# *Debre Berhan University*
# *Applied mathematics One(Math 1041)*
# *Lecture note: version 1.1*

"Mathematics is the most beautiful and most powerful creation of the human spirit."

Stefan Banach

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# Contents

**1 Vectors and vector spaces** **2**
1.1 Geometry and Distance . . . . . 2
1.2 Vectors & dot product . . . . . 7
1.3 Direction angles and direction cosine . . . . . 18
1.4 Cross Product . . . . . 19
1.5 Lines and planes in space . . . . . 24
1.6 Vector space and Subspace . . . . . 31
1.7 Linear dependence, independence and Basis . . . . . 35

# Chapter 1

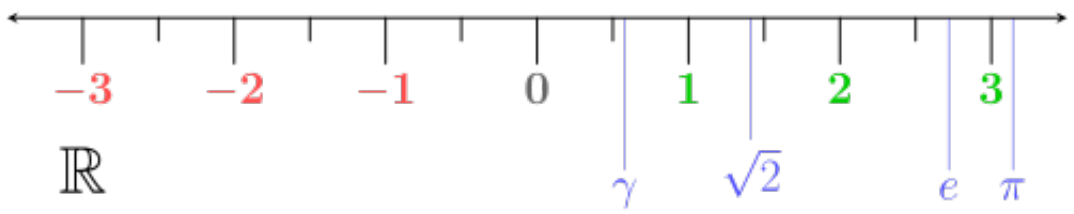
# Vectors and vector spaces

## 1.1 Geometry and Distance

The collection of all real numbers is denoted by $\mathbb{R}$. Thus $\mathbb{R}$ includes the integers

$$\cdots, -2, -1, 0, 1, 2, \cdots$$

the rational numbers, $p/q$, where $p$ and $q$ are integers ($q \neq 0$), and the irrational numbers, like $\sqrt{2}, \pi, e$, etc. Members of $\mathbb{R}$ may be visualized as points on the real-number line as shown in




We write $a \in \mathbb{R}$ to mean $a$ is a member of the set $\mathbb{R}$. In other words, $a$ is a real number. Given two real numbers $a$ and $b$ with $a < b$, the closed interval $[a, b]$ consists of all $x$ such that $a \leq x \leq b$, and the open interval $(a, b)$ consists of all $x$ such that $a < x < b$. Similarly, we may form the half-open intervals $[a, b)$ and $(a, b]$. The absolute value of a number $a \in \mathbb{R}$ is written as $|a|$ and is defined as

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

**Definition 1.1**

*A point in the* ***plane*** $\mathbb{R}^2$ *has two* ***coordinates*** $P = (x, y)$. *A point in space* $\mathbb{R}^3$ *is determined by three coordinates* $P = (x, y, z)$. *The signs of the coordinates define* ***4 quadrants*** *in* $\mathbb{R}^2$ *or* ***8 octants*** *in* $\mathbb{R}^3$. *These regions intersect at the* ***origin*** $O = (0, 0)$ *or* $O = (0, 0, 0)$ *and are bound by* ***coordinate axes*** $\{y = 0\}$ *and* $\{x = 0\}$ *or* ***coordinate planes*** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In $\mathbb{R}^2$ its custom to orient the x-axis to the "east" and the y-axis "north". In $\mathbb{R}^3$, the most common coordinate system has the xy-plane as the "ground" and the z-coordinate axes pointing "up".

**Example 1.1**

1. *$P = (-2, -3)$ is in the third quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -8)$ is located on the negative $z$ axis. The point $P = (1, 2, -3)$ is below the xy-plane. Can you spot the point $Q$ on the xy-plane which is closest to $P$?*

2. ***Problem.*** *Find the midpoint $M$ of $P = (7, 2, 5)$ and $Q = (-15, 4, 7)$.* ***Answer.*** *The midpoint is the average of each coordinate $M = (P + Q)/2 = (-4, 3, 6)$.*

3. *In computer graphics or photography, the xy-plane represents the* ***retina*** *or film plate and the z-coordinate measures the distance towards the viewer. In this* ***photographic coordinate system****, your eyes and chin define the plane $z = 0$ and the nose points in the positive $z$ direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0), (-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.*

**Definition 1.2**

*The Euclidean distance between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.*

**Example 1.2**

***Problem:*** *Find the distance $d(P, Q)$ between $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$.* ***Answer:*** *The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.*

**Remarks.**

1. A distance can be defined by taking any non-negative function $d(P, Q) = d(Q, P)$ which satisfies the **triangle inequality** $d(P, Q) + d(Q, R) \geq d(P, R)$ and which has the property that $d(P, Q) = 0$ if and only if $P = Q$. A set $X$ equipped with such a distance function d is called a **metric space**.

2. **Symmetry** distinguishes the Euclidean distance from other distances. It is characterized by the property $d((1, 0, 0), (0, 0, 0)) = 1$ together with the requirement of rotational and translational and scaling symmetry $d(\lambda P, \lambda Q) = \lambda d(P, Q)$.

3. We usually work with a **right handed coordinate system**, where the $x, y, z$ axes can be matched with the thumb, pointing and middle finger of the **right hand**. The photographers coordinate system is an example of a **left handed coordinate system**, where we use the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some fundamental laws in particle physics related to the weak force are different when observed in a mirror. Coordinate systems with different parity can not be rotated into each other.

4. When dealing with geometric problems in the plane, we leave the z-coordinate away and have $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2}$, where $P = (x, y), Q = (a, b)$. Its important to work in $\mathbb{R}^2$ without referring to a possible $\mathbb{R}^3$ in which it might be embedded.

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**Points, curves, surfaces** and **solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus next on spheres or circles.

**Definition 1.3**

*A **circle** of radius $r \geq 0$ centered at $P = (a, b)$ is the collection of points in $\mathbb{R}^2$ which have distance $r$ from $P$.*

**Definition 1.4**

*A **sphere** of radius $\rho$ centered at $P = (a, b, c)$ is the collection of points in $\mathbb{R}^3$ which have distance $\rho \geq 0$ from $P$. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.*

**Definition 1.5**

*An **ellipse** is the collection of points $P$ in $\mathbb{R}^2$ for which the sum $d(P, A) + d(P, B)$ of the distances to two points $A, B$ is a fixed constant $l > 0$ larger than $d(A, B)$. This allows to draw the ellipse with a string of length $l$ attached at $A, B$. When 0 is the midpoint of $A, B$, an algebraic description is the set of points which satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*

**Example 1.3**

1. ***Problem:*** *Is the point $(3, 4, 5)$ outside or inside the sphere $(x-2)^2 + (y-6)^2 + (z-2)^2 = 16$?* ***Answer:*** *The distance of the point to the center of the sphere is $\sqrt{1+4+9}$. Since this is smaller than 4 the radius of the sphere, the point is inside.*

2. ***Problem:*** *Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1, 0)$ and $B = (-1, 0)$ is equal to 3.* ***Answer:*** *Square the equation $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b$ can be computed as follows: because $P = (a, 0)$ satisfies this equation, $d(P, A) + d(P, B) = (a-1) + (a+1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0, b)$ satisfying it gives $d(Q, A) + d(P, B) = 2\sqrt{b^2+1} = 3$ or $b = \sqrt{5}/2$.*

**Definition 1.6**

*When completing the square of an equation $x^2 + bx + c = 0$, we add $(b/2)^2 - c$ on both sides of the equation in order to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for $x$ gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.*

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**Example 1.4**

1. *The equation* $2x^2 - 10x + 12 = 0$ *is equivalent to* $x^2 + 5x = -6$. *Adding* $(5/2)^2$ *on both sides gives* $(x + 5/2)^2 = 1/4$ *so that* $x = 2$ *or* $x = 3$.

2. *The equation* $x^2 + 5x + y^2 - 2y + z^2 = -1$ *is after completion of the square* $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ *or* $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. *We see a sphere* ***center*** $(5/2, 1, 0)$ *and* ***radius*** $5/2$.

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## Exercise 1.1

1. (a) a) *Find its center and radius of the sphere* $S$ *given by* $x^2+y^2+z^2-20x+12y+4z=29$.
   (b) *Find the distance from the center of* $S$ *defined in a) to* $x^2+y^2+z^2=900$.
   (c) *Find the minimal distance between the spheres. This is the minimal distance between two points where each is in one sphere.*

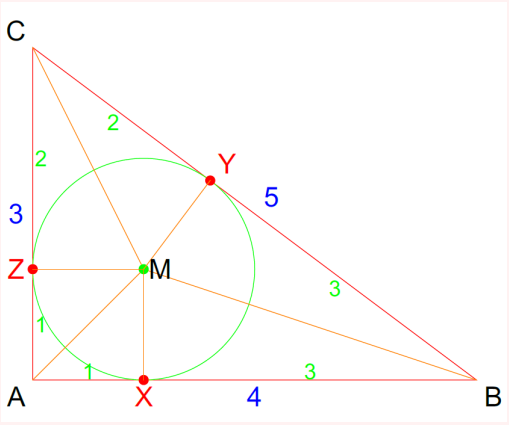
2. (a) *Find the distance* $a$ *from* $P=(-12,-4,-6)$ *to* $x=0$.
   (b) *Find the distance* $b$ *from* $P$ *to the x-axes.*
   (c) *Find the distance* $c$ *from* $P$ *to the origin* $O=(0,0,0)$.
   (d) *Take a general point* $P=(x,y,z)$ *and repeat the computation for the values* $a,b,c$ *as in* $a-c$). *What is* $ar+b^2-c^2$?

3. (a) *Find an equation of the largest sphere with center* $(4,11,9)$ *that is contained in the first octant* $\{x\leq 0, y\leq 0, z\geq 0\}$.
   (b) *Find the equation for the sphere centered at* $(6,10,8)$ *which passes through the center* $(4,11,9)$*of the sphere in* $a$).

