Vector & Transformation

1.Revision

2. Vector & its Applications

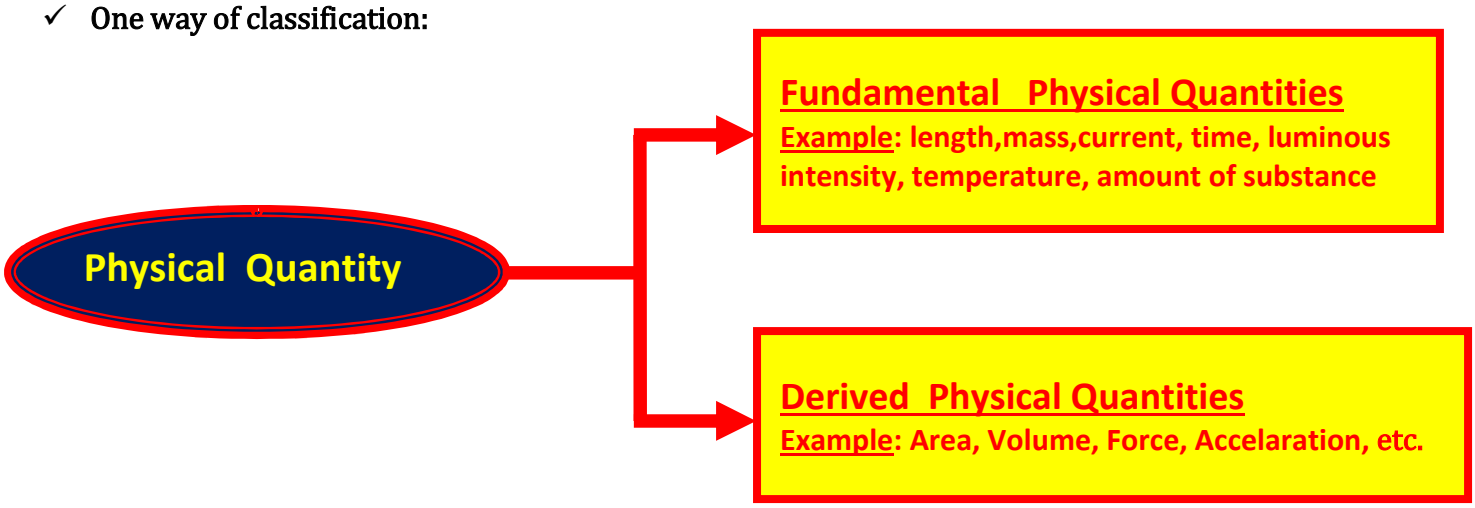
3.Transformation of the Plane



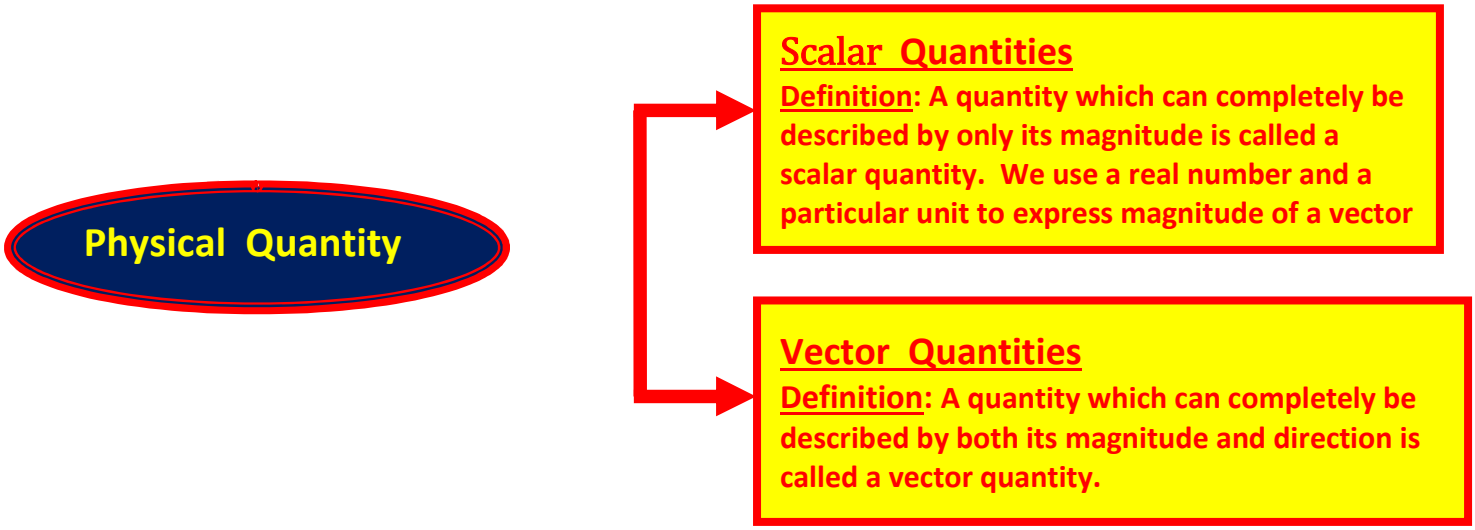
Prepared and Compiled by Bisrat Teshome

I. Revision

- Physical Quantities: can be classified in different ways
  - ✓ One way of classification:



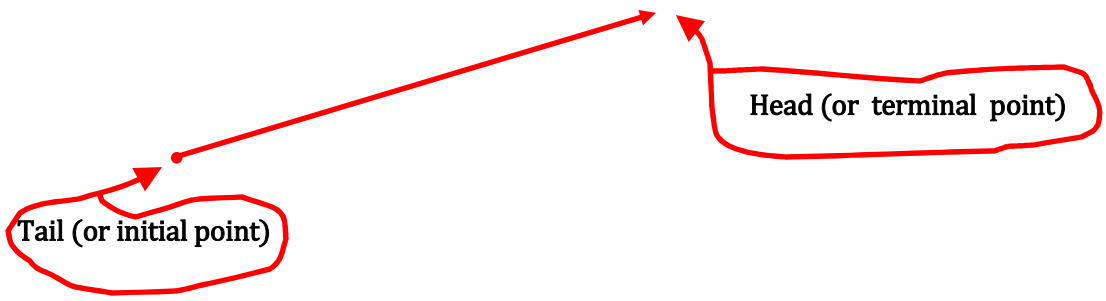
  - ✓ Another way of classification:



Examples of 1. Scalar quantities: time, mass, speed, density, area ,
2. Vector quantities: Force, displacement, velocity, acceleration

- Diagrammatic representation of vectors

Vectors are represented diagrammatically by a directed line segment as shown in the following figure.



➢ Naming Vectors: Vectors are named in different ways.

Method I: We can name vectors using a small letter with an arrow written on its head.

Example 1: symbols such as $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$, etc.



Method II: We can also name a vector using is tail and head.



Name of the vector is $\overrightarrow{PQ}$ and is read as "vector PQ"

➢ Magnitude (or Norm) of a vector

The magnitude of the vector $\overrightarrow{PQ}$ is denoted by $|\overrightarrow{PQ}|$ *or* $\|\overrightarrow{PQ}\|$. Similarly the magnitude of the vector $\vec{v}$ is denoted by $|\vec{v}|$ *or* $\|\vec{v}\|$

Example 2:

1. 

magnitude: $|\vec{u}| = 15cm$

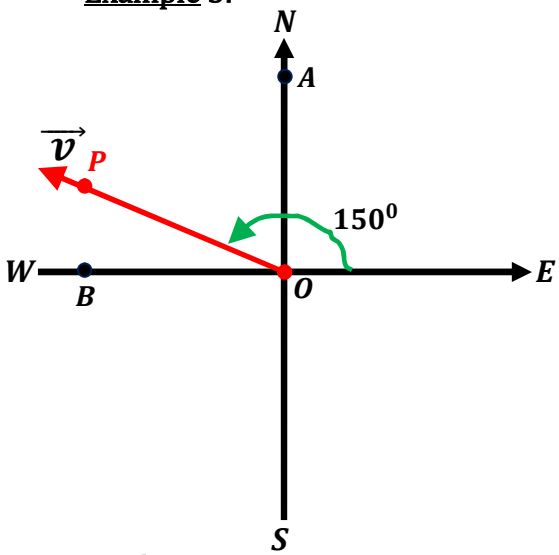
2. 

magnitude: $|\overrightarrow{AB}| = 20\ N$

➢ Direction of a vector

Direction of a vector can also be given in many different ways.

Example 3:



The direction of $\overrightarrow{v}$ can be given in any one of the following ways.

(i) direction of $\vec{v} = 150^0$ (the angle must be measured from the positive $x-axis$.)

(ii) $m(A\hat{O}P) = 150^0 - 90^0 = 60^0$. Therefore,

direction$(\overrightarrow{v})$ = $\underline{N60^0W}$ = $\underline{60^0\ West\ of\ North}$

(iii) $m(B\hat{O}P) = 180^0 - 150^0 = 30^0$. Therefore,

Direction of$(\overrightarrow{v})$ = $\underline{W30^0N}$ = $\underline{30^0\ North\ of\ West.}$

Example 4:

1) The direction of a vector $\overrightarrow{u}$ is -20^0. Which of the following directions is not the same as direction of $\overrightarrow{u}$?

(A) $E20^0S$ (C) $S70^0E$

(B) 70^0 East of South (D) 20^0 East of South

2) $\overrightarrow{F}$ is a force whose direction is 40^0 north of west. Which of the following is not the same as the given direction.

(A) 50^0 west of north (B) -140^0 (C) 140^0 (D) -220^0

Question numbers 3 up to 5 refer to the following information

A traffic light hangs from a cable tied to two other cables fastened to a support, as shown in Figure-A. The upper cables make angles of 37^0 *and* 53^0 with the horizontal.
Figure-B is free-body diagram of the traffic light. Figure-C is free-body diagram for the knot where the three cables are joined

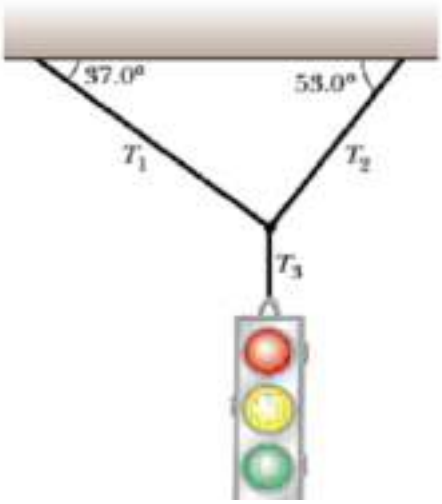


Figure – A



Figure – B

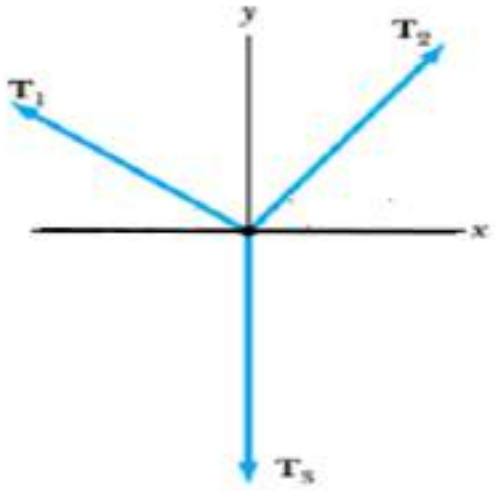


Figure – C

3. In figure C, which of the following cannot be the direction of the tension $\overrightarrow{T_2}$?
   (A) 53^0 (B) 37^0 east of north (C) $E37^0N$ (D) 53^0 north of east

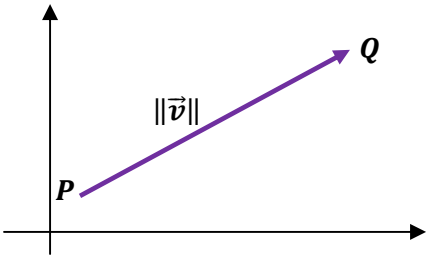
4. In figure C, which of the following cannot be the direction of the tension $\overrightarrow{T_3}$?
   (A) 37^0 (B) 143^0 (C) $W37^0N$ (D) 53^0 west of north

5. Which of the following cannot be the direction of the weight $\overrightarrow{F_g}$ of the traffic light ?
   (A) -90^0 (B) 270^0 (C) South (D) 90^0

Magnitude of a vector

Notation: The magnitude of a vector $\vec{v}$ is denoted by $|\vec{v}|$ or $\|\vec{v}\|$.
If the tail and head of a vector $\vec{v}$ are at points $P(x_1, y_1)$ and $Q(x_2, y_2)$ then its magnitude $\|\vec{v}\|$ is given by:

$$\|\vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 5:

Find the magnitude of vector $\overrightarrow{PQ}$

1. $P = (-1, 3)$ *and* $Q = (3, 4)$

2. $p = (2, -5)$ *and* $Q = (-1, -6)$

Solution

1. $P = (-1, 3)$ *and* $Q = (3, 4)$
   (x_1, y_1) (x_2, y_2)

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3 + 1)^2 + (4 - 3)^2} = \sqrt{17}$$

2. $P = (2,-5)$ *and* $Q = (-1,-6)$

(x_1, y_1) (x_2, y_2)

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1-2)^2 + (-6+5)^2} = \sqrt{10}$$

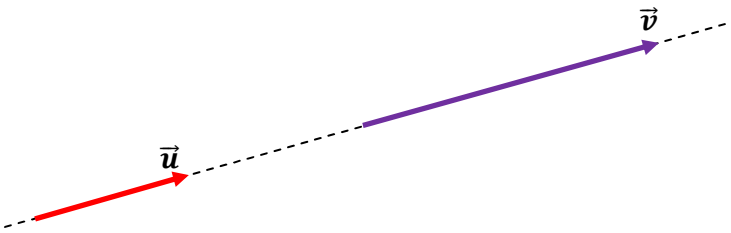
Parallel Vectors

Definition

Two vectors are parallel if and only if they lie on the same line or on two distinct parallel lines.

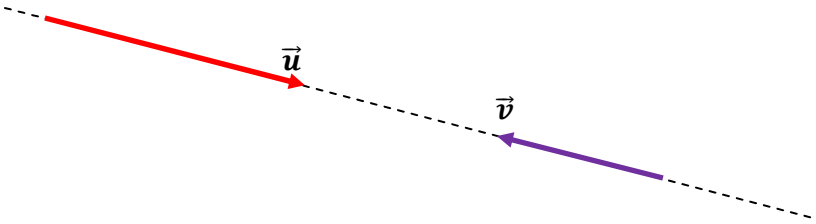
Example 6:

1.



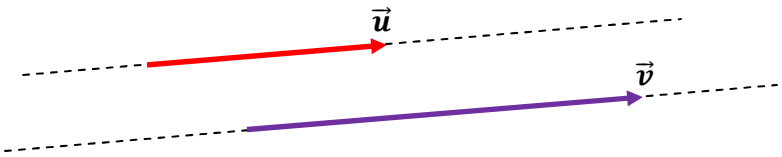
$\vec{u}$ and $\vec{v}$ lie on the same line
$\therefore \vec{u} \parallel \vec{v}$
Note also that the two vectors have same direction

2.



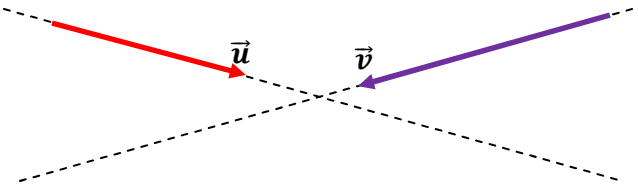
$\vec{u}$ and $\vec{v}$ lie on the same line
$\therefore \vec{u} \parallel \vec{v}$
Note also that the two vectors have opposite directions.

3.



$\vec{u}$ and $\vec{v}$ lie on two distinct parallel lines
$\therefore \vec{u} \parallel \vec{v}$
Note also that the two vectors have same direction.

4.



$\vec{u}$ and $\vec{v}$ lie neither on the same line nor on parallel lines.
$\therefore \vec{u}$ & $\vec{v}$ are not parallel vectors
These two vectors have neither same nor opposite direction.

Note

As you can see above, parallel vectors have either same direction or opposite directions.

Example 6:

Decide analytically whether the vectors $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{RT}$ are parallel vectors or not

1. $P = (-3, 4)$, $Q = (1, 0)$, $R = (2, -5)$ *and* $T = (3, -6)$
2. $P = (2, -5)$, $Q = (4, 4)$, $R = (3, 3)$, *and* $T = (-1, 1)$
3. $P = (2, 1)$, $Q = (5, 7)$, $R = (-1, -5)$ *and* $T = (3, 3)$

Solution

Let l_1 and l_2 be the lines that contain vectors $\vec{u}$ and $\vec{v}$ respectively. Let the respective slopes of the lines be m_1 and m_2. Note also that $l_1 = \overleftrightarrow{PQ}$ *and* $l_2 = \overleftrightarrow{RT}$

1. $m_1 = \frac{0-4}{1-(-3)} = \frac{-4}{4} = -1$

   $m_2 = \frac{-6-(-5)}{3-2} = \frac{-1}{1} = -1$

   $\therefore\ m_1 = m_2$

   Hence $l_1 \parallel l_2$ or l_1 & l_2 represent same line

   $\therefore \vec{u} \parallel \vec{v}$

2. $m_1 = \frac{4-(-5)}{4-2} = \frac{9}{2}$

   $m_2 = \frac{1-3}{-1-3} = \frac{-2}{-4} = \frac{1}{2}$

   $\therefore\ m_1 \neq m_2$

   Hence l_1 & l_2 are two distinct non − parallel lines

   $\therefore \vec{u} \nparallel \vec{v}$

3. $m_1 = \frac{7-1}{5-2} = \frac{6}{3} = 2$

   $m_2 = \frac{3-(-5)}{3-(-1)} = \frac{8}{4} = 2$

   $\therefore\ m_1 = m_2$

   Hence $l_1 \parallel l_2$ or l_1 & l_2 represent same line

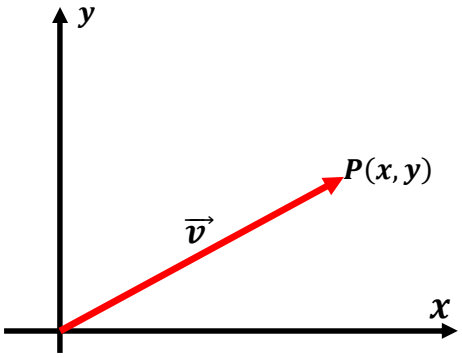
   $\therefore \vec{u} \parallel \vec{v}$

Coordinate Form of Vectors in Plane

Definition

Let $\vec{v}$ be a vector whose tail (i.e. initial point) and head (i.e., terminal point) are at the origin and at $P(x, y)$. respectively. Then, the coordinate form of $\vec{v}$ is given by:

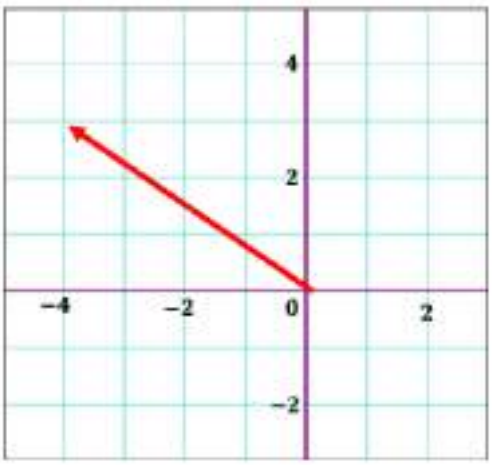
$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \ (or\ (x, y)\ or\ \langle x, y \rangle)$$



Example – 8

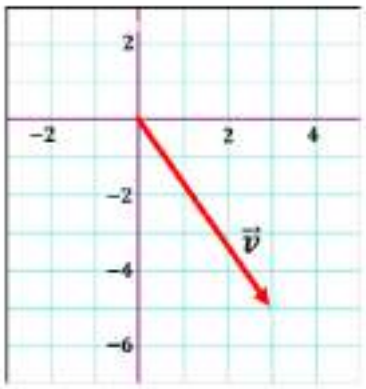
DIAGRAMS | **Coordinate Forms**

1.



$$\vec{v} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} = (-4, 3) = \langle -4, 3 \rangle$$

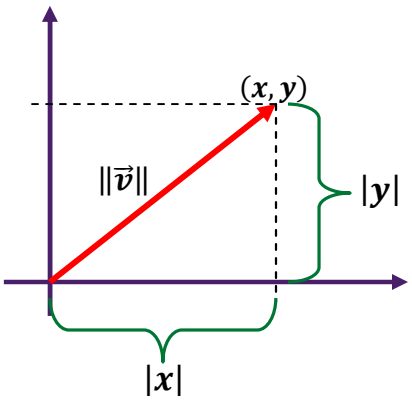
2.



$\vec{v} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} = (3, -5) = \langle 3, -5 \rangle$

Magnitude of Vectors in Coordinate Form

Consider a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$



By Pythagoras theorem

$$\|\vec{v}\|^2 = |x|^2 + |y|^2$$
$$\|\vec{v}\|^2 = x^2 + y^2$$
$$\therefore \|\vec{v}\| = \sqrt{x^2 + y^2}$$

Example – 9

1. Let $\vec{v} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$. Find $\|\vec{v}\|$. 2. Let $\vec{v}$ be a vector . If $\|\vec{v}\| = 0$, then what can you conclude about vector $\vec{v}$?

Solution

1. $\|\vec{v}\| = \sqrt{12^2 + (-5)^2}$
$= \sqrt{169} = 13$

2. Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$
Given : $\|\vec{v}\| = 0$
$\Rightarrow \sqrt{x^2 + y^2} = 0$
$\Rightarrow x^2 + y^2 = 0$

The abo e is True only if both x & y are zeros.
That is, $x = 0$ & $y = 0$
$\Rightarrow \vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ =zero vector

NOTE

The magnitude of a vector equals zero if and only if the vector is the zero vector

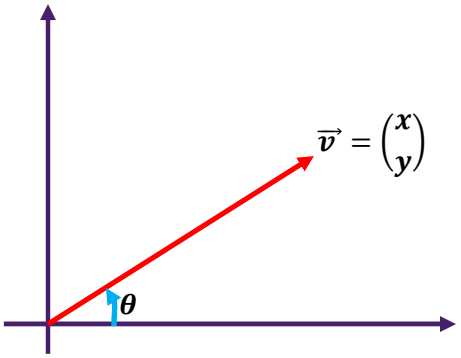
Direction of Vectors in Coordinate Form

The only vector which has no direction is the zero vector.

Vector of the form $\vec{v} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, $where\ y \neq 0$ has direction North if $y > 0$. Its direction is South if $y < 0$

Vector of the form $\vec{v} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $where\ x \neq 0$ has direction due East if $x > 0$. Its direction is due West if $x < 0/$

Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, $where\ x \neq 0\ \&\ y \neq 0$



Direction of $\vec{v} = \theta$

$$\Rightarrow \tan\theta = \frac{y}{x}$$

$If\ x > 0 \wedge y > 0, then\ \theta \in Quadrant\ I$

$If\ x < 0 \wedge y > 0, then\ \theta \in Quadrant\ II$

$If\ x < 0 \wedge y < 0, then\ \theta \in Quadrant\ III$

$If\ x > 0 \wedge y < 0, then\ \theta \in Quadrant\ IV$

Example – 10 : Find the direction of each of the following vectors

1. $\vec{v} = \begin{pmatrix} -2 \\ \sqrt{12} \end{pmatrix}$ 2. $\vec{v} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

Solutions

1. $\vec{v} = \begin{pmatrix} -2 \\ \sqrt{12} \end{pmatrix}$

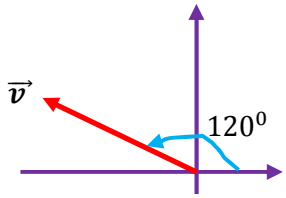
$\Rightarrow x = -2 < 0$

$y = \sqrt{12} = 2\sqrt{3} > 0$

$\therefore\ \theta\ is\ in\ Quadrant\ II$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

$Hence\ \theta = 120^0$



2. $\vec{v} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

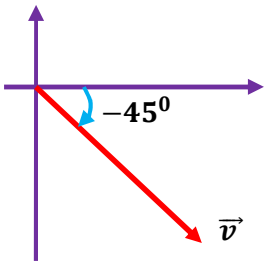
$\Rightarrow x = 4 > 0$

$y = -4 < 0$

$\therefore\ \theta\ is\ in\ Quadrant\ IV$

$\tan\theta = \frac{y}{x} = \frac{-4}{4} = -1$

$Hence\ \theta = -45^0$



Equality of Vectors

Definition

Two non-zero vectors are equal if and only if they have same direction and equal magnitude.

Example – 11

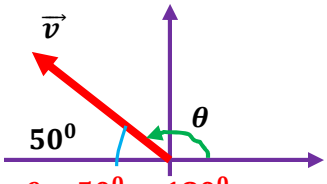
1. Determine whether the pair of vectors (i.e., $\vec{u}\ and\ \vec{v}$) given in each question letter are equal or not

Qn. Letter	Vectors	Magnitude	Direction
A	$\vec{u}$	$12N$	130^0
	$\vec{v}$	$12N$	$W50^0N$
B	$\vec{u}$	$124cm$	-55^0
	$\vec{v}$	$120cm$	-55^0
C	$\vec{u}$	$3\ m/s^2$	-140^0
	$\vec{v}$	$3\ m/s^2$	$S40^0W$
D	$\vec{u}$	$50m$	220^0
	$\vec{v}$	$50m$	$W40^0S$

2. Let $x_2 > x_1 > 0$ *and* $y_2 > y_1 > 0$. Consider the vector $\overrightarrow{PQ}$ with $P = (x_1, y_1)$ *and* $Q = (x_2, y_2)$. Let $\vec{v}$ be another vector whose initial point is at the origin and terminal point is at $(x_2 - x_1, y_2 - y_1)$. Show that $\overrightarrow{PQ}$ and $\vec{v}$ are equal vectors.

