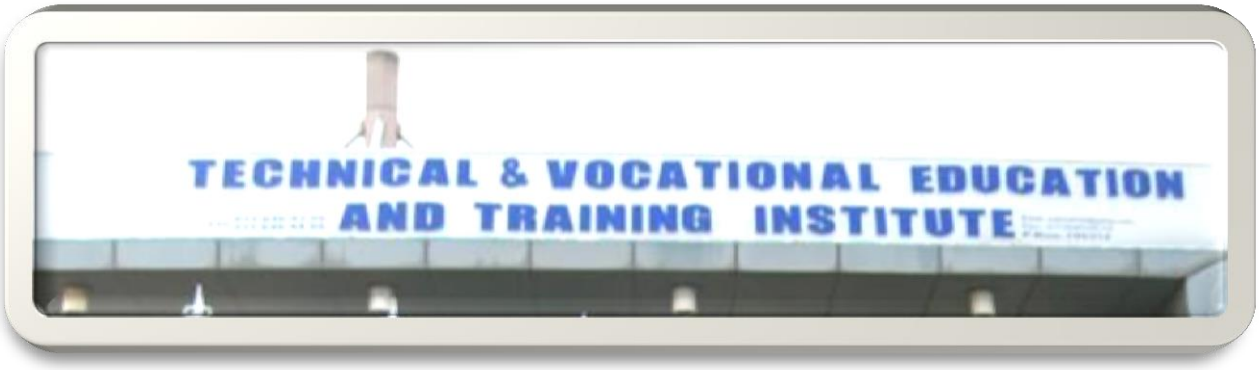
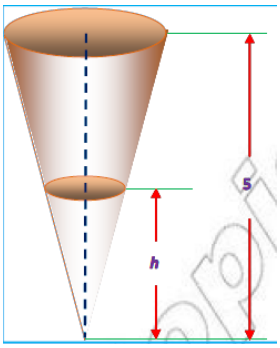
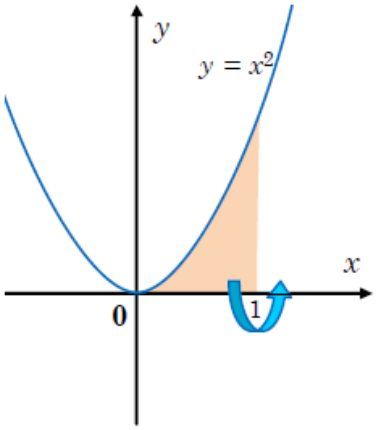
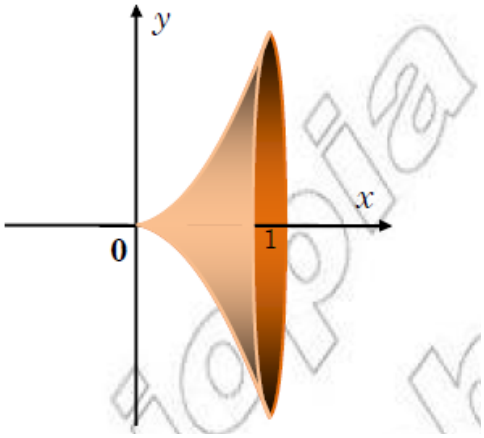


DEPARTMENT OF LANGUAGE AND BASIC SCIENCES

LEARNING GUIDE FOR APPLIED MATHEMATICS –I (MATH-102)







Course Description:

This course covers basic elements of vectors, matrices, determinants, solving systems of linear equations, concepts and applications of differential and integral calculus of one variable.

Course objectives:

On completion of the course successful students will be able to:

- understand the basic ideas of vector algebra,
- apply scalar and vector products ,
- determine direction angles and direction cosines of a vector,
- write equations of lines and planes,
- understand matrix algebra,
- Know and apply elementary row operations,
- solve systems of linear equations,
- determine the determinants,
- determine inverse of a matrix,
- understand the concepts of limit and continuity
- Understand the intuitive and formal definitions of limit,
- Understand the basic limit theorems,
- Understand definition of derivative,
- apply derivatives,
- understand the concepts of integration,
- evaluate integrals,
- Know different techniques of integration,
- apply integrals,

Course Contents

Chapter 1: Vectors

1.1 Definition of points in n-space

1.2 vectors in a plane and their geometrical interpretation .

1.3 Vectors in n-space $(\mathbb{R}^n)$

1.4 Scalar product, Orthogonal Projection and Directions cosine.

1.5 The Vector Product .

1.6 Application on Area and Volume

1.7 Lines and Planes

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5.2. Techniques of integration

5.2.1 Integration by substitution, by parts and by partial fraction

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5.2.3 Integration by trigonometric substitution

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Chapter - I

Vectors

Introduction

Many physical quantities such as mass, length, area, density, volume, etc that possess only magnitude are called $scalars$. While other physical quantities such as force, displacement, velocity, acceleration, etc that has both magnitude and direction are called $vectors$.

Therefore, this chapter deals with the concept of vectors and vector operations in n-space, norm of a vector, scalar product, vector product, orthogonal projections, applications of vectors and equations of lines and planes.

Chapter Objectives:

At the end of this chapter students will be able to:-

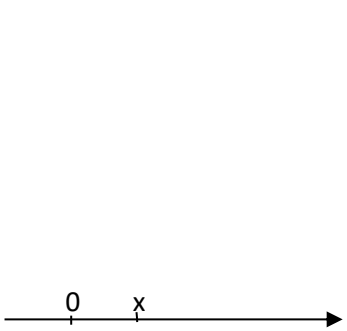
- Define a vector.
- Identify properties of vectors.
- Add vectors and multiply a vector by a scalar.
- Determine the length and direction of vectors.
- List properties of scalar product and vector product.
- Determine the dot and cross product of vectors.
- Determine the volume of a parallelepiped generated by three vectors.
- Determine equation of a line in space.
- Identify the different types of equation of a line in space.
- Determine equation of a plane in space.

1.1 Definition of points in n-space

It is known that a real number x can be used to represent a point on a number line once a unit length is selected.(see figure1.1).

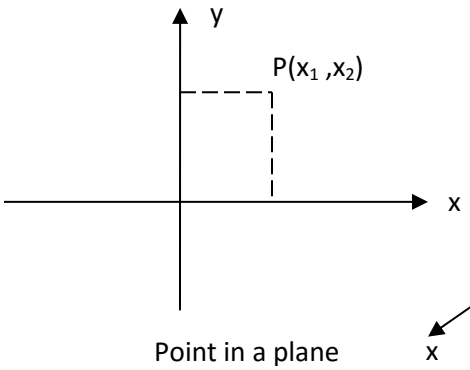
A pair of numbers (x_1, x_2) can be used to represent a point in the plane.(see figure1.2).

Similarly a triple of numbers (x_1, x_2, x_3) can be used to represent a point in three dimensional space.(see figure1.3).



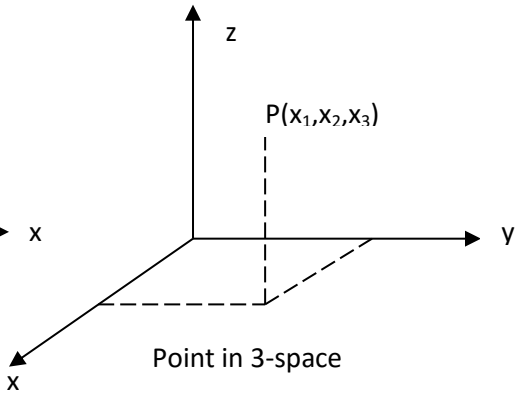
Point on a line

Figure1.1



Point in a plane

Figure1.2



Point in 3-space

Figure1.3

Generally, an n-tuple of numbers $(x_1, x_2, x_3, \ldots, x_n)$is used to represent a point in n-dimensional space. The numbers $x_1, x_2, x_3 \ldots x_n$ are called the coordinates of the point.

If $A = (x_1, x_2, x_3, \ldots, x_n)$ and $B = (y_1, y_2, y_3, \ldots, y_n)$ are points in the same space $\mathbb{R}^n$ and if λ is a real number then,

1. A and B are equal (or represent the same point) if $x_1 = y_1, x_2 = y_{2,\ldots\ldots,}$and $x_n = y_n$
2. $A + B = (x_1 + y_1,\ x_2 + y_2, \ldots, x_n + y_n)$ and
3. $\lambda A=(\lambda x_1, \lambda x_2,\ldots, \lambda x_n)$

1.2 vectors in a plane and their geometrical interpretation

In physics and engineering, a vector is characterized by two quantities **length** and **direction**. Thus a vector is a directed line segment that corresponds to a displacement from one point to another point.

Now we can visualize this directed line segment geometrically as follows.

Geometrical representation of vectors

Vectors can be represented geometrically as a directed line segments or arrows in a plane. Suppose P and Q be two points in a plane. The vector from P to Q is denoted by $\overrightarrow{PQ}$, where the point P is called its initial point (tail), and the point Q is called its terminal point (head) as shown in the figure1.4.

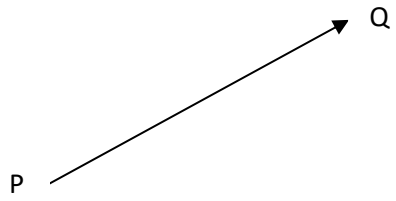


Figure1.4

A vector is also denoted by single boldface, lowercase letters such as **u**, **v**, **w**, etc.

Note: The length (magnitude) of the vector $\overrightarrow{PQ}$ is denoted by $\left|\overrightarrow{PQ}\right|$and its direction is from point P to point Q as shown in the above fig.

Question: What is the length and direction of the vector $\overrightarrow{PP}$?

Definition 1.2.1:- Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equal if and only if they have the same magnitude and the same direction.

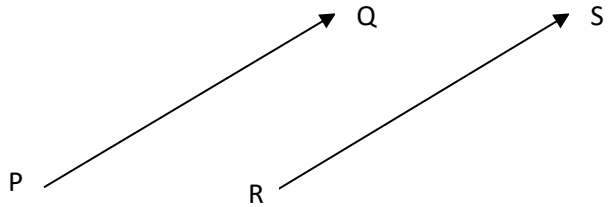


Figure1.5

Coordinate representation of vectors in the plane

We say that a vector in the plane is represented geometrically by a directed line segment. Vectors with initial points at the origin of the coordinate system are said to be vectors in standard position. The vector **u** in the figure1.6 below is a standard position vector with terminal point(x, y).

The ordered pair (x, y) is the coordinate representation of the vector **u,** and it can be written as $\mathbf{u} = (x, y)$. The coordinates x and y are called the **components** of the vector **u.**

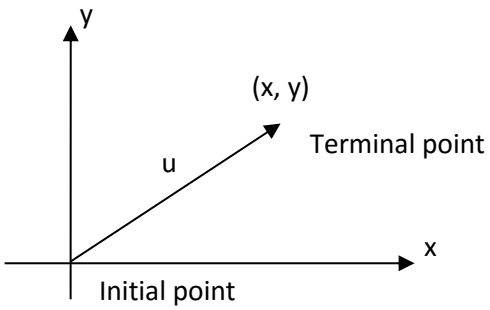


Figure1.6

Two position vectors $\mathbf{u} = (x_1, x_2)$ and $\mathbf{v} = (y_1, y_2)$ in the plane are equal If and only if

$x_1 = y_1$ and $x_2 = y_2$.

It is known that two given points in a plane determine a line segment. Let (x_1,x_2) and (y_1,y_2) be the coordinate representation of the given points A and B respectively in a plane. Since a vector is a directed line segment, then $\overrightarrow{AB}$ is a vector determined by the points A and B, and this vector is given by B - A=(y_1-x_1, y_2-x_2), that is the coordinates of B minus corresponding coordinates of A. Therefore, for such types of vectors in a plane, there exists a unique position vector $\overrightarrow{OP}$ such that $\overrightarrow{OP}$= $\overrightarrow{AB}$=B - A, where O is the origin of the coordinate system.

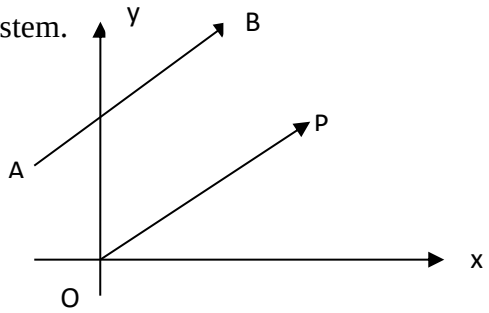


Figure1.7

Example1.2.1: Let $A = (1,3)$ and $B = (4,5)$ be two points on a plane. Find the terminal point of the vector $\overrightarrow{OP}$ such that $\overrightarrow{AB} = \overrightarrow{OP}$.

$\mathbf{0} = (0,0)$and $P = B - A = (4,5) - (1,3) = (4 - 1,5 - 3) = (3,2)$, which is the terminal point of the vector $\overrightarrow{OP}$.

Vector operations in a plane

A) Vector Addition

Now, let **u** and **v** be vectors with initial points at $P$and $P'$and terminal points at Q and Q' respectively. To find the sum **u** + **v**, geometrically connect the two vectors head-to-tail as shown in figure1.6.

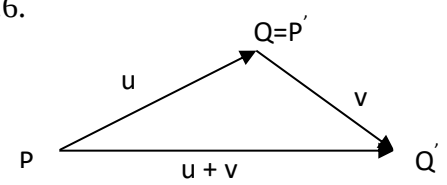


Figure1.8

On the other hand, let **u** and **v** be vectors in a plane with coordinate $\mathbf{u} = (u_1, u_2)$ and

$\mathbf{v} = (v_1, v_2)$ then the sum of the two vectors is given by:

$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1\ ,\ u_2 + v_2)$

Example1.2.2: let $\mathbf{u} = (-3,5)$ and $\mathbf{v} = (6,-1)$ then $\mathbf{u} + \mathbf{v} = (-3,5) + (6,-1)$

$= (-3 + 6\ ,\ 5 + (-1))$

$= (3,4)$

Question: Is it possible for three nonzero vectors to have sum zero?

Properties of Vector Addition

1. Vector addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$**,** for any vectors **u** and **v.**

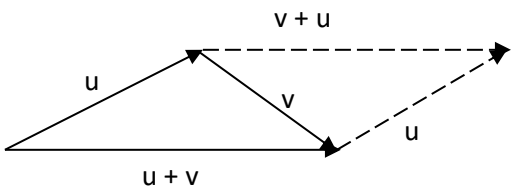


Figure1.9

2. Vector addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$, for any vectors **u**, **v** and **w**.

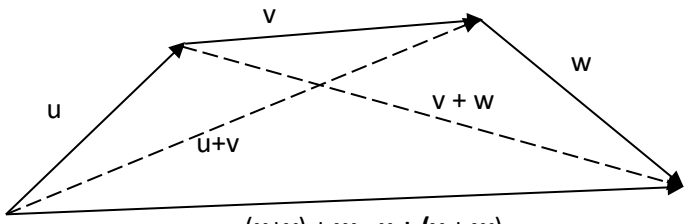


Figure1.10

3. Existence of identity element: the zero vector, denoted by **0,** is the identity element for vector addition. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$, for any vector **u**.

Question: What can you say about the magnitude and direction of the zero vector?

4. Existence of additive inverse:- The additive inverse of a vector **u** has the same magnitude as **u** but opposite direction to **u**, and is denoted by $-\mathbf{u}$. Hence $\mathbf{u}+(-\mathbf{u})=\mathbf{0}=(-\mathbf{u})+\mathbf{u}$

B) Scalar multiplication

Let **u** be a vector and $\lambda \in$ □ . The scalar multiple of **u,** denoted by $\lambda\mathbf{u}$, is a vector with magnitude$|\lambda||\mathbf{u}|$.

If $\lambda < 0$ then the vectors **u** and $\lambda\mathbf{u}$ have opposite direction.
If $\lambda > 0$ then the vectors **u** and $\lambda\mathbf{u}$ have the same direction.
If $\lambda = 0$ then the vectors $\lambda\mathbf{u}$ is the zero vector.

Properties of scalar multiplication

Let **u** and **v** be vectors and $\lambda, \mu \in$ □ , then

1. $(\lambda+\mu)\mathbf{u}=\lambda\mathbf{u}+\mu\mathbf{u}$.

2. $\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$
3. $(\lambda\mu)\mathbf{u}=\lambda(\mu\mathbf{u})=\mu(\lambda\mathbf{u})$.
4. $1\mathbf{u}=\mathbf{u}$.

1.3 Vectors in n-space ($\mathbb{R}^n$)

The discussion of vectors in the plane can now extended to a discussion of vectors in n-space.

Definition1.3.1: An ordered n-tuple of numbers $(x_1, x_2, x_3, \ldots, x_n)$ represent a vector in $\mathbb{R}^n$ .The numbers $x_1, x_2, x_3 \ldots x_n$ are called components of the vector.

Remark: From definition of points in n-space (section1.1) and definition1.3.1, we can conclude that an ordered n-tuple of numbers $(x_1, x_2, x_3, \ldots, x_n)$ represent either a point or a position vector.

The set of all n-tuples is called n-space and is denoted by $\mathbb{R}^n$.

That is: $\mathbb{R}^1$ =1-space=set of all real numbers.

$\mathbb{R}^2$ =2-space=set of all ordered pairs of real numbers.

$\mathbb{R}^3$ =3-space=set of all ordered triples of real numbers.

⋮

$\mathbb{R}^n$ =n-space=set of all ordered n-tuples of real numbers.

Similar to that of equality of two vectors in 2-space, two vectors in n-space are equal if and only if corresponding components are equal.

The sum of two vectors and the scalar multiple of a vector in $\mathbb{R}^n$ are defined as follows.

1. Let $\mathbf{u}=(u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v}=(v_1, v_2, v_3, \ldots, v_n)$ be vectors in $\mathbb{R}^n$ and λ be a real number then the sum of **u** and **v** is defined as the vector,

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,u_3,\ \dots\ ,u_n)+(v_1,v_2,v_3,\ \dots\ ,v_n)$$

$$=(u_1+v_1,u_2+v_2,u_3+v_3,\ \dots\ ,u_n+v_n)$$

2. The scalar multiple of **u** by λ is defined as the vector

$$\lambda\mathbf{u}=\lambda(u_1,u_2,u_3,\ \dots\ ,u_n)=(\lambda u_1,\lambda u_2,\lambda u_3,\ \dots\ ,\lambda u_n).$$

Notice that $u_1,u_2,u_3,\ \dots\ ,u_n$ and $v_1,v_2,v_3,\ \dots\ ,v_n$ are the components of the vectors **u** and **v** respectively.

Example1.3.1: let $\mathbf{u}=(-1,3,2)$ *and* $\mathbf{v}=(2,-1,5)$ be two vectors in $\mathbb{R}^3$, then find

a) $\mathbf{u}-3\mathbf{v}$

b) find a vector **w** such that $\mathbf{w}-2\mathbf{u}=\mathbf{v}-3\mathbf{w}$.

a) $\mathbf{u}-3\mathbf{v}=(-1,3,2)-3(2,-1,5)=(-1,3,2)-(6,-3,15)$

$$=(-1-6,3-(-3),2-15)=(-7,6,-13)$$

b) $\mathbf{w}-2\mathbf{u}=\mathbf{v}-3\mathbf{w}$ Rewriting this we get

$4\mathbf{w}=2\mathbf{u}+\mathbf{v}=2(-1,3,2)+(2,-1,5)=(0,5,9)$, now multiplying both sides by $\frac{1}{4}$ we get $\mathbf{w}=\frac{1}{4}(0,5,9)=(0,\frac{5}{4},\frac{9}{4})$.

Properties of vector addition and scalar multiplication in $\mathbb{R}^n$

Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ (n-space) and let λ and μ be scalars, then

1. Closure property under addition
   $\mathbf{u}+\mathbf{v}$ is a vector in $\mathbb{R}^n$.
2. Commutative property of additive
   $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. Associative property of addition
   $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

4. Additive identity property

   $\mathbf{u} + \mathbf{0} = \mathbf{u}$.-

5. Additive inverse property

   $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. Closure property under multiplication

   $\lambda\mathbf{u}$ is a vector in $\mathbb{R}^n$

7. Distributive property

   $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$.

8. Distributive property

   $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

9. Associative property of multiplication

   $(\lambda\mu)\mathbf{u} = \lambda(\mu\mathbf{u}) = \mu(\lambda\mathbf{u})$.

10. Multiplicative identity property

    $1\mathbf{u} = \mathbf{u}$.

Norm of a vector

Definition1.3.2 The length (magnitude) of a vector $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ in $\mathbb{R}^n$ is

called the norm of **u,** denoted by $\|\mathbf{u}\|$, and is defined by $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

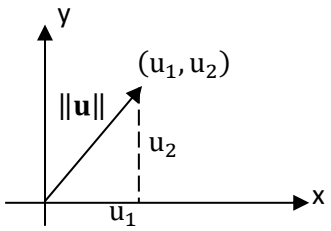


Figure1.11

If $\mathbf{u} = (u_1, u_2)$ is in 2-space, as shown in the figure,

Then by Pythagoras theorem $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$.

Example1.3.2: Find the norm of the vector a) $\mathbf{u} = (2, -3)$

b) $\mathbf{u} = (2,1,5)$

a) $\|\mathbf{u}\| = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

b) $\|\mathbf{u}\| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{4+1+25} = \sqrt{30}$

Definition1.3.3: A vector **u** is said to be a unit vector if and only if $\|\mathbf{u}\| = 1$.

Question: Can you find a unit vector along any given non-zero vector **u?** (how?)

The answer is yes, because given any non-zero vector **u**, we can always obtain a unit vector along the given vector, and this unit vector is given by: $\frac{\mathbf{u}}{\|\mathbf{u}\|}$.

Question: let $\mathbf{u} = (-3,0,4)$ be a given vector, can you find two unit vectors, one is in the direction of **u** and the other is in the opposite direction of **u**?

Definition1.3.4: Two non-zero vectors **u** and **v** are said to be parallel if and only if $\mathbf{u} = \lambda\,\mathbf{v}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

Question: Find at least four vectors that are parallel to the vector u= (2,-3,0, 8).

Angle between two vectors

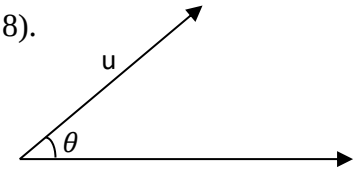


Figure1.12

The angle between two non-zero vectors **u** and **v** is defined as

the angle θ $(0 \le \theta \le \pi)$ formed by the vectors **u** and **v**

with the same initial point.

Activity1.3

1. If **u** and **v** are two vectors with the same initial point and θ is the angle between them.

   What can you say about the directions of the two vectors **u** and **v** if:-

$$i)\ \theta = 0 \qquad ii)\ \theta = \pi \qquad iii)\ \theta = \frac{\pi}{2}$$

2. Let **u**=(1,3,-5), **v**=(0,0,2) and **w**=(-2,0,3).

a) find the components of i) **u-3v** ii) **3(u+v-2w)**

b) find the scalars a, b, c such that a**u** + b**v** - c**w**=(1,0,2).

3. Find the vector **u** and **v** so that **u** + **v** = (4,5,6) and **u** – **v** = (2,5,8).

4. Find a unit vector u in the direction of **v** = (3,4,5).

5. Find the norm of the vector **u**= $(2, -3, \sqrt{5}, \frac{2}{3})$.

1.4 Scalar product, Orthogonal Projection and Directions cosine.

Definition 1.4.1: Let **u** and **v** be two vectors and θ be the angle between **u** and **v**. Then the scalar product of **u** and **v** is denoted by **u** . **v** and is defined by

$$\mathbf{u}\,.\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|cos\theta\,, \qquad 0 \le \theta \le \pi$$

If **u** and **v** are non-zero vectors, the angle between them is given by: $cos\theta = \frac{\mathbf{u}\,.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Example 1.4.1: Let **u** and **v** be vectors with lengths 2 and 5 units respectively. If the angle between **u** and **v** is 45^0 then the scalar product of **u** and **v** is

$$\mathbf{u}\,.\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|cos\theta = (2)(5)cos45^0 = 10\left(\frac{\sqrt{2}}{2}\right) = 5\sqrt{2}$$

Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in n-space. Then the scalar product of **u** and **v** is defined by:

$$\mathbf{u}\,.\,\mathbf{v} = (u_1, u_2, u_3, \ldots, u_n).(v_1, v_2, v_3, \ldots, v_n) = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

Example 1.4.2: Let $\mathbf{u} = (1,0,1)$ and $\mathbf{v} = (0,1,1)$ be two vectors in 3-space. Then the angle between them is

$$cos\theta = \frac{\mathbf{u}\,.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1,0,1).(0,1,1)}{\left(\sqrt{1^2+0^2+1^2}\right)\left(\sqrt{0^2+1^2+1^2}\right)} = \frac{1}{(\sqrt{2})(\sqrt{2})} = \frac{1}{2}.$$

Hence the angle between them is $\theta = cos^{-1}\left(\frac{1}{2}\right) = 60^0$

Example 1.4.3: Let $\mathbf{u} = (-1,3,2,5)$ and $\mathbf{v} = (3,0,-4,1)$ then

$$\mathbf{u}\,.\mathbf{v} = (-1,3,2,5).(3,0,-4,1)$$

$$= (-1)(3) + 3(0) + 2(-4) + 5(1) = -3 + 0 + (-8) + 5 = 6$$

Question: 1. Can you determine the angle between **u** and **v** in example 1.4.3?

2. Find $\hat{\boldsymbol{\imath}}.\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}\,.\hat{\boldsymbol{\jmath}}$ where $\hat{\boldsymbol{\imath}}$**=(1,0,0)** and $\hat{\boldsymbol{\jmath}}$**=(0,1,0)**

Properties of scalar product

Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}$, then

1. $\mathbf{u}\,.\mathbf{v} = \mathbf{v}.\mathbf{u}$……Comutativity property.
2. $\mathbf{u}\,.(\mathbf{v}+\mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$…….Associativity property.
3. $\mathbf{u}\,.(\lambda\mathbf{v}) = (\lambda\mathbf{u}).\mathbf{v} = \lambda(\mathbf{u}.\mathbf{v})$……..Relationship between scalar product and scalar multiplication.
4. $\mathbf{u}.\mathbf{u} \geq 0$ and $\mathbf{u}.\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Remark: Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ be a vector. Using scalar product we can also determine the length of the vector **u** as follows.

$$\mathbf{u}\,.\mathbf{u} = \|\mathbf{u}\|\|\mathbf{u}\|cos\theta$$

$$= \|\mathbf{u}\|^2 cos\theta\,, since\ \theta = 0^0$$

$$= \|\mathbf{u}\|^2$$

Then, $\|\mathbf{u}\| = \sqrt{\mathbf{u}.\mathbf{u}} = \sqrt{(u_1, u_2, u_3, \ldots, u_n).(u_1, u_2, u_3, \ldots, u_n)}$

$= \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}$, as stated in definition1.3.2.

Theorem1.4.1: Let **u** be a vector in n-space and let λ be a scalar. Then

a) $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

b) $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$

Proof: a) Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$

then $\|\mathbf{u}\| = \sqrt{\mathbf{u}.\mathbf{u}} = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}$

Since ${u_i}^2 \geq 0$ for all $i = 1,2, \ldots, n$, $\|\mathbf{u}\| \geq 0$.

$\|\mathbf{u}\| = 0$ if and only if ${u_i}^2 = 0$ for all $i = 1,2, \ldots, n$, and

${u_i}^2 = 0$ if and only if $u_i = 0$ for all $i = 1,2, \ldots, n$,

Therefore $\mathbf{u} = \mathbf{0}$

b) $\|\lambda\mathbf{u}\| = \sqrt{\lambda\mathbf{u}.\lambda\mathbf{u}} = \sqrt{\lambda^2(\mathbf{u}.\mathbf{u})} = \sqrt{\lambda^2}\sqrt{\mathbf{u}.\mathbf{u}} = |\lambda|\|\mathbf{u}\|$

Orthogonal vectors

It is known that the concept of perpendicularity is fundamental to geometry; consequently the concept of orthogonality is fundamental to Linear Algebra. Thus the perpendicularity or orthogonality of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 1.4.2: Two non -zero vectors **u** and **v** are said to be perpendicular (or orthogonal) if and only if the angle between them is 90^0.

Question: 1**.** Is the zero vector orthogonal to every vector? Why?

2. Let $\mathbf{u} = (5, -2, 3)$ be a given vector. Then how many nonzero vectors are there that are orthogonal to **u**?

Theorem1.4.2: Let **u** and **v** be two non-zero vectors. **u** and **v** are orthogonal if and only if

a) **u.v**=0

b) $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$

Proof: a) (i) Suppose **u** and **v** are orthogonal. We need to show that **u.v**=0.

$\mathbf{u}\,.\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

$= \|\mathbf{u}\|\|\mathbf{v}\|\cos 90^{0}$`

$= 0$

(ii) Suppose $\mathbf{u}\,.\,\mathbf{v} = 0$. We want to show that **u** and **v** are orthogonal.

$0 = \mathbf{u}\,.\,\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$, dividing both sides by $\|\mathbf{u}\|\|\mathbf{v}\|$ we get

$\cos\theta = 0$ and $\theta = 90^{0}$ since $0 \le \theta \le \pi$

Therefore, **u** and **v** are orthogonal.

b) (i) Suppose **u** and **v** are orthogonal.

$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v}).(\mathbf{u}-\mathbf{v})$

$= (\mathbf{u}-\mathbf{v}).\mathbf{u} - (\mathbf{u}-\mathbf{v}).\mathbf{v}$

$= (\mathbf{u}.\mathbf{u}) - (\mathbf{v}.\mathbf{u}) - (\mathbf{u},\mathbf{v}) + (\mathbf{v}.\mathbf{v})$

$= \|\mathbf{u}\|^2 - 2\mathbf{u}.\mathbf{v} + \|\mathbf{v}\|^2$

$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, since $\mathbf{u}\,.\,\mathbf{v} = 0$

$= \|\mathbf{u}\|^2 + \mathbf{2u}.\mathbf{v} + \|\mathbf{v}\|^2$ (why?)

$= \|\mathbf{u}+\mathbf{v}\|^2$

Hence, $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$.

(ii) Suppose $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$. Then squaring both sides we get

$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}+\mathbf{v}\|^2$, expanding it we also get

$\|\mathbf{u}\|^2 - \mathbf{2u}.\mathbf{v} + \|\mathbf{v}\|^{\mathbf{2}} = \|\mathbf{u}\|^{\mathbf{2}} + \mathbf{2u}.\mathbf{v} + \|\mathbf{v}\|^2$, after simplification we get

$4\mathbf{u}.\mathbf{v} = 0$ and then $\mathbf{u}.\mathbf{v} = 0$

Therefore, **u** and **v** are orthogonal vectors.

Definition1.4.3: Let P and Q be points in $\mathbb{R}^n$. The distance between P and Q denoted by

$\mathrm{d}(\mathrm{P},\mathrm{Q})$ is defined as $\mathrm{d}(\mathrm{P},\mathrm{Q}) = \|\mathrm{P}-\mathrm{Q}\|$. ($P-Q$ is a vector).

Example1.4.3: Find the distance between the points $P = (3,-2,6)$ and $Q = (1,2,-7)$.

Solution: $\mathrm{d}(\mathrm{P},\mathrm{Q}) = \|\mathrm{P}-\mathrm{Q}\| = \|(3,-2,6)-(1,2,-7)\|$

$= \|(\mathbf{2},-4,13)\|$

$= \sqrt{2^2+(-4)^2+(13)^2}$

$= \sqrt{4+16+169}$

$= \sqrt{189}$

Theorem1.4.2: Let **u** and **v** be vectors in $\mathbb{R}^n$, then

a) $|\mathbf{u}.\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ …….(Schwartz inequality)

b) $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$…….(Triangle inequality)

Proof: a) If either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then the theorem is true.

If $\|\mathbf{u}\| \neq \mathbf{0}$ and $\|\mathbf{v}\| \neq \mathbf{0}$, we need to prove $-\|\mathbf{u}\|\,\|\mathbf{v}\| \le \mathbf{u}.\mathbf{v} \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

This statement is true if and only if $-1 \le \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \le 1$.

Now , $\frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$, where $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ and $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ are both unit vectors.

Thus, it is enough to show that $-1 \le \boldsymbol{a}.\boldsymbol{b} \le 1$ for any unit vectors **a** and **b.**

The term $\mathbf{a}.\mathbf{b}$ occurs as part of the expansion of $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})$ as well as part of

$(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})$.

The first expansion gives $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) = \|\mathbf{a}+\mathbf{b}\|^2 \geq 0$

$\Rightarrow \boldsymbol{a}.\boldsymbol{a} + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} \geq 0$

$\Rightarrow \|\mathbf{a}\|^2 + 2\boldsymbol{a}.\boldsymbol{b} + \|\mathbf{b}\|^2 \geq 0$

$\Rightarrow 1 + 2\boldsymbol{a}.\boldsymbol{b} + 1 \geq 0$

$\Rightarrow \boldsymbol{a}.\boldsymbol{b} \geq -1$

Similarly the second expansion gives $(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) = \|\mathbf{a}-\mathbf{b}\|^2 \geq 0$

$\Rightarrow \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} \geq 0$

$\Rightarrow \|\mathbf{a}\|^2 - 2\boldsymbol{a}.\boldsymbol{b} + \|\mathbf{b}\|^2 \geq 0$

$\Rightarrow 1 - 2\boldsymbol{a}.\boldsymbol{b} + 1 \geq 0$

$\Rightarrow \boldsymbol{a}.\boldsymbol{b} \leq 1$

Hence $-1 \leq \boldsymbol{a}.\boldsymbol{b} \leq 1$.

b) $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v})$

$= \|\mathbf{u}\|^2 + 2\mathbf{u}.\mathbf{v} + \|\mathbf{v}\|^2$

$\leq \|\mathbf{u}\|^2 + 2|\mathbf{u}.\mathbf{v}| + \|\mathbf{v}\|^2$

$\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2$ …….Schwartz inequality.

$= [\|\mathbf{u}\| + \|\mathbf{v}\|]^2$

Therefore, $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Orthogonal projection of a vector

The general concept of scalar product of vectors together with their orthogonality property is still important in dealing with orthogonal projection of vectors.

Definition 1.4.4: Let $\mathbf{u}$ and $\mathbf{v}$ be two non zero vectors with the same initial point at A and θ

be the angle between **u** and **v,**($0 \leq \theta \leq \pi$). The vector obtained by droping a perpendicular from the head of **u** onto **v** or on the extention of **v** with initial at A is called an orthogonal projection of **u** onto **v.**

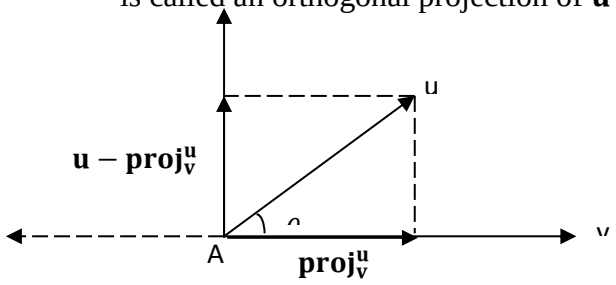


Figure1.13

The projection of **u** onto **v** is a vector denoted by $\text{proj}_{\mathbf{v}}^{\mathbf{u}}$ and the length of the vector $\text{proj}_{\mathbf{v}}^{\mathbf{u}}$ is given by $\|\mathbf{u}\|cos\theta$. Thus the vector $\text{proj}_{\mathbf{v}}^{\mathbf{u}} = \|\mathbf{u}\|cos\theta\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)$

$$= \frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}(\cos\theta)\mathbf{v}$$

$$= \frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}\left(\frac{\text{u.v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)\mathbf{v}$$

$$= \frac{\mathbf{u.v}}{\|\mathbf{v}\|^2}(\mathbf{v})$$

Example1.4.4: Let $\mathbf{u} = (4,3,-2)$ and $\mathbf{v} = (1,-1,1)$. Find

a) $\text{proj}_{\mathbf{v}}^{\mathbf{u}}$ b) $\text{proj}_{\mathbf{u}}^{\mathbf{v}}$ c) $\text{proj}_{\mathbf{2v}}^{(\mathbf{u}-\mathbf{v})}$

a) $\text{proj}_{\mathbf{v}}^{\mathbf{u}} = \frac{\mathbf{u.v}}{\|\mathbf{v}\|^2}(\mathbf{v}) = \frac{(4,3,-2).(1,-1,1)}{\left(\sqrt{1^2+(-1)^2+1^2}\right)^2}(\mathbf{1},-\mathbf{1},\mathbf{1}) = \frac{-1}{3}(\mathbf{1},-\mathbf{1},\mathbf{1}) = (\frac{-1}{3},\frac{1}{3},\frac{1}{3})$.

