

Worksheet on Integration

1. Suppose $f''(x) = x - \frac{1}{x^2}$ such that $f(-1) = 4$ and $f'(2) = 3$, find an expression for $f(x)$.

2. Evaluate the following indefinite integrals

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| a) $\int (x^3 + 2x^2 + \frac{x}{2} + 4)dx$ | h) $\int \frac{\sin(\ln x)}{x}dx$ | o) $\int \frac{1}{4x^2-4x+5}dx$ |
| b) $\int (xe^{(x^2+1)} + x)dx$ | i) $\int \frac{1}{16-9x^2}dx$ | p) $\int \frac{3x+2}{(1+x)^2(x-2)}dx$ |
| c) $\int x^2 \cos(x^3)dx$ | j) $\int \frac{x^3}{\sqrt{4-x^2}}dx$ | q) $\int \frac{x^3+2x^2}{(x-1)(x^2+2x+2)}dx$ |
| d) $\int \frac{x}{\sqrt{x^2+1}}dx$ | k) $\int \frac{x^3}{x^2-2x-3}dx$ | r) $\int \frac{2x-1}{x^4+x^2+1}dx$ |
| e) $\int \sin^3 x \cos x dx$ | l) $\int \sin 2x \cos 3x dx$ | s) $\int \frac{\cosh 2x}{1+\sinh 2x}dx$ |
| f) $\int x^2 e^x dx$ | m) $\int \frac{1}{x(1-\ln x)}dx$ | t) $\int e^x \cosh x dx$ |
| g) $\int \frac{1}{x^2+3x+2}dx$ | n) $\int \frac{\sin 3x}{\cos x}dx$ | u) $\int \frac{\sinh(\frac{1}{x})}{x^2}dx$ |

3. Evaluate the following definite integrals

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|---|--|---|
| a) $\int_0^1 \frac{x}{\sqrt{1+x^4}}dx$ | h) $\int_0^1 \sin^{-1} x dx$ | o) $\int_1^2 (\sqrt{x} + \frac{1}{\sqrt{x}})dx$ |
| b) $\int_0^2 x^2 + 1 dx$ | i) $\int_0^4 \sqrt{16-x^2}dx$ | p) $\int_0^\pi \sin^2 x \cos^3 x dx$ |
| c) $\int_{-\pi/2}^{\pi/2} x^2 \cos x dx$ | j) $\int_{-1}^2 \frac{x}{x^2+2x+10}dx$ | q) $\int_0^1 \sin^{-1} x dx$ |
| d) $\int_2^5 x-3 dx$ | k) $\int_0^2 (x^2+1)dx$ | r) $\int_0^4 \sqrt{16-x^2}dx$ |
| e) $\int_{-1}^1 x(1-x^2)^5 dx$ | l) $\int_{-\pi/2}^{\pi/2} x^2 \cos x dx$ | s) $\int_{-1}^2 \frac{x}{x^2+2x+10}dx$ |
| f) $\int_1^2 (\sqrt{x} + \frac{1}{\sqrt{x}})dx$ | m) $\int_2^5 x-3 dx$ | t) $\int_a^x e^t dt$ |
| g) $\int_0^\pi \sin^2 x \cos^3 x dx$ | n) $\int_{-1}^1 x(1-x^2)^5 dx$ | |

4. Show that $\int_0^\pi \frac{x}{1+\sin x} dx = \pi$

5. Evaluate the following improper integrals whenever converges.

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| a) $\int_3^4 \frac{1}{(x-3)^2}dx$ | d) $\int_0^\infty \frac{1}{e^x+1}dx$ | g) $\int_{\pi/4}^{\pi/2} \sec x dx$ |
| b) $\int_0^2 \frac{x}{\sqrt{4-x^2}}dx$ | e) $\int_0^1 \frac{1}{e^x-e^{-x}}dx$ | h) $\int_{-\infty}^\infty x e^{-x^2} dx$ |
| c) $\int_0^1 \frac{1}{x^3+x}dx$ | f) $\int_2^\infty \frac{1}{x(\ln x)^3}dx$ | i) $\int_{-\infty}^\infty \frac{1}{x^2-6x+10}dx$ |

