−1	1	
0	5	← indicates a relative maximum
1	3	
2	1	← indicates a relative minimum
3	5	



Look at the table of values and the graph.

- The values of $f(x)$ change signs between $x = -2$ and $x = -1$, indicating a zero of the function.
- The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.
- The value of $f(x)$ at $x = 2$ is less than the surrounding points, so it is a relative minimum.

The graph of a polynomial function can reveal trends in real-world data.

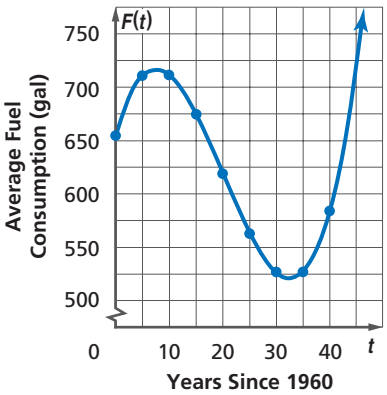
Example 4 Graph a Polynomial Model

ENERGY **The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation $F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654$, where t is the number of years since 1960.**

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

t	$F(t)$
0	654
5	710.88
10	711.5
15	674.63
20	619
25	563.38
30	526.5
35	527.13
40	584



b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as t increases, $F(t)$ increases.

c. What trends in fuel consumption does the graph suggest?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise.

More About . . .

Energy

Gasoline and diesel fuels are the most familiar transportation fuels in this country, but other energy sources are available, including ethanol, a grain alcohol that can be produced from corn or other crops.

Source: U.S. Environmental Protection Agency

A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

Graphing Calculator Investigation

Maximum and Minimum Points

You can use a TI-83 Plus to find the coordinates of relative maxima and relative minima. Enter the polynomial function in the Y= list and graph the function. Make sure that all the turning points are visible in the viewing window. Find the coordinates of the minimum and maximum points, respectively.

KEYSTROKES: *Refer to page 293 to review finding maxima and minima.*

(continued on the next page)

Think and Discuss

1. Graph $f(x) = x^3 - 3x^2 + 4$. Estimate the x-coordinates of the relative maximum and relative minimum points from the graph.
2. Use the maximum and minimum options from the CALC menu to find the exact coordinates of these points. You will need to use the arrow keys to select points to the left and to the right of the point.
3. Graph $f(x) = \frac{1}{2}x^4 - 4x^3 + 7x^2 - 8$. How many relative maximum and relative minimum points does the graph contain? What are the coordinates?

Check for Understanding

Concept Check

1. **Explain** the Location Principle in your own words.
2. **State** the number of turning points of the graph of a fifth-degree polynomial if it has five distinct real zeros.
3. **OPEN ENDED** Sketch a graph of a function that has one relative maximum point and two relative minimum points.

Guided Practice

Graph each polynomial function by making a table of values.

4. $f(x) = x^3 - x^2 - 4x + 4$
5. $f(x) = x^4 - 7x^2 + x + 5$

Determine consecutive values of x between which each real zero of each function is located. Then draw the graph.

6. $f(x) = x^3 - x^2 + 1$
7. $f(x) = x^4 - 4x^2 + 2$

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur.

8. $f(x) = x^3 + 2x^2 - 3x - 5$
9. $f(x) = x^4 - 8x^2 + 10$

Application

CABLE TV **For Exercises 10–12, use the following information.**
The number of cable TV systems after 1985 can be modeled by the function $C(t) = -43.2t^2 + 1343t + 790$, where t represents the number of years since 1985.

10. Graph this equation for the years 1985 to 2005.
11. Describe the turning points of the graph and its end behavior.
12. What trends in cable TV subscriptions does the graph suggest?

Practice and Apply

Homework Help

For Exercises	See Examples
13–26	1, 2, 3
27–35	4

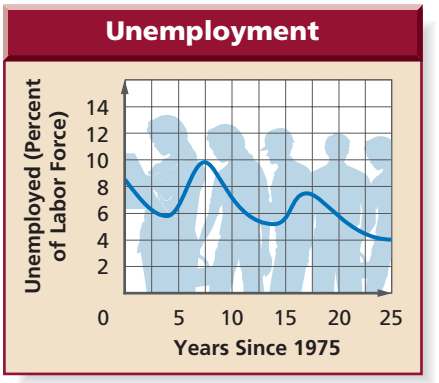
Extra Practice
See page 842.

For Exercises 13–26, complete each of the following.

a. **Graph each function by making a table of values.**
b. **Determine consecutive values of x between which each real zero is located.**
c. **Estimate the x-coordinates at which the relative maxima and relative minima occur.**

13. $f(x) = -x^3 - 4x^2$
14. $f(x) = x^3 - 2x^2 + 6$
15. $f(x) = x^3 - 3x^2 + 2$
16. $f(x) = x^3 + 5x^2 - 9$
17. $f(x) = -3x^3 + 20x^2 - 36x + 16$
18. $f(x) = x^3 - 4x^2 + 2x - 1$
19. $f(x) = x^4 - 8$
20. $f(x) = x^4 - 10x^2 + 9$
21. $f(x) = -x^4 + 5x^2 - 2x - 1$
22. $f(x) = -x^4 + x^3 + 8x^2 - 3$
23. $f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$
24. $f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5$
25. $f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$
26. $f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6$

EMPLOYMENT **For Exercises 27–30, use the graph that models the unemployment rates from 1975–2000.**



27. In what year was the unemployment rate the highest? the lowest?

28. Describe the turning points and end behavior of the graph.

29. If this graph was modeled by a polynomial equation, what is the least degree the equation could have?

30. Do you expect the unemployment rate to increase or decrease from 2001 to 2005? Explain your reasoning.

Online Research **Data Update** What is the current unemployment rate? Visit www.algebra2.com/data_update to learn more.

CHILD DEVELOPMENT **For Exercises 31 and 32, use the following information.** The average height (in inches) for boys ages 1 to 20 can be modeled by the equation $B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25$, where x is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation $G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26$.

31. Graph both equations by making a table of values. Use $x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ as the domain. Round values to the nearest inch.

32. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls?

More About...

Child Devolpment

As children develop, their sleeping needs change. Infants sleep about 16–18 hours a day. Toddlers usually sleep 10–12 hours at night and take one or two daytime naps. School-age children need 9–11 hours of sleep, and teens need at least 9 hours of sleep.

Source: www.kidshealth.org

PHYSIOLOGY **For Exercises 33–35, use the following information.** During a regular respiratory cycle, the volume of air in liters in the human lungs can be described by the function $V(t) = 0.173t + 0.152t^2 - 0.035t^3$, where t is the time in seconds.

33. Estimate the real zeros of the function by graphing.

34. About how long does a regular respiratory cycle last?

35. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air.

CRITICAL THINKING **For Exercises 36–39, sketch a graph of each polynomial.**

36. even-degree polynomial function with one relative maximum and two relative minima

37. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative

38. even-degree polynomial function with four relative maxima and three relative minima

39. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

40. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

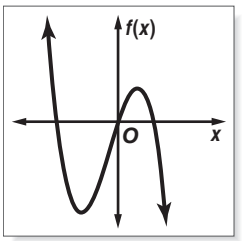
How can graphs of polynomial functions show trends in data?

Include the following in your answer:

- a description of the types of data that are best modeled by polynomial equations rather than linear equations, and
- an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.

Standardized Test Practice

41. Which of the following could be the graph of $f(x) = x^3 + x^2 - 3x$?

Ⓐ


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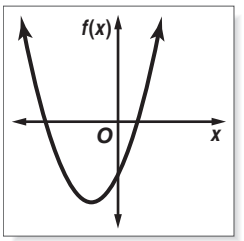

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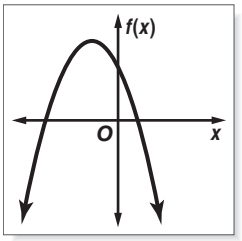

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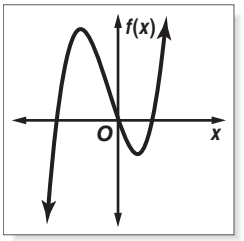


42. The function $f(x) = x^2 - 4x + 3$ has a relative minimum located at which of the following x values?

Ⓐ -2 Ⓑ 2 Ⓒ 3 Ⓓ 4

Graphing Calculator

Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

43. $f(x) = x^3 + x^2 - 7x - 3$

44. $f(x) = -x^3 + 6x^2 - 6x - 5$

45. $f(x) = -x^4 + 3x^2 - 8$

46. $f(x) = 3x^4 - 7x^3 + 4x - 5$

Maintain Your Skills

Mixed Review

If $p(x) = 2x^2 - 5x + 4$ and $r(x) = 3x^3 - x^2 - 2$, find each value. *(Lesson 7-1)*

47. $r(2a)$

48. $5p(c)$

49. $p(2a^2)$

50. $r(x - 1)$

51. $p(x^2 + 4)$

52. $2[p(x^2 + 1)] - 3r(x - 1)$

Graph each inequality. *(Lesson 6-7)*

53. $y > x^2 - 4x + 6$

54. $y \le -x^2 + 6x - 3$

55. $y < x^2 - 2x$

Solve each matrix equation or system of equations by using inverse matrices. *(Lesson 4-8)*

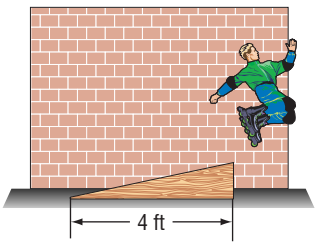
56. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$

57. $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

58. $3j + 2k = 8$
$j - 7k = 18$

59. $5y + 2z = 11$
$10y - 4z = -2$

60. **SPORTS** Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of $\frac{1}{4}$, how tall should they make the ramp? *(Lesson 2-3)*



Getting Ready for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.**
*(To review **factoring polynomials**, see Lesson 5-4.)*

61. $x^2 - x - 30$

62. $2b^2 - 9b + 4$

63. $6a^2 + 17a + 5$

64. $4m^2 - 9$

65. $t^3 - 27$

66. $r^4 - 1$

Graphing Calculator Investigation

A Follow-Up of Lesson 7-2

Modeling Real-World Data

You can use a TI-83 Plus to model data whose curve of best fit is a polynomial function.

Example

The table shows the distance a seismic wave can travel based on its distance from an earthquake's epicenter. Draw a scatter plot and a curve of best fit that relates distance to travel time. Then determine approximately how far from the epicenter the wave will be felt 8.5 minutes after the earthquake occurs.

Source: University of Arizona

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

Step 1 Enter the travel times in L1 and the distances in L2.

KEYSTROKES: *Refer to page 87 to review how to enter lists.*

Step 2 Graph the scatter plot.

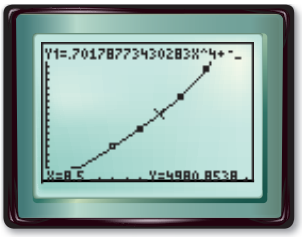
KEYSTROKES: *Refer to page 87 to review how to graph a scatter plot.*

Step 3 Compute and graph the equation for the curve of best fit. A quartic curve is the best fit for these data.

KEYSTROKES: [STAT] [▶] 7 [2nd] [L1] [,] [2nd] [L2] [ENTER] [Y=] [VARS] 5 [▶] [▶] 1 [GRAPH]

Step 4 Use the [CALC] feature to find the value of the function for $x = 8.5$.

KEYSTROKES: *Refer to page 87 to review how to find function values.*



[0, 15] scl: 1 by [0, 10000] scl: 500

After 8.5 minutes, you would expect the wave to be felt approximately 5000 kilometers away.

Exercises

Use the table that shows how many minutes out of each eight-hour work day are used to pay one day's worth of taxes.

Year	Minutes
1940	83
1950	117
1960	130
1970	141
1980	145
1990	145
2000	160

Source: Tax Foundation

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the year. Try LinReg, QuadReg, and CubicReg.
2. Write the equation for the curve that best fits the data.
3. Based on this equation, how many minutes should you expect to work each day in the year 2010 to pay one day's taxes?

7-3 Solving Equations Using Quadratic Techniques

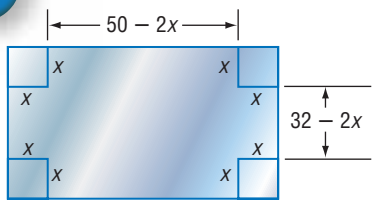
What You'll Learn

- Write expressions in quadratic form.
- Use quadratic techniques to solve equations.

Vocabulary

- quadratic form

How can solving polynomial equations help you to find dimensions?



The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner. The volume of the box depends on the side length x of the cut squares. It is given by $V(x) = 4x^3 - 164x^2 + 1600x$. You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

QUADRATIC FORM In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$ the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form**.

Key Concept — Quadratic Form

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

Example 1 Write Expressions in Quadratic Form

Write each expression in quadratic form, if possible.

a. $x^4 + 13x^2 + 36$

$x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36$ $\quad (x^2)^2 = x^4$

b. $16x^6 - 625$

$16x^6 - 625 = (4x^3)^2 - 625$ $\quad (x^3)^2 = x^6$

c. $12x^8 - x^2 + 10$

This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

d. $x - 9x^{\frac{1}{2}} + 8$

$x - 9x^{\frac{1}{2}} + 8 = \left(x^{\frac{1}{2}}\right)^2 - 9\left(x^{\frac{1}{2}}\right) + 8$ $\quad x^1 = \left(x^{\frac{1}{2}}\right)^2$

SOLVE EQUATIONS USING QUADRATIC FORM In Chapter 6, you learned to solve quadratic equations by using the Zero Product Property and the Quadratic Formula. You can extend these techniques to solve higher-degree polynomial equations that can be written using quadratic form or have an expression that contains a quadratic factor.

Example 2 Solve Polynomial Equations

Solve each equation.

a. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + 36 = 0$	Original equation
$(x^2)^2 - 13(x^2) + 36 = 0$	Write the expression on the left in quadratic form.
$(x^2 - 9)(x^2 - 4) = 0$	Factor the trinomial.
$(x - 3)(x + 3)(x - 2)(x + 2) = 0$	Factor each difference of squares.

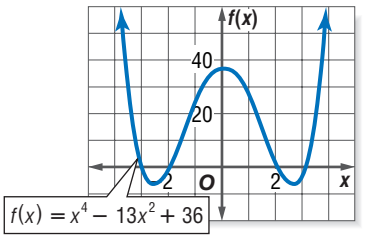
Use the Zero Product Property.

$x - 3 = 0$ or $x + 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ $\quad x = -3$ $\quad x = 2$ $\quad x = -2$

The solutions are -3, -2, 2, and 3.

CHECK The graph of $f(x) = x^4 - 13x^2 + 36$ shows that the graph intersects the x-axis at -3, -2, 2, and 3. ✓



b. $x^3 + 343 = 0$

$x^3 + 343 = 0$	Original equation
$(x)^3 + 7^3 = 0$	This is the sum of two cubes.
$(x + 7)[x^2 - x(7) + 7^2] = 0$	Sum of two cubes formula with $a = x$ and $b = 7$
$(x + 7)(x^2 - 7x + 49) = 0$	Simplify.
$x + 7 = 0$ or $x^2 - 7x + 49 = 0$	Zero Product Property

The solution of the first equation is -7. The second equation can be solved by using the Quadratic Formula.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic Formula
$= \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)}$	Replace a with 1, b with -7, and c with 49.
$= \dfrac{7 \pm \sqrt{-147}}{2}$	Simplify.
$= \dfrac{7 \pm i\sqrt{147}}{2}$ or $\dfrac{7 \pm 7i\sqrt{3}}{2}$	$\sqrt{147} \times \sqrt{-1} = 7i\sqrt{3}$

Thus, the solutions of the original equation are -7, $\dfrac{7 + 7i\sqrt{3}}{2}$, and $\dfrac{7 - 7i\sqrt{3}}{2}$.

Study Tip

Look Back
To review the formula for factoring the **sum of two cubes**, see Lesson 5-4.

Some equations involving rational exponents can be solved by using a quadratic technique.

Example 3 Solve Equations with Rational Exponents

Solve $x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$.

$x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$	Original equation
$\left(x^{\frac{1}{3}}\right)^2 - 6\left(x^{\frac{1}{3}}\right) + 5 = 0$	Write the expression on the left in quadratic form.

(continued on the next page)

Study Tip

Substitution
To avoid confusion, you can substitute another variable for the expression in parentheses.

