

Tutorial Notes

Example 1.2.3: Prove that

$$\text{a) } \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dy \right\} dx = \frac{1}{2} \quad \text{b) } \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dx \right\} dy = -\frac{1}{2}$$

and comment why this happens.

Solution:

$$\begin{aligned} \text{a) } \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dy \right\} dx &= \int_0^1 \left\{ \int_0^1 \frac{2x-(x+y)}{(x+y)^3} dy \right\} dx = \int_0^1 \left\{ \int_0^1 \left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right) dy \right\} dx \\ &= \int_0^1 \left(\frac{-x}{(x+y)^2} + \frac{1}{x+y} \right) \Big|_{y=0}^1 dx = \int_0^1 \frac{dx}{(x+1)^2} = -\frac{1}{x+1} \Big|_0^1 = \frac{1}{2}. \end{aligned}$$

b) This follows at once on formally interchanging x and y in (a) to obtain

$$\int_0^1 \left\{ \int_0^1 \frac{y-x}{(x+y)^3} dx \right\} dy = \frac{1}{2} \text{ and then multiplying both sides by } -1.$$

This example shows that interchanging the order of integration may not always produce equal results. A sufficient condition under which the order may be interchanged is that the double integral over the

corresponding region exists. In this case $\iint_R \frac{x-y}{(x+y)^3} dx dy$, where $\mathbf{R}$ is the region

$0 \leq x \leq 1, 0 \leq y \leq 1$ fails to exist because of the discontinuity of the integrand at $(0, 0)$. The Integral is actually an improper double integral.

Example 1.2.4: Compute the following integral in two different ways

$$\iint_R x^2 y dA,$$

where $\mathbf{R}$ is the first-quadrant region bounded by the two curves $y = \sqrt{x}$ and $y = x^3$.

Solution:

Before attempting to solve the double integral, it is better to find the points where the two curves intersect. As indicated in **Fig 1.2.2** and **Fig 1.2.3**, the given region is both vertically simple and horizontally simple.

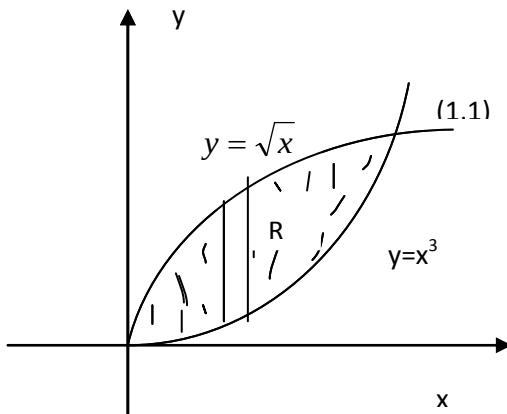


Fig1.2.2 the vertically simple region of example 1.2.4

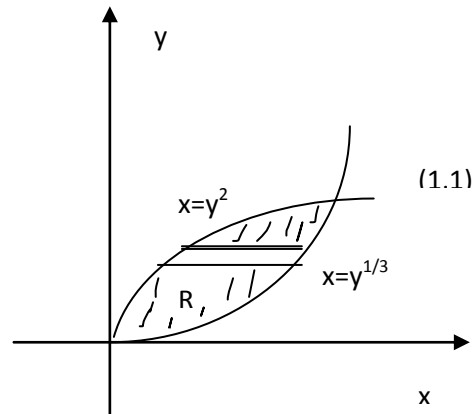


Fig1.2.3 the horizontally simple region of example 1.2.4

$$\iint_R x^2 y dA = \int_{x=0}^1 \left\{ \int_{y=x^3}^{\sqrt{x}} x^2 dy \right\} dx$$

$$= \int_{x=0}^1 \left\{ \frac{x^2 y^2}{2} \right\} \bigg|_{y=x^3}^{\sqrt{x}} dx = \int_{x=0}^1 \left(\frac{x^3 - x^8}{2} \right) dx = \left\{ \frac{x^4}{8} - \frac{x^9}{18} \right\} \bigg|_0^1 = \frac{5}{72}.$$

When we solve for x to the equations $y = \sqrt{x}$ and $y = x^3$ respectively are $x = y^2$ and $x = \sqrt[3]{y}$.