Similarly, you can do for (b) and (c).

Note: $\text{proj}_{\mathbf{v}}^{\mathbf{u}}$ and **v** are parallel vectors while **u**-$\text{proj}_{\mathbf{v}}^{\mathbf{u}}$ and **v** are perpendicular vectors**.(** verify it).

Direction cosines

Let $\hat{\mathbf{i}}, \hat{j}$ and $\hat{k}$ be unit vectors along the positive x, y and z- axes of the rectangular coordinate system. Let $\mathbf{u} = a\hat{\mathbf{i}} + b\hat{j} + c\hat{k} = (a, b, c)$ be a nonzero vector. If α, β and θ are angles between $\mathbf{u}$ and the vector $\hat{\mathbf{i}}, \hat{j}$ and $\hat{k}$ respectively as shown in the figure,

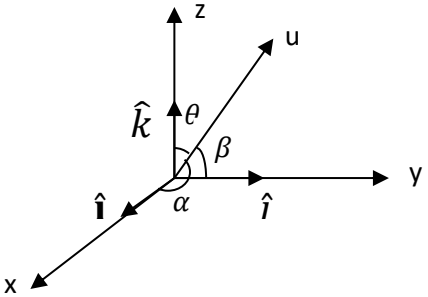


Figure1.14

Then $\cos\alpha = \frac{\mathbf{u}.\hat{\mathbf{i}}}{\|\mathbf{u}\|\|\hat{\mathbf{i}}\|} = \frac{a}{\|\mathbf{u}\|}$

$\cos\beta = \frac{u.\hat{j}}{\|\mathbf{u}\|\|\hat{j}\|} = \frac{b}{\|\mathbf{u}\|}$ and

$\cos\theta = \frac{u.\hat{k}}{\|\mathbf{u}\|\|.\hat{k}\|} = \frac{c}{\|\mathbf{u}\|}$

The angles α, β and θ are called directions angles of **u,** and the numbers $\cos\alpha$, $\cos\beta$ and $\cos\theta$ are called direction cosines of **u**.

Activity 1.4

1. Determine whether the following pairs of vectors are **equal** or **parallel** or **orthogonal**.

a) $\mathbf{u} = (3,1,-3)$ and $\mathbf{v} = (4,-6,2)$

b) $\mathbf{u} = (-3,1,7)$ and $\mathbf{v} = (-9,3,21)$

c) $\mathbf{u} = (5,-6,7)$ and $\mathbf{v} = (-5,6,-7)$

d) $\mathbf{u} = (2,0,-5)$ and $\mathbf{v} = (5,0,-2)$

2. For vectors $\mathbf{u}$ and $\mathbf{v}$ in R^2 *or* R^3 specify the condition(s) for which $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

3. Let $\mathbf{u} = (4, c)$ and $\mathbf{v} = (3,5)$find c such that

a) **u** and **v** are orthogonal.

b) the angle between **u** and **v** is $\frac{\pi}{3}$

4. Let $\mathbf{u} = (3,0,4)$ and $\mathbf{v} = (2,3,3)$. Find

a) $\mathbf{proj}_{\mathbf{v}}^{\mathbf{u}}$

b) the angle between **u** and **v**

c) a unit vector orthogonal to the vector **v** by projecting **u** onto **v**.

d) the direction cosines of **u**.

5. Let θ be the angle between **u** and **v,** then identify the direction of the vectors $\mathbf{proj}_{\mathbf{v}}^{\mathbf{u}}$ and **v** if

a) $0 \leq \theta < \frac{\pi}{2}$ b) $\theta = \frac{\pi}{2}$ c) $\frac{\pi}{2} < \theta \leq \pi$

1.5 The Vector Product

In the previous section we have seen that the scalar product of two vectors in $\mathbb{R}^n$ is a **scalar**. However, in this section our discussion of vector operation is restricted only in 3-space, and this type of vector multiplication produces a **vector**.

Definition1.5.1: Let $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ be vectors in $\mathbb{R}^3$.

The vector product (or cross product) of **u** and **v** is denoted by $\mathbf{u} \times \mathbf{v}$ is a vector in $\mathbb{R}^3$

and defined by : $\mathbf{u} \times \mathbf{v} = (bz - cy,\ -(az - cx),\ ay - bx).$

Example1.5.1: Find the vector product of the vectors

a) $\mathbf{u} = (1, -1,2)$ and $\mathbf{v} = (3,1,2)$

b) $\mathbf{u} = (4,0, -2)$ and $\mathbf{v} = (2,0, -1)$

a) $\mathbf{u} \times \mathbf{v} = \big((-1)(2) - 2(1), -\big(1(2) - 2(3)\big), 1(1) - (-1)(3)\big)$

$= (-2-2, -(2-6), 1+3)$

$= (-4,4,4).$

b) $\mathbf{u} \times \mathbf{v} = \big(0-0, -\big(-4-(-4)\big), 0-0\big)$

$= (0,0,0)$

Standard Unit Vectors

The vectors $\hat{\mathbf{i}} = (1,0,0), \hat{\mathbf{j}} = (0,1,0)$ and $\hat{\mathbf{k}} = (0,0,1)$ are unit vectors that lie on the positive x, y and z-axes respectively, and are called standard unit vectors in 3-space.

Every vector $\mathbf{u} = (x, y, z)$ in $\square^3$ can be expressed in terms of the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. That is $\mathbf{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Example1.5.2: Write the vector $\mathbf{u} = (2,3,-1)$ interms of the standard unit vectors.

$$\mathbf{u} = (2,3,-1) = 2(1,0,0) + 3(0,1,0) + -1(0,0,1) = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Remark:- $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0 \qquad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$

$\hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0 \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$

$\hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$

Now if $\mathbf{u} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and and $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ then

$\mathbf{u} \times \mathbf{v} = \big(a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}\big) \times \big(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\big) = ax\big(\hat{\mathbf{i}} \times \hat{\mathbf{i}}\big) + ay\big(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\big) + az\big(\hat{\mathbf{i}} x \hat{\mathbf{k}}\big) + bx\big(\hat{\mathbf{j}} \times \hat{\mathbf{i}}\big) + by\big(\hat{\mathbf{j}} \times \hat{\mathbf{j}}\big) + bz\big(\hat{\mathbf{j}} \times \hat{\mathbf{k}}\big) + cx\big(\hat{\mathbf{k}} \times \hat{\mathbf{i}}\big)\, cy\big(\hat{\mathbf{k}} \times \hat{\mathbf{j}}\big) + cz\big(\hat{\mathbf{k}} \times \hat{\mathbf{k}}\big)$

$= ay\hat{\mathbf{k}} + az(-\hat{\mathbf{k}}) + bx(-\hat{\mathbf{k}}) + bz\hat{\mathbf{i}} + cx\hat{\mathbf{j}} + cy(-\hat{\mathbf{i}})$

$= (bz - cy)\hat{\mathbf{i}} + (cx - az)\hat{\mathbf{j}} + (ay - bx)\hat{\mathbf{k}}$

$= (bz - cy, -(az - cx), ay - bx)$, as given in definition1.5.1.

Theorem1.5.1: (properties of vector product)

Let **u**, **v** and **w** be vectors in $\mathbb{R}^3$ and $\lambda \in \mathbb{R}$, then

a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

b) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

c) $\mathbf{u} \times \mathbf{0} = \mathbf{0}$

d) $\mathbf{u}.(\mathbf{u} \times \mathbf{v}) = \mathbf{v}.(\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

e) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

f) $\lambda (\mathbf{u} \times \mathbf{v}) = (\lambda \mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\lambda \mathbf{v})$

Proof: Let $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ then

$$\mathbf{u} \times \mathbf{v} = (a, b, c) \times (x, y, z)$$

$$= (bz - cy, -(az - cx), ay - bx)$$

$$= -(cy - bz, az - cx, bx - cy$$

$$= -(\mathbf{v} \times \mathbf{u})$$

To prove the other properties take the vectors $\mathbf{u} = (a, b, c)$ and $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (r, s, t)$ and apply the definition of cross product.

Remark: 1. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors **u** and **v.**

2**.** If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then **u** and **v** are parallel vectors.

3**.** Since the scalar product of **u** and $\mathbf{u} \times \mathbf{v}$ is zero then **u** and $\mathbf{u} \times \mathbf{v}$ are orthogonal vectors.

Example1.5.3: Let $\mathbf{u} = (\mathbf{1}, -2, 2)$ and $\mathbf{v} = (1, 3, 0)$ then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors **u** and **v**.

$$\mathbf{u} \times \mathbf{v} = (1, -2, 3) \times (1, 3, 0) = (-6, 2, 5)$$

Thus $\mathbf{u}.(\mathbf{u} \times \mathbf{v}) = (1, -2, 2).(-6, 2, 5) = -6 - 4 + 10 = 0$

$$\mathbf{v}.(\mathbf{u}\times\mathbf{v}) = (1,3,0).(-6,2,5) = -6+6+0 = 0$$

Therefore, $\mathbf{u}\times\mathbf{v}$ is orthogonal to both vectors **u** and **v.**

Theorem1.5.2: Let **u** and **v** be vectors in R^3 if θ is the angle between **u** and **v**. Then

a) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}.\mathbf{v})^2$

b) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|sin\theta$

proof: Let **u**=(a,b,c) and **v**=(x,y,z) be vectors in R^3

i) $\|\mathbf{u}\times\mathbf{v}\|^2 = (\mathbf{u}\times\mathbf{v}).(\mathbf{u}\times\mathbf{v})$

$= (bz-cy, cx-az, ay-bx).(bz-cy, cx-az, ay-bx)$

$= (bz-cy)^2 + (cx-az)^2 + (ay-bx)^2$

$= (bz)^2 - 2bzcy + (cy)^2 + (cx)^2 - 2cxaz + (az)^2 + (ay)^2 - 2aybx + (bx)^2$

$= (bz)^2 + (cy)^2 + (cx)^2 + (az)^2 + (bx)^2 - 2bzcy - 2cxaz - 2aybx$

$= (a^2+b^2+c^2)(x^2+y^2+z^2) - (ax)^2 - (by)^2 - (cz)^2 - 2bzcy - 2cxaz - 2aybx$

$= (a^2+b^2+c^2)(x^2+y^2+z^2) - (ax+by+cz)^2$

$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}.\mathbf{v})^2$

ii) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}.\mathbf{v})^2$

$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta)^2$

$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2[1-\cos^2\theta]$

$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta$

$= [\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta]^2$

Therefore, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$, $0 \le \theta \le \pi$.

Activity 1.5

1. Let $\mathbf{u} = \mathbf{3i} + \mathbf{5j} - \mathbf{2k}$ and $\mathbf{v} = \mathbf{4i} - \mathbf{8j} + \mathbf{7k}$. Then find:-

a) $\mathbf{u} + \mathbf{v}$ **b)** $\mathbf{u}.\mathbf{v}$ and c) $\mathbf{u} \times \mathbf{v}$

2. Let $\mathbf{u} = (1,2,3), \mathbf{v} = (3,0,1)$ and $\mathbf{w} = (1,5,-4)$ be three vectors in $\mathbb{R}^3$, then find

a) $\mathbf{u} \times \mathbf{v}$

b) $\mathbf{u} \times (\mathbf{v} - \mathbf{2w})$

c) an orthogonal vector to both **v** and **w**.

d) a unit vector perpendicular to both vectors u and w.

e) the angle between the vectors u and v.

f) the volume of the parallelepiped generated by the three vectors u, v and w.

1.6 Application on Area and Volume

Area: Let **u** and **v** be nonzero vectors in $\mathbb{R}^3$ and θ be the angle between **u** and **v**. Then

$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| sin\theta$, for $0 \leq \theta \leq \pi$, is the area of the parallelogram determined by the vectors **u** and **v**.

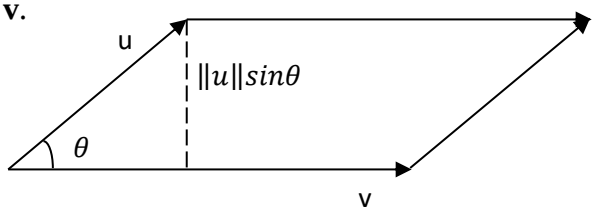


Figure1.15

Area = (base)(height)=$\|\mathbf{v}\|\|\mathbf{u}\| sin\theta = \|\mathbf{u}\|\|\mathbf{v}\| sin\theta = \|\mathbf{u} \times \mathbf{v}\|$

Example1.6.1:- Find the area of the parallelogram determined by the vectors

$\mathbf{u} = (2,3,0)$ and $\mathbf{v} = (-1,2,-2)$

Area=$\|\mathbf{u}\times\mathbf{v}\|$

$= \|(-\mathbf{6},\mathbf{4},\mathbf{7})\| = \sqrt{(-6)^2+4^2+7^2} = \sqrt{101}$ square units.

Example1.6.2: Find the area of the triangle with vertices P=(2,6,-1), Q=(1,4,1) and R=(3,6,2).

Let $\mathbf{u} = \mathrm{Q} - \mathrm{P} = (-1,-2,2)$ and $\mathbf{v} = \mathrm{R} - \mathrm{P} = (1,0,3)$ be vectors with initial at P and terminal at **Q** and ***R*** respectively. Then the area A of triangle PQR is one-half the area of the parallelogram determined by the vectors **u** and **v.**

That is, $A = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\|(-\mathbf{6},\mathbf{5},\mathbf{2})\| = \frac{1}{2}\sqrt{36+25+4} = \frac{1}{2}\sqrt{65}$ square units.

To see the other application of scalar product together with vector product of vectors, let us see first the following definition.

Definition1.6.1: Let $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, then $\mathbf{u}.(\mathbf{v}\times\mathbf{w})$ is called **the scalar triple product** of $\mathbf{u},\mathbf{v}$ and **w.**

Example1.6.3: Let $\mathbf{u} = (3,2,-5), \mathrm{v} = (3,0,1)$ and $\mathrm{w} = (1,5,-4)$, then

$\mathbf{u}.(\mathbf{v}\times\mathbf{w}) = (3,2,-5).\big((3,0,1)\times(1,5,-4)\big)$

$= (3,2,5).(-5,13,15)$

$= -15+26-75$

$= -64$

Remark: For any vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, $\mathbf{u}.(\mathbf{v}\times\mathbf{w}) = \mathbf{v}.(\mathbf{w}\times\mathbf{u}) = \mathbf{w}.(\mathbf{u}\times\mathbf{v})$**.**

Volume: The geometrical interpretation of the scalar triple product $\mathbf{u}.(\mathbf{v}\times\mathbf{w})$ is as follows

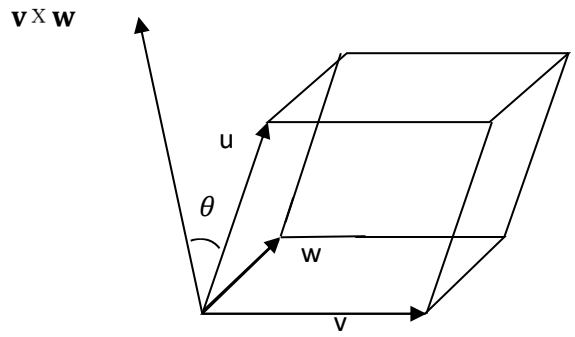


Figure1.16

$\mathbf{u}.(\mathbf{v}\times\mathbf{w}) = \|\mathbf{u}\|\|\mathbf{v}\times\mathbf{w}\|cos\theta$, where θ is the angle between u and $\mathbf{v}\times\mathbf{w}$ as shown in figure1.15**.**

$= (\|\mathbf{u}\|cos\theta)\|\mathbf{v}\times\mathbf{w}\|$, where $\|\mathbf{u}\||cos\theta|$ is height of the parallelepiped and $\|\mathbf{v}\times\mathbf{w}\|$ is the area of the base of the parallelepiped.

Therefore,$|\mathbf{u}.(\mathbf{v}\times\mathbf{w})|$ =Volume of the parallelepiped, generated by the three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

Example1.6.4: Find the volume V of the parallelepiped generated by the vectors

a) $\mathbf{u} = (3,2,-4)$, v $= (1,0,1)$ and $\mathbf{w} = (1,3,-2)$

b) $\mathbf{u} = (1,2,5)$, v $= (1,-1,1)$ and $\mathbf{w} = (0,1,-3)$

a) $V = |\mathbf{u}.(\mathbf{v}\times\mathbf{w})| = \big|(3,2,-4).\big((1,0,1)\times(1,3,-2)\big)\big|$

$= |(3,2,-4).(3,-3,3)| = |9+(-6)+(-12)| = |-9|$

$= 9$ cubic units.

b) $V = |\mathbf{u}.(\mathbf{v}\times\mathbf{w})| = \big|(1,2,5).\big((1,-1,1)\times(0,1,-3)\big)\big|$

$= |(1,2,5).(2,3,1)| = |2+6+5| = |13|$

$= 13$ cubic units

1.7 Lines and Planes

In this section we use the concept of vectors to drive the parametric, the normal and vector forms of equations of a line in $\mathbb{R}^2$ and $\mathbb{R}^3$, as well as planes.

Lines in $\mathbb{R}^2$

Suppose that l is a line in a plane through the points P and Q as shown in the figure1.16. and let **n** be a vector perpendicular to l at P, then the vectors $\mathrm{Q} - P$ and **n** are orthogonal to each other at P.

That is $\mathbf{n}.(\mathrm{Q} - \mathrm{P}) = 0.$

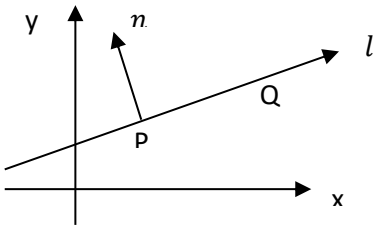


Figure1.17

Therefore, the equation $\mathbf{n}.(\mathrm{Q} - \mathrm{P}) = 0$ is called the normal form of equation of a line l in $\mathbb{R}^2$. The general form of equation of a line l is $\mathrm{ax} + \mathrm{by} = \mathrm{c}$, where $\mathbf{n} = (\mathrm{a}, \mathrm{b})$ and (x,y) is a point on the line l.

Lines in $\mathbb{R}^3$

Again let l be a line in 3-space passing through the point $P_0 = (\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ and parallel to the nonzero vector $\mathbf{v} = (\mathrm{a}, \mathrm{b}, \mathrm{c})$. Let $P = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ be an arbitrary point on l for which the vector $\overrightarrow{P_0P}$ is parallel to **v**. Then by definition1.3.4, if two vectors are parallel then there exist a scalar t such that $\overrightarrow{P_0P} = \mathrm{t}\mathbf{v}, \quad -\infty < t < \infty$ ……(1)

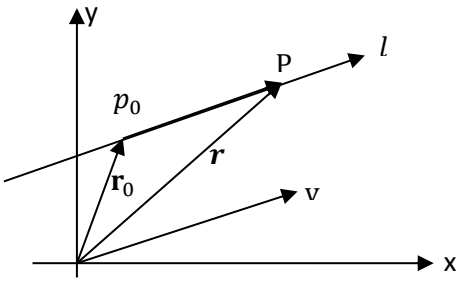


Figure1.18

But $\overrightarrow{P_0P} = P - P_0$, and then $P - P_0 = t\mathbf{v}$.

In terms of components equation (1) becomes $(\mathrm{x} - \mathrm{x}_0, \mathrm{y} - \mathrm{y}_0, \mathrm{z} - \mathrm{z}_0) = (t\mathrm{a}, t\mathrm{b}, t\mathrm{c})$ and from equality of vectors $\mathrm{x} - \mathrm{x}_0 = t\mathrm{a}$, $\mathrm{y} - \mathrm{y}_0 = t\mathrm{b}$ and $\mathrm{z} - \mathrm{z}_0 = t\mathrm{c}$.

Thus the equation $\begin{cases} \mathrm{x} = \mathrm{x}_0 + \mathrm{a}t \\ \mathrm{y} = \mathrm{y}_0 + \mathrm{b}t \\ \mathrm{z} = \mathrm{z}_0 + ct \end{cases}$ $-\infty < t < \infty$ ……(2)

is called **parametric equation** of the line $\boldsymbol{l}$ and t is called **a parameter**.

Now taking $P_0 = (\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ and $P = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ to be the terminal points of position vectors $\mathbf{r}_0 = (\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ and $\mathbf{r} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ respectively as shown in the figure1.17, we get $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r_0}$ that is, $\mathbf{r} - \mathbf{r_0} = \mathrm{t}\mathbf{v}$ or $\mathbf{r} = \mathbf{r_0} + \mathrm{t}\mathbf{v}$. $-\infty < t < \infty$……(3).

Equation (3) is called **the vector form of a line** in $\mathbb{R}^3$.

Example1.7.1: Find the vector form and parametric form of equation of the line l passing through the point $P_0 = (1,2,-1)$ and parallel to the vector $\mathbf{v} = (5,-1,3)$.

The vector form of equation of a line is $\mathbf{r} = \mathbf{r_0} + \mathrm{t}\mathbf{v}$, where $\mathbf{r}$ is a position vector whose terminal point is an arbitrary point $P = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ on the line l and $\mathbf{r}_0$ is also a position vector with terminal point $P_0 = (1,2,-1)$ on l.

Therefore, the required vector form of the equation of the line is:

$(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (1,2,-1) + t(5,-1,3)$ and the corresponding parametric equation of the line is $\mathrm{x} = 1 + 5t, \mathrm{y} = 2 - t$ and $\mathrm{z} = -1 + 3t$.

Example1.7.2: a) Find the parametric form of equation of the line, l which passes through the points $P_0 = (2,4,1)$ and $P_1 = (3,0,-2.)$.

b) Find the intersection of the line you obtain in part (a) and the xy-plane?

a) Since the vector $\overrightarrow{P_0P_1} = (1,-4,-3)$ lies on the line l, $\overrightarrow{P_0P_1}$ and l are parallel. As $P_0 = (2,4,1)$ is on l, the parametric equation of the line l is $x = 2 + t, y = 4 - 4t$ and $z = 1 - 3t$.

b) The line l intersects the xy-plane at the point where $z = 1 - 3t = 0$; that is when $t = \frac{1}{3}$.

Substituting this value of t in the parametric equation of l, we get the point of intersection $(x, y, z) = (\frac{7}{8}, \frac{8}{3}, 0)$.

Equation of a Plane

Suppose the plane passing through the point $P_0 = (x_0, y_0, z_0)$and having the nonzero normal vector $\mathbf{n} = (a, b, c)$ to the plane at P_0. Let $P = (x, y, z)$ be arbitrary point on the plane, as shown in the figure1.18. (a vector that is orthogonal to a plane is called normal vector).

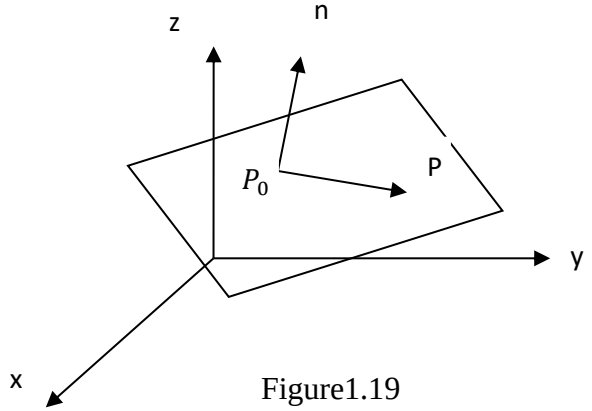


Figure1.19

n is orthogonal to the plane at P_0, hence $\mathbf{n}.\overrightarrow{P_0P} = 0$……(1)

where, $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ and $\mathbf{n} = (a, b, c)$.

Equation (1) can be written as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$……(2)

Equation (2) is called **point-normal form of equation of a plane**.

From equation (2) we get $ax + by + cz - ax_0 - by_0 - cz_0 = 0$ and letting the constant $-ax_0 - by_0 - cz_0 = d$, we get the equation $ax + by + cz + d = 0$, which is called the general form of equation of a plane, with normal vector $\mathbf{n} = (a, b, c)$.

Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{r} = (x, y, z)$ be position vectors with terminal points $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$ respectively, since the two vectors $\overrightarrow{P_0P}$ and $\mathbf{r} - \mathbf{r_0}$ represent the same vector as shown in figure 1.17, $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r_0}$**.** So equation (1) can be written as $\mathbf{n}.(\mathbf{r} - \mathbf{r_0}) = 0$. This equation is called the vector form of the equation of a plane.

Two planes in 3-space are said to be parallel if their normal vectors are parallel. They are said to be perpendicular if their normal vectors are perpendicular. The angle between two planes is defined to be the angle between their normal vectors.

Example1.7.3: Find the equation of the plane passing through the point $(-1,2,4)$ and perpendicular to the vector $\mathbf{n} = (-2,4,1)$.

Let $P = (x, y, z)$ be any point on the plane. Then the point-normal form of equation of the plane is given by: $-2(x + 1) + 4(y - 2) + 1(z - 4) = 0$. Simplifying this gives $-2x + 4y + z - 14 = 0$.

Example1.7.4: Let $P_1 = (1,2,-1)$, $P_2 = (2,3,1)$and $P_3 = (3,-1,2)$ be three points. Find the equation of the plane that pass through the points P_1, P_2 and P_3.

$\overrightarrow{P_1P_2} = (1,1,2)$ and $\overrightarrow{P_1P_3} = (2,-3,3)$ are two vectors in the plane with common initial point at P_1. The normal vector to the plane at $P_1$is

$\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (9,1,-5)$

Since P_1 is on the plane, the point- normal form of equation of the plane is

$9(x-1)+1(y-2)-5(z+1)=0$ or $9x+y-5z-16=0$,

Example1.7.5: Find the equation of the plane that pass through the point $(3,-6,7)$ and is parallel to the plane $5x-2y+z-5=0$.

Let $\mathbf{n}_1$ be a normal vector to the required plane and $\mathbf{n}_2$ be the normal vector to the plane $5x-2y+z-5=0$. Then $\mathbf{n}_2=(5,-2,1)$. Since the two planes are parallel, their normal vectors are parallel. Thus, $\mathbf{n}_1=t\mathbf{n}_2$, where t is a nonzero scalar. This implies $\mathbf{n}_1=(5t,-2t,t)$.Now if we choose $t=1$ then $\mathbf{n}_1=(5,-2,1)$, then by the point-normal form of equation of the plane, the required equation is

$5(x-3)-2(y+6)+1(z-7)=0$ or $5x-2y+z-34=0$.

Distance from a point to a plane

In this subtopic we want to drive the distance formula from a point to a plane using the concept of vectors.

Theorem1.7.1: The distance D from a point$P_0=(x_0,y_0,z_0)$, not on the plane, and the plane $ax+by+cz+d=0$ is given by: $D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

where P_0 is not on the plane,

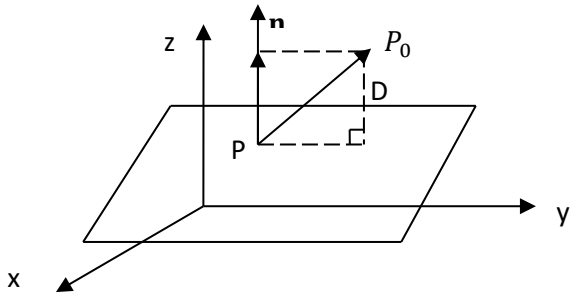


Figure1.20

Proof: let $P=(x_1,y_1,z_1)$ be any point in the plane. Let $\mathbf{n}=(a,b,c)$be a normal vector whose

initial point is at P.

$$D = \left\| proj_{\mathbf{n}}^{\overrightarrow{PP_0}} \right\| = \left| \frac{\mathbf{n}.\overrightarrow{PP_0}}{\|\mathbf{n}\|^2}(\mathbf{n}) \right| = \frac{|\mathbf{n}.\overrightarrow{PP_0}|}{\|\mathbf{n}\|},$$

But $\mathbf{n}.\overrightarrow{PP_0} = (a,b,c).(x_0 - x_1, y_0 - y_1, z_0 - z_1) = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$

$= ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1.$

$\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$. Thus $D = \frac{|ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1|}{\sqrt{a^2 + b^2 + c^2}}$.

Since $P = (x_1, y_1, z_1)$ lies on the plane it satisfies the equation of the plane. That is

$ax_1 + by_1 + cz_1 + d = 0$ or $d = -ax_1 - by_1 - cz_1$.

Therefore, $= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Examples1.7.6: Find the distance between the point $(0,3,2)$ and the plane

$$2x + y - 3z = 5.$$

$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$, where a=2,b=1,c=-3 and d=-5

$$D = \frac{|2(0) + 1(3) + (-3)(-2) + (-5)|}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{6}{\sqrt{14}}$$

Example1.7.7: Find the distance between the parallel planes: $x + 2y - 2z = 3$ and

$$2x + 4y - 4z = 7$$

Since corresponding normal vectors $\mathbf{n}_1$=(1,2,-2) and $\mathbf{n}_2$=(2,4,-4) are parallel, the two planes are parallel. Thus take any point on one of the plane and find the distance between the point of your choice and the other plane. For instance, the point $(1,1,0)$ is on the plane $x + 2y - 2z = 3$.

Therefore, the distance between the plane containing the point $(1,1,0)$ and the other plane $2x + 4y - 4z - 7 = 0$ is: $D = \frac{|2(1)+4(1)+-4(0)+(-7)|}{\sqrt{2^2+4^2+(-4)^2}} = \frac{|-1|}{\sqrt{36}} = \frac{1}{6}$.

Activity1.7

1. Let l be a line through the points $P_1(1,-2,3)$ and $P_2(-2,3,-3)$. Find the vector form and the parametric equation of l.

2. Determine the equation of the xy-plane.

3. Determine the acute angles between the planes $x + y + z = 1$ and $x - 2y - z = 3$.

4. Find the distance from the point $(-1,3,2)$ to

   a) the plane $z = -\frac{1}{2}x + 5$

   b) the line $x - 1 = \frac{y-2}{3}$.

Check list

Evaluate yourself with the following questions whether you attain the objectives stated at the beginning of this chapter.

Can I:

1. Define a vector?
2. Find the scalar product of two vectors?
3. Find the angle between two nonzero vectors?
4. Specify the conditions for two vectors to be orthogonal?
5. State and prove Schwarz inequality and triangular inequality?
6. Find two vectors one is parallel and the other is perpendicular to the given vector?

7. Define cross product of vectors and state its properties?

8. Explain the difference between dot product and cross product of vectors?

9. Find the area of a parallelogram generated by two vectors?

10. The volume of the parallelepiped generated by three vectors?

11. Find the equations of a line and a plane in 3-space?

12. Find the distance between a point and a plane?

If your answer to each of these questions is absolutely yes, you can proceed to the self-test exercise and then to the next chapter. However if you are hesitated to a certain extent, please review your lessons once more.

Self-test exercise

1. Find the non-zero vector u with initial point at (-1,3,-5)and having the same direction as the vector (2,-1,4).
2. Let $\mathbf{u} = (1, -3,2)$ and $\mathbf{v} = (-1,4,3)$. then find$\left\|-2\mathbf{u} + \sqrt{2}\mathbf{v}\right\|$.
3. Using vector method, show that the line joining the mid points of the two sides of a triangle is parallel to the third side and has half the length.
4. Let **u**= (2,-1) and **v**= (-1,2) be given. Find a vector **w** such that **w.u**=1 and **w.v**=0.
5. If $\|\mathbf{u}\|$=**5** and $\|\mathbf{u}\|$=**3.** What are the smallest and the largest values of
   a) $\mathbf{u}.\mathbf{v}$ b) $\|\mathbf{u} - \mathbf{v}\|$.
6. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in R^n such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$ and$\|\mathbf{u}\| = 3$, $\|\mathbf{v}\| = 5$, $\|\mathbf{w}\| = 7$. Then find the angle between $\mathbf{u}$ and $\mathbf{v}$.
7. If θ is the angle between **u** and **v** and **u.v**$\neq$ 0, then prove that $tan\theta = \frac{\|\mathbf{uxv}\|}{\mathbf{u}.\mathbf{v}}$.
8. Let **u**=(x, y+2, 1, 2) and **v**=(3, 1, x+y, x-1) be vectors. For what values of x and y the vectors **u** and **v** are orthogonal.
9. If $\mathbf{u}.(\mathbf{v}\times\mathbf{w}) = 5$ then find the values of $\mathbf{u}.(\mathbf{w}\times\mathbf{v})$ and $\mathbf{v}.(\mathbf{u}\times\mathbf{w})$.
10. If the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and **r,** lies in the same plane then prove that $(\mathbf{u}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{r}) = \mathbf{0}$.

11. Verify that the triangle with vertices P(2,3,-4), Q(3,1,2) and R(7,0,1) is a right angled triangle.
12. Find the interior angles and area of the parallelogram generated by the vectors $(3,1,1)$ & $(2,-2,5)$.
13. Find the angle between a diagonal of a cube and one of its adjacent edges?
14. Find the parametric equation of the line passing through the point $P=(3,-1,2)$ and Parallel to the vector $\mathbf{v}=(1,-2,3)$.
15. Determine whether the following lines and planes are parallel or perpendicular or neither.
    a. $x=5-4t,\ y=1-t,\ z=3+2t$; $x+2y+3z-9=0$
    b. $x=2+t,\ y=1-t,\ z5+3t$; $6x+6y-7z=0$
    c. $x=9-5t,\ y=-1-t,\ z=3+t$; $2x-3y+4z+7=0$
16. Find the parametric equation for the line of intersection of the planes $2x+3y-5x=0$ And $y=0$.
17. Find the equation of the plane that pass through the origin and is parallel to the plane $7x+4y-2z+3=0$.
18. Find the intersection of the line with parametric equation $x=2+2t$, $y=1-2t, z=3+t$ and the plane $x+2y-z=7$.
19. Find the distance from the point $P=(1,2,3)$ to the line l through the point $Q=(1,-1,-2)$and is parallel to the vector $\mathbf{v}=(2,0,-2)$.

CHAPTER TWO

Matrices and Determinants

Introduction

The concepts of matrices had its origin in various types of linear problems, the most important of which concerns the nature of solutions of any given system of linear equation. Matrices are also useful in organizing and manipulating large amounts of data.

Today, the subject of matrices is one of the most important and powerful tools in Mathematics which has found applications to a very large number of disciplines such as Engineering, Business and Economics, Statistics etc.

Chapter Objectives

At the end of this chapter the readers should be able to:

- Define the term matrix.
- Identify the properties of a matrix.
- Define the inverse, transpose and determinant of a matrix.
- Find the inverse, transpose and determinant of a matrix.
- Identify the properties of a determinant.
- Determine echelon form of a given matrix.
- Solve systems of linear equations.

2.1 Definition of a Matrix and Basic Operations

Definition 2.1: A matrix is any rectangular array of real numbers or variables of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The numbers or the variables in the matrix are called entries or elements of the matrix. If a matrix has m rows and n columns then we say that its size is m by n (mxn) matrix .An nxn matrix is called a square or a matrix of order n. A 1x1 matrix is simply a real number .Matrices will be denoted by capital bold –faced letters **A**, B etc or by (a_{ij}) or (b_{ij}).

For instance if

$$\mathbf{A}=\begin{pmatrix}1 & 4 & 5\\ 0 & 8 & 0\\ 4 & 7 & 9\end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix}1 & 0 & 3\\ 6 & \sqrt{3} & \frac{1}{2}\end{pmatrix} \qquad \text{……..} \qquad \textbf{(2)}$$

then **A** is 3x3 square matrix or a matrix order 3 while **B** is 2x3 matrix .The entries in the i^{th} row and j^{th} column of mxn matrix of **B** is written a_{ij} .For an nxn square matrix ,the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal element .The the main diagonal entries for the matrix A (2) are 1,8,9.