For example, $\left(x^{\frac{1}{3}}\right)^2 - 6\left(x^{\frac{1}{3}}\right) + 5 = 0$ could be written as $u^2 - 6u + 5 = 0$. Then, once you have solved the equation for u, substitute $x^{\frac{1}{3}}$ for u and solve for x.

$$\left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} - 5\right) = 0$$ Factor the trinomial.

$$x^{\frac{1}{3}} - 1 = 0 \quad \text{or} \quad x^{\frac{1}{3}} - 5 = 0$$ Zero Product Property

$$x^{\frac{1}{3}} = 1 \qquad x^{\frac{1}{3}} = 5$$ Isolate x on one side of the equation.

$$\left(x^{\frac{1}{3}}\right)^3 = 1^3 \qquad \left(x^{\frac{1}{3}}\right)^3 = 5^3$$ Cube each side.

$$x = 1 \qquad x = 125$$ Simplify.

CHECK Substitute each value into the original equation.

$$x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0 \qquad\qquad x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5 = 0$$

$$1^{\frac{2}{3}} - 6(1)^{\frac{1}{3}} + 5 \stackrel{?}{=} 0 \qquad\qquad 125^{\frac{2}{3}} - 6(125)^{\frac{1}{3}} + 5 \stackrel{?}{=} 0$$

$$1 - 6 + 5 \stackrel{?}{=} 0 \qquad\qquad 25 - 30 + 5 \stackrel{?}{=} 0$$

$$0 = 0 \ \checkmark \qquad\qquad 0 = 0 \ \checkmark$$

The solutions are 1 and 125.

To use a quadratic technique, rewrite the equation so one side is equal to zero.

Example 4 Solve Radical Equations

Solve $x - 6\sqrt{x} = 7$.

$$x - 6\sqrt{x} = 7$$ Original equation

$$x - 6\sqrt{x} - 7 = 0$$ Rewrite so that one side is zero.

$$\left(\sqrt{x}\right)^2 - 6\left(\sqrt{x}\right) - 7 = 0$$ Write the expression on the left in quadratic form.

You can use the Quadratic Formula to solve this equation.

$$\sqrt{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$\sqrt{x} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)}$$ Replace a with 1, b with -6, and c with -7.

$$\sqrt{x} = \frac{6 \pm 8}{2}$$ Simplify.

$$\sqrt{x} = \frac{6 + 8}{2} \quad \text{or} \quad \sqrt{x} = \frac{6 - 8}{2}$$ Write as two equations.

$$\sqrt{x} = 7 \qquad \sqrt{x} = -1$$ Simplify.

$$x = 49$$

Since the principal square root of a number cannot be negative, the equation $\sqrt{x} = -1$ has no solution. Thus, the only solution of the original equation is 49.

Study Tip

Look Back
To review **principal roots**, see Lesson 5-5.

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form.
2. **Explain** how the graph of the related polynomial function can help you verify the solution to a polynomial equation.
3. **Describe** how to solve $x^5 - 2x^3 + x = 0$.

Guided Practice

Write each expression in quadratic form, if possible.

4. $5y^4 + 7y^3 - 8$

5. $84n^4 - 62n^2$

Solve each equation.

6. $x^3 + 9x^2 + 20x = 0$

7. $x^4 - 17x^2 + 16 = 0$

8. $x^3 - 216 = 0$

9. $x - 16x^{\frac{1}{2}} = -64$

Application

10. POOL The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are x feet deep by $7x - 6$ feet wide by $9x - 2$ feet long. How deep is the pool?

Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1
17–28	2–4
29–36	2

Extra Practice
See page 842.

Write each expression in quadratic form, if possible.

11. $2x^4 + 6x^2 - 10$

12. $a^8 + 10a^2 - 16$

13. $11n^6 + 44n^3$

14. $7b^5 - 4b^3 + 2b$

15. $7x^{\frac{1}{9}} - 3x^{\frac{1}{3}} + 4$

16. $6x^{\frac{2}{5}} - 4x^{\frac{1}{5}} - 16$

Solve each equation.

17. $m^4 + 7m^3 + 12m^2 = 0$

18. $a^5 + 6a^4 + 5a^3 = 0$

19. $b^4 = 9$

20. $t^5 - 256t = 0$

21. $d^4 + 32 = 12d^2$

22. $x^4 + 18 = 11x^2$

23. $x^3 + 729 = 0$

24. $y^3 - 512 = 0$

25. $x^{\frac{1}{2}} - 8x^{\frac{1}{4}} + 15 = 0$

26. $p^{\frac{2}{3}} + 11p^{\frac{1}{3}} + 28 = 0$

27. $y - 19\sqrt{y} = -60$

28. $z = 8\sqrt{z} + 240$

29. $s^3 + 4s^2 - s - 4 = 0$

30. $h^3 - 8h^2 + 3h - 24 = 0$

31. GEOMETRY The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

Career Choices

Designer

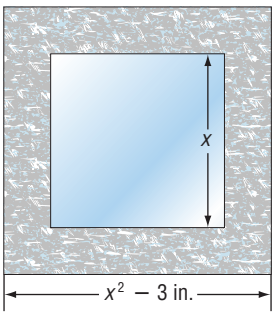
Designers combine practical knowledge with artistic ability to turn abstract ideas into formal designs. Designers usually specialize in a particular area, such as clothing, or home interiors.

Online Research

For information about a career as a designer, visit: www.algebra2.com/careers

DESIGN **For Exercises 32–34, use the following information.**

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.



32. Write a polynomial equation that models the area of the frame.

33. What are the dimensions of the glass piece?

34. What are the dimensions of the frame?

PACKAGING **For Exercises 35 and 36, use the following information.**

A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation $V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24$, where h is the height of the original piece.

35. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

36. How much did each dimension of the packaging increase for the new foam piece?

37. **CRITICAL THINKING** Explain how you would solve $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

38. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can solving polynomial equations help you to find dimensions?

Include the following items in your answer:

- an explanation of how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches, and
- an explanation of why there can be more than one square that can be cut to produce the same volume.

Standardized Test Practice

39. Which of the following is a solution of $x^4 - 2x^2 - 3 = 0$?

(A) $\sqrt[4]{2}$ (B) 1 (C) -3 (D) $\sqrt{3}$

40. **EXTENDED RESPONSE** Solve $18x + 9\sqrt{2x} - 4 = 0$ by first rewriting it in quadratic form. Show your work.

Maintain Your Skills

Mixed Review

Graph each function by making a table of values. *(Lesson 7-2)*

41. $f(x) = x^3 - 4x^2 + x + 5$

42. $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

Find $p(7)$ and $p(-3)$ for each function. *(Lesson 7-1)*

43. $p(x) = x^2 - 5x + 3$

44. $p(x) = x^3 - 11x - 4$

45. $p(x) = \frac{2}{3}x^4 - 3x^3$

For Exercises 46–48, use the following information.
Triangle ABC with vertices $A(-2, 1)$, $B(-3, -3)$, and $C(3, -1)$ is rotated 90° counterclockwise about the origin. *(Lesson 4-4)*

46. Write the coordinates of the triangle in a vertex matrix.

47. Find the coordinates of $\triangle A'B'C'$.

48. Graph the preimage and the image.

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each quotient.**
*(To review **dividing polynomials**, see Lesson 5-3.)*

49. $(x^3 + 4x^2 - 9x + 4) \div (x - 1)$

50. $(4x^3 - 8x^2 - 5x - 10) \div (x + 2)$

51. $(x^4 - 9x^2 - 2x + 6) \div (x - 3)$

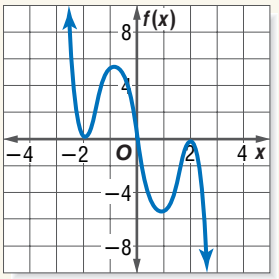
52. $(x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1)$

Practice Quiz 1

Lessons 7-1 through 7-3

1. If $p(x) = 2x^3 - x$, find $p(a - 1)$. *(Lesson 7-1)*

2. Describe the end behavior of the graph at the right. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeros. *(Lesson 7-1)*



3. Graph $y = x^3 + 2x^2 - 4x - 6$. Estimate the x-coordinates at which the relative maxima and relative minima occur. *(Lesson 7-2)*

4. Write the expression $18x^{\frac{1}{3}} + 36x^{\frac{2}{3}} + 5$ in quadratic form. *(Lesson 7-3)*

5. Solve $a^4 = 6a^2 + 27$. *(Lesson 7-3)*

7-4 The Remainder and Factor Theorems

What You'll Learn

- Evaluate functions using synthetic substitution.
- Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

Vocabulary

- synthetic substitution
- depressed polynomial

How can you use the Remainder Theorem to evaluate polynomials?

The number of international travelers to the United States since 1986 can be modeled by the equation $T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9$, where x is the number of years since 1986 and $T(x)$ is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function for $x = 20$, or you can use synthetic substitution.

Study Tip

Look Back
To review **dividing polynomials** and **synthetic division**, see Lesson 5-3.

SYNTHETIC SUBSTITUTION Synthetic division is a shorthand method of long division. It can also be used to find the value of a function. Consider the polynomial function $f(a) = 4a^2 - 3a + 6$. Divide the polynomial by $a - 2$.

Method 1 Long Division

$$\begin{array}{r} 4a + 5 \\ a - 2\overline{)4a^2 - 3a + 6} \\ \underline{4a^2 - 8a} \\ 5a + 6 \\ \underline{5a - 10} \\ 16 \end{array}$$

Method 2 Synthetic Division

$$\begin{array}{r|rrr} 2 & 4 & -3 & 6 \\ & & 8 & 10 \\ \hline & 4 & 5 & \boxed{16} \end{array}$$

Compare the remainder of 16 to $f(2)$.

$$\begin{aligned} f(2) &= 4(2)^2 - 3(2) + 6 && \text{Replace } a \text{ with 2.} \\ &= 16 - 6 + 6 && \text{Multiply.} \\ &= 16 && \text{Simplify.} \end{aligned}$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This illustrates the **Remainder Theorem**.

Key Concept — Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

$$\underbrace{f(x)}_{\text{Dividend}} \underbrace{=}_{\text{equals}} \underbrace{q(x)}_{\text{quotient}} \underbrace{\cdot}_{\text{times}} \underbrace{(x - a)}_{\text{divisor}} \underbrace{+}_{\text{plus}} \underbrace{f(a),}_{\text{remainder.}}$$

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

Example 1 Synthetic Substitution

If $f(x) = 2x^4 - 5x^2 + 8x - 7$, find $f(6)$.

Method 1 Synthetic Substitution

By the Remainder Theorem, $f(6)$ should be the remainder when you divide the polynomial by $x - 6$.

6	2	0	−5	8	−7
		12	72	402	2460
	2	12	67	410	2453

Notice that there is no x^3 term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, $f(6) = 2453$.

Method 2 Direct Substitution

Replace x with 6.

$f(x) = 2x^4 - 5x^2 + 8x - 7$ Original function

$f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7$ Replace x with 6.

$= 2592 - 180 + 48 - 7$ or 2453 Simplify.

By using direct substitution, $f(6) = 2453$.

FACTORS OF POLYNOMIALS

Divide $f(x) = x^4 + x^3 - 17x^2 - 20x + 32$ by $x - 4$.

4	1	1	−17	−20	32
		4	20	12	−32
	1	5	3	−8	0

Study Tip

Depressed Polynomial
A *depressed polynomial* has a degree that is one less than the original polynomial.

The quotient of $f(x)$ and $x - 4$ is $x^3 + 5x^2 + 3x - 8$. When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial**. From the results of the division and by using the Remainder Theorem, we can make the following statement.

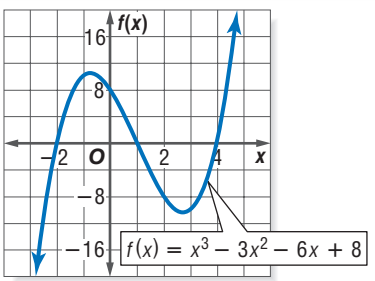
$$\underbrace{x^4 + x^3 - 17x^2 - 20x + 32}_{\text{Dividend}} \underbrace{=}_{\text{equals}} \underbrace{(x^3 + 5x^2 + 3x - 8)}_{\text{quotient}} \underbrace{\cdot}_{\text{times}} \underbrace{(x - 4)}_{\text{divisor}} \underbrace{+}_{\text{plus}} \underbrace{0}_{\text{remainder.}}$$

Since the remainder is 0, $f(4) = 0$. This means that $x - 4$ is a factor of $x^4 + x^3 - 17x^2 - 20x + 32$. This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

Key Concept — **Factor Theorem**

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Suppose you wanted to find the factors of $x^3 - 3x^2 - 6x + 8$. One approach is to graph the related function, $f(x) = x^3 - 3x^2 - 6x + 8$. From the graph at the right, you can see that the graph of $f(x)$ crosses the x-axis at −2, 1, and 4. These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.



CONTENTS

$$f(x) = [x - (-2)](x - 1)(x - 4)$$
$$= (x + 2)(x - 1)(x - 4)$$

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed and once graphed, the exact zeros are often difficult to determine.

The Factor Theorem can help you find all factors of a polynomial.

Example 2 Use the Factor Theorem

Show that $x + 3$ is a factor of $x^3 + 6x^2 - x - 30$. Then find the remaining factors of the polynomial.

The binomial $x + 3$ is a factor of the polynomial if -3 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

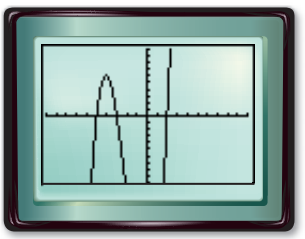
-3	1	6	-1	-30
		-3	-9	30
	1	3	-10	0

Since the remainder is 0, $x + 3$ is a factor of the polynomial. The polynomial $x^3 + 6x^2 - x - 30$ can be factored as $(x + 3)(x^2 + 3x - 10)$. The polynomial $x^2 + 3x - 10$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 + 3x - 10 = (x - 2)(x + 5)$ Factor the trinomial.

So, $x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5)$.

CHECK You can see that the graph of the related function $f(x) = x^3 + 6x^2 - x - 30$ crosses the x-axis at -5, -3, and 2. Thus, $f(x) = [x - (-5)][x - (-3)](x - 2)$. ✓

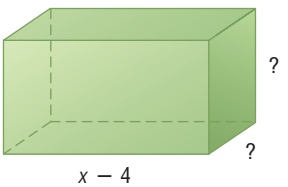


Study Tip

Factoring
The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

Example 3 Find All Factors of a Polynomial

GEOMETRY **The volume of the rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures.**



The volume of a rectangular prism is $\ell \times w \times h$.

You know that one measure is $x - 4$, so $x - 4$ is a factor of $V(x)$.

4	1	3	-36	32
		4	28	-32
	1	7	-8	0

The quotient is $x^2 + 7x - 8$. Use this to factor $V(x)$.

$V(x) = x^3 + 3x^2 - 36x + 32$ Volume function

$= (x - 4)(x^2 + 7x - 8)$ Factor.

$= (x - 4)(x + 8)(x - 1)$ Factor the trinomial $x^2 + 7x - 8$.

So, the missing measures of the prism are $x + 8$ and $x - 1$.

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of a polynomial function that has a remainder of 5 when divided by $x - 4$.

2. **State** the degree of the depressed polynomial that is the result of dividing $x^5 + 3x^4 - 16x - 48$ by one of its first-degree binomial factors.

3. **Write** the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 6 & 32 \\ & & -2 & 4 & -20 \\ \hline & 1 & -2 & 10 & \boxed{12} \end{array}$$

Guided Practice

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

4. $f(x) = x^3 - 2x^2 - x + 1$
5. $f(x) = 5x^4 - 6x^2 + 2$

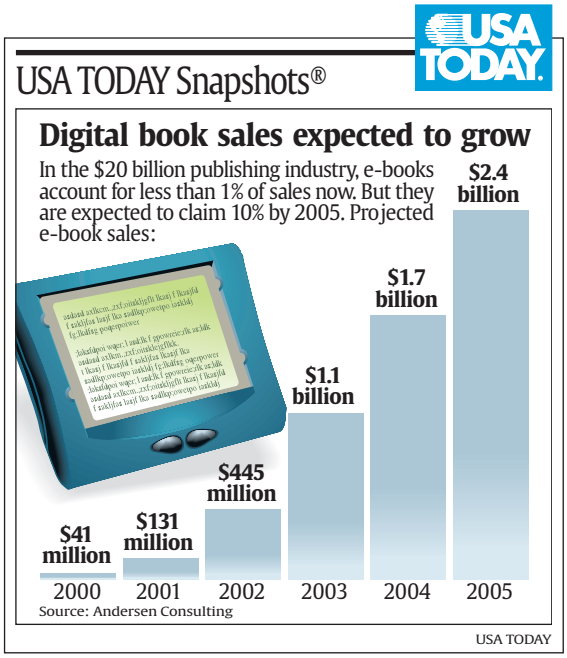
Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. $x^3 - x^2 - 5x - 3;\ x + 1$
7. $x^3 - 3x + 2;\ x - 1$
8. $6x^3 - 25x^2 + 2x + 8;\ 3x - 2$
9. $x^4 + 2x^3 - 8x - 16;\ x + 2$

Application

For Exercises 10–12, use the graph at the right.
The projected sales of e-books can be modeled by the function $S(x) = -17x^3 + 200x^2 - 113x + 44$, where x is the number of years since 2000.