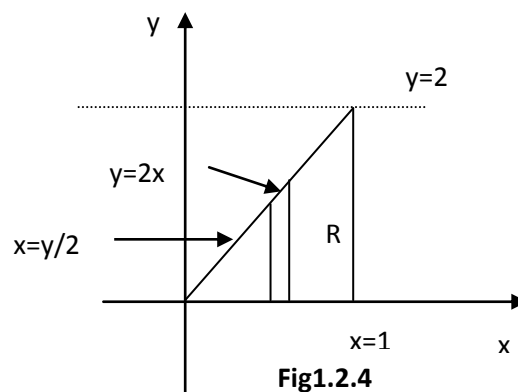
Hence,

$$\begin{aligned} \iint_R x^2 y dA &= \int_{y=0}^1 \left\{ \int_{x=y^2}^{y^{\frac{1}{3}}} x^2 y dx \right\} dy = \int_{y=0}^1 \left\{ \frac{x^3 y}{3} \right\} \bigg|_{x=\sqrt{y^2}}^{y^{\frac{1}{3}}} dy \\ &= \int_0^1 \left(\frac{y^2 - y^7}{3} \right) dy = \left\{ \frac{y^3}{9} - \frac{y^8}{24} \right\} \bigg|_0^1 = \frac{1}{9} - \frac{1}{24} = \frac{5}{72}. \end{aligned}$$

Although the region R is a practical problem in both vertically simple and horizontally simple, it may be easier to integrate in one order rather than the other because of the shape of R . We naturally prefer the easier route. The choice of the preferable order of integration may be influenced also by the nature of the function $f(x, y)$. It may be difficult – or even impossible to compute a given iterated integral but easy to do so after we reverse the order of integration. The next example is a typical one.

Example 1.2.5

Evaluate $\int_0^2 \int_{\frac{y}{2}}^1 ye^{x^3} dx dy$



Solution: Any one can't integrate first with

respect to x , as indicated, because $\exp(x^3)$

is known to have no elementary antiderivative.

So we try to evaluate the integral by first reversing the order of integration. To do so, we sketch the region of integration specified by the limits in the given iterated integral.

The region R is determined by the inequalities $\frac{1}{2}y \leq x \leq 1$ and $0 \leq y \leq 2$. Thus all points

(x, y) of R lie between the horizontal lines $y = 0$ and $y = 2$ and between the two lines and

$x = 1$. We draw the four lines $y = 0$, $y = 2$, $x = \frac{y}{2}$ and $x = 1$ and find that the region of integration is the shaded triangle as shown in **Fig1.2.4**.

Integrating first with respect to y , from $y_1(x) = 0$ to $y_2(x) = 2x$, we obtain

$$\begin{aligned} \int_0^2 \int_{y/2}^1 ye^{x^3} dx dy &= \int_0^1 \int_0^{2x} ye^{x^3} dy dx = \int_0^1 \left\{ \int_0^{2x} ye^{x^3} dy \right\} dx \\ &= \int_0^1 \left[\frac{1}{2} y^2 e^{x^3} \right]_{y=0}^{2x} dx = \int_0^1 2x^2 e^{x^3} dx = \frac{2}{3} e^{x^3} \Big|_0^1 = \frac{2}{3}(e-1). \end{aligned}$$