Solutions

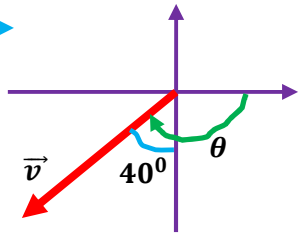
1. (A) Magnitudes
$\|\vec{v}\| = 12N = \|\vec{u}\|$
Directions
Direction of $\vec{u} = 130^0$
Direction of $\vec{v} = W50^0N$



$\theta + 50^0 = 180^0$
$\Rightarrow \theta = 130^0$
$\therefore \vec{u}$ & $\vec{v}$ have same direction
Hence, $\vec{u} = \vec{v}$

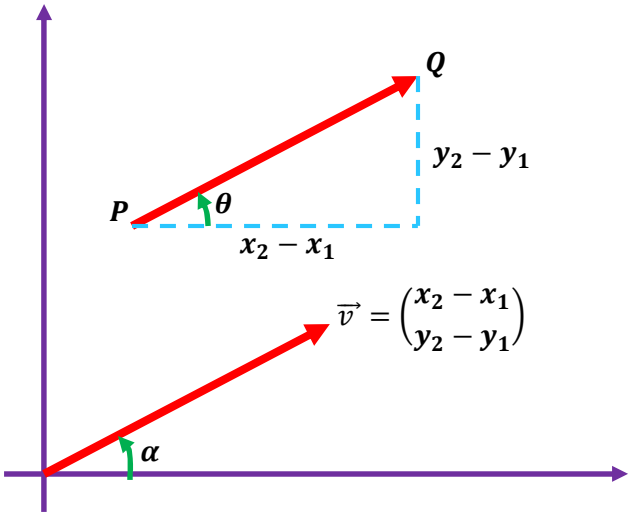
(B) Magnitudes
$\|\vec{u}\| = 124cm$
$\|\vec{v}\| = 120cm$
$\therefore \|\vec{u}\| \neq \|\vec{v}\|$
Hence, $\vec{u} \neq \vec{v}$

(C) Magnitudes
$\|\vec{u}\| = \|\vec{v}\| = 3\ m/s^2$
Directions
Direction of $\vec{u} = -130^0$



Direction of $\vec{v} = S50^0W$
$= \theta$
$= -90^0 + -40^0$
$= -130^0$
$=$ direction of $\vec{u}$
Hence, $\vec{u} = \vec{v}$

2.



The given conditions in the problem imply that $x_2 - x_1$ & $y_2 - y_1$ are positive and also α & θ are acute angles.
$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha \Rightarrow \theta = \alpha$ which means,
$\vec{v}$ & $\overrightarrow{PQ}$ have same direction
$\|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \|\vec{v}\|$
Which means the two vectors have equal magnitude
$\overrightarrow{PQ} = \vec{v} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

You will get the same result as we get in number 2, considering x_1, x_2, y_1 & y_2 satisfying conditions other than the ones given in the example. For instance try to see when $(i) x_1 < x_2 < 0$ & $y_1 > y_2 > 0$ *or* $(ii) x_1 > 0 > x_2$ & $y_1 < 0 < y_2$ and so on.

NOTE

The coordinate form of any vector $\overrightarrow{PQ}$ in plane with $P = (x_1, y_1)$ & $Q = (x_2, y_2)$ is given by:

$$\overrightarrow{PQ} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Equality of Vectors in Coordinate Form

Definition

For two non-zero vectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ & $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{v} = \vec{u}$ if and only if $(x = a) \wedge (y = b)$

Exercise:

Try to show analytically that the above definition is the same as the definition given on page 7.

Example – 12

1. Find the coordinate form of the vector $\overrightarrow{PQ}$, if

   (A) $P = (-1, 6)$ & $Q = (-5, 1)$ (B) $P = (-5, 1)$ & $Q = (-1, 6)$ (C) $P = (5, -6)$ & $Q = (8, -6)$

2. Consider the vector $\overrightarrow{PQ} = \binom{3}{-4}$. In each question letter try to find the missing coordinates of the points P, *or* Q, *or both*, if:

   (A) $P = (-1, 3)$ (B) $Q = (1, -8)$ (C) $P = (2x + 1, x)$ & $Q = (8, -2)$ (D) $P = (2x, y - 5)$ & $Q = (-7, 2x)$

Solution To Selected Problems

2. Given : $\overrightarrow{PQ} = \binom{3}{-4}$

(A) Given: $P = (-1, 3)$

Required: $Q = (?, ?)$

Let $Q = (x, y)$

$\overrightarrow{PQ} = \binom{3}{-4}$

$\Rightarrow \binom{x-(-1)}{y-3} = \binom{3}{-4}$

$(x + 1 = 3)$ *and*

$(y - 3 = -4)$

$(x = 2) \wedge (y = -1)$

$(x = 2) \wedge (y = -1)$

$\therefore Q = (x, y) = (2, -1)$

(D) Given

$P = (2x, y - 5)$ and

$Q = (-7, 2x)$

$\overrightarrow{PQ} = \binom{3}{-4}$

$\binom{-7-2x}{2x-(y-5)} = \binom{3}{-4}$

$\Rightarrow -7 - 2x = 3$ and

$2x - y + 5 = -4$

$\Rightarrow x = -5$ and

$2(-5) - y + 5 = -4$

$\Rightarrow x = -5 \wedge y = -1$

$\therefore P = (2x, y - 5)$

$= (-10, -6)$

Similarly,

$Q = (-7, 2x)$

$= (-7, -10)$

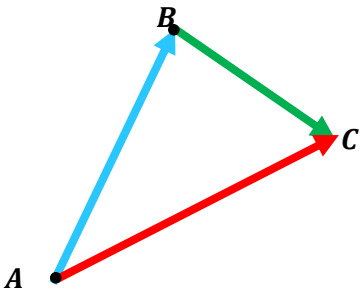
Answers: 1. (A) $\overrightarrow{PQ} = \binom{-4}{-5}$ (B) $\overrightarrow{PQ} = \binom{4}{5}$ (C) $\overrightarrow{PQ} = \binom{3}{0}$ 2. (B) $P = (-2, -4)$ (C) $P = (5, 2)$

Adding Vectors

Definition

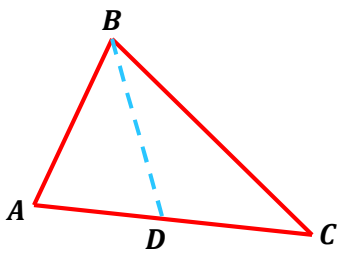
Let $\vec{u}$ *and* $\vec{v}$ be vectors in plane represented by the directed line-segments $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ respectively, as shown in the adjoining figure. Then their sum $\vec{u} + \vec{v}$ is given by :

$$\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \overrightarrow{AC}$$



Example – 13

1.

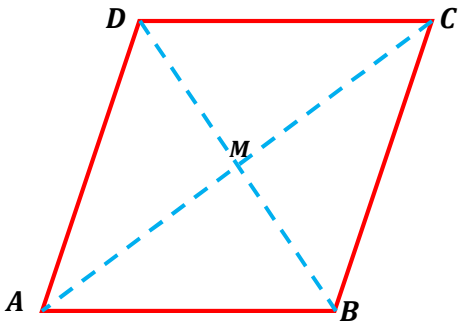


In the adjoining figure, you are given ΔABC. D is the midpoint of $\overline{AC}$. Determine whether each of the following statements is True or False.

(A) $\overrightarrow{AD} = \overrightarrow{CD}$ (B) $\overrightarrow{AD} = \overrightarrow{DC}$ (C) $\overrightarrow{DB} + \overrightarrow{BA} = \overrightarrow{DA}$

(D) $\overrightarrow{AD} + \overrightarrow{CB} = \overrightarrow{DB}$

2. In the following figure you are given a parallelogram. $\overline{AC}$ *and* $\overline{BD}$ are diagonals of the parallelogram. Determine whether each statement given beside the figure is True or False.



(A) $\overrightarrow{MD} + \overrightarrow{DC} = \overrightarrow{AM}$
(B) $\overrightarrow{AB} + \overrightarrow{CM} = \overrightarrow{MB}$
(C) $\overrightarrow{AM} + \overrightarrow{MB} = \overrightarrow{BC}$
(D) $\overrightarrow{AC} + \overrightarrow{DA} = \overrightarrow{AB}$

Solutions

1. (A) False
Because, $\overrightarrow{AD}$ & $\overrightarrow{CD}$ have different directions
$\therefore \overrightarrow{AD} \neq \overrightarrow{CD}$

(B) True
Because
(i) $\overrightarrow{AD}$ & $\overrightarrow{DC}$ have same direction.
(ii) $\|\overrightarrow{AD}\| = \frac{1}{2}\|\overrightarrow{AC}\| = \|\overrightarrow{DC}\|$
$\therefore \overrightarrow{AD} = \overrightarrow{DC}$

(C) True
$\overrightarrow{DB}$ & $\overrightarrow{BC}$ are connected
Head to Tail
$\therefore \overrightarrow{DB} + \overrightarrow{BA} = \overrightarrow{DA}$

(D) True
$\overrightarrow{AD} + \overrightarrow{CB}$
$= \overrightarrow{DC} + \overrightarrow{CB}$ since $\overrightarrow{AD} = \overrightarrow{DC}$
$= \overrightarrow{DB}$

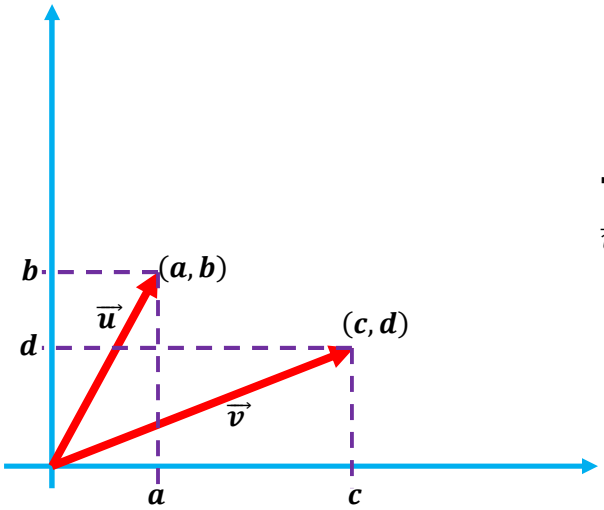
2. In order to decide the truth values of the given statements you need to recall the following facts about parallelograms from your elementary geometry lessons.
(i) Both pairs of opposite sides of a parallelogram are parallel and congruent.
(ii) The diagonals of a parallelogram bisect each other.
(A) $\overrightarrow{MD} + \overrightarrow{DC} = \overrightarrow{MC}$
$= \overrightarrow{AM}$
$\therefore$ ***A is True***

(B) $\overrightarrow{AB} + \overrightarrow{CM} = \overrightarrow{DC} + \overrightarrow{CM}$
$= \overrightarrow{DM} = \overrightarrow{MB}$
$\therefore$ ***B is True***

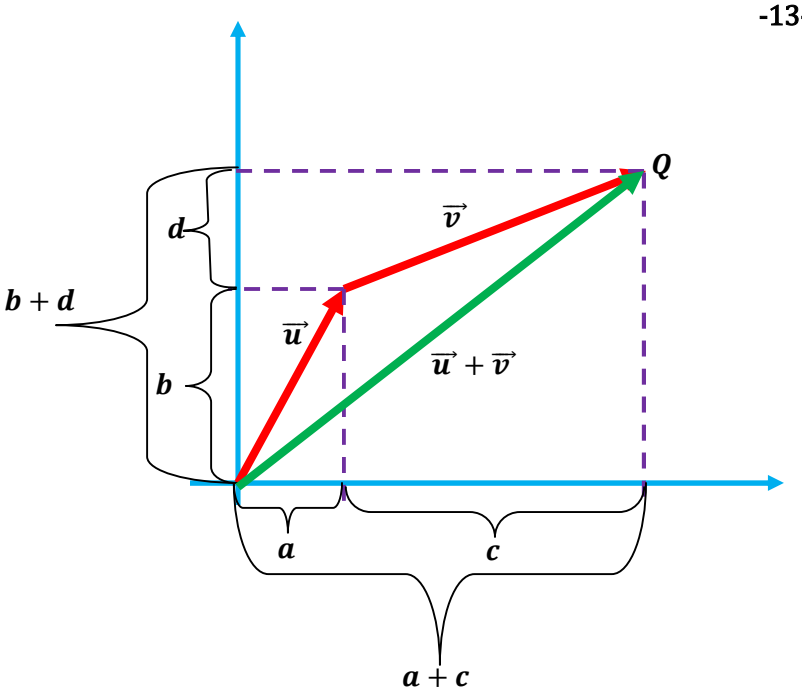
(C) $\overrightarrow{AM} + \overrightarrow{BM} = \overrightarrow{AM} + \overrightarrow{MD}$
$= \overrightarrow{AD} = \overrightarrow{BC}$
$\therefore$ ***C is True***

(D) $\overrightarrow{AC} + \overrightarrow{DA} = \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{AB}$
$\therefore$ ***D is True***

Adding Vectors in Coordinate Form



The figure shows two vectors,
$\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$



$$Q = (a + c, b + d)$$
$$\vec{u} + \vec{v} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

NOTE

If $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ then $\vec{u} + \vec{v} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$

Example – 14

If $\vec{u} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, then $\vec{u} + \vec{v} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 8+(-4) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Scalar Multiplication

Definition

For $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and scalar $k \in \mathbb{R}$, the scalar multiple $k\vec{u}$ is defined by

$$k\vec{u} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

Example – 15

1. Let $\vec{u} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $k = 2$, $m = \frac{1}{2}$, $n = -2$, $r = -\frac{1}{2}$, $t = 3$. Compute the vector:

   (A) $k\vec{u}$ (B) $r\vec{u}$ (C) $k\vec{u} - t\vec{v}$ (D) $5(\vec{v} + \vec{u})$ (E) $5\vec{v} + 5\vec{u}$ (F) $(\vec{u} + \vec{v}) + \vec{w}$ (G) $\vec{u} + (\vec{v} + \vec{w})$
   (H) $k(\vec{u} + \vec{w})$ (I) $k\vec{u} + k\vec{w}$ (J) $k(\vec{u} - \vec{w})$ (K) $k\vec{u} - k\vec{w}$ (L) $6[\vec{u} - 2(\vec{w} - 5\vec{v})]$
   (M) $(k + t)\vec{u}$ (N) $k\vec{u} + t\vec{u}$ (O) $5k\vec{u} - 15\vec{w} - 4\vec{v} - \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

2. Using your results in number 1, determine whether each of the following statements is True or False.

   (A) $5(\vec{v} + \vec{u}) = 5\vec{v} + 5\vec{u}$ .....(see $1.D$ & $1.E$) (B) $k(\vec{u} + \vec{w}) = k\vec{u} + k\vec{w}$.....(see 1.H & 1.I)
   (C) $k(\vec{u} - \vec{w}) = k\vec{u} - k\vec{w}$ .....(see 1.J & 1.K) (D) $(k + t)\vec{u} = k\vec{u} + t\vec{u}$..... (1.M & 1.N)

3. Draw the following pairs of vectors in the same coordinate plane. Then write what you can generalize from the figure you have drawn.

   (A) $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *and* $\vec{v} = 2\vec{u}$ (B) $\vec{u} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ *and* $\vec{v} = -2\vec{u}$ (C) $\vec{u} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$ *and* $\vec{v} = \frac{1}{2}\vec{u}$
   (D) $\vec{u} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ *and* $\vec{v} = -\frac{1}{2}\vec{u}$

4. Let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ *and* $\vec{O}$ be the zero vector (i.e., $\vec{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$). Then find

   (A) $\vec{v} + \vec{O}$ (B) $0\vec{v}$ (C) $0\vec{O}$

Solutions To Selected Problems

1. (A) $k\vec{u} = 2\begin{pmatrix}-2\\6\end{pmatrix} = \begin{pmatrix}-4\\12\end{pmatrix}$

(C) $k\vec{u} - t\vec{v} = 2\begin{pmatrix}-2\\6\end{pmatrix} - 3\begin{pmatrix}-3\\-4\end{pmatrix}$
$= \begin{pmatrix}-4\\12\end{pmatrix} + \begin{pmatrix}9\\12\end{pmatrix} = \begin{pmatrix}5\\24\end{pmatrix}$

(H) $k(\vec{u} + \vec{w}) = 2\left[\begin{pmatrix}-2\\6\end{pmatrix} + \begin{pmatrix}-1\\3\end{pmatrix}\right]$
$= 2\begin{pmatrix}-3\\9\end{pmatrix} = \begin{pmatrix}-6\\18\end{pmatrix}$

(L) $6[\vec{u} - 2(\vec{w} - 5\vec{v})]$
$= 6\left\{\begin{pmatrix}-2\\6\end{pmatrix} - 2\left[\begin{pmatrix}-1\\3\end{pmatrix} + -5\begin{pmatrix}-3\\-4\end{pmatrix}\right]\right\}$
$= 6\left\{\begin{pmatrix}-2\\6\end{pmatrix} + -2\left[\begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}15\\20\end{pmatrix}\right]\right\}$
$= 6\left[\begin{pmatrix}-2\\6\end{pmatrix} + -2\begin{pmatrix}14\\23\end{pmatrix}\right]$
$= 6\left[\begin{pmatrix}-2\\6\end{pmatrix} + \begin{pmatrix}-28\\-46\end{pmatrix}\right]$
$= 6\begin{pmatrix}-30\\-40\end{pmatrix} = \begin{pmatrix}-180\\-240\end{pmatrix}$

(O) $5k\vec{u} - 15\vec{w} - 4\vec{v} - \begin{pmatrix}7\\1\end{pmatrix}$
$= 10\begin{pmatrix}-2\\6\end{pmatrix} + -15\begin{pmatrix}-1\\3\end{pmatrix} + -4\begin{pmatrix}-3\\-4\end{pmatrix} - \begin{pmatrix}7\\1\end{pmatrix}$
$= \begin{pmatrix}-20\\60\end{pmatrix} + \begin{pmatrix}15\\-45\end{pmatrix} + \begin{pmatrix}12\\16\end{pmatrix} + \begin{pmatrix}-7\\-1\end{pmatrix}$
$= \begin{pmatrix}-20+15+12+-7\\60\pm-45+16\pm-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Answers of the other questions in number 1

(B) $\begin{pmatrix}1\\-3\end{pmatrix}$ (C) $\begin{pmatrix}5\\24\end{pmatrix}$ (D) $\begin{pmatrix}-25\\10\end{pmatrix}$ (E) $\begin{pmatrix}-25\\20\end{pmatrix}$ (F) $\begin{pmatrix}-6\\5\end{pmatrix}$ (G) $\begin{pmatrix}-6\\5\end{pmatrix}$ (H) $\begin{pmatrix}-6\\18\end{pmatrix}$ (I) $\begin{pmatrix}-6\\18\end{pmatrix}$
(J) $\begin{pmatrix}-2\\6\end{pmatrix}$ (K) $\begin{pmatrix}-2\\6\end{pmatrix}$ (M) $\begin{pmatrix}-10\\30\end{pmatrix}$ (N) $\begin{pmatrix}-10\\30\end{pmatrix}$

Properties

1. Commutative Property of Vector Addition
For any two vectors $\vec{u}$ & $\vec{v}$
$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2. Associative Property of Vector Addition
For any three vectors $\vec{u}, \vec{v}$ & $\vec{w}$
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3. Distributivity of Scalar Multiplication over Vector Addition
For any two vectors $\vec{u}$ & $\vec{v}$ and any scalar k,
$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

4, For any two scalars m & n and any vector $\vec{v}$
$(m + n)\vec{v} = m\vec{v} + n\vec{v}$

5. If $\vec{O}$ is the zero vector and k is any scalar, then
$k\vec{O} = \vec{O}$

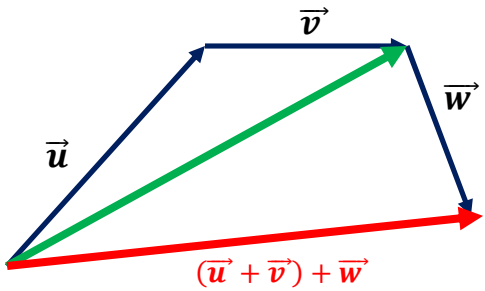
6. For any vector $\vec{v}$, $0\vec{v} = \vec{O}$

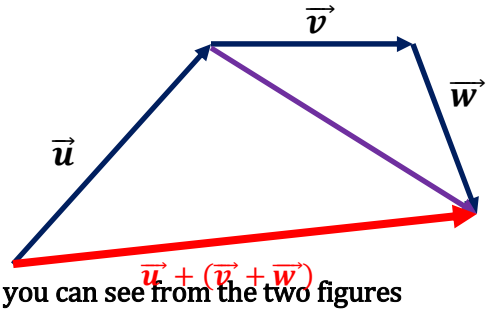
Proofs of the Properties

We can prove the theorems in different ways. For instance, we can do our proof geometrically using head to tail connection of vectors or analytically using the coordinate form of vectors and real number properties you have learnt in lower grades.

Proof of Property 2

Method I: Geometrical





As you can see from the two figures

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Method II: Analytical Method

Let $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}, \vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ *and* $\vec{w} = \begin{pmatrix} r \\ t \end{pmatrix}$, where $a, b, c, d, r, t \in \mathbb{R}$

$\therefore (\vec{u} + \vec{v}) + \vec{w}$

$= \left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right] + \begin{pmatrix} r \\ t \end{pmatrix}$

$= \begin{pmatrix} a+c \\ b+d \end{pmatrix} + \begin{pmatrix} r \\ t \end{pmatrix}$ ...see the Note on Page 13

$= \begin{pmatrix} (a+c)+r \\ (b+d)+t \end{pmatrix}$ .... see the Note on Page 13

$= \begin{pmatrix} a+(c+r) \\ b+(d+t) \end{pmatrix}$....Associativity of addition in $\mathbb{R}$

$= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c+r \\ d+t \end{pmatrix}$... see the Note on Page 13

$= \begin{pmatrix} a \\ b \end{pmatrix} + \left[\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} r \\ t \end{pmatrix}\right]$ ...see the Note on Page 13

$= \vec{u} + (\vec{v} + \vec{w})$

You can prove the other properties in a similar way and are left as an exercise to you.

Unit Vector

Definition

A unit vector is a vector whose magnitude is 1 unit long.

That is, a vector $\vec{v}$ is said to be a unit vector if and only if $\|\vec{v}\| = 1$

Example – 16

Check whether each of the following vectors is a unit vector or not.

1. $\vec{u} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ 2. $\vec{v} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$ 3. $\vec{w} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$

Solution

1. $\vec{u} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$\|\vec{u}\| = \sqrt{(1/2)^2 + (1/2)^2}$

$\Rightarrow \|\vec{u}\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}}$

$\therefore \|\vec{u}\| \neq 1$

Hence $\vec{u}$ is not a unit vector.

2. $\vec{v} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$

$\|\vec{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$

$\Rightarrow \|\vec{v}\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1}$

$\therefore \|\vec{v}\| = 1$

Hence $\vec{v}$ is a unit vector

3. $\vec{w} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$

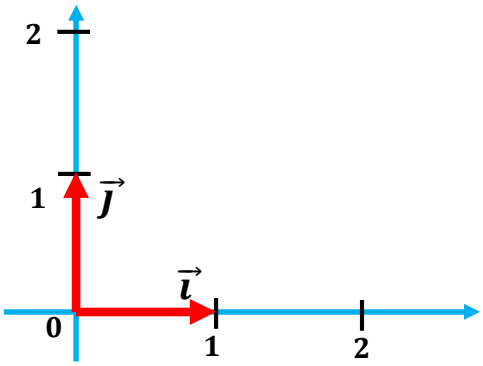
$\|\vec{w}\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$\|\vec{w}\| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

Hence $\vec{w}$ is a unit vector.

Standard Unit Vectors in Plane

The two vectors in the directions of positive $x-axis$ and positive $y-axis$, denoted by $\vec{\imath}$ and $\vec{\jmath}$, where $\vec{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called standard unit vectors.

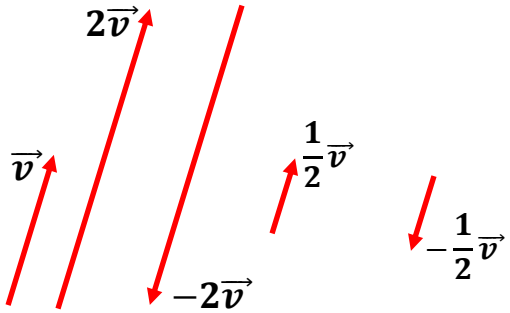


Effect of Multiplying a Vector by a non-zero Scalar

Let $\vec{v}$ be a non-zero vector and k be a non-zero real number. Then the scalar multiple $k\vec{v}$ is a vector whose magnitude is $|k|$ times the magnitude of $\vec{v}$, and which has

(i) the same direction as $\vec{v}$, if $k > 0$ and

(ii) opposite direction to that of $\vec{v}$, if $k < 0$



NOTE

1. If $\vec{v}$ is a non-zero vector and k is a non-zero real number then the vectors $\vec{v}$ and $k\vec{v}$ are parallel vectors.
2. For two vectors $\vec{v}$ and $\vec{w}$, $\vec{w} \parallel \vec{v}$ if and only if $\vec{w} = k\vec{v}$ for some $k \in \mathbb{R}$

Example – 17

1. Let $\vec{v}$ and $\vec{w}$ be vectors and $k \in \mathbb{R}$ such that $\vec{w} = k\vec{v}$. Prove that $\|\vec{w}\| = |k|\|\vec{v}\|$
2. Let $\vec{v} = \binom{2}{4}$. For each vector $\vec{w}$ given below, find express $\vec{w}$ in coordinate form and then draw both $\vec{v}$ & $\vec{w}$ in the same coordinate plane. Finally, explain what you have observed.

(A) $\vec{w} = 2\vec{v}$ (B) $\vec{w} = -3\vec{v}$ (C) $\vec{w} = -\frac{1}{2}\vec{v}$

Solution

1. Given: $\vec{w} = k\vec{v}$

Required: W.T.S that $\|\vec{w}\| = |k|\|\vec{v}\|$

Let $\vec{v} = \binom{a}{b}$ for $a, b \in \mathbb{R}$

$\Rightarrow \|\vec{v}\| = \sqrt{a^2 + b^2}$ .....(*)

$\vec{w} = k\vec{v}$

$\Rightarrow \vec{w} = k\binom{a}{b}$

$\Rightarrow \vec{w} = \binom{ka}{kb}$

$\Rightarrow \|\vec{w}\| = \sqrt{(ka)^2 + (kb)^2}$

$= \sqrt{k^2a^2 + k^2b^2}$

$= \sqrt{k^2(a^2 + b^2)}$

$= \sqrt{k^2} \times \sqrt{a^2 + b^2}$

$= |k|\|\vec{v}\|$ ..... by (*)

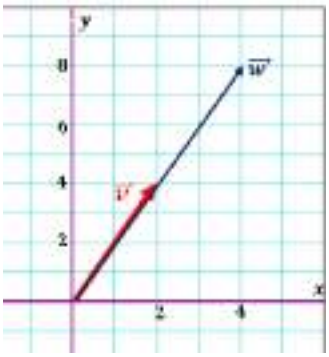
2. Given: $\vec{v} = \binom{2}{4}$

(A) Given: $\vec{w} = 2\vec{v}$

$\vec{w} = 2\vec{v}$

$\Rightarrow \vec{w} = 2\binom{2}{4}$

$\Rightarrow \vec{w} = \binom{4}{8}$

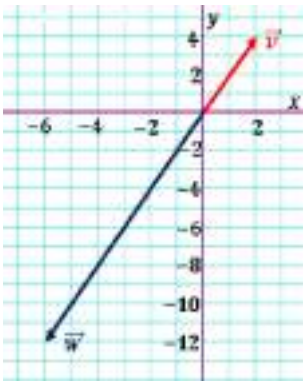


(B) Given: $\vec{w} = -3\vec{v}$

$\vec{w} = -3\vec{v}$

$\Rightarrow \vec{w} = -3\binom{2}{4}$

$\Rightarrow \vec{w} = \binom{-6}{-12}$

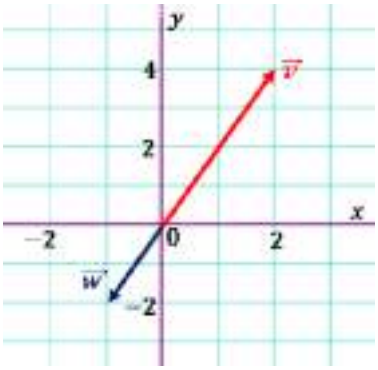


(C) Given: $\vec{w} = -\frac{1}{2}\vec{v}$

$\vec{w} = -\frac{1}{2}\vec{v}$

$\Rightarrow \vec{w} = -\frac{1}{2}\binom{2}{4}$

$\Rightarrow \vec{w} = \binom{-1}{-2}$



Question: Given two non-zero vectors in coordinate form, how do we check whether the two vectors are parallel or not?