4. (a) *Describe the surface* $x^2-(z^2+4z)=12$ *in* $\mathbb{R}^3$.
   (b) *What is the surface* $x^2=z^2$ *in three dimensional space* $\mathbb{R}^3$.
   (c) *Draw the surfaces of* $a$) *and* $b$) *and their intersection.*

5. *You play billiard in the table* $\{(x,y)|0\leq x\leq 4, 0\leq y\leq 8\}$.
   (a) *Hit the ball at* $(3,2)$ *to reach the hole* $(4,8)$ *bouncing* 3 *times at the left wall and three times at the right wall and no other walls. Find the length of the shot.*
   (b) *Hit from* $(3,2)$ *to reach the hole* $(4,0)$ *after hitting twice the left and twice the right wall as well as the top wall* $y=8$ *once. What is the length of the trajectory?*

6. *Verify that the radius of the inscribed circle in a* $3:4:5$ *triangle is* 1. *Use the picture with the triangle* $ABC$ *given by:*




$A=(0,0), B=(4,0), C=(0,3)$, *introduce* $M=(1,1)$ *then get the coordi* *the points* $X,Y,Z$, *then compute the distances.*

**Definition 1.7**

*A physical quantity is a physical property of a phenomenon, body, or substance, that can be quantified by measurement.*

**Definition 1.8**

*Quantities which can be described by magnitude only are called scalars. Scalars are represented by a single letter, such as a. Some examples of scalar quantities are mass (five kilograms), temperature (twenty-two degrees Celsius), and numbers without units (such as three).*

## 1.2 Vectors & dot product

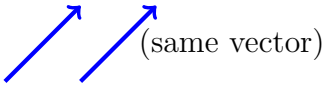
### Geometric view :

**Definition 1.9**

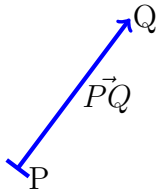
*A vector is defined as having a magnitude and a direction. We represent it by an arrow in the plane or in space. The length of the arrow is the vector's magnitude and the direction of the arrow is the vector's direction.*



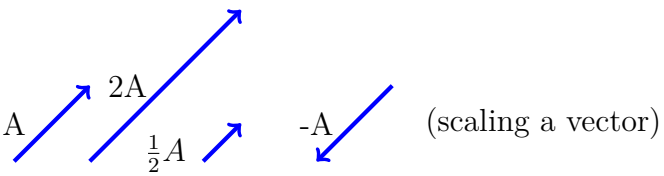

In this way, two arrows with the same magnitude and direction represent the same vector.




We will refer to the start of the arrow as the tail and the end as the tip or head.

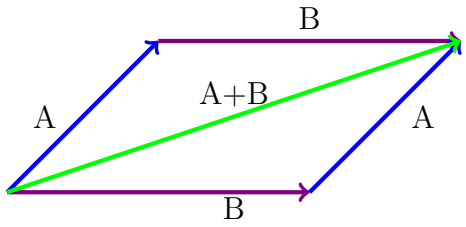



To distinguish vectors from points, it is custom to write $[2,3,4]^T$ or $\langle 2,3,4\rangle$ for vectors and $(2,3,4)$ for points.

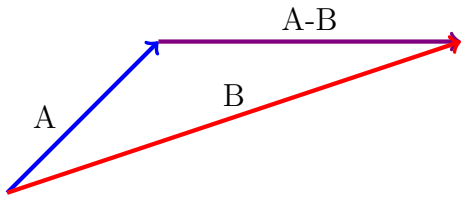



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Because we use numbers to scale a vector we will often refer to real numbers as scalars. You add vectors by placing them head to tail. As the figure shows, this can be done in either order




It is often useful to think of vectors as **displacements**. In this way, $A + B$ can be thought of as the displacement $A$ followed by the displacement $B$.
You subtract vectors either by placing the tail to tail or by adding $A + (-B)$.




Thought of as displacements $A - B$ is the displacement from the end of $B$ to the end of $A$.

## Algebraic view:

**Definition 1.10**

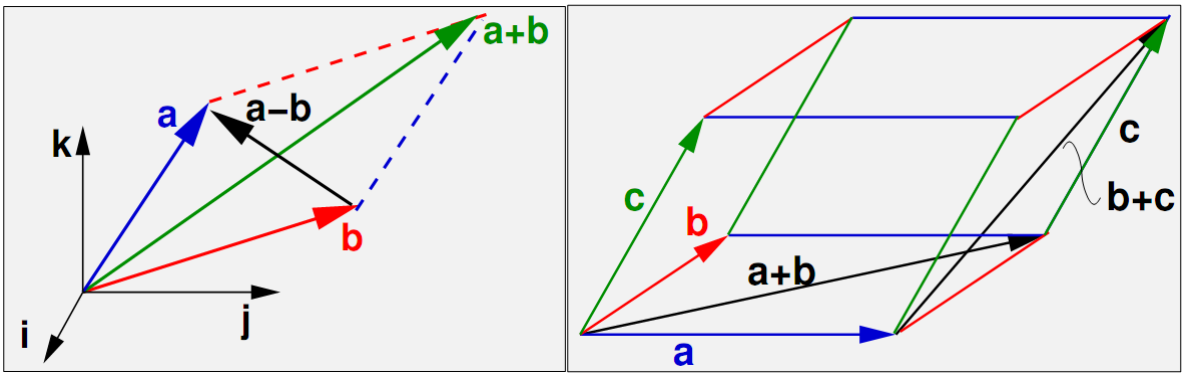
*Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in $\mathbb{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$ which we simply write also as $[x-a, y-b, z-c]^T$ or $\langle x-a, y-b, z-c\rangle$ to save space. It is custom in linear algebra to call $[3, 4, 5]$ a row vector and its transpose $[3, 4, 5]^T = \langle 3, 4, 5\rangle$ a **column vector**. As it goes from $P$ to $Q$ we write $\vec{v} = \vec{PQ}$. The real numbers numbers $p, q, r$ in $\vec{v} = [p, q, r]^T$ are called the **components** of $\vec{v}$.*

Vectors can be drawn anywhere in space. But two vectors with the same components are considered equal. Vectors can be translated into each other if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]^T$ heads to the point $(p, q, r)$. One can therefore identify points $P = (a, b, c)$ with vectors $\vec{v} = [a, b, c]^T$ attached to the origin. For clarity, we often draw an arrow on top of vectors and if $\vec{v} = \vec{PQ}$ then $P$ is the "tail" and $Q$ is the "head" of the vector.

**Definition 1.11**

*The **sum** of two vectors is $\vec{A} + \vec{B} = [a_1, a_2]^T + [b_1, b_2]^T = [a_1 + b_1, a_2 + b_2]^T$. The **scalar multiple** is $\lambda\vec{A} = \lambda[a_1, a_2]^T = [\lambda a_1, \lambda a_2]^T$ . The **difference** $\vec{A} - \vec{B}$ can best be seen as the addition of $\vec{A}$ and $(-1)\vec{B}$.*

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On can check commutativity, associativity, or distributivity rules for vectors as for numbers.

**Definition 1.12**

*The vectors* $\vec{i} = [1,0]^T, \vec{j} = [0,1]^T$ *are called the* ***standard basis vectors*** *in the plane. In space, one has the standard basis vectors* $\vec{i} = [1,0,0]^T, \vec{j} = [0,1,0]^T, \vec{k} = [0,0,1]^T$ .

Every vector $\vec{v} = [p,q]^T$ in the plane can be written as a combination $\vec{v} = p\vec{i} + q\vec{j}$ of standard basis vectors. Every vector $\vec{v} = [p,q,r]^T$ in space can be written as $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$. Vectors appear everywhere in applications. For example in mechanics: if $r(\vec{t}) = [f(t), g(t)]^T$ is a point in the plane which depends on time $t$, then $\vec{v} = [f'(t), g'(t)]^T$ is the **velocity vector** at $r(\vec{t})$. Here $f'(t), g'(t)$ are the derivatives. In physics, where we often want to determine forces acting on objects, these forces are represented as vectors. In particular, electromagnetic or gravitational fields or velocity fields in fluids are described by vectors. Vectors appear also in computer science: the "scalable" vector graphics "format" is a standard for the web for describing graphics. In quantum computation, rather than working with bits, one deals with **qbits**, which are vectors. Finally, color can be written as a vector $\vec{v} = [r,g,b]^T$ , where $r$ is **red**, $g$ is **green** and $b$ is **blue** component of the color vector. An other coordinate system for color is $\vec{v} = [c,m,y]^T = [1-r, 1-g, 1-b]^T$ , where $c$ is **cyan**, $m$ is **magenta** and $y$ is **yellow**. Vectors appear also in probability theory and statistics. On a finite probability space for example, a random variable is nothing else than a vector.

The addition and scalar multiplication of vectors satisfy the laws known from arithmetic.

1. **commutativity** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,
2. **associativity** $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}, r * (s * \vec{v}) = (r * s) * \vec{v}$
3. **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$
4. And $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$, where * denotes multiplication with a scalar.

**Definition 1.13**

*The length* $|\vec{v}|$ *of a vector* $\vec{v} = \vec{PQ}$ *is defined as the distance* $d(P,Q)$ *from* $P$ *to* $Q$*. A vector of length* 1 *is called a unit vector. If* $\vec{v} \neq \vec{0}$*, then* $\frac{\vec{v}}{|\vec{v}|}$ *is called a unit vector.*

**Example 1.5**

$|[3,4]^T| = 5$ *and* $|[3,4,12]^T| = 13$. *Examples of unit vectors are* $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$ *and* $[3/5, 4/5]^T$ *and* $[3/13, 4/13, 12/13]^T$ . *The only vector of length 0 is the zero vector* $|\vec{0}| = 0$.

**Example 1.6**

1. *Add the following two vectors*
   (a) $\vec{u} = (-1, 2, 1)$ *and* $\vec{v} = (2, -5, 0)$
   (b) $\vec{u} = i + 4j - 2k$ *and* $\vec{v} = 2i - 5j$
2. *Let* $\vec{u} = 3i + 4j - k$, *then find*
   (a) $\frac{2}{3}\vec{u}$
   (b) $-1\vec{u}$

***solution***:

1. (a)

$$\begin{aligned}\vec{u} + \vec{v} &= (-1, 2, 1) + (2, -5, 0)\\ &= (-1 + 2, 2 - 5, 1 + 0)\\ &= (1, -3, 1)\end{aligned}$$

(b)

$$\begin{aligned}\vec{u} + \vec{v} &= (i + 4j - k) + (2i - 5j)\\ &= (1 + 2)i + (4 - 5)j + (-1 + 0)k\\ &= 3i - j - k\end{aligned}$$

2. (a)

$$\begin{aligned}\frac{2}{3}\vec{u} &= \frac{2}{3}(3i + 4j - k)\\ &= 2i + \frac{8}{3}j - \frac{2}{3}k\end{aligned}$$

(b)

$$\begin{aligned}-1\vec{u} &= -(3i + 4j - k)\\ &= -3i - 4j + k\end{aligned}$$

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# Dot Product

The dot product is one way of combining ("multiplying") two vectors. The output is a scalar (a number). It is called the dot product because the symbol used is a dot. Because the dot product results in a scalar it, is also called the scalar product.
The geometric definition of the dot product says that the dot product between two vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta,$$

where $\theta$ is the angle between vectors $\vec{a}$ and $\vec{b}$. Although this formula is nice for understanding the properties of the dot product, a formula for the dot product in terms of vector components would make it easier to calculate the dot product between two given vectors.