Example 1.4.2: Evaluate the integral

$\iint_R \ln(x^2 + y^2) dx dy$, where R is the annular region bounded by the circles

$$x^2 + y^2 = 4 \text{ and } x^2 + y^2 = e^2 \text{ (See Fig 1.4.3).}$$

Solution:

$$\iint_R \ln(x^2 + y^2) dx dy = \iint_{R'} \ln(r^2) r dr d\theta = \int_0^{2\pi} \int_2^{e^2} \ln(r^2) r dr d\theta$$

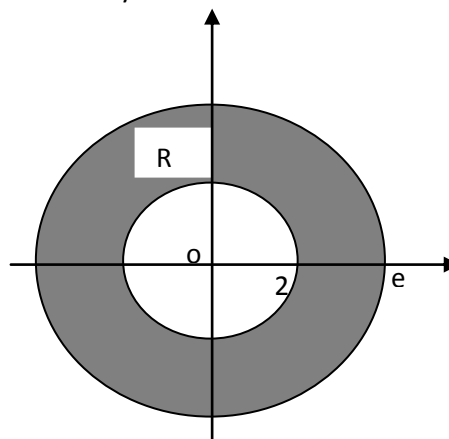


Fig 1.4.3

$$= \int_0^{2\pi} \int_4^{e^2} \frac{1}{2} \ln u du d\theta = \frac{1}{2} \int_0^{2\pi} u(\ln u - 1) \Big|_4^{e^2} d\theta$$

$$= \pi \{ e^2(2-1) - 4(2\ln 2 - 1) \} = \pi(e^2 + 4 - 8\ln 2), \text{ (with the}$$

help of integration by parts).

Example 1.4.3: Calculate $\iint_R r^3 dr d\theta$ where **R** is the region bounded by the circles

$$r = 2\sin\theta \text{ and } r = 4\sin\theta.$$

Solution:

As shown **Fig1.4.6** region of integration **R** is

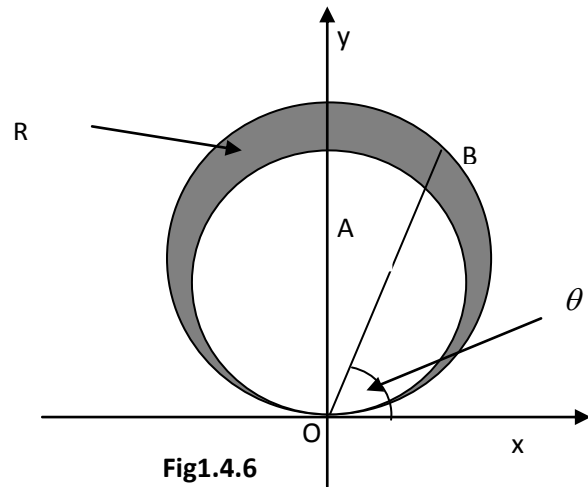
bounded by the circles $r = 2\sin\theta$ and

$r = 4\sin\theta$ where $0 \leq \theta \leq \pi$. If we

integrate first with respect to r , then its limits

are from $A(r = 2\sin\theta)$ to $B(r = 4\sin\theta)$ and to

cover the whole region **R**, θ varies from 0 to π .