10. Use synthetic substitution to estimate the sales for 2006 in billions of dollars.
11. Evaluate $S(6)$.
12. Which method—synthetic division or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer.



Practice and Apply

Homework Help

For Exercises	See Examples
13–20	1
21–36	2
37–44	3

Extra Practice See page 843.

Use synthetic substitution to find $g(3)$ and $g(-4)$ for each function.

13. $g(x) = x^2 - 8x + 6$
14. $g(x) = x^3 + 2x^2 - 3x + 1$
15. $g(x) = x^3 - 5x + 2$
16. $g(x) = x^4 - 6x - 8$
17. $g(x) = 2x^3 - 8x^2 - 2x + 5$
18. $g(x) = 3x^4 + x^3 - 2x^2 + x + 12$
19. $g(x) = x^5 + 8x^3 + 2x - 15$
20. $g(x) = x^6 - 4x^4 + 3x^2 - 10$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

21. $x^3 + 2x^2 - x - 2;\ x - 1$
22. $x^3 - x^2 - 10x - 8;\ x + 1$
23. $x^3 + x^2 - 16x - 16;\ x + 4$
24. $x^3 - 6x^2 + 11x - 6;\ x - 2$

25. $2x^3 - 5x^2 - 28x + 15; x - 5$

26. $3x^3 + 10x^2 - x - 12; x + 3$

27. $2x^3 + 7x^2 - 53x - 28; 2x + 1$

28. $2x^3 + 17x^2 + 23x - 42; 2x + 7$

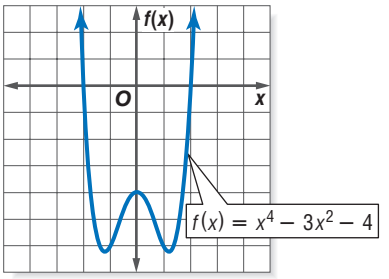
29. $x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1$

30. $16x^5 - 32x^4 - 81x + 162; x - 2$

WebQuest

Changes in world population can be modeled by a polynomial function. Visit www.algebra2.com/webquest to continue work on your WebQuest project.

31. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial.



32. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors.

Find values of k so that each remainder is 3.

33. $(x^2 - x + k) \div (x - 1)$

34. $(x^2 + kx - 17) \div (x - 2)$

35. $(x^2 + 5x + 7) \div (x + k)$

36. $(x^3 + 4x^2 + x + k) \div (x + 2)$

ENGINEERING **For Exercises 37 and 38, use the following information.**
When a certain type of plastic is cut into sections, the length of each section determines its strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

37. Show that $x - 5$ is a factor of the polynomial function.

38. Are there other lengths of plastic that are extremely weak? Explain your reasoning.

ARCHITECTURE **For Exercises 39 and 40, use the following information.**
Elevators traveling from one floor to the next do not travel at a constant speed. Suppose the speed of an elevator in feet per second is given by the function $f(t) = -0.5t^4 + 4t^3 - 12t^2 + 16t$, where t is the time in seconds.

39. Find the speed of the elevator at 1, 2, and 3 seconds.

40. It takes 4 seconds for the elevator to go from one floor to the next. Use synthetic substitution to find $f(4)$. Explain what this means.

41. **CRITICAL THINKING** Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

PERSONAL FINANCE **For Exercises 42–45, use the following information.**
Zach has purchased some home theater equipment for \$2000, which he is financing through the store. He plans to pay \$340 per month and wants to have the balance paid off after six months. The formula $B(x) = 2000x^6 - 340(x^5 + x^4 + x^3 + x^2 + x + 1)$ represents his balance after six months if x represents 1 plus the monthly interest rate (expressed as a decimal).

42. Find his balance after 6 months if the annual interest rate is 12%. (*Hint*: The monthly interest rate is the annual rate divided by 12, so $x = 1.01$.)

43. Find his balance after 6 months if the annual interest rate is 9.6%.

44. How would the formula change if Zach wanted to pay the balance in five months?

45. Suppose he finances his purchase at 10.8% and plans to pay \$410 every month. Will his balance be paid in full after five months?

46. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you use the Remainder Theorem to evaluate polynomials?

Include the following items in your answer:

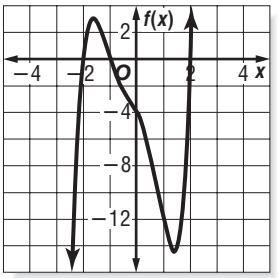
- an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution, and
- evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

Standardized Test Practice

47. Determine the zeros of the function $f(x) = x^2 + 7x + 12$ by factoring.

(A) 7, 12 (B) 3, 4 (C) −5, 5 (D) −4, −3

48. **SHORT RESPONSE** Using the graph of the polynomial function at the right, find all the factors of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$.



Maintain Your Skills

Mixed Review

Write each expression in quadratic form, if possible. *(Lesson 7-3)*

49. $x^4 - 8x^2 + 4$ 50. $9d^6 + 5d^3 - 2$ 51. $r^4 - 5r^3 + 18r$

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. *(Lesson 7-2)*

52. $f(x) = x^3 - 6x^2 + 4x + 3$ 53. $f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4$

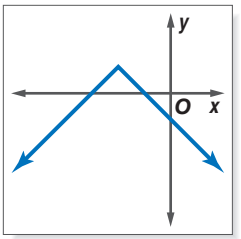
54. **PHYSICS** A model airplane is fixed on a string so that it flies around in a circle. The formula $F_c = m\left(\frac{4\pi^2 r}{T^2}\right)$ describes the force required to keep the airplane going in a circle, where m represents the mass of the plane, r represents the radius of the circle, and T represents the time for a revolution. Solve this formula for T. Write in simplest radical form. *(Lesson 5-8)*

Solve each matrix equation. *(Lesson 4-1)*

55. $\begin{bmatrix} 7x \\ 12 \end{bmatrix} = \begin{bmatrix} 28 \\ -6y \end{bmatrix}$ 56. $\begin{bmatrix} 5a + 2b \\ a - 7b \end{bmatrix} = \begin{bmatrix} -17 \\ 4 \end{bmatrix}$

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. *(Lesson 2-6)*

57.



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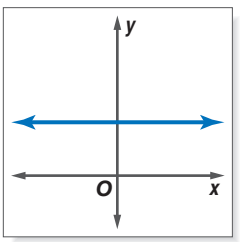

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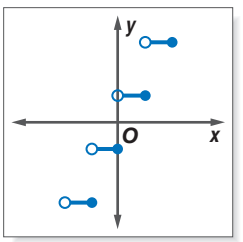


Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the exact solutions of each equation by using the Quadratic Formula.** *(For review of the **Quadratic Formula**, see Lesson 6-5.)*

60. $x^2 + 7x + 8 = 0$ 61. $3x^2 - 9x + 2 = 0$ 62. $2x^2 + 3x + 2 = 0$

7-5 Roots and Zeros

What You'll Learn

- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

How can the roots of an equation be used in pharmacology?

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time.

Suppose the equation $M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

TYPES OF ROOTS You have already learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

Concept Summary — Zeros, Factors, and Roots

Let $f(x) = a_nx^n + \ldots + a_1x + a_0$ be a polynomial function. Then

- c is a zero of the polynomial function $f(x)$,
- $x - c$ is a factor of the polynomial $f(x)$, and
- c is a root or solution of the polynomial equation $f(x) = 0$.

In addition, if c is a real number, then $(c, 0)$ is an intercept of the graph of $f(x)$.

Study Tip

Look Back
For review of **complex numbers**, see Lesson 5-9.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra**.

Key Concept — Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Example 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. $x + 3 = 0$

$x + 3 = 0$ Original equation

$x = -3$ Subtract 3 from each side.

This equation has exactly one real root, -3.

b. $x^2 - 8x + 16 = 0$

$x^2 - 8x + 16 = 0$ Original equation

$(x - 4)^2 = 0$ Factor the left side as a perfect square trinomial.

$x = 4$ Solve for x using the Square Root Property.

Since $x - 4$ is twice a factor of $x^2 - 8x + 16$, 4 is a double root. So this equation has two real roots, 4 and 4.

c. $x^3 + 2x = 0$

$x^3 + 2x = 0$ Original equation

$x(x^2 + 2) = 0$ Factor out the GCF.

Use the Zero Product Property.

$x = 0$ or $x^2 + 2 = 0$

$x^2 = -2$ Subtract two from each side.

$x = \pm\sqrt{-2}$ or $\pm i\sqrt{2}$ Square Root Property

This equation has one real root, 0, and two imaginary roots, $i\sqrt{2}$ and $-i\sqrt{2}$.

d. $x^4 - 1 = 0$

$$x^4 - 1 = 0$$
$$(x^2 + 1)(x^2 - 1) = 0$$
$$(x^2 + 1)(x + 1)(x - 1) = 0$$

$x^2 + 1 = 0$ or $x + 1 = 0$ or $x - 1 = 0$

$x^2 = -1$ $\quad x = -1$ $\quad x = 1$

$x = \pm\sqrt{-1}$ or $\pm i$

This equation has two real roots, 1 and -1, and two imaginary roots, i and $-i$.

Study Tip

Reading Math

In addition to double roots, equations can have triple or quadruple roots. In general, these roots are referred to as *repeated roots*.

More About . . .

Descartes

René Descartes (1596–1650) was a French mathematician and philosopher. One of his best-known quotations comes from his *Discourse on Method*: "I think, therefore I am."

Source: *A History of Mathematics*

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

Key Concept — Corollary

A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.

Similarly, a polynomial function of nth degree has exactly n zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

Key Concept — Descartes' Rule of Signs

If $P(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable,

- the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

Example 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$.

Since $p(x)$ has degree 5, it has five zeros. However, some of them may be imaginary. Use Descartes' Rule of Signs to determine the number and type of real zeros. Count the number of changes in sign for the coefficients of $p(x)$.

$$p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$$

yes (+ to −) no (− to −) yes (− to +) yes (+ to −) yes (− to +)

Since there are 4 sign changes, there are 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$\begin{aligned} p(x) &= (-x)^5 - 6(-x)^4 - 3(-x)^3 + 7(-x)^2 - 8(-x) + 1 \\ &= -x^5 - 6x^4 + 3x^3 + 7x^2 + 8x + 1 \end{aligned}$$

no (− to −) yes (− to +) no (+ to +) no (+ to +) no (+ to +)

Since there is 1 sign change, there is exactly 1 negative real zero.

Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero. Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	1	0	$4 + 1 + 0 = 5$
2	1	2	$2 + 1 + 2 = 5$
0	1	4	$0 + 1 + 4 = 5$

Study Tip

Zero at the Origin
Recall that the number 0 has no sign. Therefore, if 0 is a zero of a function, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

FIND ZEROS We can find all of the zeros of a function using some of the strategies you have already learned.

Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 - 4x^2 + 6x - 4$.

Since $f(x)$ has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = x^3 - 4x^2 + 6x - 4 \qquad f(-x) = -x^3 - 4x^2 - 6x - 4$$

yes yes yes no no no

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of $f(-x)$, $f(x)$ has no negative real zeros. Thus, $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and $f(0)$ is -4, begin by evaluating $f(x)$ for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find $f(a)$ for several values of a.

(continued on the next page)

Study Tip

Finding Zeros
While direct substitution could be used to find each real zero of a polynomial, using synthetic substitution provides you with a depressed polynomial that can be used to find any imaginary zeros.

x	1	−4	6	−4
1	1	−3	3	−1
2	1	−2	2	**0**
3	1	−1	3	5
4	1	0	6	20

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = 2$. Since the depressed polynomial of this zero, $x^2 - 2x + 2$, is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, $x^2 - 2x + 2 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

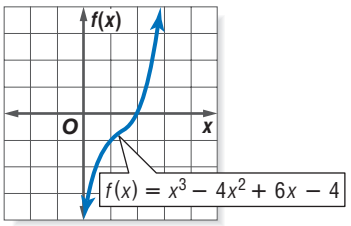
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$ Replace a with 1, b with −2, and c with 2.

$$= \frac{2 \pm \sqrt{-4}}{2}$$ Simplify.

$$= \frac{2 \pm 2i}{2}$$ $\sqrt{4} \times \sqrt{-1} = 2i$

$$= 1 \pm i$$ Simplify.

Thus, the function has one real zero at $x = 2$ and two imaginary zeros at $x = 1 + i$ and $x = 1 - i$. The graph of the function verifies that there is only one real zero.



In Chapter 6, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

Key Concept — **Complex Conjugates Theorem**

Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Standardized Test Practice A B C D

Example 4 Use Zeros to Write a Polynomial Function

Short-Response Test Item

> Write a polynomial function of least degree with integral coefficients whose zeros include 3 and $2 - i$.

Read the Test Item

- If $2 - i$ is a zero, then $2 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x - 3$, $x - (2 - i)$, and $x - (2 + i)$ are factors of the polynomial function.

Solve the Test Item

- Write the polynomial function as a product of its factors.

  $f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$

- Multiply the factors to find the polynomial function.

$$\begin{aligned}
f(x) &= (x-3)[x-(2-i)][x-(2+i)] && \text{Write an equation.}\\
&= (x-3)[(x-2)+i][(x-2)-i] && \text{Regroup terms.}\\
&= (x-3)[(x-2)^2-i^2] && \text{Rewrite as the difference of two squares.}\\
&= (x-3)[x^2-4x+4-(-1)] && \text{Square } x-2 \text{ and replace } i^2 \text{ with } -1.\\
&= (x-3)(x^2-4x+5) && \text{Simplify.}\\
&= x^3-4x^2+5x-3x^2+12x-15 && \text{Multiply using the Distributive Property.}\\
&= x^3-7x^2+17x-15 && \text{Combine like terms.}
\end{aligned}$$

$f(x) = x^3 - 7x^2 + 17x - 15$ is a polynomial function of least degree with integral coefficients whose zeros are 3, $2 - i$, and $2 + i$.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write a polynomial function $p(x)$ whose coefficients have two sign changes. Then describe the nature of its zeros.
2. **Explain** why an odd-degree function must always have at least one real root.
3. **State** the least degree a polynomial equation with real coefficients can have if it has roots at $x = 5 + i$, $x = 3 - 2i$, and a double root at $x = 0$.

Guided Practice

Solve each equation. State the number and type of roots.

4. $x^2 + 4 = 0$
5. $x^3 + 4x^2 - 21x = 0$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

6. $f(x) = 5x^3 + 8x^2 - 4x + 3$
7. $r(x) = x^5 - x^3 - x + 1$

Find all of the zeros of each function.

8. $p(x) = x^3 + 2x^2 - 3x + 20$
9. $f(x) = x^3 - 4x^2 + 6x - 4$
10. $v(x) = x^3 - 3x^2 + 4x - 12$
11. $f(x) = x^3 - 3x^2 + 9x + 13$

Standardized Test Practice

A B C D

12. **SHORT RESPONSE** Write a polynomial function of least degree with integral coefficients whose zeros include 2 and $4i$.

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	1
19–24, 41	2
25–34, 44–48	3
35–40, 42, 43	4

Extra Practice
See page 843.

Solve each equation. State the number and type of roots.

13. $3x + 8 = 0$
14. $2x^2 - 5x + 12 = 0$
15. $x^3 + 9x = 0$
16. $x^4 - 81 = 0$
17. $x^4 - 16 = 0$
18. $x^5 - 8x^3 + 16x = 0$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

19. $f(x) = x^3 - 6x^2 + 1$
20. $g(x) = 5x^3 + 8x^2 - 4x + 3$
21. $h(x) = 4x^3 - 6x^2 + 8x - 5$
22. $q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$
23. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$
24. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

Find all of the zeros of each function.

25. $g(x) = x^3 + 6x^2 + 21x + 26$

26. $h(x) = x^3 - 6x^2 + 10x - 8$

27. $h(x) = 4x^4 + 17x^2 + 4$

28. $f(x) = x^3 - 7x^2 + 25x - 175$

29. $g(x) = 2x^3 - x^2 + 28x + 51$

30. $q(x) = 2x^3 - 17x^2 + 90x - 41$

31. $f(x) = x^3 - 5x^2 - 7x + 51$

32. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40$

33. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13$

34. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156$

Write a polynomial function of least degree with integral coefficients that has the given zeros.

35. $-4, 1, 5$

36. $-2, 2, 4, 6$

37. $4i, 3, -3$

38. $2i, 3i, 1$

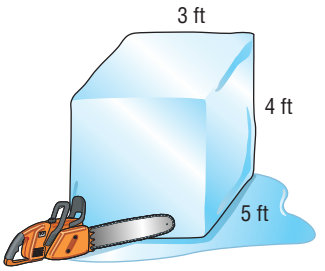
39. $9, 1 + 2i$

40. $6, 2 + 2i$

41. Sketch the graph of a polynomial function that has the indicated number and type of zeros.

a. 3 real, 2 imaginary **b.** 4 real **c.** 2 imaginary

SCULPTING **For Exercises 42 and 43, use the following information.**
Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet.