Here is the answer

Let $\vec{v}$ & $\vec{w}$ be two non zero vectors. Then, by the note given at the beginning of this page $\vec{w} \parallel \vec{v}$ if and only if there exists $k \in \mathbb{R}$ such that

$\vec{w} = k\vec{v}$

$\Rightarrow \|\vec{w}\| = \|k\vec{v}\|$

$\Rightarrow \|\vec{w}\| = |k|\|\vec{v}\|$

$\Rightarrow |k| = \frac{\|\vec{w}\|}{\|\vec{v}\|}$

$\Rightarrow k = \pm\frac{\|\vec{w}\|}{\|\vec{v}\|}$

$\therefore \vec{w} \parallel \vec{v}$ if and only if

$\vec{w} = \pm\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v}$

OR

$\vec{v} = \pm\frac{\|\vec{v}\|}{\|\vec{w}\|} \times \vec{w}$

Example – 18 : Check whether $\vec{w}$ & $\vec{v}$ are parallel vectors or not.

1. $\vec{v} = (-2, 5)$ *and* $\vec{w} = (6, -15)$ 2. $\vec{v} = (6, -8)$ *and* $\vec{w} = (9, -12)$ 3. $\vec{v} = (9, 12)$ & $\vec{w} = (6, 18)$

Solution

1. $\vec{v} = (-2, 5)$ &
$\vec{w} = (6, -15)$

$\therefore \|\vec{v}\| = \sqrt{(-2)^2 + 5^2}$
$= \sqrt{29}$

Similarly,

$\|\vec{w}\| = \sqrt{6^2 + (-15)^2}$
$= \sqrt{3^2 2^2 + 3^2(-5)^2}$
$= \sqrt{3^2 \times 29} = 3\sqrt{29}$

$\therefore -\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v}$
$= -\frac{3\sqrt{29}}{\sqrt{29}} \times (-2, 5)$
$= -3(-2, 5)$
$= (6, -15) = \vec{w}$

Hence, we have shown that $\vec{w} = -\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v}$

$\therefore \vec{w} \parallel \vec{v}$

2. $\vec{v} = (6, -8)$ *and*
$\vec{w} = (9, -12)$

$\therefore \|\vec{v}\| = \sqrt{6^2 + (-8)^2}$
$= \sqrt{100} = 10$

Similarly,

$\|\vec{w}\| = \sqrt{9^2 + (-12)^2}$
$= \sqrt{3^2 3^2 + 3^2(-4)^2}$
$= \sqrt{3^2(25)} = 15$

$\therefore \frac{\|\vec{w}\|}{\|\vec{v}\|} = \frac{15}{10} = \frac{3}{2}$

Hence,

$\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v} = \frac{3}{2}(6, -8)$
$= (9, -12) = \vec{w}$

$\therefore \vec{w} = \frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v}$

$\Longrightarrow \vec{w} \parallel \vec{v}$

3. $\vec{v} = (9, 12)$ &
$\vec{w} = (6, 18)$

$\therefore \|\vec{v}\| = \sqrt{9^2 + 12^2}$
$= 15$

Similarly,

$\|\vec{w}\| = \sqrt{6^2 + 18^2}$
$= \sqrt{6^2 + 6^2 3^2}$
$= \sqrt{6^2 \times 10} = 6\sqrt{10}$

$\therefore \frac{\|\vec{w}\|}{\|\vec{v}\|} = \frac{6\sqrt{10}}{15} = \frac{2\sqrt{10}}{5}$

Hence,

$\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v} = \frac{2\sqrt{10}}{5}(9, 12)$
$= \left(\frac{18\sqrt{10}}{5}, \frac{24\sqrt{10}}{5}\right)$
$\neq \vec{w}$

In a similar way you can show that

$\vec{w} \neq -\frac{\|\vec{w}\|}{\|\vec{v}\|} \times \vec{v}$

$\Longrightarrow \vec{w} \nparallel \vec{v}$

Linear Combination of Vectors

Definition

Let $\vec{u}$ *and* $\vec{v}$ be vectors in plane , and k & m be real numbers. Then the expression $k\vec{u} + m\vec{v}$ is called linear combination of the vectors $\vec{u}$ & $\vec{v}$.

Example – 19

1. Let $\vec{u} = (-1, 3)$ *and* $\vec{v} = (3, -2)$. In each question letter a vector $\vec{w}$ is expressed as a linear combination of the vectors $\vec{u}$ & $\vec{v}$. Find the coordinate form of $\vec{w}$.
   (A) $\vec{w} = 2\vec{u} + 3\vec{v}$ (B) $\vec{w} = 4\vec{u} - 2\vec{v}$
2. Prove that any vector $\vec{v}$ in plane (i.e., in $\mathbb{R}^2$) can be expressed as a linear combination of the two standard unit vectors $\vec{\imath}$ & $\vec{J}$
3. Express each of the following as a linear combination of $\vec{\imath}$ & $\vec{J}$.
   (A) $\vec{v} = (2, 5)$ (B) $\vec{v} = (-3, 0)$ (C) $\vec{v} = (0, -8)$
4. Let $\vec{u} = (-2, 6)$ & $\vec{v} = (9, -6)$. Express each vector $\vec{w}$ given below as a linear combination of $\vec{u}$ & $\vec{v}$.
   (A) $\vec{w} = (-10, 6)$ (B) $\vec{w} = (42, -42)$ (C) $\vec{w} = (-7, 7)$ (D) $\vec{w} = (0, 42)$
5. Let $\vec{v} = (2, -4)$ and $\vec{u}$ be an unknown vector whose magnitude is $\|\vec{u}\| = 6\sqrt{5}$. Find $\vec{u}$ in coordinate form if, $\vec{u}$ & $\vec{v}$ have (A) same direction (B) opposite direction
6. Suppose $\vec{u}$ & $\vec{v}$ have opposite directions, $\vec{v} = (4, -6)$ and $\|2\vec{u}\| = \|-3\vec{v}\|$. Find $\vec{u}$

Solution

1. Given :
$\vec{u} = (-1, 3)$ and
$\vec{v} = (3, -2)$

(A) $\vec{w} = 2\vec{u} + 3\vec{v}$
$= 2\begin{pmatrix}-1\\3\end{pmatrix} + 3\begin{pmatrix}3\\-2\end{pmatrix}$
$= \begin{pmatrix}-2\\6\end{pmatrix} + \begin{pmatrix}9\\-6\end{pmatrix}$
$= \begin{pmatrix}7\\0\end{pmatrix}$ or $(7, 0)$

(B) $\vec{w} = 4\vec{u} - 2\vec{v}$
$= 4\begin{pmatrix}-1\\3\end{pmatrix} + -2\begin{pmatrix}3\\-2\end{pmatrix}$
$= \begin{pmatrix}-4\\12\end{pmatrix} + \begin{pmatrix}-6\\4\end{pmatrix}$
$= \begin{pmatrix}-10\\16\end{pmatrix}$

2. Proof

Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ be any vector in $\mathbb{R}^2$. Then

$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$

$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= a\vec{\imath} + b\vec{\jmath}$

Hence, we have shown that any vector $\vec{v}$ can be expressed as a linear combination of $\vec{\imath}$ & $\vec{\jmath}$

3. (A) $\vec{v} = (2, 5)$

$= 2\vec{\imath} + 5\vec{\jmath}$

(B) $\vec{v} = (3, 0)$

$= 3\vec{\imath} + 0\vec{\jmath}$

$= 3\vec{\imath}$

(C) $\vec{v} = (0, -8)$

$= 0\vec{\imath} + -8\vec{\jmath}$

$= -8\vec{\jmath}$

4. Given :

$\vec{u} = (-2, 6)$ *and* $\vec{v} = (9, -6)$

(A) Given :

$\vec{w} = (-10, 16)$

Let $m, n \in \mathbb{R}$ such that

$\vec{w} = m\vec{u} + n\vec{v}$

$\Rightarrow (-10, 16) = m(-2, 6) + n(9, -6)$

$\Rightarrow (-10, 16) = (-2m, 6m) + (9n, -6n)$

$\Rightarrow (-10, 16) = (-2m + 9n, 6m - 6n)$

$\Rightarrow \begin{cases} -2m + 9n = -10 \\ 6m - 6n = 16 \end{cases}$

Multiplying the first equation by 3 we get

$\begin{cases} -6m + 27n = -30 \\ 6m - 6n = 16 \end{cases}$ now add the equations

$21n = -14 \Rightarrow n = -\frac{2}{3}$

$6m - 6n = 16$ & $n = -\frac{2}{3}$ give

$6m + 4 = 16 \Rightarrow m = 2$

$\therefore \vec{w} = 2\vec{u} - \frac{2}{3}\vec{v}$

Answers to B, C, and D

(B) $\vec{w} = -3\vec{u} + 4\vec{v}$

(C) $\vec{w} = \frac{1}{2}\vec{u} - \frac{2}{3}\vec{v}$

(D) $\vec{w} = 9\vec{u} + 2\vec{v}$

5. Given : $\vec{v} = (2, -4)$ & $\|\vec{u}\| = 6\sqrt{5}$

(A) Given :

$\vec{u}$ & $\vec{v}$ have same direction

$\vec{v} = (2, -4)$

$\Rightarrow \|\vec{v}\| = \sqrt{2^2 + (-4)^2}$

$= \sqrt{20} = 2\sqrt{5}$

$\therefore \vec{u} = \frac{\|\vec{u}\|}{\|\vec{v}\|} \times \vec{v}$

$= \frac{6\sqrt{5}}{2\sqrt{5}}(2, -4)$

$= 3(2, -4)$

$= (6, -12)$

(B) Given :

$\vec{u}$ & $\vec{v}$ have opposite directions

$\therefore \vec{u} = -\frac{\|\vec{u}\|}{\|\vec{v}\|} \times \vec{v}$

$= -\frac{6\sqrt{5}}{2\sqrt{5}}(2, -4)$

$= -3(2, -4)$

$= (-6, 12)$

6. Given :

- $\vec{v} = (4, -6)$
- $\|2\vec{u}\| = \|-3\vec{v}\|$
- $\vec{u}$ & $\vec{v}$ have opposite *directios.*

$\vec{v} = (4, -6)$

$\Rightarrow \|\vec{v}\| = \sqrt{4^2 + (-6)^2}$

$= 2\sqrt{13}$

$\|2\vec{u}\| = \|-3\vec{v}\|$

$\Rightarrow |2|\|\vec{u}\| = |-3|\|\vec{v}\|$

$\Rightarrow \|\vec{u}\| = \frac{3}{2}\|\vec{v}\|$

$\Rightarrow \|\vec{u}\| = \sqrt{13}$

$\therefore \vec{u} = -\frac{\|\vec{u}\|}{\|\vec{v}\|}\vec{v}$

$= -\frac{\sqrt{13}}{2\sqrt{13}}(4, -6)$

$= -\frac{1}{2}(4, -6) = (-2, 3)$

Unit Vector Parallel to a Given non-zero Vector

Let $\vec{v}$ be a given non-zero vector and $\vec{u}$ be a unit vector parallel to $\vec{v}$

- $\vec{u}$ is a unit vector $\Rightarrow \|\vec{u}\| = 1$
- $\vec{u} \parallel \vec{v} \Rightarrow \vec{u} = k\vec{v}$ for some k

$\Rightarrow \|\vec{u}\| = \|k\vec{v}\|$

$\Rightarrow 1 = |k|\|\vec{v}\|$

$\Rightarrow |k| = \frac{1}{\|\vec{v}\|}$

$\Rightarrow k = \pm\frac{1}{\|\vec{v}\|}$

If $= \frac{1}{\|\vec{v}\|}$, then $k > 0$

$\therefore$ the unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by

$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

If $= -\frac{1}{\|\vec{v}\|}$, then $k < 0$

$\therefore$ the unit vector $\vec{u}$ opposite to $\vec{v}$ is given by

$\vec{u} = -\frac{1}{\|\vec{v}\|}\vec{v}$

Example – 20

1. Find the unit vector in the direction of each of the following vectors.
   (A) $\vec{v} = (-3, 4)$ (B) $\vec{v} = (3, 3)$ (C) $\vec{v} = (5, 0)$
2. Let $\vec{v} = -\vec{\imath} + 4\vec{\jmath}$ and $\vec{w} = 2\vec{\imath}$. Find the unit vector in the direction of (A) $\vec{v} - \vec{w}$ (B) $2\vec{v} + 3\vec{w}$
3. Find the unit vector opposite to the vector (A) $\vec{v} = (-3, 4)$ (B) $\vec{v} = 2\vec{\imath} - 3\vec{\jmath}$
4. Let $\vec{v}$ *and* $\vec{w}$ be vectors with opposite directions. If $\|\vec{v}\| = 3\|\vec{w}\|$, then find the value of the vector $2\vec{v} - \vec{w}$ in terms of (A) $\vec{v}$ (B) $\vec{w}$

Solution

1. Let $\vec{u}$ be the unit vector in the direction of $\vec{v}$

(A) $\vec{v} = (-3, 4)$

$\Rightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{(-3)^2+4^2}}\vec{v}$

$\Rightarrow \vec{u} = \frac{1}{5}(-3, 4)$

$\Rightarrow \vec{u} = \left(-\frac{3}{5}, \frac{4}{5}\right)$

(B) $\vec{v} = (3, 3)$

$\Rightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

$\Rightarrow \vec{u} = \frac{1}{\sqrt{3^2+3^2}}\vec{v}$

$\Rightarrow \vec{u} = \frac{1}{3\sqrt{2}}(3, 3)$

$\Rightarrow \vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(C) Answer:

$\vec{u} = 5\vec{\imath}$

2. Given

$\vec{v} = -\vec{\imath} + 4\vec{\jmath}$

$\vec{w} = 2\vec{\imath}$

(A) Let $\vec{u}$ be the unit vector in the direction of $\vec{v} - \vec{w}$

$\vec{v} - \vec{w} = -\vec{\imath} + 4\vec{\jmath} - 2\vec{\imath}$

$= -3\vec{\imath} + 4\vec{\jmath}$

$\therefore \|\vec{v} - \vec{w}\|$

$= \sqrt{(-3)^2 + 4^2}$

$= 5$

$\therefore \vec{u} = \frac{1}{\|\vec{v} - \vec{w}\|} \times (\vec{v} - \vec{w})$

$\Rightarrow \vec{u} = \frac{1}{5}(-3\vec{\imath} + 4\vec{\jmath})$

$\Rightarrow \vec{u} = -\frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath}$

(B) Let $\vec{u}$ be the unit vector in the direction of $2\vec{v} + 3\vec{w}$

$2\vec{v} + 3\vec{w}$

$= 2(-\vec{\imath} + 4\vec{\jmath}) \times 3(2\vec{\imath})$

$= 4\vec{\imath} + 8\vec{\jmath}$

$\therefore \|2\vec{v} + 3\vec{w}\|$

$= \|4\vec{\imath} + 8\vec{\jmath}\|$

$= \sqrt{4^2 + 8^2}$

$= \sqrt{4^2(1 + 2^2)}$

$= 4\sqrt{5}$

$\therefore \vec{u} = \frac{1}{\|2\vec{v}+3\vec{w}\|}(2\vec{v} + 3\vec{w})$

$= \frac{1}{4\sqrt{5}}(4\vec{\imath} + 8\vec{\jmath})$

$= \frac{1}{\sqrt{5}}\vec{\imath} + \frac{2}{\sqrt{5}}\vec{\jmath}$

3. Let $\vec{u}$ be the unit vector opposite to $\vec{v}$

(A) $\vec{v} = (-3, 4)$

$\Rightarrow \vec{u} = -\frac{1}{\|\vec{v}\|}\vec{v}$

$\Rightarrow \vec{u} = -\frac{1}{\sqrt{(-3)^2+4^2}}\vec{v}$

$\Rightarrow \vec{u} = -\frac{1}{5}(-3, 4)$

$\Rightarrow \vec{u} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

(B) $\vec{v} = 2\vec{\imath} - 3\vec{\jmath}$

$\Rightarrow \vec{u} = -\frac{1}{\|\vec{v}\|}\vec{v}$

$\Rightarrow \vec{u} = -\frac{1}{\sqrt{2^2+(-3)^2}}\vec{v}$

$\Rightarrow \vec{u} = -\frac{1}{\sqrt{13}}(2\vec{\imath} - 3\vec{\jmath})$

$\Rightarrow \vec{u} = -\frac{2}{\sqrt{13}}\vec{\imath} + \frac{3}{\sqrt{13}}\vec{\jmath}$

Answers (to number 4) (A) $2\vec{v} - \vec{w} = \frac{7}{3}\vec{v}$ (B) $2\vec{v} - \vec{w} = -7\vec{w}$

NATIONAL EXAMINATION QUESTIONS

1. If $\vec{u}$ is a vector in the direction as vector $\vec{v} = (-3, 4)$ having length 3 times the length of $\vec{v}$, what is the length of the vector $\vec{u} - \frac{3}{2}\vec{v} - 2\vec{\jmath}$, where $\vec{\jmath}$ is the standard unit vector in the direction of positive $y - axis$.

   (A) $\frac{1}{2}\sqrt{185}$ (B) $\frac{1}{2}\sqrt{145}$ (C) $\frac{21}{2}$ (D) $\frac{17}{2}$

2. If $\vec{A} = (3, -3)$ and $\vec{B} = (1, -3)$, then what is the unit vector in the direction of $\vec{C} = 3\vec{A} - \vec{B}$?

   (A) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ (B) $\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$ (C) $\left(\frac{4}{5}, -\frac{3}{5}\right)$ (D) $\left(-\frac{6}{10}, \frac{8}{10}\right)$

3. Let $ABCD$ be a parallelogram and $\overline{AC}$ and $\overline{BD}$ be its diagonals. Which of the following statements is not true?

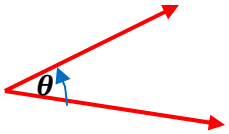
   (A) $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC}$ (B) $\overrightarrow{AB} = \overrightarrow{BC} - \overrightarrow{BD}$ (C) $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$ (D) $\overrightarrow{CD} + \overrightarrow{BC} = \overrightarrow{AD}$

4. Let $A=(0,1)$, $B=(2,7)$ *and* $C=(a,b)$ be points in the coordinate plane. What is (a,b) so that $\overrightarrow{AC}=\overrightarrow{AB}+2\overrightarrow{CA}$?

(A) $\left(\frac{1}{3},2\right)$ (B) $(1,3)$ (C) $(2,3)$ (D) $\left(\frac{2}{3},3\right)$

Answers : 1) B 2) C 3) D 4) D

Scalar (Dot or Inner) Product of Vectors



Angle Between Two Vectors

The angle between two vectors is the angle formed when the vectors are connected end to end.

Scalar Product (or Dot Product ,or Inner Product)

Definition

Let $\vec{u}$ *and* $\vec{v}$ be vectors and θ be the angle between them. Then the scalar product of the vectors $\vec{u}$ & $\vec{v}$ is given by

$$\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$$

NOTE

1. If $\vec{u}$ & $\vec{v}$ are orthogonal vectors (i.e., perpendicular vectors) then $\theta=90^0$ and hence $\vec{u}\cdot\vec{v}=0$
2. If $\vec{u}$ & $\vec{v}$ have same direction then $\theta=0^0$. Hence $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|$.
3. If $\vec{u}$ & $\vec{v}$ have opposite directions, then $\theta=180^0$. Hence $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|$.
4. If $\vec{u}\parallel\vec{v}$, then $\vec{u}\cdot\vec{v}=\pm\|\vec{u}\|\|\vec{v}\|$
5. Let $\vec{u}$ *and* $\vec{v}$ be non zero vectors and θ be the angle between them. Then

$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

6. The standard unit vectors $\vec{\imath}$ & $\vec{\jmath}$ are orthogonal. Hence $\vec{\imath}\cdot\vec{\jmath}=0$

Notation

Fon any vector $\vec{u}$, $\vec{u}\cdot\vec{u}$ can shortly be written as $\vec{u}^2$

Example – 21

1. Let $\vec{u}$ & $\vec{v}$ be vectors with $\|\vec{u}\|=3$ & $\|\vec{v}\|=4$. Let θ be the angle between the two vectors. Find $\vec{u}\cdot\vec{v}$, if (A) $\theta=\frac{\pi}{4}$ (B) $\theta=\pi$ (C) $\theta=\frac{2\pi}{3}$ (D) $\theta=90^0$
2. Let $\vec{u}$ & $\vec{v}$ be vectors satisfying the conditions given in each question letter. Compute the measure of the angle between the two vectors.

(A) $\vec{u}\cdot\vec{v}=5$, $\|\vec{u}\|=\sqrt{5}$ & $\|\vec{v}\|=2\sqrt{5}$ (B) $\vec{u}\cdot\vec{v}=-24$, $\|\vec{u}\|=3\sqrt{2}$ & $\|\vec{v}\|=2\sqrt{8}$

3. For any vector $\vec{v}$ prove that $\vec{v}^2=\|\vec{v}\|^2$
4. For the standard unit vectors $\vec{\imath}$ & $\vec{\jmath}$ compute (A) $\vec{\imath}\cdot\vec{\imath}$ (B) $\vec{\jmath}^2$
5. If the measure of the angle between the vectors $\vec{u}$ & $\vec{v}$ is θ, then find the measure of the angle between each of the following pair of vectors in terms of θ. (A) $\vec{u}$ & $-\vec{v}$ (B) $-\vec{u}$ & $-\vec{v}$

Solution

1. Given: $\|\vec{u}\|=3$ & $\|\vec{v}\|=4$

(A) Given: $\theta=\frac{\pi}{4}$

$\therefore\ \vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\left(\frac{\pi}{4}\right)$

$=3(4)\left(\frac{\sqrt{2}}{2}\right)=6\sqrt{2}$

(B) Given: $\theta=\pi$

$\therefore\ \vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\pi$

$=3(4)(-1)=-12$

(C) Given: $\theta=\frac{2\pi}{3}$

$\therefore\ \vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\left(\frac{2\pi}{3}\right)$

$=3(4)\left(-\frac{1}{2}\right)=-6$

2. Let θ be the angle between the given pair of vectors.

(A) Given: $\vec{u} \cdot \vec{v} = 5$

$\|\vec{u}\| = \sqrt{5}$

$\|\vec{v}\| = 2\sqrt{5}$

$\therefore \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\Rightarrow \cos\theta = \dfrac{5}{(\sqrt{5})(2\sqrt{5})}$

$\Rightarrow \cos\theta = \dfrac{1}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3} \ (or \ 60^0)$

(B) Given: $\vec{u} \cdot \vec{v} = -24$

$\|\vec{u}\| = 3\sqrt{2}$

$\|\vec{v}\| = 2\sqrt{8}$

$\therefore \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\Rightarrow \cos\theta = \dfrac{-24}{(3\sqrt{2})(2\sqrt{8})}$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pi \ (or \ 180^0)$

3. Proof

Let $\vec{v}$ be any arbitrary vector.

The angle between a vector & itself is 0^0.

$\therefore \vec{v}^2 = \vec{v} \cdot \vec{v}$

$= \|\vec{v}\|\|\vec{v}\| \cos 0^0$

$= \|\vec{v}\|\|\vec{v}\|(1)$

$= \|\vec{v}\|^2$

4. (A) $\vec{\imath}$ a standard unit vector.

$\therefore \|\vec{\imath}\| = 1$

$\vec{\imath} \cdot \vec{\imath} = \|\vec{\imath}\|\|\vec{\imath}\| \cos 0^0$

$= 1(1)(1) = 1$

(B) $\vec{\jmath}$ a standard unit vector.

$\therefore \|\vec{\jmath}\| = 1$

$\vec{\jmath} \cdot \vec{\jmath} = \|\vec{\jmath}\|\|\vec{\jmath}\| \cos 0^0$

$= 1(1)(1) = 1$

Properties

1. Commutative Property of Scalar Product

For any two vectors $\vec{u}$ & $\vec{v}$

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2. For any two vectors $\vec{u}$ & $\vec{v}$

(A) $\vec{u} \cdot (-\vec{v}) = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v}$

(B) $(-\vec{u}) \cdot (-\vec{v}) = \vec{u} \cdot \vec{v}$

3. Associativity with Scalars

Let $\vec{u}$ & $\vec{v}$ be vectors and k & m be scalars.

Then,

(A) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

(B) $(k\vec{u}) \cdot (m\vec{v}) = km(\vec{u} \cdot \vec{v})$

4. Distributive Property of Dot Product over Vector Addition

For any three vectors $\vec{u}$, $\vec{v}$ *and* $\vec{w}$

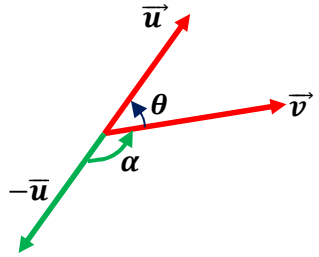
$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(The proof of Property 4 needs the concept of projection of a vector onto another vector. The proof is given at the end of this unit)

Proof of Some Properties

Let the angle between vectors $\vec{u}$ & $\vec{v}$ be θ.

Angle between $-\vec{u}$ & $\vec{v}$



Angle between $-\vec{u}$ & $\vec{v} = \alpha = \pi - \theta$

Similarly angle between $\vec{u}$ & $-\vec{v} = \pi - \theta$

In a similar way as shown above you can easily determine that the measure of the angle between $-\vec{u}$ & $-\vec{v} = \theta$

Proof of Property 2-A

Let the angle between $\vec{u}$ & $\vec{v}$ *be* θ. Then,

- $\vec{u} \ \ \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$
- $\Rightarrow -(\vec{u} \ \ \vec{v}) = -\|\vec{u}\|\|\vec{v}\| \cos\theta$.....(*)
- $\vec{u} \ \ (-\vec{v}) = \|\vec{u}\|\|-\vec{v}\| \cos(\pi - \theta)$

$= \|\vec{u}\|\|\vec{v}\|(-\cos\theta)$

$= -\|\vec{u}\|\|\vec{v}\| \cos\theta$

- $= -(\vec{u} \ \ \vec{v})$ .....by step (*)

Similarly $(-\vec{u}) \cdot \vec{v} = -(\vec{u} \ \ \vec{v})$

Hence $\vec{u} \ \ (-\vec{v}) = -(\vec{u} \ \ \vec{v}) = (-\vec{u}) \ \ \vec{v}$

Try to prove Property 3 by yourself.