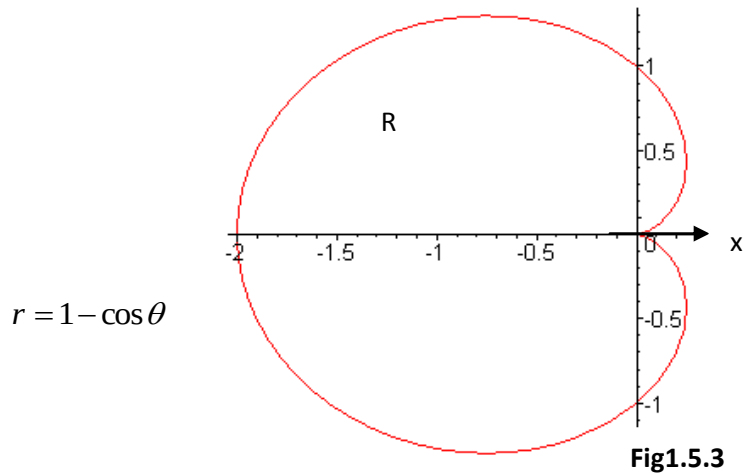


$$\iint_R r^3 dr d\theta = \int_{\theta=0}^{\pi} \left\{ \int_{r=2\sin\theta}^{4\sin\theta} r^3 dr \right\} d\theta = \int_0^{\pi} \left\{ \frac{r^4}{4} \right\}_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= 60 \int_0^{\pi} \sin^4 \theta d\theta = 120 \int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta = 120 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{45}{2} \pi.$$

As shown **Fig1.5.3**, Calculate area of the region **R** bounded by the Cardioid $r = 1 - \cos\theta$.





Solution:

$$\begin{aligned}
 A &= \iint_R 1 \cdot dA = \int_0^{2\pi} \left\{ \int_0^{1-\cos\theta} r dr \right\} d\theta \\
 &= \int_0^{2\pi} \left(\frac{r^2}{2} \right) \bigg|_0^{1-\cos\theta} d\theta = \int_0^{2\pi} \frac{(1-\cos\theta)^2}{2} d\theta = \int_0^{2\pi} \frac{1-2\cos\theta+\cos^2\theta}{2} d\theta = \frac{3}{2}\pi.
 \end{aligned}$$

Example 1.5.7

Find the surface area cut from the paraboloid $Z = r^2$ by the cylinder $r = 1$.

Solution:

From equation (14), the surface area is

$$A = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + r^2(2r)^2} dr d\theta = 2\pi \int_0^1 r \sqrt{1+4r^2} dr$$

$$= 2\pi \left[\frac{2}{3} \cdot \frac{1}{8} (1 + 4r^2)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{6} (5\sqrt{5} - 1).$$

Example 2.2.2

Evaluate $\int_{-3-\sqrt{9-x^2}}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{(x^2+y^2)^2}^1 y^2 dz dy dx$.

Solution:

The limits of integrations -3 and 3 in the first integral and $-\sqrt{9-x^2}$ and $\sqrt{9-x^2}$ on the second integral tell us that those two integrals are taken over the region of the disk

$x^2 + y^2 \leq 9$ or $r \leq 3$, in the xy-plane. It follows that

$$\int_{-3-\sqrt{9-x^2}}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{(x^2+y^2)^2}^1 y^2 dz dy dx = \int_0^{2\pi} \int_0^3 \int_{r^4}^1 (r \sin \theta)^2 r dz dr d\theta$$

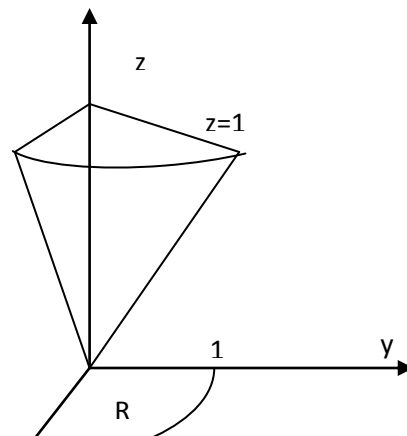
$$= \int_0^{2\pi} \int_0^3 [r^3 z \sin^2 \theta]_{r^4}^1 dr d\theta = \int_0^{2\pi} \int_0^3 [r^3 - r^7] \sin^2 \theta dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^8}{8} \right]_0^3 \sin^2 \theta d\theta = -\frac{6318}{8} \int_0^{2\pi} \sin^2 \theta d\theta$$

$$= -\frac{6318}{16} \int_0^{2\pi} (1 - \cos 2\theta) d\theta = -\frac{3159}{8} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = -\frac{3159}{4} \pi$$

Example 2.2.3

Evaluate the integral $\iiint_D yz dv$, where **D** is



the solid region in the first octant bounded

above by the plane $z = 1$ and below by

the cone $z = \sqrt{x^2 + y^2}$.

