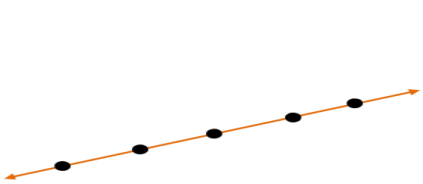
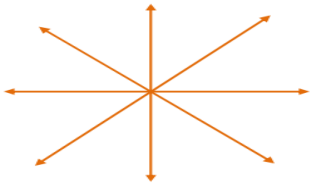
Chapter 3: Geometry and Measurement

3.1. Theorems on triangles

Three or more points that lie on one line are called collinear points. Three or more lines that pass through one point are called concurrent lines.



Collinear points



Concurrent lines

A median of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side.

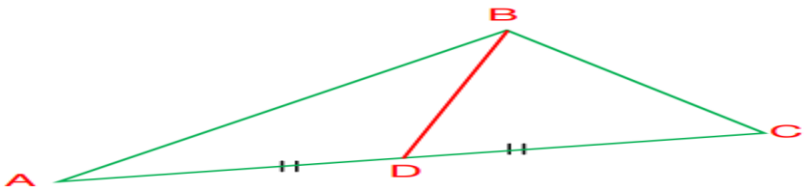


Figure 6.2

$\overline{BD}$ is a median of triangle ABC.

Theorem 3.1: The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

Example 1 In Figure 6.6, $\overline{AN}$, $\overline{CM}$ and $\overline{BL}$ are medians of ΔABC. If $AN = 12$ cm, $OM = 5$ cm and $BO = 6$ cm, find BL, ON and OL.

Solution:

By Theorem 6.1,

$$BO = \frac{2}{3} BL \text{ and } AO = \frac{2}{3} AN$$

Substituting $6 = \frac{2}{3} BL$ and $AO = \frac{2}{3} \times 12$

So $BL = 9$ cm and $AO = 8$ cm.

Since $BL = BO + OL$,

$$OL = BL - BO = 9 - 6 = 3 \text{ cm}.$$

Now, $AN = AO + ON$ gives

$$ON = AN - AO = 12 - 8 = 4 \text{ cm}$$

$\therefore BL = 9$ cm, $OL = 3$ cm and $ON = 4$ cm

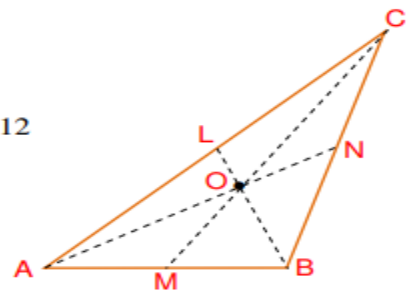


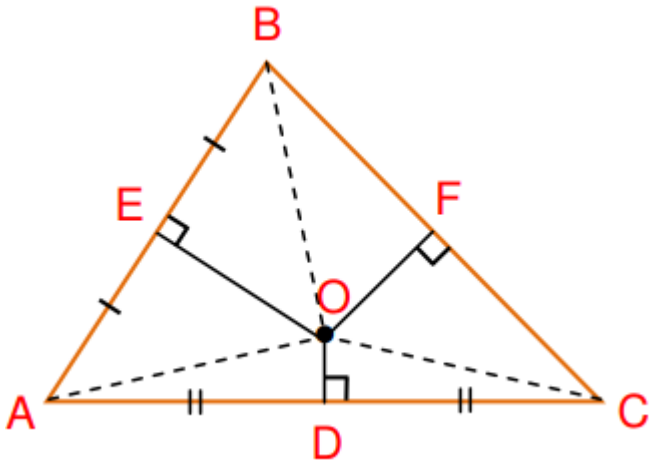
Figure 6.6

Note: The point of intersection of the medians of a triangle is called the centroid of the triangle.

Definition: The altitude of a triangle is a line segment drawn from a vertex, perpendicular to the opposite side, or to the opposite side produced.

Theorem 3.2: The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

$$i.e\ AO = OC = OB$$



Note: The point of intersection of the perpendicular bisectors of a triangle is called circumcenter of the triangle.

Theorem 3.3: The altitudes of a triangle are concurrent.

Note: The point of intersection of the altitudes of a triangle is called orthocenter of the triangle.

Theorem 3.4: The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

Note: The point of intersection of the bisectors of the angles of a triangle is called the incentre of the triangle.

Example 2 In a right angle triangle ABC, $\angle C$ is a right angle, $AB = 8$ cm and $CA = 6$ cm. Find the length of $\overline{CO}$ where O is the point of intersection of the perpendicular bisectors of ΔABC.

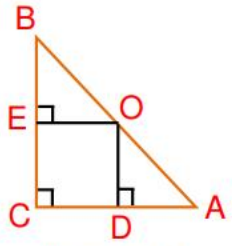


Figure 6.11

Solution: The perpendicular bisector of $\overline{CA}$ is parallel to $\overline{CB}$.

Hence, O is on $\overline{AB}$.

Therefore, $AO = 4$. (By theorem 6.2, $AO = BO$)

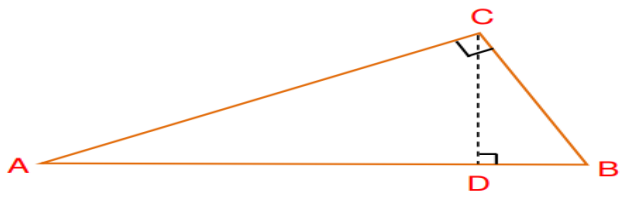
By theorem 6.2, O is equidistant from A, B and C

Therefore, $CO = AO = 4$ cm.

Theorem 3.5 Altitude theorem

In a right angled triangle *ABC* with altitude *CD* to the hypotenuse *AB*, $\frac{AD}{DC} = \frac{CD}{DB}$

$$i.e\ (AD)(DB) = CD^2$$



Example 3 In ΔABC, $\overline{CD}$ is the altitude to the hypotenuse $\overline{AB}$, $AD = 9$ cm and $BD = 4$ cm. How long is the altitude $\overline{CD}$? *See* Figure 6.12.

Solution Let $h = CD$. From the Altitude Theorem, $(CD)^2 = (AD)\,(BD)$

Substituting, $h^2 = 9 \times 4 = 36$ cm ^{2}

So, $h = 6$ cm.

The length of the altitude is 6 cm.

3.2. Special quadrilaterals

In this section, we consider the following special quadrilaterals: trapezium, parallelogram, rectangle, rhombus and square.

Definition 3.1: A trapezium is a quadrilateral where only two of the sides are parallel.

In Figure 6.22, the quadrilateral *ABCD* is a trapezium. The sides $\overline{AD}$ and $\overline{BC}$ are non-parallel sides of the trapezium *ABCD*.

Note that if the sides $\overline{AD}$ and $\overline{BC}$ of trapezium *ABCD* are congruent, then the trapezium is called an isosceles trapezium.

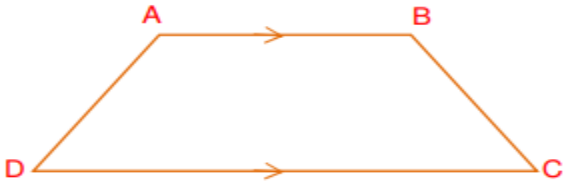
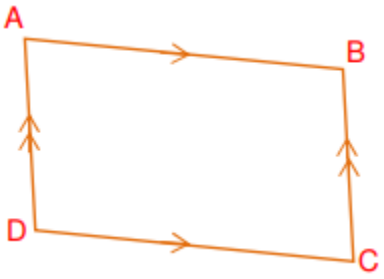


Figure 6.22

Definition 3.2: A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

In Figure 6.23, the quadrilateral *ABCD* is a parallelogram. $\overline{AB} \,//\, \overline{DC}$ and $\overline{AD} \,//\, \overline{BC}$



Theorem 3.7

a. The opposite sides of a parallelogram are congruent.

b. The opposite angles of a parallelogram are congruent.

c. The diagonals of a parallelogram bisect each other.

d. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

e. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

f. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Definition 3.3: A rectangle is a parallelogram in which one of its angles is a right angle.

In Figure 6.27, the parallelogram *ABCD* is a rectangle whose angle *D* is a right angle. What is the measure of each of the other angles of the rectangle *ABCD*?

Some properties of a rectangle

i A rectangle has all properties of a parallelogram.

ii Each interior angle of a rectangle is a right angle.

iii The diagonals of a rectangle are congruent.

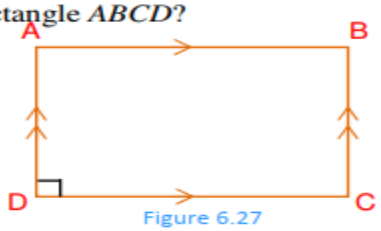


Figure 6.27

Definition 3.4: A rhombus is a parallelogram which has two congruent adjacent sides.

In Figure 6.28, the parallelogram *ABCD* is a rhombus.

Some properties of a rhombus

i A rhombus has all the properties of a parallelogram.

ii A rhombus is an equilateral quadrilateral.

iii The diagonals of a rhombus are perpendicular to each other.

iv The diagonals of a rhombus bisect its angles.

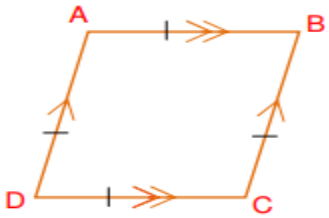


Figure 6.28

Definition 3.5: A square is a rectangle which has congruent adjacent sides.

In Figure 6.29, the rectangle *ABCD* is a square.

Some properties of a square

i A square has the properties of a rectangle.

ii A square has all the properties of a rhombus.

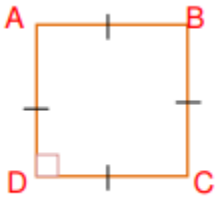


Figure 6.29

Theorem 3.8: If the diagonals of a quadrilateral are congruent and are perpendicular bisectors of each other, then the quadrilateral is a square.

