

CC 230 : Calculus I

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1. Limit at Infinity

Definition 1.1. Let f be a function such that $f(x)$ is defined for sufficiently large x. Suppose $L \in \mathbb{R}$ satisfying the following condition:
(*) $f(x)$ is arbitrarily close to L if x is sufficiently large.
Then we say that L is the limit of f at infinity and write

$$\lim_{x\to\infty} f(x) = L.$$

The following rules apply:

1. $\lim_{x\to\infty} k = k$, (where k is a constant)
2. $\lim_{x\to\infty} \frac{1}{x^p} = 0$, (where p is a positive constant),
3. $\lim_{x\to\infty} \frac{1}{a^x} = 0$, (where $a > 1$ is a constant),
4. All the limit rules(sum, difference, constant, product, quotient and power rules) hold as $x \to \infty$,
5. The above rules apply as $x \to -\infty$ as well.

Example 1.2.

$$\begin{aligned}\lim_{x\to\infty}(2^{-3x}+2) &= \lim_{x\to\infty} e^{-3x} + \lim_{x\to\infty} 2\\ &= \lim_{x\to\infty} \frac{1}{e^{3x}} + 2\\ &= 0 + 2\\ &= 2\end{aligned}$$



Example 1.3.

$$\begin{aligned}\lim_{x\to\infty}\frac{2x^2+5x}{3x^3+4x-1} &= \lim_{x\to\infty}\frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{4}{x^2}-\frac{1}{x^3}}\\ &= \frac{\lim_{x\to\infty}(\frac{2}{x}+\frac{5}{x^2})}{\lim_{x\to\infty}(3+\frac{4}{x^2}+\frac{1}{x^3})}\\ &= \frac{0}{0}=0.\end{aligned}$$

Theorem 1.4. *The Leading Term Rule:*
$\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}$.

Remark 1.5.

- If $n=m$, the limit is $\frac{a_n}{b_n}$,
- if $n<m$, the limit is $\frac{a_n}{b_m}\lim_{x\to\infty}\frac{1}{x^{m-n}}=0$,
- if $n>m$, the limit is $\frac{a_n}{b_m}\lim_{x\to\infty}x^{n-m}$, which does not exist because as x increases indefinitely, x^{n-m} increases indefinitely.
- The Leading Terms Rule can also be applied to "functions similar to rational functions", for example,
$$\lim_{x\to\infty}\frac{x+2\sqrt{x}+3}{5x+6\sqrt{x}+7}=\lim_{x\to\infty}\frac{x}{5x}=\frac{1}{5}(Why?)$$
- The Leading Terms Rule cannot be applied to limits of rational functions at a point: $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $a\in\mathbb{R}$.



Remark 1.6. When $\lim_{x\to\pm\infty} f(x) = L$, the graph off gets closer and closer to the horizontal line $y = L$ as x gets larger and larger. Then the line $y = L$ is called a horizontal asymptote of the graph.

Example 1.7. Find the horizontal asymptote of $f(x) = \dfrac{x+2}{\sqrt{x^2-3x+2}}$

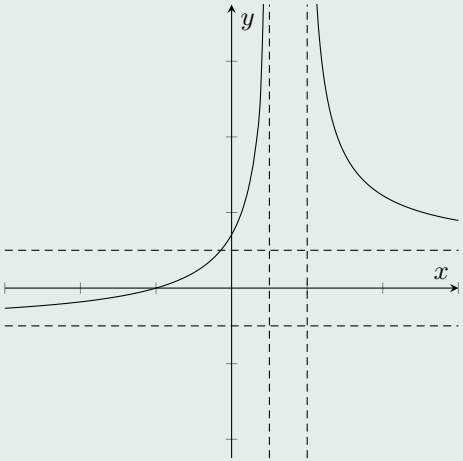
Solution: By the Leading Term Rule, for x very large in magnitude, $\dfrac{x+2}{\sqrt{x^2-3x+2}}$ can be approximated by $\dfrac{x}{\sqrt{x^2}} = \dfrac{x}{|x|}$. Thus,