Solution:

Using cylindrical coordinates

$$\begin{aligned}
 \iiint_D yz \, dv &= \int_0^{\pi/2} \int_0^1 \int_r^1 z \cdot r \sin \theta \cdot r \, dz \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left\{ \int_0^1 r^2 \left[\int_r^1 z \, dz \right] dr \right\} \sin \theta \, d\theta \\
 &= \int_0^{\pi/2} \left\{ \int_0^1 r^2 \left(\frac{z^2}{2} \right) \Big|_r^1 dr \right\} \sin \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \left\{ \int_0^1 (r^2 - r^4) dr \right\} \sin \theta \, d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \left\{ \left(\frac{r^3}{3} - \frac{r^5}{5} \right) \Big|_0^1 \right\} \sin \theta \, d\theta = \frac{1}{15} \int_0^{\pi/2} \sin \theta \, d\theta = \frac{1}{15}.
 \end{aligned}$$

Example 2.3.3

Evaluate $\iiint_D dv$, where D is the solid region

between the spheres $x^2 + y^2 + z^2 = 1$,

$x^2 + y^2 + z^2 = 9$, and the upper nappe of the cone

$z^2 = 3(x^2 + y^2)$ (Fig 2.3.2).

Solution: In spherical coordinates the equations of

r, θ, ϕ

ρ, θ, ϕ

ρ, θ, ϕ

the given spheres are $\rho = 1$ and $\rho = 3$. Next, recall

$$\text{that } z^2 = \rho^2 \cos^2 \phi \text{ and } x^2 + y^2 = r^2 = \rho^2 \sin^2 \phi.$$

Therefore, the equation $z^2 = 3(x^2 + y^2)$ becomes

$$\rho^2 \cos^2 \phi = 3\rho^2 \sin^2 \phi.$$

It follows that $\rho = 0$ or $\cos^2 \phi = 3\sin^2 \phi$, so that if $\rho \neq 0$, then $\tan^2 \phi = \frac{1}{3}$.

Since **D** lies above the upper nappe of the cone $z^2 = 3(x^2 + y^2)$, we have $0 \leq \phi \leq \frac{\pi}{2}$, so $\tan \phi = \frac{1}{\sqrt{3}}$.

Thus, $\phi = \frac{\pi}{6}$. Consequently, **D** is the collection of points with spherical coordinates (ρ, ϕ, θ) such

that $1 \leq \rho \leq 3$, $0 \leq \phi \leq \frac{\pi}{6}$ and $0 \leq \theta \leq 2\pi$.

$$\text{Therefore, } \iiint_D dv = \int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \left[\frac{\rho^3}{3} \right]_1^3 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/6} \frac{26}{3} \sin \phi \, d\phi \, d\theta$$

$$= \frac{26}{3} \int_0^{2\pi} (-\cos \phi) \Big|_0^{\pi/6} d\theta = \frac{26}{3} \left(1 - \frac{\sqrt{3}}{2} \right) \int_0^{2\pi} d\theta = \frac{26}{3} \pi (2 - \sqrt{3}).$$

Example 2.4.2

Find the volume **V** of the region in the first octant

bounded by the planes $z = 10 - x - y$, $y = 2 - x$, $y = x$, $z = 0$ and $x = 0$.

Solution:

The region **R** of the solid bounded in the xy -plane is as shown **Fig 2.4.1**. The line $y = 2 - x$ and $y = x$ intersect at $(1, 1)$. Hence, $x \leq y \leq 2 - x$ as $0 \leq x \leq 1$.