Example – 22

1. If $\vec{u} = (a, b)$ & $\vec{v} = (c, d)$, then prove that $\vec{u} \cdot \vec{v} = ac + bd$
2. For any two vectors $\vec{u}$ & $\vec{v}$; if θ is the angle between them, then prove that
   (A) $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| \cos\theta + \|\vec{v}\|^2$ (B) $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta + \|\vec{v}\|^2$
   (C) $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$
3. Find $\vec{u} \cdot \vec{v}$ (A) $\vec{u} = 2\vec{\imath} - 3\vec{\jmath}$ & $\vec{v} = 4\vec{\imath} + \vec{\jmath}$ (B) $\vec{u} = 3\vec{\imath}$ & $\vec{v} = 2\vec{\jmath}$
4. Let $\theta = \frac{\pi}{3}$ be the measure of the angle between the vectors $\vec{u}$ *and* $\vec{v}$. If $\|\vec{u}\| = 6$ *and* $\|\vec{v}\| = 4$, then compute:
   (A) $\|3\vec{v} - 2\vec{u}\|$ (B) $\|\vec{u} + 3\vec{v}\|$ (C) $\vec{u} \cdot (\vec{u} + 2\vec{v})$ (D) $(\vec{u} + 3\vec{v}) \cdot (2\vec{u} - \vec{v})$
5. Let $\vec{A} = 5\vec{\imath} - 12\vec{\jmath}$ *and* $\vec{B}$ be a vector in the $xy-$*plane* such that the angle between $\vec{A}$ & $\vec{B}$ is $\frac{\pi}{3}$. If $\vec{u}$ is the unit vector in opposite direction of $\vec{B}$, then find the value of $\vec{A} \cdot (\vec{A} - 4\vec{u})$
6. Let θ be the angle between vectors $\vec{u}$ & $\vec{v}$. If $\vec{u} = (x, 3)$, $\vec{v} = (2, 2)$ *and* $\cos\theta = \frac{1}{\sqrt{5}}$, then find the value of x.
7. Let $\vec{u}$ be a vector in plane. If $\vec{u} \cdot (\vec{\imath} + \vec{\jmath}) = 1$ *and* $\vec{u} \cdot (\vec{\imath} + \vec{\jmath} - 2\vec{u}) = -9$, the find the value of $\vec{u}$ (two answers)
8. Let $\vec{u}$ & $\vec{v}$ be non-parallel unit vectors. If $\vec{w} = \vec{u} - (\vec{u} \cdot \vec{v})\vec{v}$ and θ is the angle between $\vec{u}$ *and* $\vec{v}$, then which of the following equals to $\|\vec{w}\|$?
   (A) $\sin\theta$ (B) $\cos 2\theta$ (C) $\cos\theta$ (D) $\sin 3\theta$
9. If $\vec{u}, \vec{v}$ & $\vec{w}$ are three non-zero & unequal vectors, then the equation $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ necessarily implies that :
   (A) $\vec{u}$ is orthogonal to both $\vec{v}$ & $\vec{w}$ (C) $\vec{u}$ is orthogonal to $\vec{v} - \vec{w}$
   (B) $\vec{v} = -2\vec{w}$ (D) $\vec{v}\ \vec{w} = 1$
10. The sum to two unit vectors is a unit vector. The magnitude of the difference of the two unit vectors is:
    (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{5}$ (D) 2
11. Let $\vec{u}, \vec{v}$ & $\vec{w}$ be vectors such that $\|\vec{u}\| = 3$, $\|\vec{v}\| = 4$ *and* $\|\vec{w}\| = 2$. If $\vec{u} + \vec{v} + \vec{w} = \vec{O}$, where $\vec{O}$ is the zero vector, then which of the following equals to $\vec{v} \cdot \vec{w} + \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{v}$?
    (A) $\frac{19}{2}$ (B) $\frac{29}{2}$ (C) $-\frac{29}{2}$ (D) $-\frac{19}{2}$

Solution

1. Given: $\vec{u} = a\vec{\imath} + b\vec{\jmath}$ & $\vec{v} = c\vec{\imath} + d\vec{\jmath}$

$\vec{u} \cdot \vec{v} = (a\vec{\imath} + b\vec{\jmath}) \cdot (c\vec{\imath} + d\vec{\jmath})$
$= ac(\vec{\imath} \cdot \vec{\imath}) + ad(\vec{\imath} \cdot \vec{\jmath}) + bc(\vec{\jmath} \cdot \vec{\imath}) + bd(\vec{\jmath} \cdot \vec{\jmath})$
$= ac\|\vec{\imath}\|^2 + ad(0) + bc(0) + bd\|\vec{\jmath}\|^2$
$= ac + bd$

2. A) $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$
$= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$
$= \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| \cos\theta + \|\vec{v}\|^2$

(B) $\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$
$= \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$
$= \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta + \|\vec{v}\|^2$

(C) $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u}^2 - \vec{v}^2$
$= \|\vec{u}\|^2 - \|\vec{v}\|^2$

3. (A) $\vec{u} \cdot \vec{v} = (2\vec{\imath} - 3\vec{\jmath}) \cdot (4\vec{\imath} + \vec{\jmath})$
$= 2(4) + (-3)(1) = 5$

(B) $\vec{u}\ \vec{v} = (2\vec{\imath} + 0\vec{\jmath}) \cdot (0\vec{\imath} + 2\vec{\jmath})$
$= 2(0) + 0(2) = 0$

4. Given: $\|\vec{u}\| = 6, \|\vec{v}\| = 4$ & $\theta = \frac{\pi}{3}$

(A) $\|3\vec{v} - 2\vec{u}\|^2 = (3\vec{v} - 2\vec{u})^2$
$= 9\vec{v}^2 - 6\vec{u} \cdot \vec{v} + 4\vec{u}^2$
$= 9\|\vec{v}\|^2 - 6\|\vec{u}\|\|\vec{v}\| \cos\left(\frac{\pi}{3}\right) + 4\|\vec{u}\|^2$
$\Rightarrow \|3\vec{v} - 2\vec{u}\|^2 = 9(16) - 6(6)(4) + 4(36)$
$= 144$
$\Rightarrow \|3\vec{v} - 2\vec{u}\| = \sqrt{144} = 12$

NOTE

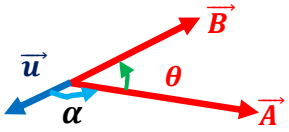
To find the scalar product of two vectors in coordinate form or expressed in terms of the standard vectors we simply add the product of the corresponding components of the two vectors.

5. Given:

✓ $\vec{A} = 5\vec{\imath} - 12\vec{\jmath}$

✓ Angle b/n $\vec{A}$ & $\vec{B}$: $\theta = \frac{\pi}{3}$

✓ $\vec{u}$ is a unit vector opposite to $\vec{B}$



$\|\vec{A}\| = \sqrt{5^2 + (-12)^2}$

$= 13$

Angle b/n $\vec{A}$ & $\vec{u}$

$= \alpha = \pi - \theta$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore \vec{A} \cdot (\vec{A} - 4\vec{u})$

$= \vec{A} \cdot \vec{A} - 4\vec{A} \cdot \vec{u}$

$= \|\vec{A}\|^2 - 4\|\vec{A}\|\|\vec{u}\| \cos\left(\frac{2\pi}{3}\right)$

$= 13^2 - 4(13)(1)\left(-\frac{1}{2}\right)$

$= 169 + 26 = \underline{195}$

6. Given:

✓ $\vec{u} = (x, 3)$

✓ $\vec{v} = (2, 2)$

✓ $\cos\theta = \frac{1}{\sqrt{5}}$

$\|\vec{u}\| = \sqrt{x^2 + 3^2}$

$\|\vec{v}\|\sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\vec{u} \cdot \vec{v} = 2(x) + 3(2)$

$= 2x + 6$

But,

$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

$\Rightarrow 2x + 6 = \left(\sqrt{x^2 + 9}\right)(2\sqrt{2})\frac{1}{\sqrt{5}}$

$\Rightarrow 4(x+3)^2 = (x^2 + 9)\frac{8}{5}$

$\Rightarrow 5(x^2 + 6x + 9) = 2(x^2 + 9)$

$\Rightarrow 3(x+1)(x+9) = 0$

$\Rightarrow x = -1 \; or \; x = -9$

But if $x = -9$, $\cos\theta < 0$

$\therefore x = -1$

7. Given

✓ $\vec{u} \cdot (\vec{\imath} + \vec{\jmath}) = 1$...(*)

✓ $\vec{u} \cdot (\vec{\imath} + \vec{\jmath} - 2\vec{u}) = -9$...(**)

Let $\vec{u} = x\vec{\imath} + y\vec{\jmath}$

Now from equation (*) we get

$x + y = 1$.....(A)

From Equation (**)

$\vec{u} \cdot (\vec{\imath} + \vec{\jmath}) - 2\vec{u} \cdot \vec{u} = -9$

$1 - 2\|\vec{u}\|^2 = -9$

$\Rightarrow \|\vec{u}\|^2 = 5$

$\Rightarrow x^2 + y^2 = 5$...(B)

Solving (A) & (B)

simultaneously we get

$x^2 + (1-x)^2 = 5$

$\Rightarrow 2x^2 - 2x - 4 = 0$

$2(x-2)(x+1) = 0$

$\Rightarrow x = 2 \; or \; x = -1$

$x = 2 \Rightarrow y = 1 - x = -1$

$\Rightarrow \vec{u} = x\vec{\imath} + y\vec{\jmath}$

$= 2\vec{\imath} - \vec{\jmath}$

$x = -1 \Rightarrow y = 1 - x = 2$

$\Rightarrow \vec{u} = x\vec{\imath} + y\vec{\jmath}$

$= -\vec{\imath} + 2\vec{\jmath}$

8. Given

✓ $\vec{u}$ & $\vec{v}$ are unit vectors

✓ $\vec{u} \nparallel \vec{v}$

✓ θ = angle b/n $\vec{u}$ & $\vec{v}$

✓ $\vec{w} = \vec{u} - (\vec{u} \cdot \vec{v})\vec{v}$

$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

$= (1)(1)\cos\theta = \cos\theta$

$\vec{w} = \vec{u} - (\vec{u} \cdot \vec{v})\vec{v}$

$\Rightarrow \vec{w} = \vec{u} - \vec{v}\cos\theta$

$\Rightarrow \|\vec{w}\|^2 = \vec{w}^2$

$= (\vec{u} - \vec{v}\cos\theta)^2$

$= \vec{u}^2 - 2(\vec{u} \cdot \vec{v})\cos\theta + \vec{v}^2$

$= \|\vec{u}\|^2 - 2\cos^2\theta + \|\vec{v}\|^2\cos^2\theta$

$= 1 - 2\cos^2\theta + \cos^2\theta$

$\Rightarrow \|\vec{w}\|^2 = 1 - \cos^2\theta = \sin^2\theta$

$\Rightarrow \|\vec{w}\| = \sin\theta$

9. Given

✓ $\vec{u}, \vec{v}, \vec{w}$ are non-zero vectors

✓ $\vec{u}, \vec{v}, \vec{w}$ are non-equal vectors

✓ $\vec{u} \cdot \vec{v} = \vec{u} \;\; \vec{w}$

$\Rightarrow \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w} = 0$

$\Rightarrow \vec{u} \cdot (\vec{v} - \vec{w}) = 0$

$\Rightarrow \vec{u} \perp (\vec{v} - \vec{w})$

$\therefore \vec{u}$ & $\vec{v} - \vec{w}$ are orthogonal

10. Given:

✓ $\vec{u}$ & $\vec{v}$ are unit vectors

✓ $\vec{u} + \vec{v}$ is also a unit vector

Required: $\|\vec{u} - \vec{v}\| = ?$

$\|\vec{u} + \vec{v}\| = 1$

$\Rightarrow \|\vec{u} + \vec{v}\|^2 = 1$

$\Rightarrow (\vec{u} + \vec{v})^2 = 1$

$\Rightarrow \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 = 1$

$\Rightarrow 2 + 2\vec{u} \;\; \vec{v} = 1$

$\Rightarrow \vec{u} \cdot \vec{v} = -\frac{1}{2}$

$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v})^2$

$= \|\vec{u}\|^2 - 2\vec{u} \;\; \vec{v} + \|\vec{v}\|^2$

$\Rightarrow \|\vec{u} - \vec{v}\|^2 = 2 - 2\left(-\frac{1}{2}\right)$

$\Rightarrow \|\vec{u} - \vec{v}\|^2 = 3$

$\Rightarrow \|\vec{u} - \vec{v}\| = \sqrt{3}$

11. Given

✓ $\|\vec{u}\| = 3, \|\vec{v}\| = 5$

✓ $\|\vec{w}\| = 2$

✓ $\vec{u} + \vec{v} + \vec{w} = \vec{O}$

✓ $\vec{O}$ is the zero vector

$\vec{u} + \vec{v} + \vec{w} = \vec{O}$

$\Rightarrow \|\vec{u} + \vec{v} + \vec{w}\| = \|\vec{O}\|$

$\Rightarrow (\vec{u} + \vec{v} + \vec{w})^2 = 0$

$\Rightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 + \|\vec{w}\|^2 +$

$2\vec{v} \cdot \vec{w} + 2\vec{u} \cdot \vec{w} + 2\vec{u} \cdot \vec{v} = 0$

$\Rightarrow 9 + 16 + 4 + 2\vec{v} \;\; \vec{w} +$

$2\vec{u} \cdot \vec{w} + 2\vec{u} \cdot \vec{v} = 0$

$\Rightarrow \vec{v} \cdot \vec{w} + \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{v} = -\frac{29}{2}$

NATIONAL EXAMINATION QUESTIONS

1. Let $\overrightarrow{PQ}$ be a vector with initial point $P = (1, 5)$ and terminal point $Q = (4, 0)$. If $\vec{v} = x\,\vec{\imath} + 2\,\vec{\jmath}$ is parallel to $\overrightarrow{PQ}$, what is the value of x?

   (A) $-\frac{6}{5}$ (B) $-\frac{2}{5}$ (C) -3 (D) 3

2. For two non – zero vectors $\vec{a}$ and $\vec{b}$, if $\|\vec{a} + \vec{b}\| = \|\vec{a}\|$, then which of the following is necessarily true?

   (A) $2\,\vec{a}\ \vec{b} = \vec{b}\ \vec{b}$
   (B) $2\,\vec{a} + \vec{b}$ & $\vec{b}$ are perpendicular
   (C) $2\,\vec{a} + \vec{b}$ & $\vec{a}$ are parallel
   (D) $\vec{a}\ \vec{b} - \vec{b}\ \vec{b} = 0$

3. If $\vec{V} = \overrightarrow{AB} + 3\,\overrightarrow{BA}$, where A and B are distinct points in the coordinate plane, then which of the following equals to $3\,\vec{V}$?

   (A) $6\,\overrightarrow{AB}$ (B) $-6\,\overrightarrow{AB}$ (C) $12\,\overrightarrow{AB}$ (D) $-12\,\overrightarrow{AB}$

4. If $\vec{u}$ is the unit vector in the direction of $\vec{A}$ and $\|\vec{A}\| = 4$, then $\vec{A}\ \vec{u}$ is equal to _____.

   (A) $\frac{1}{4}$ (B) 4 (C) $\frac{1}{2}$ (D) 2

5. If $\vec{A}$ and $\vec{B}$ are parallel vectors with opposite directions and $\|\vec{B}\| = \|2\,\vec{A}\|$, then $\vec{B} - \vec{A}$ is equal to:

   (A) $\vec{A}$ (B) $-\vec{A}$ (C) $-3\,\vec{A}$ (D) $3\,\vec{A}$

6. If $A = (1, -2)$, $B = (-3, 2)$ and $\vec{V}$ is a position vector such that $2\,\vec{V} + \overrightarrow{AB} = \vec{O}$, where $\vec{O}$ is the zero vector, then $\vec{V}$ is equal to _____.

   (A) $(2, 0)$ (B) $(-1, 0)$ (C) $(-2, 2)$ (D) $(2, -2)$

7. If $\vec{A} = 3\,\vec{\imath} - 4\,\vec{\jmath}$ and $\vec{u}$ is a unit vector such that $\|\vec{A} + \vec{u}\|^2 = 27$, then the cosine of the angle between $\vec{A}$ and $\vec{u}$ is equal to:

   (A) 0.1 (B) 0.3 (C) -0.3 (D) 0.4

8. Suppose $\vec{A} = 3\,\vec{\imath} - 4\,\vec{\jmath}$ and $\vec{B}$ is a vector in the $xy-$plane such that the angle between $\vec{A}$ & $\vec{B}$ is $\frac{\pi}{3}$. If $\vec{u}$ is the unit vector in the direction of $\vec{B}$, then $\vec{A}\ (\vec{A} - 2\,\vec{u})$ is equal to _____

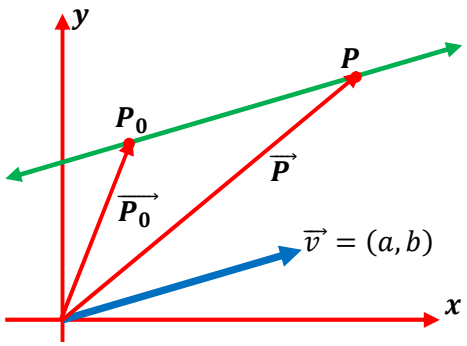
   (A) 20 (B) 5 (C) 15 (D) 30

Answers

1. A 2. B 3. B 4. B 5. C 6. D 7. A 8. A

Vectors and Lines

Consider a line l passing through the point $P_0 = (x_0, y_0)$ and parallel to the vector $\vec{v} = (a, b)$. Suppose $P = (x, y)$ is any arbitrary point on l (as shown in the figure).



$\overrightarrow{P_0} + \overrightarrow{P_0P} = \overrightarrow{P}$

$\therefore\ \overrightarrow{P_0P} = \overrightarrow{P} - \overrightarrow{P_0}$

$l \parallel \vec{v}$

$\Rightarrow \overrightarrow{P_0P} \parallel \vec{v}$

$\Rightarrow \overrightarrow{P_0P} = t\,\vec{v}$ for some $t \in \mathbb{R}$

$\Rightarrow \overrightarrow{P} - \overrightarrow{P_0} = t\,\vec{v}$

$\Rightarrow \overrightarrow{P} = \overrightarrow{P_0} + t\,\vec{v}$

The above is Vector Equation of l.

$\overrightarrow{P} = \overrightarrow{P_0} + t\,\vec{v}$

$\Rightarrow (x, y) = (x_0, y_0) + t(a, b)$

$\Rightarrow (x, y) = (x_0, y_0) + (ta, tb)$

$\Rightarrow (x, y) = (x_0 + ta, y_0 + tb)$

Hence,

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

The above is called Parametric Equation. The parameter of the Equation is t.

Solving each equation, in the above system of equations, for t we get;

$\frac{x-x_0}{a} = t \;\&\; \frac{y-y_0}{b} = t;\; ab \neq 0$

$\Rightarrow \frac{y-y_0}{b} = \frac{x-x_0}{a}$ or

$y - y_0 = \frac{b}{a}(x - x_0)$ for $a \neq 0$

The above is called Standard Equation of l.

If $a = 0$, the standard equation is $x = x_0$

Example – 23

A line l passes through the point $(3, 2)$ and is parallel to the vector $\vec{v} = (1, 2)$. Find

(A) the vector equation of l. (B) the parametric equation of l. (C) the standard equation of l.

Solution

Given: $P_0 = (3, 2)$

$\vec{v} = (1, 2)$

Let $P = (x, y)$ be a point on l.

(A) $\vec{P} = \vec{P_0} + t\vec{v}, t \in \mathbb{R}$

$(x, y) = (3, 2) + t(1, 2)$

(B) $(x, y) = (3, 2) + t(1, 2)$

$(x, y) = (3 + t, 2 + 2t)$

$\therefore \begin{cases} x = 3 + t \\ y = 2 + 2t \end{cases}$

(C) $x = 3 + t \Rightarrow x - 3 = t$

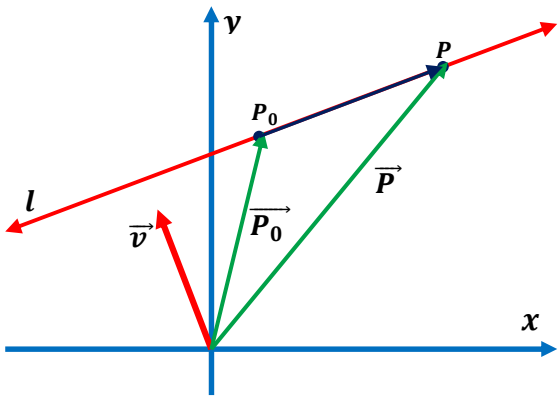
$y = 2 + 2t \Rightarrow \frac{y-2}{2} = t$

$\therefore \frac{y-2}{2} = x - 3$

$\Rightarrow y - 2 = 2x - 6$

$\Rightarrow y = 2x - 4$

Lines Perpendicular to a Given Vector



Let l be a line passing through $P_0 = (x_0, y_0)$ and perpendicular to the vector $\vec{v}$ as shown in the adjoining figure. Take any arbitrary point $P = (x, y)$ on l.

$\therefore \vec{P_0} + \vec{P_0P} = \vec{P}$

$\Rightarrow \vec{P_0P} = \vec{P} - \vec{P_0}$

Since $l \perp \vec{v}$, then

$\vec{v} \perp \vec{P_0P}$

$\Rightarrow \vec{v} \cdot \vec{P_0P} = 0$

$\Rightarrow \vec{v} \cdot (\vec{P} - \vec{P_0}) = 0$

The above is vector Equation of l

$\therefore (a, b) \cdot [(x, y) - (x_0, y_0)] = 0$

$\Rightarrow (a, b) \cdot (x - x_0, y - y_0) = 0$

$\Rightarrow a(x - x_0) + b(y - y_0) = 0$

$\Rightarrow ax + by = ax_0 + by_0$

The above is standard equation of l.

Example – 24

1. Let $\vec{v} = (3, 2)$. Find the equation of the line l that passes through $(1, 5)$ and which is also perpendicular to the vector $\vec{v}$.
2. Find the vector equation of the line through the two points $(-1, -3)$ *and* $(1, 1)$.
3. The vector equation of a line l is: $(x, y) = (-1, 1) + t(1, 2)$. Determine whether each of the following points lie on l or doesn't. (A) $P = (2, 7)$ (B) $P = (3, 1)$
4. Using the same line in number 3, find the coordinates of the point on l when the parameter t equals:

(A) 0 (B) -4 (C) 3

Solution

1. Given

✓$\vec{v} = (3, 2) = (a, b)$

✓$l \perp \vec{v}$

✓$(1, 5) = (x_0, y_0)$ *is on* l

Equation of l

$ax + by = ax_0 + by_0$

$\Rightarrow 3x + 2y = 3(1) + 2(5)$

$\Rightarrow 3x + 2y = 13$

2. Let $P_0 = (-1, -3)$

$Q = (1, 1)$

then $\vec{v} = \vec{P_0Q} = (2, 4)$

$\Rightarrow \vec{v} \parallel l$

If $P = (x, y)$ is any arbitrary point on l, then $\vec{P} = \vec{P_0} + t\vec{v}$, this becomes $(x, y) = (-1, -3) + t(2, 4)$

If we take $P_0 = (1, 1)$ the vector equation becomes $(x, y) = (1, 1) + t(2, 4)$

3. We can solve this problem in two ways; (i) using the given vector equation itself. (ii) by first change the given vector equation to standard form.

Given

✓ $(x, y) = (-1, 1) + t(1, 2)$

(A)Given: $P = (2, 7)$

If P lies on l the position vector $\vec{P}$ satisfies the given vector equation:

$\vec{P} = (-1, 1) + t(1, 2)$

$\Rightarrow (2, 7) = (-1, 1) + t(1, 2)$

$\Rightarrow (2, 7) = (-1 + t, 1 + 2t)$

$\Rightarrow 2 = -1 + t \ \wedge 7 = 1 + 2t$

$\Rightarrow t = 3 \ \wedge t = 3 \Rightarrow t = 3$

$\therefore P = (2, 7)$ *is on* l

Method II

First write the given equation in standard form.

$(x, y) = (-1, 1) + t(1, 2)$

$\Rightarrow x = t - 1 \wedge y = 2t + 1$

$\Rightarrow \dfrac{y - 1}{2} = x + 1$

$\Rightarrow y = 2x + 3$

now check whether $P = (2, 7)$ satisfies the above equation

$y = 2x + 3$

$\Rightarrow 7 = 2(2) + 3$

$\Rightarrow 7 = 7$ *True*

$\therefore P$ *lies on* l

(B) Given: $P = (3, 1)$

Method II

$y = 2x + 3 \Rightarrow 1 = 2(3) + 3$

$\Rightarrow 1 = 7$ *False*

$\therefore P = (3, 1)$ doesn't lie on l

Answers

4. (A) $(-1, 1)$ (B) $(-5, -7)$ (C) $(2, 7)$

5. Let $\vec{v} = (3, 2)$. Find the equation of the line l that passes through $(1, 5)$ and which is also perpendicular to the vector $\vec{v}$.

Solution

5. Given

✓$\vec{v} = (3, 2) = (a, b)$

✓$P_0 = (1, 5) = (x_0, y_0)$

✓$\vec{v} \perp l$

$\therefore$ *the equation of* l *will be*:

$ax + by = ax_0 + by_0$

$3x + 2y = 3(1) + 2(5)$

$3x + 2y = 13$

NATIONAL EXAMINATION QUESTIONS

1. A line is given by the vector equation $(x, y) = (-t, 6 + 2t)$, is tangent to a circle at point $(1, 4)$. What is the radius of the circle if its center is on the $y - axis$?

(A) $\sqrt{5}$ (B) $\frac{1}{2}\sqrt{5}$ (C) $2\sqrt{5}$ (D) $\sqrt{10}$

2. What is the work done (in Joules) when a force of 50 Newton is used to pull a crate 20 meters long along a level path , if the force is at an angle of 60^0 ? (Remember that $Work = Force \times Distance = \vec{F} \bullet \vec{S}$)

(A) 360 (B) 500 (C) 760 (D) $1500\sqrt{2}$

3. What is the equation of a line that passes through the point $(-1, 2)$ and parallel to the vector $(1, -1)$?

(A) $2x - y = 1$ (B) $x + y - 1 = 0$ (C) $x - 2y = 3$ (D) $y - 2x + 1 = 0$

4. If l is the line passing through $(0, 2)$ and parallel to $\vec{v} = \vec{\imath} + 3\vec{\jmath}$, which of the following is true about l and and the circle $(x - 2)^2 + (y - 1)^2 = 5$?

(A) l is tangent to the circle at $(0, 2)$

(B) l is tangent to the circle at some point P, where $P \neq (0, 2)$

(C) l intersects the circle at two distinct points

(D) The distance between l and the center of the circle is greater than $\sqrt{5}$

5. A line given by the vector equation $r(t) = (0, 3) + t(1, 1)$ is tangent to a circle at point $(0, 3)$. If the radius of the circle is $\sqrt{2}$, which of the following is the center of the circle ?