$\lim_{x\to\infty} \dfrac{x+2}{\sqrt{x^2-3x+2}} = \lim_{x\to\infty} \dfrac{x}{|x|} = 1.$

Similarly, $\lim_{x\to-\infty} \dfrac{x+2}{\sqrt{x^2-3x+2}} = \lim_{x\to\infty} \dfrac{x}{|x|} = -1$. This means $y = 1 and y = -1$ are horizontal asymptotes. (See the graph below)



$$f(x) = \frac{x+2}{\sqrt{x^2-3x+2}}$$



Theorem 1.8. *(Sandwich Theorem) Let f, g and h be functions such that $f(x), g(x)$ and $h(x)$ are defined for sufficiently large x. Suppose that $f(x) \leq g(x) \leq h(x)$ if x is sufficiently large and that both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} h(x)$ exist and are equal (with common limit denoted by L). Then we have*

$$\lim_{x\to\infty} g(x) = L.$$

Remark 1.9. The condition "$f(x) \leq g(x) \leq h(x)$ if x is sufficiently large" means that there is a real number r such that the inequalities are true for all $x > r$.

Example 1.10. Find $\lim_{x\to\infty} \dfrac{\sin(x)}{x}$, if it exists.

Solution: The given function can be written as a product of two functions: $\sin(x)$ and $\dfrac{1}{x}$. For

the second function, its limit at infinity is 0. However, since we have $-1 \leq \sin(x) \leq 1$ for the first function,

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}.$$

Moreover, $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$

Thus applying the Sandwich(Squeeze) Theorem, $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0.$

Exercise 1.11. [1] Calculate the limits

1. $\lim_{x\to\infty} \frac{(9x^8 - 6x^5 + 4)^{1/4}}{(64x^{12} + 14x^7 - 7)^{1/3}}$
2. $\lim_{x\to\infty} \sqrt{x}(\sqrt{x+3} - \sqrt{x-2})$
3. $\lim_{x\to-\infty} \frac{(2x^4 - 137)^5}{(x^2 + 429)^{10}}$
4. $\lim_{x\to\infty} f(x)$, if for all $x > 1$, $\frac{10e^x - 21}{2e^x} < f(x) < \frac{5 * \sqrt{x}}{\sqrt{x-1}}$.

Definition 1.12. Let f be a function such that $f(x)$ is defined for sufficiently large x. Suppose that
(*) $f(x)$ is arbitrarily large if x is sufficiently large.
Then, we write $\lim_{x\to\infty} f(x) = \infty$.

Remark 1.13.

- Because ∞ is not a real number, $\lim_{x\to\infty} f(x) = \infty$ does not mean the limit exists. In fact, it only indicates why the limit does not exist.
- The limit can also be stated for $x \to \infty$ in a similar fashion.

Example 1.14. $\lim_{x\to\infty} (1 + \ln(x)) = \infty.$

[1] Answers: 1) 0; 2) $\frac{5}{2}$; 3)32; 4) 5

2. Infinite Limits

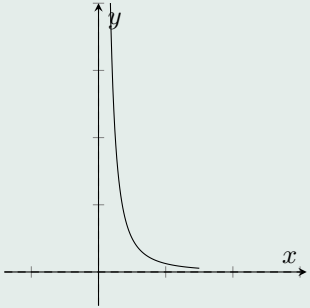
If the value of a function f increases without bound as x decreases to a real number a we write

$$\lim_{x \to a^+} f(x) = +\infty$$

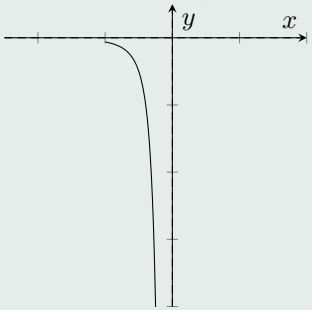
. $\lim_{x \to a^-} f(x) = +\infty$, $\lim_{x \to a^+} f(x) = -\infty$ and $\lim_{x \to a^-} f(x) == \infty$ have analogous meanings.