Therefore,

$$\begin{aligned}
 V &= \iiint_X dv = \int_{x=0}^1 \int_{y=x}^{2-x} \int_{z=0}^{10+x+y} dz \, dy \, dx \\
 &= \int_0^1 \left\{ \int_x^{2-x} \left[\int_0^{10+x+y} dz \right] dy \right\} dx = \int_0^1 \left\{ \int_x^{2-x} (10x + x + y) dy \right\} dx \\
 &= \int_0^1 \{ 22 - 20x - 2x^2 \} dx = 2 \left[11x - 5x^2 - \frac{x^3}{3} \right]_0^1 = \frac{34}{3}.
 \end{aligned}$$

Example 2.4.3

Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = y + 2$ (**Fig 2.4.2**).

Solution

The given region **D** is z -simple, but its projection in to the xy -plane is bounded by the graph of the equation $x^2 + y^2 = y + 2$, which is a translated circle. It would be possible to integrate first with respect to z , but perhaps another choice will yield a simpler integral.

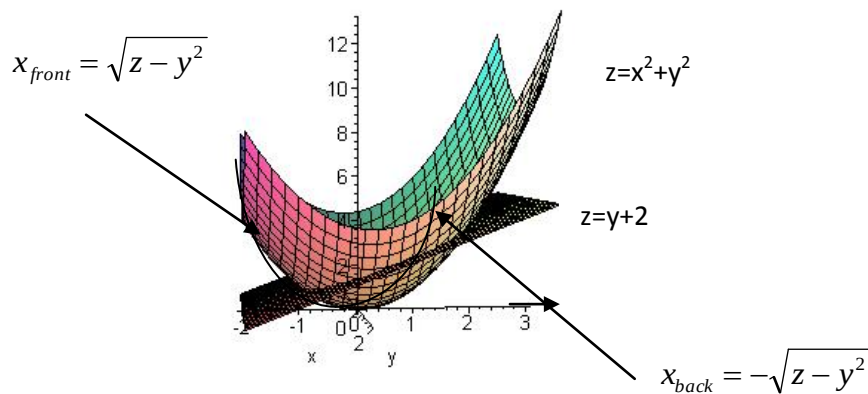


Fig 2.4.2

The region D is also x -simple, so we may integrate first with respect to x . The projection of D in to the yz -plane is bounded by the line $z = y + 2$ and the parabola $z = y^2$, which intersect at the points $(-1, 1)$ and $(2, 4)$ (**Fig 2.4.3**). The end points of a line segment in D parallel to the x -axis have x -coordinates $x = \pm\sqrt{z - y^2}$. Because D is symmetric with respect to the yz -plane, we can integrate from $x = 0$ to $x = \sqrt{z - y^2}$ and double the result.