Since the standard unit vectors are orthogonal, we immediately conclude that the dot product between a pair of distinct standard unit vectors is zero:

$$i \cdot j = i \cdot k = j \cdot k = 0.$$

The dot product between a unit vector and itself is also simple to compute. In this case, the angle is zero and $\cos\theta = 1$. Given that the vectors are all of length one, the dot products are

$$i \cdot i = j \cdot j = k \cdot k = 1.$$

The second step is to calculate the dot product between two three-dimensional vectors

$$a = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$$
$$b = (b_1, b_2, b_3) = b_1 i + b_2 j + b_3 k.$$

To do this, we simply assert that for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, and any scalar $\lambda$,

$$(\lambda a) \cdot b = \lambda(a \cdot b)$$

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

**Definition 1.14**

*The dot product of two vectors $\vec{v} = [a, b, c]^T$ and $\vec{w} = [p, q, r]^T$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.*

**Remarks.**

1. Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, the **Dirac notation** $[\vec{v}|\vec{w}]^T$ is used in quantum mechanics or the **Einstein notation** $v_i w^i$ or more generally $g_{ij} v_i w_j$ in general relativity. The dot product is also called **scalar product** or **inner product**.

2. Any product $g(v, w)$ which is linear in $v$ and $w$ and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \leq 0$ and $g(v, v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v, w) = 3v_1 w_1 + 2v_2 w_2 + v_3 w_3$.

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**Theorem 1.1**

*The dot product determines distances and distances determines the dot product.*

***Proof:*** *Write $v = \vec{v}$. Using the dot product one can express the length of $v$ as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v+w)\cdot(v+w) = v\cdot v + w\cdot w + 2(v\cdot w)$ can be solved for $v \cdot w$:*

$$v \cdot w = (|v+w|^2 - |v|^2 - |w|^2)/2.$$

**Theorem 1.2**

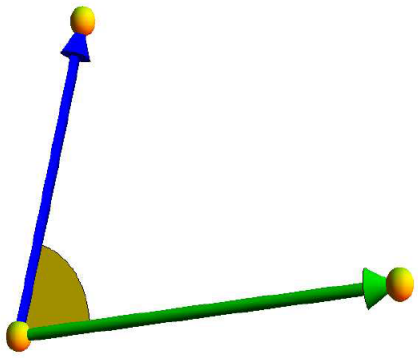
*The Cauchy-Schwarz inequality tells $|\vec{v}\cdot\vec{w}| \leq |\vec{v}||\vec{w}|$.*

***Proof.*** *If $|w| = 0$, there is nothing to show. Otherwise, assume $|w| = 1$ by dividing the equation by $|w|$. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw)\cdot(v - aw)$ to get $0 \leq (v-(v\cdot w)w)\cdot(v-(v\cdot w)w) = |v|^2 + (v\cdot w)^2 - 2(v\cdot w)^2 = |v|^2 - (v\cdot w)^2$ which means $(v\cdot w)^2 \leq |v|^2$.*

Having established this, it is possible to give a definition of what an angle is, without referring to any geometric pictures:

**Definition 1.15**

*The angle between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cos(\alpha)$. Since cos maps $[0,\pi]$ in a $1:1$ manner to $[-1,1]$, this is well defined.*



**Note:**

1. The cosine of the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$\implies \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$$

2. If an angle between two vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$ then $\vec{a}\cdot\vec{b} = 0$ and the vectors $\vec{a}$ and $\vec{b}$ are known as orthogonal vectors. In general the dot product of two non-zero vectors is zero only if they are orthogonal.

**Remark**:- Suppose $\vec{a}$ and $\vec{b}$ are two non-zero vectors and $\theta$ is the angle between $\vec{a}$ and $\vec{b}$:

i) if $\theta$ is acute, then $\vec{a} \cdot \vec{b} > 0$.

ii) if $\theta$ is obtuse, then $\vec{a} \cdot \vec{b} < 0$.

iii) if $\theta$ is right angle, then $\vec{a} \cdot \vec{b} = 0$.

**Theorem 1.3: Al Kashi's theorem:**

*If $a, b, c$ are the side lengths of a triangle $ABC$ and $\alpha$ is the angle at the vertex $C$, then $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.*

***Proof.*** *Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v}\vec{w}$, We know $\vec{v}\vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.*

The angle **definition** works in any space with a dot product. In statistics one works with vectors of $n$ components. They are called data or random variables and $\cos(\alpha)$ is called the **correlation** between two random variables $\vec{v}, \vec{w}$ of zero **expectation** $E[\vec{v}] = (v_1 + \cdots + v_n)/n$. The dot product $(v_1 w_1 + \cdots + v_n w_n)/n$ is then the **covariance**, the scaled length $|v|/\sqrt{n}$ is the **standard deviation** and denoted by $\sigma(v)$. The formula $Corr[v, w] = Cov[v, w]/(\sigma(v)\sigma(w))$ for the correlation is the familiar angle formula. Statistics shows that geometry in arbitrary dimensions can be useful.

**Theorem 1.4**

*The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.*
***Proof:*** $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

**Example 1.7**

*Calculate the dot product of $a = (1, 2, 3)$ and $b = (4, -5, 6)$. Do the vectors form an acute angle, right angle, or obtuse angle?*
***Solution**: Using the component formula for the dot product of three-dimensional vectors,*

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

*we calculate the dot product to be*

$$a \cdot b = 1(4) + 2(-5) + 3(6) = 4 - 10 + 18 = 12.$$

*Since a.b is positive, we can infer from the geometric definition, that the vectors form an acute angle.*

**Example 1.8**

*Calculate the dot product of* $c = (-4, -9)$ *and* $d = (-1, 2)$ *. Do the vectors form an acute angle, right angle, or obtuse angle?*

***Solution****: Using the component formula for the dot product of two-dimensional vectors,*

$$a \cdot b = a_1 b_1 + a_2 b_2,$$

*we calculate the dot product to be*

$$c \cdot d = -4(-1) - 9(2) = 4 - 18 = -14.$$

*Since c.d is negative, we can infer from the geometric definition, that the vectors form an obtuse angle.*

**Example 1.9**

*If* $a = (6, -1, 3)$*, for what value of* $c$ *is the vector* $b = (4, c, -2)$ *perpendicular to* $a$*?*

***Solution****: For* $a$ *and* $b$ *to be perpendicular, we need their dot product to be zero. Since*

$$a \cdot b = 6(4) - 1(c) + 3(-2) = 24 - c - 6 = 18 - c,$$

*the number* $c$ *must satisfy* $18 - c = 0$*, or* $c = 18$*. You can double-check that the vector* $b = (4, 18, -2)$ *is indeed perpendicular to* $a$ *by verifying that*

$$a \cdot b = (6, -1, 3).(4, 18, -2) = 0.$$

1. *Find the value of* $\boldsymbol{k}$ *such that the vectors* $(2, 3, -2)$ *and* $(5, k, 2)$ *are perpendicular.*
2. *Find the values of* $\beta$ *such that the vectors* $(\beta, -3, 1)$ *and* $(\beta, \beta, 2)$ *are perpendicular.*

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**Example 1.10**

1. *Find the component form and the length of vector with initial point and terminal point* $p = (3, -7, 1)$ *and* $Q = (-2, 5, 3)$
2. *Find the unit vector in the direction of* $\vec{v} = (-2, 5, 3)$
3. *Let* $\vec{u} = (2, 5, -1)$ *and* $\vec{v} = (-1, 2, 3)$,*the find the scalar product of* $\vec{u}$ *and* $\vec{u}$

***Solution***

1. $\vec{v} = \vec{PQ} = (-2-3, 5+7, 3-1) = (-5, 12, 2)$
   *and* $\|\vec{v}\| = \sqrt{(-5)^2 + (12)^2 + 2^2} = \sqrt{173}$

2. $\|\vec{v}\| = \sqrt{-2^2 + 5^2 + 3^2} = \sqrt{38}$
   $\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{38}}(-2, 5, 3)$
   $= (\frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}})$

3. $\vec{u} \cdot \vec{v} = (2, 5, -1) \cdot (-1, 2, 3)$
   $= 2. - 1 + 5.2 + -1.3$
   $= -2 + 10 - 3$
   $= 5$

**Exercise 1.2**

1. *Find the cosine of the angle and the angle between the following vectors*
   (a) $\vec{u} = 2i - 5j + k$ *and* $\vec{v} = i + j + 2k$
   (b) $\vec{v} = (-\sqrt{3}, 1)$ *and* $\vec{w} = (0, 5)$
   (c) $\vec{v} = (0, -1, 0)$ *and* $\vec{w} = (0, 0, 3)$

**Definition 1.16**

*Two vectors are called* ***orthogonal*** *or* ***perpendicular*** *if* $\vec{v} \cdot \vec{w} = 0$. *The zero vector* $\vec{0}$ *is orthogonal to any vector. For example,* $\vec{v} = [2, 3]^T$ *is orthogonal to* $\vec{w} = [-3, 2]^T$ .

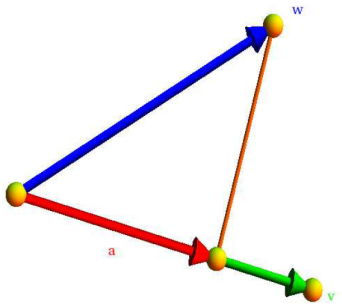
**Theorem 1.5**

***Pythagoras theorem:*** *if* $\vec{v}$ *and* $\vec{w}$ *are orthogonal, then* $|v - w|^2 = |v|^2 + |w|^2$.
***Proof:*** $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

# Projection

Suppose that two non-zero vectors $\vec{v}$ and $\vec{w}$ are positioned as in the figure below and that the light casts a shadow on the line containing the vector $\vec{v}$. Informally we think of the shadow as

determining a vector parallel to vector $\vec{w}$, we call this vector the projection of vector $\vec{w}$ on to vector $\vec{v}$ which is denoted by $proj_{\mathbf{v}}\ ^{\mathbf{w}}$ and defined as $proj_v\ ^w = (\frac{v \cdot w}{\|v\|^2})v$.




