

1 Pointwise Convergence

Pointwise convergence of a sequence of functions

Let $A \subseteq \mathbb{R}$ be a nonempty set and suppose that for each $n \in \mathbb{N}$ there is a function

$$f_n : A \rightarrow \mathbb{R}.$$

Then $\langle f_n(x) \rangle$ is called a **sequence of real-valued functions on A** . Note that for each $x \in A$, such a sequence gives a sequence of real numbers, namely,

$$\langle f_n(x) \rangle \quad \dots (1).$$

This sequence of real numbers in (1) may converge or diverge. Depending on the choice of the point $x \in A$, for which the sequence in (1) converges, there is a uniquely determined real number, namely,

$$\lim_{n \rightarrow \infty} f_n(x).$$

This gives rise to a function

$$f : A_0 \rightarrow \mathbb{R} \quad \text{defined by} \quad f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

whose domain A_0 consists of all real numbers $x \in A$ for which the sequence in (1) converges. We call this function f the pointwise limit of the sequence $\langle f_n(x) \rangle$ on A_0 . Formally, we have

Definition 1 Let $\langle f_n(x) \rangle$ be a sequence of functions defined on a set $A_0 \subseteq \mathbb{R}$ such that the $\lim_{n \rightarrow \infty} f_n(x)$ exists for each $x \in A_0$. The function $f : A_0 \rightarrow \mathbb{R}$ defined by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in A_0$$

is called the pointwise limit of the sequence $\langle f_n(x) \rangle$ on A_0 , and the sequence $\langle f_n(x) \rangle$ is said to be pointwise convergent to f on A_0 , written as

$$f_n \rightarrow f \quad \text{pointwise on } A_0.$$

Example 1 For each $x \in \mathbb{R}$, let us define $g_n(x) = x^n$ for every $n \in \mathbb{N}$. Then, for $x = 0$, we have $g_n(x) = 0$ for every n , so that

$$g_n(0) \rightarrow 0.$$

For $x = 1$, we have $g_n(x) = 1 \quad \forall n \in \mathbb{N}$, so

$$g_n(1) \rightarrow 1.$$

Let $0 < x < 1$. Writing $x = \frac{1}{1+y}$, $y > 0$, and using the Bernoulli inequality, we see that

$$0 < x^n < \frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, we get $\lim_{n \rightarrow \infty} x^n = 0$. Therefore,

$$g_n(x) \rightarrow 0 \text{ pointwise on } (0, 1).$$

Let $-1 < x < 0$. Then we write $x = -z$, where $0 < z < 1$, to obtain that $\lim_{n \rightarrow \infty} x^n = 0$. Thus, we have

$$g_n(x) \rightarrow 0 \text{ pointwise on } (-1, 0).$$

For $x = -1$, we have $\langle x^n \rangle = \langle (-1)^n \rangle$, which is a divergent sequence. Finally, for $|x| > 1$, the sequence $\langle x^n \rangle$ is not bounded and therefore divergent. Now, if we define

$$g(x) = \begin{cases} 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } x = 1, \end{cases}$$

then the sequence $\langle g_n(x) \rangle$ is pointwise convergent to $g(x)$ on $(-1, 1]$.

Example 2 For each $x \in \mathbb{R}$, let us define $f_n(x) = \frac{x}{n}$ for every $n \in \mathbb{N}$. Then, for $x = 0$, we have $f_n(x) = 0$ for every n , and for each fixed $x \in \mathbb{R}$, $x \neq 0$, we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \left(\frac{x}{n}\right) = 0.$$

Therefore, if we define $f(x) = 0 \forall x \in \mathbb{R}$, then $\langle f_n(x) \rangle$ is pointwise convergent to f on $\mathbb{R}$.

Example 3 Let $h_n(x) = \frac{x^2+nx}{n}$ for all $x \in \mathbb{R}$, and $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} h_n(x) = \lim_{n \rightarrow \infty} \frac{x^2+nx}{n} = \lim_{n \rightarrow \infty} \left(\frac{x^2}{n} + x\right) = x,$$

we have that the sequence $\langle h_n(x) \rangle$ converges pointwise on $\mathbb{R}$ to the function h defined by $h(x) = x \forall x \in \mathbb{R}$.

Example 4 Consider the function $k_n(x) = \frac{\sin(nx+n)}{n}$ for all $x \in \mathbb{R}$, and $n \in \mathbb{N}$. Since for all $x \in \mathbb{R}$,

$$0 \leq |k_n(x)| = \left| \frac{\sin(nx+n)}{n} \right| = \left| \frac{1}{n} \right| \cdot |\sin(nx+n)| \leq \frac{1}{n}$$

and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we obtain that $k_n(x) \rightarrow 0$ pointwise on $\mathbb{R}$.

Example 5 Consider the function $s_n(x) = nxe^{-nx^2}$ for all $x \in \mathbb{R}$, and $n \in \mathbb{N}$. For $x = 0$, we have $s_n(x) = 0$ for all $n \in \mathbb{N}$. Also, for all $x \in \mathbb{R}$, $x \neq 0$, we have

$$\lim_{n \rightarrow \infty} s_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{e^{nx^2}} = \lim_{n \rightarrow \infty} \frac{x}{x^2 e^{nx^2}} = \lim_{n \rightarrow \infty} \frac{1}{xe^{nx^2}} = 0.$$

Therefore, $s_n(x) \rightarrow 0$ pointwise on $\mathbb{R}$.

Pointwise convergence of a series of functions

Let $A \subseteq \mathbb{R}$ be a nonempty set and suppose that for each $n \in \mathbb{N}$ there is a function

$$f_n : A \rightarrow \mathbb{R}.$$

Then the sum

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{or} \quad \sum f_n(x) \quad \text{or simply} \quad \sum f_n$$

is called a **series of functions on A** . Note that for each $x \in A$, such a series gives a series of real numbers, namely,

$$\sum_{n=1}^{\infty} f_n(x). \quad \dots (2)$$

This series in (2) may converge or diverge. Depending on the choice of the point $x \in A$, for which the series in (2) converges, there is a uniquely determined real number, namely,

$$\sum_{n=1}^{\infty} f_n(x).$$

This gives rise to a function

$$f : D \rightarrow \mathbb{R} \quad \text{defined by} \quad f(x) = \sum_{n=1}^{\infty} f_n(x)$$

whose domain D consists of all real numbers $x \in A$ for which the series in (2) converges. We call this function f the pointwise limit of the series $\sum f_n$ on D , and say that the series $\sum f_n$ is pointwise convergent to f on D , i.e.,

$$\sum f_n \rightarrow f \quad \text{pointwise on } D.$$

We may also define the pointwise convergence of a series of functions on D by considering the pointwise convergence of the sequence of partial sums $\langle s_n(x) \rangle$ of the infinite series $\sum f_n$ defined on D as follows:

$$\begin{aligned} s_1(x) &:= f_1(x), \\ s_2(x) &:= s_1(x) + f_2(x), \\ s_3(x) &:= s_2(x) + f_3(x), \\ &\dots \quad \dots \\ s_n(x) &:= s_{n-1}(x) + f_n(x) \\ &\dots \quad \dots \end{aligned}$$

Definition 2 Let $\langle f_n(x) \rangle$ be a sequence of functions defined on a set $D \subseteq \mathbb{R}$ such that the $\lim_{n \rightarrow \infty} s_n(x)$ of the sequence of partial sums exists for each $x \in D$. Then the function $s : D \rightarrow \mathbb{R}$ defined as

$$s(x) = \lim_{n \rightarrow \infty} s_n(x) \quad \forall x \in D$$

is called the pointwise limit of the series $\sum f_n$ on D , and the series $\sum f_n$ is said to be pointwise convergent to s on D .

Example 6 Consider the geometric series $\sum x^n$, where $x \in \mathbb{R}$. We know that this series is convergent for each x , $|x| < 1$, and divergent for each x , $|x| \geq 1$. Also, when $|x| < 1$, this geometric series has the sum

$$s(x) = \frac{1}{1-x}.$$

Thus, the geometric series $\sum x^n \rightarrow s(x)$ pointwise on $D = (-1, 1)$.