(A) $(1, 4)$ (b) $(1, -4)$ (C) $(-1, 2)$ (D) $(1, 2)$

Answers : 1. B 2. B 3. B 4. C 5. D

TRANSFORMATION OF THE PLANE

Transformation is a motion that maps a plane onto itself. That is a transformation takes every point of the plane to a point of the plane.

We can use transformation to solve many problems and also to describe difficulties in simpler forms.

Example-25

1. Given a circle with center at the origin , by moving this circle 2 $units$ to the $right$ and 3 $units$ down we get another Circle with same radius but whose center is at $(2,-3)\backslash$
2. Given a horizontal ellipse, by rotating every point of the ellipse through 30^0 about the origin in clock wise direction we get another ellipse whose axis are neither horizontal nor vertical

NOTE

- Some transformations maintain direction , some do not.
- Some transformations preserve distance but some do not.

NOTE

Transformations can be classified in to two categories as

- Rigid Motion : is motion that preserves distance
- Non-rigid Motion : is motion that doesn't preserve distance

Notations

If T is a transformation that moves the point $P=(x,y)$ to the ***image point*** $P'=(x',y')$, then we can use any of the following notations to express this fact.

- $T: P \to P'$ .....()
- $T:(x,y) \to (x',y')$
- $T((x,y))=(x',y')$ .....(T is a function of two variables)
- T *maps point* P *to point* P'
- *Image of* P *under* T *is* P'
- T *maps point* (x,y) *to point* (x',y')
- *Image of* (x,y) *under* T *is* (x',y')

NOTE

When do we say that a transformation preserves distance ?

Let T be a transformation. Then T is said to preserve distance if and only if for any two points P & Q, whose images under T are P' & Q' respectively $PQ=P'Q'$. That is, if distance between any two pre-image points equals to distance between their images.

Example-26

1. Let T be a transformation defined by $T((x,y))=(2x,2y)$
   (A) What is the image P', of the point $P=(-1,3)$ under T ?
   (B) What is the image Q', of the point $Q=(2,-1)$ under T ?
   (C) Find the distance between P & Q. (D) Find the distance between P' & Q'
   (E) Is $PQ=P'Q'$? (F) Is T a rigid motion ?
2. Let T be a transformation defined by $T((x,y))=(x-3,y+4)$
   (A) What is the image P', of the point $P=(2,-5)$ under T ?
   (B) What is the image Q', of the point $Q=(-3,7)$ under T ?
   (C) Find the distance between P & Q. (D) Find the distance between P' & Q' (F) Is T a rigid motion ?

Solution

1. Given : $T((x,y)) = (2x, 2y)$

(A) Given : $P = (-1, 3)$

$\therefore\ T((x,y)) = (2x, 2y)$

$\Rightarrow T((-1,3)) = (2(-1), 2(3))$

$= (-2, 6) = P'$

(B) Given : $Q = (2, -1)$

$\therefore\ T((x,y)) = (2x, 2y)$

$\Rightarrow T((2,-1)) = (2(2), 2(-1))$

$= (4, -2) = Q'$

(C) $P = (-1, 3)$ & $Q = (2, -1)$

$\therefore PQ = \sqrt{(2+1)^2 + (-1-3)^2}$

$= 5$

(D) $P' = (-2, 6)$ & $Q' = (4, -2)$

$\therefore P'Q' = \sqrt{(4+2)^2 + (-2-6)^2}$

$= 10$

(E) No, $PQ \neq P'Q'$

(F) *T is not a rigid motion*

2. Given: $T((x,y)) = (x-3, y+4)$

(A) Given : $P = (2, -5)$

$T((x,y)) = (x-3, y+4)$

$\Rightarrow T((2,-5)) = (2-3, -5+4)$

$= (-1, -1) = P'$

(B) Given : $Q = (-3, 7)$

$T((x,y)) = (x-3, y+4)$

$\Rightarrow T((-3,7)) = (-3-3, 7+4)$

$= (-6, 11) = Q'$

(C) $P = (2, -5)$ & $Q = (-3, 7)$

$\therefore PQ = \sqrt{(-3-2)^2 + (7+5)^2}$

$= 13$

(D) $P' = (-1, -1)$ & $Q' = (-6, 11)$

$\therefore PQ = \sqrt{(-6+1)^2 + (11+1)^2}$

$= 13$

(E) In C and D we got that $PQ = P'Q'$. But in order to conclude that T is a rigid motion we need to give a general proof.

Let $P = (a, b)$ & $Q = (c, d)$ be any two points in plane with images P' & Q', respectively. Then

$P' = T((a,b)) = (a-3, b+4)$

$Q' = T((c,d)) = (c-3, d+4)$

$PQ = \sqrt{(c-a)^2 + (d-b)^2}$ ..(*)

$P'Q'$

$= \sqrt{(c-3-a+3)^2 + (d+4-b-4)^2}$

$= \sqrt{(c-a)^2 + (d-b)^2} = PQ$

$\therefore PQ = P'Q'$

Hence *T is a rigid motion*

Three rigid motions are discussed in this unit: Namely;

(1) Translation (or Parallel Motion) (2) Reflection and (3) Rotation

Translation

Definition

A translation is a mapping of a plane onto itself (or a function from a plane onto itself) in such a way that every point of the plane is moved through the same distance and in the same direction. A translation is usually denoted by T .

If a translation T maps the point $P = (x, y)$ to the point $P' = (x', y')$, that is $T: P(x, y) \rightarrow P'(x', y')$, then the vector $\overrightarrow{PP'} = (x' - x, y' - y)$ is called ***translation vector*** .

If $\vec{v} = (a, b)$ is a translation vector, then it moves every point through $|a|$ ***units horizontally*** and $|b|$ ***units vertically***. If $a > 0$, it moves the point to the right. If $a < 0$, it moves the point to the left. Similarly, if $b > 0$ then it moves the point upward . If $b < 0$ it moves down.

PROBLEM-I Given Pre-image and Image to Find the Translation Vector.

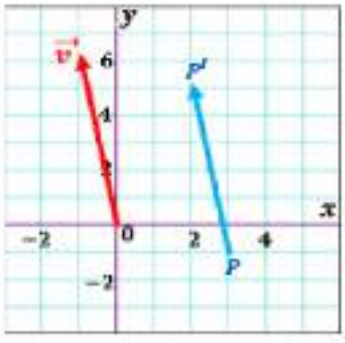
Example-27

A translation T moves P to P'. Find the translation vector $\vec{v}$. Then draw the vectors $\vec{v}$ *and* $\overrightarrow{PP'}$

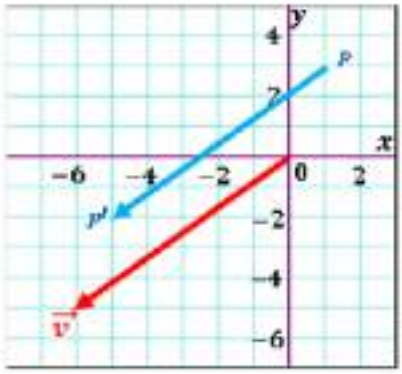
1. $P = (3, -1)$ *and* $P' = (2, 5)$ 2. $P = (1.3)$ *and* $P' = (-5, -2)$

Solution

1. $\vec{v} = \overrightarrow{PP'} = (2-3, 5+1) = (-1, 6)$



2. $\vec{v} = \overrightarrow{PP'} = (-5-1, -2-3) = (-6, -5)$



PROBLEM-II Given Translation Vector and Pre-image Point to Find The Image Point

NOTE

Let $\vec{v} = (a, b)$ be a translation vector of a translation T and $P = (x, y)$ a pre-image point. Then, the image P' of the point P is computed by the formula

$$T: P(x, y) \xrightarrow{\vec{v} = (a, b)} P'(x + a, y + b)$$

Or $T\big((x, y)\big) = (x + a, y + b)$

That is, we simply add the corresponding components of the pre-image and the vector to get the coordinates of the image point.

Example-28

1. Let T be a translation by the vector $\vec{v} = (-2, 5)$. Find
   (A) image of the point $P = (3, -1)$ (B) $T(Q)$,where $Q = (2, -5)$ (C) $T(A)$, where $A = (4, 2)$
2. Using diagram show how T moves the points P, Q *and* A in number 1
3. Does T move the three points through the same distance and in the same direction ?

Solution

1. Given : $\vec{v} = (-2, 5)$

(A) Given : $P = (3, -1)$

$\therefore P' = T(P)$

$= T\big((3, -1)\big)$

$= (3 + -2, -1 + 5)$

$= (1, 4)$

(B) Given : $Q = (2, -5)$

$\therefore Q' = T(Q)$

$= T\big((2, -5)\big)$

$= (2 + -2, -5 + 5)$

$= (0, 0)$

(C) Given : $A = (4, 2)$

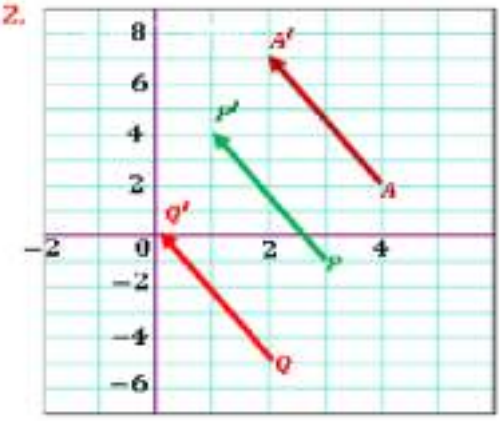
$\therefore A' = T(A)$

$= T\big((4, 2)\big)$

$= (4 + -2, 2 + 5)$

$= (2, 7)$

2.



3. Yes, T moves each of the three points through the same distance and in the same direction.

$$distance = \left\|\overrightarrow{PP'}\right\| = \left\|\overrightarrow{QQ'}\right\| = \left\|\overrightarrow{AA'}\right\| = \sqrt{29}$$

Every point is moved 2 *units to the left* and 5 *units upward.*

PROBLEM-III Given Image and Translation Vector to Find The Pre-image Point

NOTE

Let $\vec{v} = (a, b)$ be a translation vector of a translation T. Let also $P' = (x', y')$ be the image of the point P. If the coordinates of $P = (x, y)$, then

$P' = T(P)$
$P' = T((x, y))$
$\Rightarrow P' = (x + a, y + b)$
But $P' = (x', y')$

$\therefore x' = x + a$
$\Rightarrow x = x' - a$
Similarly, $y = y' - b$
$\therefore P = (x' - a, y' - b)$

Hence, in order to get the coordinates of the pre-image we simply subtract the components of the vector from the corresponding components of the image point

Example-29

Let T be a translation by the vector $\vec{v} = (5, -3)$ and P' be the image of the point P under T. Find the coordinates of the point P, if P equals to (1) $(6, 5)$ (2) $(-3, -4)$

Answers : (1) $P = (1, 8)$ (2) $P = (-8, -1)$

PROBLEM-IV Combination of Translations

NOTE

Let T_1 and T_2 be translations whose translation vector s are $\vec{v}_1 = (a, b)$ & $\vec{v}_2 = (c, d)$, respectively. The image of the point $P = (x, y)$ after it is transformed with T_1 followed by T_2 can be obtained using the following three methods.

Method – I : Step by Step Method

Step – 1 : Find image of P under T_1.
Let P' be the image of $P = (x, y)$ under T_1
Then $P' = T_1((x, y))$
$\Rightarrow P' = (x + a, y + b)$

Step - 2 : Find the image of P' under T_2
Let P'' be the desired image of P'
$\therefore P'' = T_2(P') = T_2((x + a, y + b))$
$= (x + a + c, y + b + d)$

Method – II : Composition Method

In Method I, to find the image of P'' of the point $P = (x, y)$ we did as follows :
Step – 1 : $P' = T_1(P)$
Step – 2 : $P'' = T_2(P')$
$\Rightarrow P'' = T_2(T_1(P))$
$\Rightarrow P'' = (T_2 \circ T_1)(P)$

To get the image of P when transformed by
(A) T_1 followed by T_2
$P'' = (T_2 \circ T_1)(P)$
(B) T_2 followed by T_1
$P'' = (T_1 \circ T_2)(P)$

Method – III : Shortest Method

For translations T_1 and T_2 with vectors $\vec{v}_1$ & $\vec{v}_2$, respectively ; the image P'' obtained after P is transformed by T_1 followed by T_2 is given by
$P'' = (x - comp, y - comp)$, where

$x - comp = sum\ of\ x - components\ of\ P, \vec{v}_1\ and\ \vec{v}_2$
$y - comp = sum\ of\ y - components\ of\ P, \vec{v}_1\ and\ \vec{v}_2$

Example-30

1. Let T_1 & T_2 be translations by vectors $\vec{v}_1 = (2,-1)$ *and* $\vec{v}_2 = (-5,3)$, respectively. Find the image of the point $P = (6,5)$ after it is transformed by the transformations T_1 *followed by* T_2 :
   (A) using Method-I (Step by Step Method)
   (B) using Method-II (Composition Method)
   (C) using Method-III (Shortest Method)

2. Considering the translations T_1 & T_2 and the point $P = (6,5)$ given in number 1, find the image of the point P after it is mapped by the translations T_2 *followed by* T_1 using Composition Method.

3. (A) Are the results in number s 1 and 2 the same ? (B) What does this indicate ?

Solutions

1. Given :
✓$\vec{v}_1 = (2,-1)$
✓$\vec{v}_2 = (-5,3)$
✓$P = (6,5)$

(A) Method – I
Step-1:
$P' = T_1(P)$
$= T_1((6,5))$
$= (6+2, 5+-1)$
$= (8,4)$

Step – 2
$P'' = T_2(P')$
$= T_2((8,4))$
$= (8+-5, 4+3)$
$= (3,7)$

(B) Method-II
$P'' = T_2(T_1(P))$
$= T_2(T_1((6,5)))$
$= T_2((6+2, 5-1))$
$= T_2((8,4))$
$= (8-5, 4+3)$
$= (3,7)$

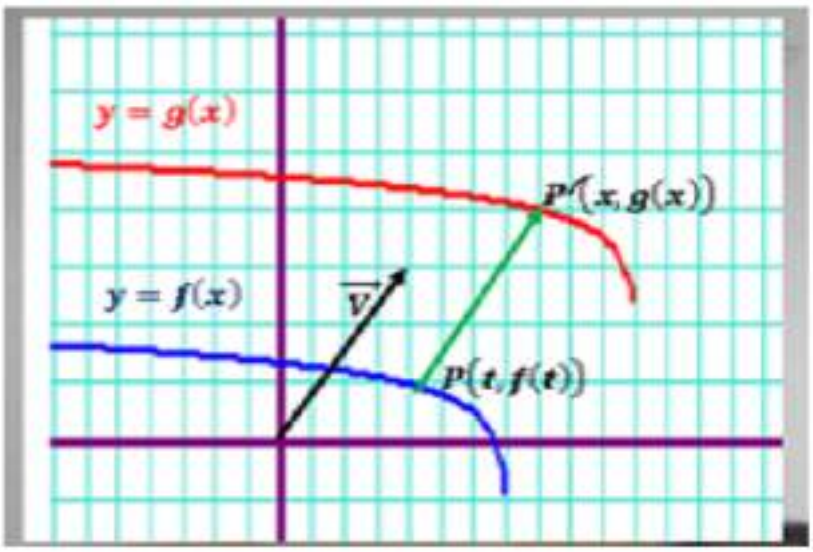
(C) Method-III
x-components
of $\vec{v}_1, \vec{v}_2$ & $P = 2,-5,6$
y-components
of $\vec{v}_1, \vec{v}_2$ & $P = -1,3,5$
$\therefore\ P'' = (2-5+6, -1+3+5)$
$= (3,7)$

2. $P'' = (T_1 \circ T_2)(P)$
$= T_1(T_2(P))$
$= T_1(T_2((6,5)))$
$= T_1((6-5, 5+3))$
$= T_1((1,8))$
$= (1+2, 8-1)$
$= (3,7)$

3. (A) Yes , the results are the same.

(B) $T_1 \circ T_2 = T_2 \circ T_1$

Images of Functions Under Translation



In the figure, $y = g(x)$ is the image of $y = f(x)$ under a translation T by the vector $\vec{v} = (a,b)$.

Point $P = (t, f(t))$ is on the graph of the function $y = f(x)$, and its image $P' = (x, g(x))$ is on the graph of the function $y = g(x)$

(See on next page)

Since $\vec{v} = (a, b)$
$P' = T(P)$
$\Rightarrow (x, g(x)) = T\big((t, f(t))\big)$

$\Rightarrow (x, g(x)) = (t + a, f(t) + b)$
$\Rightarrow x = t + a \;\&\; g(x) = f(t) + b$
But $x = t + a \Rightarrow t = x - a$

$\therefore g(x) = f(t) + b$ becomes
$y = f(x - a) + b$
$\Rightarrow y - b = f(x - a)$

The above means, to find the image of the function $y = f(x)$ under a translation T by the vector $\vec{v} = (a, b)$ we simply subtract b *from* y and we subtract a *from* x, then finally we simplify the resulting equation.

Example-31

1. Find the image of the function $y = 2x^2 - 7x + 3$ under the translation T by the vector $\vec{v} = (-2, 3)$

Solution

Given :
✓Pre-image: $y = 2x^2 - 7x + 3$
✓Vector : $\vec{v} = (-2, 3)$
Required : Image function
Pre-image: $y = 2x^2 - 7x + 3$

$\Rightarrow$ *Image*:
$y - 3 = 2(x + 2)^2 - 7(x + 2) + 3$
$\Rightarrow y - 3 = 2x^2 + 8x + 8 - 7x - 11$
$\Rightarrow y = 2x^2 + x$

2. In question number 1:
   - the given pre-image function is : $y = 2x^2 - 7x + 3$
   - the given vector is : $\vec{v} = (-2, 3)$
   - the image function obtained is : $y = 2x^2 + x$

   (A) Does the point $P = (1, -2)$ lie on the graph of the given pre-image function: $= 2x^2 - 7x + 3$?
   (B) What are the coordinates of the image P' of the point P (in A) under the translation ?
   (C) Does the image point P' lie on the graph of the image function $y = 2x^2 + x$?

Solution

(A) $P = (1, -2)$
$\quad x \quad y$
$y = 2x^2 - 7x + 3$
$\Rightarrow -2 = 2(1^2) - 7(1) + 3$
$\Rightarrow -2 = -2 \equiv True$
$\therefore P$ satisfies the equation
Hence , the point P lies on the pre-image function

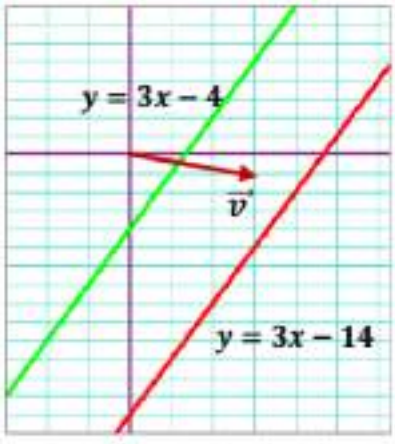
(B) $P' = T(P)$
$= T\big((1, -2)\big)$
$= (1 - 2, -2 + 3)$
$= (-1, 1)$

(C) $P' = (-1, 1)$
Image-function :

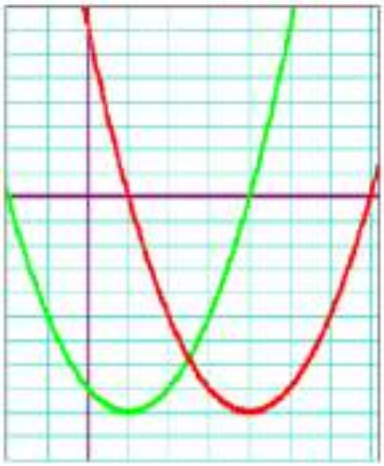
$y = 2x^2 + x$
$1 = 2\big[(-1)^2\big] + -1$
$\Rightarrow 1 = 2 - 1$
$\Rightarrow 1 = 1 \equiv True$
$\therefore P'$ lies on the graph of the image function

3. Find the images of each of the following functions under the translation T by the vector $\vec{v} = (3, -1)$
   (A) $y = 3x - 4$ (B) $2y - 3x + 3 = 0$ (C) $y = x^2 - 2x - 8$
4. For each question letter of number 3, sketch the graph of the given function, the graph of its image and the translation vector in the same coordinate plane

4. A)



C)

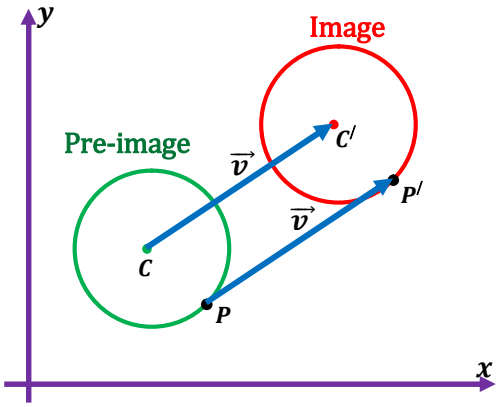


Images of Conic Sections Under Translation

Images of Circles

The image of a circle under a translation can be obtained in the following two ways.

Method - I



Sliding (or translating) a circle doesn't change its size and shape. Hence, the radius of the image circle equals to the radius of the pre-image circle. But their centers are distinct points (i.e., C & C'). C' is the image of the center C under the same translation used to move the circle.

How do we get the equation of the image circle if we are given the equation of the pre-image circle and the translation vector ? Getting the equation of any circle is very free from any pain if we know the length of its radius and the coordinates of its center. Hence, in order to get the equation of the image circle we need first to find its radius (r') and the coordinates of its center , $C' = (h', k')$. Below are listed few steps we may follow to obtain the equation of the image circle.

Given : (i) Equation of pre-image circle

(ii) Translation vector : $\vec{v} = (m, n)$

Aim : To find equation of image circle

Steps ***Activities***

1. ..... Find the center $C = (h, k)$ and $radius = r$ of the pre – image circle.
2. ..... Radius of image circle : $r' = r$
3. ..... Center of image circle : $C' = T(C) = T\big((h, k)\big) = (h + m, k + n)$
4. ..... Now find the equation of the image circle :

$$(x - h - m)^2 + (y - k - n)^2 = r^2$$

Example-32

Find the equation of the image of each of the following circles under a translation T whose translation vector is $\vec{v}(2, 4)$

1. $(x - 3)^2 + (y + 1)^2 = 16$ 2. $x^2 + y^2 + 4x - 6y + 4 = 0$

Solution

Vector : $\vec{v} = (2, 4)$

1. Pre-image circle

$(x - 3)^2 + (y + 1)^2 = 16$

$(x - h)^2 + (y - k)^2 = r^2$

Center:

$C = (h, k) = (3, -1)$

Radius : $r = \sqrt{16} = 4$

Image Circle

Radius : $r' = r = 4$

Center :

$C' = T(C)$

$= T\big((3, -1)\big)$

$= (3 + 2, -1 + 4)$

$= (5, 3)$

Hence equation of image circle will be

$(x - 5)^2 + (y - 3)^2 = 16$

2. Review Note

> If equation of a circle is $x^2 + y^2 + Dx + Ey + F = 0$, where $D, E, F \in \mathbb{R}$, then its center $C = \left(-\frac{D}{2}, -\frac{E}{2}\right)$ and $r = \sqrt{\left(-\frac{D}{2}\right)^2 + \left(-\frac{E}{2}\right)^2 + -F}$

Pre-image circle

$x^2 + y^2 + 4x - 6y + 4 = 0$

Center :

$C = \left(-\frac{4}{2}, \frac{6}{2}\right) = (-2, 3)$

Radius

$r = \sqrt{(-2)^2 + 3^2 - 4} = 3$

Image Circle

Radius : $r' = r = 3$

Center :

$C' = T(C)$

$= T\big((-2, 3)\big)$

$= (-2 + 2, 3 + 4)$

$= (0, 7)$

Equation of Image

$(x - 0)^2 + (y - 7)^2 = 3^2$

$x^2 + y^2 - 14y + 49 = 9$

$x^2 + y^2 - 14y + 40 = 0$

Method II

Given the equation of a pre-image circle to obtain the equation of the pre-image circle and a translation vector $\vec{v} = (m, n)$ to compute the equation of the image circle, we simply insert $x - m$ in place of x and $y - n$ in place of y in the given equation of the pre-image circle. Finally, simplify the resulting equation.

Example-33

Compute the equation of the image of each of the following circles under a translation T by the vector $\vec{v} = (2, 4)$

1. $(x-3)^2 + (y+1)^2 = 16$ 2. $x^2 + y^2 + 4x - 6y + 4 = 0$

Solution

Given : $\vec{v} = (2, 4)$
We insert $x - 2$ in place of x and $y - 4$ in place of y
1. $(x-3)^2 + (y+1)^2 = 16$
Image:
$(x - 2 - 3)^2 + (y - 4 + 1)^2 = 16$

$(x-5)^2 + (y-3)^2 = 16$

2. $x^2 + y^2 + 4x - 6y + 4 = 0$
Image:
$(x-2)^2 + (y-4)^2 + 4(x-2) - 6(y-4) + 4 = 0$
$x^2 - 4x + 4 + y^2 - 8y + 16 + 4x - 8 - 6y + 24 + 4 = 0$
$x^2 + y^2 - 14y + 40 = 0$

Example-34

1. Find the vector of a translation that maps :
   (A) the circle $(x-2)^2 + (y-5)^2$ to the circle $(x+1)^2 + (y-11)^2 = 12$
   (B) the circle $x^2 + y^2 + 4x - 6y + 4 = 0$ to the circle $x^2 + y^2 - 14y + 40 = 0$

2. A translation T maps the line $y = 2x + 3$ to $y = 2x$. If the translation vector $\vec{v} = (m, n)$ and $m + n = 3$, then find (A) the value of the vector $\vec{v}$ (B) the image of the point $(6, -8)$ under this translation.