Definition 1.2: [Column and row vectors]

An nx1matrix $\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}$ is called a column vector. A 1xn matrix $(a_1, a_2, .., a_n)$ is called a row vector.

2.1.1 Special Matrices

In matrix theory there are many special kinds of matrices that are important because they posses certain properties .The following is a list of some of these matrices.

a) A matrix that consists of all zero is called a zero matrix and is denoted by O_{mxn}

For example $O_{2x1}=\begin{pmatrix}0\\ 0\end{pmatrix}$ $O_{2x2}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ $O_{3x3}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

b) An nxn matrix A is said to be a triangular matrix if all its entries below the main diagonal are zeros or if all its entries above the main diagonal are zeros,[in other words a square matrix **A**

is triangular if $a_{ij} = 0\ for\ i > j\ or\ a_{ij} = 0, i < j$].More specially ,in the first case the matrix is called upper triangular and in the second case the matrix is called lower triangular .The following matrices

$$\begin{pmatrix} 2 & 5 & 6 & 11 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 10 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 & 0 \\ -5 & 6 & 0 & 0 \\ 4 & 9 & 5 & 0 \\ 7 & 3 & 9 & 2 \end{pmatrix}$$

Upper triangular matrix Lower triangular

An nxn matrix A is said to be a **diagonal matrix** if all its entries not on the main diagonal are zeros. In terms symbolism D= $(d_{ij})_{nxn}$, D is diagonal matrix if $d_{ij} = 0\ for i \neq j$ .The matrix D thus is given by

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \quad \text{..... (3)}$$

(c) If in (3) if all the diagonal elements are equal, it is referred to as a **scalar matrix** $\boldsymbol{S_n}$, and if these elements are equal to 1, we have a unity or identity matrix $\boldsymbol{I_n}$of order$n$.

Thus, $$S_n = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix} \qquad I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

are respectively a scalar and identity matrix.

2.1.2 Operation on Matrices

Defination1.3 [Equality of Matrices]

Two matrices $\boldsymbol{A} = (\boldsymbol{a_{ij}})_{\boldsymbol{m\times n}}$ and $\boldsymbol{B} = (\boldsymbol{b_{ij}})_{\boldsymbol{m\times n}}$ are equal if $a_{ij} = b_{ij}$ for each $i\ and\ j$. In other words, two matrixes are equal if and only if they have the same size and their corresponding entries are equal.

Matrix Addition

When two matrices A and B are of the size we can add them by adding their corresponding entries

Definition 1.4 If A and B are mxn matrices, then their sum is

$$A+B=(a_{ij}+b_{ij})_{mxn}.$$

Example 1: Addition of two matrices.

$$a)\ A=\begin{pmatrix}2 & -1 & 3\\ 0 & 4 & 6\\ -6 & 10 & -5\end{pmatrix} \quad and \begin{pmatrix}4 & 7 & -8\\ 9 & 3 & 5\\ 1 & -1 & 2\end{pmatrix},\text{then}$$

$$A+B=\begin{pmatrix}2+4 & -1+7 & 3+-8\\ 0+9 & 4+3 & 6+5\\ -6+1 & 10+(-1) & -5+2\end{pmatrix}=\begin{pmatrix}6 & 6 & -5\\ 9 & 7 & 11\\ -5 & 9 & -3\end{pmatrix}$$

b) The sum of

$$A=\begin{pmatrix}1 & 3 & 4\\ 2 & 5 & 7\end{pmatrix} \quad and \quad B=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$$

is not define, since A and B are of different sizes.

Definition 1.5 [Scalar Multiplication of a matrix]

If k is a real number, then the scalar multiple of a matrix A is

$$kA=\begin{pmatrix}ka_{11} & ka_{12} & \cdots & ka_{1n}\\ ka_{21} & ka_{22} & \cdots & ka_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ ka_{m1} & ka_{m2} & \cdots & ka_{mn}\end{pmatrix}=\left(ka_{ij}\right)_{mxn}$$

In other words, to compute kA, we simply multiply each.

For instance, from definition 1.5

$$5\begin{pmatrix}2 & 0 & 5\\ 1 & 3 & 4\\ 3 & 6 & 1\end{pmatrix}=\begin{pmatrix}5.2 & 5.0 & 5.5\\ 5.1 & 5.3 & 5.4\\ 5.3 & 5.6 & 5.1\end{pmatrix}=\begin{pmatrix}10 & 0 & 25\\ 5 & 15 & 20\\ 15 & 30 & 5\end{pmatrix}$$

The difference of two mxn matrices defined in usual $A-B=A+(-B)\ where\ -1B=-B$

2.1.3 Properties of Matrix Addition and Scalar Multiplication

Suppose $A, B\ and\ C$ are mxn matrices and $\alpha\ and\ \beta$ are scalars .Then

i) $\boldsymbol{A+B=B+A}$ $(Commutative\ law\ of\ addition)$

ii) $(\boldsymbol{A+B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$ $(Associative\ law\ of\ addition)$

iii) $\alpha(A+B)=\alpha A+\alpha B$

iv) $(\alpha+\beta)A=\alpha A+\beta A$

v) $(\alpha\beta)A=\alpha(\beta A)$

vi) $1A=A$

Note: Each of the above six properties can be proved by using definition 1.4 and 1.5

2.2 Product of Matrices and Some Algebraic properties, Transpose

Definition 1.6: Let the number of columns in matrix A be the same as the number of rows in matrix B, then the matrix product AB exists and the element in row i and column j of AB is obtained by multiplying the corresponding elements of row i of A and column j of B and adding the product.

In other words if matrix A has n column and matrix B has n rows then the i[th] row of A is $(a_{i1}, a_{i2},..., a_{in})$ and j^{th} column of B $\begin{pmatrix}b_{1j}\\ b2\,j\\ .\\ .\\ .\\ bnj\end{pmatrix}$.Thus if C=AB then

$$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{1n} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Moreover the number of rows and the number of columns of C are equal to the number of rows of A and the number of column of B, respectively .Thus

$$\underset{m\times n}{\mathbf{A}} \quad \underset{n\times r}{\mathbf{B}} = \underset{m\times r}{\mathbf{C}}$$

Example 1:if $A=\begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Then $AB=\begin{pmatrix} 3+12+2-1 & 6+16+0+1 \\ 1+6+3-1 & 2+8+0+1 \\ 0+3+2-3 & 0+4+0+3 \end{pmatrix}=\begin{pmatrix} 16 & 23 \\ 9 & 11 \\ 2 & 7 \end{pmatrix}$

We note here that the size of A is 3x4 and the size of B is 4x2 consequently the size of AB is 3x2.

2.2.1 Properties of Matrix Multiplication

In defining the properties of matrix multiplication below, the matrix A, B *and* $\boldsymbol{C}$ are assumed to be of compatible dimensions for the operations in which they appear.

Property I: Matrix multiplication is, in general, not commutative. That is $\boldsymbol{AB \neq BA}$**.** Observe that in Example 1 of section $\boldsymbol{BA}$ is not even define because the first matrix in this case B does not have the same numbers of columns as the number of rows of the second matrix **A.**

Property II: From **AB=0**, it does not follow that either **A=0** or **B=0**.Here **O**$^{'S}$ are null matrix of appropriate order.

Example 2: For matrices **A** and **B** are given by

$$\mathbf{A}=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$$

We have, $\mathbf{AB}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ is a null matrix even though **A** or **B** is not a null matrix.

Property III: The relation AB=AC or BA=CA does not imply that B=C.The cancelation law does not hold in general as in a real numbers.

Example 4 For the matrices

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 2\\ -1 & 4 & 3\end{pmatrix} \quad B=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & -1\\ 2 & 2 & 2\end{pmatrix}, \quad C=\begin{pmatrix}2 & 3 & 4\\ 2 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}$$

We have, by direct multiplication

$$AB=\begin{pmatrix}9 & 10 & 7\\ 6 & 7 & 6\\ 9 & 8 & -1\end{pmatrix}=AC, although B\neq C$$

Property IV: Matrix multiplication is associative That is

A (BC) = (AB) C

Property V: The multiplication of matrices is distributive with respect to addition i.e

A(B+C)=AB+AC (B+C)A=BA+CA

Example 5

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \quad \mathbf{B}=\begin{pmatrix}2 & 1\\ -3 & 2\end{pmatrix} \quad \text{and} \quad C=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

Verify that **A (BC**) = (**AB) C** and A **(B+C) =AB+AC**

Solution:

$$A(BC)=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}4 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}6 & 5\\ 16 & 11\end{pmatrix}$$

$$(AB)C=\begin{pmatrix}-4 & 5\\ -6 & 11\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}=\begin{pmatrix}6 & 5\\ 16 & 11\end{pmatrix}$$

Thus

$$A(BC)=\begin{pmatrix}6 & 5\\ 16 & 11\end{pmatrix}=(AB)C$$

$$A(B+C)=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & 1\\ -1 & 3\end{pmatrix}=\begin{pmatrix}1 & 7\\ 5 & 15\end{pmatrix}$$

$$AB+AC=\begin{pmatrix}-4 & 5\\ -6 & 11\end{pmatrix}+\begin{pmatrix}5 & 2\\ 11 & 4\end{pmatrix}=\begin{pmatrix}1 & 7\\ 5 & 15\end{pmatrix}$$

Therefore

A(B+C)=AB+AC.

Notation: Since A (BC)=(AB)C, one may simply omit the parentheses and write ABC.

The same is true for a product of or more matrices .In the case where an nxn matrix is multiplied by itself a number of times, it is convenient to use exponential notation. Thus, if k is a positive, then

$$A^k = \underbrace{AA\cdots A}_{k\ times}$$

Example 6 *if* $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

Then $A^2=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$

$$A^3 = AAA = AA^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

And in general

$$A^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

Example 7 Simplify the following matrix expression

$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB$

Solution: Using the properties of matrix we get

$$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB = A^2+2AB+6BA-3B^2-A^2+7B^2-5AB$$
$$= -3AB+6BA+4B^2$$

2.2.2 Transpose of a Matrix

Definition 1.7 The transpose of a matrix A, denoted A^T, is the matrix whose column are the rows of the given matrix A. Symbolically the transpose of an mxn matrix $A=(a_{ij})_{mxn}$ is nxm matrix $A^T=(a_{ij}{}^T)_{nxm} = (a_{ji})_{nxm}$ Where $a_{ij}{}^T = a_{ji}$.

For example, if

Theorem 1.1 Suppose A and B are matrices and k a scalar. Then

i) $(A^T)^T = A$ *ii*) $(A+B)^T = A^T + B^T$
iii) $(AB)^T = B^T A^T$ *iv*) $(kA)^T = kA^T$

Proof: We give here the proof of iii) here the rest is left as exercise

Note that

$$A = (a_{ik})_{mxn}, \qquad B = (b_{kj})_{nxr}$$

then

$$B^T A^T = \left(b_{ik}{}^T\right)_{rxn}\left(a_{kj}{}^T\right)_{nxm} = \left(\sum_{k=1}^{n} b_{ik}{}^T \; a_{kj}{}^T\right)_{rxm} = \left(\sum_{k=1}^{n} a_{jk} b_{ki}\right)_{rxm} \qquad (1)$$

and the last step follows from the definition of a transpose .Also

$$AB = \left(a_{ik}\right)_{mxn}\left(b_{kj}\right)_{nxr} = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)_{mxr}$$

Which on being transpose (i.e., on interchanging the subscript $i \; and \; j$) gives

$$(AB)^T = \left(\sum_{k=1}^{n} a_{jk} b_{ki}\right)_{rxm} \qquad (2)$$

Now iii) follows from (1) and (2)

The remaining properties can e proved similarly.

Definition 1.8 An nxn matrix A=$\left(a_{ij}\right)$ is said to be

i) Symmetric if $a_{ij} = a_{ji}$ for all i and j, that is if $\mathbf{A^T = A.}$

ii) skew –symmetric if $a_{ij} = -a_{ji}$ for all i and j ,that is $\mathbf{A^T = A}$

The following are examples of symmetric matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{pmatrix}$$

Activity 2.1

1. *If* $A = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & -4 & 5 \\ 1 & 0 & 0 \end{pmatrix}$

Then find

a) **A+C** b**) B – C** c) **AB** d) **AC** e) $\mathbf{B^TA^T}$ f) $\mathbf{(AB)^T}$

g) Determine the following elements of D = AB + 2C, without computing the complete matrix

d_{12} *and* d_{21}

2. Let **A** be 3 × 5 matrix**, B** be 5 × 2 matrix, **C** be 3 × 4 matrix, **D** be 4 × 2 matrix, **E** be 4 × 5 matrix, give the size of **2(EB) + DA b) CD – 2(CE)B.**

3. Simplify **A(A – 4B) + 2B(A + B) – A2+ 7B2 + AB.**

4. Show that for a square matrix $\mathbf{A=(a_{ij})}$

i) $\mathbf{A + A^T}$ is a symmetric matrix

ii) $\mathbf{A – A^T}$ is skew-symmetric matrix

iii) $\mathbf{AA^T}$ and $\mathbf{A^TA, A^2}$ are symmetric matrices**.**

2.3 Elementary Row Operations and Echelon Form

We use matrices to describe systems of linear equations. There are two important matrices associated with every system of linear equations. The coefficients of the variables form a matrix called the matrix of coefficients of the system. The coefficients, together with the constant terms, form a matrix called the augmented matrix of the system. For example, the matrix of coefficients and the augmented matrix of the following system of linear equations are as shown.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned} \qquad \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}}_{matrix\ of\ coefficient} \qquad \underbrace{\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{pmatrix}}_{augmented\ matrix}$$

Observe that the matrix of coefficients is a sub matrix of the augmented matrix. The augmented matrix completely describes the system.

Transformations called elementary transformations can be used to change a system of linear equation into another system of linear equations that has the same solution. These

transformations are used to solve systems of linear equations by eliminating variables. In practice it is simpler to work in terms of matrices using equivalent transformations called elementary row operations. These transformations are as follows:

Elementary Transformations

1. Interchanging two equations
2. Multiplying both sides of an equation by a nonzero constant
3. Add a multiple of one equation on to another equation.

Elementary Row Operations

1. Interchanging two rows of a matrix
2. Multiply the elements row by a nonzero constant
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Systems of equations that are related through elementary transformations, and thus have the same solutions, are called equivalent systems. The symbol ≅ is used to indicate equivalent system of equations. The next example compares the elementary transformation with elementary row operations.

Example 1: Solve the following system of linear equations.

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 - 2x_3 = -6 \end{cases}$$

Solution: Elimination Methods

$$\text{Initial system} \quad \begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Eliminate x_1 from the 2nd and 3rd equations

$$\cong \begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -6 \end{aligned} \qquad \begin{aligned} eq(2) &\to 2eq(1) - eq(2) \\ eq(3) &\to eq(3) - eq(1) \end{aligned}$$

Eliminate x_2 from the 1st and 3rd equations

$$\begin{array}{r} x_1 + 2x_3 = 3 \\ \cong\ x_2 - x_3 = -1 \\ -5x_3 = -10 \end{array} \qquad \begin{array}{l} eq(1) \to eq(1) - eq(2) \\ eq(3) \to eq(3) + 2eq(2) \end{array}$$

Make coefficient of x_3 in 3rd

$$\begin{array}{r} x_1 + 2x_3 = 3 \\ \cong\ x_2 - x_3 = -1 \\ x_3 = 2 \end{array} \qquad eq(3) \to \frac{1}{5} eq(3)$$

Eliminate x_3 from 1st and 2nd equations

$$\begin{array}{r} x_1 = -1 \\ \cong\ x_2 = 1 \\ x_3 = 2 \end{array}$$

The solution is $x_1 = -1, x_2 = 1, x_3 = 2$

Matrix Method

$$\text{Augmented matrix} \quad \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{pmatrix}$$

We refer to the first row as the pivot row, and then we have:

$$\cong \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{pmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

Create appropriate zeros in column 2

$$\cong\begin{pmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -1 & -1\\ 0 & 0 & -5 & -10\end{pmatrix} \qquad R_1 \to R_1 - R_2, R_3 \to R_3 - 2R_2$$

Make the (3,3) element 1

$$\cong\begin{pmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 2\end{pmatrix}, R_3 \to \frac{-1}{5}R_3$$

Create zeros in column 3

$$\cong\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}, R_1 \to R_1 - 2R_3, R_2 \to R_2 + R_3$$

Matrix corresponding to the system

$$x_1 = -1$$
$$x_2 = 1$$
$$x_3 = 2$$

Thus the solution is x_1=-1,x_2=1,x_3=2

Exercise 1: Solve the following system of linear equations.

$$x_1 - 2x_2 + 4x_3 = 12$$
$$2x_1 - x_2 + 5x_3 = 18$$
$$-x_1 + 3x_2 - 3x_3 = -8$$

2.3.1 Reduced Row Echelon form and Elementary Row Operations:

In above motivating example, the key to solve a system of linear equations is to transform the original augmented matrix to some matrix with some properties via a few elementary row operations. As a matter of fact, we can solve any system of linear equations by transforming the

associate augmented matrix to a matrix in some form. The form is referred to as the reduced row echelon form.

2.3.2 Definition of a Matrix in Reduced Row Echelon Form:

A matrix in reduced row echelon form has the following properties:

1. All rows consisting entirely of 0 are at the bottom of the matrix.

2. For each nonzero row, the first entry is 1. The first nonzero entry is called a leading 1.

3. For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.

4. If a column contains a leading 1, then all other entries in that column are 0.

Note: a matrix is in row echelon form as the matrix has the first 3 properties.

Example $\begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

are the matrices in reduced row echelon form.

The matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 9 & 6 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not in reduced row echelon form but in row echelon form since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not 0.

The matrix $\begin{pmatrix} 1 & 0 & 8 & 9 \\ 0 & 1 & -6 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are not in row echelon form (also not in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not 0.

2.3.3 Definition of Elementary Row(Or Column) Operation

Elementary row (or column) operation on a matrix A inculding the following :

i) Interchanging any two rows (or column)

i.e $R_i \leftrightarrow R_j \; or \;\; C_i \leftrightarrow C_j$ ii) Multipling a row (or column) by a non zero scalar.

i.e $R_i \rightarrow \alpha R_i \; or \; C_j \rightarrow \alpha C_j$

iii) Replacing the i^{th} row (or column) by α times the jth row (or column) plus the i^{th} row (or column).

i.e $R_i \rightarrow \alpha R_j + R_i \;\; or \; C_i \rightarrow \alpha C_j + C_i$

Defination 1.10 A matrix A is said to be row (or colunm) equivalent to a matrix B if B can be obtained by appling a finite sequence of elemntary row (or column) operations to A.

Example : The following four matrices are row equivalent .

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{pmatrix} \cong B = \begin{pmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \cong C = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{pmatrix} \cong D = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{pmatrix}$$

Because of

- B is obtained from A by interchanging row 1and row 3 .(i.e $R_1 \leftrightarrow R_3$)
- C is obtained from A multiply the third row of A by $\frac{1}{3}$. (i.e $R_3 \rightarrow \frac{1}{3} R_3$)

- D is also obtained from A multiply the second of A by -2. (i.e $R_2 \to -2R_2$)

Note: 1) Every nonzero m×n matrix can be transformed to a unique matrix in reduced row echelon form via elementary row operations.

2) If the augmented matrix [A:b] can be transformed to the matrix in reduced row echelon form [C:d] via elementary row operations, then the solutions for the linear system corresponding to [C:d] is exactly the same as the one corresponding to [A :b]

Activity 2.2

1. Reduce the following matrices to row echelon and reduced row echelon forms.

$$a)\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad b)\begin{pmatrix} 2 & 2 & 4 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \qquad c)\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 3 \end{pmatrix}$$

2.4 Rank of a Matrix Using Elementary Row Operations

Defination 2.11: Let A be mxn matrix and U be an echelon or the reduced echelon form of A.The rank of A is denoted by Rank(A) and is define as the number of non-zero rows of U.

i.e if a matrix A is carried to a row-echelon matrix U by elementary row operations, then the number of leading 1s in U is called the rank of A.

Example: Find the rank of each of the following matrices.

$$A=\begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \qquad B=\begin{pmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

Solution : We transform the matrix A in to row –echelon form by using elementary row operations.

$$A=\begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & -2 \\ 0 & -4 & 5 \\ 0 & 4 & 2 \end{pmatrix}, R_2 \to -3R_1 + R_2 \; and \; R_3 \to R_1 + R_3 \quad ,$$

$$\cong \begin{pmatrix} 1 & 1 & -2 \\ 0 & -4 & -5 \\ 0 & 0 & -3 \end{pmatrix}, R_3 \to R_2 + R_3$$

$$\cong \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & \frac{4}{5} \\ 0 & 1 & 1 \end{pmatrix}, R_2 \to -\frac{1}{4}R_2 \; and \; R_3 \to -\frac{1}{3}R_3$$

is row echelon form.Therefore Rank(A)=3

$$B = \left(\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right) \cong \begin{pmatrix} 3 & -2 & 1 & -2 \\ 0 & -5 & 10 & 13 \\ 0 & 5 & 2 & -1 \end{pmatrix}, R_2 \to 3R_2 + R_1 \; and \; R_3 \to 3R_3 - R_1$$

$$\cong \begin{pmatrix} -3 & -2 & 1 & -2 \\ 0 & -5 & 10 & 13 \\ 0 & 0 & 12 & 12 \end{pmatrix}, \; R_3 \to R_2 + R_3$$

$$\cong \begin{pmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ 0 & 1 & -2 & \frac{-13}{5} \\ 0 & 0 & 1 & 1 \end{pmatrix} R_1 \to \frac{-1}{3}R_1, R_2 \to -\frac{1}{5}R_2 \; and \; R_3 \to -\frac{1}{12}R_3$$

$$\cong \begin{pmatrix} 1 & \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ 0 & 1 & -2 & \frac{-13}{5} \\ 0 & 0 & 0 & \frac{-3}{5} \end{pmatrix}, R_3 \to 2R_3 + R_2$$

is row of echelon form and also the number of leading $1^{'s}$ is 3.Therefore Rank(B)=3

2.5 Inverse of a Matrix and its Properties

We motivate the idea of the inverse of a matrix by looking at the multiplicative inverse of a real number. If number b is the inverse of a, then

ab=1 and ba=1

for example $\frac{1}{2}$ is the inverse of 2 and we have

$$2(\frac{1}{2})=(\frac{1}{2})2=1$$

These are the ideas we extend to matrices.

Definition 1.12 An n×n matrix A is said to be nonsingular or invertible if there exists a matrix B such that $AB=BA=I_n$. The matrix B is said to be the multiplicative inverse of A. Note: If B and C are both multiplicative inverses of A, then

$$B = BI_n = B(AC) = (BA)C = {}_{InC} = C.$$

Thus an invertible matrix has a unique inverse.

Example 1: Show that the matrix B is the inverse of matrix A if

$$A=\begin{pmatrix} 2 & 0- & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad B=\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Solution: Observe that

$$AB=\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I_3$$

And

$$BA=\begin{pmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{pmatrix}\begin{pmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{pmatrix}=I_3$$

Thus AB=BA=I_3,which shows matrix Bis the inverse of A.

Note:i) Inverse of a matrix B is only defined for square matrices.

ii)A matrix may not be invertable even if it is square matrix.

For example ,let $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$,then if A is invertable ,then there exists a matrix say $B=\begin{pmatrix}x & x'\\ y & y'\end{pmatrix}$ $such\ that\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x & x'\\ y & y'\end{pmatrix}=\begin{pmatrix}y & y'\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Implying 0=1, which is absurd. Thus A is not invertible.

Definition 1.13: An n×n matrix is said to be singular if it does not have a multiplicative inverse.

Notation: Let A be an invertible matrix. We denote its inverse by $\mathbf{A^{-1}}$.

Gauss-Jordan Elimination for finding the inverse of a matrix

Let A be an n×n matrix.

1. Adjoin the identity n×n matrix I_n to A to form the augmented matrix (A: I_n)
2. Compute the reduced echelon form of (A: I_n). If the reduced echelon form is of the type

(I_n: B), then B is the inverse of A. If the reduced echelon form is not of the type (I_n: B), in that the first n×n submatrix is not I_n, then A has no inverse.

Example 2: Determine the inverse of the matrix

$$A=\begin{pmatrix}1 & -1 & -2\\ 2 & -3 & -5\\ -1 & 3 & 5\end{pmatrix}$$

Solution:Applying the method of Gauss-Jordan Elimnation ,we get

$$(A:I_3)=\begin{pmatrix}1&-1&-2&1&0&0\\2&-3&-5&0&1&0\\-1&3&5&0&0&1\end{pmatrix}\cong\begin{pmatrix}1&-1&-2&1&0&0\\0&-1&-1&-2&1&0\\0&2&3&1&0&1\end{pmatrix},\begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3+R_1\end{matrix}$$

$$\cong\begin{pmatrix}1&-1&-2&1&0&0\\0&1&1&2&-1&0\\0&2&3&1&0&1\end{pmatrix},R_2\to -R_2$$

$$\cong\begin{pmatrix}1&0&-1&3&-1&0\\0&1&1&2&-1&0\\0&0&1&-3&2&1\end{pmatrix},R_1\to R_1+R_2,\quad R_3\to R_3-2R_2$$

$$\cong\begin{pmatrix}1&0&0&0&1&1\\0&1&0&5&-3&1\\0&0&1&-3&2&1\end{pmatrix},R_1\to R_1+R_3,\quad R_2\to R_2-R_3$$

Thus

$$A^{-1}=\begin{pmatrix}0&1&1\\5&-3&-1\\-3&2&1\end{pmatrix}$$

The following example illustrates the application of the method for a matrix that does not have an inverse. Letter on in this chapter we devise more effective method to decide whether a matrix invertible.

Example 3: Determine the inverse of the matrix below, if it exists.

$$A=\begin{pmatrix}1&1&5\\1&2&7\\2&-1&4\end{pmatrix}$$

Solution: Applying the method of Gauss-Jordan Elimnation we get

$$(A:I_3)=\begin{pmatrix}1&1&5&1&0&0\\1&2&7&0&1&0\\2&-1&4&0&0&1\end{pmatrix}\cong\begin{pmatrix}1&1&5&1&0&0\\0&1&2&-1&1&0\\0&-3&-6&-2&0&1\end{pmatrix}\ R_2\to R_2-R_1\ and\ R_3\to R_3-2R_1$$

$$\cong\begin{pmatrix}1&0&3&2&-1&0\\0&1&2&-1&1&0\\0&0&0&-5&3&1\end{pmatrix},R_1\to R_1-R_2\ and\ R_3\to R_3-3R_2$$

There is no need to proceed further. The reduced echelon from cannot have a one in the (3,3) location. That is the reduced echelon form cannot be of the form (In: B). Thus A-1does not exist.

Activivity 2.3

1. Find the inverse of the matrix below if it exists.

$$a)\begin{pmatrix}1&1\\2&3\end{pmatrix}\qquad b)\begin{pmatrix}2&1&3\\0&2&1\\1&1&2\end{pmatrix}\qquad c)\begin{pmatrix}1&2&-3\\1&-2&1\\5&-2&-3\end{pmatrix}$$

Properties of Inverse Matrices

Let A and B be invertible matrices and c a nonzero scalar. Then

1. $\left(A^{-1}\right)^{-1}=A$
2. $(cA)^{-1}=\frac{1}{c}A^{-1}$
3. $(AB)^{-1}=B^{-1}A^{-1}$
4. $(A^n)^{-1}=(A^{-1})^n$
5. $(A^T)^{-1}=(A^{-1})^T$

we verify the 1st and 3rd results to illustrate the techniques involved leaving for the reader the remaining results to verify.

i) $(A^{-1})^{-1}$ =A this results followes directly from the defination of inverse of a matrix .Since A^{-1} is the inverse of A ,we have

$$\mathbf{AA^{-1}=A^{-1}A=I_n}$$

This statement also tells us that A is the inverse of A^{-1}.Thus $\mathbf{(A^{-1})^{-1}=A}$

iii)$(AB)^{-1}=B^{-1}A^{-1}$ we want to show that the matrix $\mathbf{B^{-1}A^{-1}}$ is the inverse of the matrix AB.We get,using the properties of matrices,

$$\begin{aligned} AB(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} = I_n \end{aligned}$$

Similarly ,it can be shown that $\mathbf{(B^{-1}A^{-1})(AB)=I_n}$.

Thus $\mathbf{(B^{-1}A^{-1})}$ is the inverse of the matrix **AB.**

Activety 2.4

1. *If* $A=\begin{pmatrix}5 & 1\\ 9 & 2\end{pmatrix}$, *then* $A^{-1}=\begin{pmatrix}2 & -1\\ -9 & 5\end{pmatrix}$

Use this information to determine

$a)(2A^T)^{-1}$ $b)\ A^{-3}$ $c)\ (AA^T)^{-1}$

2.6 Determinants and its Properties

To every square matrix $A=[a_{ij}]_{nxn}$ is associated a number or an expression called the determinant of **A** and is denoted by |A| or det (A).

2.6.1 Determinant of Order One

Let $A=[a_{11}]$ be a square matrix of order one. Then det (A) = a_{11}.By definition, if A is invertible, then $a_{11}\neq 0$ and so det A≠0. Also, conversely if det (A)≠0, then $a_{11}\neq 0$ and so, A is invertible.

2.6.2 Determinant of Order Two

The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1: Determine the determinant of the following matrices

a. $\begin{pmatrix} 1 & 5 \\ 7 & 10 \end{pmatrix}$ *b.* $\begin{pmatrix} \frac{1}{9} & -6 \\ -\frac{1}{3} & 3 \end{pmatrix}$

Solution:a. $\det\left(\begin{pmatrix} 1 & 5 \\ 7 & 10 \end{pmatrix}\right) = 1.10 - 7.5 = 10 - 35 = -25$

b. $\det\left(\begin{pmatrix} \frac{1}{9} & -6 \\ \frac{-1}{3} & 3 \end{pmatrix}\right) = \frac{1}{9}.3 - 6\frac{1}{3} = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6}$

Defination 2.14: Let $A=[a_{ij}]$ and M_{ij} be the (n-1)x(n-1) matrix obtained from A by deleting the I row and j column containing a_{ij}.The $\det(M_{ij})$ is called the minor of a_{ij}.We define the cofactor C_{ij} of a_{ij} by $C_{ij}=(-1)^{i+j}\det(M_{ij})$

Example 2 Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A = \begin{pmatrix} 1 & 8 & 0 \\ 5 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:Applying the above defination we get the following.

Minor of a_{11}:

$\det(M_{11})=\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2$,By deletining the 1^{st} row 1^{st} column.

And the cofactor of $a_{11:}$

$C_{11}=(-1)^{1+1}\det(M_{11})=2$

Minor of $a_{32}=\det(M_{32})=\begin{vmatrix}1 & 0\\ 5 & 3\end{vmatrix}=3-0=3$

The cofactor of a_{32}:

$C_{32}=(-1)^{3+2}\det(M_{32})=-3$

Theorem2.2 : If A is an nxn matrix with $n\geq 3$,then det(A) can be expressed as the a cofactor expansion using any row or column of A.

$$i.e.,\quad \det(A)=a_{i1}C_{i1}+a_{i2}C_{i2}+\ldots+a_{in}C_{in}.\ or$$
$$=a_{1j}C_{1j}+a_{2j}C_{2j}+\ldots+a_{nj}C_{nj}$$

for i=1,. . .,n and j=1,. . .,n.

Note: One can often save work by expanding along the row or column that contains the most zeros.There is a useful rule that can be used to give the sign part $(-1)^{i+j}$, of the cofacotors in these expansion.The rule is sumarized in the following array.

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \end{pmatrix}$$

If, for example ,one expands in terms of the second row ,the signs will be - + - etc.The signs alternate as one goes along any row or column.

Example 3: To evaluate the determinant of

$$\begin{pmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

Solution:One would exxpand down the first column.The first three terms will drop out,leaving

$$-2\det\begin{pmatrix}2&3&0\\4&5&0\\1&0&3\end{pmatrix}=-2.3\det\begin{pmatrix}2&3\\4&5\end{pmatrix}=-2.3(-2)=12$$

Example 4: Caluculate $\det\begin{pmatrix}2&-1&4&9\\3&-2&7&3\\0&0&0&0\\-4&3&-2&5\end{pmatrix}$

Solution: Choose row 3 ,since it has the most zeros.

$$\det\begin{pmatrix}2&-1&4&9\\3&-2&7&3\\0&0&0&0\\-4&3&-2&5\end{pmatrix}=0.\det(\ )+0.\det(\ )+0.\det(\ )+0.\det(\ )=0$$

This example illustrate es the fact that if amatrix has a row (or column) containing all zeros the determinant is zero.

Example 5: Calculate $\det\begin{pmatrix}2&7&3&2\\0&-3&4&3\\0&0&6&4\\0&0&0&10\end{pmatrix}$

Solution: Choose column 1 since it has the most zeros.

$$\det\begin{pmatrix}2&7&3&2\\0&-3&4&3\\0&0&6&4\\0&0&0&10\end{pmatrix}=2\det\begin{pmatrix}-3&4&3\\0&6&4\\0&0&10\end{pmatrix}, \text{expand by column 1.}$$

$$=2(-3)\det\begin{pmatrix}6&4\\0&10\end{pmatrix}=2(-3)(6)(10)=-360$$

Another look at Example 3 shows us that the determinant of the given matrix was the product of the diagonal element .Although this does not happen for all matrices ,it does if the matrix is upper or lower triangular matrix.

Theorem2.3 If A be a nxn matrix is upper (or lower) triangular,then det(A)= $a_{11}a_{22}...a_{nn}$.

Proof: Let us use the principle of mathematical induction. The proposition P(n) is as follows: An n × n upper triangular matrix A has determinanta_{11} a_{22} ... a_{nn} First, we check p(n).When n=2

$$A=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

and by definition det(A)=$a_{11}a_{22}$. The proposition is true for n=2 . For the induction hypothesis we suppose that P(k) is true. That is, suppose that if $A_{k\times k}$ is upper triangular then

$$\det(A_{k\times k})= a_{11}a_{22}...a_{kk} .$$

To complete the proof ,we must show that $\det(A_{k+1xk+1}) = a_{11}a_{22}...a_{(k+1)(k+1)}$.

Writing $A_{k+1k+1}=\begin{pmatrix} a_{11} & a_{12} & ... & a_{1k} & a_{1k+1} \\ 0 & a_{22} & ... & a_{2k} & a_{2k+1} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & \\ 0 & & & \ddots & \vdots \\ 0 & 0 & 0 & a_{kk} & a_{kk+1} \\ 0 & 0 & 0 & 0 & a_{k+1k+1} \end{pmatrix}$

We compute $\det(A_{k+1xk+1})$ by using row k+1 to find

$$\det(A_{k+1k+1}) = (-1)^{2k+2} a_{k+1k+1} \det(A_{kxk})$$
$$\underbrace{By\, induction}_{hypothesis} = a_{k+1k+1}(a_{11}\cdots a_{kk})$$
$$= a_{11}\cdots a_{kk}a_{k+1k+1}$$

Thus by the principle of mathematical induction ,the propostion is true for all n.

So, if A is upper or lower triangular, the determinant is easy to calculate. To use this fact, we can row-reduce a matrix to upper or lower triangular form, calculate the determinant of the resulting matrix, and then relate that determinant to the determinant of the original matrix.

Activty2.5

1.Let $A=\begin{pmatrix} 2 & 0 & 3 & 9 \\ 4 & 0 & 1 & 5 \\ 5 & 8 & 2 & 5 \\ 0 & -3 & 6 & 7 \end{pmatrix}$,then find the following minors and cofactors of A.

a. M_{12} and C_{12} b) M_{43} and C_{43}.