Properties of Different Types of Quadrilaterals

Type of Quadrilateral	Properties	Formulas
Trapezium	• Only one pair of opposite sides are parallel • Non-parallel sides are called legs • Diagonals intersect each other • The length of the mid-segment is equal to half the sum of the parallel bases	Area of trapezium = (½) (a + b)h = (½) (Sum of two parallel sides) × HeightPerimeter = Sum of all the sides
Parallelogram	• The opposite sides are parallel and equal • The opposite angles are equal • The sum of two consecutive angles is 180 degrees • If any one of the angles is a right angle, then all the other angles will be right angles • Diagonals bisect each other • Each diagonal bisects the parallelogram into two congruent triangles	Area = Base × Height Perimeter = Sum of all the side
Rectangle	• The opposite sides are parallel and equal • Each interior angle is a right angle • The diagonals bisect each other and have the same length	Area = Length × Breadth Perimeter = 2(Length + Breadth)
Rhombus	• All sides of the rhombus are congruent, and the opposite sides are parallel • Opposite angles of a rhombus are congruent • Diagonals bisect each other at right angles • Diagonals bisect the interior angles of a rhombus • The sum of two adjacent angles is equal to 180 degrees	Area = (½) ($d_1 \times d_2$) = (½) (Product of the length of diagonals) Perimeter = 4(side length)
Square	• All four sides of the square are equal, and the opposite sides of the square are parallel to each other • The measure of each interior angle is 90 degrees • The diagonals of the square are equal to each other and bisect each other at right angles	Area = $(side)^2$ Perimeter = 4(side)

3.3. Circles

A circle is a plane figure, all points of which are equidistant from a given point called the center of the circle.

A line segment through the center of a circle with end points on the circle is called a diameter.

A chord of a circle is a line segment whose end points lie on the circle.

In Figure 3.3. 2 PQ is a chord of the circle with center O. AB is a chord (diameter) AXC is an arc of the circle.

If A and C are not end-points of a diameter, AXC is a minor arc.

∠BOC is a central angle. arc AXC is said to subtend ∠AOC or ∠AOC intercepts arc AXC

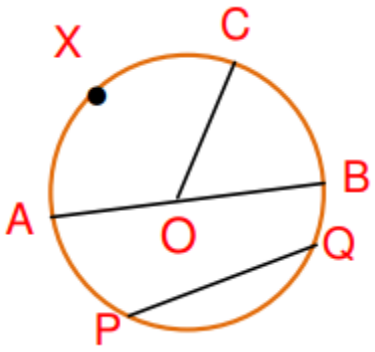


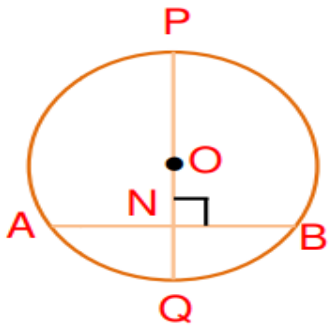
Figure 3.3. 1

Theorem 3.8 :

a. The line segment joining the center of a circle to the mid-point of a chord is perpendicular to the chord.

b. The line segment drawn from the center of a circle perpendicular to a chord bisects the chord.

$$AN = NB$$



Theorem 3.9

a. If two chords of a circle are equal, then they are equidistant from center

b. If two chords of a circle are equidistant from the center then their lengths are equal.

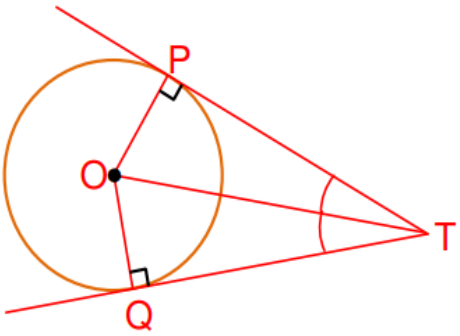
Theorem 3.10

If two tangent segments are drawn to a circle from an external point, then,

i. the tangents are equal in length $TP = TQ$

ii. The line segment joining the center to the external point bisects the angle between the tangents.

$m(\angle OTP) = m(\angle OTQ)$



A radius is perpendicular to a tangent at the point of tangency

Inscribed Angles

In a circle, an inscribed angle is an angle whose vertex lies on the circle and whose sides are chords of the circle.

A central angle of a circle is an angle whose vertex is at the center of the circle and whose sides are radii of the circle.

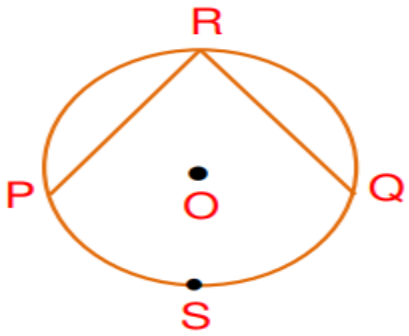


Figure 6.33

In Figure 6.33, angle PRQ is inscribed in the circle. We also say that $\angle PRQ$ is inscribed in the arc PSQ and $\angle PRQ$ is subtended by arc PSQ (or $\overset{\frown}{PSQ}$).

Measure of a central angle: Note that the measure of a central angle is the measure of the arc it intercepts.

So, $m(\angle POQ) = m(\overset{\frown}{PXQ})$

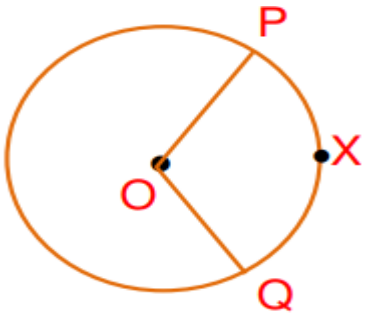
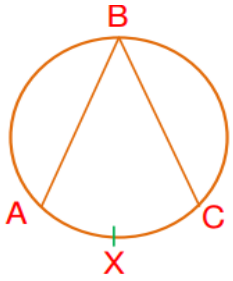
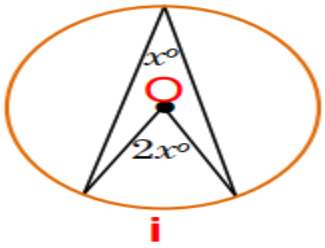


Figure 6.34

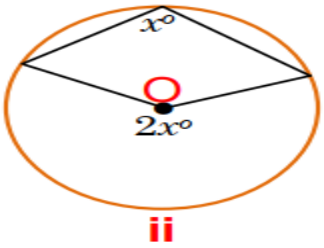
Theorem 3.11: The measure of an angle inscribed in a circle is half the measure of the arc subtending it.

$$m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$$



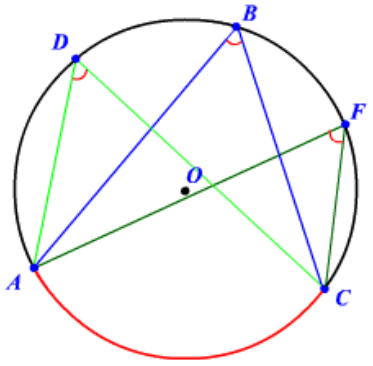


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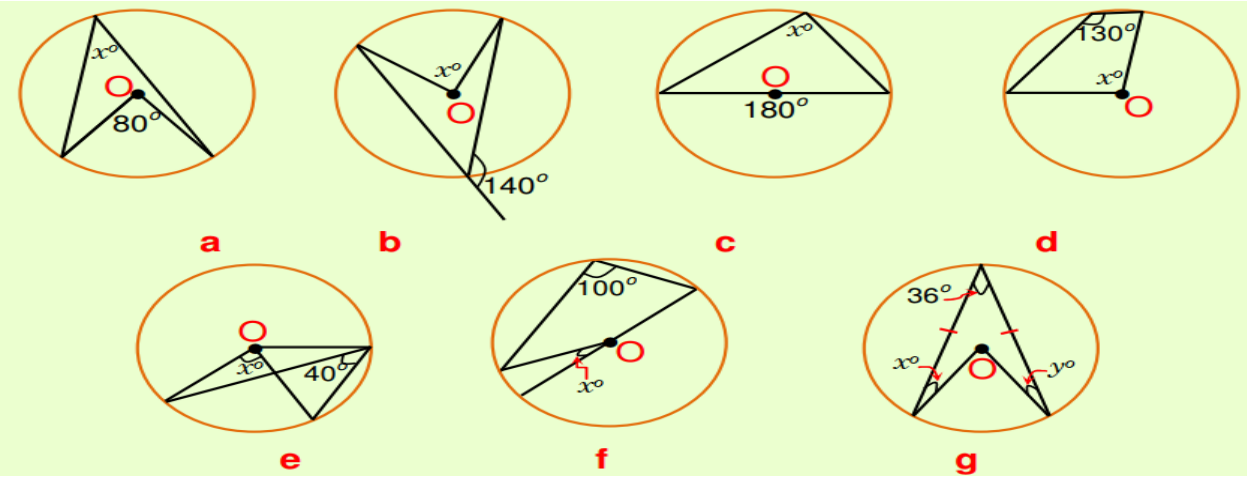
ii

Corollary : Angles inscribed in the same arc of a circle (*i.e., subtended by the same arc*) are equal.

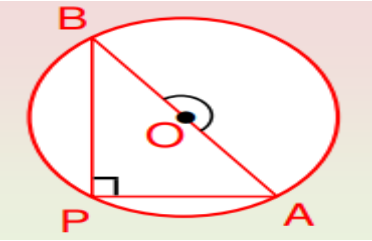


Here, $\angle ADC \cong \angle ABC \cong \angle AFC$.

Example: In each of the following figures, O is the center of the circle. Calculate the measure of the angles marked *x*.