Example 7 Consider the series $\sum f_n(x)$, where for every $n \in \mathbb{N}$,

$$f_n(x) = \frac{x^2}{(1+x^2)^{n-1}} \quad \forall x \in \mathbb{R}.$$

At $x = 0$, we have $f_n(0) = 0$ for every $n \in \mathbb{N}$, so that the sum of the series is $s(0) = 0$. For $x \neq 0$, the given series is a geometric series with common ratio $(1+x^2)^{-1} < 1$, and therefore, the sum of the series is

$$g(x) = \frac{x^2}{1 - (1+x^2)^{-1}} = 1 + x^2 \quad \forall x \in \mathbb{R}.$$

Thus, the given series is pointwise convergent to $s(x)$ on $\mathbb{R}$, where

$$s(x) = \begin{cases} 1 + x^2, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Example 8 Suppose we have a series a functions $\sum u_n(x)$, where

$$u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \quad \forall x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

We use the sequence $\langle s_n(x) \rangle$ of partial sums to discuss the pointwise convergence of the given series. Here, we have

$$\begin{aligned} u_1(x) &= \frac{x}{1+x^2} - 0, \\ u_2(x) &= \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}, \\ u_3(x) &= \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}, \\ &\vdots \\ u_n(x) &= \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \end{aligned}$$

Therefore, we have

$$s_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) = \frac{nx}{1+n^2x^2}.$$

Since

$$\lim_{n \rightarrow \infty} s_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{1+n^2x^2} = \lim_{n \rightarrow \infty} \frac{x}{1/n + nx^2} = 0 \quad \forall x \in \mathbb{R},$$

we get that $s_n(x) \rightarrow 0$ pointwise on $\mathbb{R}$. Hence, the series $\sum u_n(x)$ converges pointwise to 0 on $\mathbb{R}$.

Example 9 Suppose we want to find the pointwise limit of the series

$$\sum_{n=1}^{\infty} xe^{-(n-1)x}, \quad x \in [0, 1].$$

The n^{th} -term of the sequence of partial sums $\langle s_n(x) \rangle$ is the sum of first n terms of a geometric series, so that

$$s_n(x) = x + xe^{-x} + xe^{-2x} + \cdots + xe^{-(n-1)x} = \frac{x(1 - e^{(1-n)x})}{1 - e^{-x}}.$$

Now, at $x = 0$, we have $s_n(x) = 0$ for all $x \in [0, 1]$. Also, for $x \neq 0$, we have

$$\lim_{n \rightarrow \infty} s_n(x) = \lim_{n \rightarrow \infty} \left(\frac{xe^x}{e^x - 1} \left(1 + \frac{1}{e^{(n-1)x}} \right) \right) = \frac{xe^x}{e^x - 1}.$$

So, the function

$$s(x) = \begin{cases} \frac{xe^x}{e^x-1}, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

is the required pointwise limit and, therefore, the series $\sum_{n=1}^{\infty} xe^{-(n-1)x}$ is pointwise convergent to $s(x)$ on $[0, 1]$.

Example 10 Consider the series $\sum u_n(x)$, where the n^{th} -term of the series is

$$u_n(x) = \frac{x}{[(n-1)x+1][nx+1]} \quad \forall x \in \mathbb{R}.$$

The sequence of partial sums $\langle s_n(x) \rangle$ of the given series is defined by

$$\begin{aligned} s_n(x) &= u_1(x) + u_2(x) + \cdots + u_n(x) \\ &= \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \cdots + \left(\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right) \\ &= 1 - \frac{1}{nx+1}. \end{aligned}$$

Thus, for $x \neq 0$, we have $\lim_{n \rightarrow \infty} s_n(x) = 1$, and for $x = 0$, $s_n(x) = 0 \quad \forall n \in \mathbb{N}$.

Hence, the given series is pointwise convergent to $s(x)$ on $\mathbb{R}$, where

$$s(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

2 Uniform convergence

In some of the preceding examples, notice that there are certain properties belonging to the terms of the sequence (respectively, the series) of functions which are not transferred to the corresponding pointwise limit function of the sequence (respectively, the series) of functions. For example,

- (i) In Example 6, each term of the geometric series $\sum x^n$ is bounded in $D = (-1, 1)$, but the sum function $s(x) = \frac{1}{1-x}$ is unbounded in D .
- (ii) In Example 7, each term of the series $\sum \frac{x^2}{(1+x^2)^{n-1}}$ is continuous at $x = 0$, whereas the sum function

$$s(x) = \begin{cases} 1 + x^2, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

is not continuous at $x = 0$. Similarly, in Example 1, every term of the sequence $\langle x^n \rangle$ is continuous in the interval $(-1, 1]$, but the limit function is discontinuous at $x = 1$.

(iii) Consider the sequence of functions $\langle f_n(x) \rangle$, where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \quad \forall x \in \mathbb{R}.$$

For each $x \in \mathbb{R}$, since $0 \leq \left| \frac{\sin nx}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$, we have that

$$f_n(x) \rightarrow f(x) \text{ pointwise on } \mathbb{R}, \text{ where } f(x) = 0.$$

Therefore, $f'(x) = 0$ and so $f'(0) = 0$. But, for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$f'_n(x) = \cos nx \cdot \frac{n}{\sqrt{n}} = \sqrt{n} \cdot \cos nx.$$

So, $f'_n(0) = \sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$. Thus, at $x = 0$, the sequence $\langle f'_n(x) \rangle$ diverges whereas $f'(x) = 0$. Thus, we have

$$\lim_{n \rightarrow \infty} \left(\left| \frac{d}{dx} f_n(x) \right|_{x=0} \right) \neq \left| \frac{d}{dx} \left(\lim_{n \rightarrow \infty} f_n(x) \right) \right|_{x=0}.$$

This shows that for a sequence of functions, the limit of differentials at a point may not be equal to the differential of the limit at that point.

(iv) Consider the sequence of functions $\langle g_n(x) \rangle$, where

$$g_n(x) = nx(1 - x^2)^n, \quad \forall x \in [0, 1].$$

At $x = 0$ and $x = 1$, each function $g_n = 0$, so that $\lim_{n \rightarrow \infty} g_n = 0$. For $0 < x < 1$,

$$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{(1 - x^2)^n} = 0.$$

Thus, the sequence $\langle g_n(x) \rangle$ converges to $g(x) = 0 \quad \forall x \in [0, 1]$, and

$$\int_0^1 g(x) dx = 0.$$

Also, for every $n \in \mathbb{N}$, we have

$$\int_0^1 g_n(x) dx = \int_0^1 nx(1 - x^2)^n dx = \frac{n}{n+2},$$

so that $\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx = \lim_{n \rightarrow \infty} \frac{n}{n+2} = \frac{1}{2}$. Hence, we get

$$\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx \neq \int_0^1 \lim_{n \rightarrow \infty} g_n(x) dx = \int_0^1 g(x) dx.$$

This shows that the limit of the integrals may not be equal to the integral of the limit.

Remark 1 *The above observations clearly indicate that the limit of a convergent sequence (or the series) of continuous (resp. differentiable, integrable) functions may not be continuous (resp. differentiable, integrable). The concept of uniform convergence of the sequence (resp. the series) is the sufficient condition to guarantee that the limit of a sequence of continuous functions is continuous and the limit of a sequence of integrable functions is integrable. Further, the limit of the integral is equal to the integral of the limit. A similar result for a sequence (or the series) of differentiable functions holds.*

Now, we discuss the concept of uniform convergence and apply it to prove the preceding remark. Recall that a sequence $f_n \rightarrow f$ pointwise on A_0 means the following: “for each $\epsilon > 0$ and each $x \in A_0$, there exists a natural number N (depending on both ϵ and $x \in A_0$) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.”$$

That the value of N depends on both ϵ and $x \in A_0$ has an intuitive interpretation. It says that the convergence of the sequence is “significantly faster” at some points than it is at others. However, if for each $\epsilon > 0$, we could find N so that

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in A_0 \text{ and } n \geq N, \quad \dots(3)$$

then the convergence of the sequence would be uniform, that is, the values $f_n(x)$ would be “uniformly” close to the values $f(x)$. Note that here N would depend on ϵ but not on $x \in A_0$. Such an N would exist if the set $\{N_\alpha : \alpha \in \mathcal{I}\}$ (it is an uncountable set) is bounded above with N as its supremum, where the natural number N_α is obtained as follows: “given $\epsilon > 0$, each point $x_\alpha \in A_0$ determines a natural number N_α such that

$$|f_n(x_\alpha) - f(x_\alpha)| < \epsilon \quad \forall n \geq N_\alpha.”$$

It is this subtle difference that distinguishes the notion of “pointwise convergence” of a sequence of functions and the notion of “uniform convergence”. Formally, we have

Definition 3 *A sequence $\langle f_n \rangle$ of real valued functions on $A \subseteq \mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ if for each $\epsilon > 0$, there is a natural number N (depending on ϵ but not on $x \in A_0$) such that*

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in A_0 \text{ and } \forall n \geq N. \quad \dots(4)$$

We denote it by $f_n \Rightarrow f$ on A_0 , or $f_n \rightarrow f$ uniformly on A_0 .