6. Find the area of the region bounded by each of the given equations

a) $y = x, x = 2$ and $y = 0$

e) $y = x^3, y = x$

b) $y = x^2, x = -2, x = 2$ and $y = 0$

f) $y = x^2 - 4, y = 4 - x^2$

c) $y = \sin x, x = -\pi, x = \pi$ and $y = 0$

g) $y = x(x - 2)^2, y = 0$

d) $y = x^2, y = x$

h) $y^2 = 6x, x^2 = 6y$

7. Find the area of the region between the graphs of

a) $f(x) = x^2 + x$ and $g(x) = 3x^2 - x$

b) $y^2 = x$ and $y = x - 2$

8. Determine the area of the region enclosed by the ellipse $b^2x^2 + a^2y^2 = (ab)^2$.

Worksheet 1

1. Find the domain, range and sketch the domain of

a) $f(x, y) = \frac{1}{x^2 + y^2}$

b) $f(x, y) = \frac{2}{\sqrt{9 - x^2 - y^2}}$

2. Identify the level curves and sketch some of them.

a) $f(x, y) = \frac{1}{x - y^2}$

b) $f(x, y) = \frac{1}{x^2 + y^2}$

c) $f(x, y) = xy$

d) $f(x, y) = |y|$

3. Sketch the graph of f

a) $f(x, y) = x + 2y$

b) $f(x, y) = \sqrt{4 - x^2 - y^2}$

c) $f(x, y) = e^{-x}$

4. Sketch the level surface $f(x, y, z) = c$.

a) $f(x, y, z) = 2x - 4y + z; c = 1$

c) $f(x, y, z) = 4x^2 + 4y^2 + z^2; c = 1$

b) $f(x, y, z) = x^2 + y^2 + z^2; c = 2$

5. The given surfaces intersect in a space curve C. Determine the the projection of C onto the xy -plane.

a) The sphere $x^2 + y^2 + (z - 1)^2 = \frac{7}{4}$ and the paraboloid $x^2 + y^2 - z = 1$.

b) The cone $x^2 + y^2 = z^2$ and the plane $y + z = 2$.

c) The paraboloid $x^2 + y^2 + z = -2$ and the surface $x^2 - 3y^2 = z$.

6. Check whether or not the limit exists. If it exists, evaluate it.

a) $\lim_{(x,y) \rightarrow (-2,1)} \frac{x^3 + 2x^2y - xy - 2y^2}{x + 2y}$

c) $\lim_{(x,y) \rightarrow (3,1)} \frac{x^2 - xy - 6y^2}{\sqrt{x - y} - \sqrt{2y}}$

b) $\lim_{(x,y) \rightarrow (0,2)} \frac{(y - 2) \cos x - y + 2}{x^2y^2 - 4x^2}$

d) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$

7. Using the definition show that

a) $\lim_{(x,y) \rightarrow (0,0)} \frac{x(x-2y)}{\sqrt{x^2+y^2}} = 0$

b) $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{y \sin(x^2+y^2+z^2)}{x^2+y^2+z^2} = 0$

8. Prove that f is not continuous at $(0, 0)$

a) $f(x, y) = \begin{cases} \frac{\sin xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$

b) $f(x, y) = \begin{cases} \frac{x^2 y^3}{x^{12}+y^4} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$

9. Find the first and second partial derivatives of f .

a) $f(x, y) = \ln\left(\frac{x}{x-y}\right)$

b) $f(x, y) = \int_y^x e^{-t^2} dt$

10. A function is said to be harmonic if it satisfies Laplace's equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Show that the following functions are harmonic:

a) $f(x, y) = x^3 - 3xy^2$

b) $f(x, y) = e^x \sin y$

c) $f(x, y) = \ln \sqrt{x^2 + y^2}$

11. Verify $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ given that

a) $f(x, y) = g(x) + h(y)$, where g and h are differentiable functions.

b) $f(x, y) = g(x)h(y)$, where g and h are differentiable functions.

12. Let g be a twice differentiable function of one variable. If $h(x, y) = g(x+y) - g(x-y)$, then prove that $\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial y^2}$.

13. a) Find $\frac{dz}{dt}$ where $z = xy^2 + x^2y$, $x = at^2$ and $y = 2at$, a a constant.

b) If $z = f(x, y)$ with $x = e^u + e^{-v}$; $y = e^{-u} + e^v$. Then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.

14. Find the equation of the tangent plane to each of the following at the indicated point.

a) $f(x, y) = 3x^2 + 4y^2$; at $P(-2, 1, 16)$

b) $e^{xy} \cos(zy) = 1$; at $(1, 0, 1)$.

15. Find the directional derivative of $f(x, y, z) = x^3 - \tan^{-1}(y + 2z)$ at $(1, 3, -1)$ in the direction of $v = 2i + j - k$.

16. Find the directions in which f increases and decreases most rapidly at the given point. Find the rate of changes in these directions.

a) $f(x, y) = \arctan(x - y)$; $P(2, -2)$.

b) $f(x, y) = 2x^2 + 3xy + 4y^2$; $P(1, 2)$.

17. The temperature at each point of a metal plate is given by the function

$T(x, y) = e^x \cos y + e^y \cos x$. In what direction does the temperature increase most rapidly at point $(0, 0)$? What is this maximum rate?

18. Find the local extreme points and values of

- a) $z = x^2 + xy + y^2 - 2x - y$
 b) $f(x, y) = (y - 2) \ln(xy)$

19. Find extreme values in the given set D.

- a) $f(x, y) = 2x^2 + y^2 + 2x$; $D = \{(x, y) | x^2 + y^2 \leq 1\}$.
 b) $f(x, y) = e^{x^2 - y^2}$; D is the region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.
 c) $f(x, y) = x^2 + y^2 xy$; D is the triangular region with vertices at (0,0), (2,0) and (0,2).