2.Find the determinant of the following matrix using as little computation as possible.

$$B=\begin{pmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 5 & 0 \\ 7 & -3 & 8 & 4 \\ -3 & 9 & 3 & 0 \end{pmatrix}$$

2.6.3 Properties of a Determinant

The following theorem tells us how elementary row operation affect determinants. It also tells us that these operations can be extended to columns.

Theorem 2.4 Let A be an n × n matrix and c be a nonzero scalar.

a) If a matrix B is obtained from A by multiplying the elements of a row (column) by c then |B| = c|A|.

b) If a matrix B is obtained from A by interchanging two rows (column) then |B| = –|A|

c) If a matrix B is obtained from A by adding a multiple of one row (column) to another row(column), then |B| = |A|.

The proof is left as exercise.

Example 6: Evalute the following determinant

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$$

Solution: We examine the rows and columns of the determinant to see if we can create zeros in a row column the above operations. Note that we can create zeros in the second column by adding twice the third column to it:

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix}, C_2 \to C_2 + 3C_3$$

Expand this determinant in terms of the second column to take adevantage of the zeros.

$$= (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = (-3)(9-2) = -21.$$

We shall find that matrices that have zero determinant play a significant role in theory of matrices.

Defination2.14: A square matrix A is said to be singular if $|A| = 0$ .A is nonsingular if $|A| \neq 0$.

The following theorem gives information about some of the circumstance under which we can expect a matrix to be singular.

Theorem 2.5 Let A be a square matrix. A is singular if

a) all the elements of a row (column) are zero.
b) two rows (columns) are equal.
c) Two rows(columns) are proportional.

Example 7 :Show that the following matrices are singular.

a) $A = \begin{pmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{pmatrix}$ *b*) $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$

Solution:

a) All the elements in column 2 of A are zero. Thus |A| = 0.

b) Observe that every element in row 3 of B is twice the corresponding element in row 2. We write

(row 3) = 2(row 2)

Row 2 and row 3 are proportional. Thus |B| = 0.

The following theorem tells us how determinants interact with various matrix operations. The examples following it demonstrate the theorem in use.

Theorem 2.6 Let A and B be n × n matrices and c be a nonzero scalar.

a) Determinant of a scalar multiple : $|cA|=c^n|A|$

b) Determinant of a product: |AB| = |A||B|

c) Determinant of a transpose: $|A^t| = |A|$

d) Determinant of an inverse: $\left|A^{-1}\right| = \frac{1}{|A|}$ (assuming A^{-1} exists).

Example 8 If A is a 2×2 matrix with |A| = 4, use Theorem 2.22 to compute the following determinants.

a) |3A| b) |A| c) $\left|5A^t A^{-1}\right|$,assuming A^{-1} exists.

Solution:a) $|3A|=(3)^2|A| = 9x4=36$

a) $|A^2| = |AA| = |A||A| = (4).(4) = 16$

b) $\left|5A^t A^{-1}\right| = 5^2\left|A^t A^{-1}\right| = 25|A|\frac{1}{|A|} = 25$

Exampe 9: Prove that $\left|A^{-1}A^t A\right| = |A|$

Solution: by the properties of matrices ,determinants, and real numbers we get

$$\left|A^{-1}A^t A\right| = \left|(A^{-1}A^t)A\right| = \left|A^{-1}A^t\right||A| = \left|A^{-1}\right|\left|A^t\right||A| = \frac{1}{|A|}|A||A| = |A|.$$

Activity 2.5

1.Find all the values of x that make the following determinant zero. $\begin{pmatrix} x-1 & -2 \\ x-2 & x-1 \end{pmatrix}$

2.If $A=\begin{pmatrix}1 & -1 & -3\\ 2 & 0 & -4\\ -1 & 1 & 2\end{pmatrix}$ then |A| = –2. Use this information, together with the properties of determinants, to compute the determinant of the following matrices.

$$a)\begin{pmatrix}1 & -1 & -3\\ 2 & 0 & -4\\ -2 & 2 & 4\end{pmatrix} \quad b)\begin{pmatrix}2 & 0 & -4\\ 1 & -1 & -3\\ -1 & 1 & 2\end{pmatrix} \quad c)\begin{pmatrix}1 & -1 & -3\\ 4 & -2 & -10\\ -1 & 1 & 2\end{pmatrix}$$

3.If A and B are 3 × 3 matrices and |A|= – 3, |B| = 2, compute the following determinants.

$a)\ |AB| \qquad b)\ |AA^T| \qquad c)\ |(AB^{-1})^T|$

2.7 Determinant Method of Finding Inverse Matrices

We first introduce tools necessary for developing a formula for the inverse of nonsingular matrix.

Definition 2.23: Let A be an n × n matrix andC_{ij} be the cofactor of a_{ij}.The matrix whose (i, j)th element is C_{ij} is called the matrix of cofactors. The transpose of this matrix is called the adjoint of A and is denoted adj(A).

$$\underbrace{\begin{pmatrix}C_{11} & C_{12} & \cdots & C_{1n}\\ C_{21} & C_{22} & \cdots & C_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ C_{n1} & C_{n2} & \cdots & C_{nn}\end{pmatrix}}_{matrix\ of\ cofactors},then\ \begin{pmatrix}C_{11} & C_{12} & \cdots & C_{1n}\\ C_{21} & C_{22} & \cdots & C_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ C_{n1} & C_{n2} & \cdots & C_{nn}\end{pmatrix}^T = \underbrace{\begin{pmatrix}C_{11} & C_{21} & \cdots & C_{n1}\\ C_{12} & C_{22} & \cdots & C_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ C_{1n} & C_{2n} & \cdots & C_{nn}\end{pmatrix}}_{adjoint\ matrix}$$

Example 10 : Find the matrix of cofactors and the adjoint matrix of the matrix $A=\begin{pmatrix}-2 & -1 & 3\\ -4 & 5 & 2\\ -3 & 1 & 4\end{pmatrix}$

Solution: The cofactors of **A** are

$$C_{11}=\begin{vmatrix}5 & 2\\ 1 & 4\end{vmatrix}=18 \qquad C_{12}=-\begin{vmatrix}-4 & 2\\ -3 & 4\end{vmatrix}=10 \qquad C_{13}=\begin{vmatrix}-4 & 5\\ -3 & 1\end{vmatrix}=11$$

$$C_{21}=\begin{vmatrix}-1 & 3\\ 1 & 4\end{vmatrix}=7 \qquad C_{22}=\begin{vmatrix}-2 & 3\\ -3 & 4\end{vmatrix}=1 \qquad C_{23}=\begin{vmatrix}-2 & -1\\ -3 & 1\end{vmatrix}=5$$

$$C_{31}=\begin{vmatrix}-1 & 3\\ 5 & 2\end{vmatrix}=-17 \qquad C_{32}=\begin{vmatrix}-2 & 3\\ -4 & 2\end{vmatrix}=-8 \qquad C_{33}=\begin{vmatrix}-2 & -1\\ -4 & 5\end{vmatrix}=-14$$

Thus the matrix of cofactor of this matrix is

$$\begin{pmatrix}18 & 10 & 11\\ 7 & 1 & 5\\ -17 & -8 & -14\end{pmatrix}$$

and the adjoint of A is the transpose of this matrix

$$adj(A)=\begin{pmatrix}18 & 7 & -17\\ 10 & 1 & -8\\ 11 & 5 & -14\end{pmatrix}$$

Theorem 2.7 Let be a square matrix with $\det(A)\neq 0$. A is invertable with

$$A^{-1}=\frac{1}{|A|}adj(A)$$

Proof: as exersise

Example 11: For a 2x2 matrix $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$,the adjoint $Adj(A)=\begin{pmatrix}a_{22} & -a_{12}\\ a_{21} & a_{11}\end{pmatrix}$

If A is nonsingular ,then $A^{-1}=\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{pmatrix}a_{22} & -a_{12}\\ a_{21} & a_{11}\end{pmatrix}$

Example 12: use the result of Theorem 1.25 to compute the inverse of the matrix $A=\begin{pmatrix}-2 & -1 & 3\\ -4 & 5 & 2\\ -3 & 1 & 4\end{pmatrix}$

Solution: $|A|$ is computed and found to be -13.This matrix was discussed in example 10.There we found that

$$adj(A)=\begin{pmatrix}18 & 7 & -17\\ 10 & 1 & -8\\ 11 & 5 & -14\end{pmatrix}$$

The formula for the inverse of a matrix gives

$$A^{-1}=\frac{-1}{13}adj(A)=\begin{pmatrix}\frac{-18}{13} & \frac{-7}{13} & \frac{17}{13}\\ \frac{-10}{13} & \frac{-1}{13} & \frac{8}{13}\\ \frac{-11}{13} & \frac{-5}{13} & \frac{14}{13}\end{pmatrix}$$

Activity 2.6

1. Determine whether the following matrices have inverse. If a matrix has an inverse, find the inverse using the formula for the inverse of a matrix.

a) $\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}$ b) $\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 4 & 8 & 7\end{pmatrix}$

We now discuss the relationship between the existence and uniqueness of the solution to a system of n linear equations in n variables and the determinant of the matrix of coefficient of the system.

2.8 System of Linear Equations and Characterization of Solutions

Theorem 2.8: (Cramer's Rule). Let A be an n × n nonsingular matrix and let B ∈ R^n. Let A_i be the matrix obtained by replacing the ith column of A by B. If X is the unique solution to AX =B, then

$$x_i=\frac{|A_i|}{|A|}\ ,\ for\ i=1,2,...,n$$

Theorem 1.9 Let **AX = B** be a system of n linear equations in n variables.

i) If $|A| \neq 0$, then AX = B has a unique solution. The system has a trivial solution that is **X=0** if **B = 0.**

ii) If $|A| = 0$, and at least one of the A_is is nonzero the system has no solution. For, if $|A| = 0$ and $|A_1| \neq 0$, then X $|A| = |A_1|$ leads to a contradiction. Such systems are called inconsistent

iii) If $|A| = 0$ and $|A_i| = 0$, i =1,2,. . . ,n the system my ave an infinite number of solutions or may not have a solution. A system having an infinite number of solutions is called dependent.

Definition 2.14 If AX = 0 then the system of equations is said to be homogeneous.

Example 1: Solve

$$\begin{aligned} 3x_1 - x_2 + x &= 3 \\ x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 - x_3 &= -4 \end{aligned}$$

by using Cramer's rule.

Solution: First we calculte det(A).

$$\det(A) = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix}, R_3 \to R_2 + R_3$$

$$= 2\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -4$$

Now substute $B = \begin{pmatrix} 3 \\ 6 \\ -4 \end{pmatrix}$ for column 1 and calculate

$$x_1 = \frac{\det\begin{pmatrix} 4 & -1 & 1 \\ 6 & 1 & 1 \\ -4 & -1 & 1 \end{pmatrix}}{-4} = \frac{\det\begin{pmatrix} 4 & -1 & 2 \\ 6 & 1 & 0 \\ -4 & -1 & 0 \end{pmatrix}}{-4}, C_3 \to -C_2 + C_3 = \frac{-4}{-4} = 1$$

Similarly

$$x_2 = \frac{\det\begin{pmatrix} 3 & 4 & 1 \\ 1 & 6 & 1 \\ 1 & -4 & 1 \end{pmatrix}}{-4} = \frac{\det\begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}}{-4}, C_2 \to -4C_2 + C_3$$

$$= \frac{-8}{-4} = 2$$

$$x_3 = \frac{\det\begin{pmatrix} 3 & -1 & 4 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{pmatrix}}{-4} = \frac{\det\begin{pmatrix} 3 & -1 & 4 \\ 4 & 0 & 10 \\ 2 & 0 & 2 \end{pmatrix}}{-4}, R_2 \to R_2 + R_1 \; and \; R_3 \to R_2 + R_3$$

$$= \frac{-12}{-4} = 3$$

The answer checks, by direct substitution.

The following two systems of linear equations, each of which has a singular matrix of coefficients, illustrate that there may be many or no solutions.

$$\underbrace{\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 4x_2 + 5x_3 &= 3 \\ 2x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}}_{many \quad solution}$$

$$x_1 = t + 1, x_2 = 2t, x_3 = t$$

$$\underbrace{\begin{pmatrix} x_1 + 2x_2 + 3x_3 = 3 \\ 2x_1 + x_2 + 3x_3 = 3 \\ x_1 + x_2 + 2x_3 = 0 \end{pmatrix}}_{no \; solution}$$

Example 2: Determine values of r for which the following system of equations has nontrivial solutions. Find the solutions for each value of r.

$$\begin{aligned} (r+2)x_1 + (r+4)x_2 &= 0 \\ 2x_1 + (r+1)x_2 &= 0 \end{aligned}$$

Solution: This system is a homogeneous system of linear equations. It thus has the trivial solution by Theorem 2.26 (i). The same theorem part (iii) tells us that there is the possibility of other solution only if the determinant of the matrix of coefficients is zero.

Equating this determinant to zero, we get

$$\begin{vmatrix} r+2 & r+4 \\ 2 & r+1 \end{vmatrix} = 0$$

$$(r+2)(r+1) - 2(r+4) = 0$$

$$r^2 + r - 6 = 0$$

$$(r-2)(r+3) = 0$$

Thus the determinant is zero if r=-3 or r=2.

r=-3 result in system

$$-x_1 + x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

This system has infinitely many solutions $x_1 = m, x_2 = m, \text{ for } m \in \Re$

r=2 the result of system

$$4x_1 + 6x_2 = 0$$
$$2x_1 + 3x_2 = 0$$

This system has many solutions

$$x_1 = \frac{-3m}{2}, x_2 = m, \text{ for } m \in \Re$$

Cramer's rule gives us a convenient method for writing down the solution to an n × n system of equations in terms of determinants. In this method we can solve for any one of the x_is with out solving the solution of the entire system. However to compute the solution of the system as a whole, one must evaluate n + 1 determinants of order n. Evaluating even two of these determinants generally involves more computation than solving the system using Gaussian Elimination that we are going to see below.

2.8.1 The Gaussian Elimination Method

Let AX = B be a linear system of equations then

1. Write down the augmented matrix of the system of linear equations.
2. Find an echelon form of the augmented matrix using elementary row operations.
3. Write down the system of equations corresponding to the echelon form.
4. Use back substitution to arrive at the solution.

Example 3 Solve the following system of linear equations using the method of Gaussian elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_3 &= 4 \end{aligned}$$

Solution: Solving the augmented matrix, create zeros below the pivot in the first column.

$$[A:b] = \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{pmatrix} \cong \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{pmatrix}, R_2 \to R_2 + R_1, R_3 \to R_3 - 2R_2$$

$$\cong \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix}, R_3 \to R_3 - 2R_2$$

$$\cong \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, R_3 \to \frac{1}{2}R_3$$

We have arrived at the echelon form.

The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_2 = -1$$
$$x_3 + 3x_4 = 1$$
$$x_4 = 2$$

The system is now solved by ***back substitution*** i.e. the value of x_4 is substituted into the second equation to give x_3. x_3 and x_4 are then substituted into the first equation to get x_1.

We get

$$x_3 + 3(2) = 1$$
$$x_3 = -5$$

Substituting x_4=2 and x_3=-5 into the first equation we have

$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$
$$x_1 + 2x_2 = 10$$
$$x_1 = -2x_2 + 10$$

Let $x_2 = t$, hence the system has infinitely many solutions.

Therefore the solution set is **$(-2t+10, t, -5, 2)^t$**

Example 4 Determine the value of k so that the following system of unknown x, y, z has

(i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

$$x + y - z = 1$$
$$2x + 3y + kz = 3$$
$$x + ky + 3y = 2$$

Solution: The augmented matrix determined by the system is

$$[A:b]=\begin{pmatrix}1 & 1 & -1 & 1\\ 2 & 3 & -k & 3\\ 1 & k & 3 & 2\end{pmatrix}\cong\begin{pmatrix}1 & 1 & -1 & 1\\ 0 & 1 & k+2 & 1\\ 0 & k-1 & 4 & 1\end{pmatrix},\ R_2\to R_2-2R_1, R_3\to R_3-R_2$$

$$\cong\begin{pmatrix}1 & 1 & -1 & 1\\ 0 & 1 & k+2 & 1\\ 0 & 0 & -(k-1)(k+2)+4 & 2-k\end{pmatrix}, R_3\to R_3-(k-1)R_2$$

$$\cong\begin{pmatrix}1 & 1 & -1 & 1\\ 0 & 1 & k+2 & 1\\ 0 & 0 & (2-k)(k+3) & 2-k\end{pmatrix}$$

The system has a unique solution if the solution if the coefficient of z in the third equation is not zero; that is, if k ≠ 2 and k ≠ -3. In case k = 2, the third equation reduces to 0 = 0 and the system has infinite equation reduces to 0 = 5 and the system has no solution.

Summarizing (i) k ≠ 2 and k ≠ 3, (ii) k = -3, (iii) k

Activity 2.5

1. Solve the following systems of equations using
   a. Cramer's rule (if possible)
   b. Gaussian elimination method

$$2x_1+7x_2+3x_3=7$$
$$i)\quad x_1+2x_2+x_3=3$$
$$x_1+5x_2+2x_3=5$$

$$x_1+6x_2-x_3=3$$
$$ii)\ x_1-2x_2+3x_3=2$$
$$4x_1-2x_2+5x_3=5$$

Self test

1. Find the inverse of the following matrices if possible

$(a)\ \begin{pmatrix}1 & 3\\ 8 & 4\end{pmatrix}$ $(b)\begin{pmatrix}1 & 2 & 3\\ -3 & 1 & 2\\ 9 & 5 & 6\end{pmatrix}$ $(c)\begin{pmatrix}-1 & 3 & 7 & 5\\ -1 & 2 & -1 & 3\\ 2 & 0 & 1 & 4\\ 1 & -1 & -1 & 3\end{pmatrix}$ $(d)\begin{pmatrix}1 & 0 & 1\\ 2 & 1 & 2\\ 1 & 2 & 1\end{pmatrix}$

2.Find the values of x for which the matrix A=$\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 1 & 2 & x\end{pmatrix}$ is invertible. In that case give A^{-1}.

3. Find the determinants of the matrices in 1above.

4. What is the determinant of the product matrix below?

$$\begin{pmatrix}1 & -2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & \frac{1}{3}\\ 1 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

5. Given that $\begin{pmatrix}1 & -2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A\begin{pmatrix}1 & 2 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=5I_3$,what is det(A)?

6. Find the determinant and inverse of the following matrix

$(a)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ a & b & c & d\end{pmatrix}$ $(b)\begin{pmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{pmatrix}$ $(c)\begin{pmatrix}x+m & x & x\\ x & xm & x\\ x & x & x\end{pmatrix}$

7. For what values of k wills the system

$$x_1 + x_2 + kx_3 = 0$$
$$x_1 + kx_2 + x_3 = 0$$
$$kx_1 + x_2 + x_3 = 0$$

have a non-trivial solution? In each case what are these solutions?

8. Solve each of the following system of linear equations by using:

i) Gaussian elimination method

ii) Cramer's rule, whenever possible

$$(a)\ \begin{matrix} 2x-5y=1 \\ 3x-2y=-4 \end{matrix} \qquad (b)\ \begin{matrix} 3x+2y-z=-1 \\ -x+2y-9z=9 \end{matrix} \qquad c)\ \begin{matrix} 2x+y+6z=6 \\ 3x+2y-2z=-2 \\ x+y+2z=4 \end{matrix} \qquad d)\ \begin{matrix} 3x+4y+7z=0 \\ y-2z=3 \\ x+3y-z=-5 \end{matrix}$$

$$e)\ \begin{matrix} x+y+z+w=-4 \\ 2y+z+3w=4 \\ 2x+y-z+2w=5 \\ x-y+w=4 \end{matrix}$$

9. A man refused to tell anyone his age, but he likes to drop hints about it. He then remarks that twice his mother's age add up to 140 and also that his age plus his father's age add up to 105 Furthermore, he says that the sum of his age and his mother's age is 30 more than his father's age. Calculate the man's age or show that his hints contradict one a

CHAPTER THREE

3. Limit and Continuity

Introduction:

The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is fundamental to finding the tangent to a curve or the velocity of an object.

In this chapter we develop the limit first intuitively and then formally. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produces only small changes in $f(x)$. Other functions can have values that jump or vary erratically. The notion of limit gives a precise way to distinguish between these behaviors. The geometric application of using limits to define the tangent to a curve leads at once to the important concept of the derivative of a function. The derivative, which we investigate thoroughly in chapter 4, quantifies the way a functions value's change.

Chapter Objectives

At the end of this chapter the readers should be able to:

- Understand the concept of limit.
- Study and use a formal definition of limit.
- Evaluate a limit using properties of limits.
- Understand one-sided limit.
- Evaluate one-sided limit.
- Determine limit at infinity and infinite limits.
- Determine the horizontal and the vertical asymptote of the graph of a function.
- Determine continuity at a point and continuity on an interval.
- Understand and use the Intermediate Value Theorem.

3.1 Definitions of Limit

Until now we have been evaluating the limit of a function by using its intuitive definition. That is we have said that limit of $f(x)$ as x approaches to a is L and write

$$\lim_{x\to a} f(x) = L$$

if we can make $f(x)$ close enough to L by choosing x close enough to a but distinct from a. Although this intuitive definition is sufficient for solving limit problems it is not prices enough. In this section we see the formal definition of limit, which we call *the* $\varepsilon - \delta$ definition of limit.

Definition 3.1(Formal definition of limit)

The limit of $f(x)$ as x approaches a is L, written

$$\lim_{x\to a} f(x) = L$$

if every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

In definition 3.1 above we should not that:

i. The absolute value symbol is read as "the distance between" for instance $|x - a|$ is the distance between x and a.
ii. Notice that $|x - a| > 0$. In other words x is not equal to a.

So with this in mind we can read the definition as:

"The distance between $f(x)$ and L can be made smaller than any positive number ε, whenever the distance between x and a is less than some number δ and x does not equal a." Fig 3.1 below represents this idea pictorially as:

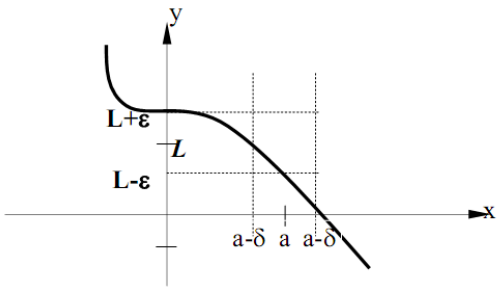


Fig 3.1

If we wish to use a form of Definition 3.1 that does not contain absolute value symbols we can have the following alternative definition of limit.

Definition 3.2: $\lim_{x\to a} f(x) = L$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is in the open interval $(a-\delta, a+\delta)$ and $x \neq a$ then $f(x)$ is in the open interval $(L-\varepsilon, L+\varepsilon)$.

Using either of the definitions of limit given above we can prove the following theorem.

Theorem 3.3: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

The above theorem tells us that if a limit of a function $f(x)$ at a exists then it must be *unique*.

3.2 Examples on Limit

Even if it is very difficult to us the formal definition of limit to handle all limit problems, let us see how we can use it for evaluating some important limits that may help us in developing rules by the way of which we can evaluate limits without using the formal definition.

Example 1: Assume that $\lim_{x\to 2}(5x-7) = 3$. By using properties of inequalities, determine a $\delta > 0$ such that

if $0 < |x-2| < \delta$ then $|(5x-7)-3| < 0.01$

Solution: By considering $|(5x-7)-3| < 0.01$, we can see that

$$|(5x-7)-3| < 0.01 \Leftrightarrow |5x-10| < 0.01$$
$$\Leftrightarrow 5|x-2| < 0.01$$
$$\Leftrightarrow |x-2| < 0.002$$

So now it is clear that if we choose $\delta = 0.002$ the statement holds, but to check our result holds we proceed as follows:

$$\text{If } 0 < |x-2| < \delta \text{ then } |x-2| < 0.002$$

$$\Rightarrow |x-2| < \frac{0.01}{5}$$

$$\Rightarrow 5|x-2| < 0.01$$

$$\Rightarrow |5x-10| < 0.01$$

$$\Rightarrow |(5x-7)-3| < 0.01$$

Thus we have shown that the choice of $\delta = 0.002$ satisfies the statement

if $0 < |x-2| < \delta$ then $|(5x-7)-3| < 0.01$

This example is for the specific $\varepsilon = 0.01$. The general case can be seen as follows.

Example 2: Show that $\lim_{x\to 2}(5x-7) = 3.$

Solution:

We need to show that given $\varepsilon > 0$ then there exists $\delta > 0$ such that

$$\text{if } 0 < |x-2| < \delta \text{ then } |(5x-7)-3| < \varepsilon$$

To choose an appropriate δ we start with $|(5x-7)-3| < \varepsilon$ then we have

$$|5x-10| < \varepsilon \Rightarrow 5|x-2| < \varepsilon$$

$$\Rightarrow 5|x-2| < \frac{\varepsilon}{5}$$

Hence, we let $\delta = \frac{\varepsilon}{5}$

This proves that $\lim_{x\to 2}(5x-7) = 3.$

Example 3: Prove that $\lim_{x\to 5} 7 = 7.$

Solution:

Begin by letting $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < |x-5| < \delta$ then $|f(x)-7| < \varepsilon$

i.e. $|7-7| < \varepsilon$ i.e. $|0| < \varepsilon$. But this trivial inequality is always true, no matter what value is chosen for δ. For example $\delta = \frac{1}{2}$ will work.

Thus, if $0 < |x-5| < \delta$ then it follows that $|f(x)-7| < \varepsilon$.

This completes the proof.

A similar proof as example 2 shows us that for any number a and k

$$\lim_{x \to k} k = k. \quad \text{..... (1)}$$

Example 4: Prove that $\lim_{x \to a} kx = ka$ for any real number k.

Solution: from (1) it is clear that if $k = 0$

$$\lim_{x \to a} kx = \lim_{x \to a} 0 = 0 = 0.a = k.a$$

If $k \neq 0$, letting $\varepsilon > 0$ we must find a $\delta > 0$ so that

$$0 < |x - a| \Rightarrow |kx - ka| < \varepsilon$$

Since,

$$|kx - ka| < \varepsilon \Rightarrow |k||x - a| < \varepsilon$$

$$\Rightarrow |x - a| < \frac{\varepsilon}{|k|}$$

Now, choose $\delta = \frac{\varepsilon}{|k|}$

Thus, $\lim_{x \to a} kx = ka$ for any real number$k$.

Example 5: Prove that $\lim_{x \to 1}\left(x^2 + 3\right) = 4$

Solution:

Begin by letting $\varepsilon > 0$ be given. Find $\delta > 0$ (which depends on ε) so that

If $0 < |x - 1| < \delta$, then $|f(x) - 4| < \varepsilon$. Begin with $|f(x) - 4| < \varepsilon$ and "solve for" $|x - 1|$. Then

$$|f(x) - 4| < \varepsilon \Leftrightarrow |(x^2 + 3) - 4| < \varepsilon$$

$$\Leftrightarrow |x^2 - 1| < \varepsilon$$

$$\Leftrightarrow |(x - 1)(x + 1)| < \varepsilon$$

$$\Leftrightarrow |x - 1||x + 1| < \varepsilon$$

We will now replace the term $|x + 1|$ with an appropriate constant and keep the term $|x - 1|$, since this is the term we wish to "solve for." To do this, we will arbitrary assume that $\delta \leq 1$

[This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of a δ also work].

Then

$$|x-1| < \delta \leq 1 \Rightarrow -1 < x-1 < 1$$
$$\Rightarrow 0 < x < 2$$
$$\Rightarrow 1 < |x+1| < 3$$

It follows that

$$|x-1||x+1| < |x-1|(3) < \varepsilon$$
$$\Leftrightarrow 3|x-1| < \varepsilon$$
$$\Leftrightarrow |x-1| < \frac{\varepsilon}{3}$$

Now choose $\delta = min\left\{1, \frac{\varepsilon}{3}\right\}$

Thus, if $0 < |x-1| < \delta$, it follows that $|f(x)-4| < \varepsilon$.

Hence, $\lim_{x\to 1}\left(x^2+3\right)=4$

Example 6: Prove that $\lim_{x\to 3}\frac{2}{x+3}=\frac{1}{3}$

Solution: Begin by letting $\varepsilon > 0$ be given. Find $\delta > 0$ (which depends on ε) so that

If $0 < |x-3| < \delta$, then $\left|f(x)-\frac{1}{3}\right| < \varepsilon$. Begin with $\left|f(x)-\frac{1}{3}\right| < \varepsilon$ and "solve for" $|x-3|$. Then

$$\left|f(x)-\frac{1}{3}\right| < \varepsilon \Leftrightarrow \left|\frac{2}{x+3}-\frac{1}{3}\right| < \varepsilon$$
$$\Leftrightarrow \left|\frac{3}{3}\frac{2}{x+3}-\frac{1}{3}\frac{x+3}{x+3}\right| < \varepsilon$$
$$\Leftrightarrow \left|\frac{6-(x+3)}{3(x+3)}\right| < \varepsilon$$
$$\Leftrightarrow \frac{|3-x|}{|3||x+3|} < \varepsilon$$
$$\Leftrightarrow \frac{|x-3|}{|3||x+3|} < \varepsilon$$

$$\Leftrightarrow \frac{1}{3}\frac{|x-3|}{|x+3|} < \varepsilon$$

$$\Leftrightarrow \frac{1}{3}|x-3|\frac{1}{|x+3|} < \varepsilon$$

We will now replace the term $|x+3|$ with an appropriate constant and keep the term $|x-3|$, since this is the term we wish to "solve for." To do this, we will arbitrary assume that $\delta \leq 1$ [This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of a δ also work].

Then

$$|x-3| < \delta \leq 1 \Rightarrow -1 < x-3 < 1$$

$$\Rightarrow 2 < x < 4$$

$$\Rightarrow 5 < |x+3| < 7$$

$$\Rightarrow \frac{1}{7} < \frac{1}{|x+3|} < \frac{1}{5}$$

It follows that

$$\frac{1}{3}|x-3|\frac{1}{|x+3|} < \frac{1}{3}|x-3|\frac{1}{5} < \varepsilon$$

$$\Leftrightarrow \frac{1}{15}|x-3| < \varepsilon$$

$$\Leftrightarrow |x-3| < 15\varepsilon$$

Now choose $\delta = min\{1, 15\varepsilon\}$

Thus, if $0 < |x-3| < \delta$, it follows that $\left|f(x) - \frac{1}{3}\right| < \varepsilon$.

Hence, $\lim\limits_{x\to 3}\dfrac{2}{x+3} = \dfrac{1}{3}$

Example 7: Prove that $\lim\limits_{x\to 9}\left(2+\sqrt{x}\right) = 5$

Solution:

Begin by letting $\varepsilon > 0$ be given. Find $\delta > 0$ (which depends on ε) so that

If $0 < |x-9| < \delta$, then $|f(x)-5| < \varepsilon$. Begin with $|f(x)-5| < \varepsilon$ and "solve for" $|x-9|$. Then

$$|f(x)-5| < \varepsilon \Leftrightarrow \left|\left(2+\sqrt{x}\right)-5\right| < \varepsilon$$

$$\Leftrightarrow |\sqrt{x}-3| < \varepsilon$$

$$\Leftrightarrow \left|(\sqrt{x}-3)\frac{\sqrt{x}+3}{\sqrt{x}+3}\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{x-9}{\sqrt{x}+3}\right| < \varepsilon$$

$$\Leftrightarrow |x-9|\frac{1}{|\sqrt{x}+3|} < \varepsilon$$

We will now replace the term $|\sqrt{x}+3|$ with an appropriate constant and keep the term $|x-9|$, since this is the term we wish to "solve for." To do this, we will arbitrary assume that $\delta \leq 1$ [This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of a δ also work].

Then

$$|x-9| < \delta \leq 1 \Rightarrow -1 < x-9 < 1$$

$$\Rightarrow 8 < x < 10$$

$$\Rightarrow \sqrt{8} < \sqrt{x} < \sqrt{10}$$

$$\Rightarrow \sqrt{8}+3 < |\sqrt{x}+3| < \sqrt{10}+3$$

$$\Rightarrow \frac{1}{\sqrt{10}+3} < \frac{1}{|\sqrt{x}+3|} < \frac{1}{\sqrt{8}+3}$$

It follows that

$$|x-9|\frac{1}{|\sqrt{x}+3|} < |x-9|\frac{1}{\sqrt{8}+3} < \varepsilon$$

$$\Leftrightarrow |x-9| < (\sqrt{8}+3)\,\varepsilon$$

Now choose $\delta = min\{1, (\sqrt{8}+3)\varepsilon\}$

Thus, if $0 < |x-9| < \delta$, it follows that $|f(x)-5| < \varepsilon$.

Hence, $\lim_{x\to 9}\left(2+\sqrt{x}\right)=5$

Example 8: Prove that $\lim_{x\to 0} 3x\sin\frac{1}{x}=0$

Solution:

We need to show that given $\varepsilon > 0$, then there exists $\delta > 0$ such that

$0 < |x - 0| < \delta$ implies $\left|3x\sin\frac{1}{x} - 0\right| < \varepsilon$. For" $|x - 9|$.

Looking for δ

$$\left|3x\sin\frac{1}{x}\right| < \varepsilon$$

$$\Leftrightarrow |3x|\left|\sin\frac{1}{x}\right| < \varepsilon$$

$$\Leftrightarrow 3|x|\left|\sin\frac{1}{x}\right| \leq 3|x|.1 < \varepsilon; \quad \text{since } \left|\sin\frac{1}{x}\right| \leq 1$$

$$\Leftrightarrow |x| < \frac{\varepsilon}{3}$$

Hence we let $\delta = \frac{\varepsilon}{3}$

Thus, $\lim_{x\to 0} 3x\sin\frac{1}{x} = 0$

Negation of the Existence of a Limit

Next we present an example of a function that does not have a limit at a certain point. For a function f not to have real number L, the statement "L is the limit at a" is false. What does it mean for that statement to be false? By definition 3.1, " is the limit of f at a" means that:

For every $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

For this statement to be false, there must be some $\varepsilon > 0$ such that for every $\delta > 0$ it is false that

if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$ …….… (2)

But, to say that (2) is false is the same as to say that there must be a number x such that

$0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$

Thus to say that the statement $\lim_{x\to a} f(x) = L$ is false is the same as to say that there is some $\varepsilon > 0$such that for every $\delta > 0$ there is a number x satisfying

$0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$

Example 9: Let f be defined by

$$f(x) = \begin{cases} x^2 & ; \quad for\ x > 0 \\ -1; & for\ x \le 0 \end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does not exist

Solution:

Let L be any number. We will prove that the statement "L is the limit of f at 0" is false by letting $\varepsilon = \frac{1}{2}$ and showing that for any $\delta > 0$ there is an x satisfying

$0 < |x - a| < \delta$ and $|f(x) - L| \ge \frac{1}{2} = \varepsilon$

Let δ be any positive number.

If $L \le -\frac{1}{2}$, then we let $x = \frac{\delta}{2}$ and note that $f(x) = x^2$, so that

$$|f(x) - L| = \left|\frac{\delta^2}{4} - L\right| \ge \left|\frac{\delta^2}{4} + \frac{1}{2}\right| > \frac{1}{2} = \varepsilon$$

If $L \ge -\frac{1}{2}$, then we let $x = -\frac{\delta}{2}$ and note that $f(x) = -1$, so that

$$|f(x) - L| = |-1 - L| = |1 + L| \ge \left|1 - \frac{1}{2}\right| = \frac{1}{2} = \varepsilon$$

In either case we have shown that for any $\delta > 0$ there is an x satisfying

$0 < |x - a| < \delta$ and $|f(x) - L| \ge \frac{1}{2} = \varepsilon$

Therefore, f has no limit at 0. $\left(i.e.\ \lim_{x\to 0} f(x)\ does\ not\ exist\right)$

Activity 3.1

1. Using the $\varepsilon - \delta$ definition of limit, prove that

   a. $\lim_{x\to 1}(2x-1) = 1$

   b. $\lim_{x\to 2}\sqrt{x-1} = 1$

   c. $\lim_{x\to -2} x^2 = 4$

   d. $\lim_{x\to 1}(2x-1) \ne 3$

3.3 One-Sided Limit

The notion of limit discussed in the preceding sections can be extended to one-sided limit as we can see from the definition below.