The uniform convergence for a series of functions $\sum f_n(x)$ is defined in a similar way by considering the uniform convergence of its sequence $\langle s_n(x) \rangle$ of partial sums.

Definition 4 A series $\sum f_n(x)$ of real valued functions on $A \subseteq \mathbb{R}$ **converges uniformly on** $D \subseteq A$ to a function $s : D \rightarrow \mathbb{R}$ if its sequence of partial sums $\langle s_n(x) \rangle$ is uniformly convergent on D , that is for each $\epsilon > 0$, there is a natural number N (depending on ϵ but not on $x \in D$) such that

$$|s_n(x) - s(x)| < \epsilon \quad \forall x \in D \text{ and } \forall n \geq N. \quad \dots(5)$$

We denote it by $\sum f_n \Rightarrow f$ on D , or $\sum f_n \rightarrow f$ uniformly on D .

Remark 2 If a sequence $\langle f_n(x) \rangle$ is uniformly convergent on A_0 , then it is pointwise convergent as well. For, there exists a natural number N (depending on ϵ but not on $x \in A_0$) such that the condition (4) holds and therefore this same N satisfies that for each $x \in A_0$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.$$

Similarly, if a series is uniformly convergent on D , then it pointwise convergent on D . However, pointwise convergence of a sequence or a series does not imply that it is uniformly convergent as well, but non-pointwise convergence does imply non-uniform convergence.

Theorem 1 (Cauchy's criterion for uniform convergence)

A sequence of functions $\langle f_n \rangle$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for each $\epsilon > 0$ and for all $x \in [a, b]$, there exists a positive integer N such that

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq N, p \geq 1. \quad \dots(1)$$

Proof: Let $\epsilon > 0$. First, suppose that the sequence $\langle f_n \rangle$ converges uniformly on $[a, b]$ to the limit function f . Then, by definition of uniform convergence, there exist positive integers m_1, m_2 such that

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall n \geq m_1 \text{ and } \forall x \in [a, b], \quad \dots(2)$$

$$|f_{n+p}(x) - f(x)| < \epsilon/2 \quad \forall n \geq m_2, p \geq 1 \text{ and } \forall x \in [a, b]. \quad \dots(3)$$

Let $N = \max\{m_1, m_2\}$. Then, by using (2) and (3) together with the triangle inequality, we get that $\forall x \in [a, b], \forall n \geq N$ and $p \geq 1$,

$$|f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Conversely, suppose that the given condition (1) holds. Then, by Cauchy's general principal of convergence for the sequence of numbers, the sequence $\langle f_n \rangle$ converges for each $x \in [a, b]$ to a limit, say f . So, the sequence $\langle f_n \rangle$ is pointwise convergent to f . To show that the convergence is uniform, let us choose N such that (1) holds. Then, for each $x \in [a, b]$,

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq N, p \geq 1.$$

Fix n and let $p \rightarrow \infty$. Since $f_{n+p} \rightarrow f$ as $p \rightarrow \infty$, we obtain that

$$|f(x) - f_n(x)| < \epsilon \quad \forall n \geq N \text{ and } \forall x \in [a, b].$$

This completes the proof. ■

Corollary 1 *A series of functions $\sum f_n$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\epsilon > 0$ and for all $x \in [a, b]$, there exists a positive integer N such that*

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \epsilon \quad \forall n \geq N, p \geq 1.$$

Proof: Consider the sequence of partial sums $s_n(x) = \sum_{i=1}^n f_i(x)$, $x \in [a, b]$. By Cauchy's criterion for uniform convergence of a sequence of functions in the preceding theorem, the sequence $\langle s_n \rangle$ is uniformly convergent on $[a, b]$ if and only if for all $x \in [a, b]$, there exists a positive integer N such that

$$|s_{n+p}(x) - s_n(x)| = |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \epsilon \quad \forall n \geq N, p \geq 1.$$

By definition of uniform convergence, the proof is complete. ■

The following is a necessary and sufficient condition for a sequence $\langle f_n \rangle$ to fail to converge uniformly to f on A_0 .

Lemma 1 (A test for non-uniform convergence)

A sequence $\langle f_n \rangle$ of real valued functions on $A \subseteq \mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ if and only if for some $\epsilon_0 > 0$ there is a subsequence $\langle f_{n_k} \rangle$ of $\langle f_n \rangle$ and a sequence $\langle x_k \rangle$ in A_0 such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \quad \forall k \in \mathbb{N}.$$

Proof: The negation of the statement that defines uniform convergence of the sequence $\langle f_n \rangle$ completes the proof. ■

Example 11 Consider the sequence $\langle f_n(x) \rangle$ defined by $f_n(x) := x/n$ for $n \in \mathbb{N}$, and let $f(x) = 0$ for all $x \in \mathbb{R}$. Example 1 shows that the sequence $\langle f_n \rangle$ is convergent pointwise to f on $\mathbb{R}$. If we let $n_k := k$ and $x_k := k$, then $f_{n_k}(x_k) = 1$ so that $|f_{n_k}(x_k) - f(x_k)| = |1 - 0| = 1$. Thus, taking $\epsilon_0 = 1/2$ and applying the preceding lemma, we obtain that the sequence $\langle f_n \rangle$ is not uniformly convergent to f on $\mathbb{R}$.

Example 12 Consider the sequence $\langle g_n(x) \rangle$ defined by $g_n(x) := x^n$ for $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$. Example 5 shows that this sequence is pointwise convergent to $g(x)$ defined on the set $A_0 = (-1, 1]$ by the equation

$$g(x) = \begin{cases} 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

If we let $n_k := k$ and $x_k := (\frac{1}{2})^{1/k}$, then $g_{n_k}(x_k) = 1/2$ so that

$$|g_{n_k}(x_k) - g(x_k)| = |1/2 - 0| = 1/2.$$

Thus, taking $\epsilon_0 = 1/3$ and applying the preceding lemma, we obtain that the sequence $\langle g_n \rangle$ is not uniformly convergent to g on $(-1, 1]$.

Example 13 Consider the sequence $\langle h_n(x) \rangle$ defined by $h_n(x) := \frac{(x^2 + nx)}{n}$ for $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, and let $h(x) := x$ for all $x \in \mathbb{R}$. Example 2 shows that $\langle h_n(x) \rangle \rightarrow h(x)$ pointwise on $\mathbb{R}$. If we let $n_k := k$ and $x_k := -k$, then $h_{n_k}(x_k) = 0$ and $h(x_k) = -k$ so that

$$|h_{n_k}(x_k) - h(x_k)| = |0 - (-k)| = k.$$

Thus, taking $\epsilon_0 < k$ and applying the preceding lemma, we obtain that the sequence $\langle h_n \rangle$ is not uniformly convergent to h on $\mathbb{R}$.

Definition 5 Let $A \subseteq \mathbb{R}$ and $\varphi : A \rightarrow \mathbb{R}$ be a function which is bounded on A . Then the **uniform norm of φ on A** is defined by

$$\|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}.$$

Note that it follows by definition that if $\epsilon > 0$, then we have

$$\|\varphi\|_A \leq \epsilon \quad \Leftrightarrow \quad |\varphi(x)| \leq \epsilon \quad \forall x \in A.$$

Theorem 2 (Norm test for uniform convergence)

A sequence $\langle f_n \rangle$ of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly to f on A if and only if $\|f_n - f\|_A \rightarrow 0$.