Solution

1. (A) Given
✓Pre-image:
$(x-2)^2 + (y-5)^2 = 12$
✓Image :
$(x+1)^2 + (y-11)^2 = 12$
Pre-image center:
$C = (2, 5)$
Image Center :
$C' = (-1, 11)$
T maps C to C'

$C = (2,5) \xrightarrow{\vec{v} = (m.n)} C' = (-1, 11)$
$m + 2 = -1 \Rightarrow m = -3$
$n + 5 = 11 \Rightarrow n = 6$
$\therefore \vec{v} = (-3, 6)$

B) Given
✓Pre-image
$x^2 + y^2 + 4x - 6y + 4 = 0$

✓Image
$x^2 + y^2 + 0x - 14y + 40 = 0$
Pre-image center
$C = \left(-\frac{4}{2}, \frac{6}{2}\right) = (-2, 3)$
Image center
$C' = \left(\frac{0}{2}, \frac{14}{2}\right) = (0, 7)$
T maps C to C'

$C = (-2, 3) \xrightarrow{\vec{v} = (m,n)} C' = (0, 7)$
$\therefore m - 2 = 0 \Rightarrow m = 2$
$n + 3 = 7 \Rightarrow n = 4$
$\therefore \vec{v} = (2, 4)$

2. Given:
✓$\vec{v} = (m, n)$
✓Pre-image : $y = 2x + 3$
✓Image: $y = 2x$
✓$m + n = 3$

Image of $y = 2x + 3$ is
$y - n = 2(x - m) + 3$
$\Rightarrow y = 2x - 2m + n + 3$
This line must be the same as the given image line $y = 2x$
$\therefore -2m + n + 3 = 0$
$\Rightarrow 2m - n = 3$
Solve this equation simultaneously with the given equation
$\begin{cases} 2m - n = 3 \\ m + n = 3 \end{cases}$ add the equations
$3m = 6 \Rightarrow m = 2$
$m + n = 3 \Rightarrow n = 1$
$\therefore \vec{v} = (2, 1)$

(B) $T((6, -8)) = (6 + 2, -8 + 1)$
$= (8, -7)$

NATIONAL EXAM QUESTIONS

1. What is the image of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$, when the origin is shifted to the point $(1, 1)$ after translation of axes ?
   (A) $x^2 + y^2 - 4x - 2y + 3 = 0$
   (B) $x^2 + y^2 - 4x - 6y + 3 = 0$
   (C) $x^2 + y^2 + 6x + 8y - 23 = 0$
   (D) $x^2 + y^2 - 6x - 8y + 23 = 0$

2. What is the standard equation of the line passing through the point $(2,3)$ and parallel to the line whose parametric equation is given by:

$$\begin{cases} x = 1 + 2\lambda \\ y = -2 - \lambda \end{cases}, \lambda \in \mathbb{R} \ ?$$

(A) $\frac{x-2}{2} = \frac{y-3}{-1}$ (B) $\frac{x-2}{-1} = \frac{y-3}{2}$ (C) $\frac{x-1}{2} = \frac{y-3}{-2}$ (D) $\frac{x-1}{1} = \frac{y-3}{-2}$

3. If a translation T takes the circle $x^2 + y^2 - 2x + 6y + 3 = 0$ into the circle whose equation is $(x+2)^2 + (y-4)^2 = 7$, then what is the image of the origin under T ?

(A) $(-3,7)$ (B) $(1,2)$ (C) $(1,-3)$ (D) $(-2,4)$

4. Which of the following is a vector equation of the line tangent to the circle $x^2 + y^2 + 2x - 7 = 0$ at $(1,2)$?

(A) $(x,y) = (0,3) + \lambda(-1,2)$ (C) $(x,y) = (0,3) + \lambda(1,-1)$
(B) $(x,y) = (1,2) + \lambda(2,-1)$ (D) $(x,y) = (1,2) + \lambda(-1,2)$

5. A translation takes $x^2 + (y+1)^2 = 5$ to $(x-2)^2 + y^2 = 5$. What is the image of the point $(1,3)$ under this translation ?

(A) $(3,4)$ (B) $(3,2)$ (C) $(-1,2)$ (D) $(-2,3)$

Answers : 1. D 2. A 3. A 4. C 5. A

Images of Other Conic Sections

Given equation of the pre-image conic section and a translation vector $\vec{v} = (m,n)$ to find the equation of the image conic section we can employ different methods.

Method – I

To get the equation of the image conic section, we simply insert $x - m$ in place of x and $y - k$ in the place of y in the equation of the pre-image conic section and then simplify the resulting equation.

Example-35

Find the equation of the image of each conic section given below under the translation T by the vector $\vec{v} = (-1,3)$

1. $\frac{(x+2)^2}{3} - \frac{(y-1)^2}{6} = 1$ 2. $(y+4)^2 = 8(x-3)$ 3. $x^2 + 4y^2 - 4x + 16y + 16 = 0$
4. $y^2 - 3x^2 - 18x - 2y - 27 = 0$

Solutions

Given : $\vec{v} = (-1,3)$
(m,n)

1. Pre-image :
$\frac{(x+2)^2}{3} - \frac{(y-1)^2}{6} = 1$
$\therefore$ ***Image*** =
$\frac{(x-m+2)^2}{3} - \frac{(y-n-1)^2}{6} = 1$
$\Rightarrow \frac{(x+1+2)^2}{3} - \frac{(y-3-1)^2}{6} = 1$
$\Rightarrow \frac{(x+3)^2}{3} - \frac{(y-4)^2}{6} = 1$

2. Pre-image :
$(y+4)^2 = 8(x-3)$
$\therefore$ ***Image*** =
$(y-n+4)^2 = 8(x-m-3)$
$\Rightarrow (y-3+4)^2 = 8(x+1-3)$
$\Rightarrow (y+1)^2 = 8(x-2)$

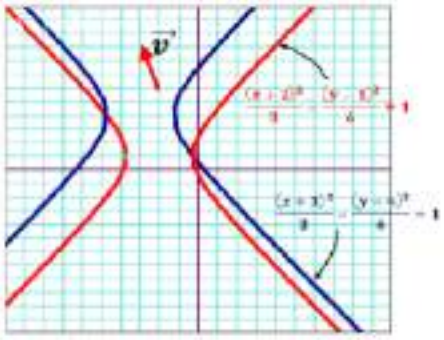
3. Pre-image :
$x^2 + 4y^2 - 4x + 16y + 16 = 0$

$\therefore$ ***Image*** =
$(x-m)^2 + 4(y-n)^2 - 4(x-m) + 16(y-n) + 16 = 0$
$\Rightarrow (x+1)^2 + 4(y-3)^2 - 4(x+1) + 16(y-3) + 16 = 0$
$\Rightarrow x^2 + 2x + 1 + 4(y^2 - 6y + 9) - 4x - 4 + 16y - 48 + 16 = 0$
$\Rightarrow x^2 + 4y^2 - 2x - 8y + 1 = 0$

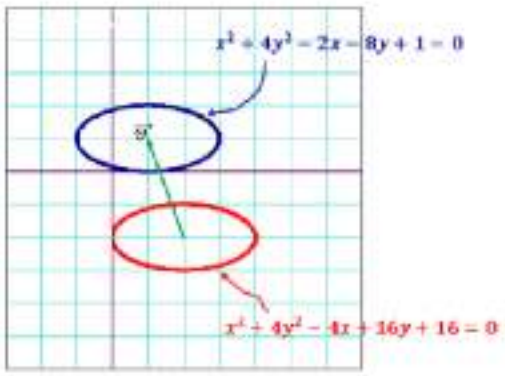
3. Pre-image :
$y^2 - 3x^2 - 18x - 2y - 27 = 0$
$\therefore$ ***Image*** =
$(y-n)^2 - 3(x-m)^2 - 18(x-m) - 2(y-n) - 27 = 0$
$\Rightarrow (y-3)^2 - 3(x+1)^2 - 18(x+1) - 2(y-3) - 27 = 0$
$\Rightarrow y^2 - 6y + 9 - 3(x^2 + 2x + 1) - 18x - 18 - 2y + 6 - 27 = 0$
$\Rightarrow y^2 - 3x^2 - 24x - 8y - 33 = 0$

Study the graphs of the pre-image and image conic section of #1 and 3 on next page

1.



3.



Method – II (For each of the remaining three conic sections)

Images of Parabolas

NOTE

1. A vertical parabola is translated to a vertical parabola. Similarly, a horizontal parabola is translated to a horizontal parabola.
2. A parabola open to the right is translated to a parabola open to the right and the same is true for the parabola open to the left.
3. A parabola open upward is translated to a parabola open upward . Similarly, a parabola open down is translated to a parabola open down.

Given a translation vector $\vec{u} = (m, n)$ and the equation of a pre-image parabola , to get the equation of the image parabola we do as follows.

Steps ***Activities***

1.....First find the focal length p and the coordinates of the vertex V of the pre-image parabola.

2.....Determine the type of and the direction towards which the pre-image parabola is opened.

3.....Since translation preserves distance the focal length p' of the image parabola $= p$

4.....Then, translate the vertex V of the pre-image parabola in order to get the vertex V' of the image parabola. Hence, $V' = T(V)$

5.....Finally compute the equation of the image parabola using your results in the above steps.

Example-36

Find the equation of the image of the parabola given below under the translation T by the vector $\vec{u} = (-1, 3)$

1. $(y+4)^2 = 8(x-3)$ 2. $x^2 - 4x + 12y + 40 = 0$

Solutions

Given : $\vec{u} = (-1, 3)$

1. Pre-image : $(y+4)^2 = 8(x-3)$

Type of parabola :

Horizontal & open to the right

Vertes : $V = (3, -4)$

Focal length : $4p = 8 \Rightarrow p = 2$

Image Parabola

Focal lengh : $p' = p = 2$

Vertex : $V' = T(V)$

$= T((3, -4))$

$= (3 + -1, -4 + 3)$

$= (2, -1)$

$\therefore$ ***Equation of image***

$(y+1)^2 = 4(2)(x-2)$

$\Rightarrow (y+1)^2 = 8(x-2)$

2. Pre-image parabola :

$x^2 - 4x + 12y + 40 = 0$

Change the above to standard form

$x^2 - 4x + 12y + 40 = 0$

$\Rightarrow x^2 - 4x + 4 - 4 = -12y - 40$

$\Rightarrow (x-2)^2 = -12(y+3)$

Type of parabola :

vertical and open down

Focal length: $-4p = -12 \Rightarrow p = 3$

Vertex : $V = (2, -3)$

Image Parabola :

Focal length : $p' = p = 3$

Vertex : $V' = T(V)$

$= T((2, -3))$

$= (2 + -1, -3 + 3)$

$= (1, 0)$

$\therefore$ ***Equation of image***

$(x-1)^2 = -4(3)(y-0)$

$\Rightarrow x^2 - 2x + 12y + 1 = 0$

Images of Ellipses and Hyperbolas

In the previous parts we have discussed how to compute the equations of the images of circles and parabolas under translation. We will use the same methods discussed therein to get images of ellipses and hyperbolas.

Example-37

1. Find the equations of the image of each of the following under the translation T by the vector $\vec{u} = (-2, 2)$.

   (A) $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{3} = 1$ (B) $2x^2 + y^2 + 4x + 4y + 2 = 0$

2. A translation T moves the hyperbola $3x^2 - 4y^2 - 6x - 24y - 45 = 0$ to the hyperbola $3x^2 - 4y^2 + 6x - 8y - 13 = 0$. Find (A) the translation vector (B) $T((2,-5))$

Solutions

1. Given : $\vec{u} = (-2, 2)$

Method - I

Pre-image : $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{3} = 1$

Type : Horizontal

Center : $C = (1, -3)$

Semi transverse axis :

$a = \sqrt{4} = 2$

Semi conjugate axis :

$b = \sqrt{3}$

Image hyperboal

Type : Horizontal

Center : $C' = T(C)$

$= T((1,-3))$

$= (1 + -2, -3 + 2)$

$= (-1, -1)$

Semi transverse axis :

$a' = a = 2$

Semi conjugate axis :

$b' = b = \sqrt{3}$

Equation of Image :

$\frac{(x+1)^2}{4} - \frac{(y+1)^2}{3} = 1$

Method – II

Pre-image: $\frac{(x-1)^2}{4} - \frac{(y+3)^2}{3} = 1$

Insert $x + 2$ in place of x and $y = 2$ in place of y

Image: $\frac{(x+2-1)^2}{4} - \frac{(y-2+3)^2}{3} = 1$

$\Rightarrow \frac{(x+1)^2}{4} - \frac{(y+1)^2}{3} = 1$

(B) Pre-image

$2x^2 + y^2 + 4x + 4y + 2 = 0$

Image

$2(x+2)^2 + (y-2)^2 + 4(x+2) + 4(y-2) + 2 = 0$

$\Rightarrow 2x^2 + y^2 + 8x + 4x - 4y + 4y + 14 = 0$

$\Rightarrow 2x^2 + y^2 + 12x + 14 = 0$

2. Given

✓ Pre-image

$3x^2 - 4y^2 - 6x - 24y - 45 = 0$

✓ *Image*

$3x^2 - 4y^2 + 6x - 8y - 13 = 0$

Standard Equation of Pre-image

$3x^2 - 4y^2 - 6x - 24y - 45 = 0$

$\Rightarrow 3(x^2 - 2x) - 4(y^2 + 6y) = 45$

$\Rightarrow 3(x^2 - 2x + 1 - 1) - 4(y^2 + 6y + 9 - 9) = 45$

$\Rightarrow 3(x-1)^2 - 4(y+3)^2 = 45 - 33$

$\Rightarrow \frac{3(x-1)^2}{12} - \frac{4(y+3)^2}{12} = 1$

$\Rightarrow \frac{(x-1)^2}{4} - \frac{(y+3)^2}{3} \Rightarrow Center : C = (1, -3)$

Standard Equation of Image

$3x^2 - 4y^2 + 6x - 8y - 13 = 0$

$\Rightarrow 3(x^2 + 2x) - 4(y^2 + 2y) = 13$

$\Rightarrow 3(x^2 + 2x + 1 - 1) - 4(y^2 + 2y + 1 - 1) = 13$

$\Rightarrow 3(x+1)^2 - 4(y+1)^2 = 13 - 1$

$\Rightarrow \frac{3(x+1)^2}{12} - \frac{4(y+1)^2}{12} = 1$

$\Rightarrow \frac{(x+1)^2}{4} - \frac{(y+1)^2}{3} \Rightarrow Center : C' = (-1, -1)$

Let $\vec{u} = (m, n)$

Since $T(C) = C'$

$\Rightarrow T((1,-3)) = (-1, -1)$

$\Rightarrow (1 + m. -3 + n) = (-1, -1)$

$\Rightarrow 1 + m = -1 \; and \; -3 + n = -1$

$\Rightarrow m = -2 \; \& \; n = 2 \Rightarrow \vec{u} = (-2, 2)$

(B) Translation vector : $\vec{u} = (-2, 2)$

$\therefore T((2,-5)) = (2 + -2, -5 + 2) = (0, -3)$

National Exam Question

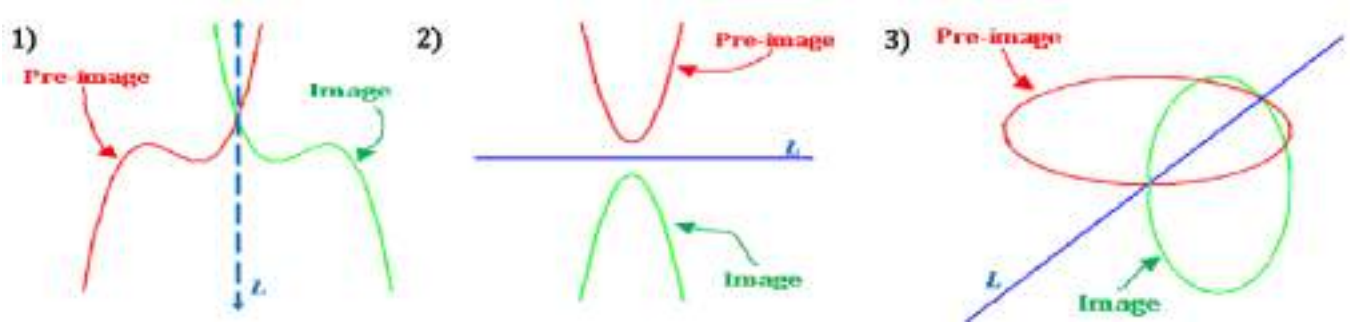
1. What is the translation vector $\vec{u} = (h, k)$ so that the equation $x^2 + 2y^2 + 6x - 8y + 15 = 0$ is transformed to an equation of the form $x^2 + 2y^2 + d = 0$, where d is a constant ?

   (A) $\vec{u} = (-3, 2)$ (B) $\vec{u} = (3, -2)$ (C) $\vec{u} = (-2, 3)$ (D) $\vec{u} = (2, -3)$

Answer : *B*

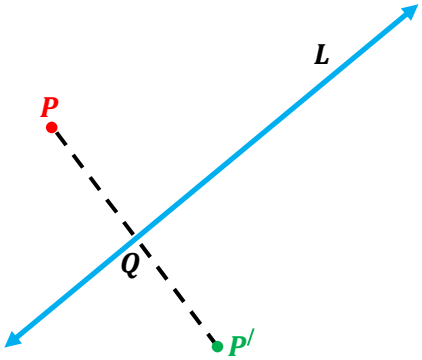
REFLECTION

In each figure given below, consider line L as mirror; we are also given a pre-image figure (in red) and its image (green)



Definition

Let L be a fixed line in a plane. Then, a reflection M about the line L is a transformation of the plane onto itself which carries every point P of the plane into a point P' of the plane in such a way that L is the perpendicular bisector of the segment $\overline{PP'}$.



$\overline{P\,P'} \perp L$ and $PQ = Q\,P'$

NOTE

1. The line L in the above definition is called ***axis of reflection*** or ***line of reflection.***
2. The image of every point on the line of reflection is the point itself.

Notations

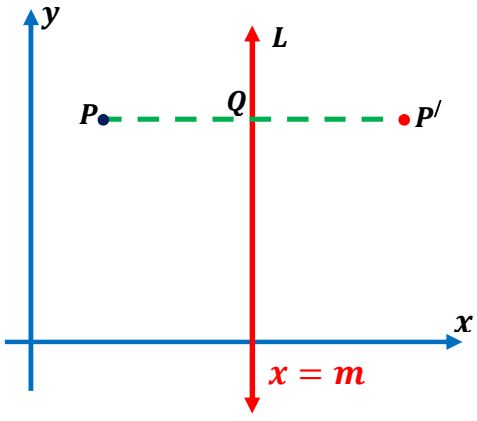
If M is a reflection about the line L and P' is the image of point P under this reflection, then we use the following notations to represent this fact.

$M: P \to P'$ (which means ***M maps point P to P'***) or
$P' = M(P)$

Vertical Line of Reflection

(I) ***Images of Points***

Let L be a line with equation $x = m$. Let $P' = (x', y')$ be the image of the point $P = (x, y)$ under a reflection about L. Since Q is on L its x-coordinate is . $\overline{PP'} \perp L$ implies that $\overline{PP'}$ is a horizontal segment. Hence the y-coordinates of P', P & Q are equal. Therefore, $P' = (x', y)$ & $Q = (m, y)$. But Q is mid-point of $\overline{PP'}$. Hence $m = \frac{x'+x}{2} \Rightarrow x' = 2m - x$.



$\therefore M(P) = P'$

$\Rightarrow M((x, y)) = (x', y') = (2m - x, y)$

Example-38

1. Find the coordinates of the image point P' of each point P given below, under the reflection M about the line $x = 4$.
   (A) $P = (-1, 3)$ (B) $P = (4, -2)$ (C) $P = (8, 6)$ (D) $P = (-4, 4)$

2. In each of the following question letters you are given the coordinates of the points P & P'. If P' is the image of P under a reflection M about a line L, then find the equation of L.
   (A) $P = (2, -8)$ & $P' = (10. -8)$ (B) $P = (5, 2)$ & $P' = (-1, 2)$

3. Let M_1 & M_2 be reflections about the lines $L_1: x = 3$ *and* $L_2: x = 7$, respectively. Find :
   (A) $M_2[M_1((1, 4))]$ (B) $M_2[M_1((1, 4))]$ (C) $M_2[M_1((-5, 1))]$ (D) $M_1[M_2((-5, 1))]$ (E) $M_2[M_1((a, b))]$

4. Let T be a translation by the vector $\vec{u} = (1, 3)$ and M be a reflection about the line $L: x = 4$. Find :
   (A) $M[T((2, -1))]$ (B) $T[M((2, -1))]$ (C) $M[T((a, b))]$ (D) $T[M((a, b))]$

5. Let T be a translation by the vector $\vec{u} = (r, t)$ and M be a reflection about the axis $x = k$. When will the statement $T \circ M = M \circ T$ be true ?

Solutions

1. Given : axis $\rightarrow x = 4$
(A) Given : $P = (-1, 3)$
$\therefore P' = M(P)$
$= M((-1, 3))$
$= (2(4) + 1, 3)$
$= (9, 3)$

(B) Given : $P = (4, -2)$
$\therefore P' = M(P)$
$= M((4, -2))$
$= (2(4) - 4, -2)$
$= (4, -2)$
Here the image of
P *is itself*. Why ?

(C) Given : $P = (8, 6)$
$\therefore P' = M(P)$
$= M((8, 6))$
$= (2(4) - 8, 6)$
$= (0, 6)$

1. P & P' have same
$y -$ *coordinates*.
$\therefore \overline{PP'}$ is horizontal.
$\Rightarrow L$ is vertical and
passes through the
mid point of $\overline{PP'}$
(A) Mid-point of $\overline{PP'}$
$= \left(\frac{2+10}{2}, \frac{-8+-8}{2}\right)$
$= (6, -8)$
$\Rightarrow$ *Equation of* L
$x = 6$

(B) Mid-Point of $\overline{PP'}$
$= \left(\frac{5+-1}{2}, \frac{2+2}{2}\right)$
$= (2, 2)$
$\Rightarrow$ *Equation of* L :
$x = 2$

3. Given
$\checkmark L_1: x = 3$
$\checkmark L_2: x = 7$
(A) $M_2[M_1((1, 4))]$
$= M_2[(2(3) - 1, 4)]$
$= M_2((5, 4))$
$= (2(7) - 5, 4)$
$= (9, 4)$

(B) $M_1[M_2((1, 4))]$
$= M_2[(2(7) - 1, 4)]$
$= M_2((13, 4))$
$= (2(3) - 13, 4)$
$= (-7, 4)$

(E) $M_2[M_1((a, b))]$
$= M_2[(2(3) - a, b)]$
$= M_2((6 - a, b))$
$= (2(7) - 6 + a, b)$
$= (a + 8, b)$

Answers :
C) $(3, 1)$ D) $(-13, 1)$
F) $(a - 8, b)$

4. Given :
$\checkmark \vec{u} = (1, 3)$
$\checkmark L: x = 4$
(A) $M[T((2, -1))]$
$= M(2 + 1, -1 + 3)$
$= M(3, 2)$
$= (2(4) - 3, 2)$
$= (5, 2)$

(B) $T[M((2, -1))]$
$= T(2(4) - 2, -1)$
$= T(6, -1)$
$= (6 + 1, -1 + 3)$
$= (7, 2)$

(C) $M[T((a, b))]$
$= M(a + 1, b + 3)$
$= (2(4) - a - 1, b + 3)$
$= (7 - a, b + 3)$
Answer :
D) $(9 - a, b + 3)$

5. $(T \circ M)(a, b)$
$= T(2k - a, b)$
$= (2k - a + r, b + t)$
$(M \circ T)(a, b)$
$= M(a + r, b + t)$
$(2k - a - r, b + t)$
$\therefore T \circ M = M \circ T$
$\Rightarrow 2k - a + r = 2k - a - r$
$\Rightarrow r = 0$
Hence $T \circ M = M \circ T$
when $r = 0$

II. Images of Functions

In the figure, the image of the function $y = f(x)$ under a reflection M about the line $x = k$ is the function $y = g(x)$. The point $P(t, f(t))$ is on the pre-image function and its image $P'(x, g(x))$ is on the image function. Hence,

$P' = (x, g(x)) = M(P)$

$= M\big((t, f(t))\big)$

$= (2k - t, f(t))$

$\therefore (x, g(x)) = (2k - t, f(t))$

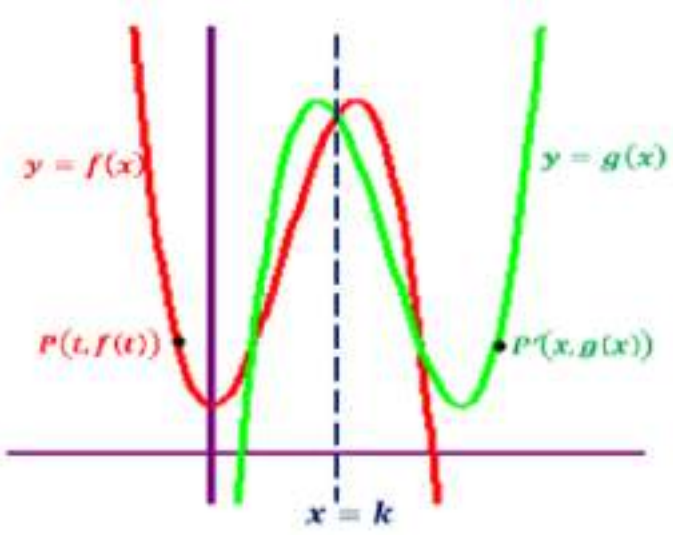
$\Rightarrow x = 2k - t \ \& \ g(x) = f(t)$

$\Rightarrow t = 2k - x$

$\therefore g(x) = f(t)$

$\Rightarrow g(x) = f(2k - x)$

Therefore, the image of the function $y = f(x)$ is the function $y = f(2k - x)$. This means, in order to get the image function we simply insert $2k - x$ in place of x in the pre-image function and then simplify the resulting expression.



Example-39

1. Find the image of each of the following functions under the reflection M about the line $x = 2$

A) $f(x) = 5x - 3$ B) $f(x) = x^2 - 3x + 4$ C) $f(x) = \frac{2x-1}{3x+4}$

Solutions

1. Given : axis $\rightarrow x = 2$

A) Pre-image

$f(x) = 5x - 3$

Image

$y = f(2(2) - x)$

$\Rightarrow y = f(4 - x)$

$\Rightarrow y = 5(4 - x) - 3$

$\Rightarrow y = 17 - 5x$

(see the graphs of the pre-image & image in figure-A)

B) Pre-image

$f(x) = x^2 - 3x + 4$

Image

$y = f(2(2) - x)$

$\Rightarrow y = f(4 - x)$

$\Rightarrow y = (4 - x)^2 - 3(4 - x) + 4$

$\Rightarrow y = x^2 - 8x + 16 - 12 + 3x + 4$

$\Rightarrow y = x^2 - 5x + 8$

(see the graphs of the pre-image & image in figure-B)

C) Pre-image

$f(x) = x^2 - 3x + 4$

Image

$y = f(2(2) - x)$

$\Rightarrow y = f(4 - x)$

$\Rightarrow y = \frac{2(4-x)-1}{3(4-x)+4}$

$\Rightarrow y = \frac{7-2x}{16-3x}$

$\Rightarrow y = \frac{2x-7}{3x-16}$

Figure – A

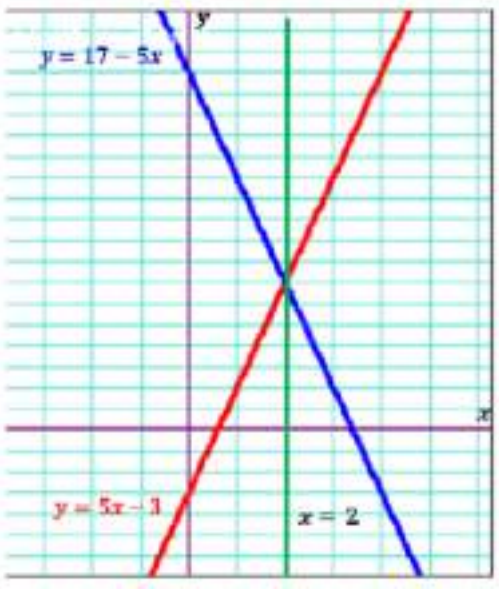
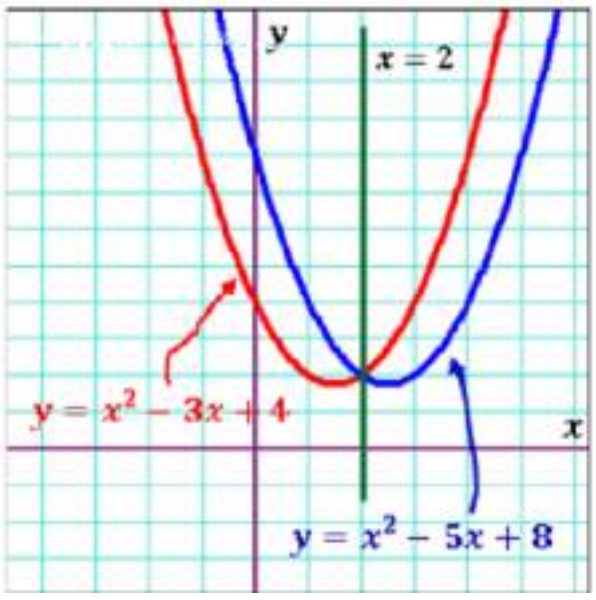


Figure – B



2. Let T be a translation by the vector $\vec{u} = (-1, 2)$ and M be a reflection about the line $x = 1$. Find the image of the function $y = 4x + 3$ when it is transformed

(A) first by T followed by M. (B) first by M followed by T.

