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DOCUMENTS

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THE MATHEMATICS OF MAP COLORING*

H. S. M. Coxeter**

Looking at a colored map of the United States in almost any atlas, we see that neighboring states are distinguished by being differently colored, and the total number of colors used is five or six. Apparently the artist did not realize that four colors would have sufficed. (It is understood that two states may be colored alike if they merely have a point in common, as in the case of Arizona and Colorado.) The map-coloring problem becomes slightly more complicated if we include seas and lakes and insist on making them all blue. For instance, we need three different colors (say green, red, and yellow) for Belgium, France, and Germany, which all touch one another; Holland may have the same color as France, but Luxembourg must be blue, like the sea, if no fifth color is available. With this slight departure from practical cartography, we now formulate the mathematical question:

Can every conceivable map (on a sphere or a plane) be colored with four colors, or does some particular map really need five?

There is no reason to suppose that anyone asked this question before 1852, when Francis Guthrie put it to his younger brother Frederick [1]. Francis, a graduate student at University College, London, noticed the sufficiency of four colors for distinguishing the counties in a map of England. He attempted a proof but considered it unsatisfactory. Frederick was a student of Augustus De Morgan, whose letter of October 23, 1852 to Sir William Rowan Hamilton includes the following passage [2]:

> 'A student of mine asked me today to give him a reason for a fact which I did not know was a fact, and do not yet. He says, that if a figure be anyhow divided, and the compartments differently coloured, so that figures with any portion of common boundary *lines* are differently coloured—four colours may be wanted, but no more. Query: cannot a necessity for five or more be invented?... Now, it does seem that drawing three compartments with common boundary, two and two, you cannot make a fourth take boundary from all, except by enclosing one. But it is tricky work, and I am not sure of all convolutions. What do you say? And has it, if true, been noticed?... The more I think of it, the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did.'

Neither the Guthrie brothers nor De Morgan nor Hamilton made any progress toward solving the problem. Francis Guthrie, its inventor, became professor of mathematics at the South African College, Cape Town, where he wrote papers also on geology and botany. Frederick wrote several books and founded the Physical Society of London in 1873. De Morgan's *Budget of Paradoxes* does not seem to make any mention of map coloring. Hamilton is universally acclaimed as one of the founders of modern algebra, but his only contribution to topology is his invention of the 'Hamiltonian Circuit' [3], which is interesting but not helpful. His reply to De Morgan was prompt and brief:

'I am not likely to attempt your "quaternion of colours" very soon.'

In fact, the problem was entirely neglected till 1878, when Arthur Cayley told the members of the London Mathematical Society that he had been unable to solve it. His challenge was taken up by A. B. Kempe and P. G. Tait, who published 'proofs' in 1879 and 1880, respectively.

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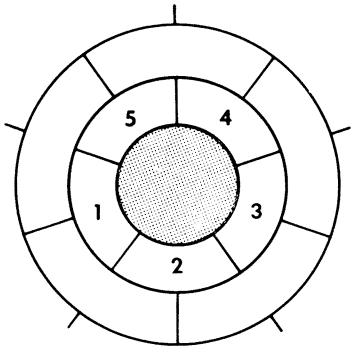


Fig. 1.

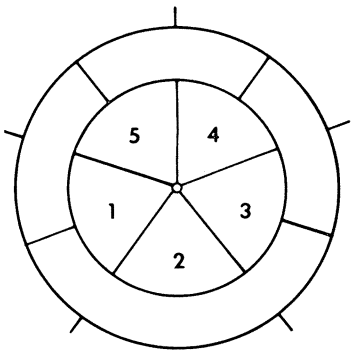


Fig. 2.

To give something of the flavor of the subject, let us prove, in the manner of Heawood[4], the sufficiency of six colors, and then the sufficiency of five, taking for granted a theorem of topology which states that, in any map, *at least one state has five or fewer neighbors* [5].