Proof: Suppose $\langle f_n \rangle$ converges uniformly to f on A . Then given any $\epsilon > 0$ there exists a natural number $N(\epsilon)$ such that for every $n \geq N(\epsilon)$ and $x \in A$, we have

$$|f_n(x) - f(x)| \leq \epsilon.$$

Therefore, we have

$$\|f_n - f\|_A := \sup\{|f_n(x) - f(x)| : x \in A\} \leq \epsilon \quad \text{whenever } n \geq N(\epsilon).$$

Since ϵ is arbitrary, this implies that $\|f_n - f\|_A \rightarrow 0$.

Conversely, let $\|f_n - f\|_A \rightarrow 0$. Then, given $\epsilon > 0$ there exists a positive integer $K(\epsilon)$ such that if $n \geq K(\epsilon)$, we have $\|f_n - f\|_A \leq \epsilon$. It follows by definition therefore that

$$|f_n(x) - f(x)| \leq \epsilon \quad \forall x \in A \text{ and } n \geq K(\epsilon).$$

Thus, $\langle f_n \rangle$ converges uniformly to f on A . ■

Remark 3 *Two equivalent useful forms of norm test are as follows:*

- (a) A sequence of bounded functions $f_n \rightarrow f$ uniformly on A if and only if there exists a natural number $N(\epsilon)$ such that

$$\|f_n - f\| < \epsilon \quad \forall n \geq N(\epsilon).$$

- (b) (**M_n -test**) Let $M_n = \sup\{|f_n(x) - f(x)| : x \in A\} = \|f_n - f\|_A$ so that M_n is the maximum value of $|f_n(x) - f(x)|$ on A , if it exists. Then $f_n \rightarrow f$ uniformly on A if and only if $M_n \rightarrow 0$ as $n \rightarrow \infty$.

The preceding lemma says that given a sequence $\langle f_n \rangle$ of functions, which converges pointwise to f on $A \subseteq \mathbb{R}$, the norm test can be applied to test its uniform convergence on A only if the function $f_n(x) - f(x)$ is bounded on A . For example, for the sequence of functions defined on $\mathbb{R}$ given in Example 11, norm test cannot be applied to test its uniform convergence on $\mathbb{R}$ because $f_n(x) - f(x) = x/n$ is not bounded on $\mathbb{R}$. However, if there is a set $B \subseteq \mathbb{R}$ such that the function $f_n(x) - f(x)$ is bounded on B , then the norm test can be applied to test the uniform convergence of $\langle f_n \rangle$ on B .

Example 14 Let $A := [0, 1]$ and consider the sequence $\langle f_n \rangle$ of functions defined on A as $f_n(x) = x/n$ for $n \in \mathbb{N}$. Observe that

$$\|f_n - f\| = \sup\{|x/n - 0| : x \in A\} = 1/n,$$

so that $\|f_n - f\| \rightarrow 0$. Therefore, by norm test, $\langle f_n \rangle$ is uniformly convergent on A .

Example 15 Consider the sequence $\langle g_n(x) \rangle$ defined by $g_n(x) := x^n$ for $n \in \mathbb{N}$ and for all $x \in A := [0, 1]$. Let $g(x) := 0$ for all $x : 0 \leq x < 1$ and $g(1) := 1$. Example 5 shows that $g_n \rightarrow g$ pointwise on A . Now, the functions $g_n(x) - g(x)$ are bounded on A and since for all $n \in \mathbb{N}$

$$\|g_n - g\| = \sup \begin{cases} x^n, & \text{for } 0 \leq x < 1 \\ 0, & \text{for } x = 1, \end{cases}$$

we get that $\|g_n - g\|_A = 1$ for all $n \in \mathbb{N}$. Thus, $\|g_n - g\|_A \not\rightarrow 0$, so by norm test, we infer that the sequence $\langle g_n \rangle$ is not uniformly convergent on A .

Example 16 Let $B := [0, 8]$ and $h_n(x) = (x^2 + nx)/n$ for all $n \in \mathbb{N}$ and for all $x \in B$. Then, Example 13 gives $h_n(x) \rightarrow h(x) := x$ pointwise on B , and

$$\|h_n - h\|_B = \sup \left\{ \frac{x^2}{n} - x : x \in B \right\} = \frac{64}{n}.$$

Therefore, $\|h_n - h\|_B \rightarrow 0$ so that by norm test, the sequence $\langle h_n(x) \rangle$ converges to $h(x)$ uniformly on B .

Example 17 Consider the sequence $\langle f_n \rangle$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$. Since

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{1+n^2x^2} = 0 \quad \forall x \in \mathbb{R},$$

the sequence $f_n \rightarrow 0$ pointwise on $\mathbb{R}$. Now, for each $n \in \mathbb{N}$, the function $f_n(x)$ attains its maximum value at $x = 1/n$. Also, the maximum value of $f_n(x)$ is $1/2$, so we get that

$$M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup\{|f_n(x)| : x \in \mathbb{R}\} = \frac{1}{2}.$$

Thus, $M_n \not\rightarrow 0$ as $n \rightarrow \infty$. Since $x = 1/n \rightarrow 0$ as $n \rightarrow \infty$, we get that by M_n -test, 0 is the point of non-uniform convergence. Hence, the sequence is not uniformly convergent on any interval which contains the origin.

Example 18 Consider the sequence $\langle f_n \rangle$, where $f_n(x) = \frac{x}{1+nx^2}$ $\forall x \in \mathbb{R}$. Since

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x}{1+nx^2} = 0 \quad \forall x \in \mathbb{R},$$

the sequence $f_n \rightarrow 0$ pointwise on $\mathbb{R}$. Now, for each $n \in \mathbb{N}$, the function $f_n(x)$ attains its maximum value at $x = 1/\sqrt{n}$. Also, the maximum value of $f_n(x)$ at $x = 1/\sqrt{n}$ is $1/2\sqrt{n}$, so we get that

$$M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup\{|f_n(x)| : x \in \mathbb{R}\} = \frac{1}{2\sqrt{n}}.$$

But then $\lim_{n \rightarrow \infty} M_n = 0$. Hence, by M_n -test, the sequence $\langle f_n \rangle$ is uniformly convergent on $\mathbb{R}$.

Example 19 Let $f_n(x) = nxe^{-nx^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, k]$, $k > 0$. We claim that the sequence $\langle f_n \rangle$ is not uniformly convergent on $[0, k]$. Here, the pointwise limit is

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{e^{nx^2}} = 0 \quad \forall x \in [0, k].$$

Now, for each $n \in \mathbb{N}$, the function $f_n(x) = nxe^{-nx^2}$ attains its maximum value at $x = 1/\sqrt{2n}$. Also, the maximum value of $f_n(x)$ at $x = 1/\sqrt{2n}$ is $\sqrt{n/2e}$, so we get that

$$M_n = \sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = \sup\{|f_n(x)| : x \in \mathbb{R}\} = \sqrt{\frac{n}{2e}}.$$

But then $\lim_{n \rightarrow \infty} M_n = \infty$. Since $x = 1/\sqrt{2n} \rightarrow 0$ as $n \rightarrow \infty$, 0 is the point of non-uniform convergence. Hence, by M_n -test, the sequence $\langle f_n \rangle$ is not uniformly convergent on $[0, k]$.