Definition 3.4

a. A number L is the **right-hand limit of f at a** denoted by $\lim_{x\to a^+} f(x) = L$

if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$

b. A number L is the **left-hand limit of f at a** denoted by $\lim_{x\to a^-} f(x) = L$

if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$

Example 9: Show that $\lim_{x\to 1^+} \sqrt{x-1} = 0$

Solution: Let $\varepsilon > 0$ be given. We need to show that there is a $\delta > 0$ such that

if $0 < x - 1 < \delta$, then $\left|\sqrt{x-1} - 0\right| < \varepsilon$

from $\left|\sqrt{x-1}\right| < \varepsilon$ squaring both sides we get $0 < x - 1 < \varepsilon^2$

Hence choose $\delta = \varepsilon^2$

Then

if $0 < x - 1 < \delta$, then $\left|\sqrt{x-1} - 0\right| = \sqrt{x-1} < \sqrt{\delta} = \varepsilon$

★ Below we give a theorem that relates one-sided limit with the general limit the student can see Robert Ellis and Danny Gulick for the proof of the theorem.

Activity 3.2

1. Show that

a. $\lim_{x\to 0^+} \sqrt{x} = 0$ b. $\lim_{x\to 4^-} \sqrt{4-x} = 0$

Theorem 3.5: $\lim_{x\to a} f(x)$ exists and is equal to L if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$ both exists and both are equal to L.

Example 10: Observe that in Example 1 even if the right hand side of f at 1 exists since the left side limit of f at 1 does not exist, as the function is not defined for $x < 1$ then

$\lim_{x\to 1}\sqrt{x-1}$ does not exist.

3.4 Infinite Limits and Infinite Limits at Infinity

According to Definition 3.1 if a function f has a limit L at a then L is a real number, so if the value of a function f becomes larger and larger in absolute value as x approaches a from the right or from the left of a the f has no limit at a. Now we introduce a definition that addresses such a case.

Infinite Limits

Definition 3.6 Let f be defined on some open interval (a, c).

a. If $\forall N, \exists \delta > 0$ such that

if $0 < x - a < \delta$ then $f(x) > N$

Then $\lim_{x\to a^+} f(x) = \infty$

b. If $\forall N, \exists \delta > 0$ such that

if $0 < x - a < \delta$ then $f(x) < N$

Then $\lim_{x\to a^+} f(x) = -\infty$

c. In either case (a) or (b) the vertical line $x = a$ is called a **vertical asymptote** of the graph of f, and we say that f has an **infinite right- hand limit at a.**

There are analogous definitions for the limits

$$\lim_{x\to a^-} f(x) = \infty \text{ and } \lim_{x\to a^+} f(x) = -\infty$$

Note if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} = \infty$ then we write simply $\lim_{x\to a} = \infty$ for the common expression and say that the limit of $f(x)$ as x approaches a is ∞ and that f has an infinite limit at a.

Example 10: Show that $\lim_{x\to 0}\frac{1}{x^2} = \infty$. Show also that the line $x = 0$ is a vertical asymptote of the graph of $\frac{1}{x^2}$.

Solution: Observe that for any $N > 0$,

if $0 < x < \frac{1}{\sqrt{N}}$, then $\frac{1}{x^2} > N$

Thus $\lim_{x\to 0^+}\frac{1}{x^2}=\infty$, and that the line $x=0$ is a vertical asymptote of the graph of $\frac{1}{x^2}$.

Once more for any $N>0$, if $-\frac{1}{\sqrt{N}}<x<0$, then $\frac{1}{x^2}>N$

Thus $\lim_{x\to 0^-}\frac{1}{x^2}=\infty$, and that the line $x=0$ is a vertical asymptote of the graph of $\frac{1}{x^2}$.

Finally since $\lim_{x\to 0^+}\frac{1}{x^2}=\infty=\lim_{x\to 0^-}\frac{1}{x^2}$, we have $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

Limits at Infinity

Until now the limits we have seen have been limits of a function f at a number a. Now we consider the limit of f as x becomes larger and larger in absolute value.

Definition 3.7

a. $\lim_{x\to\infty} f(x)=L$ if for every $\varepsilon>0$ there is a number M such that

if $x>M$, then $|f(x)-L|<\varepsilon$

b. $\lim_{x\to-\infty} f(x)=L$ if for every $\varepsilon>0$ there is a number M such that

if $x<M$, then $|f(x)-L|<\varepsilon$

c. If either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$, then we call the horizontal line $y=L$ a horizontal asymptote of the graph of f.

Example 11: Show that $\lim_{x\to\infty}\frac{1}{x^2}=0$ and $\lim_{x\to-\infty}\frac{1}{x^2}=0$

Solution: Let $\varepsilon>0$. To show that $\lim_{x\to\infty}\frac{1}{x^2}=0$ we must find an M such that

if $x>M$, then $\left|\frac{1}{x^2}-0\right|=\left|\frac{1}{x^2}\right|=\frac{1}{x^2}<\varepsilon$

But then

If $x>\frac{1}{\sqrt{\varepsilon}}$, then $\frac{1}{x^2}>\varepsilon$

Therefore we let $M = \frac{1}{\sqrt{\varepsilon}}$ and conclude that $\lim_{x\to\infty}\frac{1}{x^2} = 0$.

To show that $\lim_{x\to-\infty}\frac{1}{x^2} = 0$, we simply choose $M = -\frac{1}{\sqrt{\varepsilon}}$

Then $M < 0$, and thus

if $x < M$, then $\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \frac{1}{M^2} = \varepsilon$

This proves that $\lim_{x\to-\infty}\frac{1}{x^2} = 0$.

Note here that the line $y = 0$ is the horizontal asymptote of the graph of $\frac{1}{x^2}$.

Activity 3.3

1. Find the vertical asymptote (if any) of the graph of the following functions.

a. $f(x) = \frac{x^2}{x^2-4}$

b. $f(x) = \frac{x-1}{x^2+1}$

c. $f(x) = \frac{e^{-2x}}{x-1}$

d. $f(x) = \frac{\ln(x^2+1)}{x+2}$

e. $f(x) = \frac{t}{\sin t}$

f. $f(x) = 1 - \frac{4}{x^2}$

2. Find the horizontal asymptote (if any) of the graph of the following functions.

a. $f(x) = \frac{|x|}{x+1}$

b. $f(x) = \frac{\sqrt{9x^2-2}}{2x+1}$

c. $f(x) = \frac{3x}{\sqrt{x^2+2}}$

d. $f(x) = \frac{|3x+2|}{x-2}$

Infinite Limits at Infinity

We now see the last possible formal definition of limit that is not considered yet.

Definition 3.8 $\lim_{x\to\infty} f(x) = \infty$ if for any real number N there is some number M such that

If $x > M$, then $f(x) > N$

Note: the definition of

$\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are completely analogous.

Example 12: Show that $\lim_{x\to\infty} x^3 = \infty$

Solution: We use the fact that $x^3 > x$ for $x > 1$.

For any N, choose M so that $M > 1$ and $M > N$. Then it follows that

If $x > M$, then $x^3 > x > M > N$

Therefore by **definition 3.8** , we have $\lim_{x\to\infty} x^3 = \infty$

Similarly, we conclude that for any positive integer n, $\lim_{x\to\infty} x^n = \infty$.

3.5 Basic Limit Theorems

Even if we have developed important techniques of solving limit problems by using the formal definition, I hope by now we have realized that it is not that easy to use this definition to solve each and every problem. Nevertheless the student had encountered in his or her earlier studies of calculus rather easy ways of evaluating limits by the help of different rules. Here we state and prove some of them by using **Definition 3.1** and use them to evaluate more complex limit cases.

Theorem 3.9 Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, and c is a constant. Then

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
2. $\lim_{x\to a} cf(x) = cL$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = L.M$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$; provided that $M \neq 0$.

Proof: Here we proof (1). Statement $(2), (3), and\ (4)$ are left as exercise.

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

We need to show for every $\varepsilon > 0$ there is some $\delta > 0$ such that

if $0 < |x - a| < \delta$, then $|f(x) + g(x) - (L + M)| < \varepsilon$

Observe that $\lim_{x\to a} f(x) = L$ if and only if for every $\frac{\varepsilon}{2} > 0$ there is some $\delta_1 > 0$ such that

if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\varepsilon}{2}$

Similarly $\lim_{x\to a} g(x) = M$ if and only if for every $\frac{\varepsilon}{2} > 0$ there is some $\delta_2 > 0$ such that

if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\varepsilon}{2}$

Let $\delta = min\{\delta_1, \delta_2\}$. Then we can see that

if $0 < |x - a| < \delta$, then $|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$

In addition to these rules you have also seen that for instance if f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Now let as quickly go through some important limit finding techniques that would require a little bit of caution before applying the rules in **Theorem 3.9**.

Example 13: Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$

Solution: Direct substitution of 2 in $\frac{x^2-4}{x-2}$ implies that we have $0/0$ which is indeterminate. Thus we cannot use Theorem 3.9(4) but for $x \neq 0$ simplification of the rational expression would lead us to

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

Thus, $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}[x+2] = 4$

Example 14: Find $\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+1}-1}$

Solution: Again here we cannot use Theorem 3.9(4), as we get from direct substitution the indeterminate $0/0$. But for $x \neq 0$ rationalizing the denominator we have:

$$\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+1}-1} = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+1}-1} \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$$

$$= \lim_{x\to 0} \frac{x^2\left[\sqrt{x^2+1}+1\right]}{\left[x^2+1\right]-1}$$

$$= \lim_{x\to 0} \frac{x^2\left[\sqrt{x^2+1}+1\right]}{x^2}$$

$$= \lim_{x\to 0} x^2\left[\sqrt{x^2+1}+1\right] = 2$$

Example 15: Find $\lim_{x\to 0} x|x|$

Solution: Observe that

$$x|x| = \begin{cases} x^2, & if\ x \geq 0 \\ -x^2, & if\ x < 0 \end{cases}$$

Since $x|x| = x^2$ for $x > 0$, we have

$$\lim_{x\to 0^+} x|x| = \lim_{x\to 0^+} x^2 = 0$$

And also $x|x| = -x^2$ for $x < 0$, we have

$$\lim_{x\to 0^-} x|x| = \lim_{x\to 0^-} -x^2 = 0$$

Therefore, we conclude that

$$\lim_{x\to 0} x|x| = 0.$$

Example 16: Prove that $\lim_{x\to -1} \frac{x+1}{|x+1|}$ does not exist.

Solution:

$$\lim_{x\to -1^+} \frac{x+1}{|x+1|} = \lim_{x\to -1^+} \frac{x+1}{x+1} = \lim_{x\to -1^+} 1 = 1 \quad and \quad \lim_{x\to -1^-} \frac{x+1}{|x+1|} = \lim_{x\to -1^-} \frac{x+1}{-(x+1)} = \lim_{x\to -1^-} -1 = -1$$

Consequently

$$\lim_{x\to -1^+} \frac{x+1}{|x+1|} \neq \lim_{x\to -1^-} \frac{x+1}{|x+1|}$$

Thus, $\lim_{x\to -1} \frac{x+1}{|x+1|}$ does not exist.

Example 17: Find $\lim_{x\to\infty}\frac{x-2x^2}{x^2-1}$ and $\lim_{x\to-\infty}\frac{x-2x^2}{x^2-1}$

Solution: Dividing the numerator and the denominator of $\frac{x-2x^2}{x^2-1}$ by x^2 in the limit we have

$$\lim_{x\to\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to\infty}\frac{\frac{1}{x}-2}{1-\frac{1}{x}}=-2$$

Similarly

$$\lim_{x\to-\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to-\infty}\frac{\frac{1}{x}-2}{1-\frac{1}{x}}=-2$$

Observe that $y=2$ is the horizontal asymptote of the graph of $f(x)=\frac{x-2x^2}{x^2-1}$.

Example 18: Let $f(x)=\frac{x-2x^2}{x^2-1}$. Find all vertical asymptotes of the graph of f.

Solution: Since f is not defined at $x=1$ and $x=-1$ they are the possible vertical asymptotes but to confirm our claim we use limit:

$$\lim_{x\to1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=-\infty \text{ and}$$

$$\lim_{x\to-1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to-1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=\infty$$

It follows that $x=1$ and $x=-1$ are the vertical asymptotes of the graph of f.

The next theorems give two additional properties of limits. For their proofs the student may refer any major calculus books.

Theorem 3.10 If $f(x)\le g(x)$ for all x in an open interval that contains a (except possibly at a) and the limits of f and g both exist as x approaches$a$, then

$$\lim_{x\to a}f(x)\le g(x)$$

Theorem 3.11 (The squeezing Theorem)

If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then

$$\lim_{x\to a} g(x) = L.$$

I don't think the student is new for these theorems and for the special limit that is the Consequence of especially the Squeezing Theorem. i.e.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\left[since, \cos x \le \frac{\sin x}{x} \le 1 \quad for \left(\frac{-\pi}{2}, \frac{\pi}{2} \right). Taking both sides \lim it as\, x \to 0, \, we\, get \lim_{x\to 0} \frac{\sin x}{x} = 1 \right]$$

(you can see the detail proof on the book Robert Ellis)

Example 19: Find $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

Solution: Since $-1 \le \sin\frac{1}{x} \le 1, \ \forall x \ne 0$, we have

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2, \ \forall x \ne 0$$

Moreover $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$

Thus, by **squeezing theorem** we have

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Example 20: Find $\lim_{x\to\infty} \frac{x^4 - x^2}{x+1}$.

Solution: Simplifying $\frac{x^4 - x^2}{x+1}$ we can evaluate the limit as below

$$\lim_{x\to\infty} \frac{x^4 - x^2}{x+1} = \lim_{x\to\infty} \frac{x^2\left[x^2 - 1\right]}{x+1} = \lim_{x\to\infty} \frac{x^2(x+1)(x-1)}{x+1} = \lim_{x\to\infty} x^2\left[x-1\right] = \infty$$

Activity 3.4

1. Evaluate each of the following limits, if it exists.

a. $\lim_{x\to -1}\dfrac{x^2-x-2}{x+1}$

b. $\lim_{x\to 0}\dfrac{\sin 2x}{3x}$

c. $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$

d. $\lim_{x\to 1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

e. $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^3, & if\ x<1 \\ (x-2)^2, & if\ x>1 \end{cases}$

f. $\lim_{x\to \frac{\pi}{4}}\dfrac{1-\tan x}{\sin x-\cos x}$

3.6 Continuity of a Function and the Intermediate Value Theorem

Definition 3.11 A function f is continuous at a number a in its domain if

$$\lim_{x\to a} f(x)=f(a)$$

f is said to be discontinuous at a if f is not continuous at a.

Notice that **Definition 3.11** implicitly requires three things if f is continuous at a:

i. $f(a)$ is defined (that is, a is in the domain f).

ii. $\lim_{x\to a} f(x)$ exists (so f must be defined on an open interval that contains a).

iii. $\lim_{x\to a} f(x)=f(a)$.

Example 21: Let $f(x)=\lim_{x\to -1}\dfrac{x^2-x-2}{x+1}$. Determine the number at which f is discontinuous.

Solution: Notice that f is a rational function. Since the denominator of f is 0 for $x=-1$, f is defined for all x except at -1. Thus f is discontinuous only at $x=-1$ else where it is continuous in its domain.

Example 22: If we redefine the function f in **Example 21** as:

$$f(x)=\begin{cases} \dfrac{x^2-x-2}{x+1}, & if\ x\neq -1 \\ -3, & if\ x=-1 \end{cases}$$

Then since

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{x^2-x-2}{x+1}=\lim_{x\to -1}\frac{(x-2)(x+1)}{x+1}=-3$$

and hence $\lim_{x\to -1} f(x) = -3 = f(-1)$

Thus, f is continuous.

Notice that we are able to make f in **Example 21** to be continuous by redefining it at -1 as in **Example 22**. Such discontinuity points like -1 in our example are called **removable discontinuities** because we can remove the discontinuity of the function by redefining the function just at the discontinuity point. Otherwise it is called **non-removable discontinuity.**

Example 23: Let $f(x) = \frac{1}{x^2}$ and $g(x) = \begin{cases} 1, & if\ x \le 0 \\ 0, & if\ x \ge 0 \end{cases}$. Then we can see that, f is not defined at 0 and $\lim_{x\to 0} f(x) = \infty$, g is defined at 0 but $\lim_{x\to 0} g(x)$ does not exist as $\lim_{x\to 0^-} g(x) = 1$ and $\lim_{x\to 0^+} g(x) = 0$.

Thus, both functions are not continuous at 0.

We say we have infinite discontinuity at 0 in case of f while we say we have jump discontinuity at 0 in case of g

Clearly combinations of continuous functions follow immediately from the corresponding results for limits.

Theorem 3.12 If f and g are continuous at a and c is a constant, then the following functions are also continuous at a.

i. $f \pm g$ iii. fg

ii. cf iv. $\frac{f}{g}$, if $g(a) \neq 0$

So using **Theorem 3.12** we can show that every polynomial function is continuous over $\mathbb{R}$. Every rational function is continuous every where except at numbers where the denominator is 0. Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

Theorem 3.13 If f continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f\big(g(x)\big) = f(b) = f\Big(\lim_{x\to a} g(x)\Big)$$

The following theorem tells us that the composition of two continuous functions at a given number is continuous.

Theorem 3.14 If g is continuous at a and f is continuous at $g(a)$, then

$(f \circ g)(x) = f(g(x))$ is continuous at a.

Activity 3.5

1. Where are the following functions continuous?

a. $f(x) = |x|$ b. $g(x) = \dfrac{1}{\sqrt{x^2+3}-2}$

3.6.1 One-Sided Continuity

Definition 3.15:

-a function f is continuous is continuous from the right at a point a in its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

-a function f is continuous is continuous from the left at a point a in its domain if

$$\lim_{x \to a^-} f(x) = f(a)$$

Example 24: The step function $g(x) = \begin{cases} 1, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ is continuous from he left at 0.

Since $\lim_{x \to 0^-} g(x) = 1 = g(0)$ but it is not continuous from the right at 0 as $\lim_{x \to 0^+} g(x) \neq g(0)$. (verify!)

3.6.2 Continuity on an Interval

Definition 3.16:

a) a function f is continuous on an open interval (a,b), if it is continuous at every point in (a,b).

b) a function f is continuous on a closed interval $[a,b]$, if it is continuous at every point in (a,b) and is also continuous from he right at a and continuous from the left at b.

Example: Discuss the continuity of $f(x) = \sqrt{1-x^2}$.

Solution: The domain of f is the closed interval $[-1,1]$.

At all points in the open interval $(-1,1)$, f is continuous.

Moreover,

$\lim_{x\to -1^+}\sqrt{1-x^2}=0=f(-1)$ …… continuous from the right

and

$\lim_{x\to 1^-}\sqrt{1-x^2}=0=f(1)$ ……. Continuous from he left

Thus, f is continuous on the closed interval $[-1,1]$.

Activity 3.6

Let $f(x)=\sqrt{16-x^2}$. Show that f is continuous on $[-4,4]$.

An important property of continuous functions is expressed by the following theorem.

Theorem 3.17 [The Intermediate Value Theorem (IVT)]

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exists a number c in (a,b) such that $f(c)=N$.

In other words, if f is continuous on the closed interval $[a,b]$ and if $f(a)$ and $f(b)$ differ or opposite in sign , then there exists atleast one real root c in (a,b) such that $f(c)=0$.

Example 24: Show that there is a root of the equation

$$4x^3-6x^2+3x-2=0 \text{ on } [1,2].$$

Solution: Let $f(x)=4x^3-6x^2+3x-2$.

We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c)=0$.

Therefore we take $a=1,\ b=2,\ and\ N=0$ in Theorem 3.17. We have

$$f(1)=4-6+3-2=-1<0 \text{ and } f(2)=32-24+6-2=12>0$$

Thus, $f(1)<0<f(2)$, that is, $N=0$ is a number between $f(1)$ and $f(2)$.

Also f is continuous, since it is a polynomial.

Hence, by the IVT, there is a number c between 1 and 2 such that $f(c)=0$.

In other words, the equation $4x^3-6x^2+3x-2=0$ has a root c in the interval $(1,2)$.

Activity 3.7

1. Find the constant a, or the constant b, or the constant c such that the function is continuous on the entire real line.

a. $f(x)=\begin{cases} x^2-2\ , & if\ x<1 \\ Ax-4\ , & if\ x\geq 1 \end{cases}$

c. $f(x)=\begin{cases} 2\ , & if\ x\leq -1 \\ ax+b, & if\ -1<x<3 \\ -2\ , & if\ x\geq 3 \end{cases}$

b. $g(x)=\begin{cases} \dfrac{4\sin x}{x}\ , & if\ x<0 \\ a-2x\ , & if\ x\geq 0 \end{cases}$

d. $f(x)=\begin{cases} x^2+bx+c\ , & if\ |x-2|\geq 1 \\ x+1\ , & if\ 1<x<3 \end{cases}$

2. Demonstrate that the equation $\cos x+x=0$ has atleast one solution.
3. Verify that the Intermediate Value Theorem applies to the give interval and find the value of c guaranteed by the theorem.

a. $f(x)=x^2+x-1,\quad [0,5],\quad f(c)=11$

b. $f(x)=\dfrac{x^2+x}{x-1},\quad \left[\dfrac{5}{2},4\right],\quad f(c)=6$

Self Test Exercises

1. Find the limit L. Then use the $\varepsilon-\delta$ definition to prove that the limit is L.

a. $\lim\limits_{x\to -1}(x-1)$

c. $\lim\limits_{x\to 2}(1-x^2)$

b. $\lim\limits_{x\to 9}\sqrt{x}$

d. $\lim\limits_{x\to -2}(3-2x)$

2. Evaluate the following limits (if it exists)

a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin x}$

f. $\lim\limits_{x\to 4}\dfrac{\sqrt{x-3}-1}{x-4}$

b) $\lim\limits_{x\to 3}\dfrac{x^2-9}{3-x}$

g. $\lim\limits_{x\to -2}\dfrac{x^2-4}{x^3+8}$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}$

h. $\lim\limits_{x\to 0}\dfrac{\sin[\pi/6+x]-1/2}{x}$

d) $\lim\limits_{x\to 0}\dfrac{\left(1/\sqrt{1+x}\right)-1}{x}$

i. $\lim\limits_{x\to 2}\dfrac{\ln(x-1)^2}{\ln|(|x-1)}$

e) $\lim_{x\to 1}\frac{1-\sqrt[3]{x}}{x-1}$

j. $\lim_{x\to 1}\frac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

3. Evaluate the following limits (if it exists)

a) $\lim_{x\to\infty}\frac{\sin 2x}{x}$

c) $\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right)$

b) $\lim_{x\to\infty}\left[\frac{5}{2}+\ln\left(\frac{x^2+1}{x^2}\right)\right]$

d) $\lim_{x\to\infty}\left(2x-\sqrt{4x^2+1}\right)$

4. Find the one-sided limit (if it exists)

a. $\lim_{x\to -1^-}\frac{x+1}{x^4-1}$

d. $\lim_{x\to -2^-}\frac{2x^2+x+1}{x+2}$

b. $\lim_{x\to 0^+}\frac{\sin 4x}{5x}$

e. $\lim_{x\to 0^-}12e^{-2/x}$

c. $\lim_{x\to -1^+}\frac{x^2-2x+1}{x+1}$

f. $\lim_{x\to -1^+}\frac{x+1}{x^3+1}$

5. Find the values of the constants a *and* b such that

$$\lim_{x\to 0}\frac{\sqrt{a+bx}-\sqrt{3}}{x}=\sqrt{3}$$

6. Determine the value of c such that the function is continuous on the entire real number line

$$f(x)=\begin{cases}x+3, & \text{if } x\le 2\\ cx+6, & \text{if } x>2\end{cases}$$

7. Determine the values of a *and* b such that the function is continuous on the entire real number line

$$f(x)=\begin{cases}x+1, & \text{if } 1<x<3\\ x^2+ax+b, & \text{if } |x-2|\ge 1\end{cases}$$

8. Determine all values of the constant a such that the following function is continuous for all real numbers

$$f(x)=\begin{cases}\frac{ax}{\tan x}, & \text{if } x\ge 0\\ a^2-2, & \text{if } x<0\end{cases}$$

9. Determine the intervals on which the following functions are continuous

a) $f(x) = -3x^2 + 7$

b) $f(x) = \begin{cases} \dfrac{3x^2 - x - 2}{x - 1}, & if \quad x \neq 1 \\ 0, & if \quad x = 1 \end{cases}$

c) $f(x) = \sqrt{49 - x^2}$

d) $f(x) = \sqrt{\dfrac{x+1}{x}}$

10. Find the vertical and horizontal asymptotes of the graph of the following functions (if it exists)

a) $f(x) = 1 + \dfrac{2}{x}$

b) $f(x) = \dfrac{4x}{4 - x^2}$

c) $f(x) = \ln(25 - x^2)$

d) $f(x) = 7e^{-3/x}$

11. Use the Intermediate Value Theorem to show that

$f(x) = 2x^3 - 3$ has a zero in the interval $[1,2]$.

CHAPTER FOUR

Derivatives and Application of Derivatives

Introduction

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as in the previous chapter. This special type of limit is called a derivative and we will see that it can be interpreted as a rate of change of any of the sciences or engineering. We also learn how derivatives are used to find extreme values of functions, to determine and analyze the shapes of graghs, to calculate limits of fractions whose numerators and denominators both approach zero or infinity, and to find numerically where a function equals zero. We also consider the process of recovering a function from its derivative. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in chapter 5.

In this chapter we discuss general properties of inverses and their derivatives. Also recall that the forms $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ are called indeterminate forms because they do not guarantee that a limit exists, nor do the indicate what the limit is, if one does exist. In this chapter we develop a mechanism for evaluating such limit problems.

Chapter Objectives

At the end of this chapter the readers should be able to:

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.
- Explain different properties of differentiation rules.
- Find the derivative of a function using the differentiation rules.
- Find the derivative of a composite function using the chain rule.
- Distinguish between functions written in implicit form and explicit form.
- Use implicit differentiation to find the derivative of a function.

- Define extrema of a function on an interval.
- Understand and use Rolle's theorem.
- Understand and use the Mean Value Theorem.
- Determine intervals on which a function is increasing or decreasing.
- State and apply the first and second derivative test to find relative extrema of a function.
- Evaluate the the relative extrema of a function.
- Determine intervals on which a function is concave upward or concave down ward.
- Find any points of inflection of the gragh of a function.
- Analyze and sketch the graph of a function.
- State L 'Hôpital's rule.
- Recognize limits that produce indeterminate forms.
- Apply L 'Hôpital's rule to evaluate a limit.

The Tangent Lines

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at point P is the line that is perpendicular to the radial line at point P, as shown in figure 4.1. For a general curve, however, the problem is more difficult. For example, how would you define the tangent lines shown in figure 4.2?

You might say that line is tangent to a curve at point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve shown in figure 4.2, but not for the second. Or you might say that a line is tangent to a curve if the line touches or intersects at exactly one point-----this definition would work for a circle but not for more general curves, as the third curve in figure 4.2 shows.

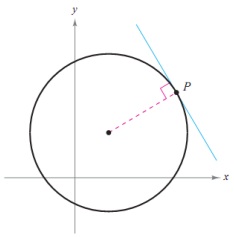


Figure 4.1

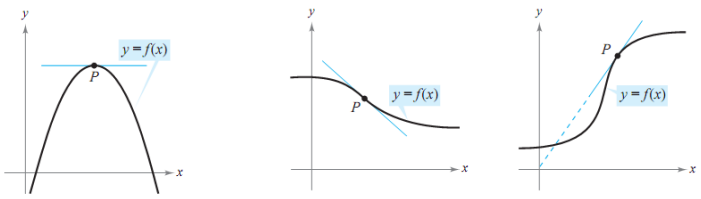


Figure 4.2

Tangent line to a circle Tangent line to a curve at a point

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the **slope** of the tangent line at P. You can approximate this slope using a **secant line** through the point of tangency and a second point on the curve, as shown in figure 4.3. If $(c, f(c))$ is the point of tangency and $(c+\Delta x, f(c+\Delta x))$ is a second point on the graph of f, then the slope of the secant line through the two points is given by

$$m_{\text{sec}} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x} \leftarrow \frac{change\ in\ y}{change\ in\ x} \leftarrow \text{Slope of secant line}$$

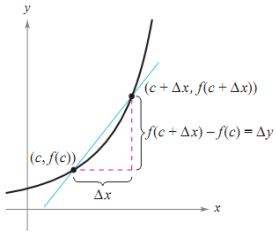
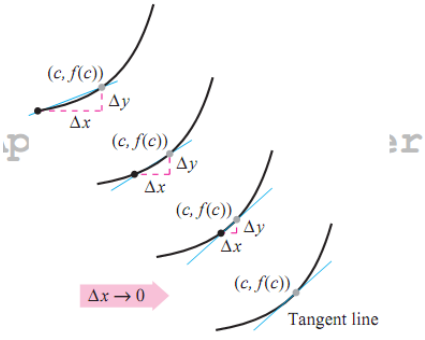


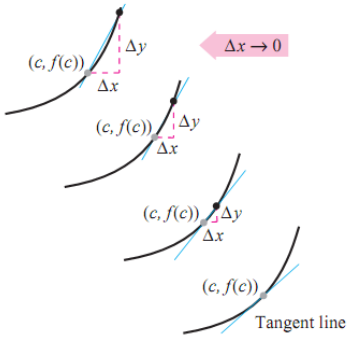
Figure 4.3

The secant line through $(c, f(c))$ and $(c+\Delta x, f(c+\Delta x))$

You can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in figure 4.4.



As $\Delta x \to 0$ from the left, the secant line approaches the tangent line.



As $\Delta x \to 0$ from the right, the secant line approaches the tangent line.

Figure 4.4

4.1 Definition of Derivatives

Definition: If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=m$$

exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$.

Example 1: Find the slope of the graph of $f(x)=2x-5$ at the point $(3,1)$.

Solution: To find the slope, m, of the graph of f when $c=3$, we can apply the definition of the slope of the tangent lines, as follows

$$m=\lim_{\Delta x\to 0}\frac{f(3+\Delta x)-f(3)}{\Delta x}=\lim_{\Delta x\to 0}\frac{[2(3+\Delta x)-5]-[2(3)-5]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{6+2\Delta x-5-6+5}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x}$$

$$=2$$

Thus, the slope of f at $(3,1)$ is $m=2$.

Example 2: Find the equation of the tangent line to the graph of $f(x)=x^2-7$ at the point $(2,-3)$.

Solution: First we have to find the slope, m, of the graph of f when $c=2$, we can apply the definition of the slope of the tangent lines, as follows

$$m=\lim_{\Delta x\to 0}\frac{f(2+\Delta x)-f(2)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\left[(2+\Delta x)^2-(-3)\right]-\left[(2)^2-(-3)\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{4+4\Delta x+(\Delta x)^2+3-4-3}{\Delta x}=\lim_{\Delta x\to 0}\frac{4\Delta x+(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0}(4 + \Delta x) = 4$$

Hence, the equation of the tangent line to the graph of f at the point $(2,-3)$ is

$$y - f(c) = m(x - c) \Rightarrow y - (-3) = 4(x - 2)$$

This becomes, $y = 4x - 11$.

Definition: The derivative of a function f at a number a, denoted by $f'(a)$, is defined as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{……...……..…..} \ (1)$$

provided that this limit exists.

If we write $x = a + h$, then $x - a = h$ and x approaches a if and only if h approaches to 0.

Thus, equivalent way of stating the definition of the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{……} \ (2)$$

This last definition is more convenient for finding the derivative of a function.

Note: In addition to $f'(x)$, other notations are used to denote the derivative of $y = f(x)$. The most common are:

$$f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}[f(x)],\ D_x[y]$$

The notation $\frac{dy}{dx}$ is read as "the derivative of y with respect to x." Using limit notation, we can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

Example 1: Find the derivative of the function $f(x)=x^2+3x+2$ at -1.

Solution: By definition, we have

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

Thus,

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to o}\frac{\left[(-1+h)^2+3(-1+h)+2\right]-\left[(-1)^2+3(-1)+2\right]}{h}$$

$$=\lim_{h\to o}\frac{1-2h+h^2-3+3h+2}{h}$$

$$=\lim_{h\to o}\frac{h+h^2}{h}=\lim_{h\to 0}(1+h)=1$$

I hope the student remembers that the slope of the tangent line to the graph of the function f at a point $(a, f(a))$ is given by the derivative of f at a i.e $f'(a)$. Consequently, using the Point-slope form of the equation of a line, we have the equation of the tangent line to the curve $y=f(x)$ at a point $(a, f(a))$ is given by

$$y-f(a)=f'(a)(x-a).$$

For instance, the equation of the tangent line to the graph of $f(x)=x^2+3x+2$ at $(-1,0)$ in our Example 1 is given by

$$y-f(-1)=f'(-1)(x-(-1))\Rightarrow y-0=1(x+1)\Rightarrow y=x+1$$

Given a function f, we associate with it a new function f', is called the derivative of f, defined by:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example 2: Find the derivative of the function $f(x)=x+\sqrt{x+1}$.

Solution: By definition of derivative, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Thus,

$$f'(x)=\lim_{h\to 0}\frac{\left[x+h)+\sqrt{x+h+1}\right]-\left(x+\sqrt{x+1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{h+\sqrt{(x+h)+1}-\sqrt{x+1}}{h}$$

$$=\lim_{h\to 0}\frac{h+\left[\sqrt{(x+h)+1}-\sqrt{x+1}\right].\left[\sqrt{(x+h)+1}+\sqrt{x+1}\right]}{h\left[\sqrt{(x+h)+1}+\sqrt{x+1}\right]}$$, rationalize the radicals.

$$=\lim_{h\to 0}\left[1+\frac{1}{\sqrt{(x+h)+1}+\sqrt{x+1}}\right]$$

$$=1+\frac{1}{2\sqrt{x+1}}$$

Definition 4.2: A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** $(a,b)\left[or\ (a,\infty)\ or\ (-\infty,a)\ or\ (-\infty,\infty)\right]$ if it is differentiable at every number in the interval.

Example 3: Show that $f(x)=|x|$ is not differentiable at 0.

Solution: Observe that

$$|x|=\begin{cases} x, & if\ x\geq 0 \\ -x, & if\ x<0 \end{cases}$$

Then, for $x>0$ using (1) we have

$$\lim_{x\to 0^+}\frac{|x|-|0|}{x-0}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}1=1$$

and for $x>0$, we have

$$\lim_{x\to 0^-}\frac{|x|-|0|}{x-0}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}-1=-1$$

Hence, $\lim_{x\to 0}\frac{|x|-|0|}{x-0}$ does not exist.

Thus, $f(x)=|x|$ is not differentiable at 0.

Theorem 4.3 If f is differentiable at a, then f is continuous at a.

Proof: To prove that f is continuous at a, we have to show that $\lim_{x\to a}f(x)=f(a)$.

We do this by showing that the difference $f(x)-f(a)$ approaches 0.

For $x\neq a$ we can divide and multiply by $x-a$.

We did this in order to involve the difference quotient.

Thus, we can use the product law of limits to write

$$\lim_{x\to a}[f(x)-f(a)]=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)$$

$$=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right].\lim_{x\to a}(x-a)$$

$$=f'(a).0=0$$

Therefore,

$$\lim_{x\to a}f(x)=\lim_{x\to a}[f(a)+f(x)-f(a)]$$

$$=\lim_{x\to a}f(a)+\lim_{x\to a}[f(x)-f(a)]$$

$$=f(a)+0=f(a)$$

Hence, f is continuous at a.

Note: The converse of Theorem 4.3 is false; that is, there are functions that are continuous but not differentiable at a point.

For instance, consider the function $f(x)=|x|$.

Clearly, $f(x)=|x|$ is continuous at 0 because

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}|x|=0=f(0).$$

But, as we have seen in **Example 3** that f is not differentiable at 0.

Let us remind you some of the differentiation rules that you have developed in your high school calculus course. We advice the student to check on these results using the definition of derivative.