Fig. 1 shows a typical map, in which one of the states having five or fewer neighbors is shaded. Fig. 2 shows a modified map in which this particular state has been shrunk to a point (i.e. in which its territory has been ceded to its neighbors). If the modified map can be colored with six colors, so can the original; for, when we apply the same coloring to the non-shaded part of the original, the neighbors will use up at most five colors, and we have a sixth color to spare for the shaded state. We can reduce the modified map in a similar manner, gradually obtaining simpler and simpler maps (i.e. maps having smaller numbers of states), such that *if* the last can be colored with six colors, so can all the preceding maps. Eventually we must be left with a map having only six states, whose coloring is obvious. Hence the original map also can be colored with six colors, and we have proved the Six-Color Theorem:

To color any map on a sphere, or on a plane, requires at most six colors.

Similarly, to prove the Five-Color Theorem, we begin again with an arbitrary map, shrink the shaded region to a point, and argue that, if the modified map can be colored with *five* colors, so can the original. If the shaded state has fewer than five neighbors, or if one or two colors are repeated (so that the five neighbors use up at most four of the five colors), the previous argument can be applied with 'six' replaced by 'five.' If the shaded state has exactly five neighbors, all differently colored, we try to alter the coloring so as to eliminate one of the colors by repeating another.

At this stage it is helpful to define, for any given colored map, a *Kempe chain* to be a connected aggregate of states, extended as far as possible subject to the condition that just two particular colors occur. (For instance, Fig. 3 contains a yellow–red Kempe chain and a green–orange Kempe chain, each involving at least three states, and a red–orange Kempe chain consisting of the red state alone.) Clearly, the given map will remain properly colored when the two colors have been consistently interchanged throughout any Kempe chain.

Figure 3 is part of a typical map in which the five neighbors (numbered from 1 to 5 in Fig. 1) use up the five colors red, blue, orange, green, yellow. This map has been deliberately left unfinished so as to illustrate two alternative possibilities: states 2 and 5 may belong to different blue–yellow chains (such as in Fig. 4) or to the same blue–yellow chain (such as in Figs. 5 and 6). In the former case we can eliminate the blue by interchanging the two colors throughout 2's blue-yellow chain. In the latter case we notice that the blue–yellow chain (which now includes both 2 and 5) forms, with the shaded state, a closed belt or *zone* which separates 1 from 3 and 4, making it impossible for 1 and 4 to belong to the same red–green chain. It follows that the transposition of colors throughout 1's red–green chain will remove the red, and the number of colors surrounding the shaded state is again reduced from five to four, allowing us to use the fifth color (red) for the shaded state itself. In other words, the five-coloring of a map (such as Fig. 1) can always be made to depend on the five-coloring of a reduced map (Fig. 2). We reduce again, step by step, until we are left with a map having only five states, and the Five-Color Theorem is proved:

To color any map on a sphere, or on an ordinary plane, requires at most five colors.

The word 'ordinary' is introduced here because mathematicians often speak of another kind of plane, called the 'real projective plane,' in which a map may need six colors. In the ordinary (or 'inversive') plane that we are using here, the part outside our finite map (yellow in Fig. 4, and blue in Fig. 5) can be regarded as one more state (or an ocean covering all the rest of the planet) just like any of the obviously finite states.

The simple and complete proof of the Five-Color Theorem helps us to understand what happened when Kempe [6, 7] tried to prove the analogous Four-Color Theorem. We have seen that the transition from six colors to five was achieved by considering a coloring of the non-shaded part of a given map (Fig. 1) and changing this coloring (if necessary) so that two of the 'neighbors' 1, 2, 3, 4, 5 are colored alike, as in Fig. 4. What Kempe intended to prove is that any four-coloring of the non-shaded part of the given map can be changed (if necessary) so that the neighbors use up at most three of the four colors, and the fourth is left for the shaded state itself.