Solutions

2. Given
✓Vector: $\vec{u} = (-1, 2)$
✓Axis : $x = 1$
A) T followed by M
(i) image of $y = 4x + 3$ under T
$y - 2 = 4(x + 1) + 3$
$\Rightarrow y = 4x + 9$

(ii) image of $y = 4x + 9$ under M
$y = 4(2(1) - x) + 9$
$\Rightarrow y = 17 - 4x$

B) M followed by T
(i) image of $y = 4x + 3$ under M
$y = 4(2(1) - x) + 3$
$\Rightarrow y = 11 - 4x$
(ii) image of $y = 11 - 4x$ under T
$y - 2 = 11 - 4(x + 1)$
$\Rightarrow y - 2 = 7 - 4x$
$\Rightarrow y = 9 - 4x$

III. Images of Conic Sections(Vertical Axis of Reflection)

If we are given the equation of the pre-image conic section and also the equation of the axis of reflection, we can compute the equation of the image conic section in the following two ways.

Method-I :(Easier Method)

Given : (i) Equation of the pre-image conic section and
(ii) Axis of reflection : $x = t$

To compute the equation of the image conic section we simply insert $2t - x$ in place of x in the equation of the pre-image conic section and then simplify the resulting equation.

Example-40

Find the equation of the image of each of the following conic sections under the reflection M about the line $x = \frac{3}{2}$.

1. $x^2 + y^2 + 2x - 6y - 6 = 0$ 2. $(y - 3)^2 = -8(x + 1)$ 3. $(x + 5)^2 - \frac{(y-3)^2}{4} = 1$

Solutions

Given :Axis $\rightarrow x = \frac{3}{2}$

1. Pre-image
$x^2 + y^2 + 2x - 6y - 6 = 0$
Image
$\left(2\left(\frac{3}{2}\right) - x\right)^2 + y^2 + 2\left(2\left(\frac{3}{2}\right) - x\right) - 6y - 6 = 0$
$(3 - x)^2 + y^2 + 2(3 - x) - 6y - 6 = 0$
$x^2 - 6x + 9 + y^2 + 6 - 2x - 6y - 6 = 0$
$x^2 + y^2 - 8x - 6y_9 = 0$

2. Pre-image
$(y - 3)^2 = -8(x + 1)$
Image
$(y - 3)^2 = -8\left(2\left(\frac{3}{2}\right) - x + 1\right)$
$(y - 3)^2 = -8(4 - x)$
$(y - 3)^2 = 8(x - 4)$

3. Pre-image
$(x + 5)^2 - \frac{(y-3)^2}{4} = 1$
Image
$\left(2\left(\frac{3}{2}\right) - x + 5\right)^2 - \frac{(y-3)^2}{4} = 1$
$(x - 8)^2 - \frac{(y-3)^2}{4} = 1$

Method – II (for each conic section)

Images of Circles

Example-41

Find the image of the circle $(x - 2)^2 + (y + 3)^2 = 16$ under a reflection about the line $x = 3$.

Solution

Given
✓Axis : $x = 3$
✓Pre-image
$(x - 2)^2 + (y + 3)^2 = 16$
Center : $C = (2, -3)$

Radius : $r = \sqrt{16} = 4$
Image
Radius : $r' = r = 4$
Center : $C' = M(C)$
$= M((2, -3))$

$= (2(3) - 2, -3)$
$= (4, -3)$
$\therefore$ ***Equation of Image***:
$(x - 4)^2 + (y + 3)^2 = 16$

Images of Parabolas

NOTE

If the axis of reflection is a vertical line then
1. the image of a vertical parabola is also a vertical parabola open to the same direction as that of the pre-image parabola .
2. the image of a horizontal parabola is also a vertical parabola open to a direction opposite to that of the pre-image parabola .

Example-42

Find the equation of the image of each given parabola under a reflection about the line $x = 3$.

1. $(x+2)^2 = -8(y-1)$ 2. $y^2 - 4x - 2y - 3 = 0$

Solutions

Given : Axis → $x = 3$

1. Pre-image

$(x+2)^2 = -8(y-1)$

Type : Vertical and Open down

Vertex : $V = (-2, 1)$

Focal length :

$-4p = -8$

$\Rightarrow p = 2$

Image parabola

Type: Vertical and Open down.

Vertex: $V' = M(V)$

$= M((-2, 1))$

$= (2(3) + 2, 1)$

$= (8, 1)$

Focal length

$p' = p = 2$

Equatiion of Image

$(x-8)^2 = -4(2)(y-1)$

$(x-8)^2 = -8(y-1)$

2. Pre-image

$y^2 - 4x - 2y - 3 = 0$

$\Rightarrow y^2 - 2y = 4x + 3$

$\Rightarrow y^2 - 2y + 1 = 4x + 4$

$\Rightarrow (y-1)^2 = 4(x+1)$

Type: Horizontal & Open to the right

Vertex: $V = (-1, 1)$

Focal Length :

$4p = 4 \Rightarrow p = 1$

Image Parabola

Type: Horizontal & Open to the left

Vertex : $V' = M(V)$

$= M((-1, 1))$

$= (2(3) + 1, 1)$

$= (7, 1)$

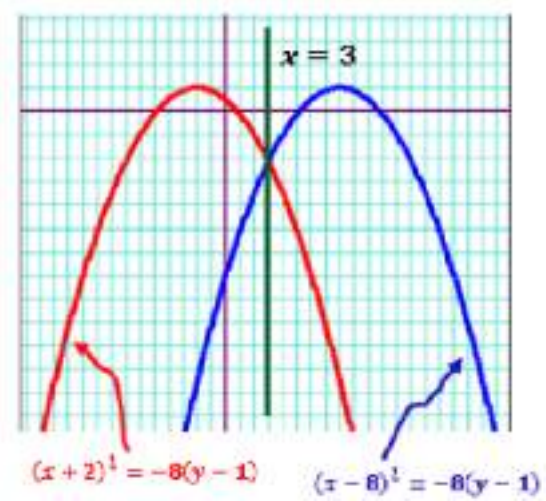
Focal length: $p' = p = 1$

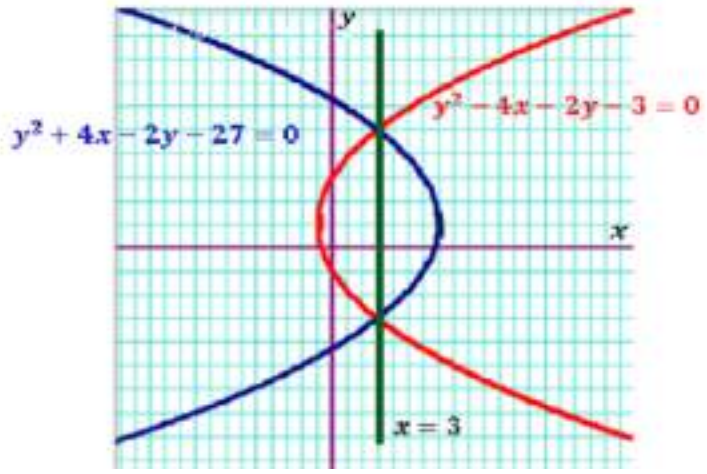
Equation of Image

$(y-1)^2 = -4(x-7)$

$y^2 + 4x - 2y - 27 = 0$

(See the graphs given below)





Images of Ellipse

Example-43

Find the equation of the image of the ellipse $\frac{(x-4)^2}{9} + \frac{(y+2)^2}{16} = 1$ under a reflection M about the line $x = 2$.

Solution

Given : Axis → $x = 2$

Pre-image

$\frac{(x-4)^2}{9} + \frac{(y+2)^2}{16} = 1$

Type : Vertical ellipse

Center: $C = (4, -2)$

Semi-major axis:

$a = \sqrt{16} = 4$

Semi-minor axis:

$b = \sqrt{9} = 3$

Image

Type: Vertical ellipse

Center: $C' = M(C)$

$= M((4, -2))$

$= (2(2) - 4, -2)$

$= (0, -2)$

Semi-major axis:

$a' = a = 4$

Semi-minor axis :

$b' = b = 3$

∴ equation of image

$\frac{(x-0)^2}{3^2} + \frac{(y+2)^2}{4^2} = 1$

$\frac{x^2}{9} + \frac{(y+2)^2}{16} = 1$

Example-44

1. In each question letter you are given a pre-image function and its image under a reflection about a vertical axis of reflection. Find the equation of the axis of reflection.

Qn. Letter	Pre-image	Image
A	$y = 2x - 5$	$y = 7 - 2x$
B	$y = x^2 - 2x - 3$	$y = x^2 + 10x + 21$

2. The image of the ellipse $x^2 + 2y^2 + 4x - 12y + 16 = 0$ under a reflection is $x^2 + 2y^2 - 8x - 12y + 28 = 0$. Find the equation of the axis of reflection.

Solutions

1. Given
Type of axis: vertical
(A) Given
✓ Pre-image
$y = 2x - 5$
✓ Image
$y = 7 - 2x$
Method-I
The axis must pass through the intersection of the two lines. Hence, we need to solve the following system.

$\begin{cases} y = 2x - 5 \\ y = 7 - 2x \end{cases}$
$\Rightarrow 2x - 5 = 7 - 2x$
$\Rightarrow 4x = 12$
$\Rightarrow x = 3$
$y = 2(3) - 5 = 1$
$\therefore$ the vertical axis of reflection passes through $(3, 1)$
Hence its equation is : $x = 3$
Method-II
Let $x = k$ be the axis.
Then image of $y = 2x - 5$ will be

$y = 2(2k - x) - 5$
$\Rightarrow y = 4k - 5 - 2x$
This must be the same as the given image $y = 7 - 2x$
$\therefore 4k - 5 = 7$
Hence , the equation of the axis is
$x = 3$

(B) Given
✓ Pre-image
$y = x^2 - 2x - 3$
✓ Image
$y = x^2 + 10x + 21$

$\Rightarrow k = 3$
The axis must pass through the intersection of the two quadratics.
$\begin{cases} y = x^2 - 2x - 3 \\ y = x^2 + 10x + 21 \end{cases}$
$\Rightarrow 10x + 21 = -2x - 3$
$\Rightarrow 12x = -24$
$\Rightarrow x = -2$
$y = (-2)^2 - 2(-2) - 3$
$\Rightarrow y = 5$
$\therefore$ the vertical axis passes through $(-2, 5)$
$\therefore$ its equation is : $x = -2$

2. Given
✓ Pre-image
$x^2 + 2y^2 + 4x - 12y + 16 = 0$
✓ Image
$x^2 + 2y^2 - 8x - 12y + 28 = 0$
Standard form of Pre-image is :
$\frac{(x+2)^2}{6} + \frac{(y-3)^2}{3} = 1$

$\therefore$ Center : $C = (-2, 3)$
Standard form of Image is :
$\frac{(x-4)^2}{6} + \frac{(y-3)^2}{3} = 1$
$\therefore$ Center : $C' = (4, 3)$
C' *is image of* C and since they have $y-$ *coordinate* they lie on a horizontal line. Therefore, the axis of reflection is a vertical line passing through the midpoint of $\overline{CC'}$.
the mid point $=$
$\left(\frac{-2+4}{2}, \frac{3+3}{2}\right) = (1, 3)$
$\therefore$ equation of the axis is :
$x = 1$

Horizontal Line of Reflection

Here you will be given only formulas. Deriving the formulas will be left for you as an exercise.

Let the axis of reflection be the horizontal line : $y = t$, where t *is a constant*

Images of Points → $M((x, y)) = (x, 2t - y)$

Images of Functions →

Pre-image Function		Image Function
$y = f(x)$	⇒	$y = 2t - f(x)$

Images of Conic Sections

Given the horizontal axis of reflection $y = t$ and the equation of the pre-image conic section to compute that of the image conic section, we simply replace y *by* $2t - y$ in the given equation of the pre-image conic section and then simplify the resulting equation.

Exercise

1. Find the image of each of the following under the reflection M about the line $y = 2$.
   (A) $P = (-3, 4)$ (B) $P = (4, 2)$ (C) $x^2 + y^2 - 6x + 2y + 1 = 0$ (D) $(x-1)^2 = 8(y-4)$
   (E) $3x^2 + 3y^2 + 6x - 4y - 7 = 0$ (F) $x^2 - 6x + 4y + 9 = 0$ (G) $y^2 - 4x - 2y - 7 = 0$
   (H) $(y+3)^2 = -4(x+1)$ (I) $f(x) = 2x^2 - 3x - 8$ (J) $g(x) = \frac{2x-1}{x+1}$

2. In the following table the equation of one of the pre-image, the image or the axis of reflection is missing. Compute the missing equation and write you answer inside the empty box of the table.

Question letter	Pre-image	Image	Axis of Reflection
A	$x^2 + y^2 - 2x + 2y - 34 = 0$	$x^2 + y^2 - 2x - 8y - 19 = 0$	
B	$x^2 - 6x - 4y + 17 = 0$		$y = 1$
C		$2x^2 + y^2 - 4x + 8y + 16 = 0$	$y = -1$
D	$4x^2 - y^2 + 24x + 2y + 31 = 0$	$4x^2 - y^2 - 8x + 2y - 1 = 0$	
E	$x^2 - 2x - 4y + 5 = 0$	$x^2 - 2x - 4y + 5 = 0$	
F	$y^2 - 4x - 2y + 9 = 0$	$y^2 - 4x - 2y + 9 = 0$	

Answers

1. (A) $(-3, 0)$ (B) $(4, 2)$ (C) $x^2 + y^2 - 6x - 10y + 25 = 0$ (D) $(x-1)^2 = -8y$
   (E) $3x^2 + 2y^2 + 6x - 12y + 9 = 0$ (F) $x^2 - 6x - 4y + 25 = 0$ (G) $y^2 - 4x - 6y + 1 = 0$
   (H) $(y-7)^2 = -4(x+1)$ (I) $y = -2x^2 + 3x + 12$ (J) $y = \frac{2x+5}{x+1}$

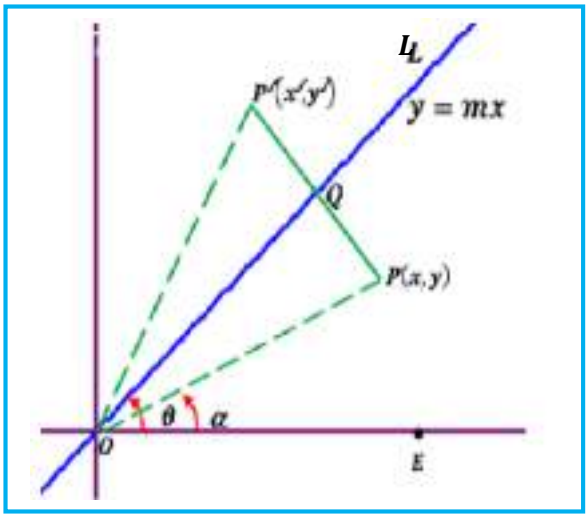
2.

Question letter	Pre-image	Image	Axis of Reflection
A	$x^2 + y^2 - 2x + 2y - 34 = 0$	$x^2 + y^2 - 2x - 8y - 19 = 0$	$y = \frac{3}{2}$
B	$x^2 - 6x - 4y + 17 = 0$	$x^2 - 6x + 4y + 9 = 0$	$y = 1$
C	$2x^2 + y^2 - 4x - 4y + 4 = 0$	$2x^2 + y^2 - 4x + 8y + 16 = 0$	$y = -1$
D	$4x^2 - y^2 + 24x + 2y + 31 = 0$	$4x^2 - y^2 - 8x + 2y - 1 = 0$	$x = -1$
E	$x^2 - 2x - 4y + 5 = 0$	$x^2 - 2x - 4y + 5 = 0$	$x = 1$
F	$y^2 - 4x - 2y + 9 = 0$	$y^2 - 4x - 2y + 9 = 0$	$y = 1$

Oblique Line of Reflection

Case-I : When the axis of reflection is the line L: $y = mx$, where $m \neq 0$.
Let θ be the angle of inclination of L

In the figure, let $P = (x, y)$ be a pre-image point and $P' = (x', y')$ be the image of P under the reflection about the line $y = mx$.
Let $OP = r$ *and* $m(\widehat{POE}) = \alpha$ (as shown in the figure)
$\Delta POQ \equiv \Delta P'OQ$...by SAS (since L is perpendicular bisector of $\overline{PP'}$)
$\therefore OP' = OP = r$ and
$m(\widehat{PO}\,\widehat{Q}) = m(\widehat{P'O}\,\widehat{Q})$ } Corresponding parts of congruent triangles
But, $m(\widehat{PO}\,\widehat{Q}) = m(\widehat{QO}\,\widehat{E}) - m(\widehat{PO}\,\widehat{E}) = \theta - \alpha$



$\therefore m(P'\widehat{O}E) = m(P'\widehat{O}Q) + m(Q\widehat{O}E)$

$= \theta - \alpha + \theta = 2\theta - \alpha$

$\sin\alpha = \frac{y}{r} \Rightarrow y = r\sin\alpha$ .....(*)

$\cos\alpha = \frac{x}{r} \Rightarrow x = r\cos\alpha$ ....(**)

Similarly,

$\sin(2\theta - \alpha) = \frac{y'}{r}$

$\Rightarrow y' = r\sin(2\theta - \alpha)$

$= r[\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha]$

$\Rightarrow y' = r\cos\alpha\sin 2\theta - r\sin\alpha\cos 2\theta$

$\Rightarrow y' = x\sin 2\theta - y\cos 2\theta$ ...by (*) & (**)

A formula we may use to compute the $y-coordinate$ of the image point P'

$\cos(2\theta - \alpha) = \frac{x'}{r} \Rightarrow x' = r\cos(2\theta - \alpha)$

$\Rightarrow x' = r[\cos\alpha\cos 2\theta + \sin\alpha\sin 2\theta]$

$\Rightarrow x' = r\cos\alpha\cos 2\theta + r\sin\alpha\sin 2\theta$

$\Rightarrow x' = x\cos 2\theta + y\sin 2\theta$ ...by (*) & (**)

A formula we may use to compute the $x-coordinate$ of the image point P'

$$x' = x\cos 2\theta + y\sin 2\theta$$
$$y' = x\sin 2\theta - y\cos 2\theta$$

1

Given the axis of reflection $L: y = mx$ and the pre-image point $P = (x, y)$ to compute the coordinates of the image point $P' = (x', y')$, we do as follows.

Steps ***Activities***

1. ..... Let θ be angle of inclination of L. Then $\tan\theta = m$ (in some problems it is easy to get the exact value of θ)
2. ..... Compute $\cos\theta$ *and* $\sin\theta$ using your result in $step-1$
3. ..... Compute $\cos 2\theta$ using any one of the following identities.

$$\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$$

4. ..... Compute $\sin 2\theta$ using the identity → $\sin 2\theta = 2\sin\theta\cos\theta$
5. .... Finally compute the values of x' *and* y' using the $formula-1$ given above.

Example-45

1. Find the coordinates of the images of the following points under the reflection M about the line $y = x$.
   (A) (x, y) (B) $(2, -3)$ (C) $(4, 4)$

2. Find the coordinates of the images of the following points under the reflection M about the line $y = -x$.
   (A) (x, y) (B) $(-3, 4)$ (C) $(-6, 6)$

3. Find the coordinates of the images of the following points under the reflection M about the line $y = 2x$.
   (A) $(-1, 4)$ (B) $(3, 6)$

Solution

1. Given :
✓ Axis of reflection: $y = x$
$\Rightarrow m = 1$
$\Rightarrow \tan\theta = 1$
$\Rightarrow \theta = 45^0$
Hence,
$\cos 2\theta = \cos 90^0 = 0$
$\sin 2\theta = \sin 90^0 = 1$

(A) Given : $P = (x, y)$
$\therefore x' = x\cos 2\theta + y\sin 2\theta$
$= x(0) + y(1) = y$
$y' = x\sin 2\theta - y\cos 2\theta$
$= x(1) - y(0) =$
Hence,
$P' = M(P)$
$= M((x.y))$

$= (x', y') = (y, x)$
Therefore, in order to get the coordinates of the image of a given point under the reflection about the line $y = x$, we simply interchange the coordinates of the given point.

(B) $M((2,-3)) = (-3,2)$
(C) $M((4,4)) = (4,4)$
In C the image of the point $(4,4)$ is itself, since the point lies on the axis of reflection .

2. Given :
✓ Axis of Reflection: $y = -x$
$\Rightarrow m = -1$
$\Rightarrow \tan\theta = -1$
$\Rightarrow \theta = 135^0$
$\therefore 2\theta = 270^0$

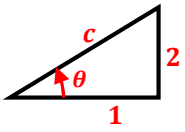
Hence,
$\cos 2\theta = 0$
$\sin 2\theta = -1$
(A) Given: $P = (x,y)$
$\therefore\ x' = x\cos 2\theta + y\sin 2\theta$
$= x(0) + y(-1) = -y$
$y' = x\sin 2\theta - y\cos 2\theta$
$= x(-1) - y(0) =$
Hence,
$P' = M(P)$
$= M((x.y))$
$= (x',y') = (-y,-x)$
Here also, in order to get

the coordinates of the image of a given point under the reflection about the line $y = -x$, we simply take the opposite of the coordinate of the given point and then interchange the resulting coordinate.

$M: (x,y) \xrightarrow{y=-x} (-y,-x)$

(B) $M((-3,4)) = (-4,3)$
(C) $M((-6,6)) = (-6,6)$

3. Given :
✓ Axis of Reflection :$y = 2x$
$\Rightarrow m = 2$
$\Rightarrow \tan\theta = 2$



By Pythagoras theorem,
$c = \sqrt{5}$
$\therefore \cos\theta = \frac{1}{\sqrt{5}}$

$\sin\theta = \frac{2}{\sqrt{5}}$
$\Rightarrow \cos 2\theta = 2\cos^2\theta - 1$
$= 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1$
$= \frac{2}{5} - 1 = -\frac{3}{5}$
$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5}$
(A) Given :$P = (-1,4)$
$\therefore\ x' = x\cos 2\theta + y\sin 2\theta$
$= -1\left(-\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{19}{5}$

$y' = x\sin 2\theta - y\cos 2\theta$
$= -1\left(\frac{4}{5}\right) - 4\left(-\frac{3}{5}\right) = \frac{8}{5}$
Hence,
$P' = M(P)$
$= M((x.y))$
$= (x',y') = \left(\frac{19}{5},\frac{8}{5}\right)$
(B) Given :$P = (3,6)$
P lies on the axis of reflection $y = 2x$.
$\therefore M((3,6)) = (3,6)$

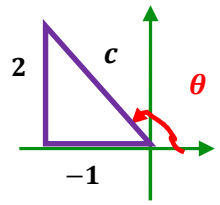
The problems in Example 45 can be solved without using Formula-1 given on page 45. See the following example.

Example-46

1. Compute the image of the point $(-3,1)$
(A) using Formula-1 (B) without using Formula-1
under the reflection M about the line $y = -2x$.
2. Did you get the same answer in letters A and B of number 1?

Solution

1.(A) Given :
✓ Axis of Reflection: $y = -2x$
$\Rightarrow m = -2$
$\Rightarrow \tan\theta = -2$

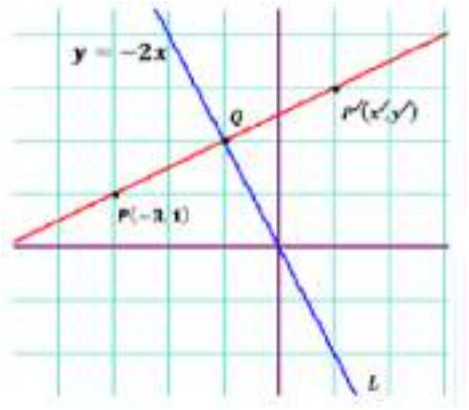


By Pythagoras theorem
$c = \sqrt{5}$

$\therefore \sin\theta = \frac{2}{\sqrt{5}}$ and
$\cos\theta = -\frac{1}{\sqrt{5}}$
$\Rightarrow \cos 2\theta = 2\cos^2\theta - 1$
$= 2\left(-\frac{1}{\sqrt{5}}\right)^2 - 1$
$= \frac{2}{5} - 1 = -\frac{3}{5}$
$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = -\frac{4}{5}$
Pre-image :$P = (-3,1)$

Image : $P' = (x',y')$ where
$x' = x\cos 2\theta + y\sin 2\theta$
$= -3\left(-\frac{3}{5}\right) + 1\left(-\frac{4}{5}\right) = 1$
$y' = x\sin 2\theta - y\cos 2\theta$
$= -3\left(-\frac{4}{5}\right) - 1\left(-\frac{3}{5}\right) = 3$
Hence,
$P' = M(P)$
$= M((x.y))$
$= (x',y') = (1,3)$

1. B)



L is perpendicular bisector of $\overline{PP'}$.

Let $m = slope\ of\ L = -2$

Since $L \perp \overleftrightarrow{PP'}$, slope of $\overleftrightarrow{PP'} = \frac{1}{2}$

$\therefore$ *equation of* $\overleftrightarrow{PP'}$ *is*:

$y - 1 = \frac{1}{2}(x + 3)$

$\Rightarrow y = \frac{1}{2}x + \frac{5}{2}$ ...(*)

Now we will find coordinate of Q by solving equation (*) and $y = -2x$ simultaneously.