The power rule: If $f(x)=x^n$ for any real number n is given by $f'(x)=nx^{n-1}$.

Derivatives of sine and cosine: $(\sin x)'=\cos x$ and $(\cos x)'=-\sin x$.

Derivatives of exponential and logarithmic functions: $(e^x)'=e^x$ and $(\ln x)'=\frac{1}{x}$.

etc.

Basic Differentiation Rules

Theorem: If f and g are differentiable , then

a. $(cf)'(x)=c\,f'(x)$, where c is an arbitrary constant.

b. $(f\pm g)'(x)=f'(x)\pm g'(x)$

c. $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$

d. $\left(\frac{f}{g}\right)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$; provided that $g(x\neq 0)$.

Activity 4.1

1. Find the derivative of each of the following functions.

a. $f(x)=x^{25}+5x^5+25$

b. $f(x)=x^2-\frac{1}{x^2}$

c. $f(x)=x^4+\sqrt[4]{x}$

d. $f(x)=x\sqrt{x}+\frac{1}{x^2\sqrt{x}}$

e. $f(x)=x\sin x$

f. $f(x)=\tan x$

g. $f(x)=\frac{\sec x}{1+\tan x}$

h. $f(x)=\csc x$

i. $f(x)=\sin x\cos x$

j. $f(x)=\frac{x^2\tan x}{\sec x}$

The Chain Rule

The rules that we have introduced till now are not enough to find composition of functions. Thus, we need to develop an appropriate technique to handle these cases.

Theorem 4.5 [The Chain Rule]: If the derivatives, $g'(x)$ and $f'(g(x))$ both exists, then

$$(f \circ g)'(x) = f'(g(x)).g'(x)$$

Example 1: Find $h'(x)$ if $h(x) = \cos 2x$.

Solution: Let $f(x) = \cos x$ and $g(x) = 2x$.

Then, $h = f \circ g$. Since also $g'(x) = 2$ and $f'(x) = -\sin x$

We conclude that

$$h'(x) = (f \circ g)'(x) = f'(g(x)).g'(x) = (-\sin 2x)(2) = -2\sin 2x$$

Example 2: Find $h'(x)$ if $h(x) = \sqrt{1+x^2}$.

Solution: Let $g(x) = 1+x^2$ and $f(x) = \sqrt{x}$ .consequently $h = f \circ g$.

Then $g'(x) = 2x$ and $f'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$.

Therefore,

$$h'(x) = f'(g(x)).g'(x) = \frac{1}{2\sqrt{1+x^2}}.2x = \frac{x}{\sqrt{1+x^2}}.$$

Activity 4.2

1. Find the first derivative of the following functions

   a. $f(x) = (x^5+2x^2+3)^{50}$ c. $y = \frac{1}{\sqrt[3]{x^6+2x+1}}$

   b. $f(x) = \cos(\sin(\tan x))$ d. $y = \sqrt{\cos(\sin^2 x)}$

2. Find the equation of the tangent line to the curve at the given point.

   a. $y = (x^3-x^2+x-1)^{10}$, $at\ (1,0)$

b. $f(x)=\sqrt{x+\frac{1}{x}}$, $at\ \left(1,\sqrt{2}\right)$

4.2 Inverse Functions and Derivatives of Inverse Trigonometric Functions

Definition 6.1 A function f with domain D and range R is one-to-one function if whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

Note: From **Definition 6.1** we see that every strictly increasing function is one-to-one, because if $a < b$, then $f(a) < f(b)$, and if $b < a$, then $f(b) < f(a)$ in short if $a \neq b$, then $f(a) \neq f(b)$. Similarly, every strictly decreasing function is one-to-one. We now give the definition of inverse functions in terms of one-to-one function.

Definition 6.2 Let f be a one-to-one function with domain D and range R. A function g with domain R and range D is the **inverse of** f, provided the following condition is true for every x in D and every y in R:

$y = f(x)$ if and only if $x = g(y)$.

If a function f has an inverse function g, we often denote g by f^{-1}. Of course we must note here that almost always $f^{-1} \neq \frac{1}{f}$.

If f is a one-o-one function with domain D and range R, then for each number y in R, there is exactly one number x in D such that $y = f(x)$. Since x is unique, we may define a function g from R to D by means of the rule $x = g(y)$. g reverses the correspondence given by f. We call g the inverse function of f.

Note: - a function f has an inverse if and only if it is one-to-one.

Theorem: Every strictly increasing and every strictly decreasing function has an inverse.

Note that: a function f is strictly increasing on I if

- $f'(x) \geq 0$ for all x in I and $f'(x) = 0$ for atmost finitely many values of x.
- $f'(x) \leq 0$ for all x in I and $f'(x) = 0$ for atmost finitely many values of x.

Example 1: Let $f(x)=2x^7+3x^5+6x-4$. Show that f has an inverse.

Solution: Since $f'(x) = 14x^6 + 15x^2 + 6 > 0$ for all x in its domain, we have

f is strictly increasing.

Thus, f has an inverse.

6.2 Properties of Inverses

Let f have an inverse. Then f and f^{-1} have the following properties.

a. Domain of f^{-1} =Range of f and Range of f^{-1} =Domain of f.

b. $\left(f^{-1}\right)^{-1} = f$.

c. $f^{-1}(f(x)) = x$ for all x in the domain of f.

d. $f\left(f^{-1}(y)\right) = y$ for all y in the range of f.

In some cases we can find the inverse of a one-to-one function by solving the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.

Guide lines for finding f^{-1}:

i. Write $y = f(x)$.

ii. Solve for x in terms of y.

iii. Write $f^{-1}(y)$ for x in step 2.

Example 2: Let $f(x) = 2x + 3$. Find the inverse of f.

Solution: Since $f'(x) = 2 > 0$, f is strictly increasing.

Thus f has an inverse.

Now let $y = 2x + 3$, from step 1.

$$\Rightarrow x = \frac{y-3}{2}, \text{ from step 2.}$$

$$\Rightarrow f^{-1}(y) = \frac{y-3}{2}, \text{ from step 3.}$$

Since we customarily use x as the independent variable, we replace y by x to obtain

$$f^{-1}(x)=\frac{x-3}{2}$$

Example 3: Let $f(x)=x^2-3$ for $x\geq 0$. Find the inverse of f.

Solution: Since $f'(x)=2x\geq 0$ for all x in is domain, f is strictly increasing.

Thus f has an inverse.

Now let $y=x^2-3$, from step 1.

$\Rightarrow x=\sqrt{y+3}$, since $x\geq 0$ and from step 2.

$\Rightarrow f^{-1}(y)=\sqrt{y+3}$, from step 3.

Since we customarily use x as the independent variable, we replace y by x to obtain

$$f^{-1}(x)=\sqrt{x+3}$$

Inverse Trigonometric Functions

Since the trigonomeric functions are not one-to-one, they do not have inverse functions. By restricting heir domains, however, we may obtain one-to-one functions that have the same values as trigonomeric funcions and that do have inverse over these restricted domains.

The Arcsine Function

If we restrict the domain of the sine function to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then the resulting function is strictly increasing (because its derivative is positive except at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$). Hence the restricted function which is called **arcsine function** has domain $[-1,1]$, and range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Its value at x is usually written $\arcsin x$ or $\sin^{-1}x$. As a consequence,

$$\arcsin x = y \text{ if and only if } \sin y = x$$

$$\text{for } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

We also see from the propery of inverse funcions that

i. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

ii. $\sin(\arcsin x) = x$ for $-1 \le x \le 1$

Example 1: Evaluate

a. $\sin\left(\arcsin\frac{1}{2}\right)$ b. $\arcsin\left(\sin\frac{\pi}{4}\right)$ c. $\arcsin\left(\sin\frac{5\pi}{6}\right)$

Solution:

a. $\sin\left(\arcsin\frac{1}{2}\right) = \frac{1}{2}$, since $-1 \le \frac{1}{2} \le 1$.

b. $\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$, since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$.

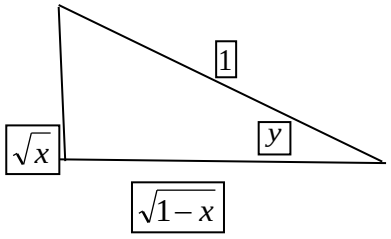
c. $\arcsin\left(\sin\frac{5\pi}{6}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

In example 1*c*) $\frac{5\pi}{6}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and hence we can not use ii. Instead we use properties of special angles to first evaluate $\sin\left(\frac{5\pi}{6}\right)$ and then find $\arcsin\left(\frac{1}{2}\right)$.

Example 2: Simplify the expression $\sec(\arcsin\sqrt{x})$.

Solution: We will evaluate $\sec(\arcsin\sqrt{x})$ by evaluating $\sec y$ for the value of y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\arcsin\sqrt{x} = y$, that is, $\sin y = \sqrt{x}$. Since $\sin y = \sqrt{x} \ge 0$, it follows that $0 \le y < \frac{\pi}{2}$.

Applying the Pythagorean Theorem to the triangle given below



We find $\sec y = \frac{1}{\sqrt{1-x}}$.

Thus, $\sec\left(\arcsin\sqrt{x}\right) = \sec y = \frac{1}{\sqrt{1-x}}$.

The Arccosine Function

If we restrict the domain of the cosine function to $[0,\pi]$, then the resulting function is strictly decreasing (because its derivative is negative except at 0 and π). Hence the restricted function which is called **arccosine function** has domain $[-1,1]$, and range $[0,\pi]$.

Its value at x is usually written arccos x or $\cos^{-1} x$. As a consequence,

$\arccos x = y$ if and only if $\cos y = x$

for $-1 \le x \le 1$ and $0 \le y \le \pi$

Since cos and *arcc* cos are inverse functions of each other, we obtain the following properties.

i. $\arccos(\cos x) = x$ for $0 \le x \le \pi$

ii. $\cos(\arccos x) = x$ for $-1 \le x \le 1$.

Example 2: Evaluate

a. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$ b. $\arccos\left(\cos\frac{2\pi}{3}\right)$ c. $\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right)$

Solution:

a. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$, since $-1 \le -\frac{1}{2} \le 1$.

b. $\arccos\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$, since $0 \le \frac{2\pi}{3} \le \pi$.

c. $\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

Note that in the *c*) part of the preceding example 2, $-\frac{\pi}{4}$ is not between 0 and π, and hence we can not use property ii above. Instead we firs evaluate $\cos\left(-\frac{\pi}{4}\right)$ and then find $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$

.

Example 3: Simplify the expression $\cos(\arctan x)$.

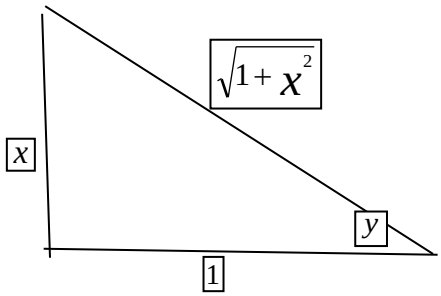
Solution: Let $y = arcanx$. Then $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$. We want to find $\cos y$. But, since $\tan y$ is known, it is easier to find $\sec y$. First:

$$\sec^2 y = 1 + \tan^2 y$$

Now, $\sec^2 y = 1 + x^2 \Rightarrow \sec y = \sqrt{1+x^2}$ (as $\sec y > 0$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$).

Thus, $\cos(\arctan x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}$.

Note that instead of using trigonometric identities as in the solution above, it is easy to use a triangular diagram. If we let $y = arcanx$, then $\tan y = x$, and using he right triangle below we can read from the figure that



$$Thus,\ \cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}.$$

The Arctangent Function

To find an inverse for the tangent function, we restrict he tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The resulting inverse function is called the **arctangent funcion**. Its domain is $(-\infty, \infty)$, and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We usually write is value at x as $\arctan x$ or $\tan^{-1} x$. As a consequence,

$\arctan x = y$ if and only if $\tan y = x$

for any x and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Thus for any x, $\arctan x$ is the number y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

As with $\arcsin x$ and $\arccos x$, we have the following properties of arctan:

i. $\arctan(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

ii. $\tan(\arctan x) = x$ for every x.

Example 4: a) $\arctan\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$, since $-\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}$.

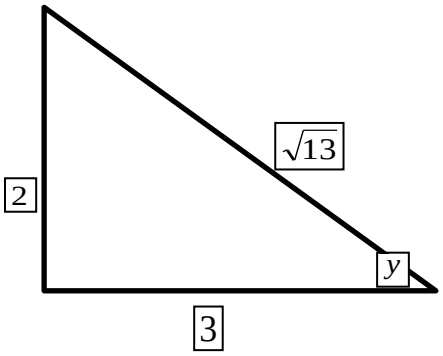
b) $\tan(\arctan 99) = 99$.

c) $\arctan(\tan\pi) = \arctan(0) = 0$.

Example 5: Evaluate $\sec\left(\arctan\frac{2}{3}\right)$.

Solution: If we let $y = \arctan\frac{2}{3}$, then $\tan y = \frac{2}{3}$. We wish o find $\sec y$.

Since $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ for every x and $\tan y > 0$, it follows that $0 < y < \frac{\pi}{2}$ and from the triangle below we obtain that



Thus, $\sec y = \frac{\sqrt{13}}{3}$

.

The remaining trigonometric functions can be done analogously and are summarized here as below:

★ $y = \csc^{-1} x$ for $|x| \geq 1$ if and only if $\csc y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

★ $y = \sec^{-1} x$ for $|x| \geq 1$ if and only if $\sec y = x$ and $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

★ $y=\cot^{-1}x$ for $x\in\Re$ if and only if $\cot y=x$ and $y\in(0,\pi)$.

Activity 4.3

1. Find the exact value of the expression, whenever it is defined.

   a. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

   b. $\arctan\left(-\sqrt{3}\right)$

   c. $\arcsin\left(\sin\frac{5\pi}{4}\right)$

   d. $\cos\left(\arctan(-\frac{3}{4})-\arcsin(\frac{4}{5})\right)$

   e. $\arccos\left(-\frac{1}{2}\right)$

   f. $\sin\left(\arcsin\frac{2}{3}\right)$

   g. $\arccos\left(\cos\frac{5\pi}{24}\right)$

   h. $\tan\left(\arctan(\frac{3}{4})+\arccos(\frac{8}{17})\right)$

2. Rewrite as an algebraic expression in x for $x>0$.

   a. $\sec\left(\arcsin(\frac{x}{3})\right)$

   b. $\cos(2\arcsin x)$

   c. $\tan\left(arc\csc(\frac{x}{2})\right)$

   d. $\sin(2\arcsin x)$

Derivatives of Inverse Trigonometric Functions

We now see the derivatives of the inverse trigonomeric functions in the following theorems.

Theorem 1:

a) $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$

b) $\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$

c) $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$

d) $\frac{d}{dx}(arc\sec x)=\frac{1}{x\sqrt{x^2-1}}$

e) $\frac{d}{dx}(arc\csc x)=-\frac{1}{x\sqrt{x^2-1}}$

f) $\frac{d}{dx}(arc\cot x)=-\frac{1}{1+x^2}$

Proof: a) To proof $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$, put $y=\arcsin x$ so that $\sin y=x$ whenever $-1<x<1$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$. Then differentiating $\sin y=x$ implicitly, we have

$$\cos y \frac{dy}{dx} = 1$$

And hence, $\frac{dy}{dx} = \frac{d}{dx}(\arcsin x) = \frac{1}{\cos y}$

Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $\cos y$ is positive and, therefore,

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

Thus, $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$; for $|x| < 1$.

Observe that he inverse sine function is not differentiable at $x = \pm 1$.

c) To proof $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$, put $y = \arctan x$ so that $\tan y = x$ whenever $x \in \Re$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Differentiating $\tan y = x$ and trigonomeric identities, we have

$$\frac{d}{dx}(\arctan x) = \frac{1}{\frac{d}{dy}\tan y} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Thus, $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$; for every $x \in \Re$.

The of the formulas can be obtained in similar fashion.

Example 1: Find

a. $\frac{d}{dx}\left(\arcsin 3x^2\right)$

b. $\frac{d}{dx}(\arccos(\ln x))$

c. $\frac{d}{dx}\left(\arctan e^{2x}\right)$

d. $\frac{d}{dx}\left(arc\sec 3x^2\right)$

Solution:

a. $\frac{d}{dx}\left(\arcsin 3x^2\right) = \frac{1}{\sqrt{1-\left(3x^2\right)^2}} \frac{d}{dx}\left(3x^2\right) = \frac{6x}{\sqrt{1-9x^4}}$.

b. $\frac{d}{dx}(\arccos(\ln x)) = -\frac{1}{\sqrt{1-(\ln x)^2}} \frac{d}{dx}(\ln x) = -\frac{1}{x\sqrt{1-(\ln x)^2}}$.

c. $\frac{d}{dx}\left(\arctan e^{2x}\right)=\frac{1}{1+\left(e^{2x}\right)^2}\frac{d}{dx}\left(e^{2x}\right)=\frac{2e^{2x}}{1+\left(e^{2x}\right)^2}$.

d. $\frac{d}{dx}\left(arcsec3x^2\right)=\frac{1}{3x^2\sqrt{\left(3x^2\right)^2-1}}\frac{d}{dx}\left(3x^2\right)=\frac{2}{x\sqrt{9x^4-1}}$.

Each of the formulas in Theorem 3.1 gives rise to an integration formula. These have the following relations:

a) $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$ if and only if $\int\frac{1}{\sqrt{1-x^2}}dx=\arcsin x+c$

b) $\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$ if and only if $\int\frac{1}{\sqrt{1-x^2}}dx=-\arccos x+c$

c) $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$ if and only if $\int\frac{1}{1+x^2}dx=\arctan x+c$

d) $\frac{d}{dx}(arcsec x)=\frac{1}{x\sqrt{x^2-1}}$ if and only if $\int\frac{1}{x\sqrt{x^2-1}}dx=-arccsc x+c$

e) $\frac{d}{dx}(arccsc x)=-\frac{1}{x\sqrt{x^2-1}}$ if and only if $\int\frac{1}{x\sqrt{x^2-1}}dx=arcsec x+c$

f) $\frac{d}{dx}(arccot x)=-\frac{1}{1+x^2}$ if and only if $\int\frac{1}{1+x^2}dx=-arccot x+c$

Note: These six properties will be considered in the next chapter when we discuss about techniques of integration.

Activity 4.4

1. Find the derivative of the following functions and simplify where possible

a. $f(x)=\arcsin(2x-1)$

b. $f(x)=\arctan\left(x-\sqrt{1+x^2}\right)$

c. $\left(\tan x\right)^{\arctan x}$

d. $f(x)=\left(1+x^2\right)\arctan x$

e. $f(x)=\cos\left(x^{-1}\right)+\left(\cos x\right)^{-1}+\cos^{-1}x$

f. $f(x)=(\arctan 4x)e^{\arctan 4x}$

4.3 Higher Order Derivatives

If f is a function, then f' is the function that assigns the number $f'(x)$ to each x a which f is differentiable. Since f' is a function, we can carry the process a step further and define $f''(a)$ by the formula

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}$$

Whenever this limit exists.

We call $f''(a)$ **the second derivative of** f **at** a

Example 1: Let $f(x) = \cos 2x$. Then find a formula for $f''(x)$.

Solution: Since $f'(x) = -2\sin 2x$, it follows that

$$f''(x) = -4\cos 2x.$$

Example 2: Let $f(x) = \sqrt{x}$. Then find a formula for $f''(x)$.

Solution: Since $f'(x) = \frac{1}{2\sqrt{x}}$, it follows that

$$f''(x) = -\frac{1}{4}x^{-3/2}.$$

In general, for any positive integer $n \geq 3$ we can define the n^{th} derivative of f at a, denoted by $f^{(n)}(a)$ as

$$f^{(n)}(a) = \lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a}$$

Whenever this limit exists.

Note: The second derivative, the third derivative, and so on are called **higher order derivatives**, to distinguish them from the first derivative.

Example 3: Let $f(x) = \frac{x+1}{x-1}$. Then find a formula for $f^{(n)}(x)$.

Solution: Using quotient rule, $f'(x) = -\frac{2}{(x-1)^2} = \frac{(-1)^1 2(1!)}{(x-1)^{1+1}}$, it follows that

$$f''(x) = \frac{4}{(x-1)^3} = \frac{(-1)^2 2(2!)}{(x-1)^{2+1}},$$

$$f'''(x) = -\frac{12}{(x-1)^4} = \frac{(-1)^3 2(3!)}{(x-1)^{3+1}},$$

$$f^{(4)}(x) = \frac{48}{(x-1)^5} = \frac{(-1)^4 2(4!)}{(x-1)^{4+1}}$$

.

.

.

Continuing this process we get

$$f^{(n)}(x) == \frac{(-1)^n 2(n!)}{(x-1)^{n+1}}; \text{ for } n \geq 1$$

For instance, $f^{(10)}(2) == \frac{(-1)^{10} 2(10!)}{(2-1)^{10+1}} = 2(10!)$.

Activity 4.5

1.Let $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$. Then find all higher derivatives of f.

2.Let $f(x) = x^8 - 2x + 3$. Then find $\lim_{x \to 2} \frac{f''(x) - f(2)''}{x-2}$.

3. Let $p(x)$ be a polynomial function of degree n with leading coefficient k, then find

a) $p^{(n-1)}(x)$ b) $p^{(n)}(x)$ c) $p^{(n+1)}(x)$

4. Find a formula for the n^{th} derivative of f, if

a. $f(x) = \ln x$ b. $f(x) = \sin x$

5. If $f^{(4)}(x) = f(x)$ for all x, then what is the relationship between $f^{(35)}(x)$ *and* $f^{(31)}(x)$

4.4 Implicit Differentiation

The following problems require the use of implicit differentiation. Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in pre-calculus involve functions y written explicitly as functions of x. For example, if

$$y = 3x^2 - \sin(7x+5)$$

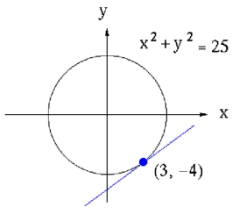
Then the derivative of y is

$$y' = \frac{dy}{dx} = 6x - 7\cos(7x+5).$$

However, some functions y are written IMPLICITLY as functions of x. A familiar example of this is the equation

$$x^2 + y^2 = 25,$$

Which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the gragh of this equation at the point $(3,-4)$.



How could we find the derivative of y in this instance? One way is to first write y explicitly as a funcion of x. Thus,

$$x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25 - x^2}.$$

Where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point $(3,-4)$ lies on the bottom semi-circle given by

$$y = -\sqrt{25 - x^2}$$

The derivative of y is

$$y' = \frac{dy}{dx} = \frac{x}{\sqrt{25-x^2}}$$

Thus, the slope , m , of the line tangent to the gragh at the point $(3,-4)$ is

$$m = y' = \frac{dy}{dx} = \frac{3}{\sqrt{25-3^2}} = \frac{3}{4}.$$

Unfortunately, not every equation involving x and y can be solved explicitly for y. For the sake of illustration we will find the derivative of y **WITHOUT** writing y explicitly as a function of x. Recall that the derivative of a function of x squared, $(f(x))^2$, can be found using the chain rule:

$$\frac{d}{dx}\left[(f(x))^2\right] = 2f(x)\frac{d}{dx}[f(x)] = 2f(x)\,f'(x)$$

Since y symbolically represents a function of x, the derivative of y^2 can be found in the same fashion:

$$\frac{d}{dx}\left[y^2\right] = 2y\frac{d}{dx}[y] = 2y\,y'.$$

Now begin with $x^2 + y^2 = 25$. Differentiate both sides of the equation, getting

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}[25] \Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

And so

$$2x + 2y\,y' = 0$$

Hence, $2y\,y' = -2x$

Thus, $\frac{dy}{dx} = y' = -\frac{x}{y}$.

Therefore, the slope of the line tangent to the gragh at the point $(3,-4)$ is

$$m = y' = \frac{dy}{dx} = \frac{-(3)}{(-4)} = \frac{3}{4}.$$

This second mehod illustrates the process of implicit differentiation. It is important to note that the derivative expression for explicit differentiation involves x only, while the derivative expression for implicit differentiation may involve BOTH x AND y.

The following problems range in difficulty from average to challenging.

Example 1: Using implici differentiation find $\frac{dy}{dx}$ for $x^3+y^3=4$.

Solution: Differentiating both sides of the equation, getting

$$\frac{d}{dx}\left[x^3+y^3\right]=\frac{d}{dx}[4]\Rightarrow\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)=\frac{d}{dx}(4)$$

So that, $3x^2+3y^2y'=0\Rightarrow 3y^2y'=-3x^2$

Thus, $y'=\frac{dy}{dx}=-\frac{x^2}{y^2}$.

Example 2: Using implici differentiation find $\frac{dy}{dx}$ for $(x-y)^2=x+y-1$.

Solution: Differentiating both sides of the equation, getting

$$\frac{d}{dx}\left[(x-y)^2\right]=\frac{d}{dx}[x+y-1]\Leftarrow\frac{d}{dx}\left[(x-y)^2\right]=\frac{d}{dx}(x)+\frac{d}{dx}(y)-\frac{d}{dx}(1).$$

Applying the chain rule on $\frac{d}{dx}\left[(x-y)^2\right]$, we get

$$\frac{d}{dx}\left[(x-y)^2\right]=2(x-y)\frac{d}{dx}[x-y]\Rightarrow\frac{d}{dx}\left[(x-y)^2\right]=2(x-y)\left(1-\frac{dy}{dx}\right)$$

So that

$$2(x-y)\left(1-\frac{dy}{dx}\right)=1+\frac{dy}{dx}-0\Rightarrow 2(x-y)-2(x-y)\frac{dy}{dx}=1+\frac{dy}{dx}$$

Hence, $2(x-y)\frac{dy}{dx}+\frac{dy}{dx}=2(x-y)-1\Rightarrow[2(x-y)+1]\frac{dy}{dx}=2(x-y)-1$

Thus, solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}=\frac{2(x-y)-1}{2(x-y)+1}.$$

Example 3: Using implici differentiation find $\frac{dy}{dx}$ for $y=\sin(3x+4y)$.

Solution: Differentiating both sides of the equation, getting

$$\frac{d}{dx}(y)=\frac{d}{dx}[\sin(3x+4y)]\Rightarrow\frac{dy}{dx}=\cos(3x+4y).\frac{d}{dx}(3x+4y)$$

So that $\frac{dy}{dx}=\cos(3x+4y)\left[\frac{d}{dx}(3x)+\frac{d}{dx}(3y)\right]\Rightarrow\frac{dy}{dx}=3\cos(3x+4y)+4\cos(3x+4y)\frac{dy}{dx}$.

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}=\frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}.$$

Example 4: Using implici differentiation find $\frac{dy}{dx}$ for $y=x^2y^3+x^3y^2$.

Solution: Differentiating both sides of the equation, getting

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(x^2y^3+x^3y^2\right)\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(x^2y^3\right)+\frac{d}{dx}\left(x^3y^2\right)$$

Using product rule, we have

$$\frac{dy}{dx}=\left[\frac{d}{dx}\left(x^2\right)y^3+x^2\frac{d}{dx}\left(y^3\right)\right]+\left[\frac{d}{dx}\left(x^3\right)y^2+x^3\frac{d}{dx}\left(y^2\right)\right]$$

$$=2xy^3+3x^2y^2\frac{dy}{dx}+3x^2y^2+2x^3y\frac{dy}{dx}$$

Factoring out $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}\left[1-3x^2y^2-2x^3y\right]=2xy^3+3x^2y^2$$

Thus, $\frac{dy}{dx}=\frac{2xy^3+3x^2y^2}{1-3x^2y^2-2x^3y}$.

Example 5: Find an equation of the tangent line to the gragh of $x^2+(y-x)^3=9$ *at* $x=1$..

Solution: Begin with $x^2+(y-x)^3=9$. If $x=1$, then

$$(1)^2+(y-1)^3=9\Rightarrow y=3$$

Hence the tangent line passes through the point $(1,3)$.

Now by using implicit differentiation, we have

$$\frac{d}{dx}\left[x^2+(y-x)^3\right]=\frac{d}{dx}(9)\Rightarrow 2x+3(y-x)^2\left[\frac{dy}{dx}-1\right]=0$$

So that $\frac{dy}{dx}\left[3(y-x)^2\right]=3(y-x)^2-2x\Rightarrow \frac{dy}{dx}=\frac{3(y-x)^2-2x}{3(y-x)^2}$.

Thus, the slope , m , of the tangent line to the gragh at $(1,3)$ is

$$m=\frac{dy}{dx}=\frac{3(3-1)^2-2(1)}{3(3-1)^2}=\frac{10}{12}=\frac{5}{6}.$$

And the equation of the tangent line is

$$y-3=\frac{5}{6}(x-1)\Rightarrow y=\frac{5}{6}x+\frac{13}{6}.$$

Example 6: Using implici differentiation find $\frac{d^2y}{dx^2}$ *or* y'' for $x^4+y^4=25$.

Solution: Differentiating both sides of the equation implicitly with respect to x, we get

$$4x^3+4y^3\frac{dy}{dx}=0\Rightarrow \frac{dy}{dx}=-\frac{x^3}{y^3}.$$

To find $\frac{d^2y}{dx^2}$ *or* y'' we differentiate this expression for $\frac{dy}{dx}$ using the quotient rule and remembering that y is a function of x:

$$\frac{d^2y}{dx^2}=y''=\frac{d}{dx}\left(-\frac{x^3}{y^3}\right)=-\frac{y^3\frac{d}{dx}(x^3)-x^3\frac{d}{dx}(y^3)}{(y^3)^2}$$

$$=-\frac{y^3(3x^2)-x^3\left(3y^2\cdot\frac{dy}{dx}\right)}{y^6}$$

If we now substitute the value of $\frac{dy}{dx}$ into this expression, we get

$$\frac{d^2y}{dx^2} = -\frac{y^3(3x^2) - x^3\left(-\frac{x^3}{y^3}\right)}{(y^3)^2}$$

$$= -\frac{3x^2(x^4 + y^4)}{y^7}$$

$$= -\frac{3x^2(25)}{y^7}, \text{ since } x^4 + y^4 = 25$$

$$= -\frac{75x^2}{y^7}.$$

Activity 4.5

1. Using implicit differentiation find $\frac{dy}{dx}$ for

   a. $\frac{y}{x^3} + \frac{x}{y^3} = x^2y^4$

   b. $\frac{x - y^3}{y + x^2} = x + 2$

   c. $x = \sqrt{x^2 + y^2}$

   d. $\cos^2 x + \cos^2 y = \cos(2x + 2y)$

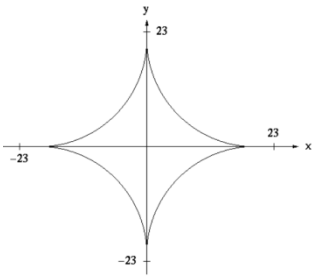
   e. $e^{xy} = e^{4x} + e^{5y}$

   f. $\sin x + \cos(y - x) = x^2$

2. Find an equation of the tangent line o he gragh of $(x^2 + y^2)^3 = 8x^2y^2$ at the point $(-1,1)$.

3. Find the slope of the gragh $x^2y + y^4 = 4 + 2x$ at the point $(-1,1)$.

4. Find all poins (x, y) on the gragh of $x^{2/3} + y^{2/3} = 8$ (see a diagram.) where lines tangen to the gragh at (x, y) have slope -1.



5. Using implicit differentiation find $\frac{d^2y}{dx^2}$ for

a. $x^2+xy+y^2=1$ c. $x^3+y^3=1$

b. $\sqrt{x}+\sqrt{y}=1$ d. $x^2+6xy+y^2=8$

4.5 Application of Derivatives

4.5.1 Extrema of a Function

Definition 1: A function f has an **absolue maximum** at c if $f(c)\geq f(x)$ for all x in D, where D is in the domain of f. The number $f(c)$ is called the **maximum value of** f on D. Similarly, f has an **absolue minimum** at c if $f(c)\leq f(x)$ for all x in D and the number $f(c)$ is called the **minimum value of** f on D. The maximum and minimum values of f are called the **extreme vales** of f.

Definition 2: A function f has a **local maximum** (or relative maximum) value at c if there is an open interval I containing c such that $f(c)\geq f(x)$ for all x in I. Similarly, f has a **local minimum** (or relative minimum) value at c if there is an open interval I containing c such that $f(c)\leq f(x)$ for all x in I.

Example 1: Consider the function $f(x)=x^2$. Then $f(x)\geq f(0)$ because $x^2\geq 0$ for all x. Therefore, $f(0)=0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola $y=x^2$. However, there is no highest point on the parabola and so this function has no maximum value.

Example 2: Consider the function $f(x)=x^3$. We see that this funcion has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extereme values either.

Theorem 1: If f has a relative (local) extremum (that is, maximum or minimum) value at c, and that $f'(c)$ exists, then $f'(c)=0.$

Definition 3: A number c in the domain of a function f is a critical number of f if eiher $f'(c)=0$ or $f'(c)$ does not exist.

Example 3: Find the critical numbers of the function $f(x)=4x^{3/5}-x^{8/5}$.

Solution: The derivative of f is given by

$$f'(x)=\frac{12}{5}x^{-2/5}-\frac{8}{5}x^{3/5}=\frac{12-8x}{5x^{\frac{2}{5}}}.$$

Therefore, $f'(c)=0$ if $12-8x=0$, that is $x=\frac{3}{2}$ and $f'(x)$ does not exist when $x=0$.

Thus the critical numbers are $x=0$ and $x=\frac{3}{2}$.

To find the absolue extreme value of a function on a closed interval a similar theorem to theorem 1 is given below.

Theorem 2: (Maximum-Minimum Theorem)

Let f be continuous on a closed interval $[a,b]$. Then f has a maximum and a minimum value on $[a,b]$.

Note that according to **Maximum-Minimum Theorem** an exreme value can be taken on more than once.

The following theorem will simplify our effort of searching for an extreme value on a closed interval.

Theorem 3: Let f be defined on $[a,b]$. If an absolute extreme value of f on $[a,b]$ occurs at a number c in (a,b) at which f has a derivative, then $f'(c)=0$.

In using theorem 2 to find the extreme value we follow the three-step procedure below:

i. Find the values of f at the critical numbers of f in (a,b).

ii. Find the values of $f(a)$ and $f(b)$.

iii. The largest of the values from steps 1 and 2 is the absolute maximum value; and the smallest of these values is the absolute minimum value.

Example 3: Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 2, \ on \ \ -\frac{1}{2} \le x \le 3.$$

Solution: Since f is continuous on $\left[-\frac{1}{2},3\right]$, we can use the procedure oulined above:

Now, $f'(x)=3x^2-6x=3x(x-2)$

Since, $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x)=0$, that is, when $x=0$ or $x=2$.

Notice that each of these critical numbers lies in the interval $\left[-\frac{1}{2},3\right]$.

The values of f at these critical numbers are

$$f(0) = 2 \ and \ \ f(2) = -2.$$

The values of f at the end points of the interval are are

$$f(-\frac{1}{2}) = \frac{1}{8} \ and \ \ f(3) = 2.$$

Comparing these four numbers, we see that

The absolute maximum value is $f(0) = f(3) = 2$, and

The absolute minimum value is $f(2) = -2$.

Activity 5.1

1. Find the critical numbers each of the following functions.

   a. $f(x) = x^3 - 6x + 1$ c. $f(x) = \cos\sqrt{x}$

   b. $f(x) = |x|$ d. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

2. Find the extreme values (if any) of the given function on the given inerval. Determine at which numbers in the interval these values occur.

   a) $f(x) = x^2 - 2x + 2, [0,3]$ c) $f(x) = x^{2/3}, [-8,8]$

b) $f(x)=x^2+\frac{2}{x}, \quad \left[\frac{1}{2},2\right]$

3. Show that 0 is a critical number of the function $f(x)=x^5$, but f doesnot have a local exremum value at 0.
4. Prove that the function $f(x)=x^{51}+x^{21}+x+1$ has neither a local maximum nor a local minimum.

4.5.2 The Rolle's Theorem and the Mean Value Theorem (MVT)

Theorem : (Rolle's Theorem)

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$ then there is atleast one number c in (a,b) such that $f'(c)=0$

Example: Let $f(x)=x^4-2x^2$. Find all value(s) of c in the interval $(-2,2)$ such that $f'(c)=0$.