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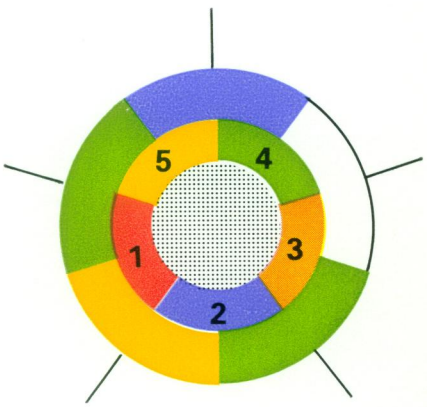


Fig. 3

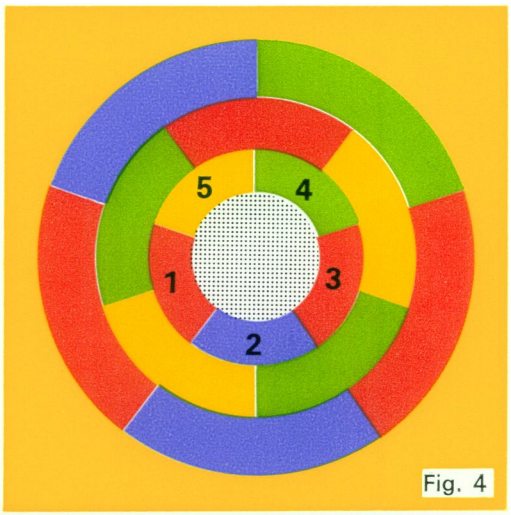


Fig. 4

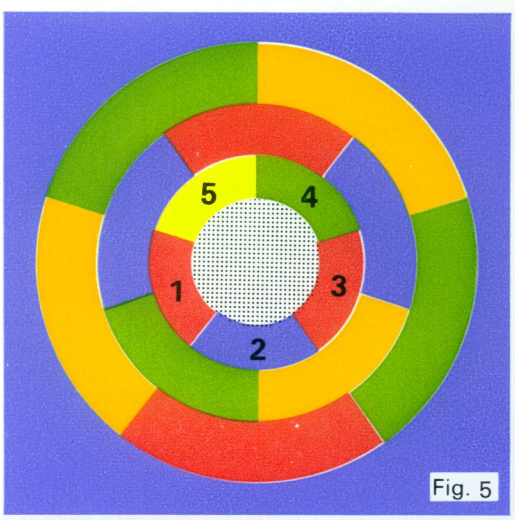


Fig. 5

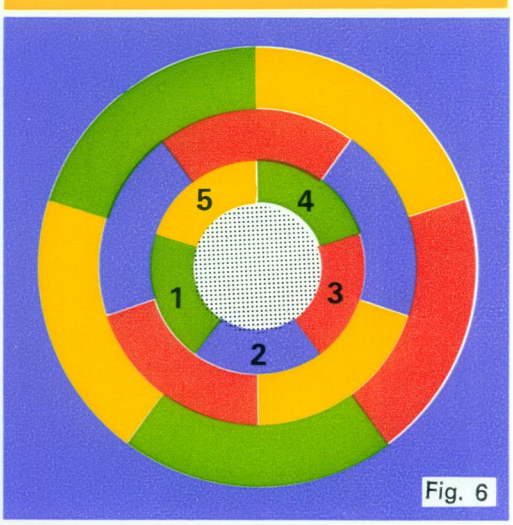


Fig. 6

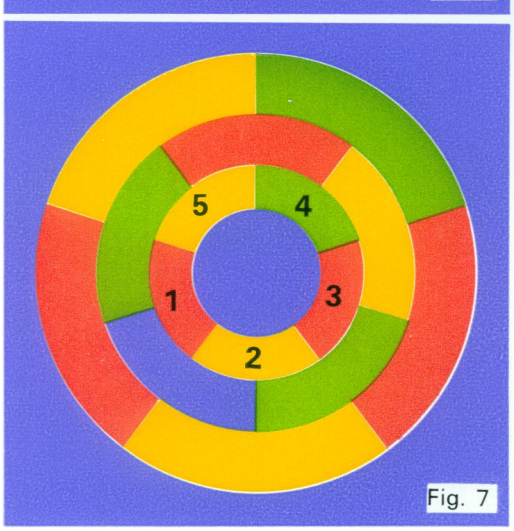


Fig. 7

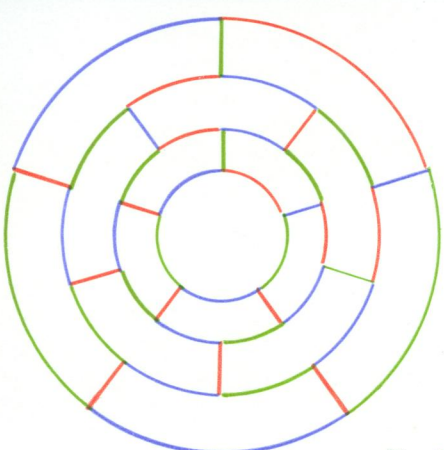


Fig. 8

Figs. 3–8.