**Definition 1.17**

*The vector* $P(\vec{v}) = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w}$ *is called the* ***projection*** *of* $\vec{v}$ *onto* $\vec{w}$*. The* ***scalar projection*** $\dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ *is a signed length of the vector projection. Its absolute value is the length of the projection of* $\vec{v}$ *onto* $\vec{w}$*. The vector* $\vec{b} = \vec{v} - P(\vec{v})$ *is a vector orthogonal to the* $\vec{w}$*-direction.*

**Example 1.11**

1. *For example, with* $\vec{v} = [0, -1, 1]^T, \vec{w} = [1, -1, 0]^T, P(\vec{v}) = [1/2, -1/2, 0]^T$ *. Its length is* $1/\sqrt{2}$*.*
2. *Projections are important in physics. If a wind force* $\vec{F}$ *affects a car driving in the direction* $\vec{w}$ *and* $P$ *denotes the projection onto* $\vec{w}$ *then* $P_{\vec{w}}(\vec{F})$ *is the force which accelerates or slows down the car.*

The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if $v$ points more towards to $w$, it is negative if $v$ points away from it. In the next lecture we use the projection to compute distances between various objects.

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**Example 1.12**

1. *Let vector* $\vec{u} = 2i - j + 3k, \quad \vec{v} = i + 2j + k$, *then find:*

   (a) $proj_{\vec{v}}\ \vec{u}$

   (b) $proj_{\vec{u}}\ \vec{v}$

   (c) $proj_{\frac{1}{2}\vec{v}}\ 3\vec{u}$

   (d) *The scalar projection of* $proj_{\vec{v}}\ \vec{u}$ *and* $proj_{\vec{u}}^{\vec{v}}$

***Solution***:-

1. (a)

$$\begin{aligned} proj_{\vec{v}}^{\vec{u}} &= \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{(i+2j+k)\cdot(2i-j+3k)}{\|i+2j+k\|^2}\right)(i+2j+k) \\ &= (\frac{2-2+3}{1^2+2^2+1^2})(i+2j+k) \\ &= \frac{3}{6}(i+2j+k) \\ &= \frac{1}{2}i + j + \frac{1}{2}k \end{aligned}$$

(b)

$$\begin{aligned} proj_{\vec{u}}^{\vec{v}} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u} \\ &= (\frac{(2i-j+3k)\cdot(i+2j+k)}{\|2i-j+3k\|^2})(2i-j+3k) \\ &= (\frac{2-2+3}{1^2+(-2)^2+3^2})(2i-j+3k) \\ &= \frac{3}{14}(2i-j+3k) \\ &= \frac{6}{14}i - \frac{3}{14}j + \frac{9}{14}k \end{aligned}$$

(c)

$$\begin{aligned} proj_{\frac{1}{2}\vec{v}}^{3\vec{u}} &= \left(\frac{\frac{1}{2}\vec{v}\cdot 3\vec{u}}{\|\frac{1}{2}\vec{v}\|^2}\right)\frac{1}{2}\vec{v} \\ &= (\frac{\frac{1}{2}(i+2j+k)\cdot 3(2i-j+3k)}{\frac{1}{4}\|i+2j+k\|^2})\frac{1}{2}(i+2j+k) \\ &= (\frac{\frac{3}{2}(2-2+3)}{\frac{1}{4}(1^2+2^2+1^2)})\frac{1}{2}(i+2j+k) \\ &= 3(i+2j+k) \\ &= 3i+6j+3k \end{aligned}$$

2.

$$\begin{aligned} \|proj_{\vec{v}}\ \vec{u}\| &= |(\frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|})| \\ &= |(\frac{(i+2j+k)\cdot(2i-j+3k)}{\|i+2j+k\|})| \\ &= |(\frac{2-2+3}{\sqrt{1^2+2^2+1^2}})| \\ &= \frac{3}{\sqrt{6}} \\ \&\|proj_{\vec{u}}\ \vec{v}\| &= |(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|})| \\ &= |(\frac{(2i-j+3k)\cdot(i+2j+k)}{\|2i-j+3k\|})| \\ &= |\frac{2-2+3}{\sqrt{1^2+(-2)^2+3^2}}| \\ &\frac{3}{\sqrt{14}} \end{aligned}$$

## 1.3 Direction angles and direction cosine

**Definition 1.18**

*Let $p$ be a vector and $\alpha, \beta$ and $\gamma$ are angles that a vector $p$ makes with the positive $x, y$ and $z$ axis respectively. These angles are called directional angles of the vector $p$ and $\cos\alpha, \cos\beta$ and $\cos\gamma$ are directional cosines of vector*

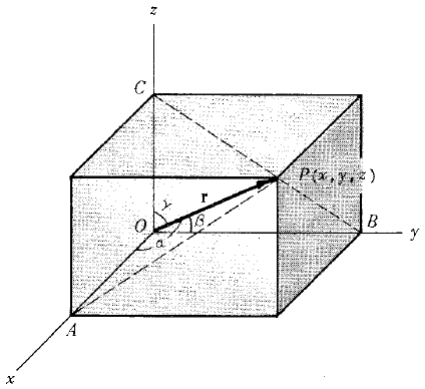



Figure 1.1: direction angeles

**Note**:-

1. $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$.
2. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

**Example 1.13**

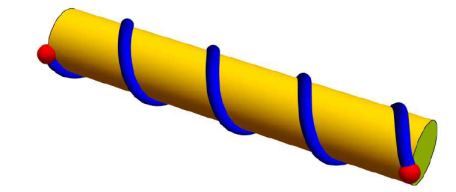
1. *consider the following vectors*
   (a) $\vec{a} = (2, 3)$
   (b) $\vec{b} = (1, 1, 1)$
   (c) $\vec{c} = (3, 5, 2)$

   *Then find the direction angles and direction cosines*
2. *If the direction angle of a vector $\vec{a}$ makes an angle of $\frac{\pi}{3}$ with the positive x-axis and an angle of $\frac{\pi}{4}$ with the positive y-axis then what angle will make with the positive z-axis.*

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**Exercise 1.3**

1. *Find a unit vector parallel to* $\vec{u} + 2\vec{v} + 4\vec{w}$ *if* $\vec{u} = [-10, 2, 9]^T$ *and* $\vec{v} = [1, 1, 3]^T$ *and* $\vec{w} = [3, 1, 1]^T$ .
2. *A* ***Alemitu*** *gets pulled with a force* $\vec{F} = \langle 7, 1, 4\rangle$. *She moves with velocity* $\vec{v} = \langle 4, -2, 1\rangle$. *The dot product of* $\vec{F}$ *with* $\vec{v}$ *is* ***power****.*
   (a) *Find the angle between the* $\vec{F}$ *and* $\vec{v}$.
   (b) *Find the* ***vector projection*** *of the* $\vec{F}$ *onto* $\vec{v}$.
   (c) *Find the* ***scalar projection*** *of* $\vec{F}$ *on* $\vec{v}$.
3. *Light shines long the vector* $\vec{a} = \langle a_1, a_2, a_3\rangle$ *and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is* $-\vec{a}$. ***Hint.*** *Reflect first at the xy-plane.*
4. *A rope is wound exactly* 5 *times around a stick of circumference* 1 *and length* 12. *How long is the rope?*



## 1.4 Cross Product

**Definition 1.19**

*The cross product of two vectors* $\vec{v} = [v_1, v_2]^T$ *and* $\vec{w} = [w_1, w_2]^T$ *in the plane is the scalar* $v_1w_2 - v_2w_1$.

To remember this, write it as a determinant of a matrix $A$ which is a $2\times 2$ array of the numbers). $\det(A)$ is the product of the diagonal entries minus the product of the side diagonal entries.

$$\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}.$$

**Definition 1.20**

*The* ***cross product*** *of two vectors* $\vec{v} = [v_1, v_2, v_3]^T$ *and* $\vec{w} = [w_1, w_2, w_3]^T$ *in space is defined as the* ***vector*** $\vec{v} \times \vec{w} = [v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1]^T$ .

To remember it we write the product as a "determinant":

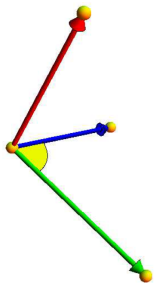
$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

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which is $\vec{i}(v_2w_3 - v_3w_2) - \vec{j}(v_1w_3 - v_3w_1) + \vec{k}(v_1w_2 - v_2w_1)$.

**Example 1.14**

1. *The cross product of* $[1,2]^T$ *and* $[4,5]^T$ *is* $5 - 8 = -3$.
2. *The cross product of* $[1,2,3]^T$ *and* $[4,5,1]^T$ *is* $[-13,11,-3]^T$ .



**Theorem 1.6**

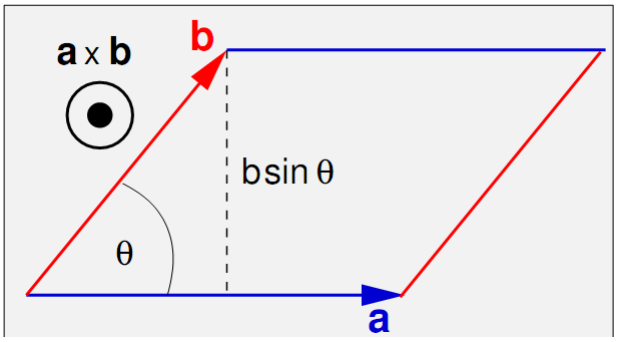
*In space, the cross product* $\vec{v} \times \vec{w}$ *is orthogonal to both* $\vec{v}$ *and* $\vec{w}$. *The product is anti-commutative.*
***Proof.*** *We verify for example that* $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ *and look at the definition.*

**Theorem 1.7**

*Length formula for the cross product:* $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.
***Proof:*** *verify first the* ***Lagrange's identity*** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ *which is also called* ***Cauchy-Binet*** *formula by direct computation . Now,* $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

**Definition 1.21**

*The absolute value respectively length* $|\vec{v} \times \vec{w}|$ *defines the* ***area of the parallelogram*** *spanned by* $\vec{v}$ *and* $\vec{w}$.