Example 2.1. Here are some examples

$$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$$



$$\lim_{x\to 0^-} \frac{1}{x^3} = -\infty$$



Definition 2.2. If for a real number a and a function f

$$\lim_{x\to a^-} f(x) = \pm\infty$$

or

$$\lim_{x\to a^+} f(x) = \pm\infty,$$

then, the line $x = a$ is called a vertical asymptote to the graph of$f$,

Example 2.3. $\lim_{x\to 2^-} \frac{x^2+2}{x+2} = -\infty$. So, $x = -2$ is a vertical asymptote to the graph of $f(x) = \frac{x^2+2}{x+2} = -\infty$

Example 2.4. $x = 2$ is a vertical asymptote to the graph of $f(x) = \frac{x+2}{\sqrt{x^2-3x+2}}$ in Example 1.7.
Note that $D_f = (-\infty, 1) \cup (2, \infty)$ and $f(x) > 0$, for all $x > 2$, giving $\lim_{x\to 2^+} \frac{x+2}{\sqrt{x^2-3x+2}} = +\infty$

Remark 2.5. Suppose

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

and

$$b = \lim_{x \to \pm\infty} (f(x) - mx),$$

then $y = mx + b$ is an oblique asymptote of the frapf of f.

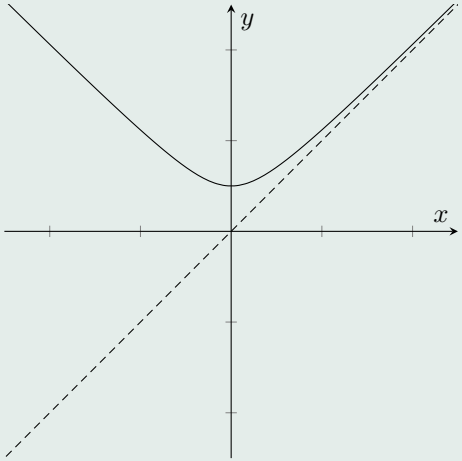
Example 2.6. $y = x$ is the oblique asymptote of the graph of $f(x) = \sqrt{x^2 + 1}$. Here

$$m = \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = 1$$

and

$$b = \lim_{x \to \infty} (f(x) - mx) = \lim_{x \to \infty} (\sqrt{x^2 + 1} - x) = 0.$$

$f(x) = \sqrt{x^2 + 1}$



Exercise 2.7. [2] Find the asymptotes for each function

1. $f(x) = x + e^{-x}$
2. $g(x) = \dfrac{3x^3 + 1}{x^2 + 2x + 1}$
3. $h(x) = e^{-2x}\sin(x)$
4. $f(x) = \dfrac{\sqrt{4x^2 - 4}}{\sqrt{3, x^3 + 1}}$.

3. Continuity

Definition 3.1. For $f : (c, d) \to \mathbb{R}$, and $c < a < d$, we say f is continuous at a iff

$$\lim_{x \to a} f(x) = f(a).$$

This means:

$$\lim_{x \to a} f(x) = f(a) = \lim_{x \to a} f(x) \tag{1}$$

Remark 3.2. In Definition 3.1 it is possible that $c = -\infty$ and $d = \infty$.

Remark 3.3. A function f can fail to be continuous at a point a, if any one of the conditions in Equation 1 fails.

Example 3.4. Consider $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

We now have, $\lim_{x \to 0^-} f(x) = -1$ while $\lim_{x \to 0^+} f(x) = 1$.

[2] Answers: 1) $y = x$; 2) $y = 3x - 6$; 3) $y = 0$; 4) $y = -\frac{2}{\sqrt{3}}$ and $y = \frac{2}{\sqrt{3}}$

This means $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Since, Equation 1 fails to hold, f is not continuous at 0. Such a discontinuity is called a *jump discontinuity* for obvious reasons (Sketch the graph and see why.)