[*facing p.* 274]

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If the five neighbors of the shaded state are colored with just four colors, two of them must be colored alike, and of course these two must not be adjacent. Hence there is no loss of generality in assuming that the colors of the states 1, 2, 3, 4, 5 are respectively red, blue, red again, green, yellow. If, as in Fig. 4, states 2 and 5 belong to different blue–yellow chains, we can eliminate the blue by interchanging the two colors throughout 2's blue–yellow chain. (The resulting complete four-color map is shown in Fig. 7.) Similarly, if states 2 and 4 belong to different blue–green chains, we can eliminate the blue equally well by interchanging these colors throughout 2's blue–green chain. However, both these devices will fail if, as in Fig. 5, states 2 and 5 belong to the same blue–yellow chain while 2 and 4 belong to the same blue–green chain. This seems to be a serious setback, but perhaps something can be salvaged. Having failed to eliminate the blue color (or the yellow or green), let us change our strategy and try to eliminate the red instead. We must expect this to be more difficult, as red occurs twice (states 1 and 3). However, we notice that the blue–yellow chain (which now includes both 2 and 5) forms, with the shaded state, a zone which separates 1 from 3 and 4, making it impossible for 1 and 4 to belong to the same red–green chain; and the blue–green chain (including 2 and 4) yields another zone which prevents 3 and 5 from belonging to the same red–yellow chain. It follows (apparently) that the transposition of colors throughout 1's red–green chain and 3's red–yellow chain *will each remove a red*, and the number of colors surrounding the shaded state is reduced from four to three, allowing us to use the fourth color (red) for the shaded state itself. In other words, the four-coloring of a map can always be made to depend on the four-coloring of a reduced map; therefore any given map can be four-colored.

This is how matters stood between 1880 and 1890. During that decade the world's mathematicians lived in a fools' paradise. Even the great Felix Klein believed that the four-color theorem had been proved. Actually, the fallacy occurs in the words that we have printed in italics. This was pointed out by Heawood [4, page 337]. 'Unfortunately,' says he, 'it is conceivable that though *either* transposition would remove a red, *both* may not remove both reds.' Figure 5 is an actual exemplification of this possibility, where either transposition prevents the other from being of any avail. Figure 6 shows how one of the reds is removed by transposing colors throughout 1's red–green chain. Now 3 and 5 belong to the *same* red–yellow chain, so that the transposition of these colors merely shifts the red from 3 to 5.

The flaw which Heawood discovered in Kempe's reasoning has never been remedied. Almost every mathematician must have experienced one glorious night when he thought he had discovered a proof, only to find the next morning that he had fallen into a similar trap.

Successful research along these lines [8, 9] has been limited to proving the four-color theorem for every map that satisfies certain stated conditions. For instance, C. E. Winn [10, 11] has proved it for every map in which at most one state has more than six neighbors, and for every map having fewer than 36 states. More recently, Ore and Stemple [12] improved Winn's result by proving the theorem for maps having fewer than 40 states.

Another approach, quite different from Kempe's but equally ineffective, was used by Tait [13]. He observed that the boundary lines and meeting points of a map may be regarded as the edges and vertices of a *graph*. This is simply an arrangement of points, certain pairs of which are joined by line segments, not necessarily straight. Graphs occur also in the structural formulas of chemistry and in electrical engineering. The chemical application makes it natural to use the term *valency* for the number of edges at a vertex, and so to call a graph (or a map) *trivalent* if every vertex belongs to three edges.