Note that we have given the definition of area, so that nothing needs to be proven. To see that the definition fits with our common intuition we have about area, note that $|\vec{w}| \sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. The area formula proves the sin-formula because the area does not depend on which pair of sides to a triangle we take. The area makes sense because it is linear in each of the vectors $\vec{v}$ and $\vec{w}$: scale one by a factor $\lambda = 2$ for example,

doubles the area.

**Theorem 1.8**

*$\vec{v} \times \vec{w}$ is zero if and only if $\vec{v}$ and $\vec{w}$ are parallel, that is if $\vec{v} = \lambda\vec{w}$ for some real $\lambda$.*
***Proof.*** *Use the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$. The cross product can therefore be used to check whether two vectors are parallel or not. Note that $v$ and $-v$ are considered parallel even so sometimes the notion anti-parallel is used.*

**Theorem 1.9**

*The trigonometric sin-formula: if $a, b, c$ are the side lengths of a triangle and $\alpha, \beta, \gamma$ are the angles opposite to $a, b, c$ then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$.*
***Proof.*** *Express the area of the triangle in three different ways:*

$$ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta).$$

*Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.*

**Example 1.15**

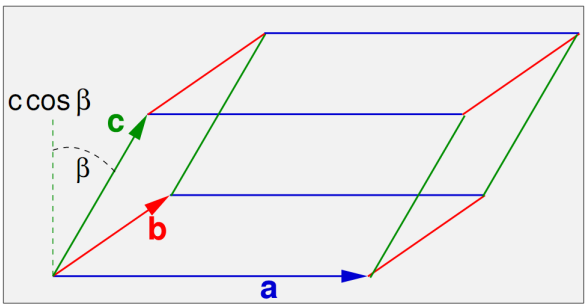
*If $\vec{v} = [a, 0, 0]^T$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]^T$ , then $v \times \vec{w} = [0, 0, ab\sin(\alpha)]^T$ which has length $|ab\sin(\alpha)|$.*

**Definition 1.22**

*The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the triple scalar product of $\vec{u}, \vec{v}, \vec{w}$.*

**Definition 1.23**

*The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.*




**Definition 1.24**

*The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors define a **right-handed** coordinate system.*

Again, there was no need to prove anything because we **defined** volume and orientation. Why does this fits with our intuition? The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped

if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector, we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with twice the volume.

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**Example 1.16**

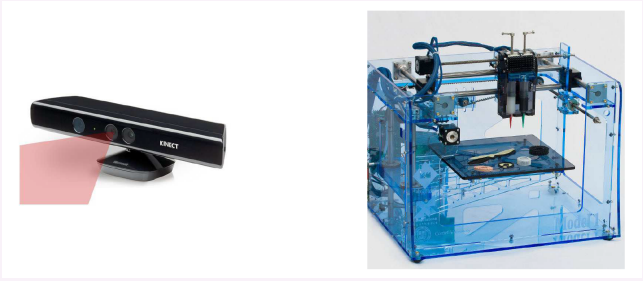
1. ***Problem:*** *You have two apples of the same shape, but one has a 3 times larger diameter. What is their weight ratio?*

   ***Answer.*** *For a parallelepiped spanned by $[a,0,0]^T[0,b,0]^T$ and $[0,0,c]^T$ , the volume is the triple scalar product abc. If $a,b,c$ are all tripled, the volume gets multiplied by a factor 27. Now cut each apple into the same amount of parallelepipeds, the larger one with slices 3 times as large too. Since each of the pieces has 27 times the volume, also the apple is 27 times heavier!*

2. ***Problem:*** *Find the volume of the parallelepiped which has the vertices $O = (1,1,0), P = (2,3,1), Q = (4,3,1), R = (1,4,1)$.* ***Answer:*** *We first see that the solid is spanned by the vectors $\vec{u} = [1,2,1]^T, \vec{v} = [3,2,1]^T$, and $\vec{w} = [0,3,1]^T$ . We get $\vec{v} \times \vec{w} = [-1,-3,9]^T$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.*

3. ***Problem.*** *A 3D scanner is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0,1,1), Q = (1,1,0)$ and $R = (1,2,3)$. Find the area of the triangle.*

   ***Solution.*** *We have to find the length of the cross product of $\vec{PQ}$ and $\vec{PR}$ which is $[1,-3,1]^T$ . The length is $\sqrt{11}$.*



4. ***Problem.*** *The scanner detects an other point $A = (1,1,1)$. On which side of the triangle is it located if the cross product of $\vec{PQ}$ and $\vec{PR}$ is considered the direction "up".*
   ***Solution.*** *The cross product is $\vec{n} = [1,-3,1]^T$ . We have to see whether the vector $\vec{PA} = [1,0,0]^T$ points into the direction of $\vec{n}$ or not. To see that, we have to form the dot product. It is 1 so that indeed, A is "above" the triangle. Note that a triangle in space a priori does not have an orientation. We have to tell, what direction is "up". That is the reason that file formats for 3D printing like contain the data for three points in space as well as a vector, telling the direction.*

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**Exercise 1.4**

1. (a) a) *Find a unit vector perpendicular to the space diagonal* $[1,1,1]^T$ *and the face diagonal* $[1,1,0]^T$ *of the cube.*

   (b) *Find the volume of the parallelepiped for which the base parallelogram is given by the points* $P=(5,2,2), Q=(3,1,2), R=(1,4,2), S=(-1,3,2)$ *and which has an edge connecting* $P$ *with* $T=(5,6,8)$.

   (c) *Find the area of the base and use (b) to get the height of the parallelepiped.*

2. (a) *Assume* $\vec{u}+\vec{v}+\vec{w}=\vec{0}$. *Verify that* $\vec{u}\times\vec{v}=\vec{v}\times\vec{w}=\vec{w}\times\vec{u}$.

   (b) *Find* $(\vec{u}+\vec{v})\cdot(\vec{v}\times\vec{w})$ *if* $\vec{u},\vec{v},\vec{w}$ *are unit vectors which are orthogonal to each other and* $\vec{u}\times\vec{v}=\vec{w}$.

3. *To find the equation* $ax+by+cz=d$ *for the plane which contains the point* $P=(1,2,3)$ *as well as the line which passes through* $Q=(3,4,4)$ *and* $R=(1,1,2)$, *we find a vector* $[a,b,c]^T$ *normal to the plane and fix* $d$ *so that* $P$ *is in the plane.*

## 1.5 Lines and planes in space

**Definition 1.25**

*A point* $P=(p,q,r)$ *and a vector* $\vec{v}=[a,b,c]^T$ *define the line*

$$L=\left\{\begin{bmatrix}p\\q\\r\end{bmatrix}+t\begin{bmatrix}a\\b\\c\end{bmatrix},t\in\mathbb{R}\right\}.$$

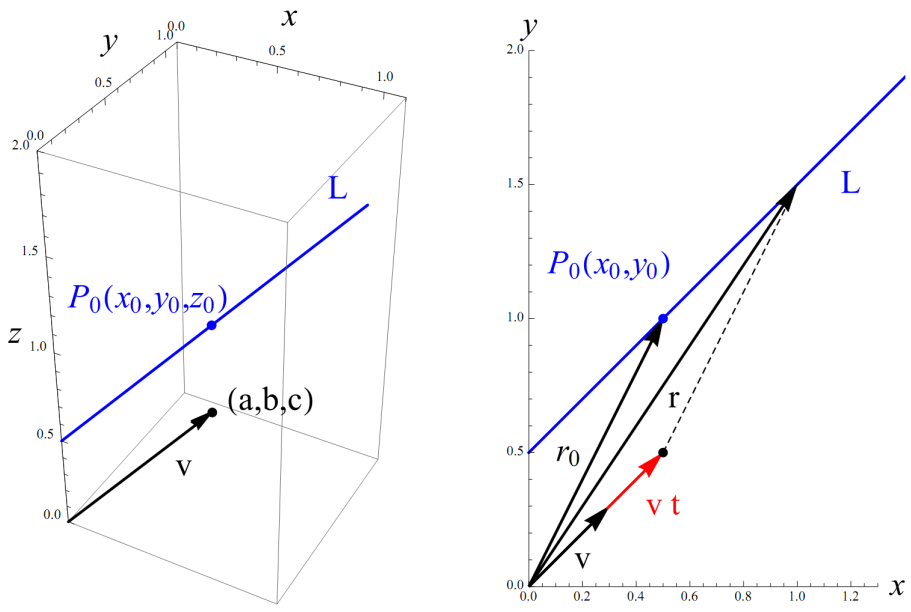



Figure 1.2: Parametric and vector equation of line respectively.

The line consists of all points obtained by adding a multiple of the vector $\vec{v} = [a, b, c]^T$ to the vector $\vec{OP} = [p, q, r]^T$ . The line contains the point $P$ as well as a copy of $\vec{v}$ attached to $P$. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter $t$ as "time". At time $t = 0$, we are at the point $P$, whereas at time $t = 1$ we are at $\vec{OP} + \vec{v}$.