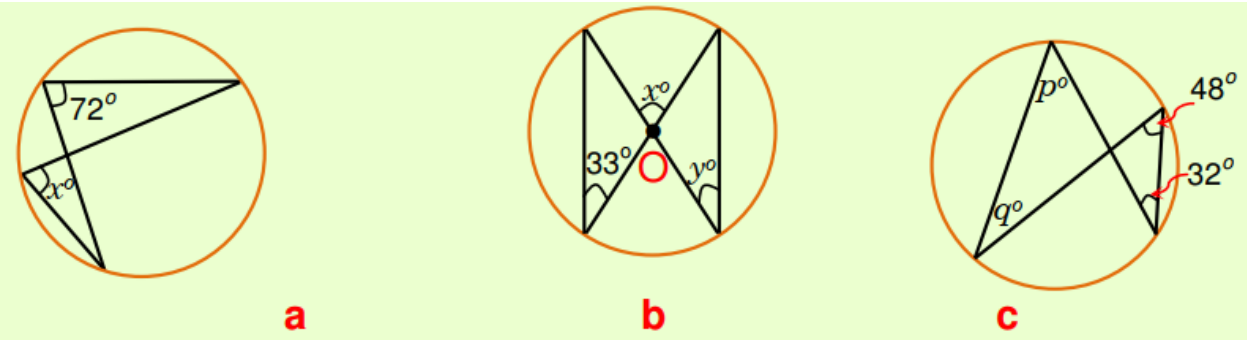


Corollary : Angle in a semicircle The angle inscribed in a semi-circle is a right angle.

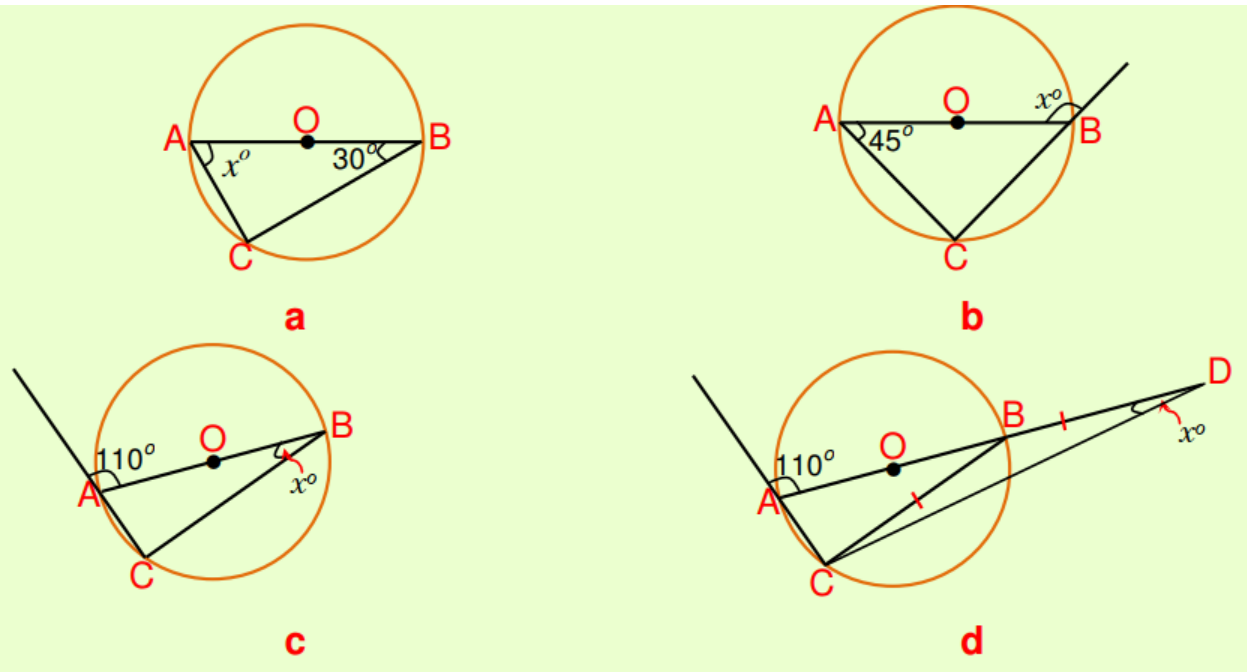


Example:

1. Calculate the marked angles in each of the following figures:



2. In each of the following figures, O is the center and $\overline{AB}$ is the diameter of the circle. Calculate the value of x in each case.



Corollary 5.13.3

Points P, Q, R and S all lie on a circle. They are called concyclic points.

Joining the points P, Q, R and S produces a cyclic quadrilateral.

The opposite angles of a cyclic quadrilateral are supplementary. i.e.,

$m(\angle P) + m(\angle R) = 180^\circ$ and $m(\angle S) + m(\angle Q) = 180^\circ$.

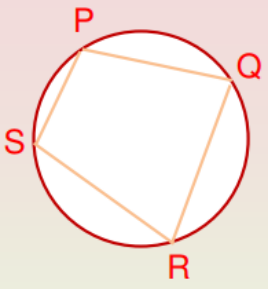
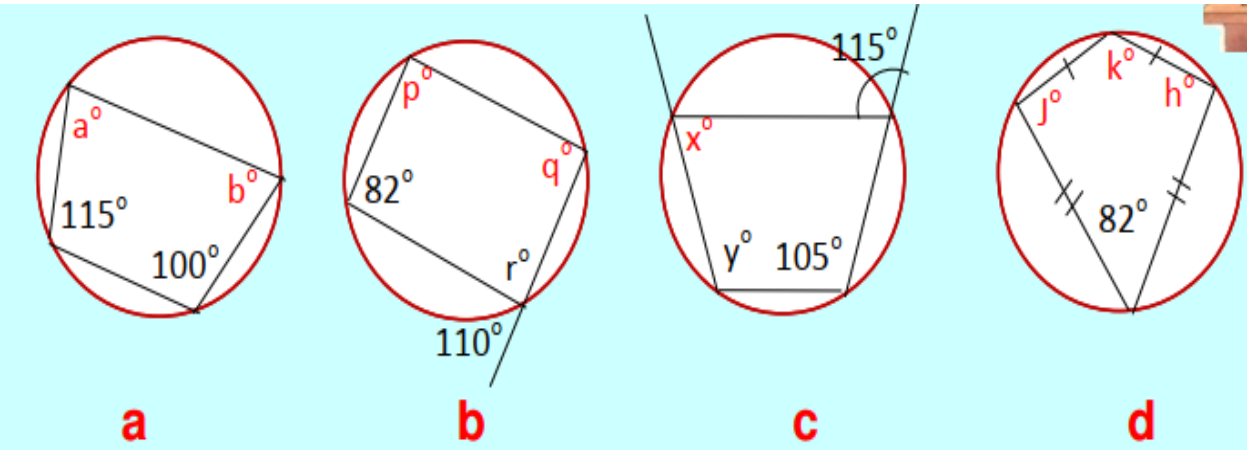
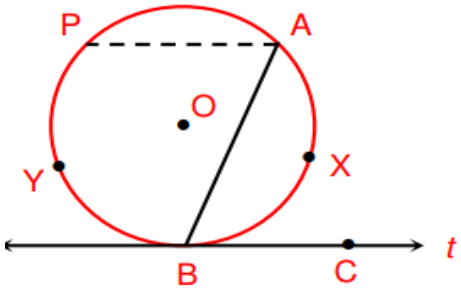


Figure 5.95

Example: Calculate the lettered angles in each of the following

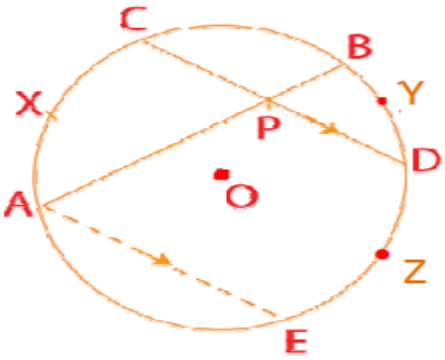


Theorem 3.12: An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.



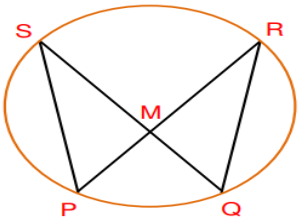
$$m(\angle ABC) = \frac{1}{2} m(\widehat{AXC})$$

Theorem 6.12: The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

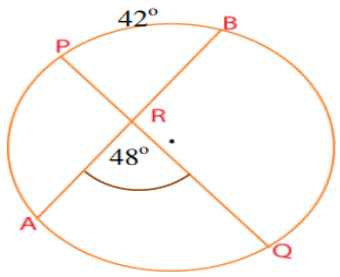


$$\boldsymbol{m(\angle BPD) = \frac{1}{2} m(\widehat{AXC}) + \frac{1}{2} m(\widehat{BYD})}$$

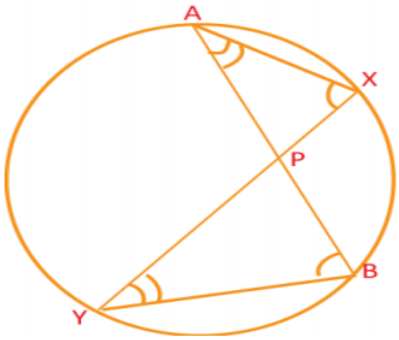
Example : In Figure below , $m(\angle MRQ) = 30^0$, and $m(\angle MQR) = 40^0$ Write down the measure of all the other angles in the two triangles, ΔPSM and ΔQMR.



Example: An angle formed by two chords intersecting within a circle is 48^0, and one of the intercepted arcs measures 42^0. Find the measures of the other intercepted arc.



Product or rectangle property of a circle: If two chords intersect in a circle as shown in Figure below , then $(AP)(PB) = (XP)(PY)$.



Example 4 In Figure 6.48, calculate the radius of the circle.

Solution: Let the radius of the circle be r units long.

Then, $OD = r$ and $PD = 2r - 2$.