Example 20 Let $f_n(x) = x^n$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$. We claim that the sequence $\langle f_n \rangle$ is uniformly convergent on $[0, k]$, $k < 1$, and only pointwise convergent on $[0, 1]$. First, note that the pointwise limit of the given sequence is

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1. \end{cases}$$

So, $f_n \rightarrow f$ pointwise on $[0, 1]$. Let $\epsilon > 0$ be given. For $0 < x < 1$, we have

$$|f_n(x) - f(x)| < \epsilon \Rightarrow |x^n - 0| < \epsilon \Rightarrow x^n < \epsilon \Rightarrow \frac{1}{x^n} > \frac{1}{\epsilon} \Rightarrow n > \frac{\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{x})}.$$

Now, as x increases in $(0, 1)$, $\ln(1/x)$ decreases so the number $\ln(1/\epsilon)/\ln(1/x)$ increases with x , its maximum value being $\ln(1/\epsilon)/\ln(1/k)$, $0 < k < 1$. If we choose $M \geq \ln(1/\epsilon)/\ln(1/k)$, then for all $n \geq M$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall x : 0 < x < 1.$$

For $x = 0$, $|f_n(x) - f(x)| = 0 < \epsilon$ for all $n \geq 1$. Thus, there exists a positive integer $N = \max\{1, M\}$ such that for all $x \in [0, k]$, $k < 1$, we have $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. This gives us that the sequence $\langle f_n \rangle$ is uniformly convergent on $[0, k]$, $k < 1$. Further, since $\ln(1) = 0$ and the number $\ln(1/\epsilon)/\ln(1/x) \rightarrow \infty$ as $x \rightarrow 1$, we have that $x = 1$ is a point of non-uniform convergence. Hence, $f_n \not\rightarrow f$ uniformly on $[0, 1]$.

Example 21 For each $n \in \mathbb{N}$, let us define $f_n(x) = \frac{1}{x+n}$ for all $x \geq 0$. We claim that the sequence $\langle f_n \rangle$ is uniformly convergent on $[0, b]$, $b > 0$. Here,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{x+n} = 0 \quad \forall x \geq 0.$$

So, the sequence $\langle f_n \rangle \rightarrow 0$ pointwise on $[0, b]$, $b > 0$. Given $\epsilon > 0$, we have

$$|f_n(x) - 0| < \epsilon \Rightarrow \left| \frac{1}{x+n} \right| < \epsilon \Rightarrow \frac{1}{x+n} < \epsilon \Rightarrow n > \frac{1}{\epsilon} - x.$$

If $g(x) = 1/\epsilon - x$, then $g'(x) = -1$ implies that $g(x)$ is decreasing function of x for all $x \geq 0$, having maximum value $1/\epsilon$. Let us choose $N \geq 1/\epsilon$ so that for all $x \geq 0$, we get $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$. By the equivalent form (a) of norm test, our claim follows.

Example 22 Consider the sequence $\langle f_n \rangle$, where $f_n(x) = \tan^{-1} nx$, $x \geq 0$. We show that this sequence is uniformly convergent on $[a, b]$, $a > 0$, but is only pointwise convergent on $[0, b]$. First, note that for all $x \geq 0$, the sequence is pointwise convergent to the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \tan^{-1} nx = \begin{cases} \frac{\pi}{2}, & \text{for } x > 0 \\ 0, & \text{for } x = 0. \end{cases}$$

Let $\epsilon > 0$. For $x > 0$, we have

$$|f_n(x) - f(x)| = |\tan^{-1} nx - \pi/2| = |\cot^{-1} nx| < \epsilon$$

if $\cot^{-1} nx < \epsilon$, or $\tan^{-1}(1/nx) < \epsilon \Rightarrow 1/nx < \tan \epsilon \Rightarrow n > \cot \epsilon/x$. Now, the function $\cot \epsilon/x$ decreases as x increases, with maximum value in $[a, b]$, $a > 0$, being $\cot \epsilon/a$. Let us choose $N \geq \cot \epsilon/a$ so we get

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N \text{ and } x \in [a, b], a > 0.$$

For $x = 0$, $|f_n(x) - f(x)| = 0 < \epsilon$ for all $n \geq 1$. Thus, there exists a positive integer $N_0 = \max\{1, N\}$ such that for all $x \in [a, b]$, $a > 0$, we have

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N_0. \quad \dots(6)$$

This gives us that the sequence $\langle f_n \rangle$ is uniformly convergent on $[a, b]$, $a > 0$. However, $\cot \epsilon/x \rightarrow \infty$ as $x \rightarrow 0$, so no integer exists such that (6) holds. Hence, the sequence $\langle f_n \rangle$ is not uniformly convergent on $[0, b]$.

Exercise 1: Test for uniform convergence the series $\sum f_n(x)$ and $\sum g_n(x)$ on the interval $[0, 1]$, where

$$f_n(x) = xe^{-nx} \quad \text{and} \quad g_n(x) = \frac{nx^2}{1+nx} - \frac{(n-1)^2x^2}{1+(n-1)x}.$$

Theorem 3 (Weierstrass's M -test for uniform convergence)

A series of functions $\sum f_n$ will converge uniformly (and absolutely) on $[a, b]$ if there exists a series $\sum M_n$ of positive numbers such that

$$(i) \quad |f_n(x)| \leq M_n \quad \forall n \in \mathbb{N} \quad \text{and} \quad \forall x \in [a, b]$$

$$(ii) \quad \sum M_n \text{ converges.}$$

Proof: Let $\epsilon > 0$. Since $\sum M_n$ is convergent, we have by Cauchy's general principle of convergence of a series of numbers that there exists a positive integer N such that

$$|M_{n+p} - M_n| = M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon \quad \forall n \geq N, \quad p \geq 1.$$

Hence, $\forall x \in [a, b]$ and $\forall n \geq N, p \geq 1$, we have by using (i) and (ii),

$$|f_{n+1}(x) + \cdots + f_{n+p}(x)| \leq |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| \leq \sum_{i=n+1}^{n+p} M_i < \epsilon.$$

By Corollary 1, this proves that $\sum f_n$ is uniformly (and absolutely) convergent on $[a, b]$. ■

Example 23 Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

We claim that this series is uniformly convergent for every $x \in \mathbb{R}$. The n^{th} -term of the series is $f_n(x) = \frac{\cos nx}{n^2}$. Since

$$\left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R},$$

and the series of numbers $\sum \frac{1}{n^2}$ is convergent, we get by M-test that the given series is uniformly convergent, and our claim follows.

Example 24 Let $0 < r < 1$. Then each of the following series

$$\sum r^n \cos nx, \quad \sum r^n \sin nx, \quad \sum r^n \cos n^2 x, \quad \sum r^n \sin n^2 x,$$

is uniformly convergent by M-test because, in each case, the n^{th} -term $f_n(x)$ satisfies $|f_n(x)| \leq r^n$ and $\sum r^n$ is convergent for $0 < r < 1$.

Example 25 The series

$$\sum \frac{\sin(n^2x + x^2)}{n(n+1)}$$

is uniformly convergent for every $x \in \mathbb{R}$ by M-test because the n^{th} -term

$$\left| \frac{\sin(n^2x + x^2)}{n(n+1)} \right| \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2},$$

and the series $\sum \frac{1}{n^2}$ is convergent.

Example 26 Both the series

$$\sum \frac{a_n x^n}{1+x^{2n}} \quad \text{and} \quad \sum \frac{a_n x^{2n}}{1+x^{2n}}$$

are uniformly convergent if the series $\sum a_n$ is convergent. This follows by using M-test as, in each series, the n^{th} -term $f_n(x)$ satisfies

$$|f_n(x)| \leq a_n \quad \forall x \in \mathbb{R}.$$

Exercise 2: Test for uniform convergence

- (i) $\sum \frac{\cos nx}{n^p} \quad (p > 1).$
- (ii) $\sum \frac{(-1)^n x^{2n}}{n^p(1+x^{2n})} \quad (p > 1).$
- (iii) $\sum \frac{1}{n^x}.$
- (iv) $\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \dots \quad -\frac{1}{2} \leq x \leq \frac{1}{2}.$

Exercise 3: Show that the series

$$\sum \frac{x}{n^p + x^2 n^q}$$

converges uniformly on any interval $[a, b]$ for

- (a) $q > 0$, $p > 1$, and
- (b) $0 < p \leq 1$, $p + q > 2$.

Theorem 4 Let $\langle f_n \rangle$ be a sequence of functions defined on A which is

(i) $f_n \rightarrow f$ uniformly on A , and

(ii) each f_n is bounded on A , that is, for each n there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$.

Then the function f is bounded on A .