**Definition 1.26**

*If $t$ is restricted to values in a **parameter interval** $[s, u]$, then $L = [p, q, r]^T + t[a, b, c]^T, s \leq t \leq u$ is a **line segment** which connects $\vec{r}(s)$ with $\vec{r}(u)$.*

**Example 1.17**

*To get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, form the vector $\vec{v} = \vec{PQ} = [1, 3, 4]^T$ and get $L = [x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T$. This can be written also as $\vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t]^T$ . If we write $[x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for $t$:*

$$L = \{(x, y, z) | (x - 1)/2 = (y - 1)/3 = (z - 2)/4\}.$$

**Theorem 1.10**

*The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]^T$ with nonzero $a, b, c$ satisfies the **symmetric equations***

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c}.$$

***Proof.*** *Each of these expressions is equal to $t$. These symmetric equations have to be modified a bit one or two of the numbers $a, b, c$ are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$. The interpretation is that the line is written as an intersection of two planes.*

**Example 1.18**

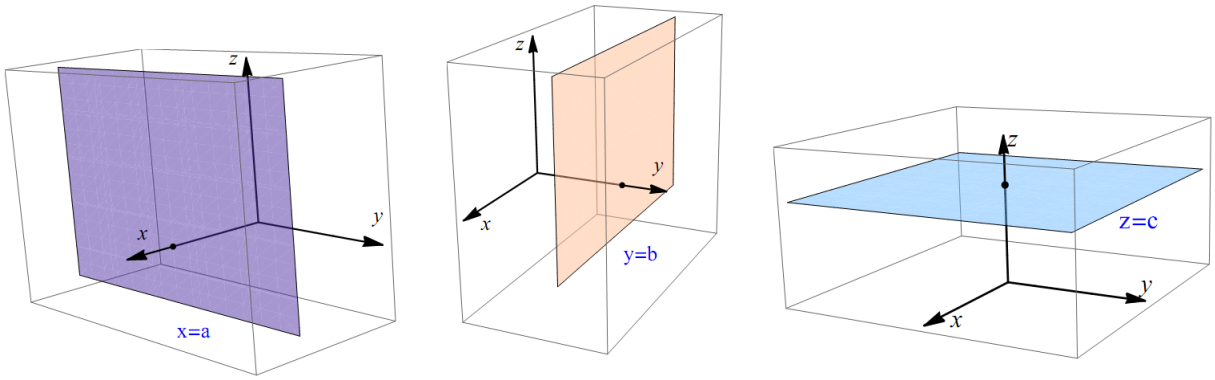
1. *Find the symmetric equations for the line through the points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$, first form the parametric equations $[x, y, z]T = [0, 1, 1]6T + t[2, 2, 3]^T$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for $t$ gives the symmetric equation $x/2 = (y - 1)/2 = (z - 1)/3$.*

2. ***Problem:** Find the symmetric equation for the $z$ axes. **Answer**: This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers $a, b, c$ are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.*

**Definition 1.27**

*A point $P$ and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \left\{\vec{OP} + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers}\right\}$.*

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**Example 1.19**

*An example is* $\sigma = \left\{[x,y,z]^T = [1,1,2]^T + t[2,4,6]^T + s[1,0,-1]^T\right\}$. *This is called the* ***parametric description*** *of a plane.*

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q-P)\cdot\vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = [a, b, c]^T$ , this means $ax + by + cz = ax_0 + by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have shown:

**Theorem 1.11**

*The equation for a plane containing* $\vec{v}$ *and* $\vec{w}$ *and a point* $P$ *is*

$$ax + by + cz = d,$$

*where* $[a, b, c]^T = \vec{v} \times \vec{w}$ *and* $d$ *is obtained by plugging in* $P$.

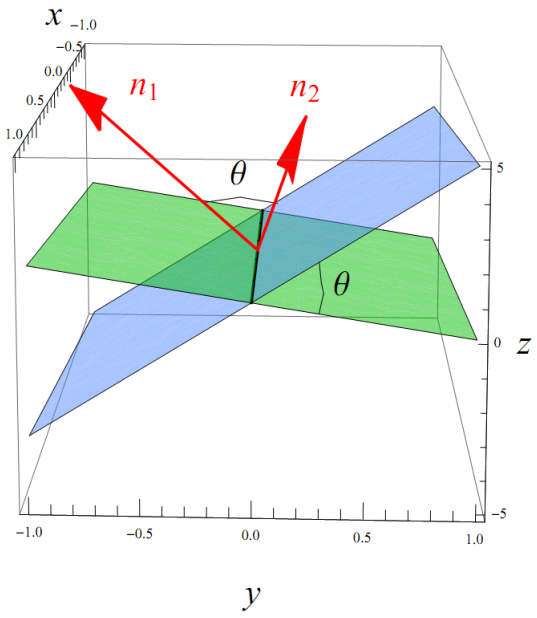
**Example 1.20**

***Problem:*** *Find the equation of a plane which contains the three points* $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
***Answer***: *The plane contains the two vectors* $\vec{v} = [1, 2, 0]^T$ *and* $\vec{w} = [2, 2, 2]^T$ . *We have* $\vec{n} = [4, -2, -2]^T$ *and the equation is* $4x - 2y - 2z = d$. *The constant* $d$ *is obtained by plugging in the coordinates of a point to the left. In our case, it is* $4x - 2y - 2z = -4$.

**Definition 1.28**

*The* ***angle between the two planes*** $ax + by + cz = d$ *and* $ex + fy + gz = h$ *is defined as the angle between the two normal vectors* $\vec{n} = [a, b, c]^T$ *and* $\vec{m} = [e, f, g]^T$ .

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**Example 1.21**

***Problem:*** *Find the angle between the planes* $x + y = -1$ *and* $x + y + z = 2$. ***Answer:*** *find the angle between* $\vec{n} = [1, 1, 0]^T$ *and* $\vec{m} = [1, 1, 1]^T$ . *It is* $arccos(2/\sqrt{6})$.

Finally, lets look at some distance formulas.

1. Sometimes it may be important to find the distance $D$, between some point $P_0$ and a plane in the form $ax + by + cz + d = 0$. The formula for the distance between a plane $\Pi : ax + by + cz + d = 0$ and a point is $P(x_0, y_0, z_0)$ is as follows:

$$D = \frac{| ax_0 + by_0 + cz_0 + d |}{\sqrt{a^2 + b^2 + c^2}}$$

**Proof:** Let $Q(x_1, y_1, z_1)$ be any point on the plane $\Pi$ and let $P_0 = (x_0, y_0, z_0)$ be our point of interest. Let $\vec{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$. Recall that this plane can be defined with a vector that is perpendicular to the plane known as a normal. Let $n = (a, b, c)$ be a normal to our plane $\Pi$. The orthogonal projection of $\vec{QP_0}$ onto $\vec{n}$ is equal to the distance between the plane $\Pi$ and $P_0$ as illustrated below: Thus it follows that

$$D = \|\text{proj}_{\vec{n}}\vec{QP_0}\|$$

When simplified and letting $d = -ax_1 - by_1 - cz_1$

$$D = \|\text{proj}_{\vec{n}}\vec{QP_0}\|$$

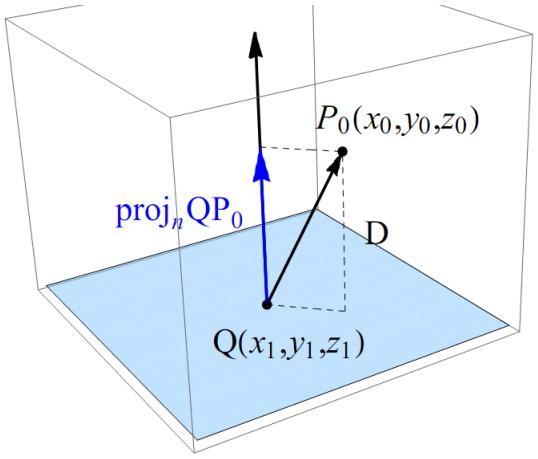
$$D = \frac{\|\vec{QP_0} \cdot \vec{n}\|}{\|\vec{n}\|}$$

$$D = \frac{| (x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a, b, c) |}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \frac{| a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) |}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \frac{| ax_0 + by_0 + cz_0 + d |}{\sqrt{a^2 + b^2 + c^2}}$$

■




If $P$ is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point $Q$, then

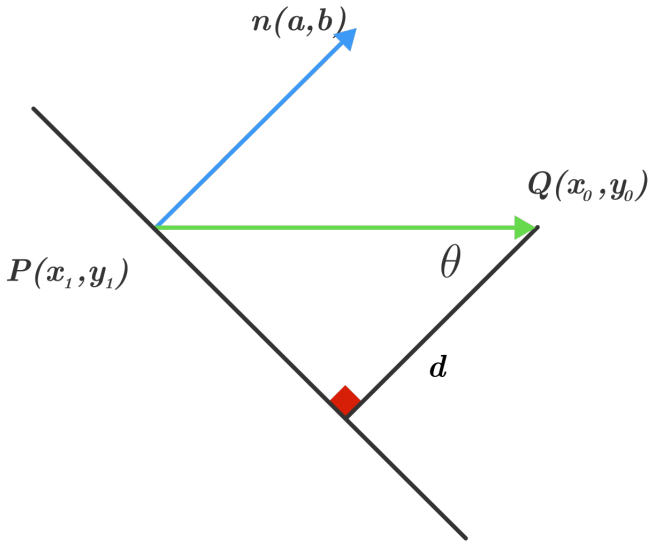
$$d(P,\Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between $P$ and the plane.

**Proof:** use the angle formula in the denominator. For example, to find the distance from $P = (7,1,4)$ to $\sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0,1,1)$ on the plane. Then compute

$$d(P,\Sigma) = \frac{|[-7,0,-3]^T \cdot [2,4,5]^T|}{|[2,4,5]^T|} = \frac{29}{\sqrt{45}}.$$

2. The distance between a point and a line, is defined as the shortest distance between a fixed point and any point on the line. It is the length of the line segment that is perpendicular to the line and passes through the point.




Let $Q$ be the point with coordinates $(x_0, y_0)$ and let the given line have equation $ax + by + c = 0$. Also, let $p = (x_1, y_1)$ be any point on this line and $n$ the vector $(a, b)$ starting at point $p$. The vector $n$ is perpendicular to the line, and the distance $d$ from point $Q$ to the line is equal to the length of the orthogonal projection of $\overrightarrow{PQ}$ on $n$. The

length of this projection is given by: $d = \dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$. Now, $\overrightarrow{PQ} = (x_0 - x_1, y_0 - y_1)$, , so $\overrightarrow{PQ} \cdot \mathbf{n} = a(x_0 - x_1) + b(y_0 - y_1)$ and $\|\mathbf{n}\| = \sqrt{a^2 + b^2}$, thus $d = \dfrac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2 + b^2}}$. Since $P$ is a point on the line, $c = -ax_1 - by_1$ , and so,

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

If $P$ is a point in space and $L$ is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between $P$ and the line $L$.

**Proof:** the area divided by base length is height of parallelogram.
For example, to compute the distance from $P = (2, 3, 1)$ to the line $\vec{r}(t) = (1, 1, 2) + t(5, 0, 1)$, compute

$$d(P, L) = \frac{|[-1, -2, 1]^T \times [5, 0, 1]^T|}{[5, 0, 1]^T} = \frac{|[-2, 6, 10]^T|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}$$

3. If $L$ is the line $\vec{r}(t) = Q + t\vec{u}$ and $M$ is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

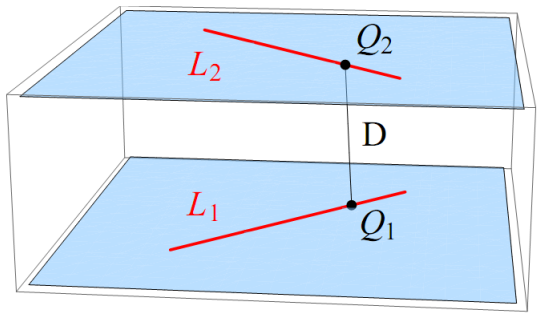
is the distance between the two lines $L$ and $M$.
**Proof:** the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines.