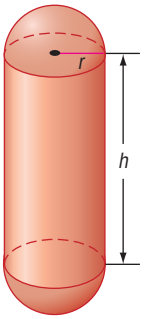


42. Write a polynomial equation to model this situation.

43. How much should he take from each dimension?

Space Exploration
A space shuttle is a reusable vehicle, launched like a rocket, which can put people and equipment in orbit around Earth. The first space shuttle was launched in 1981.
Source: kidsastronomy.about.com

SPACE EXPLORATION **For Exercises 44 and 45, use the following information.**
The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of 336π cubic meters and a height of 17 meters more than the radius of the tank. (*Hint:* $V(r) = \pi r^2 h$)



44. Write an equation that represents the volume of the cylinder.

45. What are the dimensions of the tank?

MEDICINE **For Exercises 46–48, use the following information.**
Doctors can measure cardiac output in patients at high risk for a heart attack by monitoring the concentration of dye injected into a vein near the heart. A normal heart's dye concentration is given by $d(x) = -0.006x^4 - 0.15x^3 - 0.05x^2 + 1.8x$, where x is the time in seconds.

46. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (*Hint:* Notice that 0, which is neither positive nor negative, is a zero of this function since $d(0) = 0$.)

47. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator.

48. What is the meaning of the roots in this problem?

49. **CRITICAL THINKING** Find a counterexample to disprove the following statement.

The polynomial function of least degree with integral coefficients with zeros at $x = 4$, $x = -1$, and $x = 3$, is unique.

50. **CRITICAL THINKING** If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation.

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can the roots of an equation be used in pharmacology?

Include the following items in your answer:

- an explanation of what the roots of this equation represent, and
- an explanation of what the roots of this equation reveal about how often a patient should take this medication.

Standardized Test Practice

52. The equation $x^4 - 1 = 0$ has exactly ___?___ complex root(s).

(A) 4 (B) 0 (C) 2 (D) 1

53. How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have?

(A) 3 (B) 2 (C) 1 (D) 0

Maintain Your Skills

Mixed Review

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function. *(Lesson 7-4)*

54. $f(x) = x^3 - 5x^2 + 16x - 7$

55. $f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5$

56. **RETAIL** The store Bunches of Boxes and Bags assembles boxes for mailing. The store manager found that the volume of a box made from a rectangular piece of cardboard with a square of length x inches cut from each corner is $4x^3 - 168x^2 + 1728x$ cubic inches. If the piece of cardboard is 48 inches long, what is the width? *(Lesson 7-3)*

Determine whether each function has a maximum or a minimum value. Then find the maximum or minimum value of each function. *(Lesson 6-1)*

57. $f(x) = x^2 - 8x + 3$

58. $f(x) = -3x^2 - 18x + 5$

59. $f(x) = -7 + 4x^2$

Factor completely. If the polynomial is not factorable, write *prime*. *(Lesson 5-4)*

60. $15a^2b^2 - 5ab^2c^2$

61. $12p^2 - 64p + 45$

62. $4y^3 + 24y^2 + 36y$

Use matrices A, B, C, and D to find the following. *(Lesson 4-2)*

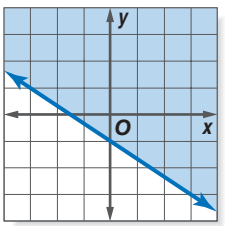
$$A = \begin{bmatrix} -4 & 4 \\ 2 & -3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 6 & -2 \end{bmatrix} \quad C = \begin{bmatrix} -4 & -5 \\ -3 & 1 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -3 & 4 \end{bmatrix}$$

63. $A + D$

64. $B - C$

65. $3B - 2A$

66. Write an inequality for the graph at the right. *(Lesson 2-7)*



Getting Ready for the Next Lesson

BASIC SKILL **Find all values of $\pm\frac{a}{b}$ given each replacement set.**

67. $a = \{1, 5\}$; $b = \{1, 2\}$

68. $a = \{1, 2\}$; $b = \{1, 2, 7, 14\}$

69. $a = \{1, 3\}$; $b = \{1, 3, 9\}$

70. $a = \{1, 2, 4\}$; $b = \{1, 2, 4, 8, 16\}$

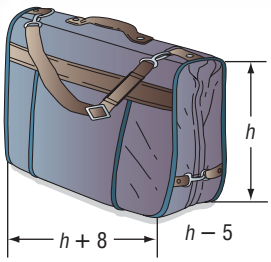
7-6 Rational Zero Theorem

What You'll Learn

- Identify the possible rational zeros of a polynomial function.
- Find all the rational zeros of a polynomial function.

How can the Rational Zero Theorem solve problems involving large numbers?

On an airplane, carry-on baggage must fit into the overhead compartment above the passenger's seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation $h(h + 8)(h - 5) = 2772$, where h is the height, $h + 8$ is the length, and $h - 5$ is the width, to find the dimensions of the overhead compartment in which your luggage must fit.



IDENTIFY RATIONAL ZEROS Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

Key Concept — Rational Zero Theorem

- **Words** Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_n and q is a factor of a_0.
- **Example** Let $f(x) = 2x^3 + 3x^2 - 17x + 12$. If $\frac{3}{2}$ is a zero of $f(x)$, then 3 is a factor of 12 and 2 is a factor of 2.

In addition, if the coefficient of the x term with the highest degree is 1, we have the following corollary.

Key Concept — Corollary (Integral Zero Theorem)

If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Example 1 Identify Possible Zeros

List all of the possible rational zeros of each function.

a. $f(x) = 2x^3 - 11x^2 + 12x + 9$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 9 and q is a factor of 2. The possible values of p are ±1, ±3, and ±9. The possible values for q are ±1 and ±2. So, $\frac{p}{q} = \pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}$, and $\pm\frac{9}{2}$.

b. $f(x) = x^3 - 9x^2 - x + 105$

Since the coefficient of x^3 is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers ±1, ±3, ±5, ±7, ±15, ±21, ±35, and ±105.

FIND RATIONAL ZEROS Once you have written the possible rational zeros, you can test each number using synthetic substitution.

Example 2 Use the Rational Zero Theorem

GEOMETRY **The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.**

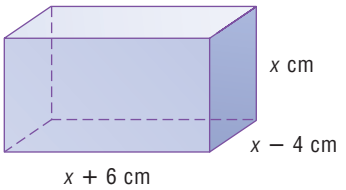
Let x = the height, $x - 4$ = the width, and $x + 6$ = the length.

Write an equation for the volume.

$x(x - 4)(x + 6) = 675$ Formula for volume

$x^3 + 2x^2 - 24x = 675$ Multiply.

$x^3 + 2x^2 - 24x - 675 = 0$ Subtract 675.



The leading coefficient is 1, so the possible integer zeros are factors of 675, ±1, ±3, ±5, ±9, ±15, ±25, ±27, ±45, ±75, ±135, ±225, and ±675. Since length can only be positive, we only need to check positive zeros. From Descartes' Rule of Signs, we also know there is only one positive real zero. Make a table and test possible real zeros.

p	1	2	−24	−675
1	1	3	−21	−696
3	1	5	−9	−702
5	1	7	11	−620
9	1	11	75	0

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are $9 - 4$ or 5 centimeters and $9 + 6$ or 15 centimeters.

CHECK Verify that the dimensions are correct. $5 \times 9 \times 15 = 675$ ✓

Study Tip

Descartes' Rule of Signs
Examine the signs of the coefficients in the equation, + + − −. There is one change of sign, so there is only one positive real zero.

You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

Example 3 Find All Zeros

Find all of the zeros of $f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60$.

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes' Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±12, ±15, ±20, ±30, ±60, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, and $\pm\frac{15}{2}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	2	−13	23	−52	60
1	2	−11	12	−40	20
2	2	−9	5	−42	−24
3	2	−7	2	−46	−78
5	2	−3	8	−12	0

(continued on the next page)

Since $f(5) = 0$, you know that $x = 5$ is a zero. The depressed polynomial is $2x^3 - 3x^2 + 8x - 12$.

Factor $2x^3 - 3x^2 + 8x - 12$.

$2x^3 - 3x^2 + 8x - 12 = 0$	Write the depressed polynomial.
$2x^3 + 8x - 3x^2 - 12 = 0$	Regroup terms.
$2x(x^2 + 4) - 3(x^2 + 4) = 0$	Factor by grouping.
$(x^2 + 4)(2x - 3) = 0$	Distributive Property

$x^2 + 4 = 0$ or $2x - 3 = 0$ Zero Product Property

$x^2 = -4$ $\quad$ $2x = 3$

$x = \pm 2i$ $\quad$ $x = \frac{3}{2}$

There is another real zero at $x = \frac{3}{2}$ and two imaginary zeros at $x = 2i$ and $x = -2i$.

The zeros of this function are 5, $\frac{3}{2}$, $2i$ and $-2i$.

Check for Understanding

Concept Check

1. **Explain** why it is useful to use the Rational Zero Theorem when finding the zeros of a polynomial function.

2. **OPEN ENDED** Write a polynomial function that has possible rational zeros of $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

3. **FIND THE ERROR** Lauren and Luis are listing the possible rational zeros of $f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6$.

Lauren	Luis
$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$	$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$

Who is correct? Explain your reasoning.

Guided Practice

List all of the possible rational zeros of each function.

4. $p(x) = x^4 - 10$

5. $d(x) = 6x^3 + 6x^2 - 15x - 2$

Find all of the rational zeros of each function.

6. $p(x) = x^3 - 5x^2 - 22x + 56$

7. $f(x) = x^3 - x^2 - 34x - 56$

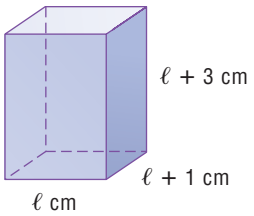
8. $t(x) = x^4 - 13x^2 + 36$

9. $f(x) = 2x^3 - 7x^2 - 8x + 28$

10. Find all of the zeros of $f(x) = 6x^3 + 5x^2 - 9x + 2$.

Application

11. **GEOMETRY** The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid.



Practice and Apply

Homework Help

For Exercises	See Examples
12–17	1
18–29, 34–41	2
30–33	3

Extra Practice
See page 843.

List all of the possible rational zeros of each function.

12. $f(x) = x^3 + 6x + 2$

13. $h(x) = x^3 + 8x + 6$

14. $f(x) = 3x^4 + 15$

15. $n(x) = x^5 + 6x^3 - 12x + 18$

16. $p(x) = 3x^3 - 5x^2 - 11x + 3$

17. $h(x) = 9x^6 - 5x^3 + 27$

Find all of the rational zeros of each function.

18. $f(x) = x^3 + x^2 - 80x - 300$

19. $p(x) = x^3 - 3x - 2$

20. $h(x) = x^4 + x^2 - 2$

21. $g(x) = x^4 - 3x^3 - 53x^2 - 9x$

22. $f(x) = 2x^5 - x^4 - 2x + 1$

23. $f(x) = x^5 - 6x^3 + 8x$

24. $g(x) = x^4 - 3x^3 + x^2 - 3x$

25. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$

26. $p(x) = x^3 + 3x^2 - 25x + 21$

27. $h(x) = 6x^3 + 11x^2 - 3x - 2$

28. $h(x) = 10x^3 - 17x^2 - 7x + 2$

29. $g(x) = 48x^4 - 52x^3 + 13x - 3$

Find all of the zeros of each function.

30. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

31. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$

32. $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

33. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

FOOD **For Exercises 34–36, use the following information.**
Terri's Ice Cream Parlor makes gourmet ice cream cones. The volume of each cone is 8π cubic inches. The height is 4 inches more than the radius of the cone's opening.

34. Write a polynomial equation that represents the volume of an ice cream cone. Use the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

35. What are the possible values of r? Which of these values are reasonable?

36. Find the dimensions of the cone.

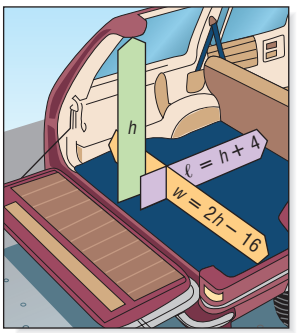
More About...

Food

The largest ice cream sundae, weighing 24.91 tons, was made in Edmonton, Alberta, in July 1988.

Source: *The Guinness Book of Records.*

AUTOMOBILES **For Exercises 37 and 38, use the following information.**
The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.



37. Write a polynomial function that represents the volume of the cargo space.

38. Find the dimensions of the cargo space.

AMUSEMENT PARKS **For Exercises 39–41, use the following information.**
An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a pyramid. Before building the new attraction, engineers must build and test a scale model.

39. If the height of the scale model is 9 inches less than its length and its base is a square, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$.

40. If the volume of the model is 6300 cubic inches, write an equation for the situation.

41. What are the dimensions of the scale model?

42. **CRITICAL THINKING** Suppose k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$. Find k and all three zeros of $f(x)$.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can the Rational Zero Theorem solve problems involving large numbers?

Include the following items in your answer:

- the polynomial equation that represents the volume of the compartment, and
- a list of all reasonable measures of the width of the compartment, assuming that the width is a whole number.

Standardized Test Practice

44. Using the Rational Zero Theorem, determine which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$.

(A) -6 (B) $\frac{3}{8}$ (C) $-\frac{2}{3}$ (D) 1

45. **OPEN ENDED** Write a polynomial with -5, -2, 1, 3, and 4 as roots.

Maintain Your Skills

Mixed Review

Given a function and one of its zeros, find all of the zeros of the function. *(Lesson 7-5)*

46. $g(x) = x^3 + 4x^2 - 27x - 90; -3$
47. $h(x) = x^3 - 11x + 20; 2 + i$
48. $f(x) = x^3 + 5x^2 + 9x + 45; -5$
49. $g(x) = x^3 - 3x^2 - 41x + 203; -7$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. *(Lesson 7-4)*

50. $20x^3 - 29x^2 - 25x + 6; x - 2$
51. $3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3$

Simplify. *(Lesson 5-5)*

52. $\sqrt{245}$
53. $\pm\sqrt{18x^3y^2}$
54. $\sqrt{16x^2 - 40x + 25}$

55. **GEOMETRY** The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? *(Lesson 3-5)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify.**

*(To review **operations with polynomials**, see Lessons 5-2 and 5-3.)*

56. $(x^2 - 7) + (x^3 + 3x^2 + 1)$
57. $(8x^2 - 3x) - (4x^2 + 5x - 3)$
58. $(x + 2)(x^2 + 3x - 5)$
59. $(x^3 + 3x^2 - 3x + 1)(x - 5)^2$
60. $(x^2 - 2x - 30) \div (x + 7)$
61. $(x^3 + 2x^2 - 3x + 1) \div (x + 1)$

Practice Quiz 2

Lessons 7-4 through 7-6

Use synthetic substitution to find $f(-2)$ and $f(3)$ for each function. *(Lesson 7-4)*

1. $f(x) = 7x^5 - 25x^4 + 17x^3 - 32x^2 + 10x - 22$
2. $f(x) = 3x^4 - 12x^3 - 21x^2 + 30x$
3. Write the polynomial equation of degree 4 with leading coefficient 1 that has roots at -2, -1, 3, and 4. *(Lesson 7-5)*

Find all of the rational zeros of each function. *(Lesson 7-6)*

4. $f(x) = 5x^3 - 29x^2 + 55x - 28$
5. $g(x) = 4x^3 + 16x^2 - x - 24$

7-7 Operations on Functions

What You'll Learn

- Find the sum, difference, product, and quotient of functions.
- Find the composition of functions.

Vocabulary

- composition of functions

Why is it important to combine functions in business?

Carol Coffmon owns a garden store where she sells birdhouses. The revenue from the sale of the birdhouses is given by $r(x) = 125x$. The function for the cost of making the birdhouses is given by $c(x) = 65x + 5400$. Her profit p is the revenue minus the cost or $p = r - c$. So the profit function $p(x)$ can be defined as $p(x) = (r - c)(x)$. If you have two functions, you can form a new function by performing arithmetic operations on them.

ARITHMETIC OPERATIONS Let $f(x)$ and $g(x)$ be any two functions. You can add, subtract, multiply, and divide functions according to the following rules.