Example 3.5. Consider $g(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

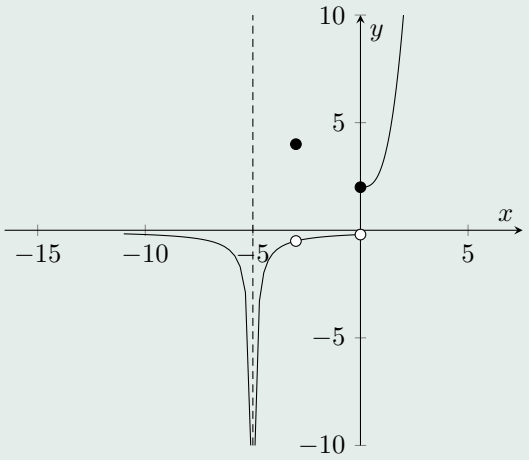
$\lim_{x\to 0} g(x) = 0 \neq 1 = g(0)$. Thus g is not continuous at 0.
Such discontinuity is called a. *removable discontinuity*. It create a "hole" in the graph of the functions.(At what point is the hole in this example?)

Example 3.6. Consider $h(x) = \dfrac{x^2+4}{x-2}$. For this function, $\lim_{x\to 2^-} f(x) = -\infty$, while $\lim_{x\to 2^+} f(x) = +\infty$. So, Equation 1 fails, and h is not continuous at $x = 2$.
Such a discontinuity is called *essential* or *infinite discontinuity*.

Here is an example that has all the three types of discontinuities.

Example 3.7. The graph of a function f is shown below. Find all values of a for which f is





discontinuous at

Solution: At $a = -5$ essential discontinuity, at $a = -3$ removable discontinuity and at $a = 0$ jump discontinuity.

Remark 3.8. If $a \notin D_f$, it is obvious that f is not continuous at a, beacuse $f(a)$ does not exist.

In the preceding discussion, we considered continuity of a function f at a point a (a real number is considered as a point on the real line). In the next definition, we consider continuity of f on an open interval. Recall that an open interval is a subset of $\mathbb{R}$ that can be written in one of the following forms:
$(\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x\beta\}$
$(\alpha, \infty) = \{x \in \mathbb{R} : \alpha < x\}$
$(-\infty, \beta) = \{x \in \mathbb{R} : x\beta\}$
$(\infty, \infty) = \mathbb{R}$
where α and β are real numbers, and for the first type, we need $\alpha < \beta$.

Definition 3.9. Let I be an open interval and let f be a function defined on I. If f is continuous at every $a \in I$, then we say that f is continuous on I.

Remark 3.10. .

- In the definition, the condition "f is a function defined on I" means that f is a function such that $f(x)$ is defined for all $x \in I$, that is, $I \subseteq D_f$.
- Since I is an open interval, we may consider continuity of f at any point a belonging to I.
- If there exists $a \in I$ such that f is not continuous at a, then f is not continuous on I.

Example 3.11. In Example 3.5, the domain of f is $\mathbb{R}$. The function is not continuous on $\mathbb{R}$ because it is not continuous at 0. For the open interval $(0, \infty)$, the function f is continuous at all a belonging to this interval. Therefore, f is continuous on $(0, \infty)$. Similarly, f is continuous on $(-\infty, 0)$.

Example 3.12. $f(x) = \dfrac{1}{x}$ is is continuous on $(-\infty, 0)$ as well as on $(0, \infty)$.