For example, to compute the distance between $\vec{r}(t) = (2, 1, 4) + t(-1, 1, 0)$ and $M$ is the line $\vec{s}(t) = (-1, 0, 2) + t(5, 1, 2)$ form the cross product of $[-1, 1, 0]^T$ and $[5, 1, 2]^T$ is $[2, 2, -6]^T$ . The distance between these two lines is

$$d(L, M) = \frac{|(3, 1, 2) \cdot (2, 2, -6)|}{|[2, 2, -6]^T|} = \frac{4}{\sqrt{44}}.$$

4. To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma, \Pi) = \frac{|e - d|}{|\vec{n}|}$$



Non-parallel planes have distance 0. **Proof:** use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

$$\frac{|8-1|}{|[5,4,3]^T|} = \frac{7}{\sqrt{50}}.$$

**Exercise 1.5**

1. *Given the three points $P = (7, 4, 5)$ and $Q = (1, 3, 9)$ and $R = (4, 2, 10)$. find the parametric and symmetric equation for the line perpendicular to the triangle $PQR$ passing through its center of mass $(P + Q + R)/3$.*

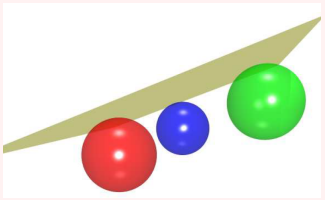
2. *Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + 4t, y = 1 - 4t, z = 8t$ and intersects this line in a point $Q$.*

3. (a) *Find the equation of the plane containing the three points $P = (1, 1, 1), Q = (4, 6, 3), R = (5, 5, 0)$.*
   (b) *Find the area of the triangle $PQR$.*

4. (a) *Compute a suitable volume to determine whether $A = (2, 2, 3), B = (4, 0, 7), C = (6, 3, 1)$ and $D = (2, -3, 11)$ are in the same plane.*
   (b) *Find the distance between the line $L$ through $A, B$ and the line $M$ through $C, D$.*

5. (a) *Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.*
   (b) *Find the distance of the plane found in (a) to the origin $(0, 0, 0)$.*

6. *Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.*



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## 1.6 Vector space and Subspace

**Definition 1.29**

*A vector space (over $\mathbb{R}$ ) consists of a set $V$ along with two operations $+$ and $\cdot$ subject to these conditions.*

1. *For any $\vec{v}, \vec{w} \in V$ , their vector sum $(\vec{v} + \vec{w})$ is an element of $V$ .*
2. *If $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ .*
3. *For any $\vec{u}, \vec{v}, \vec{w} \in V$, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$ .*
4. *There is a zero vector $\boldsymbol{0} \in V$ such that $\vec{v} + \boldsymbol{0} = \vec{v}$ for all $\vec{v} \in V$.*
5. *Each $\vec{v} \in V$ has an additive inverse $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \boldsymbol{0}$.*
6. *If $r$ is a scalar, that is, a member of $\mathbb{R}$ and $\vec{v} \in V$ then the scalar multiple $r \cdot \vec{v}$ is in $V$ .*
7. *If $r, s \in \mathbb{R}$ and $\vec{v} \in V$ then $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.*
8. *If $r \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$, then $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.*
9. *If $r, s \in \mathbb{R}$ and $\vec{v} \in V$, then $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$.*
10. *For any $\vec{v} \in V$, $1 \cdot \vec{v} = \vec{v}$.*

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**Example 1.22**

*Show that the set $\mathbb{R}^2$ is a vector space if the operations " $+$ " and " $\cdot$ " have their usual meaning.*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$
$$\&r \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}$$

***Solution***:*To prove $\mathbb{R}^2$ is a vector space we shall check all of the conditions.*
*There are five conditions in item 1.*
*For 1, closure of addition, note that for any $v_1, v_2, w_1, w_2 \in \mathbb{R}$ the result of the sum*

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

*is a column array with two real entries, and so is in $\mathbb{R}^2$.*
*For 2, that addition of vectors commutes, take all entries to be real numbers and compute*

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} \\ &= \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \end{pmatrix} \\ &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$

*(the second equality follows from the fact that the components of the vectors are real numbers, and the addition of real numbers is commutative).*
*Condition 3, associativity of vector addition, is similar.*

$$\begin{aligned} \{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} (v_1 + w_1) + u_1 \\ (v_2 + w_2) + u_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 + (w_1 + u_1) \\ v_2 + (w_2 + u_2) \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\} \end{aligned}$$

*For the fourth condition we must produce a zero element the vector of zeroes is it.*

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

*For 5, to produce an additive inverse, note that for any $v_1, v_2 \in \mathbb{R}$ we have*

$$\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

*so the first vector is the desired additive inverse of the second.*
*To check the next five conditions having to do with scalar multiplication are just as routine.*
*For 6, closure under scalar multiplication, where*

$$r, v_1, v_2 \in \mathbb{R},$$
$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix}$$

*is a column array with two real entries, and so is in $\mathbb{R}^2$ .*

**Example 1.23**

*Next, this checks 7.*

$$\begin{aligned}(r+s)\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}(r+s)v_1\\(r+s)v_2\end{pmatrix}\\ &= \begin{pmatrix}rv_1+sv_1\\rv_2+sv_2\end{pmatrix}\\ &= r\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}+s\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}\end{aligned}$$

*For 8, that scalar multiplication distributes from the left over vector addition, we have this.*

$$\begin{aligned}r\cdot(\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix}) &= \begin{pmatrix}r(v_1+w_1)\\r(v_2+w_2)\end{pmatrix}\\ &= \begin{pmatrix}rv_1+rw_1\\rv_2+rw_2\end{pmatrix}\\ &= r\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}+r\cdot\begin{pmatrix}w_1\\w_2\end{pmatrix}\end{aligned}$$

*The ninth*

$$\begin{aligned}(rs)\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix} &= \begin{pmatrix}(rs)v_1\\(rs)v_2\end{pmatrix}\\ &= \begin{pmatrix}r(sv_1)\\r(sv_2)\end{pmatrix}\\ &= r\cdot(s\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix})\end{aligned}$$

*and tenth conditions are also straightforward.*

$$1\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}1v_1\\1v_2\end{pmatrix} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$$

*In a similar way, each $\mathbb{R}^n$ where $\boldsymbol{n}\in\mathbb{N}$ is a vector space with the usual operations of vector addition and scalar multiplication. The singleton set $\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\}$ is a vector space under the operations*

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\&r\cdot\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

**Exercise 1.6**

1. *Show that* $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \,|\, a_0, \ldots, a_3 \in \mathbb{R}\}$, *the set of polynomials of degree three or less is a vector space under the operations*

   $$(a_0+a_1x+a_2x^2+a_3x^3)+(b_0+b_1x+b_2x^2+b_3x^3) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+(a_3+b_3)x^3$$

   *and*
   $r \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (ra_0) + (ra_1)x + (ra_2)x^2 + (ra_3)x^3$

2. *The set* $\mathcal{M}_{2\times2}$ *of* $2\times2$ *matrices with real number entries is a vector space under the natural entry-by-entry operations.*
   $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$ *and* $r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$

3. *The set of polynomials with real coefficients* $\{a_0 + a_1x + \cdots + a_nx^n \,|\, n \in \mathbb{N}$ *and* $a_0, \ldots, a_n \in \mathbb{R}\}$
   *makes a vector space when given the natural* " + "

   $$(a_0+a_1x+\cdots+a_nx^n)+(b_0+b_1x+\cdots+b_nx^n) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

   *and*
   $$r \cdot (a_0 + a_1x + \ldots a_nx^n) = (ra_0) + (ra_1)x + \ldots (ra_n)x^n$$

### Subspace

A subspace is a vector space that is contained within another vector space. So every subspace is a vector space in its own right, but it is also defined relative to some other (larger) vector space. We will discover shortly that we are already familiar with a wide variety of subspaces from previous sections.

**Definition 1.30**

*Suppose that* $V$ *and* $W$ *are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that* $W$ *is a subset of* $V$, $W \sqsubseteq V$. *Then* $W$ *is a subspace of* $V$.

**Proposition**:- A subset **S** of a vector space **V** is a subspace of **V** if and only if **S** is nonempty and closed under linear operations, i.e., $x, y \in \mathbf{S} \Rightarrow x + y \in \mathbf{S}$, and $x \in \mathbf{S} \Rightarrow rx \in \mathbf{S}$ for all $r \in \mathbf{R}$.

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**Example 1.24**

*The set* $P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0\}$ *is a subspace of* $\mathbb{R}^3$ *if* " $+$ " *and* " $\cdot$ " *are interpreted as* $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ *and* $r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

***Solution***:- *Since* $\mathbb{R}^3$ *is a vector space and also* $\boldsymbol{P}$ *is a subset of* $\mathbb{R}^3$ *then by the above preposition we prove that* $P$ *is closed under addition and scalar multiplication , take two elements of* $P$ $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ *and* $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ *(membership in* $P$ *means that* $x_1 + y_1 + z_1 = 0$ *and* $x_2 + y_2 + z_2 = 0$*),*

*and observe that their sum* $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ *is also in* $P$ *since its entries add* $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$*. To show that* $P$ *is closed under scalar multiplication, start with a vector from* $P$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *(so that* $x + y + z = 0$ *) and then for* $r \in \mathbb{R}$ *observe that the scalar multiple* $r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$ *satisfies that* $rx + ry + rz = r(x + y + z) = r.0 = 0$*. Thus the two closure conditions are satisfied. There fore* $\boldsymbol{P}$ *is a subspace of* $\mathbb{R}^3$*.*

**Exercise**: Show that a nonempty subset S of a real vector space is a subspace if and only if it is closed under linear combinations of pairs of vectors: whenever $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in S$ then the combination $c_1\vec{v}_1 + c_2\vec{v}_2$ is in S.

## 1.7 Linear dependence, independence and Basis

A set of vectors is linearly independent if no vector in the set **V** is a scalar multiple of another vector or a linear combination of other vectors in the set;
conversely, a set of vectors is linearly dependent if any vector in the set **V** is a scalar multiple of another vector in the set or a linear combination of other vectors in the set.