Key Concept — *Operations with Function*

Operation	Definition	Examples if $f(x) = x + 2$, $g(x) = 3x$
Sum	$(f + g)(x) = f(x) + g(x)$	$(x + 2) + 3x = 4x + 2$
Difference	$(f - g)(x) = f(x) - g(x)$	$(x + 2) - 3x = -2x + 2$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(x + 2)3x = 3x^2 + 6x$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$	$\frac{x + 2}{3x}$

Example 1 Add and Subtract Functions

Given $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$, find each function.

a. $(f + g)(x)$

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) && \text{Addition of functions}\\ &= (x^2 - 3x + 1) + (4x + 5) && f(x) = x^2 - 3x + 1 \text{ and } g(x) = 4x + 5\\ &= x^2 + x + 6 && \text{Simplify.}\end{aligned}$$

b. $(f - g)(x)$

$$\begin{aligned}(f - g)(x) &= f(x) - g(x) && \text{Subtraction of functions}\\ &= (x^2 - 3x + 1) - (4x + 5) && f(x) = x^2 - 3x + 1 \text{ and } g(x) = 4x + 5\\ &= x^2 - 7x - 4 && \text{Simplify.}\end{aligned}$$

Notice that the functions f and g have the same domain of all real numbers. The functions $f + g$ and $f - g$ also have domains that include all real numbers. For each new function, the domain consists of the intersection of the domains of $f(x)$ and $g(x)$. The domain of the quotient function is further restricted by excluded values that make the denominator equal to zero.

Example 2 Multiply and Divide Functions

Given $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$, find each function.

a. $(f \cdot g)(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$	Product of functions
$= (x^2 + 5x - 1)(3x - 2)$	$f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$
$= x^2(3x - 2) + 5x(3x - 2) - 1(3x - 2)$	Distributive Property
$= 3x^3 - 2x^2 + 15x^2 - 10x - 3x + 2$	Distributive Property
$= 3x^3 + 13x^2 - 13x + 2$	Simplify.

b. $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Division of functions}$$

$$= \frac{x^2 + 5x - 1}{3x - 2}, x \neq \frac{2}{3} \quad f(x) = x^2 + 5x - 1 \text{ and } g(x) = 3x - 2$$

Because $x = \frac{2}{3}$ makes $3x - 2 = 0$, $\frac{2}{3}$ is excluded from the domain of $\left(\frac{f}{g}\right)(x)$.

Study Tip

Reading Math
$[f \circ g](x)$ and $f[g(x)]$ are both read *f of g of x*.

COMPOSITION OF FUNCTIONS

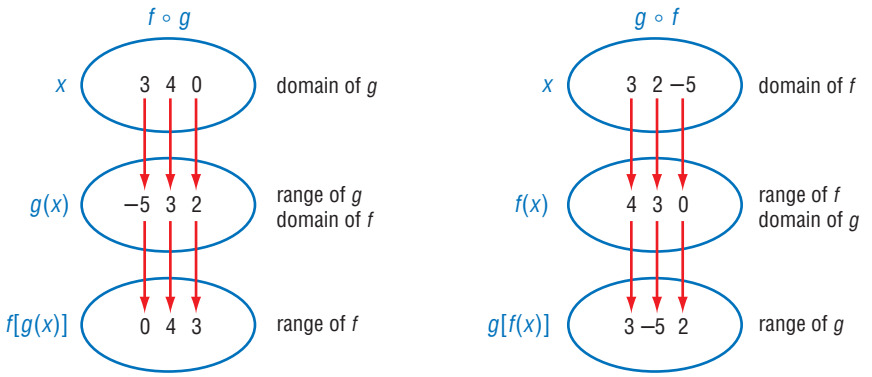
Functions can also be combined using **composition of functions**. In a composition, a function is performed, and then a second function is performed on the result of the first function. The composition of f and g is denoted by $f \circ g$.

Key Concept — Composition of Functions

Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by the equation

$$[f \circ g](x) = f[g(x)].$$

The composition of functions can be shown by mappings. Suppose $f = \{(3, 4), (2, 3), (-5, 0)\}$ and $g = \{(3, -5), (4, 3), (0, 2)\}$. The composition of these functions is shown below.



$f \circ g = \{(3, 0), (4, 4), (0, 3)\}$ $\qquad$ $g \circ f = \{(3, 3), (2, -5), (-5, 2)\}$

The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$. Similarly, $[g \circ f](x)$ is defined only if the range of $f(x)$ is a subset of the domain of $g(x)$.

Example 3 Evaluate Composition of Relations

If $f(x) = \{(7, 8), (5, 3), (9, 8), (11, 4)\}$ and $g(x) = \{(5, 7), (3, 5), (7, 9), (9, 11)\}$, find $f \circ g$ and $g \circ f$.

To find $f \circ g$, evaluate $g(x)$ first. Then use the range of g as the domain of f and evaluate $f(x)$.

$f[g(5)] = f(7)$ or 8 $\quad g(5) = 7$
$f[g(3)] = f(5)$ or 3 $\quad g(3) = 5$
$f[g(7)] = f(9)$ or 8 $\quad g(7) = 9$
$f[g(9)] = f(11)$ or 4 $\quad g(9) = 11$

$f \circ g = \{(5, 8), (3, 3), (7, 8), (9, 4)\}$

To find $g \circ f$, evaluate $f(x)$ first. Then use the range of f as the domain of g and evaluate $g(x)$.

$g[f(7)] = g(8)$ $\quad g(8)$ is undefined.
$g[f(5)] = g(3)$ or 5 $\quad f(5) = 3$
$g[f(9)] = g(8)$ $\quad g(8)$ is undefined.
$g[f(11)] = g(4)$ $\quad g(4)$ is undefined.

Since 8 and 4 are not in the domain of g, $g \circ f$ is undefined for $x = 7$, $x = 9$, and $x = 11$. However, $g[f(5)] = 5$ so $g \circ f = \{(5, 5)\}$.

Notice that in most instances $f \circ g \neq g \circ f$. Therefore, the order in which you compose two functions is very important.

Example 4 Simplify Composition of Functions

a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 3$ and $g(x) = x^2 + x - 1$.

$[f \circ g](x) = f[g(x)]$ — Composition of functions
$= f(x^2 + x - 1)$ — Replace $g(x)$ with $x^2 + x - 1$.
$= (x^2 + x - 1) + 3$ — Substitute $x^2 + x - 1$ for x in $f(x)$.
$= x^2 + x + 2$ — Simplify.

$[g \circ f](x) = g[f(x)]$ — Composition of functions
$= g(x + 3)$ — Replace $f(x)$ with $x + 3$.
$= (x + 3)^2 + (x + 3) - 1$ — Substitute $x + 3$ for x in $g(x)$.
$= x^2 + 6x + 9 + x + 3 - 1$ — Evaluate $(x + 3)^2$.
$= x^2 + 7x + 11$ — Simplify.

So, $[f \circ g](x) = x^2 + x + 2$ and $[g \circ f](x) = x^2 + 7x + 11$.

b. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = 2$.

$[f \circ g](x) = x^2 + x + 2$ — Function from part a
$[f \circ g](2) = (2)^2 + 2 + 2$ — Replace x with 2.
$= 8$ — Simplify.

$[g \circ f](x) = x^2 + 7x + 11$ — Function from part a
$[g \circ f](2) = (2)^2 + 7(2) + 11$ — Replace x with 2.
$= 29$ — Simplify.

So, $[f \circ g](2) = 8$ and $[g \circ f](2) = 29$.

Example 5 Use Composition of Functions

TAXES **Tyrone Davis has $180 deducted from every paycheck for retirement. He can have these deductions taken before taxes are applied, which reduces his taxable income. His federal income tax rate is 18%. If Tyrone earns $2200 every pay period, find the difference in his net income if he has the retirement deduction taken before taxes or after taxes.**

Study Tip

Combining Functions
By combining functions, you can make the evaluation of the functions more efficient.

Explore Let x = Tyrone's income per paycheck, $r(x)$ = his income after the deduction for retirement, and $t(x)$ = his income after the deduction for federal income tax.

Plan Write equations for $r(x)$ and $t(x)$.
$180 is deducted from every paycheck for retirement: $r(x) = x - 180$.

Tyrone's tax rate is 18%: $t(x) = x - 0.18x$.

Solve If Tyrone has his retirement deducted *before* taxes, then his net income is represented by $[t \circ r](2200)$.

$$\begin{aligned}[t \circ r](2200) &= t(2200 - 180) && \text{Replace } x \text{ with 2200 in } r(x) = x - 180.\\ &= t(2020)\\ &= 2020 - 0.18(2020) && \text{Replace } x \text{ with 2020 in } t(x) = x - 0.18x.\\ &= 1656.40\end{aligned}$$

If Tyrone has his retirement deducted *after* taxes, then his net income is represented by $[r \circ t](2200)$.

$$\begin{aligned}[r \circ t](2200) &= r[2200 - 0.18(2200)] && \text{Replace } x \text{ with 2200 in } t(x) = x - 0.18x.\\ &= r(1804)\\ &= 1804 - 180 && \text{Replace } x \text{ with 1804 in } r(x) = x - 180.\\ &= 1624\end{aligned}$$

$[t \circ r](2200) = 1656.40$ and $[r \circ t](2200) = 1624$. The difference is $1656.40 − $1624 or $32.40. So, his net pay is $32.40 more by having his retirement deducted before taxes.

Examine The answer makes sense. Since the taxes are being applied to a smaller amount, less taxes will be deducted from his paycheck.

Check for Understanding

Concept Check

1. **Determine** whether the following statement is *always*, *sometimes*, or *never* true. Support your answer with an example.
Given two functions f and g, $f \circ g = g \circ f$.

2. **OPEN ENDED** Write a set of ordered pairs for functions f and g, given that $f \circ g = \{(4, 3), (-1, 9), (-2, 7)\}$.

3. **FIND THE ERROR** Danette and Marquan are finding $[g \circ f](3)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x - 7$. Who is correct? Explain your reasoning.

Danette

$$\begin{aligned}[g \circ f](3) &= g[(3)^2 + 4(3) + 5]\\ &= g(26)\\ &= 26 - 7\\ &= 19\end{aligned}$$

Marquan

$$\begin{aligned}[g \circ f](3) &= f(3 - 7)\\ &= f(-4)\\ &= (-4)^2 + 4(-4) + 5\\ &= 5\end{aligned}$$

Guided Practice

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

4. $f(x) = 3x + 4$
$g(x) = 5 + x$

5. $f(x) = x^2 + 3$
$g(x) = x - 4$

For each set of ordered pairs, find $f \circ g$ and $g \circ f$, if they exist.

6. $f = \{(-1, 9), (4, 7)\}$
$g = \{(-5, 4), (7, 12), (4, -1)\}$

7. $f = \{(0, -7), (1, 2), (2, -1)\}$
$g = \{(-1, 10), (2, 0)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

8. $g(x) = 2x$
$h(x) = 3x - 4$

9. $g(x) = x + 5$
$h(x) = x^2 + 6$

If $f(x) = 3x$, $g(x) = x + 7$, and $h(x) = x^2$, find each value.

10. $f[g(3)]$ **11.** $g[h(-2)]$ **12.** $h[h(1)]$

Application

SHOPPING **For Exercises 13–16, use the following information.**
Mai-Lin is shopping for computer software. She finds a CD-ROM program that costs \$49.99, but is on sale at a 25% discount. She also has a \$5 coupon she can use on the product.

13. Express the price of the CD after the discount and the price of the CD after the coupon using function notation. Let x represent the price of the CD, $p(x)$ represent the price after the 25% discount, and $c(x)$ represent the price after the coupon.

14. Find $c[p(x)]$ and explain what this value represents.

15. Find $p[c(x)]$ and explain what this value represents.

16. Which method results in the lower sale price? Explain your reasoning.

Practice and Apply

Homework Help

For Exercises	See Examples
17–22	1, 2
23–28	3
29–46	4
47–55	5

Extra Practice
See page 844.

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

17. $f(x) = x + 9$
$g(x) = x - 9$

18. $f(x) = 2x - 3$
$g(x) = 4x + 9$

19. $f(x) = 2x^2$
$g(x) = 8 - x$

20. $f(x) = x^2 + 6x + 9$
$g(x) = 2x + 6$

21. $f(x) = x^2 - 1$
$g(x) = \frac{x}{x + 1}$

22. $f(x) = x^2 - x - 6$
$g(x) = \frac{x - 3}{x + 2}$

For each set of ordered pairs, find $f \circ g$ and $g \circ f$ if they exist.

23. $f = \{(1, 1), (0, -3)\}$
$g = \{(1, 0), (-3, 1), (2, 1)\}$

24. $f = \{(1, 2), (3, 4), (5, 4)\}$
$g = \{(2, 5), (4, 3)\}$

25. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
$g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

26. $f = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$
$g = \{4, 6), (2, 4), (6, 8), (8, 10)\}$

27. $f = \{(2, 5), (3, 9), (-4, 1)\}$
$g = \{(5, -4), (8, 3), (2, -2)\}$

28. $f = \{(7, 0), (-5, 3), (8, 3), (-9, 2)\}$
$g = \{(2, -5), (1, 0), (2, -9), (3, 6)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

29. $g(x) = 4x$
$h(x) = 2x - 1$

30. $g(x) = -5x$
$h(x) = -3x + 1$

31. $g(x) = x + 2$
$h(x) = x^2$

32. $g(x) = x - 4$
$h(x) = 3x^2$

33. $g(x) = 2x$
$h(x) = x^3 + x^2 + x + 1$

34. $g(x) = x + 1$
$h(x) = 2x^2 - 5x + 8$

If $f(x) = 4x$, $g(x) = 2x - 1$, and $h(x) = x^2 + 1$, find each value.

35. $f[g(-1)]$ **36.** $h[g(4)]$ **37.** $g[f(5)]$

38. $f[h(-4)]$ **39.** $g[g(7)]$ **40.** $f[f(-3)]$

41. $h\left[f\left(\frac{1}{4}\right)\right]$ **42.** $g\left[h\left(-\frac{1}{2}\right)\right]$ **43.** $[g \circ (f \circ h)](3)$

44. $[f \circ (h \circ g)](3)$ **45.** $[h \circ (g \circ f)](2)$ **46.** $[f \circ (g \circ h)](2)$

More About...

Shopping

Americans spent over $500 million on inline skates and equipment in 2000.

Source: National Sporting Goods Association

POPULATION GROWTH **For Exercises 47 and 48, use the following information.**
From 1990 to 1999, the number of births $b(x)$ in the U.S. can be modeled by the function $b(x) = -27x + 4103$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 23x + 2164$, where x is the number of years since 1990 and $b(x)$ and $d(x)$ are in thousands.

47. The net increase in population P is the number of births per year minus the number of deaths per year or $P = b - d$. Write an expression that can be used to model the population increase in the U.S. from 1990 to 1999 in function notation.

48. Assume that births and deaths continue at the same rates. Estimate the net increase in population in 2010.

SHOPPING **For Exercises 49–51, use the following information.**
Liluye wants to buy a pair of inline skates that are on sale for 30% off the original price of $149. The sales tax is 5.75%.

49. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax.

50. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

51. How much will Liluye pay for the inline skates?

TEMPERATURE **For Exercises 52–54, use the following information.**
There are three temperature scales: Fahrenheit (°F), Celsius (°C), and Kelvin (K). The function $K(C) = C + 273$ can be used to convert Celsius temperatures to Kelvin. The function $C(F) = \frac{5}{9}(F - 32)$ can be used to convert Fahrenheit temperatures to Celsius.

52. Write a composition of functions that could be used to convert Fahrenheit temperatures to Kelvin.

53. Find the temperature in Kelvin for the boiling point of water and the freezing point of water if water boils at 212°F and freezes at 32°F.

54. While performing an experiment, Kimi found the temperature of a solution at different intervals. She needs to record the change in temperature in degrees Kelvin, but only has a thermometer with a Fahrenheit scale. What will she record when the temperature of the solution goes from 158°F to 256°F?

55. **FINANCE** Kachina pays $50 each month on a credit card that charges 1.6% interest monthly. She has a balance of $700. The balance at the beginning of the nth month is given by $f(n) = f(n - 1) + 0.016\,f(n - 1) - 50$. Find the balance at the beginning of the first five months. No additional charges are made on the card. (*Hint:* $f(1) = 700$)

56. **CRITICAL THINKING** If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$.

57. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why is it important to combine functions in business?

Include the following in your answer:

- a description of how to write a new function that represents the profit, using the revenue and cost functions, and
- an explanation of the benefits of combining two functions into one function.

Standardized Test Practice

58. If $h(x) = 7x - 5$ and $g[h(x)] = 2x + 3$, then $g(x) =$

(A) $\frac{2x + 31}{7}$.
(B) $-5x + 8$.
(C) $5x - 8$.
(D) $\frac{2x + 26}{7}$.

59. If $f(x) = 4x^4 + 5x^3 - 3x^2 - 14x + 31$ and $g(x) = 7x^3 - 4x^2 + 5x - 42$, then $(f - g)(x) =$

(A) $4x^4 + 12x^3 - 7x^2 - 9x - 11$.
(B) $4x^4 - 2x^3 - 7x^2 - 19x - 11$.
(C) $4x^4 - 2x^3 + x^2 - 19x + 73$.
(D) $-3x^4 - 2x^3 - 7x^2 - 19x + 73$.