$\frac{1}{2}x + \frac{5}{2} = -2x$

$\Rightarrow x + 5 = -4x$

$\Rightarrow x = -1$

$\therefore y = -2x \Rightarrow y = 2$

Hence, $Q = (-1, 2)$

Let $P' = (x', y')$

Since Q is midpoint of $\overline{PP'}$ we will have;

$Q = \left(\frac{-3+x'}{2}, \frac{1+y'}{2}\right)$

$\Rightarrow (-1, 2) = \left(\frac{-3+x'}{2}, \frac{1+y'}{2}\right)$

$\therefore \frac{-3+x'}{2} = -1$ & $\frac{1+y'}{2} = 2$

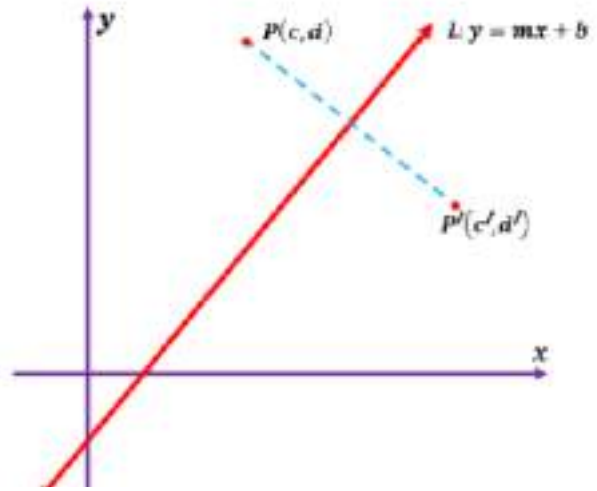
$\Rightarrow x' = 1$ & $y' = 3$

$\therefore P' = (x', y') = (1, 3)$

2. The answers in 1.A and 1.B are the same.

<u>Case-I</u> : When the axis of reflection is the line L: $y = mx$+b..(*)
where $m \neq 0$

Let $P = (c, d)$ be the pre-image point. Then to find the coordinates of the image of P under the reflection M about the line $y = mx + b$, we do as follows.



Steps ***Activities***

1. .... <u>Find the slope of L</u>. It is equal to m
2. ... <u>Compute slope of $\overleftrightarrow{PP'}$</u>. Let slope of $\overleftrightarrow{PP'} = m'$. Then since $\overleftrightarrow{PP'} \perp L$, then $m' = -1/m$
3. ... <u>Find the equation of $\overleftrightarrow{PP'}$</u>
   Using $P = (c, d)$ & $slope = m'$, then its equation is:
   $y - d = m'(x - c)$.....(**)
4. ...<u>Find the coordinates of Q</u> : Since Q is the intersection point of L *and* $\overleftrightarrow{PP'}$, then its coordinates can be obtained by solving equations (*) and (**) simultaneously.

$$\begin{cases} y = mx + b \\ y - d = m'(x - c) \end{cases}$$

5. ...<u>Compute the coordinates of the image point $P'(c', d')$</u> : Since Q is the midpoint of $\overline{PP'}$ and the coordinates of both P & Q are now known, then computing the coordinates of the image point is very free from any pain.
   $Q = \left(\frac{c+c'}{2}, \frac{d+d'}{2}\right)$. Use the coordinates of Q obtained in $step - 4$

<u>Example-47</u>

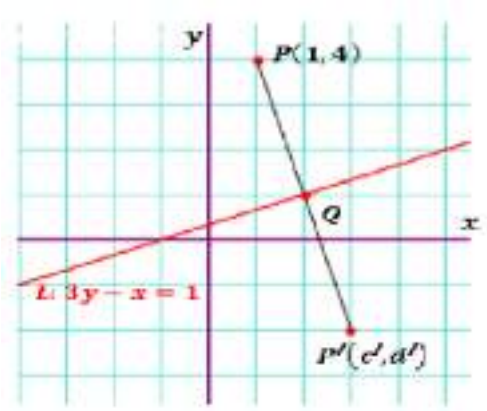
1. Find the coordinates of the image of the point $P = (1, 4)$ under the reflection M about the line $3y - x = 1$.

2. Find the coordinates of the image of the point $P = (2, -3)$ under the reflection M about the line $y + x = 1$.

1. Given

✓ Axis → $L: 3y - x = 1$..... (*)

✓ Pre-image: $P = (1, 4)$



Slope of L

$3y - x = 1 \Rightarrow y = \frac{1}{3}x + \frac{1}{3}$; $\therefore m = \frac{1}{3}$

Slope of $\overleftrightarrow{PP'}$: Let $m' = slope\left(\overleftrightarrow{PP'}\right)$

Since $\overleftrightarrow{PP'} \perp L$ then $mm' = -1$

$\therefore m' = -3$

Equation of $\overleftrightarrow{PP'}$:using $P(1, 4)$ & $m' = -3$

$y - 4 = -3(x - 1) \Rightarrow y = -3x + 7$...(**)

Coordinates of Q

Solve equations (*) & (**) simultaneously

$$\begin{cases} 3y - x = 1 \\ y = -3x + 7 \end{cases}$$

$3y - x = 1 \Rightarrow 3(-3x + 7) - x = 1$

$\Rightarrow -10x = -20 \Rightarrow x = 2$

$y = -3x + 7 \Rightarrow y = -3(2) + 7 = 1$

$\therefore Q = (2, 1)$

Coordinate of image point

Since Q is mid point of $\overline{PP'}$ & $P = (1, 4)$

then $Q = \left(\frac{1+c'}{2}, \frac{4+d'}{2}\right)$

$\Rightarrow (2, 1) = \left(\frac{1+c'}{2}, \frac{4+d'}{2}\right)$

$\Rightarrow \frac{1+c'}{2} = 2 \; and \; \frac{4+d'}{2} = 1$

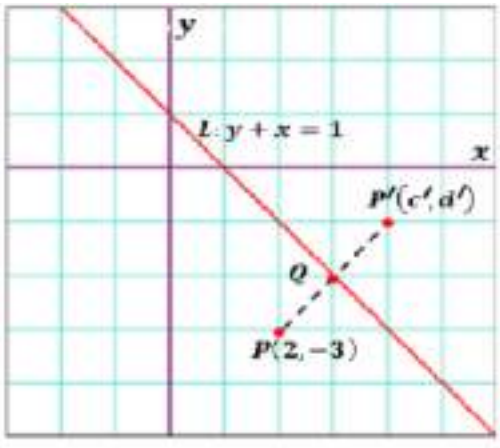
$\Rightarrow c' = 3 \; and \; d' = -2$

Hence, $P' = (c', d') = (3, -2)$

2. Given :

✓ Axis ⋯ $L: y + x = 1$.....(*)

✓ Pre-image : $P = (2, -3)$



Slope of L

$y + x = 1 \Rightarrow y = -x + 1$; $\therefore m = -1$

Slope of $\overleftrightarrow{PP'}$: Let $m' = slope\left(\overleftrightarrow{PP'}\right)$

Since $\overleftrightarrow{PP'} \perp L$ then $mm' = -1$

$\therefore m' = 1$

Equation of $\overleftrightarrow{PP'}$:using $P(2, -3)$ & $m' = 1$

$y + 3 = 1(x - 2) \Rightarrow y = x - 5$...(**)

Coordinates of Q

Solve equations (*) & (**) simultaneously

$$\begin{cases} y + x = 1 \\ y = x - 5 \end{cases}$$

$y + x = 1 \Rightarrow x - 5 + x = 1$

$\Rightarrow 2x = 6 \Rightarrow x = 3$

$y = x - 5 \Rightarrow y = 3 - 5 = -2$

$\therefore Q = (3, -2)$

Coordinate of image point

Since Q is mid point of $\overline{PP'}$ & $P = (2, -3)$

then $Q = \left(\frac{1+c'}{2}, \frac{4+d'}{2}\right)$

$\Rightarrow (3, -2) = \left(\frac{2+c'}{2}, \frac{-3+d'}{2}\right)$

$\Rightarrow \frac{2+c'}{2} = 3 \; and \; \frac{-3+d'}{2} = -2$

$\Rightarrow c' = 4 \; and \; d' = -1$

Hence, $P' = (c', d') = (4, -1)$

Exercise

1. Find the coordinates of the image of the point $P(-2, -3)$ under the reflection about the line $2y - 4x = 7$

   Answer: $P' = (-4, -2)$

NATIONAL EXAM QUESTIONS

1. What is the image of the circle $x^2+y^2-4x-6y+12=0$ when it is reflected with respect to the line $y=-x$?
   (A) $(x-2)^2+(y-3)^2=1$ (C) $(x+2)^2+(y+3)^2=1$
   (B) $(x-3)^2+(y-2)^2=1$ (D) $(x+3)^2+(y+2)^2=1$

2. What is the image of the ellipse $(x-1)^2+4y^2=1$ under a translation that takes $(1,1)$ *to* $(0,2)$ followed by a reflection through the $x-axis$?
   (A) $x^2+4(y-1)^2=1$ (B) $4x^2+(y-1)^2=1$ (C) $x^2+4(y+1)^2=1$ (D) $4x^2+(y+1)^2=1$

3. If a point $(2,5)$ is reflected under a line to the point $(-3,1)$, what is the line of reflection?
   (A) $2x+3y=7$ (B) $x+3y=7$ (C) $8y+10x=19$ (D) $2x+3y+5=0$

4. Which of the following is the reflection of the parabola $y=x^2+1$ in the line $y=-x$?
   (A) $y=-x^2-1$ (B) $y=x^2-1$ (C) $x=y^2+1$ (D) $x=-y^2-1$

Solutions

1. Given :
✓ Axis : $y=-x$
✓ Pre-image: $x^2+y^2-4x-6y+12=0$
This problem can be solved in the following two ways.

Method – I

Step – 1: Find the center & radius of the Pre-image circle.

$x^2+y^2-4x-6y+12=0$

Center : $C=\left(\frac{-(-4)}{2},\frac{-(-6)}{2}\right)=(2,3)$

Radius : $r=\sqrt{2^2+3^2-12}=1$

Step – 2: Find the center & radius of the Image circle.

Radius : $r'=r=1$

Center : $C'=M(C)=M\big((2,3)\big)=(-3,-2)$

Step – 3: Find the equation of the image

$(x+3)^2+(y+2)^2=1$

ANSWER : D

Method-II

Since the image of any point (x,y) under a reflection about the line $y=-x$ is :

$M\big((x,y)\big)=(-y,-x)$, then in order to get the image of the given circle we simply replace x by $-y$, and y *by* $-x$ in the equation of the pre-image circle.

Pre-image: $x^2+y^2-4x-6y+12=0$

Image :

$(-y)^2+(-x)^2-4(-y)-6(-x)+12=0$

$x^2+y^2+4y+6x+12=0$

$\Rightarrow x^2+6x+3^2-3^2+y^2+4y+2^2-2^2+12=0$

$\Rightarrow (x+3)^2+(y+2)^2=1$ (same answer)

(See graphs of image and pre-image on page 50)

2. Given
✓ Translation T: $(1,1)\rightarrow(0,2)$
✓ Reflection Axis: $x-axis$ $(y=0)$
✓ Pre-image $(x-1)^2+4y^2=1$

First Image

T: $(1,1)\rightarrow(0,2)\Rightarrow\vec{u}=(-1,1)$

∴ Image of the given ellipse is:

$(x+1-1)^2+4(y-1)^2=1$

$\Rightarrow x^2+4(y-1)^2=1$.....(*)

Last Image

$Axis$: $y=0$

First image: $x^2+4(y-1)^2=1$

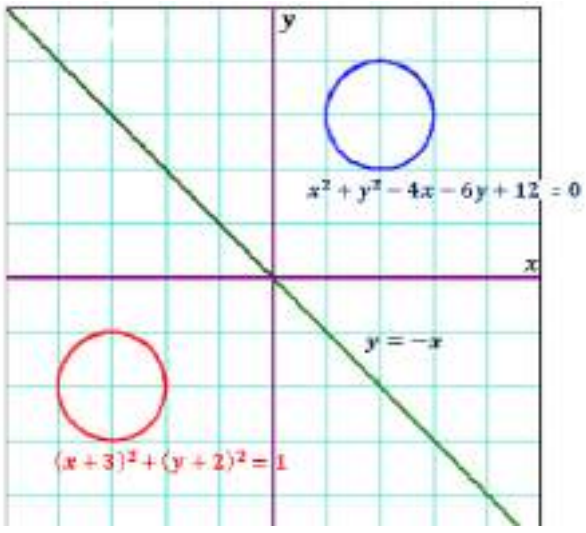
Last image: $x^2+4(2(0)-y-1)^2=1$

$\Rightarrow x^2+4(-y-1)^2=1$

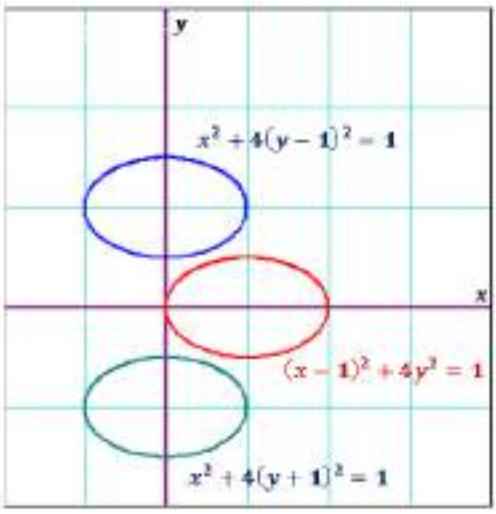
$\Rightarrow x^2+4(y+1)^2=1$

ANSWER : C

Graph For Number 1



Graph For Number 2



The axis of reflection (L) is perpendicular bisector of $\overline{PP'}$

M is midpoint . Hence,

$M=\left(\frac{2-3}{2},\frac{5+1}{2}\right)=\left(-\frac{1}{2},3\right)$

Slope of $\overleftrightarrow{PP'}=m_1=\frac{5-1}{2+3}=\frac{4}{5}$

$\therefore$ *Slope of* $L: m=-\frac{1}{m_1}=-\frac{5}{4}$

Since M is on L equation of L is:

$y-3=-\frac{5}{4}\left(x+\frac{1}{2}\right)$

$y-3=-\frac{5}{8}(2x+1)$

$\Rightarrow 8y-24=-10x-5$

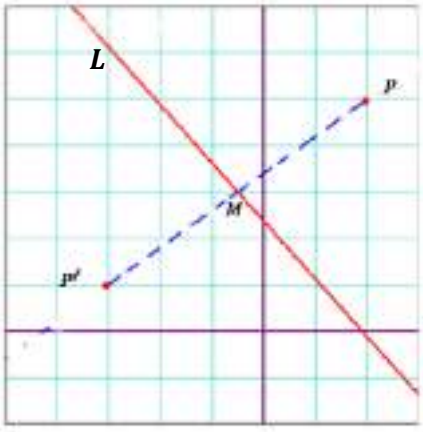
$\Rightarrow 8y+10x=19$

ANSWER : C

3. Given:

✓ Pre-image : $P=(2,5)$

✓ Image : $P'=(-3,1)$



4. The equation of the image of any given conic section under a reflection about the line $y=-x$ can easily be computed by just inserting $-x$ in place of y, and $-y$ in place of x in the equation of the pre-image conic section and then simplifying the resulting equation.

Given :

✓ Axis : $y=-x$

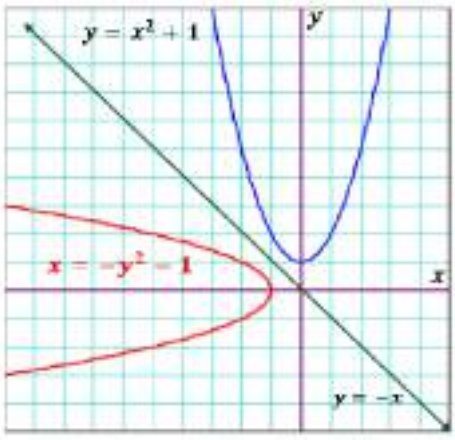
✓ Pre-image : $y=x^2+1$

$\therefore$ *the image will be*

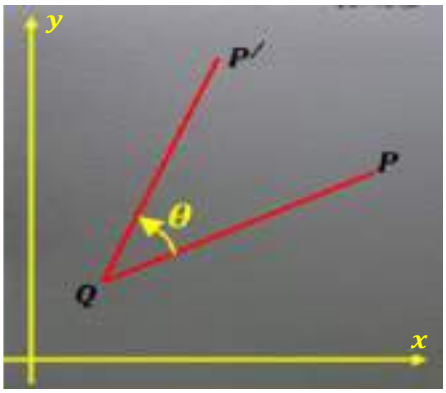
$-x=(-y)^2+1$

$\Rightarrow x=-y^2-1$

ANSWER : D



ROTATION



Definition

A rotation R about a point Q through an angle θ is a transformation is a transformation of the plane onto itself, which takes every point P of the plane into another point P' of the same plane in such a way that $m(P\hat{Q}P') = \theta$.

NOTE

1. Q *is called "Center of Rotation"*
2. θ *is called "Angle of Rotation"*
3. $\overline{QP}$ *is called "Radius of Rotation"*
   $QP = QP' = r$

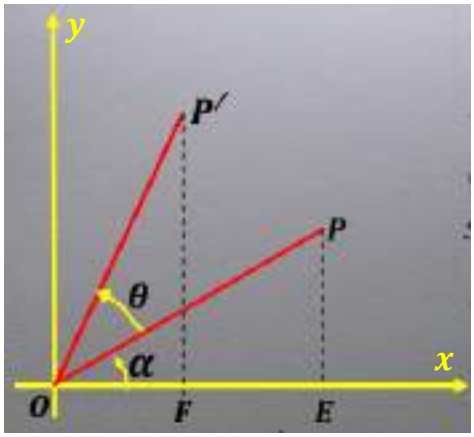
Notations

If R is a rotation that moves the point $P = (x, y)$ to the *image point* $P' = (x', y')$, then we can use any of the following notations to express this fact.

- ✓ $P' = Image\ of\ the\ point\ P$
- ✓ $(x', y') = Image\ of\ the\ point\ (x, y)$
- ✓ $P' = R(P)$
- ✓ $(x', y') = R((x, y))$
- ✓ $P' = R_\theta(P)$

Rotation About The Origin

Let $P = (x. y)$ be a point in plane and $P' = (x', y')$ be its image under a rotation R about the origin $O = (0, 0)$ through an angle θ.



Let $OP = r$ and $m(P\hat{O}E) = \alpha$

Then, $OP' = r$

$\cos\alpha = \frac{x}{r} \Rightarrow x = r\cos\alpha$.....(1)

$\sin\alpha = \frac{y}{r} \Rightarrow y = r\sin\alpha$ .....(2)

$m(P'\hat{O}F) = \alpha + \theta$

$\cos(\alpha + \theta) = \frac{x'}{r} \Rightarrow x' = r\cos(\alpha + \theta)$

$\Rightarrow x' = r[\cos\alpha\cos\theta - \sin\alpha\sin\theta]$

$\Rightarrow x' = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$\Rightarrow x' = x\cos\theta - y\sin\theta$..... by (1) & (2)

The highlighted formula helps to compute the coordinates of the image point P'.

Similarly,

$\sin(\alpha + \theta) = \frac{y'}{r}$

$y' = r\sin(\alpha + \theta)$

$y' = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$y' = y\cos\theta + x\sin\theta$

$$x' = x\cos\theta - y\sin\theta$$
$$x' = x\sin\theta + y\cos\theta$$

A

Example-48

1. Find the coordinates of the image of the point $(2, -4\sqrt{3})$ under the rotation R about the origin through 30^0.

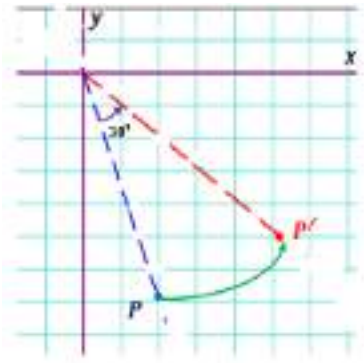
2. Find the image of the point $P = (x, y)$ under the rotation about the origin through each of the following angles.
   (A) 90^0 (B) -90^0 (C) 180^0 (D) 270^0 (E) 45^0

3. (A) What is the image of the point $P = (-2, 3)$ after it is rotated about the origin first through 30^0 followed by the rotation through 60^0 ?
   (B) What is the image of the point $P = (-2, 8)$ under the rotation R about the origin through 90^0 ?
   (C) Are the images in $question - letters\ A$ & B the same ? What do you conclude ?

Solutions

1. Given :
✓ $P = (2, -4\sqrt{3})$ & $\theta = 30^0$
Let the image point be :
$P' = (x', y')$
Then,
$x' = x\cos\theta - y\sin\theta$
$\Rightarrow x' = 2\cos 30^0 - (-4\sqrt{3})\sin 30^0$
$\Rightarrow x' = 2\left(\frac{\sqrt{3}}{2}\right) + 4\sqrt{3}\left(\frac{1}{2}\right)$
$\Rightarrow x' = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$
$y' = x\sin\theta + y\cos\theta$
$\Rightarrow y' = 2\sin 30^0 + -4\sqrt{3}\cos 30^0$
$\Rightarrow y' = 2\left(\frac{1}{2}\right) - 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$
$\Rightarrow y' = 1 - 6 = -5$
Hence the image :

Hence the image :
$P' = (x', y') = (3\sqrt{3}, -5)$



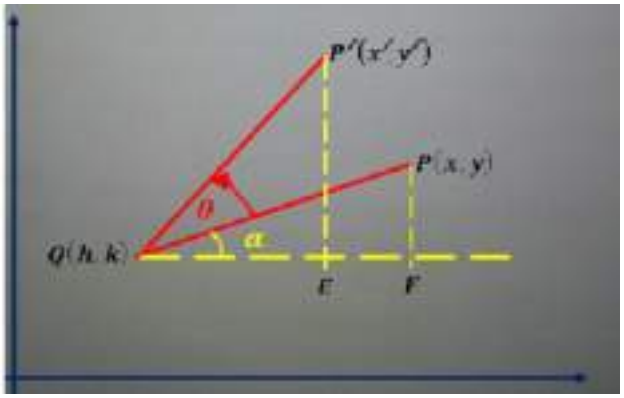
2. Given :
✓ Pre-image: $P = (x, y)$
✓ (A) Angle of rotation : $\theta = 90^0$

Let $P' = (x', y')$
$\therefore x' = x\cos\theta - y\sin\theta$
$\Rightarrow x' = x\cos 90^0 - y\sin 90^0$
$\Rightarrow x' = x(0) - y(1) = -y$
$y' = x\sin\theta + y\cos\theta$
$\Rightarrow y' = x\sin 90^0 + y\cos 90^0$
$\Rightarrow y' = x(1) + y(0) = x$
Hence,
$R((x, y)) = (-y, x)$
The above means , to get the image of the point (x, y) under a rotation through 90^0 we simply do the following two steps.
(i) interchange the coordinates of the pre-image point
(ii) change the sign of the x −coordinate of the point you have found in step-(i)

3. Given: x y
✓ (A) Pre-image : $P = (-2, 8)$
✓ Angles : $\theta_1 = 30^0$ *and* $\theta_2 = 60^0$
Let $P' = (x', y')$, be first image
$x' = x\cos\theta_1 - y\sin\theta_1$
$\Rightarrow x' = -2\cos 30^0 - 8\sin 30^0$
$\Rightarrow x' = -\sqrt{3} - 4$
$y' = x\sin\theta + y\cos\theta$
$\Rightarrow y' = -2\sin 30^0 + 8\cos 30^0$
$\Rightarrow y' = -1 + 4\sqrt{3}$
$\therefore P' = (-\sqrt{3} - 4, -1 + 4\sqrt{3})$
Let $P'' = (x'', y'')$ be image of P'
$x'' = x'\cos\theta_2 - y'\sin\theta_2$
$x'' = (-\sqrt{3} - 4)\left(\frac{1}{2}\right) - (-1 + 4\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)$
$x'' = \frac{-\sqrt{3}-4+\sqrt{3}-12}{2} = -8$
$y'' = x'\sin\theta_2 + y'\cos\theta_2$
$y'' = (-\sqrt{3} - 4)\frac{\sqrt{3}}{2} + (-1 + 4\sqrt{3})\frac{1}{2}$
$y'' = \frac{-3-4\sqrt{3}-1+4\sqrt{3}}{2} = -2$
$\therefore P'' = (-8, -2)$
Hence $R_{60^0}\left(R_{30^0}(-2, 8)\right) = (-8, -2)$
(B) $R_{90^0}((-2, 8)) = (-8, 2)$
(C) Yes , they are the same.
$\Rightarrow R_{\theta_1}R_{\theta_2} = R_{\theta_1+\theta_2}$

Rotation About Any Point in Plane

$P = (x, y)$ is a point in a plane as shown in the figure. $P' = (x', y')$ is image of P under a rotation R about point $Q = (h, k)$ through an angle θ.
In the figure $m(P\hat{Q}F) = \alpha; m(P'\hat{Q}P) = \theta$, $PQ = r = P'Q$
$QF = x - h$, $PF = y - k$
$\therefore \sin\alpha = \frac{y-k}{r} \Rightarrow y - k = r\sin\alpha$..... (*)
$\cos\alpha = \frac{x-h}{r} \Rightarrow x - h = r\cos\alpha$..... (**)



Using $\Delta P'QE$,

$QE = x' - h;\ P'E = y' - k; m(P'\hat{Q}E) = \alpha + \theta$

$\therefore \sin(\alpha + \theta) = \frac{y'-k}{r} \Rightarrow y' - k = r\sin(\alpha + \theta)$

$\Rightarrow y' - k = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$\Rightarrow y' - k = (y - k)\cos\theta + (x - h)\sin\theta$

Similarly,

$\cos(\alpha + \theta) = \frac{x' - h}{r} \Rightarrow x' - h = r\cos(\alpha + \theta)$

$\Rightarrow x' - h = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$\Rightarrow x' - h = (x - h)\cos\theta - (y - k)\sin\theta$

Hence the coordinates of the image point $P' = (x', y')$ can be computed using the formula :

$$x' - h = (x - h)\cos\theta - (y - k)\sin\theta$$
$$y' - k = (y - k)\cos\theta + (x - h)\sin\theta$$

Example-49

Find the image of the point $P = (3, -2)$ under a rotation R about the point $Q = (-1, 1)$ through:

(A) 90^0 (B) 180^0 (C) -90^0

Solution

Given : h, k

✓ Center of Rotation : $Q = (-1, 1)$

✓ Pre-image point : $P = (3, -2)$ x, y

(A) Given : $\theta = 90^0$

Let the image $P' = (x', y')$

Then,

$x' - h = (x - h)\cos\theta - (y - k)\sin\theta$

$x' + 1 = (3 + 1)\cos 90^0 - (-2 - 1)\sin 90^0$

$x' + 1 = 0 + 3 = 3 \Rightarrow x' = 2$

$y' - k = (x - h)\sin\theta + (y - k)\cos\theta$

$y' - 1 = (3 + 1)\sin 90^0 + (-2 - 1)\cos 90^0$

$y' - 1 = 4 \Rightarrow y' = 5$

Hence, image : $P' = (2, 5)$

(B) Given : $\theta = 180^0$

Let the image $P' = (x', y')$

Then,

$x' - h = (x - h)\cos\theta - (y - k)\sin\theta$

$x' + 1 = (3 + 1)\cos 180^0 - (-2 - 1)\sin 180^0$

$x' + 1 = -4 - 0 = -4 \Rightarrow x' = -5$

$y' - k = (x - h)\sin\theta + (y - k)\cos\theta$

$y' - 1 = (3 + 1)\sin 180^0 + (-2 - 1)\cos 180^0$

$y' - 1 = 3 \Rightarrow y' = 4$

Hence, image : $P' = (-5, 4)$

(C) Answer : $P' = (-4, -3)$