The problem of coloring maps is easily reduced to the problem of coloring trivalent maps. For, any map becomes trivalent if we replace each vertex of valency 4 or more by a small square or higher polygon with an edge going out from each vertex.

What Tait did was to reduce the four-color problem for maps to an equivalent three-color problem for graphs. Given a trivalent four-color map, as in Fig. 7, we can deduce a coloring of the edges with the property that the three edges meeting at any vertex all have different colors. For this purpose we specialize one of the four colors, say yellow, and agree that the edges belonging to each yellow state shall acquire the colors of the neighbor-states, while the remaining edges are blue, green or red according as they separate green and red states, red and blue, or blue and green. A comparison of Figs. 7 and 8 will make this clear.

Conversely, if a given graph (drawn on a sphere or ordinary plane without superfluous intersections) can have its edges colored blue, green, and red so that no two adjacent edges are colored alike, we can deduce a four-coloring of the map so formed, by assigning the color yellow to any one of the states, letting the neighboring states have the same colors as the edges that they share with the yellow state, letting the state beyond a blue edge of a green state be red, letting the state beyond a red edge of a red state be yellow, and so on. If we approach a particular state by two different routes, can we be sure that this rule will give it the same color both times? The answer is Yes, provided the map is drawn on a 'simply-connected' surface such as a sphere or an ordinary plane. On such a surface, every closed curve can be continuously shrunk to a single point. Since any 'two different routes' together form such a curve, either of the routes can be continuously varied until it agrees with the other.

A graph is said to be *planar* if it can be drawn on a sphere or an ordinary plane without the kind of extraneous intersections that we have indicated in Figs. 9 and 10 by little gaps in some edges near the middle. To draw a non-planar graph without any extraneous intersections, we need a multiply-connected surface such as the doughnut-shaped

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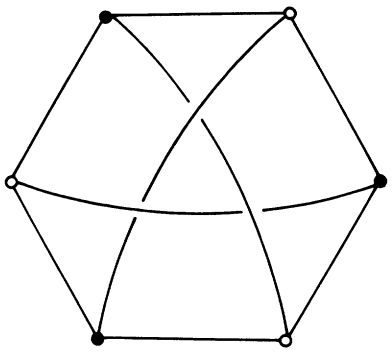


Fig. 9.

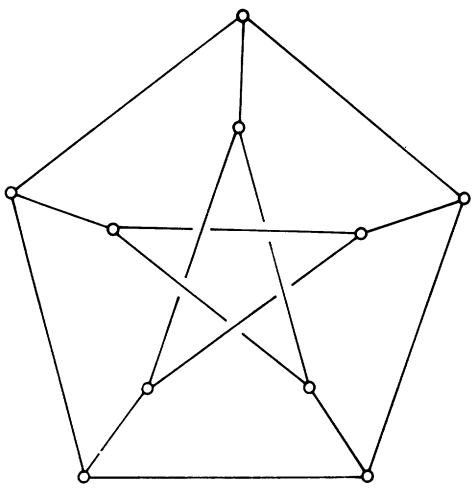


Fig. 10.

torus, on which some closed curves cannot be shrunk.

Figure 9 is the *Thomsen graph*, which is the non-planar graph having the smallest possible number of edges (namely, 9). This illustrates the impossibility of serving three houses (represented by the white vertices) with water, gas, and electricity (from stations represented by the black vertices) if we insist that the water pipes, gas pipes, and electric cables must never cross one another. Although this problem cannot be solved on a spherical planet, it could easily be solved if the planet were a torus.

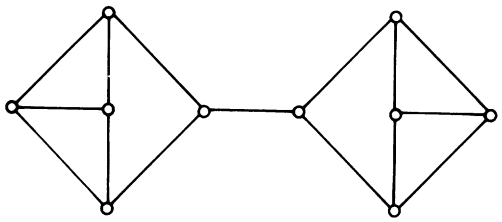


Fig. 11.