Maintain Your Skills

Mixed Review

List all of the possible rational zeros of each function. *(Lesson 7-6)*

60. $r(x) = x^2 - 6x + 8$
61. $f(x) = 4x^3 - 2x^2 + 6$
62. $g(x) = 9x^2 - 1$

Write a polynomial function of least degree with integral coefficients that has the given zeros. *(Lesson 7-5)*

63. $5, 3, -4$
64. $-3, -2, 8$
65. $1, \frac{1}{2}, \frac{2}{3}$
66. $6, 2i$
67. $3, 3 - 2i$
68. $-5, 2, 1 - i$

69. **ELECTRONICS** There are three basic things to be considered in an electrical circuit: the flow of the electrical current I, the resistance to the flow Z called impedance, and electromotive force E called voltage. These quantities are related in the formula $E = I \cdot Z$. The current of a circuit is to be $35 - 40j$ amperes. Electrical engineers use the letter j to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $430 - 330j$ volts. *(Lesson 5-9)*

Find the inverse of each matrix, if it exists. *(Lesson 4-7)*

70. $\begin{bmatrix} 8 & 6 \\ 7 & 5 \end{bmatrix}$
71. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
72. $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$
73. $\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$
74. $\begin{bmatrix} 6 & -2 \\ 9 & -3 \end{bmatrix}$
75. $\begin{bmatrix} 2 & 2 \\ 3 & -5 \end{bmatrix}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation or formula for the specified variable.**
*(To review **solving equations for a variable**, see Lesson 1-3.)*

76. $2x - 3y = 6$, for x
77. $4x^2 - 5xy + 2 = 3$, for y
78. $3x + 7xy = -2$, for x
79. $I = prt$, for t
80. $C = \frac{5}{9}(F - 32)$, for F
81. $F = G\frac{Mm}{r^2}$, for m

7-8 Inverse Functions and Relations

What You'll Learn

- Find the inverse of a function or relation.
- Determine whether two functions or relations are inverses.

Vocabulary

- inverse relation
- inverse function
- identity function
- one-to-one

How are inverse functions related to measurement conversions?

Most scientific formulas involve measurements given in SI (International System) units. The SI units for speed are meters per second. However, the United States uses customary measurements such as miles per hour. To convert x miles per hour to an approximate equivalent in meters per second, you can evaluate

$f(x) = \frac{x \text{ miles}}{1 \text{ hour}} \cdot \frac{1600 \text{ meters}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}}$ or $f(x) = \frac{4}{9}x$. To convert x meters per second to an approximate equivalent in miles per hour, you can evaluate

$g(x) = \frac{x \text{ meters}}{1 \text{ second}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \text{ meters}}$ or $g(x) = \frac{9}{4}x$.

Notice that $f(x)$ multiplies a number by 4 and divides it by 9. The function $g(x)$ does the inverse operation of $f(x)$. It divides a number by 4 and multiplies it by 9. The functions $f(x) = \frac{4}{9}x$ and $g(x) = \frac{9}{4}x$ are inverses.

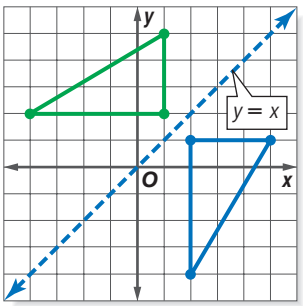
FIND INVERSES Recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each original ordered pair. The domain of a relation becomes the range of the inverse, and the range of a relation becomes the domain of the inverse.

Key Concept — Inverse Relations

- **Words** Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).
- **Example** $Q = \{(1, 2), (3, 4), (5, 6)\}$ $S = \{(2, 1), (4, 3), (6, 5)\}$
  Q and S are inverse relations.

Example 1 Find an Inverse Relation

GEOMETRY **The ordered pairs of the relation $\{(2, 1), (5, 1), (2, -4)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.**



To find the inverse of this relation, reverse the coordinates of the ordered pairs.

The inverse of the relation is $\{(1, 2), (1, 5), (-4, 2)\}$.

Plotting the points shows that the ordered pairs also describe the vertices of a right triangle. Notice that the graphs of the relation and the inverse relation are reflections over the graph of $y = x$.

Study Tip

Reading Math
f^{-1} is read *f inverse* or *the inverse of f*. Note that -1 is not an exponent.

The ordered pairs of **inverse functions** are also related. We can write the inverse of function $f(x)$ as $f^{-1}(x)$.

Key Concept — **Property of Inverse Functions**

Suppose f and f^{-1} are inverse functions. Then, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Let's look at the inverse functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$.

Evaluate $f(5)$.	Now, evaluate $f^{-1}(7)$.
$f(x) = x + 2$	$f^{-1}(x) = x - 2$
$f(5) = 5 + 2$ or 7	$f^{-1}(7) = 7 - 2$ or 5

Since $f(x)$ and $f^{-1}(x)$ are inverses, $f(5) = 7$ and $f^{-1}(7) = 5$. The inverse function can be found by exchanging the domain and range of the function.

Example 2 Find an Inverse Function

a. Find the inverse of $f(x) = \frac{x+6}{2}$.

Step 1 Replace $f(x)$ with y in the original equation.

$$f(x) = \frac{x+6}{2} \quad \Rightarrow \quad y = \frac{x+6}{2}$$

Step 2 Interchange x and y.

$$x = \frac{y+6}{2}$$

Step 3 Solve for y.

$$x = \frac{y+6}{2} \quad \text{Inverse}$$

$$2x = y + 6 \quad \text{Multiply each side by 2.}$$

$$2x - 6 = y \quad \text{Subtract 6 from each side.}$$

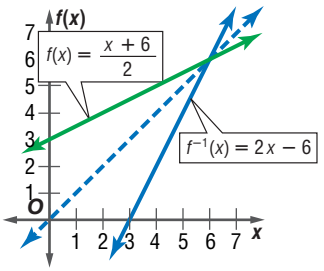
Step 4 Replace y with $f^{-1}(x)$.

$$y = 2x - 6 \qquad f^{-1}(x) = 2x - 6$$

The inverse of $f(x) = \frac{x+6}{2}$ is $f^{-1}(x) = 2x - 6$.

b. Graph the function and its inverse.

Graph both functions on the coordinate plane. The graph of $f^{-1}(x) = 2x - 6$ is the reflection of the graph of $f(x) = \frac{x+6}{2}$ over the line $y = x$.



INVERSES OF RELATIONS AND FUNCTIONS You can determine whether two functions are inverses by finding both of their compositions. If both equal the $I(x) = x$, then the functions are inverse functions.

Key Concept — **Inverse Functions**

- **Words** Two functions f and g are inverse functions if and only if both of their compositions are the identity function.
- **Symbols** $[f \circ g](x) = x$ and $[g \circ f](x) = x$

Study Tip

Inverse Functions
Both compositions of $f(x)$ and $g(x)$ must be the identity function for $f(x)$ and $g(x)$ to be inverses. It is necessary to check them both.

Example 3 Verify Two Functions are Inverses

Determine whether $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$ are inverse functions.

Check to see if the compositions of $f(x)$ and $g(x)$ are identity functions.

$$\begin{aligned}[f \circ g](x) &= f[g(x)] \\ &= f\left(\tfrac{1}{5}x - 2\right) \\ &= 5\left(\tfrac{1}{5}x - 2\right) + 10 \\ &= x - 10 + 10 \\ &= x\end{aligned}$$

$$\begin{aligned}[g \circ f](x) &= g[f(x)] \\ &= g(5x + 10) \\ &= \tfrac{1}{5}(5x + 10) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.

You can also determine whether two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images with respect to the graph of the identity function $I(x) = x$.

Algebra Activity

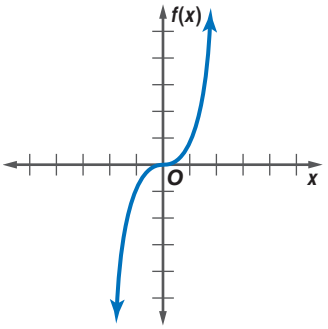
Inverses of Functions

- Use a full sheet of grid paper. Draw and label the x- and y-axes.
- Graph $y = 2x - 3$.
- On the same coordinate plane, graph $y = x$ as a dashed line.
- Place a geomirror so that the drawing edge is on the line $y = x$. Carefully plot the points that are part of the reflection of the original line. Draw a line through the points.

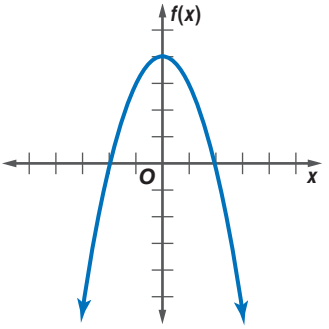
Analyze

1. What is the equation of the drawn line?
2. What is the relationship between the line $y = 2x - 3$ and the line that you drew? Justify your answer.
3. Try this activity with the function $y = |x|$. Is the inverse also a function? Explain.

When the inverse of a function is a function, then the original function is said to be **one-to-one**. To determine if the inverse of a function is a function, you can use the *horizontal line test*.



No horizontal line can be drawn so that it passes through more than one point. The inverse of this function is a function.



A horizontal line can be drawn that passes through more than one point. The inverse of this function is not a function.

Check for Understanding

Concept Check

1. **Determine** whether $f(x) = 3x + 6$ and $g(x) = x - 2$ are inverses.
2. **Explain** the steps you would take to find an inverse function.
3. **OPEN ENDED** Give an example of a function and its inverse. Verify that the two functions are inverses.
4. **Determine** the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

Guided Practice

Find the inverse of each relation.

5. $\{(2, 4), (-3, 1), (2, 8)\}$
6. $\{(1, 3), (1, -1), (1, -3), (1, 1)\}$

Find the inverse of each function. Then graph the function and its inverse.

7. $f(x) = -x$
8. $g(x) = 3x + 1$
9. $y = \frac{1}{2}x + 5$

Determine whether each pair of functions are inverse functions.

10. $f(x) = x + 7$
    $g(x) = x - 7$
11. $g(x) = 3x - 2$
    $f(x) = \frac{x - 2}{3}$

Application

PHYSICS **For Exercises 12 and 13, use the following information.**
The acceleration due to gravity is 9.8 meters per second squared (m/s^2). To convert to feet per second squared, you can use the following chain of operations:

$$\frac{9.8\ \cancel{m}}{s^2} \times \frac{100\ \cancel{cm}}{1\ \cancel{m}} \times \frac{1\ \cancel{in.}}{2.54\ \cancel{cm}} \times \frac{1\ ft}{12\ \cancel{in.}}.$$

12. Find the value of the acceleration due to gravity in feet per second squared.
13. An object is accelerating at 50 feet per second squared. How fast is it accelerating in meters per second squared?

Practice and Apply

Homework Help

For Exercises	See Examples
14–19	1
20–31, 38–43	2
32–37	3

Extra Practice
See page 844.

Find the inverse of each relation.

14. $\{(2, 6), (4, 5), (-3, -1)\}$
15. $\{(3, 8), (4, -2), (5, -3)\}$
16. $\{(7, -4), (3, 5), (-1, 4), (7, 5)\}$
17. $\{(-1, -2), (-3, -2), (-1, -4), (0, 6)\}$
18. $\{(6, 11), (-2, 7), (0, 3), (-5, 3)\}$
19. $\{(2, 8), (-6, 5), (8, 2), (5, -6)\}$

Find the inverse of each function. Then graph the function and its inverse.

20. $y = -3$
21. $g(x) = -2x$
22. $f(x) = x - 5$
23. $g(x) = x + 4$
24. $f(x) = 3x + 3$
25. $y = -2x - 1$
26. $y = \frac{1}{3}x$
27. $f(x) = \frac{5}{8}x$
28. $f(x) = \frac{1}{3}x + 4$
29. $f(x) = \frac{4}{5}x - 7$
30. $g(x) = \frac{2x + 3}{6}$
31. $f(x) = \frac{7x - 4}{8}$

Determine whether each pair of functions are inverse functions.

32. $f(x) = x - 5$
    $g(x) = x + 5$
33. $f(x) = 3x + 4$
    $g(x) = 3x - 4$
34. $f(x) = 6x + 2$
    $g(x) = x - \frac{1}{3}$
35. $g(x) = 2x + 8$
    $f(x) = \frac{1}{2}x - 4$
36. $h(x) = 5x - 7$
    $g(x) = \frac{1}{5}(x + 7)$
37. $g(x) = 2x + 1$
    $f(x) = \frac{x - 1}{2}$

NUMBER GAMES **For Exercises 38–40, use the following information.**
Damaso asked Sophia to choose a number between 1 and 20. He told her to add 7 to that number, multiply by 4, subtract 6, and divide by 2.

38. Write an equation that models this problem.

39. Find the inverse.

40. Sophia's final number was 35. What was her original number?

41. **SALES** Sales associates at Electronics Unlimited earn \$8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their income, and find how much merchandise they must sell in order to earn \$500 in a 40-hour week.

Temperature
The Fahrenheit temperature scale was established in 1724 by a physicist named Gabriel Daniel Fahrenheit. The Celsius temperature scale was established in the same year by an astronomer named Anders Celsius.
Source: www.infoplease.com

TEMPERATURE **For Exercises 42 and 43, use the following information.**
A formula for converting degrees Fahrenheit to Celsius is $C(x) = \frac{5}{9}(x - 32)$.

42. Find the inverse $C^{-1}(x)$. Show that $C(x)$ and $C^{-1}(x)$ are inverses.

43. Explain what purpose $C^{-1}(x)$ serves.

44. **CRITICAL THINKING** Give an example of a function that is its own inverse.

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are inverse functions related to measurement conversions?

Include the following items in your answer:

- an explanation of why you might want to know the customary units if you are given metric units even if it is not necessary for you to perform additional calculations, and
- a demonstration of how to convert the speed of light $c = 3.0 \times 10^8$ meters per second to miles per hour.

Standardized Test Practice

46. Which of the following is the inverse of the function $f(x) = \frac{3x - 5}{2}$?

(A) $g(x) = \frac{2x + 5}{3}$ (B) $g(x) = \frac{3x + 5}{2}$ (C) $g(x) = 2x + 5$ (D) $g(x) = \frac{2x - 5}{3}$

47. For which of the following functions is the inverse also a function?

I. $f(x) = x^3$ **II.** $f(x) = x^4$ **III.** $f(x) = -|x|$

(A) I and II only (B) I only (C) I, II, and III (D) III only

Maintain Your Skills

Mixed Review

Find $[g \circ h](x)$ and $[h \circ g](x)$. *(Lesson 7-7)*

48. $g(x) = 4x$, $h(x) = x + 5$

49. $g(x) = 3x + 2$, $h(x) = 2x - 4$

50. $g(x) = x + 4$, $h(x) = x^2 - 3x - 28$

Find all of the rational zeros of each function. *(Lesson 7-6)*

51. $f(x) = x^3 + 6x^2 - 13x - 42$

52. $h(x) = 24x^3 - 86x^2 + 57x + 20$

Evaluate each expression. *(Lesson 5-7)*

53. $16^{\frac{3}{2}}$

54. $64^{\frac{1}{3}} \cdot 64^{\frac{1}{2}}$

55. $\dfrac{3^{\frac{4}{3}}}{81^{\frac{1}{12}}}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**
*(To review **solving radical equations**, see Lesson 5-8.)*

56. $\sqrt{x} - 5 = -3$

57. $\sqrt{x + 4} = 11$

58. $12 - \sqrt{x} = -2$

59. $\sqrt{x - 5} = \sqrt{2x + 2}$

60. $\sqrt{x - 3} = \sqrt{2} - \sqrt{x}$

61. $3 - \sqrt{x} = \sqrt{x - 6}$

7-9 Square Root Functions and Inequalities

What You'll Learn

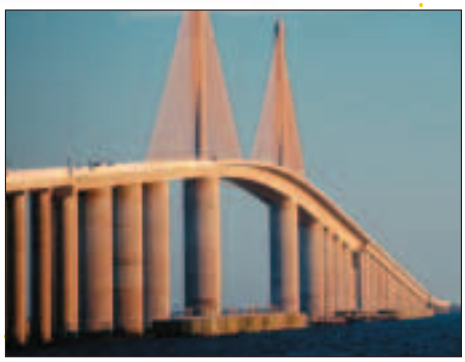
- Graph and analyze square root functions.
- Graph square root inequalities.

Vocabulary

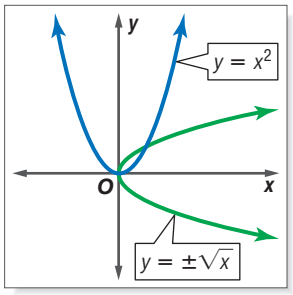
- square root function
- square root inequality

How are square root functions used in bridge design?