**Definition 1.31**

*Suppose that* **V** *is a vector space. Given n vectors* $u_1, u_2, u_3, \cdots u_n$ *and* $\boldsymbol{n}$ *scalars* $\alpha_1, \alpha_2, \alpha_3 \cdots, \alpha_n$*, their linear combination is the vector* $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n$*.*

**Definition 1.32**

*If $S = \{v_1, v_2, v_3, \cdots, v_n\}$ is a non-empty set of vectors, then the vector equation $k_1v_1 + k_2v_2 + k_3v_3 + \cdots + k_nv_n = 0$ has at least one trivial solution, namely $k_1 = k_2 = k_3 = \cdots = k_n = 0$. If this solution is the only solution of the set* **S** *then the set* **S** *is called linearly independence set.*
*if there are other solutions then the set* **S** *is called linearly dependent.*

**Example 1.25**

*Let $v_1 = (1, 0, 1)$, $v_2 = (-1, 1, 0)$ and $v_3 = (1, 2, 3)$. Express $v_3$ as a linear combination of $v_1$ and $v_2$.*
***Solution**: We must find scalars $c_1$ and $c_2$ so that $v_3 = c_1v_1 + c_2v_2$ Using our knowledge of scalar multiplication and addition of vectors, we set*

$$\begin{aligned}(1,2,3) &= c_1(1,0,1) + c_2(-1,1,0)\\ (1,2,3) &= (c_1,0,c_1) + (-c_2,c_2,0)\\ (1,2,3) &= (c_1 - c_2, c_2, c_1)\end{aligned}$$

*Equating corresponding components we have :*
$1 = c_1 - c_2,\ 2 = c_2$ & $3 = c_1$
*Since $c_1 = 3$ and $c_2 = 2$ does satisfy $c_1 - c_2 = 1$, we see that this is the solution. Hence $(1, 2, 3) = 3(1, 0, 1) + 2(-1, 1, 0)$. so we have found the required linear combination to be $v_3 = 3v_1 + 2v_2$.*

**Example 1.26**

*Determine whether $\{v_1, v_2, v_3\}$ form a linearly independent set or not, where $v_1 = (1, 1, 2, 1)$, $v_2 = (0, 2, 1, 1)$, and $v_3 = (3, 1, 2, 0)$*
***Solution**: Using the second definition, we wish to know whether $c_1v_1 + c_2v_2 + c_3v_3 = 0$ has only the trivial solution.*
*Suppose that $c_1v_1 + c_2v_2 + c_3v_3 = \mathbf{0}$.*
*Then*

$$\begin{aligned}c_1(1,1,2,1) + c_2(0,2,1,1) + c_3(3,1,2,0) &= \mathbf{0}\\ (c_1,c_1,2c_1,c_1) + (0,2c_2,c_2,c_2) + (3c_3,c_3,2c_3,0) &= \mathbf{0}\\ (c_1 + 3c_3, c_1 + 2c_2 + c_3, 2c_1 + c_2 + 2c + 3, c_1 + c_2) &= \mathbf{0}\end{aligned}$$

$$\begin{aligned}c_1 + 3c_3 &= 0\\ c_1 + 2c_2 + c_3 &= 0\\ 2c_1 + c_2 + 2c + 3 &= 0\\ c_1 + c_2 &= 0\end{aligned}$$

*has the unique solution $c_1 = c_2 = c_3 = 0$.*
*There fore we see that the set $\{v_1, v_2, v_3\}$ is linearly independent.*

**Theorem 1.12**

*Suppose that $S = \{u_1, u_2, u_3, \cdots, u_n\}$ is a set of vectors. Then* **S** *is a linearly dependent set if and only if there is an index* **t** $1 \leqslant t \leqslant n$ *such that $u_t$ is a linear combination of the vectors $u_1, u_2, u_3, \cdots u_{t-1}, u_{t+1}, \cdots, u_n$.*

**Definition 1.33**

*Let* **V** *denote a vector space and* $\boldsymbol{S} = \{u_1, u_2, u_3, \cdots, u_n\}$ *a subset of* **V**. $\boldsymbol{S}$ *is called a basis for V if the following is true:*

1. **S** *spans* **V**.
2. **S** *is linearly independent.*

**Definition 1.34**

*Let* **V** *be a vector space and let* $\boldsymbol{S} = \{v_1, v_2, \cdots, v_n\}$ *be a subset of* **V**. *We say that* **S** *spans* **V** *if every vector v in* **V** *can be written as a linear combination of vectors in* **S**. $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$

**Example 1.27**

*Show that the set* $S = \{(0,1,1), (1,0,1), (1,1,0)\}$ *spans* $R^3$ *and write the vector* $(2,4,8)$ *as a linear combination of vectors in S.*
***Solution***:- *A vector in* $R^3$ *has the form* $v = (x,y,z)$ *Hence we need to show that every such v can be written as*

$$\begin{aligned}(x,y,z) &= c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)\\ &= (c_2 + c_3, c_1 + c_3, c_1 + c_2)\end{aligned}$$

$$\begin{aligned}x &= c_2 + c_3\\ y &= c_1 + c_3\\ z &= c_1 + c_2\end{aligned}$$

$$\begin{aligned}c_3 &= x - c_2,\\ \&c_1 &= z - c_2\end{aligned}$$

$$\begin{aligned}y &= z - c_2 + x - c_2\\ &= z + x - 2c_2\end{aligned}$$

$$\begin{aligned}c_2 &= (x - y + z)/2,\\ c_1 &= (z - x + y)/2\\ c_3 &= (x + y - z)/2\end{aligned}$$

$$\therefore \text{for every } (x,y,z) = \frac{(z-x+y)}{2}(0,1,1) + \frac{(x-y+z)}{2}(1,0,1) + \frac{(x+y-z)}{2}(1,1,0).$$

*Hence* $S = \{(0,1,1), (1,0,1), (1,1,0)\}$ *spans* $R^3$
*And We have* $(2,4,8) = 5(0,1,1) + 3(1,0,1) + (-1)(1,1,0)$

**Example 1.28**

*Prove that* $\boldsymbol{S} = \{1,\ x,\ x^2\}$ *is a basis for the set of polynomials of degree less than or equal to 2.*
***Solution**: We need to prove that* **S** *spans* $P_2$ *and is linearly independent.*
**S** *spans* $P_2$. *We already did this in the section on spanning sets. A typical polynomial of degree less than or equal to* ***2*** *is* $ax^2 + bx + c$

*S is linearly independent. Here, we need to show that the only solution to* $a+bx+cx^2 = 0$ *(where* ***0*** *is the zero polynomial) is* $a = b = c = 0$.
*From algebra, we remember that two polynomials are equal if and only if their corresponding coefficients are equal. The zero polynomial has all its coefficients equal to zero. So,* $a(1) + bx + cx^2 = \boldsymbol{0}$ *if and only if* $a = 0, \quad b = 0, \quad c = 0$. *Which proves that* **S** *is linearly independent.*

**Theorem 1.13**

*Let* **V** *denote a vector space and* $S = \{u_1,\ u_2,\ u_3,\ \cdots, u_n\}$ *a basis of* **V**. *Every vector in* **V** *can be written in a unique way as a linear combination of vectors in* **S**

## Exercise 1.7

1. *A kite is pulled with a force $\vec{F} = \langle 2, 1, 4\rangle$. It has velocity $\vec{v} = \langle 1, -1, 1\rangle$. The dot product of $\vec{F}$ with $\vec{v}$ is called power.*
   (a) *Find the angle between the force and the velocity.*
   (b) *Find the vector projection of the force onto the velocity vector.*
2. *Light shines long the vector $\vec{a} = \langle a_1, a_2, a_3\rangle$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $\vec{a}$.*
   ***Hint**. Reflect first at the xy-plane and find the components of the ray after reflection. You can assume that in that case, the reflected vector is in the plane spanned by $\vec{k} = \langle 0, 0, 1\rangle$ and $\vec{a}$.*
3. *Given the vectors $\vec{a} = \langle 1, 2, 1\rangle$, $\vec{b} = \langle 1, -1, 1\rangle$, $\vec{c} = \langle 0, 1, 1\rangle$, $d = \langle 2, 3, 4\rangle$, compute all possible dot products and determine which pairs are perpendicular.*
4. (a) *Find the angle between a diagonal of a cube and the diagonal in one of its faces.*
   (b) *The hypercube or tesseract has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$*
5. (a) *Verify that if $\vec{a}$. $\vec{b}$ are nonzero, then $\vec{c} =\mid \vec{a} \mid \vec{b} + \mid \vec{b} \mid \vec{a}$ bisects the angle between $\vec{a}$. $\vec{b}$ if $\vec{c}$ is not zero.*
   (b) *Verify the parallelogram law*

$$\mid \vec{a} + \vec{a} \mid^2 + \mid \vec{a} - \vec{a} \mid^2 = 2 \mid \vec{a} \mid^2 + 2 \mid \vec{b} \mid^2 .$$

6. (a) *Find a nonzero vector orthogonal to the plane through the points $P = (-2, 3, 1)$, $Q = (1, 5, 2)$, $R = (4, 3, -1)$ and containing $P$.*
   (b) *Find the equation of this plane.*
   (c) *Find the area of the triangle $PQR$.*
7. (a) *Parametrize a line perpendicular to the plane containing $A = (1, 1, .1)$, $B = (2, 3, 4)$ and $C = (4, 5, 6)$ and passing through $A$.*
   (b) *Find the equation $ax + by + cz = d$ of the plane through $A, B, C$.*
8. (a) *Use volume to determine whether $A = (1, 1, 2)$, $B = (3, 1, 6)$, $C = (5, 2, 0)$ and $D = (1, 4, 12)$ are in the same plane.*
   (b) *Find the distance between the line $L$ through $A, B$ and the line $M$ through $C, D$.*
   (c) *How come that whenever $A, B, C, D$ are not in the same plane, then the distance between $L$ and $M$ is positive?*
9. (a) *Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.*
   (b) *Find the distance of the plane found in a) to the origin $(0, 0, 0)$.*
10. (a) *Parametrize the line $L$ through $P = (2, 1, 2)$ that intersects the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ perpendicularly.*
    (b) *What is the distance from this line $L$ to the origin $(0, 0, 0)$?*

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