Since $(AP)\ (PB) = (CP)\ (PD)$, you have

$$4 \times 4 = 2\ (2r - 2)$$

$$16 = 4r - 4$$

$$r = 5$$

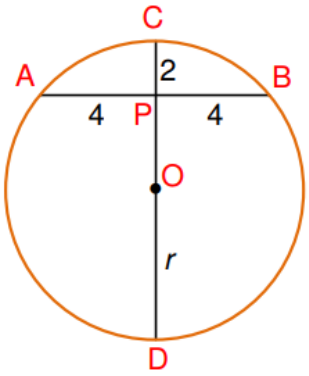
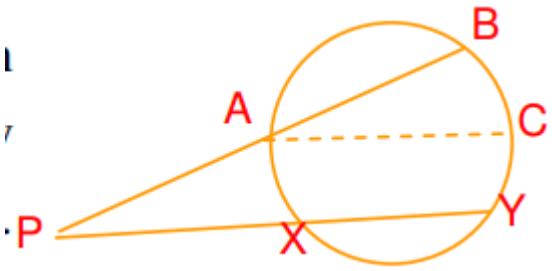


Figure 6.48

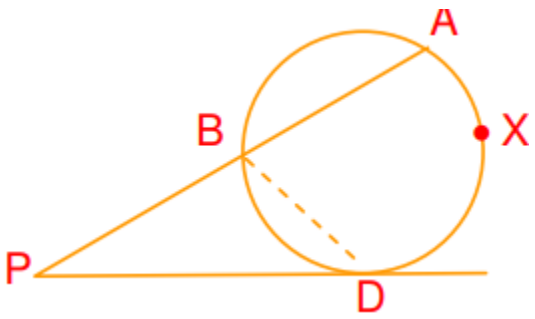
➢ Angles and Arcs Determined by Lines Intersecting Outside a Circle

Theorem 3.13: The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept.



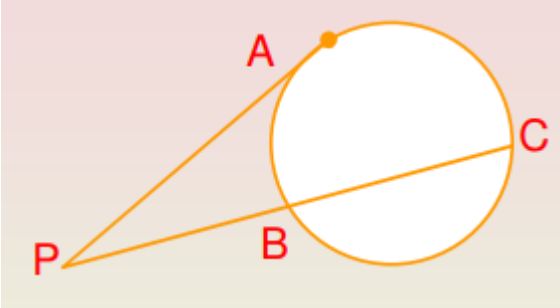
$m(\angle XPA) = \frac{1}{2}\left(m(\widehat{BY}) - m(\widehat{AX})\right)$ *and* $(PA)(PB) = (PX)(PY)$

Theorem 3.14: The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.



$$m(\angle P) = \frac{1}{2}\left(m(\overset{\frown}{AXD}) - m(\overset{\frown}{BD})\right)$$

Theorem 3.15: If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by $(PA)^2 = (PB)(PC)$.



Example 5 In Figure 6.65, from P secants $\overline{PA}$ and $\overline{PC}$ are drawn so that $m(\angle APC) = 30^\circ$; chords $\overline{AB}$ and $\overline{CD}$ intersect at F such that $m(\angle AFC) = 85^\circ$. Find the measure of arc AC, measure of arc BD and measure of $\angle ABC$.

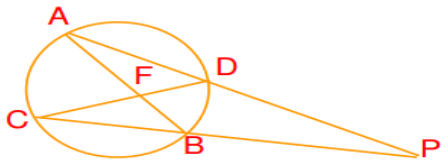


Figure 6.65

Solution: Let $m(\overset{\frown}{AC}) = x$ and $m(\overset{\frown}{DB}) = y$

$$\text{Since } m(\angle AFC) = \frac{1}{2}m(\overset{\frown}{AC}) + \frac{1}{2}m(\overset{\frown}{BD})$$

$$85^\circ = \frac{1}{2}(x + y)$$

$$x + y = 170^\circ \ldots\ldots\ldots (1)$$

$$\text{Again as } m(\angle APC) = \frac{1}{2}m(\overset{\frown}{AC}) - \frac{1}{2}m(\overset{\frown}{BD})$$

$$30^\circ = \frac{1}{2}(x - y)$$

Solving equation 1 and equation 2 simultaneously, we get

$$\begin{cases} x + y = 170^o \\ x - y = 60^o \end{cases}$$

$$2x = 230^o$$

$$x = 115^o$$

Substituting for x in equation 2,

$$115^o - y = 60^o$$

$$y = 55^o$$

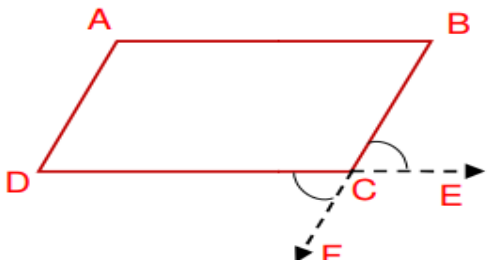
Therefore, $m(\overset{\frown}{AC}) = 115^o$ and $m(\overset{\frown}{DB}) = 55^o$.

$$m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{AC}) = \frac{1}{2}(115^o) = 57.5^o.$$

3.4. Regular Polygons

Definition: A polygon is a simple closed curve formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

Definition: An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called an exterior angle. It is formed between one side of the polygon and the extended adjacent side.



Theorem 3.16 The sum of the measures of the three interior angles of any triangle is 180^0.

Theorem 3.17: If the number of sides of a polygon is n, then the sum of the measures of all its interior angles is equal to $(n-2) \times 180^0$.

Definition : A regular polygon is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angles are equal.

For any regular n-sided polygon:

i Measure of each interior angle $= \dfrac{(n-2)180^o}{n}$.

ii Measure of each central angle $= \dfrac{360^o}{n}$.

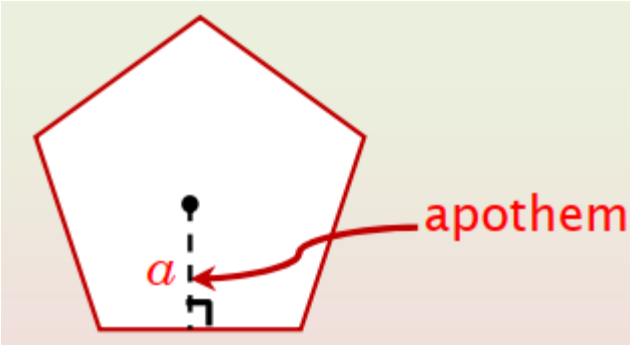
iii Measure of each exterior angle $= \dfrac{360^o}{n}$.

A polygon whose sides are tangent to a circle is said to circumscribe the circle.

A circle can always be inscribed in or circumscribed about any given regular polygon.

Definition :

The distance a from the center of a regular polygon to a side of the polygon is called the apothem of the polygon. That is, the apothem a of a regular polygon is the length of the line segment drawn from the center of the polygon perpendicular to the side of the polygon.



Theorem 5.3

Formulae for the length of side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are

1 $s = 2r \sin \dfrac{180^o}{n}$

2 $a = r \cos \dfrac{180^o}{n}$

3 $P = 2nr \sin \dfrac{180^o}{n}$

4 $A = \dfrac{1}{2}aP$

Example:

a. Find the length of the side of an equilateral triangle if its radius is $\sqrt{12}cm$.
b. Find the area of a regular hexagon whose radius is 5 cm.
c. Find the apothem of a square whose radius is $\sqrt{8}cm$.

3.5. Congruency and Similarity

Definition (Congruency): Triangles that have the same size and shape are called congruent triangles**. That is, the six parts of the triangles (three sides and three angles) are correspondingly congruent. If two triangles,** ΔABC **and** ΔDEF **are congruent like those given below, then we denote this as**

$$\Delta ABC \cong \Delta DEF$$

The notation "≅" means "is congruent to".

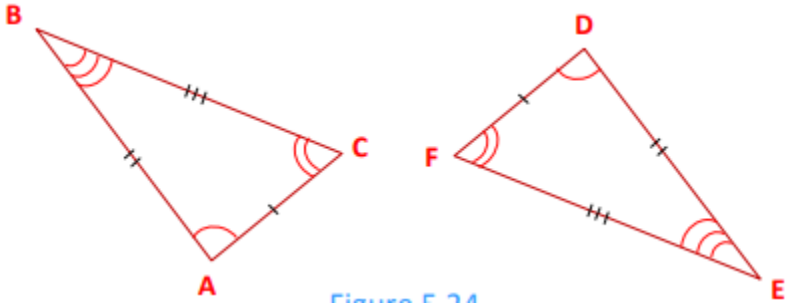
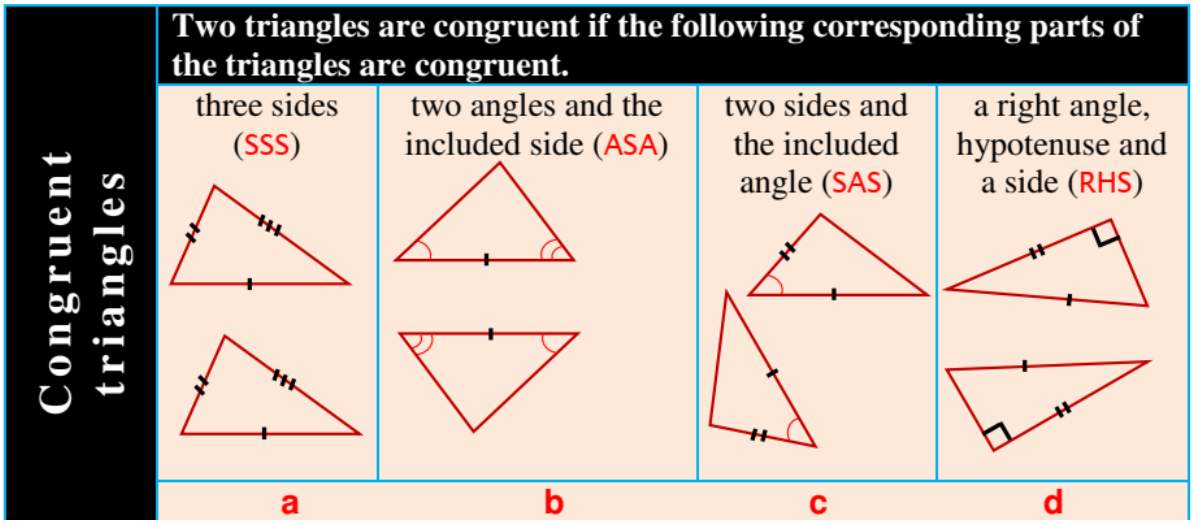


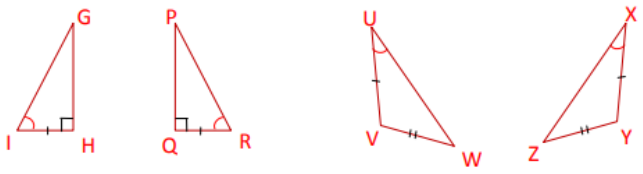
Figure 5.24

Congruent angles	Congruent sides
$\angle A \cong \angle D;\ \angle B \cong \angle E;\ \angle C \cong \angle F$	$\overline{AB} \cong \overline{DE};\ \overline{BC} \cong \overline{EF};\ \overline{AC} \cong \overline{DF}.$

Parts of congruent triangles that "match" are called corresponding parts. For example, in the triangles above, $\angle B$ corresponds to $\angle E$ and $\overline{AC}$ corresponds to $\overline{DF}$.
Note: Two triangles are congruent when all of the corresponding parts are congruent.