Proof: Since $\langle f_n \rangle$ is uniformly convergent on A , by Cauchy's criterion for uniform convergence, we have that given $\epsilon = 1$, there exists a positive integer N such that

$$|f_{n+p}(x) - f_n(x)| < 1 \quad \forall x \in A \text{ and } \forall n \geq N, p \geq 1.$$

In particular, we have $|f_n(x) - f_N(x)| < 1$ for all $x \in A$ and for all $n \geq N$, so that

$$|f_n(x)| < |f_N(x)| + 1 \quad \forall x \in A, \text{ and } \forall n \geq N. \quad \dots(1)$$

Since each f_n is bounded on A , $\|f_n\|_A \leq M_n$ for each $n \in \mathbb{N}$. If we define

$$M := \max \{M_1, M_2, \dots, M_{N-1}, M_N + 1\},$$

then, from (1), we have that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$. Hence, we obtain using (i) that for each $x \in A$,

$$\lim_{n \rightarrow \infty} |f_n(x)| = |\lim_{n \rightarrow \infty} f_n(x)| = |f(x)| \leq M.$$

Thus, $|f(x)| \leq M$ for all $x \in A$, which shows that f is bounded. ■

Remark 4 The conclusion in preceding theorem will not be true if the convergence is not uniform. For example, consider the sequence of functions $\langle f_n(x) \rangle$ defined on $[0, \infty)$ by the equation

$$f_n(x) = \frac{nx}{1 + nx^2}, \quad n \in \mathbb{N}.$$

Then, we have $0 \leq f_n(x) \leq nx \leq n$ on $[0, 1]$ and $0 \leq f_n(x) \leq 1/x \leq 1$ on $[1, \infty)$, so that each term f_n is bounded on $[0, \infty)$. Moreover, the sequence converges pointwise to $f(x)$ on $[0, \infty)$, where

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & \text{for } x = 0 \\ 1/x, & \text{for } x > 0, \end{cases}$$

which is not bounded on $[0, \infty)$. Thus, the convergence of $\langle f_n(x) \rangle$ cannot be uniform.

A result similar to the preceding theorem holds for a series of functions as well, where we use the sequence of partial sums of the series $\sum f_n$ along with the preceding theorem to prove that

Theorem 5 *Let $\sum f_n$ be a series of functions defined on A which satisfies that*

- (i) $\sum f_n \rightarrow f$ uniformly on A , and
- (ii) *each f_n is bounded on A , that is, for each n there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$.*

Then the sum function f is bounded on A .

Remark 5 Note that again the conclusion in the preceding theorem fails to hold if the convergence is not uniform. To illustrate this, consider Example 6, where we have seen that the geometric series $\sum x^n$ is pointwise convergent on $(-1, 1)$ to the sum function $f(x) = \frac{1}{1-x}$. Each term of the series is bounded on $(-1, 1)$ but the sum function $f(x)$ is unbounded on $(-1, 1)$.

Theorem 6 *Let a sequence $\langle f_n \rangle$ converge uniformly to f on A and $x_0 \in A$ be any point such that $\lim_{x \rightarrow x_0} f_n(x) = a_n$, $n = 1, 2, 3, \dots$. Then*

- (a) *the sequence $\langle a_n \rangle$ converges, and*
- (b) $\lim_{x \rightarrow x_0} f(x) = \lim_{n \rightarrow \infty} a_n$, *or equivalently*

$$\lim_{x \rightarrow x_0} \left(\lim_{n \rightarrow \infty} f_n(x) \right) = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow x_0} f_n(x) \right).$$

Proof: By Cauchy's criterion of uniform convergence, given $\epsilon > 0$, there exists a positive integer N (independent of x) such that for all $x \in A$,

$$|f_{n+p}(x) - f_n(x)| < \epsilon \quad \forall n \geq N, p \geq 1.$$

Keeping n and p fixed and letting $x \rightarrow x_0$, we get that

$$|a_{n+p} - a_n| < \epsilon \quad \forall n \geq N, p \geq 1.$$

Again, by using Cauchy's convergence criterion for convergence of a sequence of real numbers, we have that $\langle a_n \rangle$ converges, which proves (a). Let $\langle a_n \rangle \rightarrow L$ so that there exists a positive integer N_1 such that

$$|a_n - L| < \epsilon/3 \quad \forall n \geq N_1. \quad \dots(1)$$

Similarly, since $\langle f_n \rangle$ converges uniformly to f , there exists a positive integer N_2 such that for all $x \in A$,

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall n \geq N_2. \quad \dots(2)$$

Also, since $\lim_{x \rightarrow x_0} f_n(x) = a_n$ for all $n \in \mathbb{N}$, there exists a $\delta > 0$ such that for all $x \in A$ with $|x - x_0| < \delta$, we have

$$|f_n(x) - a_n| < \epsilon/3 \quad \forall n \in \mathbb{N}. \quad \dots(3)$$

Now, let $N = \max\{N_1, N_2\}$. Hence, for all $n \geq N$ and $|x - x_0| < \delta$, we have by using (1), (2), (3) that

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - a_n| + |a_n - L| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

which show that $\lim_{x \rightarrow x_0} f(x) = L = \lim_{n \rightarrow \infty} a_n$. This proves (b). $\blacksquare$

Theorem 7 *If a series $\sum f_n$ converges uniformly to f on A and $x_0 \in A$ is any point such that $\lim_{x \rightarrow x_0} f_n(x) = a_n$, $n = 1, 2, 3, \dots$. Then*

- (a) *the series $\sum a_n$ converges, and*
- (b) *$\lim_{x \rightarrow x_0} f(x) = \sum a_n$, or equivalently*

$$\lim_{x \rightarrow x_0} \left(\sum f_n(x) \right) = \sum \left(\lim_{x \rightarrow x_0} f_n(x) \right).$$

Proof: Since the series $\sum f_n$ is uniformly convergent to f on A , its sequence $\langle s_n(x) \rangle$ of partial sums is uniformly convergent to f on A . Thus, given $\epsilon > 0$, there exists an integer $m > 0$ such that for all $x \in A$, we have, by using Cauchy's uniform convergence criterion, that

$$|s_{n+p}(x) - s_n(x)| = |f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x)| < \epsilon \quad \forall n \geq m, p \geq 1.$$

Let $x \rightarrow x_0$ and keep n and p fixed, so that we get

$$|a_1 + a_2 + \dots + a_{n+p}| < \epsilon \quad \forall n \geq m, p \geq 1.$$

By Cauchy's convergence criterion, we have that the series $\sum a_n$ is convergent, which proves (a). Let $\sum a_n = S$ so that there exists an integer N_1 such that

$$\left| \sum_{r=1}^n a_r - S \right| < \epsilon/3 \quad \forall n \geq N_1. \quad \dots(1)$$

Since the sequence $\langle s_n(x) \rangle$ converges uniformly to f on A , we have by definition that there exists $N_2 > 0$ such that

$$|s_n(x) - f(x)| = \left| \sum_{r=1}^n f_r(x) - f(x) \right| < \epsilon/3 \quad \forall n \geq N_2. \quad \dots(2)$$

Again, we are given that $\lim_{x \rightarrow x_0} f_n(x) = a_n$, $(n = 1, 2, 3, \dots)$, therefore, there exists $\delta_r > 0$ such that for each x , $|x - x_0| < \delta_r$,

$$|f_r(x) - a_r| < \frac{\epsilon}{3n} \quad \forall r = 1, 2, 3, \dots$$

Let $\delta = \min \{\delta_1, \delta_2, \dots, \delta_n\}$. Then for each $r = 1, 2, 3, \dots, n$ we get

$$|f_r(x) - a_r| < \frac{\epsilon}{3n} \quad \forall x : |x - x_0| < \delta.$$

Thus, we have for each $x : |x - x_0| < \delta$, that

$$\left| \sum_{r=1}^n f_r(x) - \sum_{r=1}^n a_r \right| \leq \sum_{r=1}^n |f_r(x) - a_r| < n \cdot \frac{\epsilon}{3n} = \frac{\epsilon}{3} \quad \dots(3)$$