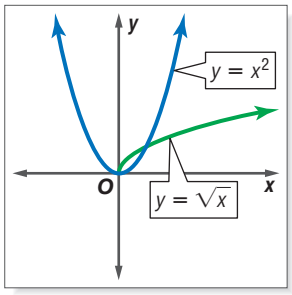
The Sunshine Skyway Bridge across Tampa Bay, Florida, is supported by 21 steel cables, each 9 inches in diameter. The amount of weight that a steel cable can support is given by $w = 8d^2$, where d is the diameter of the cable in inches and w is the weight in tons. If you need to know what diameter a steel cable should have to support a given weight, you can use the equation $d = \sqrt{\frac{w}{8}}$.



SQUARE ROOT FUNCTIONS If a function contains a square root of a variable, it is called a **square root function**. The inverse of a quadratic function is a square root function only if the range is restricted to nonnegative numbers.



$y = \pm\sqrt{x}$ is not a function.



$y = \sqrt{x}$ is a function.

In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, determine when the radicand would be negative and exclude those values from the domain.

Example 1 Graph a Square Root Function

Graph $y = \sqrt{3x + 4}$. State the domain, range, and x- and y-intercepts.

Since the radicand cannot be negative, identify the domain.

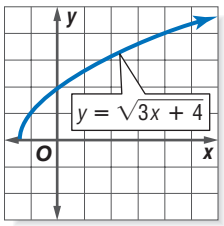
$3x + 4 \geq 0$ Write the expression inside the radicand as ≥ 0.

$x \geq -\frac{4}{3}$ Solve for x.

The x-intercept is $-\frac{4}{3}$.

Make a table of values and graph the function. From the graph, you can see that the domain is $x \geq -\frac{4}{3}$, and the range is $y \geq 0$. The y-intercept is 2.

x	y
$-\frac{4}{3}$	0
-1	1
0	2
2	3.2
4	4



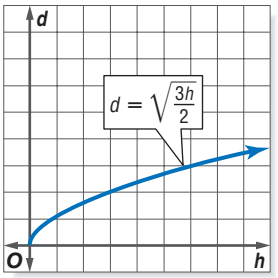
Example 2 Solve a Square Root Problem

SUBMARINES **A lookout on a submarine is h feet above the surface of the water. The greatest distance d in miles that the lookout can see on a clear day is given by the square root of the quantity h multiplied by $\frac{3}{2}$.**

a. Graph the function. State the domain and range.

The function is $d = \sqrt{\frac{3h}{2}}$. Make a table of values and graph the function.

h	d
0	0
2	$\sqrt{3}$ or 1.73
4	$\sqrt{6}$ or 2.45
6	$\sqrt{9}$ or 3.00
8	$\sqrt{12}$ or 3.46
10	$\sqrt{15}$ or 3.87



The domain is $h \geq 0$, and the range is $d \geq 0$.

b. A ship is 3 miles from a submarine. How high would the submarine have to raise its periscope in order to see the ship?

$d = \sqrt{\frac{3h}{2}}$ Original equation

$3 = \sqrt{\frac{3h}{2}}$ Replace d with 3.

$9 = \frac{3h}{2}$ Square each side.

$18 = 3h$ Multiply each side by 2.

$6 = h$ Divide each side by 3.

The periscope would have to be 6 feet above the water. Check this result on the graph.

More About . . .

Submarines

Submarines were first used by The United States in 1776 during the Revolutionary War.

Source: www.infoplease.com

Graphs of square root functions can be transformed just like quadratic functions.

Graphing Calculator Investigation

Square Root Functions

You can use a TI-83 Plus graphing calculator to graph square root functions. Use [2nd] [$\sqrt{\ }$] to enter the functions in the Y= list.

Think and Discuss

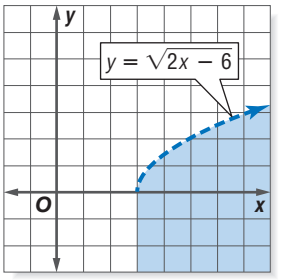
1. Graph $y = \sqrt{x}$, $y = \sqrt{x} + 1$, and $y = \sqrt{x} - 2$ in the viewing window [−2, 8] by [−4, 6]. State the domain and range of each function and describe the similarities and differences among the graphs.
2. Graph $y = \sqrt{x}$, $y = \sqrt{2x}$, and $y = \sqrt{8x}$ in the viewing window [0, 10] by [0, 10]. State the domain and range of each function and describe the similarities and differences among the graphs.
3. Make a conjecture on how you could write an equation that translates the parent graph $y = \sqrt{x}$ to the left three units. Test your conjecture with the graphing calculator.

SQUARE ROOT INEQUALITIES A **square root inequality** is an inequality involving square roots. You can use what you know about square root functions to graph square root inequalities.

Example 3 Graph a Square Root Inequality

a. Graph $y < \sqrt{2x - 6}$.

Graph the related equation $y = \sqrt{2x - 6}$. Since the boundary should not be included, the graph should be dashed.



The domain includes values for $x \geq 3$, so the graph is to the right of $x = 3$. Select a point and test its ordered pair.

Test (4, 1).

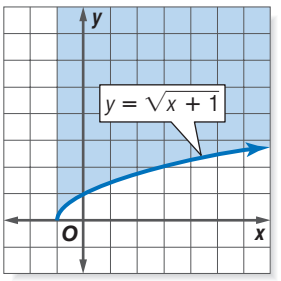
$1 < \sqrt{2(4) - 6}$

$1 < \sqrt{2}$ true

Shade the region that includes the point (4, 1).

b. Graph $y \geq \sqrt{x + 1}$.

Graph the related equation $y = \sqrt{x + 1}$.



The domain includes values for $x \geq -1$, so the graph includes $x = -1$ and the values of x to the right of $x = -1$. Select a point and test its ordered pair.

Test (2, 1).

$y \geq \sqrt{x + 1}$.

$1 \geq \sqrt{2 + 1}$

$1 \geq \sqrt{3}$ false

Shade the region that does not include (2, 1).

Check for Understanding

Concept Check

1. **Explain** why the inverse of $y = 3x^2$ is not a square root function.
2. **Describe** the difference between the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x - 4}$.
3. **OPEN ENDED** **Write** a square root function with a domain of $\{x \mid x \geq 2\}$.

Guided Practice

Graph each function. State the domain and range of the function.

4. $y = \sqrt{x} + 2$
5. $y = \sqrt{4x}$
6. $y = 3 - \sqrt{x}$
7. $y = \sqrt{x - 1} + 3$

Graph each inequality.

8. $y \leq \sqrt{x - 4} + 1$
9. $y > \sqrt{2x + 4}$
10. $y < 3 - \sqrt{5x + 1}$
11. $y \geq \sqrt{x + 2} - 1$

Application **FIREFIGHTING** **For Exercises 12 and 13, use the following information.**
When fighting a fire, the velocity v of water being pumped into the air is the square root of twice the product of the maximum height h and g, the acceleration due to gravity (32 ft/s^2).

12. Determine an equation that will give the maximum height of the water as a function of its velocity.

13. The Coolville Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 ft/s meet the fire department's need? Explain.

Practice and Apply

Homework Help

For Exercises	See Examples
14–25	1
26–31	3
32–34	2

Extra Practice
See page 844.

Graph each function. State the domain and range of each function.

14. $y = \sqrt{3x}$ **15.** $y = -\sqrt{5x}$ **16.** $y = -4\sqrt{x}$

17. $y = \frac{1}{2}\sqrt{x}$ **18.** $y = \sqrt{x + 2}$ **19.** $y = \sqrt{x - 7}$

20. $y = -\sqrt{2x + 1}$ **21.** $y = \sqrt{5x - 3}$ **22.** $y = \sqrt{x + 6} - 3$

23. $y = 5 - \sqrt{x + 4}$ **24.** $y = \sqrt{3x - 6} + 4$ **25.** $y = 2\sqrt{3 - 4x} + 3$

Graph each inequality.

26. $y \leq -6\sqrt{x}$ **27.** $y < \sqrt{x + 5}$ **28.** $y > \sqrt{2x + 8}$

29. $y \geq \sqrt{5x - 8}$ **30.** $y \geq \sqrt{x - 3} + 4$ **31.** $y < \sqrt{6x - 2} + 1$

32. **ROLLER COASTERS** The velocity of a roller coaster as it moves down a hill is $v = \sqrt{v_0^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. An engineer wants a new coaster to have a velocity of 90 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 10 feet per second, how high should the engineer make the hill?

AEROSPACE **For Exercises 33 and 34, use the following information.**
The force due to gravity decreases with the square of the distance from the center of Earth. So, as an object moves farther from Earth, its weight decreases. The radius of Earth is approximately 3960 miles. The formula relating weight and distance is

$r = \sqrt{\frac{3960^2 W_E}{W_S}} - 3960$, where W_E represents the weight of a body on Earth, W_S represents the weight of a body a certain distance from the center of Earth, and r represents the distance of an object above Earth's surface.

33. An astronaut weighs 140 pounds on Earth and 120 pounds in space. How far is he above Earth's surface?

34. An astronaut weighs 125 pounds on Earth. What is her weight in space if she is 99 miles above the surface of Earth?

More About . . .

Aerospace

The weight of a person is equal to the product of the person's mass and the acceleration due to Earth's gravity. Thus, as a person moves away from Earth, the person's weight decreases. However, mass remains constant.

35. **RESEARCH** Use the Internet or another resource to find the weights, on Earth, of several space shuttle astronauts and the average distance they were from Earth during their missions. Use this information to calculate their weights while in orbit.

36. **CRITICAL THINKING** Recall how values of a, h, and k can affect the graph of a quadratic function of the form $y = a(x - h)^2 + k$. Describe how values of a, h, and k can affect the graph of a square root function of the form $y = a\sqrt{x - h} + k$.

37. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are square root functions used in bridge design?

Include the following in your answer:

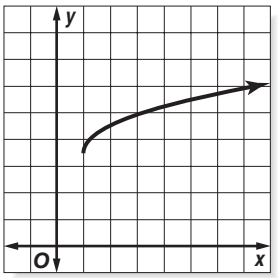
- the weights for which a diameter less than 1 is reasonable, and
- the weight that the Sunshine Skyway Bridge can support.

Standardized Test Practice

38. What is the domain of $f(x) \geq \sqrt{5x - 3}$?

(A) $\left\{x \mid x > \frac{3}{5}\right\}$ (B) $\left\{x \mid x > -\frac{3}{5}\right\}$ (C) $\left\{x \mid x \geq \frac{3}{5}\right\}$ (D) $\left\{x \mid x \geq -\frac{3}{5}\right\}$

39. Given the graph of the square root function at the right, which of the following must be true?



I. The domain is all real numbers.

II. The function is $y = \sqrt{x} + 3.5$.

III. The range is $\{y \mid y \geq 3.5\}$.

(A) I only (B) I, II, and III

(C) II and III (D) III only

Maintain Your Skills

Mixed Review

Determine whether each pair of functions are inverse functions. *(Lesson 7-8)*

40. $f(x) = 3x$
$g(x) = \frac{1}{3}x$

41. $f(x) = 4x - 5$
$g(x) = \frac{1}{4}x - \frac{5}{16}$

42. $f(x) = \frac{3x + 2}{7}$
$g(x) = \frac{7x - 2}{3}$

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. *(Lesson 7-7)*

43. $f(x) = x + 5$
$g(x) = x - 3$

44. $f(x) = 10x - 20$
$g(x) = x - 2$

45. $f(x) = 4x^2 - 9$
$g(x) = \frac{1}{2x + 3}$

46. **ENTERTAINMENT** A magician asked a member of his audience to choose any number. He said, "Multiply your number by 3. Add the sum of your number and 8 to that result. Now divide by the sum of your number and 2." The magician announced the final answer without asking the original number. What was the final answer? How did he know what it was? *(Lesson 5-4)*

Simplify. *(Lesson 5-2)*

47. $(x + 2)(2x - 8)$

48. $(3p + 5)(2p - 4)$

49. $(a^2 + a + 1)(a - 1)$

WebQuest Internet Project

Population Explosion

It is time to complete your project. Use the information and data you have gathered about the population to prepare a Web page. Be sure to include graphs, tables, and equations in the presentation.

www.algebra2.com/webquest

Chapter 7

Study Guide and Review

Vocabulary and Concept Check

Complex Conjugates Theorem (p. 374)
composition of functions (p. 384)
degree of a polynomial (p. 346)
depressed polynomial (p. 366)
Descartes' Rule of Signs (p. 372)
end behavior (p. 349)
Factor Theorem (p. 366)
Fundamental Theorem of Algebra (p. 371)
identity function (p. 391)
Integral Zero Theorem (p. 378)
inverse function (p. 391)
inverse relation (p. 390)
leading coefficients (p. 346)
Location Principle (p. 353)
one-to-one (p. 392)
polynomial function (p. 347)
polynomial in one variable (p. 346)
quadratic form (p. 360)
Rational Zero Theorem (p. 378)
relative maximum (p. 354)
relative minimum (p. 354)
Remainder Theorem (p. 365)
square root function (p. 395)
square root inequality (p. 397)
synthetic substitution (p. 365)

Choose the letter that best matches each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser y-coordinates is a _____.
2. The _____ is the coefficient of the term in a polynomial function with the highest degree.
3. The _____ says that in any polynomial function, if an imaginary number is a zero of that function, then its conjugate is also a zero.
4. When a polynomial is divided by one of its binomial factors, the quotient is called a(n) _____.
5. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in _____.
6. $f(x) = 6x - 2$ and $g(x) = \frac{x+2}{6}$ are _____ since $[f \circ g](x)$ and $[g \circ f](x) = x$.

a. Complex Conjugates Theorem
b. depressed polynomial
c. inverse functions
d. leading coefficient
e. quadratic form
f. relative minimum

Lesson-by-Lesson Review

7-1 Polynomial Functions

See pages 346–352.

Concept Summary

- The degree of a polynomial function in one variable is determined by the greatest exponent of its variable.

Example **Find $p(a + 1)$ if $p(x) = 5x - x^2 + 3x^3$.**

$p(a + 1) = 5(a + 1) - (a + 1)^2 + 3(a + 1)^3$ — Replace x with $a + 1$.

$= 5a + 5 - (a^2 + 2a + 1) + 3(a^3 + 3a^2 + 3a + 1)$ — Evaluate $5(a + 1)$, $(a + 1)^2$, and $3(a + 1)^3$.

$= 5a + 5 - a^2 - 2a - 1 + 3a^3 + 9a^2 + 9a + 3$

$= 3a^3 + 8a^2 + 12a + 7$ — Simplify.

Exercises **Find $p(-4)$ and $p(x + h)$ for each function.**

See Examples 2 and 3 on pages 347 and 348.

7. $p(x) = x - 2$
8. $p(x) = -x + 4$
9. $p(x) = 6x + 3$
10. $p(x) = x^2 + 5$
11. $p(x) = x^2 - x$
12. $p(x) = 2x^3 - 1$

7-2 Graphing Polynomial Functions

See pages 353–358.

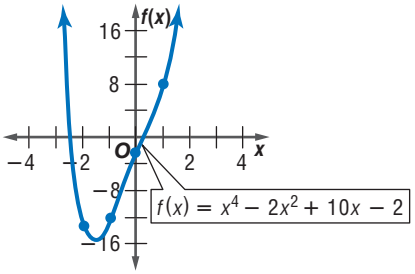
Concept Summary

- The Location Principle: Since zeros of a function are located at x-intercepts, there is also a zero between each pair of these zeros.
- Turning points of a function are called relative maxima and relative minima.

Example **Graph $f(x) = x^4 - 2x^2 + 10x - 2$ by making a table of values.**

Make a table of values for several values of x and plot the points. Connect the points with a smooth curve.

x	$f(x)$
−3	31
−2	−14
−1	−13
0	−2
1	7
2	26



Exercises **For Exercises 13–18, complete each of the following.**

a. Graph each function by making a table of values.

b. Determine consecutive values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur. *See Example 1 on page 353.*

13. $h(x) = x^3 - 6x - 9$

14. $f(x) = x^4 + 7x + 1$

15. $p(x) = x^5 + x^4 - 2x^3 + 1$

16. $g(x) = x^3 - x^2 + 1$

17. $r(x) = 4x^3 + x^2 - 11x + 3$

18. $f(x) = x^3 + 4x^2 + x - 2$

7-3 Solving Equations Using Quadratic Techniques

See pages 360–364.

Concept Summary

- Solve polynomial equations by using quadratic techniques.