Example1: Determine whether each pair of triangles is congruent. If so, write a congruence statement and state why the triangles are congruent.



Example2. Given $\Delta ABC \cong \Delta RST$.find the value of y , if $m(A) = 40^{\circ}$and $m(R) = (2y + 10)^{\circ}$.

Definition (Similar polygons): two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

Example1: Referring to figure 5.42. If $ABCDE \sim HIJKL$,then find the lengths of

a) $\overline{IL}$ b) $\overline{CD}$ c) $\overline{HL}$

Solution: Since ABCDE ~ HIJKL, we have,

$$\frac{AB}{HI} = \frac{BC}{IJ} = \frac{CD}{JK} = \frac{DE}{KL} = \frac{AE}{HL}$$

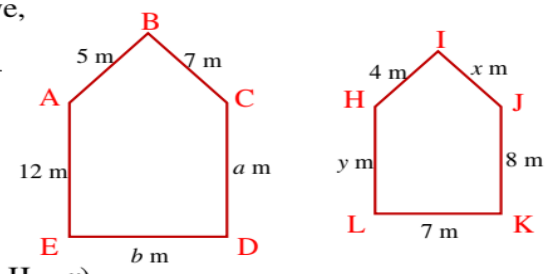


Figure 5.42

a To find the length of $\overline{IJ}$, we use

$$\frac{AB}{HI} = \frac{BC}{IJ}$$

$$\frac{5}{4} = \frac{7}{x} \quad (AB = 5, HI = 4, BC = 7, IJ = x)$$

So, $x = \frac{4 \times 7}{5} = 5.6$

Therefore, the length of side $\overline{IJ}$ is 5.6 m.

b In the same way,

$$\frac{AB}{HI} = \frac{CD}{JK}$$

$\frac{5}{4} = \frac{a}{8}$. So, $a = 10$.

Therefore, CD = 10 m.

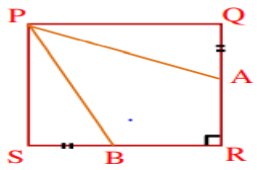
c

$$\frac{AB}{HI} = \frac{AE}{HL}. \text{ i.e., } \frac{5}{4} = \frac{12}{y}$$

So, $y = 9.6$.

Therefore, HL = 9.6 m.

Example 2: In figure below,$PQRS$ is a square. A and B are points on $\overline{QR}$&$\overline{SR}$, such that $\overline{QA} \cong \overline{SB}$. Prove that:$\angle PAQ \cong \angle PBS$.



Theorems on Similarity of Triangles

Definition (similarity of triangles): two triangles are said to be similar, if

1. Their corresponding sides are proportional (have equal ratio),and
2. Their corresponding angles are congruent.

Theorem (SSS similarity theorem): if the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.

Restatement:

Given ΔABC and ΔDEF. If $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$,

then $\Delta ABC \sim \Delta DEF$.

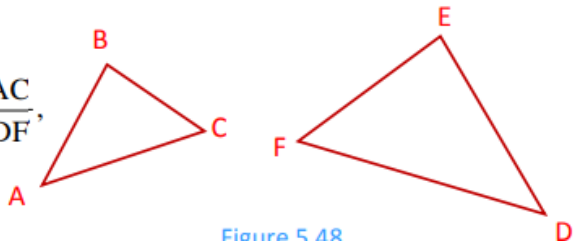
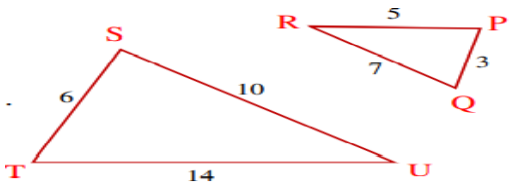


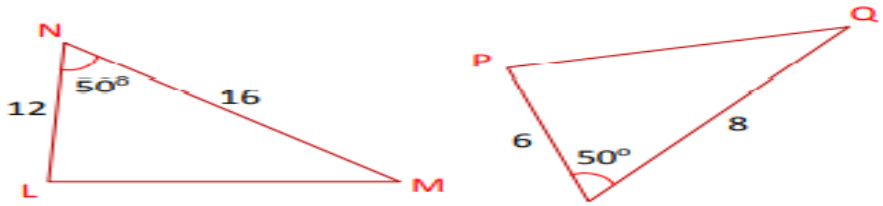
Figure 5.48

Example 1: Are the two triangles similar?



Theorem (SAS similarity): two triangles are similar, if two pairs of corresponding sides of the two triangle are proportional and if the including angle between these sides are congruent.

Example 2: Use the SAS similarity theorem to check whether the given two triangles are similar.



Theorem (AA similarity): if two angles of two triangle are congruent to two corresponding angles of another triangle, then the two triangles are similar.

Example 3: In Figure below, determine whether the two given triangles are similar.

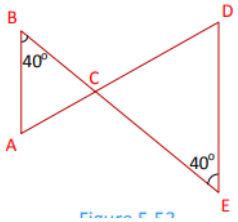


Figure 5.53

Example 4: When Ali planted a tree 5 m away from point A, the tree just blocked the view of a building 50 m away. If the building was 20 m tall, how tall was the tree?

Theorem: If the ratio of the lengths of the corresponding sides of two similar triangles is k, then

i. the ratio of their perimeters is k
ii. the ratio of their areas is k^2

Theorem: If the ratio of the lengths of any two corresponding sides of two similar polygons is k, then

i. The ratio of their perimeters is k.
ii. The ratio of their areas is k^2.

Examples

1. Let ABCD and EFGH be two quadrilaterals such that ABCD ~ EFGH. If AB = 15 cm, EF = 18 cm and the perimeter of ABCD is 40 cm, find the perimeter of EFGH.
2. Two triangles are similar. The sides of one are three times as long as the sides of the other. What is the ratio of the areas of the smaller to the larger?
3. The sides of a polygon have lengths 5, 7, 8, 11, and 19 units. The perimeter of a similar polygon is 75 units. Find the lengths of the sides of the larger polygon.

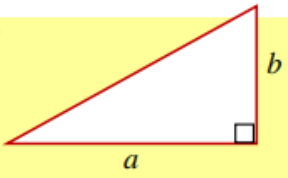
3.6. Area of a triangle and parallelogram

a) Area of a triangle

Case i uses the following fact.

The area A of a right angle triangle with perpendicular sides of length a and b is given by

$$A = \frac{1}{2}ab$$



Case ii uses the following formula.

The area A of any triangle with base b and the corresponding height h is given by

$$A = \frac{1}{2}bh$$

The base and corresponding height of a triangle may appear in any one of the following forms.

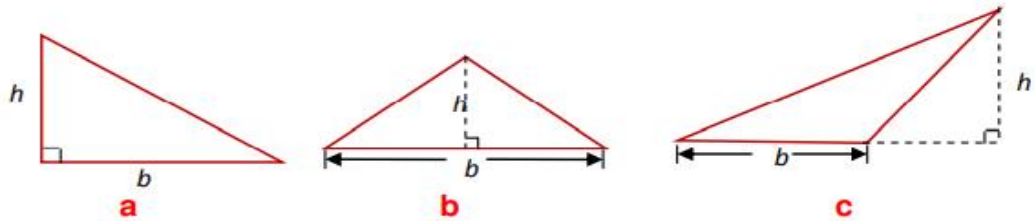


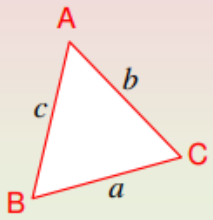
Figure 5.113

From the verification of Case iii, we come to the following formula.

The area A of any triangle with sides a and b units long and angle C ($\angle C$) included between these sides is

$$A = \frac{1}{2}ab\sin(\angle C)$$

Theorem 5.14 **Heron's formula**



The area A of a triangle with sides a, b and c units long and semi- perimeter $s=\frac{1}{2}(a+b+c)$ is given by

$$A=\sqrt{s(s-a)(s-b)(s-c)}\,.$$

Figure 5.117

Example 1 Given ΔABC. If AB = 15 units, BC = 14 units and AC = 13 units, find

a the area of ΔABC.

b the length of the altitude from the vertex A.

c the measure of $\angle B$.

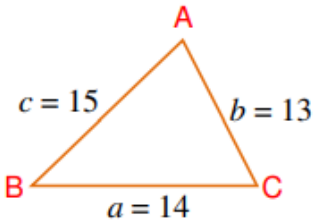


Figure 5.118

Solution:

a $a = 14$ $\quad s-a=7$

$b = 13$ $\quad s-b=8$

$c = 15$ $\quad s-c=6$

$a+b+c=42$ $\quad (s-a)+(s-b)+(s-c)=21$

Why is the sum of $s-a$, $s-b$, $s-c$ equal to s? This provides a useful check.

$$\therefore\ s=\frac{a+b+c}{2}=\frac{42}{2}=21$$

$$\therefore \text{ area of } \Delta ABC=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{21(7)(8)(6)}\ =84 \text{ unit}^2.$$

b Let the altitude from the vertex A to the corresponding base BC be h, meeting BC at D as shown.