Solution: Note that the function $f(x)=x^4-2x^2$ satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval $[-2,2]$ and differentiable on the open interval $(-2,2)$. Moreover $f(-2)=8=f(2)$.

Thus there exists atleast one number c in $(-2,2)$ such that $f'(c)=0$.

Now, $f(x)=x^4-2x^2 \Rightarrow f'(x)=4x^3-4x$

$$=4x(x^2-1)$$

$$=4x(x-1)(x+1)$$

To find c, we have

$f'(c)=0 \Rightarrow 4c(c-1)(c=1)=0$

Hence, $c=0,\ -1,\ 1$ also $0,\ -1,\ 1\in(-2,2)$

Thus all values of c that satisfies he Rolle's Theorem are $c=0,\ -1,\ 1$.

Theorem 4: Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there is a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Example 5: Let $f(x)=x^3-8x+5$. Find a number c in $(0,3)$ that satisfies the Mean Value Theorem.

Solution: Since f is coninuous on $[0,3]$ and $f'(c)$ should satisfy the condition

$$f'(c)=\frac{f(3)-f(0)}{3-0}=\frac{8-5}{3}=1$$

We seek a number c in $(0,3)$ such that $f'(c)=1$. But

$$f'(x)=3x^2-8$$

So that c must saisfy

$$3c^2-8=1\Rightarrow c^2=3$$

Hence, $c=\pm\sqrt{3}$

Since $-\sqrt{3}\notin(0,3)$, the value of c that satisfies the Mean Value Theorem in the interval $(0,3)$ is $c=\sqrt{3}$.

Applications of The Mean Value Theorem

Theorem 5:

a. Let f be continuous on an interval I. If $f'(x)=0$ for each interior point x of I, then f is constant on I.

b. Let f and g be continuous on an interval I. If $f'(x)=g'(x)$ for each interior point x of I, then $f-g$ is constant on I. In other words, there is a constant c such that $f(x)=g(x)+c$ for all x in I.

Example: Let f be such that $f'(x)=-2$ and $f(0)=2$. Determine the function f.

Solution: Since f and $-2x$ are both antiderivatives of $g(x)=-2x$, by Theorem $5(b)$ there is a constant c such that $f(x)=g(x)+c$

So that $f(x)=-2x+c$, for the appropriate constant c.

To determine c, we have the assumption $f(0)=2$, which yields

$f(0)=0 \Rightarrow -2(0)+c=2 \Rightarrow c=2$.

Thus, $f(x)=-2x+2$.

Activity 4.6

1. Verify that the function bellow satisfies the hypothesis of the Mean-Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   a. $f(x)=1-x^2;[0,3]$ c. $f(x)=\sqrt{x};[1,4]$

   b. $f(x)=3\left(x+\frac{1}{x}\right);\left[\frac{1}{3},3\right]$ d. $f(x)=(x-1)(x-2)(x-3);[1,3]$

2. Let $f(x)=|x-1|$. Show that there is no value of c such that $f(3)-f(0)=f'(c)(3-0)$. Why does this not contradict the Mean Value Theorem?
3. Determine all functions satisfying the given conditions.

   a. $f''(x)=\sin x;\ f'(\pi)=-2,\ f(0)=4$

   b. $f''(x)=0;\ f'(2)=3,\ f(-1)=1$

   c. $f^{(n)}(x)=n$; for any positive integer n.

4. Show that the equation $x^5+10x+3=0$ has exactly one real root.
5. Show that the equation $x^4+4x+c=0$ has at most two real roots.

4.5.3 First and Second Derivative Tests

We hope you remember that a function that is increasing or decreasing on an interval I is called **monotonic** on I and we used the test in the theorem bellow to identify whether a function is monotonic or not on a given interval.

Theorem 5: Suppose f is continuous on $[a,b]$ and differentiable on (a,b).

a) If $f'(x)>0$ for all x in (a,b), then f is increasing on $[a,b]$.

b) If $f'(x)<0$ for all x in (a,b), then f is decreasing on $[a,b]$.

Theorem 5 lays the bases for the proof of the first derivative test stated as follows.

Theorem 6: (The First Derivative Test)

Suppose that c is a critical number of a continuous function f.

a) If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.

b) If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.

c) If $f'(x)$ does not change sign at c (that is, $f'(x)$ is positive on both sides of c or negative on both sides of c), then f has no local extremum at c.

Example 6: Find the local extrema of $f(x)=x^{1/3}(8-x)$ and find its interval of monotonocity.

Solution: By the product rule we have

$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}(8-x)-x^{1/3}$$

$$=\frac{8-x-3x}{3x^{2/3}}=\frac{4(2-x)}{3x^{2/3}}$$

The derivative $f'(x)=0$ when $x=2$ more over $f'(x)$ does not exist when $x=0$.

Hence, the critical numbers are $0\ and\ 2$.

Below we will give the sign chart for $f'(x)$

	0	2	
$4(2-x)$	+	+	
$3x^{2/3}$	+	+	+
$f'(x)$	+	+	–

Thus, by first derivative test, f has

- Does not have an extreme value at $x=0$.
- A local maximum value at $x=2$ and the local maximum value is $f(2)=6\sqrt[3]{2}$.

Also, f is

- ✓ Increasing on $(-\infty,0)\cup(0,2]$ and
- ✓ Decreasing on $[2,\infty)$

Theorem 7: (The Second Derivative Test)

Suppose f'' is coninuous on an open inerval of a function.

a) If $f'(x)=0$ and $f''(c)>0$, then f has a local minimum value at c.

b) If $f'(x)=0$ and $f''(c)<0$, then f has a local maximum value at c.

Example 6: Use he Second Derivaive Test to find the relaive extrema of

$$f(x)=3x^4+8x^3+4$$

Solution:

$$f'(x)=12x^3+24x^2=12x^2(x+2)$$

To find the critical point, $f'(x)=12x^2(x+2)=0$

Hence, the critical numbers are $x=0$ *and* $x=-2$.

Also, $f''(x)=36x^2+48x$

Now, $f''(-2)=48>0$ and $f''(0)=0$

Thus, by the second derivative test f has a relative minimum value at $x=-2$ and he relaive minimum value is $f(-2)=-12$.

Since $f''(0)=0$, the second derivaive test fails.

In the last case, $x=0$ could still be a relative maximum, relative minimum or neither; but the Second Derivative Test fails to produce any useful information.

If you used the First Derivative Test, you would find out that $x=0$ is not relative extremum.

Definition : Let f be differentiable on an open inerval I . The gragh of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is deccreasing on the interval.

The following graghical interpretation of concavity is useful.

1. Let f be differentiable at c . If the gragh of f is concave upward at $(c, f(c))$, then the gragh of f lies above the tangent line at $(c, f(c))$ on some open interval containing c . [see figure 4.5a]
2. Let f be differentiable at c . If the gragh of f is concave downward at $(c, f(c))$, then the gragh of f lies below the tangent line at $(c, f(c))$ on some open interval containing c . [see figure 4.5b]

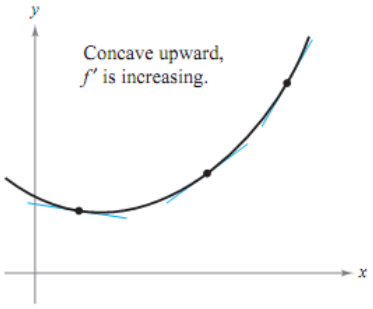


Figure 4.5a

The gragh of f lies above its tangent lines.

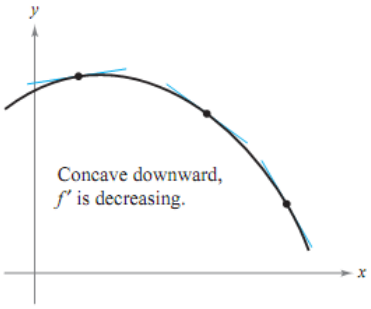


figure 4.5b

The gragh of f lies below its tangent lines

Theorem 7: (The Test for Concavity)

Suppose f is twice differentiable on an interval I.

a) If $f''(x)>0$ for all x in I, then the graph of f is concave upward on I.

b) If $f''(x)<0$ for all x in I, then the graph of f is concave downward on I.

Definition 4: A point $(c, f(c))$ on a curve is called a **point of inflection** if the curve changes from concave upward to concave downward or from concave downward to concave upward at $(c, f(c))$.

Theorem: If $(c, f(c))$ is a point of inflection of the gragh of f, then either $f''(c)=0$ or f'' is undefined at $x=c$.

Example 7: Determine the interval of concavity and the inflection point of the funcion

$$f(x)=x^3-3x+1$$

Solution: $f(x)=x^3-3x+1$, then $f'(x)=3x^2-3=3(x-1)(x+1)$

Since $f'(x)=0$ when $x^2=1$, the criical numbers are $x=\pm 1$. Also

$$f''(x)=6x$$

Hence, $f''(x)>0$ when $x>0$ and $f''(x)<0$ when $x<0$.

Thus, f is concave upward on $(0,\infty)$ and concave downward on $(-\infty,0)$.

Since concavity changes at he point $x=0$, the inflection point of f is $(0, f(0))=(0,1)$.

Example 7: Determine the interval of concavity and the inflection point(s) (if any) of the funcion $f(x)=x^4-4x^3$

Soluion: differentiating f twice we ge

$$f'(x)=4x^3-12x^2$$

$$f''(x)=12x^2-24x$$

$$=12x(x-2)$$

0 2

$12x$	$-----$	$++++$	$++++$
$x-2$	$-----$	$-----$	$++++$
$f''(x)=12x(x-2)$	$++++$	$-----$	$++++$

Thus, - f is concave upward on $(-\infty,0)\cup(2,\infty)$ and

- f is concave downward on $(0,2)$

Since concavity changes at both points $x=0$ *and* $x=2$, the inflection points are $(0, f(0))$ and $(2, f(2))$ which equals $(0,0)$ and $(2,-16)$.

Activity 4.6

1. Find the critical numbers (if any) and the intervals on which the function is increasing or decreasing

   a. $f(x)=(x-3)(x-1)^2$ c. $f(x)=(x+1)^3$

   b. $f(x)=\sqrt{x}(x-3)$ d. $f(x)=\sin x+\cos x;$ *for* $0\le x\le 2\pi$

2. Use the First or the Second Derivative Test to find any relative extrema (if any) of the following functions

   a. $f(x)=\frac{1}{4}x^4-8x$ c. $f(x)=\frac{3}{2}\sin\left(\frac{\pi x}{2}-1\right);$ *on* $[0,4]$

   b. $f(x)=x^3-9x^2+27x$ d. $f(x)=x+\frac{4}{x}$

3. Find the intervals of concavity and the inflection points(if any) of the function

   a) $f(x)=x(x-4)^3$ c) $f(x)=\frac{1}{2}x^2-\ln x$

   b) $f(x)=x^43-12x$ d) $f(x)=x\sqrt{x+3}$

4. Find $a, b, c,$ *and* d such that the cubic polynomial function $f(x)=ax^3+bx^2+cx+d$ satisfies the indicated conditions:

   Relative maximum: $(3,3)$

   Re;ative minimum: $(5,1)$

   Inflection poin: $(3,3)$

4.5.5 Curve Sketching

Now we apply the knowledge that we have developed in this chapter for sketching the graphs of different functions. The table below lists the items that are most important in graphing a function f.

Property	Test
f has y intercept c	$f(0)$=c
f has x intercept c	f©=c
symmetric with respect to the $\begin{cases} y\ axis \\ origin \end{cases}$	$f(-x) = f(x)$ $f(-x) = -f(x)$
f has a relative maximum value at c	$\begin{cases} f'(c) = 0\ and\ f'\ changes\ from\ v+\ to\ v- \\ f'(c) = 0\ and\ f''(c) < 0 \end{cases}$
f has a relative minimum value at c	$\begin{cases} f'(c) = 0\ and\ f'\ changes\ from\ v-\ to\ v+ \\ f'(c) = 0\ and\ f''(c) > 0 \end{cases}$
f is strictly increasing on an open interval I	f'>0 for all except finitely many x in I
f is strictly decreasing on an open interval I	f'<0 for all except finitely many x in I
Graph of f is concave upward on I	$f''(x) < 0 \quad \forall x \in I$
Graph of f is concave downward on I	$f''(x) > 0 \quad \forall x \in I$
(c, f©) is an inflection point	f'' changes sign at c and usually $f''(c) = 0$
f has a vertical asymptote $x = c$	$\lim_{x\to c^-} f(x) = \pm\infty$ *or* $\lim_{x\to c^-} f(x) = \pm\infty$
f has a horizontal asymptote $x = d$	$\lim_{x\to\infty} f(x) = d$ *or* $\lim_{x\to-\infty} f(x) = d$

Example: If $g(x) = \dfrac{x^2}{1-x^2}$, discuss and sketch the graph of g.

Solution:

1. **Analize the $x-intercepts\ and\ y-interceptx-intercepts\ s$**

$x-intercept = (0,0)$ and $x-intercept = (0,0)$

2. **Analize the first derivative.**

$g'(x) = \dfrac{2x}{(1-x^2)^2}$; This has a root at $x=0$. Possible local maximum or minimum here.

Notice that neither $g(x)$ nor its derivative are defined at $x=1\ and\ x=-1$.

The derivative is negative for $x<0$, except at $x=-1$, where it is not defined. It is positive for $x>0$, except at $x=1$, where it is not defined.

That is g is decreasing on $(-\infty,-1)\cup(-1,0)$ and increasing on $(0,1)\cup(1,\infty)$.

3. Analize the second derivative.

$g''(x)=\frac{2+6x^2}{(1-x^2)^3}$ **;** There are no values of x where the second derivative equals zero, so the graph of g has no inflection points.

$g''(0)=2$; At $x=0$, a critical number, the second derivative is positive, so the graph is concave up at this point, and has a local minimum.

4. Find horizontal asymptotes.

$\lim_{x\to\infty}\frac{x^2}{1-x^2}=-1$ and $\lim_{x\to-\infty}\frac{x^2}{1-x^2}=-1$

Hence, $h(x)=-1$ is a horizontal asymptote.

5. Find vertical asymptotes.

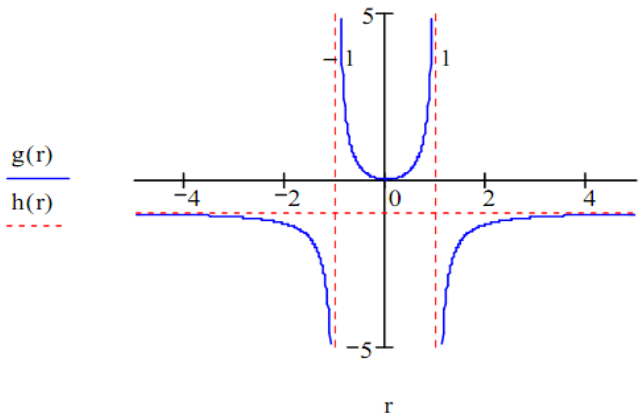
Since g is undefined at $x=1\ and\ x=-1$, examine the limits of g as x approaches these values.

$\lim_{x\to1^+}\frac{x^2}{1-x^2}=-\infty=\lim_{x\to-1^-}\frac{x^2}{1-x^2}$ and $\lim_{x\to1^-}\frac{x^2}{1-x^2}=\infty=\lim_{x\to-1^+}\frac{x^2}{1-x^2}$

Hence, g has a vertical asymptotes at $x=1\ and\ x=-1$.

6. Put it all together.

Range of $r:=-5,-4.99..5$ graghing



Notice that all the aspects of the gragh you found in your analysis are present: a local minimum at $x=0$, vertical asymptotes at $x=1\ and\ x=-1$, a horizontal asymptote at $y=-1$, downward sloping when $x<0$, upward sloping when $x>0$.

Activity 5.4

1. Discuss and sketch the gragh of f if

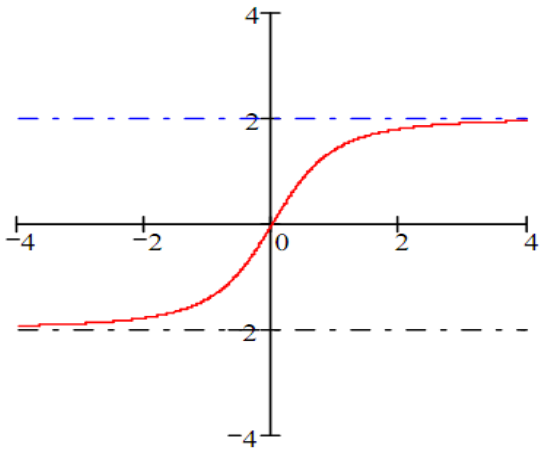
a. $f(x)=\dfrac{x^2}{x^2-x-2}$

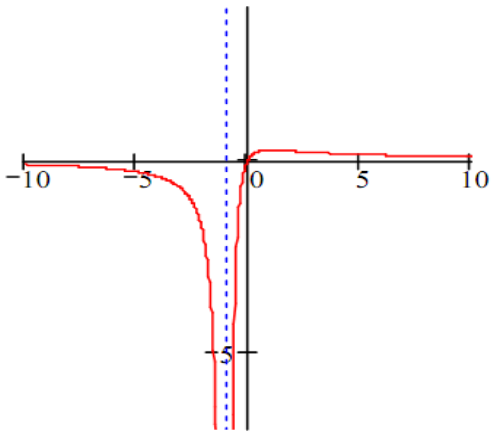
c. $f(x)=\dfrac{2x}{\sqrt{x^2+1}}$

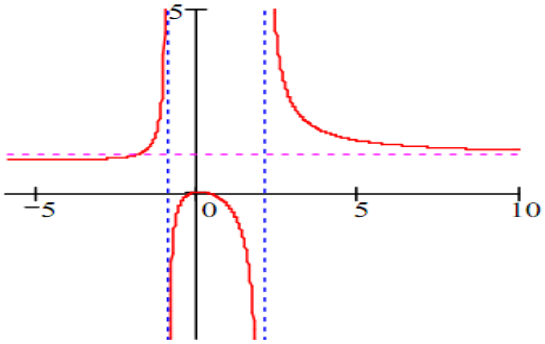
b. $f(x)=\dfrac{x}{(x+1)^2}$

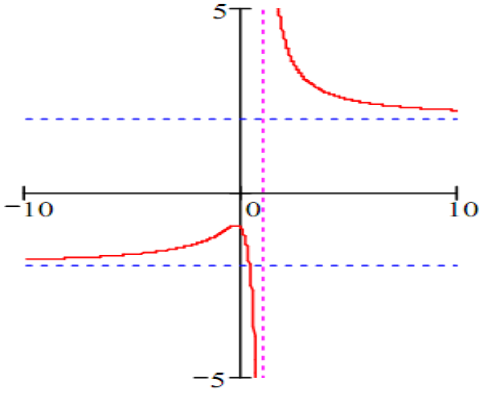
d. $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Note: Your sketches should look like one of the graphs below:









4.6 Indeteminate Forms (L'Hôpital's Rule)

While we study limits in the previous section, we considered limits of quotients such as

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \quad and \quad \lim_{x\to 0}\frac{\sin x}{x}$$

and calculated the limits by using algebraic, geometric, and trigonometric methods even if the limits have the undefined form $0/0$. In this section we develop another technique that employs the derivatives of the numerator and denominator of the quotient. His new technique is called L'Hôpitals rule. For the proof of this rule we need the following generalization of the Mean Value Theorem.

Indeterminate Forms

If $f(x)$ *and* $g(x)$ are two functions, then we know that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then the expression $\frac{f(x)}{g(x)}$ is said to have the indeterminate form $\frac{0}{0}$, at $x=a$.

If $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then the expression $\frac{f(x)}{g(x)}$ is said to have the indeterminate form $\frac{\infty}{\infty}$, at $x=a$.

The other indeterminate forms are $\infty-\infty$, $0\times\infty$, 0^0, 1^∞, ∞^0.

The Indeterminate Form $\frac{0}{0}$.

Theorem 1: (L'Hôpital's rule)

Let $f(x)$ and $g(x)$ be two functions such that

$\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, $f'(a)$ *and* $g'(a)$ exists and $g'(a)\neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}.$$

Proof: Suppose $f(x)$ and $g(x)$ satisfy the conditions of Cauchy's mean value theorem in the interval $[a,x]$. Then we have

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(c)}{g'(c)}, \text{ where } c\in(a,x).$$

Since, $f(a)=0$, $g(a)=0$ and as $x\to a, c\to a$, we get

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}$$

Hence, $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{c\to a} f'(c)}{\lim_{c\to a} g'(c)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$; (replacing c *by* x)

If $f'(a)=g'(a)=0$, then this theorem can be extended as follows:

$$\frac{\lim_{c\to a} f(x))}{\lim_{c\to a} g(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

$$= \lim_{x \to a} \frac{f'''(x)}{g'''(x)}, \text{ if } \quad f''(a) = g''(a) = 0$$

and so on.

Example 1: Evaluate $\lim_{x \to 0} \frac{1 - 3^x}{x}$.

Solution: Both the numerator and denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form $0/0$. By L'Hôpital's rule

$$\lim_{x \to 0} \frac{1 - 3^x}{x} = \lim_{x \to 0} \frac{-3^x \ln 3}{1} = -\ln 3$$

Example 2: Evaluate $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution: Both the numerator and denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form $0/0$. By L'Hôpital's rule

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{\cos x}{6}$$

$$= \frac{1}{6} \quad .$$

Example 3: Evaluate $\lim_{x \to 0} \frac{\ln(1 - x^2)}{\ln(\cos 2x)}$.

Solution: Observe that $\lim_{x \to 0} \ln(1 - x^2) = 0 = \lim_{x \to 0} \ln(\cos 2x)$

Hence the quotient has the indeterminate form $0/0$.

Thus by L'Hôpital's rule we get

$$\lim_{x\to 0}\frac{\ln\left(1-x^2\right)}{\ln(\cos 2x)}=\lim_{x\to 0}\frac{\frac{-2x}{1-x^2}}{-2\tan 2x}=\lim_{x\to 0}\left(\frac{1}{1-x^2}\cdot\frac{-2x}{-2\tan 2x}\right)$$

$$=\lim_{x\to 0}\frac{x}{\tan 2x}\text{, since }\lim_{x\to 0}\frac{1}{1-x^2}=1$$

$$=\lim_{x\to 0}\left[\frac{x}{\sin 2x}.(\cos 2x)\right]$$

$$=\lim_{x\to 0}\frac{x}{\sin 2x}.\lim_{x\to 0}(\cos 2x)$$

$$=\left(\frac{1}{2}\right)(1)=\frac{1}{2}.$$

Example 4: Evaluate $\lim_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{\frac{1}{x}}$.

Solution: Observe that $\lim_{x\to\infty}\left(\frac{\pi}{2}-\arctan x\right)=0=\lim_{x\to 0}\frac{1}{x}$, since $\lim_{x\to\infty}\arctan x=\frac{\pi}{2}$.

Hence the quotient has the indeterminate form $0/0$.

Thus by L'H*ô*pital's rule we get

$$\lim_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{-1}{1+x^2}}{\frac{-1}{x^2}}=\lim_{x\to\infty}\frac{x^2}{1+x^2}=1.$$

Example 5: Evaluate $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{2x-\sin 2x}$.

Solution: Both the numerator and denominator have the limit 0 as $x\to 0$. Hence the quotient has the indeterminate form $0/0$. By L'H*ô*pital's rule

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{2x-\sin 2x}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{2-2\cos 2x}\text{ ,}\qquad\left[\frac{0}{0}\right]\text{ form}$$

$$= \lim_{x\to 0} \frac{e^{x} - e^{-x}}{4\sin 2x}, \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x\to 0} \frac{e^{x} + e^{-x}}{8\cos 2x}, \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \frac{2}{8} = \frac{1}{4}.$$

Example 6: Evaluate $\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

Solution: Observe that $\lim_{x\to\frac{\pi}{4}} \left(\sec^2 x - 2\tan x\right) = 0 = \lim_{x\to\frac{\pi}{4}} (1+\cos 4x)$.

Hence the quotient has the indeterminate form $0/0$.

Thus by L'Hôpital's rule we get

$$\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x} = \lim_{x\to\frac{\pi}{4}} \frac{2\sec^2 x\tan x - 2\sec^2 x}{-4\sin 4x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2 x(\tan x - 1)}{2\sin 4x}, \qquad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x\to\frac{\pi}{4}} -\left\{\frac{\sec^2 x\cdot\sec^2 x + (\tan x - 1)2\sec^2 x\tan x}{8\cos 4x}\right\}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sec^4\frac{\pi}{4} + 0}{8\cos\pi}$$

$$= \frac{-4}{-8} = \frac{1}{2}.$$

The Indeterminate Form $\frac{\infty}{\infty}$.

Theorem 2: (L'Hôpital's rule)

Let $f(x)$ and $g(x)$ be two functions such that

$\lim_{x\to a} f(x) = \infty \text{ or } -\infty$ and $\lim_{x\to a} g(x) = \infty \text{ or } -\infty$, $f'(a)$ *and* $g'(a)$ exists and $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}.$$

Example 1: Evaluate $\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4\tan x}{1+\sec x}$.

Solution: Observe that the limit has the inditerminate form ∞/∞. Then by L'H*ô*pital's rule we have

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to\left(\pi/2\right)^-}\frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to\left(\pi/2\right)^-}\frac{4\sec x}{\tan x}$$

By using the trigonometric identity, we have

$$\frac{4\sec x}{\tan x} = \frac{4/\cos x}{\sin x/\cos x} = \frac{4}{\sin x}$$

Consequently

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to\left(\pi/2\right)^-}\frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to\left(\pi/2\right)^-}\frac{4\sec x}{\tan x} = \lim_{x\to\left(\pi/2\right)}\frac{4}{\sin x} = 4.$$

Example 2: Evaluate $\lim_{x\to\infty}\frac{e^{2x}}{x^2}$.

Solution: Observe that the limit has the inditerminate form ∞/∞. Then by L'H*ô*pital's rule we have

$$\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x}, \quad \left[\frac{\infty}{\infty}\right] \text{ form}$$

$$= \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{4e^{2x}}{2} = \lim_{x\to\infty}2e^{2x} = \infty.$$

Particularly in a similar fashion we can show that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$$, for every real number n.

Other Indeterminate Forms

Various inditerminate forms, such as $\infty-\infty$, $0\times\infty$, 0^0, 1^∞ *and* ∞^0, can usually be converted into the indeterminate form $0/0$ *or* ∞/∞ and then evaluaed by one of the versions of L'H*ô*pital's rule given in the above theorem.

Example 1: Evaluate $\lim_{x\to0^+}x^2\ln x$.

Solution: Since $\lim_{x\to0^+}x^2=0$ and $\lim_{x\to0^+}\ln x=-\infty$ the given limit is of the form $0\times\infty$ (more precisely, $0\times(-\infty)$). However, we can transform it into the inditerminate form ∞/∞ by writing it as

$$\lim_{x\to0^+}x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}$$

Now, applying the L'H*ô*pital's rule we get

$$\lim_{x\to0^+}x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}=\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}-\frac{x^2}{2}=0\cdot$$

Example 2: Evaluate $\lim_{x\to0^+}x^x$.

Solution: The limit evidently has the indeterminate form 0^0. But then since $x^x=e^{x\ln x}$ and consequently

$$\lim_{x\to0^+}x^x=\lim_{x\to0^+}e^{x\ln x}$$

Since the exponential function is continuous, it follows that

$$\lim_{x\to0^+}e^{x\ln x}=e^{\lim_{x\to0^+}(x\ln x)}$$

But since

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

By L'H*ô*pital's rule we get

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$$

Example 3: Show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Solution: Observe that the limit has the indetrminate form 1^∞

Since, $\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x}$, we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x}$$

But $\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+1/x\right)}{1/x}$, $\left[\frac{0}{0}\right]$ form

Thus by L'H*ô*pital's rule we get

$$= \lim_{x\to\infty}\frac{\frac{1}{(1+1/x)}\left(-\frac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{1}{1+1/x} = 1$$

Thus, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$.

Example 4: Find $\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$.

Solution: Observe that the limit has the indetrminate form $\infty-\infty$.

However, if the difference is written as a single fraction, then

$$\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}, \quad \left[\frac{0}{0}\right] \text{ form}$$

Thus by L'Hôpital's rule we get

$$\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}=\lim_{x\to 0^+}\frac{1-e^x}{xe^x+e^x-1}$$

$$=\lim_{x\to 0^+}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.$$

Hence, $\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=-\frac{1}{2}.$

Activity 4.7

1. Evaluate the following limits

a. $\lim_{x\to -1}\frac{x^6-1}{x^4-1}$

b. $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

c. $\lim_{x\to\infty}\left(\frac{x^6-1}{x^n-1}\right)$

d. $\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)$

e. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

f. $\lim_{x\to\infty}\frac{\tan\left(1/x\right)}{1/x}$

g. $\lim_{x\to -\infty}xe^x$

h. $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

i. $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

j. $\lim_{x\to 0}\frac{\sin x-x}{x^3}$

k. $\lim_{x\to 0}\frac{\arctan(2x)}{3x}$

l. $\lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)$

m. $\lim_{x\to\infty}x^2\left(1-x\sin\frac{1}{3}\right)$

n. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

o. $\lim_{x\to 0}\left(\frac{e^x-1-x-\left(x^2/2\right)}{x^3}\right)$

p. $\lim_{x\to 0}\left(\frac{e^{2x}-(1+x)^2}{x\ln(1|+x)}\right)$

2. If $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}$. Then find the values of a *and* b.

3. Why is the following application of L'Hôpital's rule invalid?

$$\frac{1}{\pi/2}=\lim_{x\to\pi/2}\frac{\sin x}{x}=\lim_{x\to\pi/2}\frac{\cos x}{1}=0.$$

Self Test Exercises

1. Find the derivative of the following functions by using the definition of derivative

   a) $f(x)=\sqrt{x}+1$ c) $f(x)=x^2-4x+5$

   b) $f(x)=\frac{6}{x}$ d) $f(x)=c0sx-\sin x$

2. Find the derivative of the following functions by using the definition of derivative at the given point c

   a) $f(x)=\frac{1}{x+4}$, *at* $c=3$ b) $f(x)=x-\sqrt{2x}$, *at* $c=8$

3. Find the derivative of the following functions

   a) $f(x)=\sqrt{x}\sin x$ c) $f(x)=\ln x\left(\frac{x-1}{x-2}\right)$

   b) $f(x)=\frac{1+\sin x}{1-\sin x}$ d) $f(x)=\frac{\cos(x-1)}{x-1}$

4. Find an equation of the tangent line to the graph of f at the given point

   a) $f(x)=\frac{x+1}{x-1}$, *at* $\left(\frac{1}{2},-3\right)$ b) $f(x)=-x\tan x$, *at* $(0,0)$

5. Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ by implicit differentiation

   a) $\sqrt{xy}=x-4y$ c) $x\sin y=y\cos x$

   b) $ye^x+xe^y=xy$ d) $x^2+3xy+y^3=10$

6. Find an equation of the tangent line to the graph of the following functions at the given point

   a) $y\ln x+y^2=0, at\ (e,-1)$ b) $x^2-y^2=20, at\ (6,4)$

7. Find a formula for the n^{th} derivative of the following functions

   a) $f(x)=2^x$ b) $f(x)=\dfrac{x-1}{x+1}$

8. Find the absolute extrema of the following functions on the given interval

   a) $f(x)=2x+5\cos x,[0,2\pi]$ b) $f(x)=\dfrac{x}{\sqrt{x^2+1}},[0,2]$ c) $f(x)=x^2+5x,[-4,0]$

9. Find the value of the number c guaranteed by the Mean Value Theorem of the following functions on the given interval

   a) $f(x)=(x-2)(x+3)^2,[-3,2]$ c) $f(x)=-\cos x,\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

   b) $f(x)=2x^2-3x+1,[0,4]$ d) $f(x)=x^{2/3},[1,8]$

10. Derermine the values $a,b,c,and\,d$ such that the function f satisfies the hypothesis of the Mean Value Theorem on the interval $[-1,2]$.

$$f(x)=\begin{cases} a, & x=-1 \\ 2, & -1<x\le 0 \\ bx^2+c, & 0<x\le 1 \\ dx+4, & 1<x\le 2 \end{cases}$$

11. Find the interval of monotonocity of the following functions

    a) $f(x)=x^2-2x-8$ b) $f(x)=x\sqrt{16-x^2}$ c) $f(x)=x+\dfrac{4}{x}$

12. Use first or second derivative test to find all the relative extrema of the following functions

    a) $f(x)=x-4\sqrt{x+1}$ d) $f(x)=x-2\cos x$, $on\ [0,4\pi]$

    b) $f(x)=2x^2(x^2-1)$ e) $f(x)=x^3-20x^2+128x-280$

    c) $f(x)=-x^4+4x^3+8x^2$ f) $f(x)=x+\dfrac{4}{x}$

13. Find the interval of concavity and the inflection point (if it exists) of the following functions

a) $f(x)=x^2-x-2$ c) $f(x)=\dfrac{x^2-1}{2x-1}$

b) $f(x)=x^5+5x^4-40x^2$ d) $f(x)=-x^3+6x^2+12x$

14. Sketch the graph of the following functions

a) $f(x)=\dfrac{1}{x-2}-3$ b) $f(x)=\dfrac{x^2-6x+12}{x-4}$ c) $f(x)=x^4-4x^3+16x$

15. Sketch the graph of a function f satisfying the following conditions:

$f(2)=f(4)=0$, $f'(x)>0$ *if* $x<3$, $f'(3)$ *does not exist*,

$f'(x)<0$ *if* $x>3$, $f''(x)>0$ *if* $x\neq 3$

16. Determine whether the following function has an inverse or not.

a) $f(x)=2-x-x^3$ c) $f(x)=|x+4|-|x-4|$

b) $f(x)=\sqrt[3]{x+1}$ d) $f(x)=\sin\left(3x/2\right)$

1. Check the existence of the inverse of the following functions. If it has, find its inverse

a) $f(x)=ax+b,\ b\neq 0$ c) $f(x)=-3$

b) $f(x)=\sqrt{x-2}$ d) $f(x)=|x-2|,\ x\leq 2$

17. Simplify the expression

a) $\csc\left(arc\cot(-\sqrt{3})\right)$ c) $\sec h(\ln\sqrt{2})$

b) $\sin(2\arcsin x)$ d) $\sinh^{-1}\left(\dfrac{4}{3}\right)$

18. Let $F(x)=f(2g(x))$ where $f(x)=x^4+x^3+1$ for $0\leq x\leq 2$, and $g(x)=f^{-1}(x)$. Find $F(3)$

19. Find $\left(f^{-1}\right)(c)$ if

a) $f(x)=x+\sqrt{x},\ c=2$ b) $f(x)=x\ln x,\ c=2e^2$

20. Differentiate the following functions

a) $f(x)=\sec h\sqrt{\ln x}$ c) $f(x)=\sinh^{-1}\left(-3x^2\right)$

b) $f(x)=\arctan\left(\frac{x+1}{x-1}\right)$ d) $f(x)=arc\sec(\ln x)$

21. Let $y=\frac{\sinh^{-1}x}{\sqrt{1+x^2}}$. Show that $\left(1+x^2\right)\frac{dy}{dx}+xy=1$.

22. Evaluate the following limits

a) $\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ f) $\lim_{x\to 0}\left(\frac{1-\cos x}{1-\sec^2 x}\right)$

b) $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ g) $\lim_{x\to 0}\left(\frac{x-\sin x}{x-\tan x}\right)$

c) $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}$ h) $\lim_{x\to 0}\left(\frac{\sec x-1}{x\sin x}\right)$

d) $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$ i) $\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\ln\sin x}$

e) $\lim_{x\to 0^+}\left(1-\frac{4}{x}\right)^{2x-1}$ j) $\lim_{x\to 4}\left(\frac{\sin^2(\pi x)}{e^{x-4}+3-x}\right)$

CHAPTER FIVE

Integration

Introduction

Integration is not as straight forward as differentiation; there are no rules that absolutely guarantee obtaining indefinite integrals of a function. In this chapter we develop techniques for using the basic integration formulas to obtain indefinite integrals of more complicated functions. Therefore we discuss a strategy for integration in this section. Also we consider integrals having an unbounded limit of integration and integrals of discontinuous functions on an interval.