Example **Solve $x^3 - 3x^2 - 54x = 0$.**

$x^3 - 3x^2 - 54x = 0$	Original equation
$x(x^2 - 3x - 54) = 0$	Factor out the GCF.
$x(x - 9)(x + 6) = 0$	Factor the trinomial.
$x = 0$ or $x - 9 = 0$ or $x + 6 = 0$	Zero Product Property
$x = 0$ $\quad$ $x = 9$ $\quad$ $x = -6$	

Exercises **Solve each equation.** *See Example 2 on page 361.*

19. $3x^3 + 4x^2 - 15x = 0$

20. $m^4 + 3m^3 = 40m^2$

21. $a^3 - 64 = 0$

22. $r + 9\sqrt{r} = -8$

23. $x^4 - 8x^2 + 16 = 0$

24. $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 20 = 0$

7-4 The Remainder and Factor Theorems

See pages 365–370.

Concept Summary

- Remainder Theorem: If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$ and $f(x) = q(x) \cdot (x - a) + f(a)$ where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.
- Factor Theorem: $x - a$ is a factor of polynomial $f(x)$ if and only if $f(a) = 0$.

Example **Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Then find any remaining factors of the polynomial.**

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

The remainder is 0, so $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Since $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$, the factors of $x^3 - 2x^2 - 5x + 6$ are $(x + 2)(x - 3)(x - 1)$.

Exercises **Use synthetic substitution to find $f(3)$ and $f(-2)$ for each function.** *See Example 2 on page 367.*

25. $f(x) = x^2 - 5$ **26.** $f(x) = x^2 - 4x + 4$ **27.** $f(x) = x^3 - 3x^2 + 4x + 8$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. *See Example 3 on page 367.*

28. $x^3 + 5x^2 + 8x + 4;\ x + 1$ **29.** $x^3 + 4x^2 + 7x + 6;\ x + 2$

7-5 Roots and Zeros

See pages 371–377.

Concept Summary

- Fundamental Theorem of Algebra: Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.
- Use Descartes' Rule of Signs to determine types of zeros of polynomial functions.
- Complex Conjugates Theorem: If $a + bi$ is a zero of a polynomial function, then $a - bi$ is also a zero of the function.

Example **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 5x^4 + 6x^3 - 8x + 12$.**

Since $f(x)$ has two sign changes, there are 2 or 0 real positive zeros.

$f(-x) = 5x^4 - 6x^3 + 8x + 12$ Two sign changes $\rightarrow$ 0 or 2 negative real zeros

There are 0, 2, or 4 imaginary zeros.

Exercises **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.** *See Example 2 on page 373.*

30. $f(x) = 2x^4 - x^3 + 5x^2 + 3x - 9$ **31.** $f(x) = 7x^3 + 5x - 1$

32. $f(x) = -4x^4 - x^2 - x + 1$ **33.** $f(x) = 3x^4 - x^3 + 8x^2 + x - 7$

34. $f(x) = x^4 + x^3 - 7x + 1$ **35.** $f(x) = 2x^4 - 3x^3 - 2x^2 + 3$

7-6 Rational Zero Theorem

See pages 378–382.

Concept Summary

- Use the Rational Zero Theorem to find possible zeros of a polynomial function.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n \neq 0$, any rational zeros of the function must be factors of a_n.

Examples **Find all of the zeros of $f(x) = x^3 + 7x^2 - 36$.**

There are exactly three complex zeros.

There is exactly one positive real zero and two or zero negative real zeros.

The possible rational zeros are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36$.

$$\begin{array}{r|rrrr} 2 & 1 & 7 & 0 & -36 \\ & & 2 & 18 & 36 \\ \hline & 1 & 9 & 18 & 0 \end{array}$$

$$\begin{aligned} x^3 + 7x^2 - 36 &= (x - 2)(x^2 + 9x + 18) \\ &= (x - 2)(x + 3)(x + 6) \end{aligned}$$

Therefore, the zeros are 2, −3, and −6.

Exercises **Find all of the rational zeros of each function.** *See Example 3 on page 379.*

36. $f(x) = 2x^3 - 13x^2 + 17x + 12$

37. $f(x) = x^4 + 5x^3 + 15x^2 + 19x + 8$

38. $f(x) = x^3 - 3x^2 - 10x + 24$

39. $f(x) = x^4 - 4x^3 - 7x^2 + 34x - 24$

40. $f(x) = 2x^3 - 5x^2 - 28x + 15$

41. $f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10$

7-7 Operations of Functions

See pages 383–389.

Concept Summary

Operation	Definition	Operation	Definition
Sum	$(f + g)(x) = f(x) + g(x)$	Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$
Difference	$(f - g)(x) = f(x) - g(x)$	Composition	$[f \circ g](x) = f[g(x)]$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	—	—

Example **If $f(x) = x^2 - 2$ and $g(x) = 8x - 1$. Find $g[f(x)]$ and $f[g(x)]$.**

$g[f(x)] = 8(x^2 - 2) - 1$ Replace $f(x)$ with $x^2 - 2$.

$= 8x^2 - 16 - 1$ Multiply.

$= 8x^2 - 17$ Simplify.

$f[g(x)] = (8x - 1)^2 - 2$ Replace $g(x)$ with $8x - 1$.

$= 64x^2 - 16x + 1 - 2$ Expand the binomial.

$= 64x^2 - 16x - 1$ Simplify.

Exercises **Find $[g \circ h](x)$ and $[h \circ g](x)$.** *See Example 4 on page 385.*

42. $h(x) = 2x - 1$, $g(x) = 3x + 4$

43. $h(x) = x^2 + 2$, $g(x) = x - 3$

44. $h(x) = x^2 + 1$, $g(x) = -2x + 1$

45. $h(x) = -5x$, $g(x) = 3x - 5$

46. $h(x) = x^3$, $g(x) = x - 2$

47. $h(x) = x + 4$, $g(x) = |x|$

7-8 Inverse Functions and Relations

See pages 390–394.

Concept Summary

- Reverse the coordinates of ordered pairs to find the inverse of a relation.
- Two functions are inverses if and only if both of their compositions are the identity function. $[f \circ g](x) = x$ and $[g \circ f](x) = x$
- A function is one-to-one when the inverse of the function is a function.

Example **Find the inverse of $f(x) = -3x + 1$.**

Rewrite $f(x)$ as $y = -3x + 1$. Then interchange the variables and solve for y.

$x = -3y + 1$ Interchange the variables.

$3y = -x + 1$ Solve for y.

$y = \frac{-x + 1}{3}$ Divide each side by 3.

$f^{-1}(x) = \frac{-x + 1}{3}$ Rewrite in function notation.

Exercises **Find the inverse of each function. Then graph the function and its inverse.** *See Example 2 on page 391.*

48. $f(x) = 3x - 4$ **49.** $f(x) = -2x - 3$ **50.** $g(x) = \frac{1}{3}x + 2$

51. $f(x) = \frac{-3x + 1}{2}$ **52.** $y = x^2$ **53.** $y = (2x + 3)^2$

7-9 Square Root Functions and Inequalities

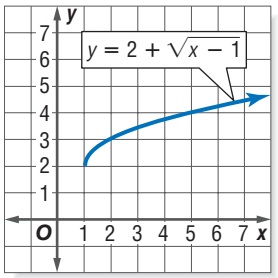
See pages 395–399.

Concept Summary

- Graph square root inequalities in a similar manner as graphing square root equations.

Example **Graph $y = 2 + \sqrt{x - 1}$.**

x	y
1	2
2	3
3	$2 + \sqrt{2}$ or 3.4
4	$2 + \sqrt{3}$ or 3.7
5	4



Exercises **Graph each function. State the domain and range of each function.** *See Examples 1 and 2 on pages 395 and 396.*

54. $y = \frac{1}{3}\sqrt{x + 2}$ **55.** $y = \sqrt{5x - 3}$ **56.** $y = 4 + 2\sqrt{x - 3}$

Graph each inequality. *See Example 3 on page 397.*

57. $y \geq \sqrt{x} - 2$ **58.** $y < \sqrt{4x - 5}$

Chapter 7

Practice Test

Vocabulary and Concepts

Match each statement with the term that it best describes.

1. $[f \circ g](x) = f[g(x)]$
2. $[f \circ g](x) = x$ and $[g \circ f](x) = x$
3. $(\sqrt{x})^2 - 2(\sqrt{x}) + 4 = 0$

a. quadratic form
b. composition of functions
c. inverse functions

Skills and Applications

For Exercises 4–7, complete each of the following.

a. Graph each function by making a table of values.
b. Determine consecutive values of x between which each real zero is located.
c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

4. $g(x) = x^3 + 6x^2 + 6x - 4$
5. $h(x) = x^4 + 6x^3 + 8x^2 - x$
6. $f(x) = x^3 + 3x^2 - 2x + 1$
7. $g(x) = x^4 - 2x^3 - 6x^2 + 8x + 5$

Solve each equation.

8. $p^3 + 8p^2 = 18p$
9. $16x^4 - x^2 = 0$
10. $r^4 - 9r^2 + 18 = 0$
11. $p^{\frac{3}{2}} - 8 = 0$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

12. $x^3 - x^2 - 5x - 3;\ x + 1$
13. $x^3 + 8x + 24;\ x + 2$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

14. $f(x) = x^3 - x^2 - 14x + 24$
15. $f(x) = 2x^3 - x^2 + 16x - 5$

Find all of the rational zeros of each function.

16. $g(x) = x^3 - 3x^2 - 53x - 9$
17. $h(x) = x^4 + 2x^3 - 23x^2 + 2x - 24$

Determine whether each pair of functions are inverse functions.

18. $f(x) = 4x - 9,\ g(x) = \frac{x - 9}{4}$
19. $f(x) = \frac{1}{x + 2},\ g(x) = \frac{1}{x} - 2$

If $f(x) = 2x - 4$ and $g(x) = x^2 + 3$, find each value.

20. $(f + g)(x)$
21. $(f - g)(x)$
22. $(f \cdot g)(x)$
23. $\left(\frac{f}{g}\right)(x)$

24. **FINANCIAL PLANNING** Toshi will start college in six years. According to their plan, Toshi's parents will save \$1000 each year for the next three years. During the fourth and fifth years, they will save \$1200 each year. During the last year before he starts college, they will save \$2000.
    a. In the formula $A = P(1 + r)^t$, A = the balance, P = the amount invested, r = the interest rate, and t = the number of years the money has been invested. Use this formula to write a polynomial equation to describe the balance of the account when Toshi starts college.
    b. Find the balance of the account if the interest rate is 6%.

25. **STANDARDIZED TEST PRACTICE** Which value is included in the graph of $y < \sqrt{2x}$?
    Ⓐ $(-2, -2)$ Ⓑ $(-1, -1)$ Ⓒ $(0, 0)$ Ⓓ None of these

Chapter 7 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If $\frac{2}{p} - \frac{4}{p^2} = -\frac{2}{p^3}$, then what is the value of p?

Ⓐ -1 Ⓑ 1 Ⓒ $-\frac{1}{2}$ Ⓓ $\frac{1}{2}$

2. There are n gallons of liquid available to fill a tank. After k gallons of the liquid have filled the tank, how do you represent in terms of n and k the percent of liquid that has filled the tank?

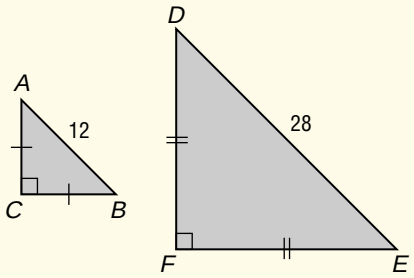
Ⓐ $\frac{100k}{n}\%$ Ⓑ $\frac{n}{100k}\%$

Ⓒ $\frac{100n}{k}\%$ Ⓓ $\frac{n}{100(n-k)}\%$

3. How many different triangles have sides of lengths 4, 9 and s, where s is an integer and $4 < s < 9$?

Ⓐ 0 Ⓑ 1 Ⓒ 2 Ⓓ 3

4. Triangles ABC and DEF are similar. The area of $\triangle ABC$ is 36 square units. What is the perimeter of $\triangle DEF$?



Ⓐ 56 units Ⓑ $28 + 28\sqrt{2}$ units

Ⓒ $56\sqrt{2}$ units Ⓓ $28 + 14\sqrt{2}$ units

5. If $2 - 3x > -1$ and $x + 5 > 0$, then x could equal each of the following *except*

Ⓐ -5. Ⓑ -4. Ⓒ -2. Ⓓ 0.

6. What is the midpoint of the line segment whose endpoints are represented on the coordinate grid by the points $(-5, -3)$ and $(-1, 4)$?

Ⓐ $\left(-3, -\frac{1}{2}\right)$ Ⓑ $\left(-3, \frac{1}{2}\right)$

Ⓒ $\left(-2, -\frac{7}{2}\right)$ Ⓓ $\left(-2, \frac{1}{2}\right)$

7. For all $n \neq 0$, what is the slope of the line passing through (n, k) and $(-n, -k)$?

Ⓐ 0 Ⓑ 1 Ⓒ $\frac{n}{k}$ Ⓓ $\frac{k}{n}$

8. Which of the following is a quadratic equation in one variable?

Ⓐ $3(x + 4) + 1 = 4x - 9$

Ⓑ $3x(x + 4) + 1 = 4x - 9$

Ⓒ $3x(x^2 + 4) + 1 = 4x - 9$

Ⓓ $y = 3x^2 + 8x + 10$

9. Simplify $\sqrt[4]{t^3} \cdot \sqrt[8]{t^2}$.

Ⓐ $t^{\frac{3}{16}}$ Ⓑ $t^{\frac{1}{2}}$ Ⓒ $t^{\frac{3}{4}}$ Ⓓ t

10. Which of the following is a quadratic equation that has roots of $2\frac{1}{2}$ and $\frac{2}{3}$?

Ⓐ $5x^2 + 11x - 7 = 0$

Ⓑ $5x^2 - 11x + 10 = 0$

Ⓒ $6x^2 - 19x + 10 = 0$

Ⓓ $6x^2 + 11x + 10 = 0$

11. If $f(x) = 3x - 5$ and $g(x) = 2 + x^2$, then what is equal to $f[g(2)]$?

Ⓐ 3 Ⓑ 6 Ⓒ 12 Ⓓ 13

12. Which of the following is a zero of $f(x) = x^3 - 7x + 6$?

Ⓐ -1 Ⓑ 2 Ⓒ 3 Ⓓ 6

Preparing for Standardized Tests
For test-taking strategies and more practice, see pages 877–892.

Part 2 Short Response/Grid In

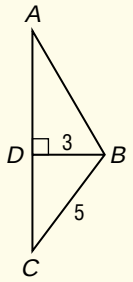
Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

13. A group of 34 people is to be divided into committees so that each person serves on exactly one committee. Each committee must have at least 3 members and not more than 5 members. If N represents the maximum number of committees that can be formed and n represents the minimum number of committees that can be formed, what is the value of $N - n$?

14. Raisins selling for \$2.00 per pound are to be mixed with peanuts selling for \$3.00 per pound. How many pounds of peanuts are needed to produce a 20-pound mixture that sells for \$2.75 per pound?

15. Jars X, Y, and Z each contain 10 marbles. What is the minimum number of marbles that must be transferred among the jars so that the ratio of the number of marbles in jar X to the number of marbles in jar Y to the number of marbles in jar Z is 1 : 2 : 3?

16. If the area of $\triangle BCD$ is 40% of the area of $\triangle ABC$, what is the measure of $\overline{AD}$?

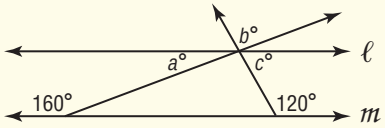


17. The mean of 15 scores is 82. If the mean of 7 of these scores is 78, what is the mean of the remaining 8 scores?

18. If the measures of the sides of a triangle are 3, 8, and x and x is an integer, then what is the least possible perimeter of the triangle?

19. If the operation ❖ is defined by the equation x ❖ $y = 3x - y$, what is the value of w in the equation w ❖ $6 = 2$ ❖ w?

20. In the figure below, $\ell \parallel m$. Find b. Assume that the figure is not drawn to scale.



Part 3 Extended Response

Record your answers on a sheet of paper. Show your work.

For Exercises 21–25, use the polynomial function $f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2$.

21. What is the degree of the function?

22. Evaluate $f(1)$, $f(-2)$, and $f(2a)$. Show your work.

23. State the number of possible positive real zeros, negative real zeros, and imaginary zeros of $f(x)$.

24. List all of the possible rational zeros of the function.

25. Find all of the zeros of the function.

26. Sketch the graphs of $f(x) = \frac{3x + 1}{2}$ and $g(x) = \frac{2x - 1}{3}$. Considering the graphs, describe the relationship between $f(x)$ and $g(x)$. Verify your conclusion.

Test-Taking Tip

Questions 13, 15, and 18
Words such as *maximum*, *minimum*, *least*, and *greatest* indicate that a problem may involve an inequality. Take special care when simplifying inequalities that involve negative numbers.