Then, area of $\Delta ABC=\frac{1}{2}\ BC\times h$

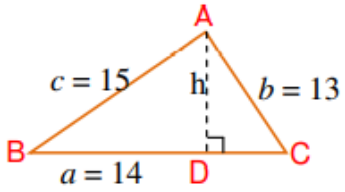


Figure 5.119

$$\therefore\ 84=\frac{1}{2}\times 14\times h=7h$$

$$\therefore\quad h=\frac{84}{7}=12 \text{ units.}$$

Therefore, the altitude of ΔABC from the vertex A is 12 units long.

c In the right angle triangle ABD shown above in Figure 5.119, we see that

$$\sin(\angle B)=\frac{AD}{AB}=\frac{h}{c}=\frac{12}{15}=0.8$$

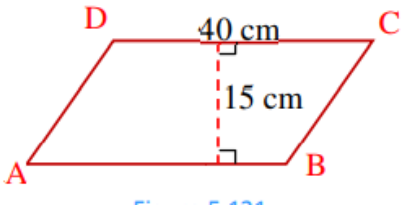
Then, from trigonometric tables, we find that the corresponding angle is 53^o.

i.e., $m(\angle B) = 53^o$.

b) Area of a parallogram

Theorem (Area of a parallogram): The area A of a parallogram with base b and perpendicular height h is $A = bh$

Example 2 If one pair of opposite sides of a parallelogram have length 40 cm and the distance between them is 15 cm, find the area of the parallelogram.



Solution: Area = 40 cm × 15 cm = 600 cm^2.

3.7. Surface area and volume of solid figures (Prism, Cylinder, Cone and Sphere)

Definition: geometrical figures that have three dimensions (length, width and height) are called solid figures. For example, cubes, prisms, cylinders, cones and pyramids are three dimensional solid figures.

Prism

- A prism is a solid figure formed by two congruent polygonal regions in parallel planes, along with three or more parallelograms, joining the two polygons. The polygons in parallel planes are called bases.
- A prism is named by its base. Thus, a prism is called triangular, rectangular, pentagonal, etc., if its base is a triangle, a rectangle, a pentagon, etc., respectively.
  In a prism,
  - ✓ the lateral edges are equal and parallel.
  - ✓ the lateral faces are parallelograms.
- A right prism is a prism in which the base is perpendicular to a lateral edge. Otherwise it is an oblique prism.
- In a right prism
  - ✓ All the lateral edges are perpendicular to both bases.
  - ✓ The lateral faces are rectangles.
  - ✓ The altitude is equal to the length of each lateral edge.

➢ A regular prism is a right prism whose base is a regular polygon.

Note:

In the prism shown in Figure 7.5,

1 $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$ are edges of the upper base.
$\overline{A'B'}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E'}$, $\overline{E'A'}$ are edges of the lower base.

2 $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$ are called lateral edges of the prism.

3 The parallelogram regions $ABB'A'$, $BCC'B'$, $AEE'A'$, $DCC'D'$, $EDD'E$ are called lateral faces of the prism.

4 The union of the lateral faces of a prism is called its lateral surface.

5 The union of its lateral faces and its two bases is called its total surface or simply its surface.

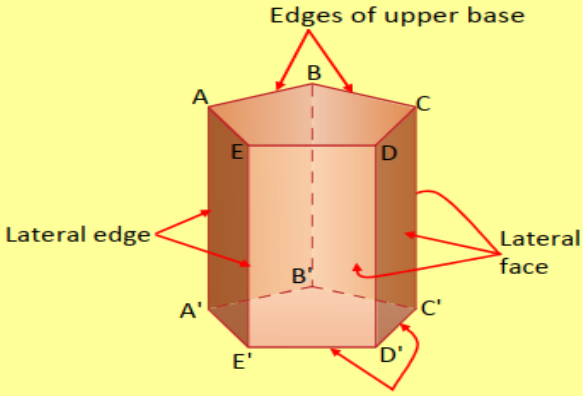


Figure 7.5

Surface area and volume of prisms

If we denote the lateral surface area of a prism by A_L, the total surface area by A_T, the area of the base by A_B and its volume by V, then

i $A_L = Ph$

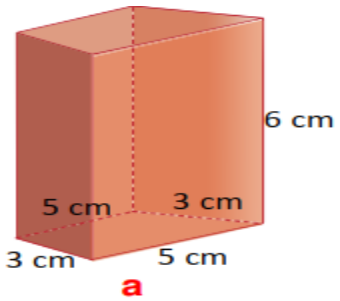
where P is the perimeter of the base and h the altitude or height of the prism.

ii $A_T = 2A_B + A_L$ iii $V = A_B\, h.$

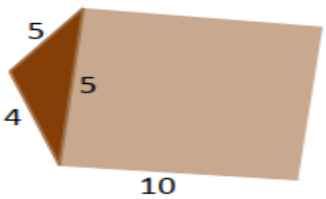
Example 1 The altitude of a rectangular prism is 4 units and the width and length of its base are 3 and 2 units respectively. Find:

a. The total surface area of the prism.
b. The volume of the prism.

Example 2: Find the lateral surface area of each of the following right prisms.



a



b

Solution:

a $A_L = Ph = (3 + 5 + 3 + 5)\text{ cm} \times 6\text{ cm} = 16\text{ cm} \times 6\text{ cm} = 96\text{ cm}^2$

b $A_L = Ph = (5 + 5 + 4) \times 10 = 14 \times 10 = 140\text{ units}^2$

Cylinder

Some important terms

For the cylinder D, the region R is called its lower base or simply base and R' is its upper base.

The line ℓ is called its directrix and the perpendicular distance between E_1 and E_2 is the altitude of D. If ℓ is perpendicular to E_1, then D is called a right cylinder, otherwise it is an oblique cylinder. If R is a circular region, then D is called a circular cylinder.

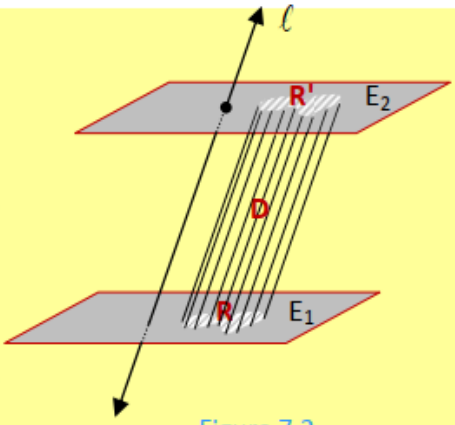
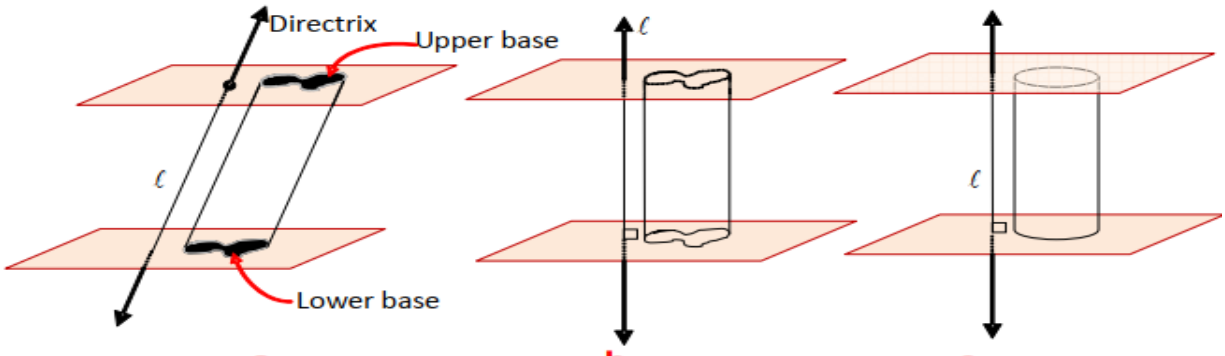


Figure 7.2

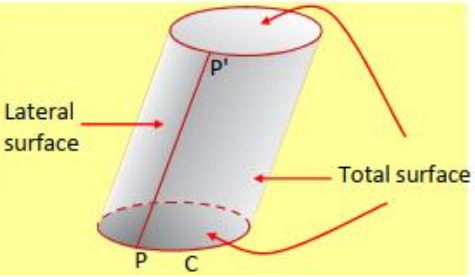


a Oblique cylinder

b Right cylinder

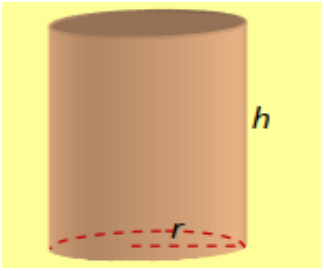
c Right circular cylinder

Let C be the bounding curve of the base region R. The union of all the elements $\overline{PP'}$ for which P belongs to C is called the lateral surface of the cylinder. The total surface is the union of the lateral surface and the bases of the cylinder.



`Surface area and volume of circular cylinders

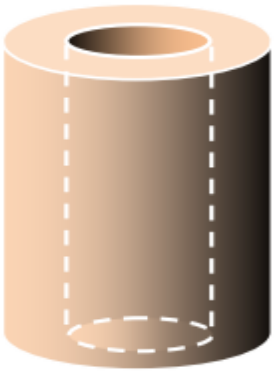
$$A_L = 2\pi rh$$
$$A_T = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$$
$$V = \pi r^2 h$$



Example: If the height of a right circular cylinder is 8 cm and the radius of its base is 5 cm find the following giving your answers in terms of π.

a. its lateral surface area
b. its total surface area c
c. its volume

Example: A circular hole of radius 2 units is drilled through the center of a right circular cylinder whose base has radius 3 units and whose altitude is 4 units. Find the total surface area of the resulting figure.



Solution: Let R be the radius of the bigger cylinder and r be the radius of the smaller cylinder then

i Area of the resulting base $= 2(\pi R^2 - \pi r^2)$

$= 2(\pi \times 3^2 - \pi \times 2^2)\ \text{unit}^2 = 10\pi\ \text{unit}^2$

ii Lateral surface area of the resulting figure = lateral surface area of the bigger cylinder + lateral surface area of inner (smaller) cylinder

$= (2\pi Rh + 2\pi rh)\ \text{unit}^2 = [\,2\pi(3)4 + 2\pi(2)4]\ \text{unit}^2$

$= 40\pi\ \text{unit}^2$

Therefore, total surface area of the resulting figure $= (10\pi + 40\pi) = 50\pi\ \text{unit}^2$.

Example: Find the volume of the cylinder whose base circumference is $12cm^2$ and whose lateral area is $288cm^2$.