Let $N = \max \{N_1, N_2\}$. Hence, by using (1), (2), (3), we get that for all $n \geq N$ and for each $x : |x - x_0| < \delta$,

$$|f(x) - S| \leq |f(x) - \sum_{r=1}^n f_r(x)| + \left| \sum_{r=1}^n f_r(x) - \sum_{r=1}^n a_r \right| + \left| \sum_{r=1}^n a_r - S \right| < 3 \cdot \frac{\epsilon}{3} = \epsilon,$$

which implies that $\lim_{x \rightarrow x_0} f(x) = S = \sum a_n$, which proves (b). ■

Remark 6 In Theorem 6 and 7, the converse is neither implied nor true, that is, the condition of uniform convergence is only sufficient, not a necessary condition. For, we have

$$1 = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow 1} x^n \right) \neq \lim_{x \rightarrow 1} \left(\lim_{n \rightarrow \infty} x^n \right) = 0,$$

and

$$0 = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow 0} \sum_{r=0}^{n-1} x e^{-nx} \right) \neq \lim_{x \rightarrow 0} \left(\lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} x e^{-nx} \right) = 1.$$

Thus, the convergence of the series $\sum_{r=0}^{n-1} x e^{-nx}$ cannot be uniform on any interval which contains 0. We have already observed in Example 12 that the sequence $\langle x^n \rangle$ is not uniformly convergent on $(-1, 1]$, or on any interval containing 1.

Theorem 8 (Uniform convergence and Continuity)

(A) Let $\langle f_n \rangle$ be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f : A \rightarrow \mathbb{R}$. Then f is continuous on A .

(B) If $f_n : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous on D for each $n \in \mathbb{N}$ and if the series $\sum f_n$ converges to f uniformly on D , then f is continuous on D .

Proof: (A) Since $\langle f_n \rangle$ is uniformly convergent to f on A , given $\epsilon > 0$, there exists a natural number N such that if $n \geq N$, then

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall x \in A.$$

In particular, we have

$$|f_N(x) - f(x)| < \epsilon/3 \quad \forall x \in A. \quad \dots(1)$$

If $c \in A$ is an arbitrary point, then (1) gives us

$$|f_N(c) - f(c)| < \epsilon/3. \quad \dots(2)$$

We show that f is continuous at c . We are given that f_n is continuous on A for each $x \in A$, so that f_N is continuous at $x = c$. Thus, there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in A$, then

$$|f_N(x) - f_N(c)| < \epsilon/3. \quad \dots(3)$$

Using (1), (2) and (3) and the Triangle inequality, we get that if $|x - c| < \delta$ and $x \in A$, then

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < 3 \cdot \epsilon/3 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that f is continuous at the arbitrary point $c \in A$. Thus, f is continuous on A .

(B) Since $\sum f_n$ converges uniformly to f on D , given $\epsilon > 0$, there exists a natural number N such that for every $x \in D$,

$$\left| \sum_{r=1}^n f_r(x) - f(x) \right| < \epsilon/3 \quad \forall n \geq N.$$

In particular, for $n = N$, we have

$$\left| \sum_{r=1}^N f_r(x) - f(x) \right| < \epsilon/3 \quad \forall x \in D. \quad \dots(1)$$

Let c be any arbitrary point in D , then inequality (1) implies that

$$\left| \sum_{r=1}^N f_r(c) - f(c) \right| < \epsilon/3. \quad \dots(2)$$

Now, for each n , f_n is continuous on D so that the sum function $\sum_{r=1}^N f_r(x)$ is continuous on D . In particular, it is continuous at $x = c$. Thus, there exists a $\delta > 0$ such that

$$\left| \sum_{r=1}^N f_r(x) - \sum_{r=1}^N f_r(c) \right| < \epsilon/3 \quad \forall x \in D \text{ with } |x - c| < \delta. \quad \dots(3)$$

By the triangle inequality,

$$|f(x) - f(c)| \leq \left| f(x) - \sum_{r=1}^N f_r(x) \right| + \left| \sum_{r=1}^N f_r(x) - \sum_{r=1}^N f_r(c) \right| + \left| \sum_{r=1}^N f_r(c) - f(c) \right|,$$

which, on using (1),(2) and (3), implies that for all $x \in D$ with $|x - c| < \delta$,

$$|f(x) - f(c)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

This shows that f is continuous at c . As c is an arbitrary element of D , we obtain that f is continuous on D . ■

Remark 7 Note that the condition of uniform convergence is only sufficient, not a necessary condition. For example, each term of the sequence $\langle f_n \rangle$ defined by $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$ is continuous on $[0, 1]$. The sequence converges to $f = 0$ on $[0, 1]$, which is continuous on $[0, 1]$. But the convergence is not uniform on $[0, 1]$, as already observed in Example 19.

On the other hand, if the limit function of a sequence or the sum function of a series is not continuous, the convergence cannot be uniform. Therefore, the preceding theorem can be used to test a sequence or a series of functions for non-uniform convergence. For example, consider the series $\sum f_n$ defined by $f_n(x) = (1 - x^2)x^n$ on $[0, 1]$. The series converges pointwise to the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} (1 - x^2)x^n = \begin{cases} x + 1, & \text{for } 0 \leq x < 1 \\ 0, & \text{for } x = 1. \end{cases}$$

For each $n \in \mathbb{N}$, each term f_n of the series is continuous on $[0, 1]$ whereas $f(x)$ is discontinuous at $x = 1$. So, the convergence of the series $\sum f_n$ cannot be uniform on $[0, 1]$. Similarly, each term of the sequence $\langle x^n \rangle$ defined on $[0, 1]$ is continuous whereas the limit function

$$f(x) = \lim_{n \rightarrow \infty} x^n = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1 \end{cases}$$

which is discontinuous. So, the sequence $\langle x^n \rangle$ cannot converge uniformly to $f(x)$ on $[0, 1]$.

Theorem 9 (Uniform convergence and Integration)

(A) Let $\langle f_n \rangle$ be a sequence of functions in $\mathcal{R}[a, b]$ and suppose that $\langle f_n \rangle$ converges uniformly to f on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ and the sequence $\langle \int_a^b f_n(t) dt \rangle$ converges uniformly to $\int_a^b f(t) dt$, i.e.

$$\int_a^b f(t) dt = \int_a^b \left(\lim_{n \rightarrow \infty} f_n(t) \right) dt = \lim_{n \rightarrow \infty} \left(\int_a^b f_n(t) dt \right).$$

(B) If $f_n \in \mathcal{R}[a, b]$ for each $n \in \mathbb{N}$ and if the series $\sum f_n$ converges to f uniformly on $[a, b]$, then $f \in \mathcal{R}[a, b]$ and the series $\sum \int_a^b f_n(t) dt$ converges uniformly to $\int_a^b f(t) dt$, i.e.

$$\int_a^b f(t) dt = \int_a^b \left(\sum_{n=1}^{\infty} f_n(t) \right) dt = \sum_{n=1}^{\infty} \left(\int_a^b f_n(t) dt \right).$$

Proof: (A) Let $\epsilon > 0$ be given. We first show that $f \in \mathcal{R}[a, b]$. Since $\langle f_n \rangle$ converges uniformly to f on $[a, b]$, there exists a natural number N such that if $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)} \quad \forall x \in [a, b]. \quad \dots(1)$$

In particular, for $n = N$, we have

$$|f_N(x) - f(x)| < \frac{\epsilon}{3(b-a)} \quad \forall x \in [a, b]$$

or equivalently

$$f(x) - \frac{\epsilon}{3(b-a)} < f_N(x) < f(x) + \frac{\epsilon}{3(b-a)} \quad \forall x \in [a, b]. \quad \dots(2)$$