20. Find the maximum and minimum values of f subject to the given constraint

- a) $f(x, y) = 4x^2 + 9y^2$, subject to $x^2 + y^2 = 1$.
 b) $f(x, y) = xy$, subject to $4x^2 + 9y^2 = 36$.
 c) $f(x, y, z) = x^2 + y^2 + z^2$; subject to $3x + 2y + z = 6$.
 d) $f(x, y, z) = xyz$ subject to $x^2 + y^2 + z^2 = 1$.

21. Use Lagrange Multipliers to

- a) find the points on the circle $x^2 + y^2 = 45$ that are closest to and furthest from (1, 2).
 b) find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to or farthest from the (4, 2, 1).
 c) find the points on the sphere $xy - z^2 = 1$ that are closest to the origin.

22. The sum of three positive numbers is 120. What is the maximum possible value of their product?

Worksheet 2

1. Evaluate the following integrals

- a) $\int_0^1 \int_0^x \cos(x^2) dy dx$ b) $\int_0^1 \int_0^y x \sqrt{x^2 + y^2} dx dy$ c) $\int_0^{\ln 3} \int_0^1 \int_0^y (z^3 - y) e^{y^2} dx dz dy$
 d) $\iint_R (x^2 + 3y^2) dA$ where $R = \{(x, y) | x^2 + y^2 \leq 1\}$.
 e) $\iiint_D y e^{-xy} dV$, where D is the cube bounded by the planes $x = 1, x = 3, y = 0, y = 2, z = -2$ and $z = 0$.
 f) $\iiint_D x(3 - y) dV$, where D is the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes.

2. By changing the order of integration, when necessary, evaluate

- a) $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(5x^2) dx dy$ c) $\int_1^4 \int_{\sqrt{x}}^2 \cos(y^3 - 3y) dy dx$
 b) $\int_0^1 \int_x^1 x^2 e^{y^4} dy dx$ d) $\int_1^3 \int_0^{\ln y} e^{x+y} dx dy$

3. Using polar or cylindrical coordinates, evaluate

- a) $\int_{-2}^2 \int_2^{\sqrt{8-x^2}} \frac{1}{\sqrt{x^2+y^2}} dy dx$ b) $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{x^2+y^2+1} dx dy$ c) $\int_0^2 \int_0^{\sqrt{2y-y^2}} \int_0^1 z \sqrt{x^2+y^2} dz dx dy$

4. Using double integrals, find the area of the region R bounded by

- a) the curves $y = x^2$ and $y = \frac{2}{1+x^2}$ b) the cardioid $r = a(1 + \sin \theta)$, $a > 0$

5. Find the surface area of

- a) the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 3$.
 b) the portion of the sphere $x^2 + y^2 + z^2 = 5$ that is inside the paraboloid $x^2 + y^2 = 4z$.

6. Find the volume of the solid D that is

- a) in the first octant bounded by the paraboloid $z = x^2 + y^2$, the plane $x + y = 1$ and the coordinate planes.
 b) inside the cylinder $x^2 + y^2 = 4x$ bounded above by the upper nappe of the cone $z^2 = x^2 + y^2$ and below by the xy-plane.
 c) inside the sphere $x^2 + y^2 + z^2 = 4$ outside the cylinder $x^2 + y^2 = 1$.
 d) bounded above by $x^2 + y^2 + z^2 = 25$ and below by $z = \sqrt{x^2 + y^2}$.

7. a) Find the volume of the solid inside the cylinder $x^2 + z^2 = 1$ bounded on both sides by the sphere $x^2 + y^2 + z^2 = 4$.
 b) Write only the iterated integrals to find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $2x + y + z = 5$.

8. Show that a) $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy dx = \frac{1}{2}$ b) $\int_{-\infty}^{\infty} \int_0^{\infty} e^{-x^2-y^2} dx dy = \frac{\pi}{2}$

9. Evaluate $\iiint_D x e^{x^2+y^2+z^2} dV$ where $D = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$.

10. Find the centroid of the solid region of constant mass density bounded below by the paraboloid $z = 2x^2 + 2y^2$ and above by the plane $z = 8$.

11. Evaluate using a suitable change of variables

- a) $\iint_R \left(\frac{x-y}{x+y} \right)^4 dA$ where R is the region bounded by the graphs of $x + y = 1$, $y = 0$ and $x = 0$.

b) $\iint_R e^{\left(\frac{2x-y}{x+y}\right)} dA$ where R is the region bounded by the graphs of $x = 2y$, $y = 2x$, $x + y = 1$ and $x + y = 2$.

c) $\iint_R x^3 y dA$ where R is bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $x, y \geq 0$.
(Hint: use $x = au \cos v$, $y = bu \sin v$)

d) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^a z \sqrt{x^2 + y^2} dz dy dx$ (use cylindrical coordinates).

e) $\iiint_D y dV$, where D is the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ (Hint: Let $x = 3u$, $y = 2v$, $z = w$).

12. Find the area of the region bounded by the curves $xy = 1$, $xy = 2$, $y = x$ and $y = 4x$ in the first quadrant. (Hint: Let $x = \frac{u}{v}$, $y = v$)