Solution: $C = 2\pi r \Rightarrow 12\pi = 2\pi r \Rightarrow r = 6 \text{ cm}$

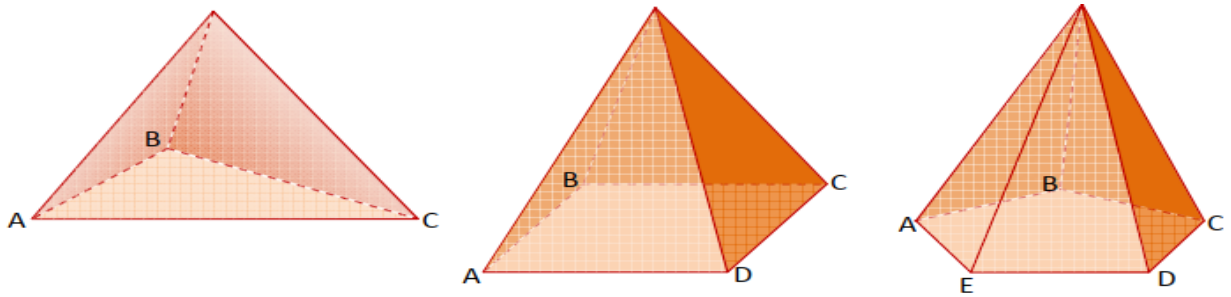
$A_L = 2\pi r\, h$

$288\pi \text{cm}^2 = 2\pi \times 6 \text{ cm} \times h \Rightarrow 288\pi \text{ cm}^2 = 12\pi \text{cm} \times h \Rightarrow h = 24 \text{ cm}$

Therefore, $V = \pi r^2 h = \pi (6 \text{ cm})^2 \times 24 \text{ cm} = 36\pi \text{cm}^2 \times 24 \text{ cm} = 864\pi \text{cm}^3$

PYRAMIDS

Definition: A pyramid is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon (See Figure 7.17).



Triangular pyramid **a** | Quadrilateral pyramid **b** | Pentagonal pyramid **c**

Figure 7.17

The altitude of a pyramid is the length of the perpendicular from the vertex to the plane containing the base.

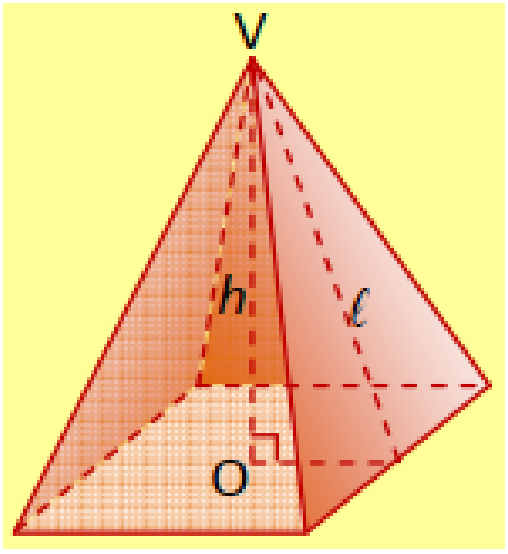
The slant height of a regular pyramid is the altitude of any of its lateral faces.

Surface area and volume

$$A_L = \frac{1}{2}pl$$

$$A_T = A_B + A_L$$

$$V = \frac{1}{3}A_B h$$



Where: A_L denotes the lateral surface area;

P denotes the perimeter of the base;

ℓ denotes the slant height.

$\boldsymbol{h}$ the altitude

Example 1 A regular pyramid has a square base whose side is 4 cm long. The lateral edges are 6 cm each.

a. What is its slant height?

b. What is the lateral surface area?

c. What is the total surface area?

d. the volume of the pyramid

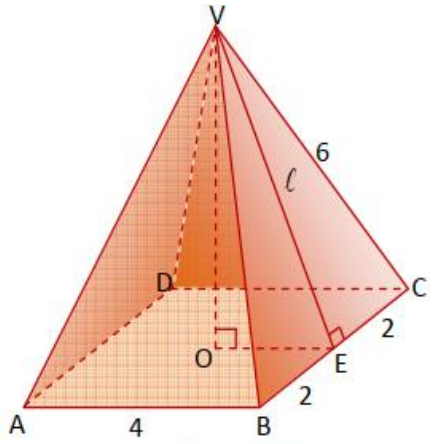


Figure 7.22

Solution: Consider Figure 7.22,

a $(VE)^2 + (EC)^2 = (VC)^2$

$\ell^2 + 2^2 = 6^2$

$\ell^2 = 32$

$\ell = 4\sqrt{2}$ cm

Therefore, the slant height is $4\sqrt{2}$ cm.

b There are 4 isosceles triangles.

Therefore,

$$A_L = 4 \times \frac{1}{2} BC \times VE = 4\left(\frac{1}{2} \times 4 \times 4\sqrt{2}\right) = 32\sqrt{2} \text{ cm}^2$$

$$\text{or } A_L = \frac{1}{2} P\ell = \frac{1}{2} (4+4+4+4)\, 4\sqrt{2} = 8 \times 4\sqrt{2} = 32\sqrt{2} \text{ cm}^2$$

c $A_T = A_L + A_B = 32\sqrt{2} + 4 \times 4$

$= 32\sqrt{2} + 16 = 16(2\sqrt{2} + 1)$ cm^2

d **)** Here, we need to find the altitude of the pyramid as shown below:

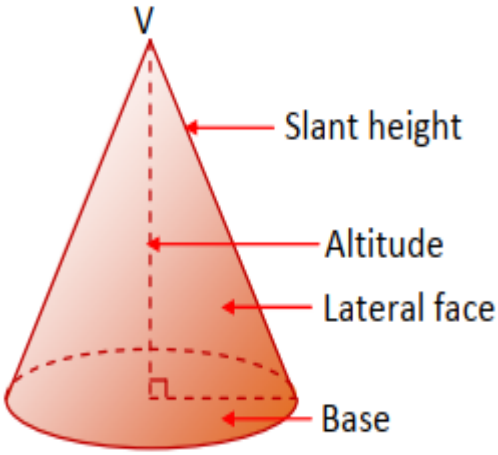
$$(VO)^2 + (OE)^2 = (VE)^2 \Rightarrow h^2 + 2^2 = (4\sqrt{2})^2$$

$$h^2 + 4 = 32$$

$$h^2 = 28 \Rightarrow h = 2\sqrt{7} \text{ cm}$$

$$V = \frac{1}{3} A_B h = \frac{1}{3} \times (4 \times 4) \times 2\sqrt{7} = \frac{32}{3}\sqrt{7} \text{ cm}^3$$

Definition: The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a cone.



A right circular cone is a cone with the foot of its altitude at the center of the base.

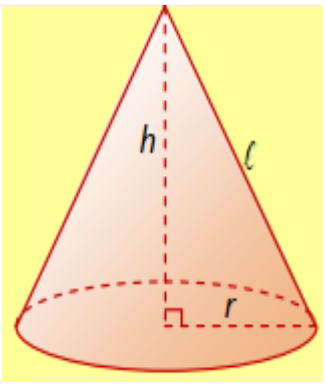
A line segment from the vertex of a cone to any point on the boundary of the base (circle) is called the slant height.

Surface area and volume

$$A_L = \frac{1}{2}pl = \frac{1}{2}(2\pi r)l = \pi rl, \quad l = \sqrt{h^2 + r^2}$$

$$A_T = A_B + A_L = \pi r^2 + \pi rl = \pi r(l + r)$$

$$V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$$



Where: A_l denotes the lateral surface area;

P denotes the perimeter of the base;

l denotes the slant height.

$\boldsymbol{h}$ the altitude

$\boldsymbol{r}$ the radius of the base

Example: The altitude of a right circular cone is 8 cm. If the radius of the base is 6 cm, then find its:

a. slant height
b. lateral surface area
c. total surface area.
d. the volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6^2 * 8) = 96\pi cm^3$

Solution: Consider Figure 7.24

a $\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = \sqrt{100}$

$\ell = 10$ cm

b $A_L = \pi r \ell = \pi \times 6 \times 10 = 60\pi \text{cm}^2$

c $A_T = \pi r\,(\ell + r) = \pi \times 6\,(10 + 6) = 6\pi \times 16$

$= 96\pi \text{cm}^2$

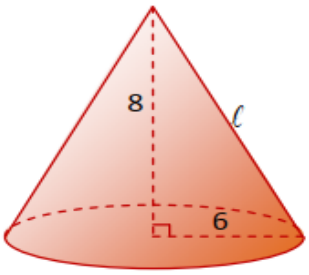


Figure 7.24

Surface area and volume of a sphere

Definition: A sphere is a closed surface, all points of which are equidistant from a point called the center.

The surface area (A) and the volume (V) of a sphere of radius r are given by

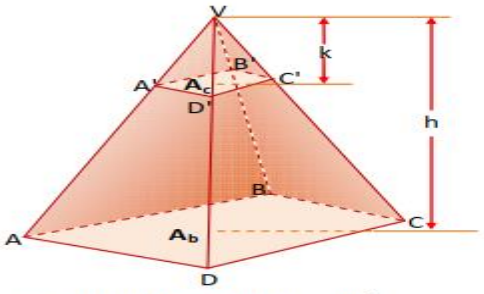
$$A = 4\pi r^2, \qquad V = \frac{4}{3}\pi r^3$$

Example: Find the surface area and volume of a spherical gas balloon with a diameter of 10 m.

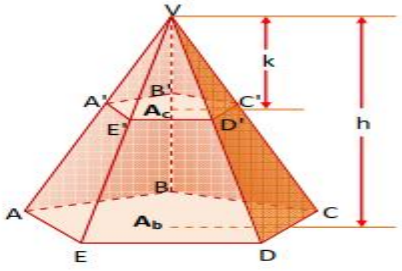
3.8. Frustums of Pyramids and Cones

Definition: If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal crosssection of the pyramid (or the cone).

Theorem: In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$ where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b}=\frac{\text{area}(A'B'C'D')}{\text{area}(ABCD)}=\frac{k^2}{h^2}$$



$$\frac{\text{area}(A'B'C'D'E')}{\text{area}(ABCD)}=\frac{k^2}{h^2}$$

Example: The area of the base of a pyramid is $90cm^2$. The altitude of the pyramid is 12 cm. What is the area of a horizontal cross-section 4 cm from the vertex?