The nine edges of the Thomsen graph can be three-colored; but this possibility has no application to maps, because the Thomsen graph is not planar. On the other hand, the fifteen edges of the planar graph shown in Figure 11 *cannot* be three-colored. This impossibility does not disprove the Four-Color Theorem, because Fig. 11 has an *isthmus* or 'wasp waist': an essentially connecting edge whose removal would have the effect of breaking the graph into two separate pieces. If we tried to turn Fig. 11 into a map, the isthmus would separate the outside state from itself, against the whole spirit of map-coloring.

Tait regarded it as obvious that every trivalent graph without an isthmus can be three-colored. This assumption was refuted by J. Petersen, whose graph (Fig. 10) is neither planar nor three-colorable. In order to complete Tait's proof of the Four-Color Theorem, we would have to prove that

Every planar trivalent graph without an isthmus can have its edges three-colored.

Tait himself apparently assumed that every planar trivalent graph without an isthmus possesses a *Hamiltonian circuit:* a polygon whose vertices are all the vertices of the graph (each 'visited' just once) while its sides are among the edges. If this were true, we could three-color the graph by coloring the sides of the Hamiltonian circuit alternately blue and green and coloring all the remaining edges red. Tutte [3] showed that Tait's assumption is false by giving a counter-example: a quite respectable graph which certainly possesses no Hamiltonian circuit. However, this counter-example does not disprove the possibility of three-coloring every planar trivalent graph that has no isthmus. The existence of a

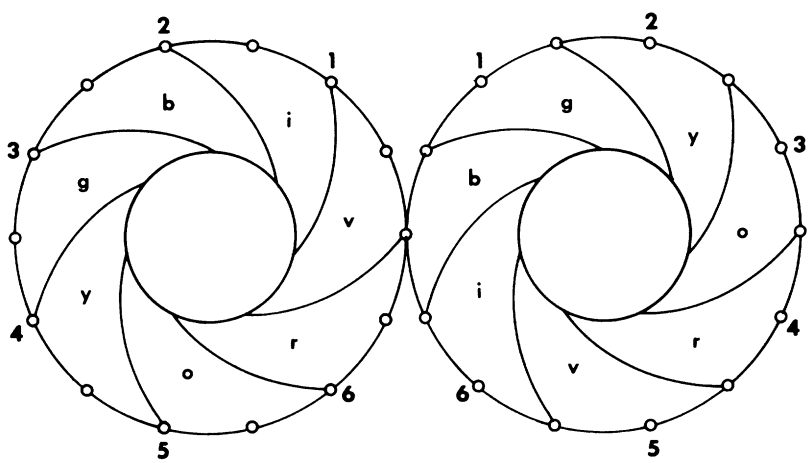


Fig. 12.

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Hamiltonian circuit is only a *sufficient* condition, not a *necessary* condition, for that possibility.

Heawood's paper of 1890 includes the important observation that the coloring problem, which has never been solved for maps on a sphere, can be completely solved for maps on any multiply-connected surface [14]. For instance, Fig. 12 shows the 'front and back' of a torus, side by side. The map drawn on this surface [15] consists of seven hexagonal states with the remarkable property that each state has all the other six for neighbors. Accordingly, this map requires seven colors. In the case of the torus, the number seven is not only necessary but also sufficient; for, Heawood's proof of the Six-Color Theorem is easily adapted to prove the Seven-Color Theorem:

To color any map on the torus requires at most seven colors.

Many people still think of mathematics as arithmetic, algebra, and geometry. The above remarks serve to illustrate the fact that the same kind of logical thinking can be applied to the different subject of topology, which has very little connection with numbers, and none with measurement. The Five-Color Theorem may reasonably be said to satisfy the criterion suggested by G. H. Hardy [16].

> 'There is a very high degree of *unexpectedness*, combined with *inevitability* and *economy* . . . A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way.'

In conclusion, I wish to express my thanks to K. O. May [17], whose historical research enabled me to name Francis Guthrie as the indisputable originator of the four-color problem, and to T. H. O'Beirne, who drew my attention to Erréra's map [18, 19] (see Fig. 5), which serves the same purpose as Heawood's with the extra virtue that each state has at least five neighbors. Moreover, Kempe's transformation will not solve the coloring problem for this particular map, even if we apply it again and again.

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