What about intervals like $[\alpha, \beta)$, $(\alpha, \beta]$ and $[\alpha, \beta]$? In this case we take one sided continuity. This means

Definition 3.13. f is continuous on $[\alpha, \beta)$, where $\alpha \in \mathbb{R}$ iff $\begin{cases} \lim\limits_{x \to a} = f(a), & if a \in (\alpha, \beta) \\ \lim\limits_{x \to \alpha^+} = f(\alpha) \end{cases}$

Definition 3.14. f is continuous on $(\alpha, \beta]$, where $\beta \in \mathbb{R}$ iff $\begin{cases} \lim\limits_{x \to a} = f(a), & if a \in (\alpha, \beta) \\ \lim\limits_{x \to \beta^-} = f(\beta) \end{cases}$

Definition 3.15. f is continuous on $[\alpha, \beta]$, where $\alpha, \beta \in \mathbb{R}$ iff $\begin{cases} \lim\limits_{x \to a} = f(a), & if a \in (\alpha, \beta) \\ \lim\limits_{x \to \alpha^+} = f(\alpha) \\ \lim\limits_{x \to \beta^-} = f(\beta) \end{cases}$

Continuity is a very important property in calculus and discontinuity of a function is a bit of a problem. Fortunately, many of the important functions are continuous in their respective domains.

Theorem 3.16. *The following functions are continuous in their respective domains:*

1. *polynomial functions*
2. *rational functions*
3. *logarithm functions*
4. *exponential functions*
5. *absolute functions*
6. *power functions*

Since continuity is related to limits, the limit rules hold for continuity as well.

Theorem 3.17. *Suppose f and g are functions with common domain such that both are continuous at $x = a$. Then, kf, $(k \in \mathbb{R}), f \pm g, fg, \frac{f}{g}$ (providd $g(a) \neq 0$) are all continuous at a.*

Theorem 3.18. *Let f be continuous at a point L and that g be a function such that $\lim_{x\to a} g(x) = L$. Then, $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L)$.*

Exercise 3.19. .

1. Give point(s) of discontinuity and state their types

   a. $f(x) = \dfrac{x^2+1}{x^2-1}$

b. $g(x) = \dfrac{|x|}{x} + 1$

c. $f(x) = \begin{cases} x, & |x| \leq 1 \\ 1, & x > 1 \\ -1, & x < -1 \end{cases}$

d. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$

2. Find the interval in which each function is continuous.

a. $f(x) = 1 - x^2$

b. $g(x) = \sqrt{x + 7}$

c. $f(x) = \sqrt{x^2 - x^4}$

d. $f(x) = \begin{cases} 5 - x, & x \geq 0 \\ x + 1, & x < 0 \end{cases}$

3. f is defined by

$$f(x) = \begin{cases} x + 1, & x < 2 \\ x^2 + kx + 1, & x \geq 2 \end{cases}$$

where k is a constant. Find k such that f is continuous at $x = 2$.

4. Can we re-define the following functions at $x = 0$ so that they will be continuous at $x = 0$?

a. $f(x) = \dfrac{2x^2 - x}{5x};\ x \neq 0$

b. $g(x) = \dfrac{5}{x}; x \neq 0$

5. Evaluate the following[3]

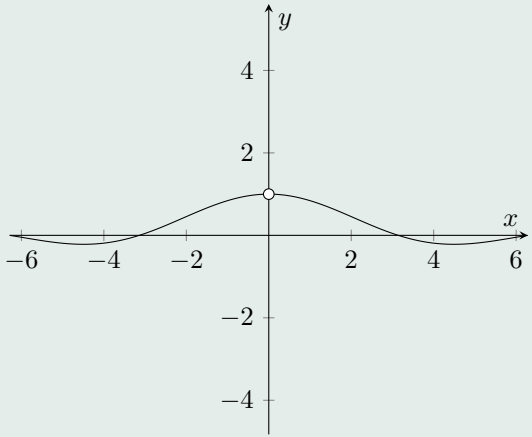
   a. $\lim_{x\to 2} \sqrt{3x+1}$

   b. $\lim_{x\to \frac{\pi}{2}} \sqrt[3]{\sin(x)}$

4. Two important limits

The following limits are very important and you need to be familiar with them. We state them without proof, but the graphs illustrate their behavior.