Now, since $f_n \in \mathcal{R}[a, b]$ for each $n \in \mathbb{N}$, f_N is integrable, so there is a partition $P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$ of $[a, b]$ such that

$$U(P, f_N) - L(P, f_N) < \epsilon/3 \quad \dots(3)$$

Let m'_i, M'_i and m_i, M_i denote the infimum and supremum of f_N and f , respectively, on $[x_{i-1}, x_i]$ for each $i = 1, 2, \dots, n$. Using second part of (2), we obtain that $m'_i \leq m_i + \frac{\epsilon}{3(b-a)}$ which implies

$$\sum_{i=1}^n m'_i \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i + \frac{\epsilon}{3(b-a)} \sum_{i=1}^n \Delta x_i \Rightarrow L(P, f_N) \leq L(P, f) + \frac{\epsilon}{3}$$

or equivalently

$$-L(P, f) \leq -L(P, f_N) + \frac{\epsilon}{3}. \quad \dots(4)$$

Similarly, by using the first part of inequality (2), we get

$$U(P, f) \leq U(P, f_N) + \frac{\epsilon}{3}. \quad \dots(5)$$

Adding (4) and (5) and using (3), we get

$$U(P, f) - L(P, f) \leq \{U(P, f_N) - L(P, f_N)\} + 2\left(\frac{\epsilon}{3}\right) < \frac{\epsilon}{3} + 2\left(\frac{\epsilon}{3}\right) = \epsilon.$$

This proves that $f \in \mathcal{R}[a, b]$. Now, by using Cauchy criterion for uniform convergence, there exists a positive integer H such that if $m > n \geq H$, then

$$-\frac{\epsilon}{b-a} \leq f_m(x) - f_n(x) \leq \frac{\epsilon}{b-a} \quad \forall x \in [a, b].$$

Integrating over $[a, b]$ and using the inequality property of integrals, we have

$$-\epsilon \leq \int_a^b f_m - \int_a^b f_n \leq \epsilon \Rightarrow \left| \int_a^b f_m - \int_a^b f_n \right| < \epsilon \quad \forall m > n \geq H.$$

Since $\epsilon > 0$ is arbitrary, the sequence $\langle \int_a^b f_n \rangle$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. Also, using (1), we obtain that for all $x \in [a, b]$ and $n \geq N$,

$$\left| \int_a^b f_n(t) dt - \int_a^b f(t) dt \right| \leq \int_a^b |f_n(t) - f(t)| dt < \int_a^b \frac{\epsilon}{(b-a)} dt \leq \epsilon$$

which shows that the sequence $\langle \int_a^b f_n(t) dt \rangle$ converges uniformly to $\int_a^b f(t) dt$ on $[a, b]$. Thus, we have

$$\lim_{n \rightarrow \infty} \left(\int_a^b f_n(t) dt \right) = \int_a^b f(t) dt = \int_a^b \left(\lim_{n \rightarrow \infty} f_n(t) \right) dt.$$

(B) Let $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ for all $x \in [a, b]$. Since each f_i ($i = 1, 2, \dots, n$) is integrable on $[a, b]$, the sequence $\langle s_n \rangle$ is integrable on $[a, b]$. Also, uniform convergence of the series $\sum f_n$ to f on $[a, b]$ is equivalent to the uniform convergence of the sequence $\langle s_n \rangle$ to f on $[a, b]$. Hence, by using part (A) of the theorem, f is integrable on $[a, b]$, and

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b s_n = \lim_{n \rightarrow \infty} \int_a^b \left(\sum_{r=1}^n f_r \right) = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\int_a^b f_r \right) = \sum_{n=1}^{\infty} \left(\int_a^b f_n \right).$$

This completes the proof. ■

Remark 8 The converse of the preceding theorem is not true, that is, a series (or a sequence) of functions may converge to a function which is integrable without itself being uniformly convergent. For example, the series

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} \quad \forall x \in (0, 1).$$

Each term of the series is integrable and

$$\int_0^1 (1 - x + x^2 - x^3 + \cdots) dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Also, the sum function is integrable and

$$\int_0^1 \frac{1}{1+x} dx = \ln 2.$$

But we know that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ so that we have

$$\int_0^1 (1 - x + x^2 - x^3 + \cdots) dx = \int_0^1 \frac{1}{1+x} dx.$$

However, the convergence of the given series is not uniform on $[0, 1]$ since at $x = 1$, the series reduces to $\sum (-1)^n$, which does not converge. So, the condition of uniform convergence is only sufficient but not necessary.

On the other hand, if the limit function is not integrable or if integrable, the integral is not equal to the limit of the integrals, the convergence cannot be uniform. For example, consider the sequence $\langle f_n \rangle$ defined by $f_n(x) = nxe^{-nx^2}$ on $[0, 1]$. We have seen in Example 19 that $\langle f_n \rangle \rightarrow f(x) = 0$ pointwise on $[0, 1]$ so that

$$\int f(x) dx = 0.$$

Also,

$$\int_0^1 f_n(x)dx = \frac{1}{2}(1 - e^{-n}) \Rightarrow \lim_{n \rightarrow \infty} \int_0^1 f_n(x)dx = \frac{1}{2} \neq 0 = \int_0^1 \lim_{n \rightarrow \infty} f_n(x)dx.$$

Thus, $\langle f_n \rangle$ is not uniformly convergent on $[0, 1]$. Thus, the preceding theorem can be used as a test for non-uniform convergence of a sequence or a series of functions. Similarly, let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined for $n \geq 2$ by

$$f_n(x) = \begin{cases} n^2x & \text{for } 0 \leq x \leq 1/n \\ -n^2(x - 2/n) & \text{for } 1/n \leq x \leq 2/n \\ 0 & \text{for } 2/n \leq x \leq 1. \end{cases}$$

Then each f_n is continuous on $[0, 1]$ and hence integrable on $[0, 1]$. Also, by using the significance of the integral as an area (or by direct calculation), we obtain

$$\int_0^1 f_n(x)dx = \frac{1}{2}(n)\left(\frac{2}{n}\right) = 1 \quad \forall n \geq 2 \Rightarrow \lim_{n \rightarrow \infty} \left(\int_0^1 f_n(x)dx \right) = 1.$$

The pointwise limit of the sequence $\langle f_n \rangle$ is $f(x) = 0$ for all $x \in [0, 1]$. But, then f is continuous and hence integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 0$. Therefore, we get

$$\int_0^1 \left(\lim_{n \rightarrow \infty} f_n(x) \right) dx = \int_0^1 f(x)dx = 0 \neq 1 = \lim_{n \rightarrow \infty} \left(\int_0^1 f_n(x)dx \right).$$

This shows that the sequence $\langle f_n \rangle$ cannot converge uniformly on $[0, 1]$.

Theorem 10 (Uniform convergence and differentiation)

(A) Let $J \subseteq \mathbb{R}$ be a bounded interval and for each $n \in \mathbb{N}$, let f_n be real-valued function on J that has a derivative f'_n on J . Suppose that

- (i) there exists $x_0 \in J$ such that sequence $\langle f_n(x_0) \rangle$ converges, and
- (ii) the sequence of derivatives $\langle f'_n \rangle$ converges uniformly on J to a function g .

Then the sequence $\langle f_n \rangle$ converges uniformly to a function f that is differentiable at every point of J and $f' = g$.

(B) For each $n \in \mathbb{N}$, let f_n be real-valued function on $J = [a, b]$ that has a derivative f'_n on J . Suppose that

- (i) there exists $x_0 \in J$ such that the series $\sum f_n(x_0)$ converges, and

(ii) the series of derivatives $\sum f'_n$ converges uniformly on J .

Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f . In addition, f has a derivative on J and

$$f' = \sum f'_n.$$

(The proof is not in the course and hence omitted).

Remark 9 The condition of uniform convergence of the sequence (or series) of differentials is only sufficient but not necessary. For example, consider the sequence $\langle f_n \rangle$ defined by $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for every $x \in \mathbb{R}$. Then this sequence converges pointwise to the function

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{\sin nx}{\sqrt{n}} \leq \lim_{n \rightarrow \infty} \left| \frac{\sin nx}{\sqrt{n}} \right| \leq \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

Therefore, $f'(x) = 0$ which gives that $f'(0) = 0$. But

$$f'_n(x) = \frac{n \cdot \cos nx}{\sqrt{n}} = \sqrt{n} \cos nx \Rightarrow f'_n(0) = \sqrt{n} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Thus, at $x = 0$, the sequence of differentials $\langle f'_n(x) \rangle$ diverges whereas $f'(x) = 0$, that is, the limit of the differentials is not equal to the differential of the limit. This shows that $\langle f_n(x) \rangle$ cannot converge uniformly on any interval containing 0.