

Addis Ababa University
Department of Mathematics
Applied mathematics II Math 2041

WORKSHEET III (on sequences)

Academic year 2021/2022

1. Using the definition verify

a) $\lim_{n \rightarrow \infty} \frac{2n-1}{n+1} = 2$ b) $\lim_{n \rightarrow \infty} \frac{3n^2}{5n^2+1} = \frac{3}{5}$ c) $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$

2. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $|a_n| \leq |b_n| \forall n \geq 1$. Show that if $\{b_n\}$ converges to 0, then $\{a_n\}$ converges to 0.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences. If $\{a_n\}$ converges to 0 and $\{b_n\}$ is bounded, show that $\{a_n b_n\}$ converges to 0.

4. Find $\lim_{n \rightarrow \infty} a_n$ if it exists

a) $a_n = \left(1 + \frac{3}{n}\right)^n$ b) $a_n = \sqrt[3]{n+2} - \sqrt[3]{n+1}$ c) $a_n = \left(\frac{n^2-1}{n}\right)^n$
d) $a_n = \frac{n}{n+1} \sin \frac{n\pi}{2}$ e) $a_n = \tanh n$ f) $a_n = \frac{n^2(n!)}{(n+2)!}$

5. Determine whether the given sequence is monotonic or not, and convergent or not

a) $\left\{\frac{3^n}{2+3^n}\right\}$ b) $\left\{\frac{n!}{n^n}\right\}$ c) $\{\cos n\pi\}$ d) $\left\{\ln \left(\frac{3n}{n+1}\right)\right\}$ e) $\left\{\sqrt[n]{3^n+4^n}\right\}$

6. Determine the convergence or divergence of the sequence $\{a_n\}$ if

a) $a_n = \frac{\ln n}{n+1}$ b) $a_n = \left(1 + \frac{1}{n^2}\right)^{3n}$ c) $a_n = \frac{1+3+5+\dots+(2n-1)}{n^2}$ d) $a_n = \sum_{k=0}^n 2^{-k}$

7. a) prove that $\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ \infty & \text{if } |r| > 1 \end{cases}$

b) using (a) find

i) $\lim_{n \rightarrow \infty} \frac{1+3^n}{2^n}$ ii) $\lim_{n \rightarrow \infty} \frac{1+(-1)^n}{e^{2n}}$

c) $\lim_{n \rightarrow \infty} \sqrt[n]{4^n+5^n}$

8. Let $\{a_n\}$ be sequence with $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$, $n \geq 1$

a) Show that $\{a_n\}$ is convergent

b) Find $\lim_{n \rightarrow \infty} a_n$

9. Let i) $a_n = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)}$. Show that $\lim_{n \rightarrow \infty} a_n = 1$

ii) $b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$. Show that $\lim_{n \rightarrow \infty} a_n = 1$

10. Find two limit points and subsequences which converge to each of these points for

a) $\left\{(-1)^n \left(2 - \frac{1}{n}\right)\right\}$ b) $\left\{\sin \frac{n\pi}{4}\right\}$

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Work sheet III (on Series)

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- 1) Determine the convergence or divergence of each of the following series. If it converges, find the sum.

a) $\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n-1}$

b) $\sum_{n=0}^{\infty} \left(-\frac{\pi}{e}\right)^n$

c) $\sum_{n=2}^{\infty} \frac{3^{2n}}{2^{3n-1}}$

d) $\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

e) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

f) $\sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{\sqrt{n}}\right)$

g) $\sum_{n=2}^{\infty} 3^{-2n}$

h) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2^{(n-1)}}}$

j) $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^5+4}}$

k) $\sum_{n=1}^{\infty} \frac{5^n - 2^n}{3^n + 4^n}$

- 2) A ball is dropped from a height of 3 meters above a smooth surface, it bounces $\frac{2}{3}$ of its previous height. Find the total length the ball moves until it gets to rest.

- 3) Show that $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$ diverges and $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$ converges.

- 4) Test the convergence or divergence of

i) Integral test

a) $\sum_{k=1}^{\infty} \frac{\arctan k}{k^2 + 1}$

b) $\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}$

c) $\sum_{n=3}^{\infty} \frac{1}{(2n+1)^3}$

d) $\sum_{n=3}^{\infty} \frac{1}{n \ln(n+1)}$

ii) Comparison test

a) $\sum_{n=3}^{\infty} \frac{\sqrt{n}-1}{10+n}$

b) $\sum_{n=3}^{\infty} \frac{3n^2+2}{4n^2+5n-1}$

c) $\sum_{n=3}^{\infty} \frac{\cos^2 n}{\sqrt{n^2+3}}$

iii) ratio test or root test

a) $\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$

b) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2-1}\right)^n$

c) $\sum_{n=1}^{\infty} \left(\frac{n-1}{4n+1}\right)^n$

d) $\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n-1}\right)^{2n-1}$

e) $\sum_{n=1}^{\infty} \frac{n^4}{3^n}$

f) $\sum_{n=1}^{\infty} \frac{n!}{1.3.5...(2n+1)}$

g) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

- 5) Find all values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

- 6) Suppose $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are both convergent, show that

i) $\sum_{k=0}^{\infty} a_k^2$ is convergent

ii) $\sum_{k=0}^{\infty} a_k b_k$ is convergent.

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7) Test the convergence or divergence using alternating series test

a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5n-2}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$ c) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n+1} \right)^n$

8. Test for absolute convergence, conditional convergence or divergence for each of the following.

a) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5n+1}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n+10)}{n(n^2+1)}$

d) $\frac{1}{3} - \frac{1}{2} + \frac{5}{9} - \dots + \frac{(-1)^{n+1} (2n-1)}{3n} + \dots$ e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+1)}}$ f) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

g) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ h) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5^n}$ i) $\sum_{n=1}^{\infty} (-1)^n \ln \left(\frac{n}{n+1} \right)$

9 Determine the convergence or divergence of

a) $\sum_{n=1}^{\infty} \frac{(n+2)!}{n! 10^n}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1.3.5 \dots (2n-1)}$ c) $\sum_{k=1}^{\infty} \frac{3.5.7 \dots (2k+1)}{2.5.8 \dots (3k-1)}$

d) $\sum_{k=1}^{\infty} \left(\frac{k!}{k^k} \right)^k$ e) $\sum_{n=1}^{\infty} n^t e^{-n}, t \in \mathbb{R}$ f) $\sum_{n=1}^{\infty} \sum_{k=1}^n 2^{-k}$

10 Show that a) $\lim_{n \rightarrow \infty} \frac{(n^n) n!}{(2n)!} = 0$ b) $\lim_{n \rightarrow \infty} n^t e^{-n} = 0, t \in \mathbb{R}.$

11. Let $a_n = \begin{cases} \frac{5}{n^2} & \text{if } n \text{ is even} \\ \frac{2}{n^2} & \text{if } n \text{ is odd} \end{cases}$ Verify that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ does not exist, but the series $\sum_{n=1}^{\infty} a_n$

converges.

12. Find the value(s) of x each of the following series converge absolutely, conditionally, or diverge.

a) $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$ b) $\sum_{n=1}^{\infty} (nx^n)^2$