Theorem 4.1. $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$



[3]Answers: 1)a) $x = -1 \& x = 1$ (infinite); b) $x = 0$ (jump); c) continuous; 2)a) $\mathbb{R}$; b)$[-7, \infty)$; c) $[-1, 1]$; d)$(-\infty, 0) \cup (0, \infty)$; 3) -1 4)a) yes(Take $f(0) = \frac{-1}{5}$ b) No 5)a)$\sqrt{7}$ b) 1

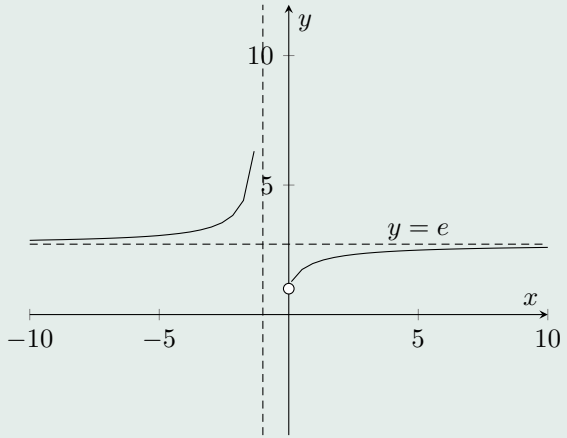
Corollary 4.2. $\lim_{x\to 0} \dfrac{1-\cos(x)}{x} = 0$

Proof. Using Theorem 4.1,

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} &= \lim_{x\to 0} \frac{1-\cos(x)}{x}\frac{1+\cos(x)}{1+\cos(x)} \\
&== \lim_{x\to 0} \frac{sin^2(x)}{x(1+cos(x))} \\
&= \lim_{x\to 0} \frac{sin(x)}{x} \lim_{x\to 0} sin(x) \lim_{x\to 0} \frac{1}{1+cos(x)} \\
&= 0
\end{aligned}$$

□

Theorem 4.3. $\lim_{x\to\pm\infty} \left(1+\dfrac{1}{x}\right)^x = e$



Remark 4.4. If we put $y = \dfrac{1}{x}$, it is to get, from Theorem 4.3, the limit

$$\lim_{y\to 0}(1+y)^{\frac{1}{y}} = e.$$

Example 4.5. Evaluate the following limits

1. $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{x}$
   Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(3x)}{x} &= \lim_{x\to 0} 3\frac{\sin(3x)}{3x}\\ &= 3\lim_{x\to 0}\frac{\sin(3x)}{3x}\\ &= 3\end{aligned}$$

2. $\lim\limits_{x\to 0}\dfrac{\cos(x)-1}{\tan(x)}$
   Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(x)-1}{\tan(x)} &= \lim_{x\to 0}\frac{\cos(x)-1}{x}\frac{x}{\sin(x)}\cos(x)\\ &= \lim_{x\to 0}\frac{\cos(x)-1}{x}\lim_{x\to 0}\frac{x}{\sin(x)}\lim_{x\to 0}\cos(x)\\ &= 0\times 1\times 1\\ &= 0\end{aligned} \qquad (2)$$

3. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+3}$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3} &= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}+\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3}\\ &= e(1+0)^3\\ &= e\end{aligned}$$

4. $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{x}$

   Solution: Let $y=\frac{x}{3}$. Then, $\frac{3}{x}=\frac{1}{y}$. This implies that $x=3y$ and $\lim_{x\to\infty} y=\infty$. Hence,

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} &= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{3y}\\ &= \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{3}\\ &= e^3\end{aligned}$$

Exercise 4.6. Evaluate the following limits.

1. $\lim_{x\to 0}\frac{\sin(2x)}{\sin(7x)}$
2. $\lim_{x\to 0}\frac{\sin(2x)}{\cos(7x)}$
3. $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$
4. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{7x}$

5. $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}$

6. $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{5x}$ [4]

Exercise 4.7. Read about Intermideate value property and Extremum values property of continuous functions.

[4] Answers: 1) $\frac{2}{7}$; 2)0; 3)$\frac{1}{2}$; 4) e^{7}; 5)$\frac{1}{e}$; 6)e^{-5}.