Solution: Let A_c be the area of the cross-section, and A_b the base area.

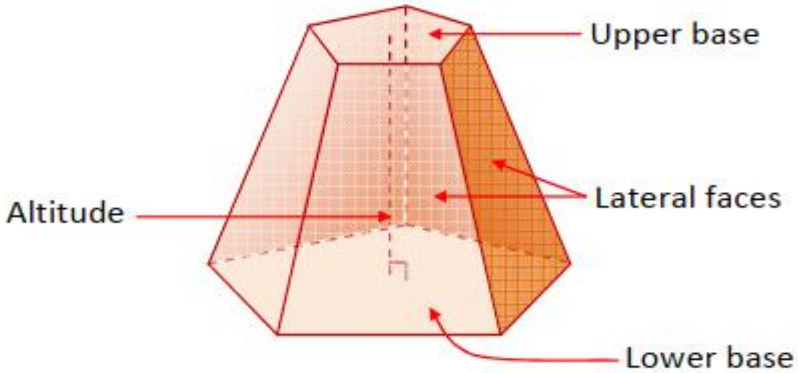
$$\text{Then, } \frac{A_c}{A_b}=\frac{k^2}{h^2}\Rightarrow\frac{A_c}{90}=\frac{4^2}{12^2}$$

$$\therefore \quad A_c=\frac{90\times16}{144}\ \text{cm}^2=10\ \text{cm}^2$$

Frustum of a pyramid

Definition: A frustum of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.

The base of the pyramid and the cross-section made by the plane parallel to it are called the bases of the frustum. The other faces are called lateral faces. The total surface of a frustum is the sum of the lateral surface and the bases. The altitude of a frustum of a pyramid is the perpendicular distance between the bases.



Note:

i. The lateral faces of a frustum of a pyramid are trapeziums.

ii. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.

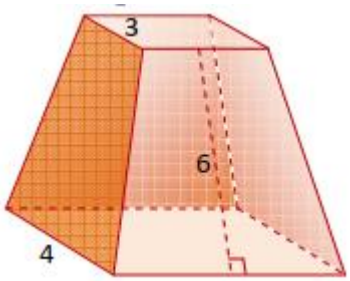
iii. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.

iv. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral face

Example: The lower base of the frustum of a regular pyramid is a square 4 cm long, the upper base is 3 cm long. If the slant height is 6 cm, find its lateral surface area.

Solution: As shown in Figure , each lateral face is a trapezium, the area of each lateral face is

$$A_L = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2} * 6(3 + 4) = 21cm^2$$



Since the four faces are congruent isosceles trapeziums, the lateral surface area is

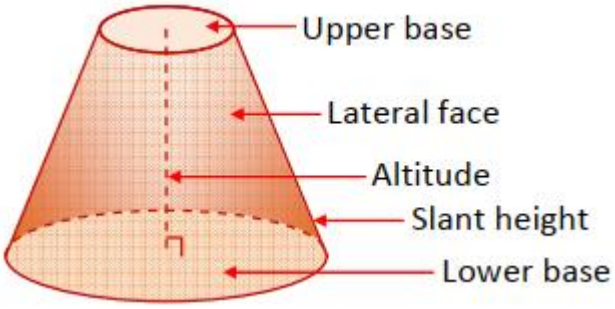
$$A_L = 4 * 21cm^2 = 84cm^2$$

Theorem: The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base. That is,

$$A_L = \frac{1}{2}l(P + P')$$

Frustum of a cone

Definition: A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.For a frustum of a cone, the bases are the base of the cone and the cross-section parallel to the base. The lateral surface is the curved surface that makes up the frustum. The altitude is the perpendicular distance between the bases.



The slant height of a frustum of a right circular cone is that part of the slant height of the cone which is included between the bases.

Theorem: For a frustum of a right circular cone with altitude h and slant height ℓ, if the circumferences of the bases are c and c', then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2}\ell\,(c + c') = \frac{1}{2}\ell\,(2\pi r + 2\pi r') = \ell\pi\,(r + r')$$

Example: A frustum formed from a right circular cone has base radii of 8 cm and 12 cm and slant height of 10 cm. Find:

a. the area of the curved surface

b. the area of the total surface. (Use $\pi \approx 3.14$).

Solution:

a $A_L = \pi\ell\,(r + r') = \pi \times 10\text{ cm}\,(8 + 12)\text{ cm} = 10\pi\text{ cm} \times 20\text{ cm}$

$= 200\pi\text{ cm}^2 = 200 \times 3.14\text{ cm}^2 = 628\text{ cm}^2$

b Area of bases:

$A_B = A_c + A_b = \pi(r')^2 + \pi r^2 = \pi(8\text{ cm})^2 + \pi(12\text{ cm})^2 = 64\pi\text{cm}^2 + 144\pi\text{cm}^2$

$= 208\pi\text{ cm}^2 \approx 208 \times 3.14\text{ cm}^2 \approx 653\text{ cm}^2$

Total surface area of the frustum:

$A_T = A_L + A_B \approx 628\text{ cm}^2 + 653\text{ cm}^2 = 1281\text{ cm}^2$

The volume of a frustum of a cone or pyramid

$$V_f = \frac{h'}{3}\left(A + A' + \sqrt{AA'}\right) = \frac{\pi}{3}h'(r^2 + (r')^2 + rr')$$

Where

V_f = the volume of the frustum

A is the lower base area, A' the upper base area and h' is the height of a frustum of a cone or pyramid.

r is the radius of the bigger (the lower base of the frustum) cone and r' is the radius of the smaller cone (upper base of the frustum).

Example: A frustum of a regular square pyramid has height 5 cm. The upper base is of side 2 cm and the lower base is of side 6 cm. Find the volume of the frustum.

Solution:

Since the upper base and lower base are squares,

$$A = (6\text{ cm})^2 = 36\text{ cm}^2$$

$$A' = (2\text{ cm})^2 = 4\text{ cm}^2$$

$$V_f = \frac{h'}{3}\left(A + A' + \sqrt{AA'}\right) = \frac{5}{3}\left(36 + 4 + \sqrt{36\times 4}\right)\text{cm}^3$$

$$= \frac{5}{3}(40 + 12)\text{ cm}^3 = \frac{5}{3}\times 52\text{ cm}^3 = \frac{260}{3}\text{ cm}^3.$$

Surface Areas and Volumes of Composed Solids

- A composite solid is a solid made of two or more well-defined solid objects.
- To find the volume of a composite solid, we identify the different parts it is made of, work out the volume of each part independently, and sum up the volumes of its parts.
- For composite prisms, where the bases are a composite shape, the area of the bases is the sum of the areas of the parts it is made of. The volume of the composite prism is then given by multiplying the whole base area by the height of the prism.

Example 1: A candle is made in the form of a circular cylinder of radius 4 cm at the bottom and a right circular cone of altitude 3 cm, as shown in Figure 7.47. If the overall height is 12 cm, find the total surface area and then volume of the candle.

Solution: Slant height of the cone is $\ell = \sqrt{3^2 + 4^2} = 5$ cm

The total surface area of the candle is the sum of the lateral surface areas of the cone, the cylinder and the area of the base of the cylinder. That is,

$$A_T = \pi r\ell + 2\pi rh + \pi r^2 = \pi(4)\,5 + 2\pi(4)\,9 + \pi(4)^2$$

$$= 20\pi + 72\pi + 16\pi = 108\pi\text{ cm}^2$$

The volume of the candle is the sum of the volumes of the cone and cylinder.

$$V_T = V_{cone} + V_{cylinder} = \frac{1}{3}\pi r^2 h_{co} + \pi r^2 h_{cy}$$

$$= \frac{1}{3}\pi\,(4)^2 \times 3 + \pi(4)^2 \times 9 = 16\pi + 144\pi = 160\pi\text{ cm}^3$$

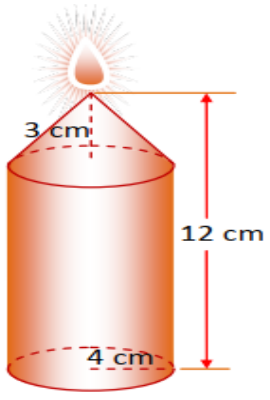
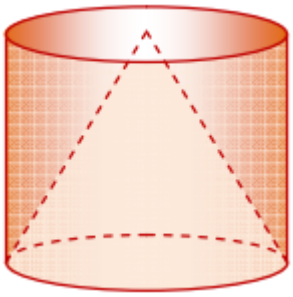


Figure 7.47

Example: A cone is contained in a cylinder so that their base radius and height are the same, as shown in Figure below Calculate the volume of the space inside the cylinder but outside the cone.



Solution: The required volume is equal to the difference between the volume of the cylinder and the cone. That is,

$$V = V_{cy} - V_{co} = \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h.$$

As $r = h$, then $V = \frac{2}{3}\pi r^3$.

Exercise

1. A hollow cylindrical metallic pipe is open at both the ends and its external diameter is 12 cm. If the length of the pipe is 70 cm and the thickness of the metal used is 1 cm, find the volume of the metal used for making the pipe.
2. Volume of a right circular cylinder is 3080 cm_3 and radius of its base is 7 cm. Find the curved surface area of the cylinder.
3. A cylindrical bucket of base diameter 28 cm and height 12 cm is full of water. This water is poured in to a rectangular tub of length 66 cm and breadth 28 cm. Find the height to which water will rise in the tub.
4. Find the curved surface area, total surface area and volume of a right circular cone whose base radius and height are respectively 5 cm and 12 cm.
5. Find the volume of a right circular cone of base area 616 cm^2 and height 9 cm.
6. Volume of a right circular cone of height 10.5 cm is 176cm^3. Find the radius of the cone.
7. Find the curved surface area of a right circular cone of volume 12936cm^3 and base diameter 42 cm.
8. When a lump of stone is submerged in a rectangular water tank whose base is 25 cm by 50 cm, the level of the water rises by 1 cm. What is the volume of the stone?
9. A frustum whose upper and lower bases are circular regions of radii 8 cm and 6 cm respectively, is 25 cm deep.