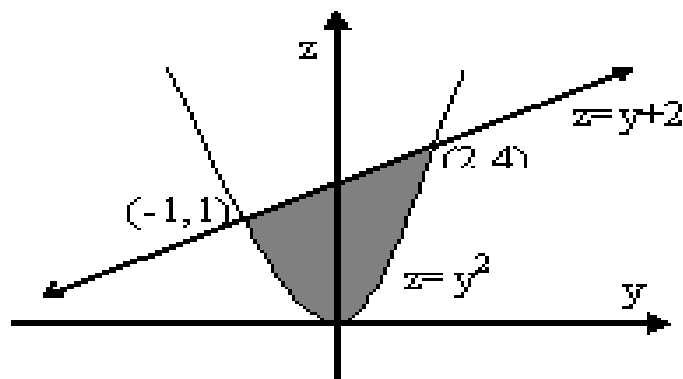


Fig. 2.4.3

Hence D has volume

$$\begin{aligned}
 V &= 2 \int_{-1}^2 \int_{y^2}^{y+2} \int_0^{\sqrt{z-y^2}} dx \, dz \, dy = 2 \int_{-1}^2 \int_{y^2}^{y+2} \left(\sqrt{z-y^2} \right) dz \, dy \\
 &= 2 \int_{-1}^2 \left[\frac{2}{3} (z-y^2)^{3/2} \right]_{z=y^2}^{y+2} dy = \frac{4}{3} \int_{-1}^2 (2+y-y^2)^{3/2} dy \\
 &= \frac{4}{3} \int_{-3/2}^{3/2} \left(\frac{9}{4} - u^2 \right)^{3/2} du \quad \left(\text{Completing the square; } u = y - \frac{1}{2} \right) \\
 &= \frac{27}{4} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta \quad \left(u = \frac{3}{2} \sin \theta \text{ and hence, } du = \frac{3}{2} \cos \theta \, d\theta \right) \\
 &= \frac{27}{4} \cdot 2 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = \frac{81}{32} \pi.
 \end{aligned}$$

Example 2.4.4

Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution:

Let OABC be the first octant of the given

ellipsoid which is bounded by the planes

OAB ($z = 0$), OBC ($x = 0$),

OCA ($y = 0$) and the surface ABC, i.e.

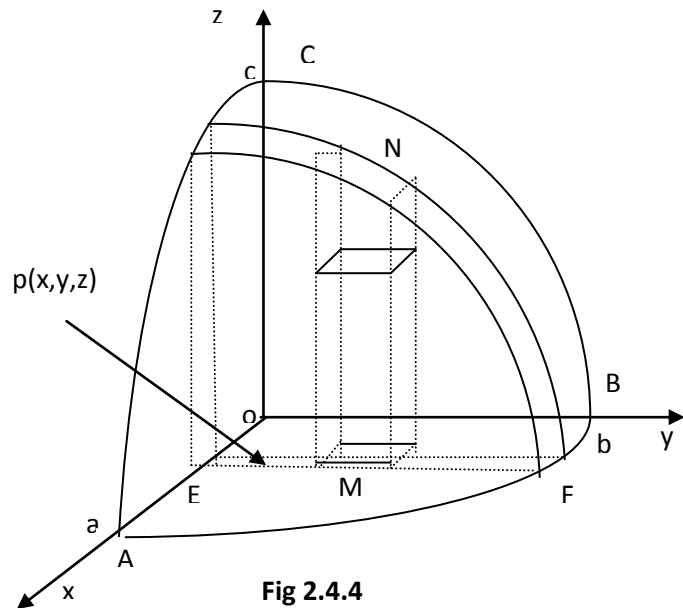


Fig 2.4.4

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region **D** in to rectangular parallelepipeds of volume $dx dy dz$. Consider such an element at P(x, y, z). (See **Fig 2.4.4**).

$$\text{Therefore, the required volume} = 8 \iiint_D dx dy dz .$$

In this region **D**,

i. Z varies from O to MN where $MN = c \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$

ii. Y varies from O to EF, where $EF = b \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$ from the equation of the ellipse

OAB, i.e. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

iii. X varies from O to OA = a

Hence, the volume of the whole ellipsoid

$$\begin{aligned} &= 8 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dx dy dz = 8 \int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} \left\{ \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz \right\} dy \right] dx \\ &= 8 \int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} \left(c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \right) dy \right] dx = 8c \int_0^a \left\{ \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dy \right\} dx \\ &= \frac{8c}{b} \int_0^a \left\{ \int_0^{\rho} \sqrt{\rho^2 - y^2} dy \right\} dx \text{ where } \rho = b\sqrt{1 - \frac{x^2}{a^2}} \end{aligned}$$

$$= \frac{8c}{b} \int_0^a \left[\frac{y\sqrt{\rho^2 - y^2}}{2} + \frac{\rho^2}{2} \sin^{-1}\left(\frac{y}{\rho}\right) \right]_0^\rho = \frac{8c}{b} \int_0^a \frac{b^2}{2} \left(1 - \frac{x^2}{a^2}\right) \frac{\pi}{2} dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left(x - \frac{x^3}{3a^2} \right) \Big|_0^a = \frac{4\pi abc}{3}.$$