Chapter Objectives

At the end of this chapter the readers should be able to:

- Know basic integration formulas of an indefinite integral.
- Find the definite and indefinite integral of different functions using different techniques of integrations.
- Define the improper integral of a function.
- Evaluate an improper integral that has an infinite limit of integration.
- Evaluate an improper integral that has an infinite discontinuity.
- Evaluate improper integrals of different functions.
- Find the area of a region between two curves using integration.
- Find the volume of a solid with known cross-sections.
- Find the volume of a solid of revolution.

5.1 Ant derivatives; indefinite Integrals

Before we see techniques of integration let us revise the integrals of important functions in the following table.

Derivative	Indefinite integral
$D_x(x)=1$	1. $\int 1dx=\int dx=x+c$
$D_x\left(\frac{x^{r+1}}{r+1}\right)=x^r\ (r\neq -1)$	2. $\int x^r dx=\frac{x^{r+1}}{r+1}+c\ \ (r\neq -1)$
$D_x(\sin x)=\cos x$	3. $\int \cos x dx=\sin x+c$
$D_x(-\cos x)=\sin x$	4. $\int \sin x dx=-\cos x+c$
$D_x(\tan x)=\sec^2 x$	5. $\int \sec^2 x dx=\tan x+c$
$D_x(-\cot x)=\csc^2 x$	6. $\int \csc^2 x dx=-\cot x+c$
$D_x(\sec x)=\sec x\tan x$	7. $\int \sec x\tan x dx=\sec x+c$
$D_x(-\csc x)=\csc x\cot x$	8. $\int \csc x\cot x dx=-\csc x+c$
$D_x(e^x)=e^x$	9. $\int e^x dx=e^x+c$
$D_x\left(\frac{a^x}{\ln a}\right)=a^x$	10. $\int a^x dx=\frac{a^x}{\ln a}+c$
$D_x(\ln\lvert x\rvert)=\frac{1}{x}$	11. $\int \frac{1}{x}dx=\ln\lvert x\rvert+c$
$D_x(\sin^{-1}\frac{x}{a})=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}D_x(\frac{x}{a})=\frac{1}{\sqrt{a^2-x^2}}$	12. $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$
$D_x(\cos^{-1}\frac{x}{a})=-\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}D_x(\frac{x}{a})=-\frac{1}{\sqrt{a^2-x^2}}$	13. $\int \frac{1}{\sqrt{a^2-x^2}}dx=-\cos^{-1}\frac{x}{a}+c$
$D_x(\frac{1}{a}\tan^{-1}\frac{x}{a})=(\frac{1}{a})\frac{1}{1+(\frac{x}{a})^2}D_x(\frac{x}{a})=\frac{1}{a^2+x^2}$	14. $\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$
$D_x\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right)=\frac{1}{x\sqrt{x^2-a^2}}$	15. $\int \frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\frac{x}{a}+c$

Table 1.0

I hope the student does not forget how to evaluate the definite integral by using the following fundamental theorem of calculus:

Theorem 5.1 (Fundamental theorem of calculus)

Suppose f is continuous on a closed interval $[a, b]$.

Part I: If the function G is defined by

$$G(x) = \int_a^x f(t)dt$$

for every x in $[a, b]$, then G is an antiderivative of f on $[a, b]$.

Part II: If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 1 Evaluate $\int_{-2}^{3}(6x^2 - 5)dx$.

Solution: An antiderivative of $6x^2 - 5$ *is* $F(x) = 2x^3 - 5x$ .Then

$$\int_{-2}^{3}(6x^2 - 5)dx = 2x^3 - 5x\Big|_{-2}^{3}$$

$$= [(3)^3 - 5(3)] - [2(-2)^3 - 5(-2)] = 45$$

5.2 Techniques of Integration

5.2.1 Integration by Substitution

The formulas for indefinite integrals in Table (1.0) are limited in scope, because we cannot use them directly to evaluate such as

$$\int \sqrt{3x-5}dx \ \ or \ \int \cos(5x)dx$$

In this section we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (1.0).

Method of Substitution

If the integral to be evaluated is of the form

$$\int f(g(x)g'(x)dx$$

We substitute $u = g(x)\ and\ du = g'(x)dx,$ then the integral becomes $\int f(u)du$

Example 1 Evaluate $\int\sqrt{2x-5}dx$.

Solution: We let u=2x-5 and du=2dx

Since du contains the factor 2, the integral is not in the proper form $\int f(u)du$ required in the method of substitution given above. However, we can introduce the factor 2 into the integrand, provided we also multiply by $\frac{1}{2}$. Doing this and property of integral we have

$$\begin{aligned}\int\sqrt{2x-5}dx &= \int\sqrt{2x-5}\frac{1}{2}2dx\\ &= \frac{1}{2}\int\sqrt{2x-5}2dx\\ &= \frac{1}{2}\int\sqrt{u}du\\ &= \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c\\ &= \frac{1}{3}u^{\frac{3}{2}}+c\\ &= \frac{1}{3}(\sqrt{2x-5})^{\frac{3}{2}}+c\end{aligned}$$

Example 2 Evaluate $\int \sin(2x)dx$

Solution: We make the substitution

u=2x,du=2dx

Since du contains the factor 2, we adjust the integrand by multiplying by 2 and compensate by multiplying the integral by $\frac{1}{2}$ before substituting:

$$\int \sin(2x)\,dx = \frac{1}{2}\int \sin(2x)\,2dx$$

$$= \frac{1}{2}\int sinudu$$

$$= \frac{1}{2}(-cosu) + c$$

$$= -\frac{1}{2}\cos(2x) + c$$

It is not always easy to decide what substitution u = g(x) is needed to transform an indefinite integral into a form that can be readily evaluated. It may be necessary to try several different possibilities before finding a suitable substitution. In most cases no substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for changing variables in indefinite integrals:

1. Decide on a reasonable substitution u = g(x).
2. Calculate $du = g'(x)dx$.
3. Using 1 and 2, try to transform the integral into a form that involves only the variable u. If necessary, introduce a constant factor **k** into the integrand and compensate by **1/k**. If any part of the resulting integrand contains the variable **x**, use a different substitution in 1.

4. Evaluate the integral obtained in 3, obtaining an antiderivative involving **u**.
5. Replace u in the antiderivative obtained in guideline 4 by g(x). The final result should Contain only the variable x.

The following examples illustrate the use of the guidelines.

Example 3 Evaluate $\int x^2(3x^3+2)^{10}dx$.

Solution: If an integrand involves an expression raised to a power, such as $(3x^3+2)^{10}$,

we often substitute u for the expression. Thus, we let

$$u = 3x^2 + 2,\ du = 9x^2dx \Leftrightarrow \frac{1}{9}du = x^2dx$$

Comparing $du = 9xdx$ with x^2 dx in the integral suggests that we introduce the factor 9 into the integrand. Doing this and compensating by multiplying the integral by 1/9, we obtain the following:

$$\int x^2(3x^3+2)^{10}dx = \int u^{10}\frac{1}{9}du$$

$$= \frac{1}{9}\int u^{10}du$$

$$= \frac{1}{9}(\frac{u^{11}}{11}) + C$$

$$= \frac{1}{99}\left(3x^3+2\right)^{11} + C$$

Example 4 Evaluate $\int x\sqrt{3x-1}dx$.

Solution: To simplify the expression $\sqrt{3x-1}$,we let

u=3x-1, so that du=3dx

Then $$\int x\sqrt{3x-1}dx = \int x\overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du}$$

Thus we still need to find x in terms of u. From the equation u=3x –1 we deduce that

$$x = \frac{1}{3}(u+1)$$

Therefore

$$\int x\sqrt{3x-1}dx = \int x\overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du} = \int \frac{(u+1)}{3}\sqrt{u}\,\frac{du}{3}$$

$$= \frac{1}{9}\int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{9}\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{9}\left(\frac{2}{5}(3x-1)^{\frac{5}{2}} + \frac{2}{3}(3x-1)^{\frac{3}{2}}\right) + C$$

Example 5 Evaluate $\int xe^{x^2}dx$.

Solution: We let

$$u = x^2, \quad du = 2xdx \Rightarrow \frac{du}{2} = xdx \text{ .Then}$$

$$\int xe^{x^2}dx = \int e^{x^2}xdx = \int e^u\,\frac{du}{2}$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^x + C$$

Example 6 Evaluate $\int \sec x dx$.

Solution: We first put the integral in the form

$$\int \sec x dx = \int \sec x \frac{(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

If we now let $u = sex + \tan x,\ du = (\sec x \tan x + \sec^2 x)dx$.Then

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Activity 5.1

Evaluate the following integrals.

1. $\int \sin^2 x dx$
2. $\int \csc x dx$
3. $\int \frac{x}{(x^2+5)^5} dx$
4. $\int \frac{1}{\sqrt[3]{1-2x^2}} dx$
5. $\int x^5 \sqrt{x^2 - 1} dx$
6. $\int_{-1}^{2} \frac{t}{\sqrt{t+2}} dt$
7. $\int_{1}^{2} \frac{e^{3/x}}{x^2} dx$
8. $\int \frac{1}{x(\ln x)^2} dx$
9. $\int \frac{3\sin x}{1+2\cos x} dx$
10. $\int_{0}^{2\sqrt{2}} \frac{x}{\sqrt{1-x^4}} dx$

5.2.2 Integration by Parts

If we try to evaluate integrals of the type

$$\int xe^x dx \text{ and } \int \ln x dx$$

by using the method of substitution we obviously fail. But don't worry the next formula will enable us to evaluate not only these, but also many other types of integrals.

Integration by Parts Formula

If $u = f(x)$ *and* $v = g(x)$ *and if* f' *and* g' are continuous ,then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int udv = uv - \int udv$$

Proof: By the product rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Or equivalently, $f(x)g'(x) = [f(x)g(x)] - g(x)f'(x)$

Integrating both sides of the last equation gives us

$$\int f(x)g'(x)dx = \int [f(x)g(x)]'dx - \int f(x)g'(x)dx$$

The first integral on the right side equals f(x)g(x)+c. Since another constant of integration

is obtained from the second integral, we may omit c in the formula; that is

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad \text{…….. (1)}$$

Since $dv = g'(x)dx \text{ and } du = f'(x)dx$ we may write the preceding formula as

$$\int udv = uv - \int udv$$

Since applying (1) involves splitting the integrand into two parts, the use of (1) is referred to as integrating by parts. A proper choice for dv is crucial. We usually let dv equal the most complicated part of the integrand that can be readily integrated. The following examples illustrate this method of integration by parts

Example 1 Evaluate $\int xe^x dx$.

Solution: The integrand xe^x can be split into two parts x and e^x . We let

$$u = x \quad and \quad dv = e^x dx$$

Then $du = x \;\; and \;\; v = \int e^x dx = e^x$

Consequently integration by parts yields

$$\int \overbrace{x}^{u}\overbrace{e^x dx}^{dv} = \overbrace{x}^{u}\overbrace{e^x}^{v} - \int \overbrace{e^x}^{v}\overbrace{dx}^{du} = xe^x - e^x + C$$

Example 2 Evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin(2x)dx.$

Solution: Let $u = x^2 \; and \; dv = \sin 2x dx$

Then, $du = 2xdx \;\; and \;\; v = \frac{-1}{2}\cos 2x$

Thus using integration by parts we have;

$$\int_0^{\frac{\pi}{2}} x^2 \sin(2x)d = \left[\frac{-1}{2}x^2 \cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\left(\frac{-1}{2}\cos 2x\right)dx$$

$$= \left[\frac{-1}{2}x^2 \cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\cos 2x dx$$

But, since

$$\left[\frac{-1}{2}x^2 \cos 2x\right]_0^{\frac{\pi}{2}} = \frac{-1}{2}\left(\frac{\pi}{2}\right)^2 \cos 2(\frac{\pi}{2}) - 0 = \frac{\pi^2}{8}$$

And

$$\int_0^{\frac{\pi}{2}} x\cos 2x dx = \left[x\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\sin 2(\frac{\pi}{2}) - 0 \right] - \frac{1}{2}\left[-\frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\cos 2(\frac{\pi}{2}) - \cos 0 \right] = \frac{1}{4}[-1-1] = \frac{-1}{2}$$

Hence,

$$\int_0^{\frac{\pi}{2}} x^2 \sin(2x) dx = \frac{\pi^2}{8} - \frac{1}{2}$$

Example 3 Evaluate $\int e^x \cos x dx$.

Solution: We could either let $dv = \cos x dx \; or \; let \; dv = e^x dx$,since each of these expression is readily integrable. Let us choose

$$u = e^x \; and \; dv = \cos x dx$$

So that $du = e^x dx \; and \; v = \sin x$

Then by integrating by parts we have;

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \qquad (1)$$

We next apply integration by parts to the integral of the right side of equation (1). Since we chose a trigonometric form for dv in the first integration by parts, we shall also choose a trigonometric form for the second. Letting

$$u = e^x \; and \quad dv = \sin x dx \; sothat$$

$$du = e^x dx \quad and v = -\cos x$$

Integrating by parts, we have

$$\int e^x \sin x dx = e^x(-\cos x) - \int (-\cos x)e^x dx$$
$$= -e^x \cos x + \int \cos x e^x dx \qquad (2)$$

If we now use equation (2) to substitute on the right side of equation (1), we obtain

$$\int e^x \cos x dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x dx]$$
$$\int e^x \cos x dx = e^x \sin x + e^x \cos x + \int e^x \cos x dx$$

Adding $\int e^x \cos x dx$ to both sides of the last equation gives us

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$

Finally, dividing both sides by 2 and adding the constant of integration yields

$$\int e^x \cos x dx = \frac{1}{2}[e^x \sin x + e^x \cos x] + C$$

We could have evaluated the given integral by using $dv = e^x dx$ for both the first and second applications of the integration by parts formula.

In conclusion we remark that integration by parts is effective with integrals involving a polynomial and either an exponential, a logarithmic, or a trigonometric function. More

Specifically, integration by parts is especially well adapted to integrals of the form

$\int (polynomial) \sin ax dx$, $\int (polynomial) \cos ax df$

$\int (polynomial) e^{ax} dx$ $\int (polynomial) \ln x dx$

In all except $\int (polynomial) \ln x dx$, the most effective choice of u is the polynomial, since the derivatives of a polynomial are simpler than other function, while the choice u=lnx is effective for $\int (polynomial) \ln x dx$.

Example 4 Evaluate $\int \sin^{-1} x\,dx$.

Solution: Let

$$u = \sin^{-1} x \;\; and \;\; dv = dx \; so that \; du = \frac{1}{\sqrt{1-x^2}}dx \; and \; v = x$$

Then

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

Now we use substitution to solve the integral to the right. That is let

$$w = \sqrt{1-x^2} \;\; or \;\; w^2 = 1-x^2 \;\; so that \; 2wdw = -2xdx$$

We then have

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{wdw}{w} = -\int dw = -w + C = -\sqrt{1-x^2} + C$$

Consequently

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$

Integration by parts may sometimes be employed to obtain reduction formulas for integrals. We now find reduction formulas of $\int \sin^n x\,dx$ $\int \cos^n x\,dx$ with the help of integration by parts.

Example 5 Find a reduction formula for $\int \sin^n x dx$.

Solution: First write $\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$

and let $u = \sin^{n-1} x \;\; and \; dv = \sin x dx \; so that$

$du = (n-1)\sin^{n-2} x \cos x dx \;\; and \; v = -\cos x$

Then using integration by parts we have:

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we may write

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

Consequently,

$$\int \sin^n x dx + (n-1)\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

The left side of the last equation reduces to n $\int \sin^n x dx$. Dividing both sides by n, we obtain

$$\int \sin^n x dx = -\frac{coxs \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$

In a similar fashion we can show the reduction formula for $\int \cos^n x dx$ is given by:

$$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Example 8 Evaluate $\int \sin^5 x dx$.

Solution: Using the reduction formula for sine with n = 5 gives us

$$\int \sin^5 x dx = -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\int \cos^3 x dx$$

A second application of the reduction formula, to $\int \sin^3 x dx$, yields

$$\begin{aligned}\int \sin^3 x dx &= -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x dx \\ &= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C\end{aligned}$$

Consequently

$$\int \sin^5 x dx = -\frac{1}{5}\cos x \sin^4 x - \frac{4}{15}\cos x \sin^2 x - \frac{8}{15}\cos x + C$$

Activity 5.2

Evaluate the following integerals.

1. $\int xe^{-x}dx$
2. $\int x\ln xdx$
3. $\int \sec^3 xdx$
4. $\int x2^x dx$
5. $\int x\tan x\sec xdx$
6. $\int_0^{\frac{\pi}{2}} 2t\sin 2tdt$
7. $\int (x+1)^{10}(x+2)dx$
8. $\int \sin(\ln x)dx; (H\,\text{int}: Let\, u = \sin(\ln x))$
9. $\int \tan^{-1} xdx$
10. $\int \cos(\sqrt{x})dx$

Evalute the integral with the help of the reduction formulas.

11. $\int \cos^3(\frac{x}{2})dx$
12. $\int \cos^7 xdx$

5.2.3 Integration by Partial Fractions

An expression for rational function is called a proper fraction if the degree of the numerator is strictly less than the degree of the denominator; otherwise it is called an improper fraction. In case of improper fraction we actually divide the numerator by the denominator and the improper fraction is expressed in terms of a polynomial and a proper fraction. For example,

$$\frac{2x-1}{x-3} = 2+\frac{7}{x-3} \quad and \quad \frac{4x^3-3x^2+2x-1}{x^2+9} = 4x-3-\frac{34x-26}{x^2+9}$$

Let us consider a proper fraction $\frac{P(x)}{Q(x)}$ where P and Q are polynomials in x, then it can

be proved that

$$\frac{P(x)}{Q(x)} = F_1 + F_2 + ... + F_n$$

Such that each term F_n of the sum has one of the forms

$$\frac{A}{(ax+b)^n} \; or \; \frac{Ax+B}{(ax+bx+c)^n}$$

for real numbers A and B and a nonnegative integer n, where ax^2+bx+c is irreducible

in the sense that this quadratic polynomial has no real zeros (that is, $b^2-4ac<0$) .In this case, ax^2+bx+c cannot be expressed as a product of two first-degree polynomials with real coefficients.

The sum $F_1 + F_2 + ... + F_k$ is the partial fraction decomposition of $\frac{P(x)}{Q(x)}$,and each F_k is a partial fraction. We state guidelines for obtaining this decomposition.

Guide lines for Partial Fraction Decompositions of P(X)/Q(X)

1. If the degree of P(x) is not lower than the degree of Q(x), use long division to obtain the proper form
2. Express Q(x) as a product of linear factors ax + b or irreducible quadratic factors $ax^2 + bx + c$, and collect repeated factors so that Q(x) is a product of different factors of the form $(ax+)^n \, or \, (ax^2 + bx + c)^n$ for nonnegative integer n.

3. Apply the following rules.

Rule a : For each factor $(ax+)^n$ with n ≥ 1, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}$$

where each numerator A_n is a real number.

Rule b: For each factor $(ax^2 + bx + c)^n$ with n $n \geq 1$,and with $ax^2 + bx + c$ irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

where each A_n and B_n is a real number.

Example: Evaluate $\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx$

Solution: We may factor the denominator of the integrand as follows:

$$x^3 + 2x^2 - 3x = x(x^2 + 2x + 3) = x(x+3)(x-1)$$

Each factor has the form stated in Rule (a) of the guideline, with n = 1. Therefore the partial fraction decomposition has the form

$$\frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} = \frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

Multiplying by the LCM of the denominators gives us

$$4x^2+13x-9=A(x+3)(x-1)+Bx(x-1)+Cx(x+3). \qquad (1)$$

In a case such as this, in which the factors are linear and non-repeated, the values of A, B and C can be found by substituting values for x that make the various factors zero. If we let x = 0 in (1), then

$$-9=-3A, or\ \ A=3$$

Letting x=1in (1) gives us

8=4C, or C=2

Finally, if x=-3 in (1), we have

-12=12B, or B=-1

The partial fraction decomposition is, therefore,

$$\frac{4x^2+13x-9}{x^3+2x^2-3x}=\frac{3}{x}+\frac{-1}{x+3}+\frac{2}{x-1}$$

Integrating and letting C denote the sum of the constants of integration we have

$$\begin{aligned}\int\frac{4x^2+13x-9}{x^3+2x^2-3x}dx&=\int\frac{3}{x}dx+\int\frac{-1}{x+3}dx+\int\frac{2}{x-1}dx\\&=3\ln|x|-\ln|x+3|+2\ln|x-1|+C\\&=\ln|x\ |^3-\ln|x+3|+\ln|x-1|^2+C\\&=\ln\left|\frac{x^3(x-1)^2}{x-1}\right|+C\end{aligned}$$

Another technique for finding A, B, and C is to expand the right-hand side of (1) and collect like powers of x as follows:

$$4x^2+13x-9=(A+B+C)x^2+(2A-B+3C)x-3A$$

We now use the fact that if two polynomials are equal, then coefficients of like powers of x are the same. It is convenient to arrange our work in the following way, which we call comparing coefficients of x.

Coefficients of x^2:A+B+C=4

Coefficients of x: 2A-B+3C=13

Constant term: -3A=-9

We may show the solution of this system of equations is A = 3, B = -1, and C = 2.

Example 2: Evaluate $\int \frac{13-7x}{(x+2)(x-1)^3}dx$

Solution: By Rule (a) of the Guidelines the partial fraction of the integrand has the form

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x+2)}$$

Multiplying both sides by $(x+2)(x-1)^3$ gives us

$$13-7x=A(x+2)(x-1)^2+B(x-1)(x+2)+C(x+2)+D(x-1)^3 \qquad (2)$$

Two of the unknown constants may be determined easily as follows.

Let x = 1 in (2) the we obtain

13-7=3C or C=2

Similarly, letting x = -2 in (2) yields,

13+14=D(-2-1)^3 or D=-1

The remaining constants may be found by comparing coefficients. So comparing the coefficients of x^3on both sides of (2), gives

0=A+D or A=-D ,A=1

And comparing the constant terms on both sides of (2), gives

13=2A-3B+2C-D or B=$\frac{1}{2}(2+4+1-13)=-3$

Therefore

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{1}{x-1}+\frac{-3}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{-1}{x+2}$$

Thus

$$\int \frac{13-7x}{(x+2)(x-1)^3}dx = \int \frac{1}{x-1}dx + \int \frac{-3}{(x-1)^2}dx + \int \frac{2}{(x-1)^3}dx + \int \frac{-1}{x+2}dx$$

$$= \ln|x-1| + \frac{3}{x-1} - \frac{1}{(x-1)^2} + \ln|x+2| + C$$

$$= \ln\left|\frac{x-1}{x+2}\right| + \frac{3}{x-1} - \frac{1}{(x-1)^2} + C$$

Example 3 Evaluate $\int \frac{x^2+2x+7}{x^3+x^2-2}dx$

Solution: The denominator of the integrand may be factored as follows:

$$x^3 + x^2 - 2 = (x-1)(x^2+2x+2)$$

Applying Rule (b) of the Guidelines to the irreducible quadratic factor x^2+2x+2 we have

$$\frac{x^2+2x+7}{x^3+x^2-2} = \frac{A}{X-1} + \frac{Bx+C}{x^2+2x+2}$$

This leads to

$$x^2+2x+7 = A(x^2+2x+2) + (Bx+C)(x-1) \quad \text{…….. (3)}$$

As in previous examples, substituting x=1 in (3) gives us

10 = A(5) or A = 2

The remaining constants may be found by combining like powers of x:

$$x^3+2x+7 = (2+B)x^2 + (4+C-B)x + (4-C) \quad \text{…….. (4)}$$

And comparing cofficients in (4)

Coefficients of x^2: 1=2+B or B=-1

Constants terms: 7=4-C orC=-3

Thus the partial fraction decomposition of the integrand is

$$\frac{x^2+2x+7}{x^3+x^2-2}=\frac{2}{x-1}+\frac{-x-3}{x^2+2x+2}$$

Consequently

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx+\int\frac{-x-3}{x^2+2x+2}dx$$
$$=2\ln|x-1|-\int\frac{x+3}{x^2+2x+2}dx$$

To evaluate the right-hand integral, we first complete the square in the denominator to obtain

$$x^2+2x+2=(x+1)^2+1$$

and substitute u=x+1 so that du = dx and x +3 = u + 2

Therefore

$$\int\frac{x+3}{x^2+2x+2}dx=\int\frac{x+3}{(x+1)^2+1}dx=\int\frac{u+2}{u^2+1}du$$
$$=\int\frac{u}{u^2+1}du+\int\frac{2}{u^2+1}du$$
$$=\frac{1}{2}\ln(u^2+1)+2\arctan u+C$$
$$=\frac{1}{2}\ln(x^2+2x+2)+2\arctan(x+1)+C$$

Hence

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx+\int\frac{-x-3}{x^2+2x+2}dx$$
$$=2\ln|x-1|-[\frac{1}{2}\ln(x^2+2x+2)]-2\arctan(x-1)+C$$

Example 4 Evaluate $\int\frac{dx}{\sin x(2+\cos^2 x)}$.

Solution: Since

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin^2 x(2+\cos^2 x)}$$

Substuting

u=cosx and du=-sinsdx,we get

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin^2 x(2+\cos^2 x)} = \int \frac{-du}{(1-u^2)(2+u^2)} = \int \frac{du}{(u^2-1)(u^2+2)}$$

But then the partial fraction representation for the integrand of the last integral has the

Form

$$\frac{1}{(u^2-1)(u^2+2)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+2}$$

Then by similar procedure as the above examples we have

$$1 = A(u+1)(u^2+2) + B(u-1)(u^2+2) + (Cu+D)(u-1)(u+1) \qquad (4)$$

Then putting u=1 gives us 1=6A or A=1/6

Putting u=-1 gives us 1=-6B or B=-1/6

We now compare coefficients to find the remaining two constants

Coefficients of x^3: 0=A+B+C or C=0

Constant terms: 1=2A- 2B-D or D=-1/3

Therefore

$$\frac{1}{(u^2-1)(u^2+2)} = \frac{\frac{1}{6}}{u-1} + \frac{\frac{-1}{6}}{u+1} + \frac{\frac{-1}{3}}{u^2+2}$$

$$\int \frac{du}{(u^2-1)(u^2+2)} = \frac{1}{6}\int \frac{du}{u-1} - \frac{1}{6}\int \frac{du}{u+1} - \frac{1}{3}\int \frac{du}{u^2+2}$$

$$= \frac{1}{6}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C$$

Consequently resubstituting cosx for u we have

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \frac{1}{6}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{\cos x}{\sqrt{2}} + C$$

Activity 5.3

1. $\int \frac{x^2}{x^2-1}dx$
2. $\int \frac{2x^2-12x+4}{x^3-4x^2}dx$
3. $\int_{-1}^{0} \frac{x^2+x+1}{x^2+1}dx$
4. $\int \frac{-x^3+x^2+x3}{(x+1)(x^2+1)^2}dx$
5. $\int \frac{x^2-1}{x^3+3x+4}dx$
6. $\int \frac{\sqrt{x}+1}{x+1}dx$
7. $\int \frac{\sin^2 x\cos x}{\sin^2 x+1}dx$
8. $\int_{0}^{\frac{\pi}{4}} \tan^3 x dx$; $(H\,\text{int}: sustuteu = \tan x)$
9. $\int \frac{e^x}{1-e^{3x}}dx$
10. $\int \frac{dx}{1+3e^x+2e^{2x}}$

5.2.4 Trigonometric Integrals

Integrals such as

$$\int \sin^5 x\cos^3 x dx,\ \int \tan^2 x\sec^3 x dx,\ and\ \int \sin 3x\cos 4x dx$$

are called trigonometric integrals because their integrands are combinations of trigonometric functions. This section is devoted to trigonometric integrals especially those in which the integrands are composed of the basic trigonometric functions.

Guidelines for evaluating integrals of the form $\int \sin^m x\cos^n dx$

1. **If m is an odd integer**: Write the integrals as $\int \sin^m x\cos^n dx = \int \sin^{m-1}\cos^n x\sin x dx$ and express $\sin^{m-1} x$ in terms of cosx by using the trigonometric identity $\sin^2 x = 1-\cos^2 x$. Make the substitution

   u=cosx, du=-sinxdx

   and evaluate the resulting integeral.
2. **If n is an odd integer**: write the integral as

$$\int \sin^m x\cos^n x dx = \int \sin^m x\cos^{n-1} x\cos x dx$$

And express $\cos^{n-1} x$ in terms of sinx by using the trigonometric identity $\cos^2 x = 1-\sin^2 x$ .Make the sustition

u=sinx, du=cosxdx

and evaluate the resulting integral.

3. **If m and n are even**: Use half-angle formulas for

$\sin^2 x = \frac{1-\cos 2x}{2}$ *and* $\cos^2 x = \frac{1+\cos 2x}{2}$ and the identity

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

to reduce the exponents by one-half.

Example 1: Evaluate $\int \sin^3 x \cos^2 x dx$

Solution: By guideline 1

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\ &= \int (1-\cos^2 x)\cos^2 x \sin x dx \\ &= \int (\cos^2 x - \cos^4 x)\sin x dx\end{aligned}$$

If we let u=cosx,then du=-sinxdx, and the integeral may be written

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int (1-u^2)u^2(-du) = \int (u^4 - u^2)du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C\end{aligned}$$

Example 2: Evaluate $\int \sin^2 x \cos^4 x dx$.

Solution: By guideline 3 we have

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int (\sin^2 x \cos^2 x)\cos^2 x dx \\ &= \int (\sin x \cos x)^2 \cos^2 x dx \\ &= \int (\frac{1}{2}\sin 2x)^2 (\frac{1+\cos 2x}{2})dx \\ &= \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cos 2x dx\end{aligned}$$

Putting $\sin^2 2x = \frac{1-\cos 4x}{2}$ and u=sin2x so that du=2cos2xdx in the first and second integerals of the right of the the last equation we get:

$$= \frac{1}{8}\int \frac{1-\cos 4x}{2}dx + \frac{1}{8}\int \frac{1}{2}u^2 du$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}u^3 + C$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

An alternative way to evaluate $\int \sin^m x \cos^n dx$ when m and n are even is to use the identity, $\sin^2 x + \cos^2 x = 1$ but this time we transform the integral into integrals of the form $\int \sin^k x dx$ or of the form $\int cox^k x dx$ which can be evaluated by the reduction formulas.

Guidelines for evaluating integrals of the form $\int \tan^m x \sec^n x dx$

1. **If m is an odd integer**: Write the integrals as

$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x dx$ *and* $\exp ressed \tan^{m-1} x$ in terms of x secx by using the trigonometric identity $\tan^2 x = \sec^2 x - 1$. Make the substitution

u=secx, du=secxtanxdx

and evaluate the resulting integral.

2. **If n is an even integer**: write the integral as

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x \sec^2 x dx$ and express $\sec^{n-2} x$ in terms of tanx by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$ .Make the substittion

u=tanx, du=sec^{2}xdx

and evaluate the resulting integral.

3. **If m is even and n is odd:** Reduce to powers of x sec alone by using the identity $\sec^2 x - 1 = \tan^2 x$

Example 3 Evaluate $\int \tan^3 x \sec^5 x dx$.

Solution: By guideline 1 above

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x (\sec x \tan x) dx$$
$$= \int (\sec^2 x - 1)\sec^4 x (\sec x \tan x) dx$$

Substituting u=secx and du=secxtanxdx,we obtain

$$\int \tan^3 x \sec^5 x dx = \int (u^2 - 1)u^4 du$$
$$= \int (u^6 - u^4) du$$
$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$
$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

Example 4 Evaluate $\int \tan^3 x \sec^4 x dx$.

Solution: By guideline 2 above

$$\int \tan^3 x \sec^4 x dx = \int \tan^3 x \sec^2 x \sec^2 x dx$$
$$= \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx$$

If we let u=tanx,then du=$\sec^2$xdx,and

$$\int \tan^3 x \sec^4 x dx = \int u^3 (1 + u^2) du$$
$$= \int (u^5 + u^3) du$$
$$= \frac{1}{6} u^6 + \frac{1}{4} u^4 + C$$
$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C$$

Integerals of the form $\int \cot^m x \csc^n x dx$may be evaluated in similar fashion.

Finally, the evaluation of integrals of the form $\int \sin ax \cos bx dx$epends on the trigonometric identity

$$\sin x \cos y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$$

With the appropriate replacements, this identity becomes

$$\sin ax \cos bx = \frac{1}{2} \sin(a - b)x + \frac{1}{2} \sin(a + b)x \qquad (*)$$

Notice that $\frac{1}{2} \sin(a - b)x$ *and* $\frac{1}{2} \sin(a + b)x$ are easy to integrate by substitution.

Example 5 Evaluate $\int \sin 4x \cos 2x dx$.

Solution: Using (*) with a =4 and b = 2, we find that

$$\int \sin 4x \cos 2x dx = \int \left(\frac{1}{2}\sin 2x + \frac{1}{2}\sin 6x \right) dx$$
$$= \frac{-1}{4}\cos 2x - \frac{1}{12}\sin 6x + C$$

Note that integrals of the form

$$\int \sin ax \sin bx dx \quad and \quad \int \cos ax \cos bx dx$$

can be found by similar techniques

Activity 5.4

Evaluate the following integral

1. $\int \sin^3 x \cos^4 x dx$
2. $\int (\tan x + \cot x)^2 dx$
3. $\int \sqrt{\sin x} \cos^3 x dx$
4. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^5 x dx$
5. $\int \tan^3 x \csc^4 x dx$
6. $\int \cot^3 x \csc^3 x dx$
7. $\int \sin^5 x \cos^2 x dx$
8. $\int_0^{\frac{\pi}{4}} \cos x \cos 5x dx$

5.2.5 Trigonometric Substitutions

Observe that the trigonometric substitution $x = \sin\theta$ simplifies the expression $\sqrt{a^2 - x^2}, with\ a > 0,$ into a trigonometric expression without radical i.e

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a\sqrt{1 - \sin^2 \theta} = a\cos\theta$$

We can use a similar procedure for $\sqrt{a^2 + x^2}, and\ \sqrt{x^2 - a^2}$ .This technique is useful for eliminating radicals from these types of integrands. The substitutions are listed in the table 1.1.

When making a trigonometric substitution we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution x=asinθ, we have $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$, In this case, $\cos\theta \ge 0$.

Trigonometric Substitutions Table

Expressions in integrand	Trigonometric substitution	Interval(s)
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2,$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2,$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta < \pi/2,\ or\ \pi \le \theta < \frac{3\pi}{2}$

Table1.1

Example 1 Evalute $\int \frac{1}{x^2\sqrt{16-x^2}}dx.$

Solution: Since $\sqrt{16-x^2} = \sqrt{4^2 - x^2}$, we substute

$$x = 4\sin\theta, so that\ dx = 4\cos\theta d\theta,\ for \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Then

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \int \frac{1}{16\sin^2\theta 4\sqrt{1-\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta$$
$$= -\frac{1}{16}\cot\theta + C$$

In order to write the answer in terms of the original variable x, we draw the triangle as fig1.1, in which x=4 sinθ .

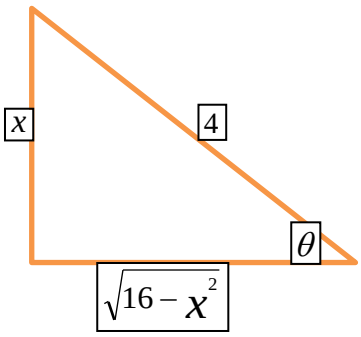


Fig 5.1

Hence, $\cot\theta = \dfrac{\sqrt{16-x^2}}{x}$

Thus

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \frac{-1}{16}\cot\theta = \frac{-\sqrt{16-x^2}}{16x} + C$$

Example 2 Evaluate $\int \dfrac{1}{x^2\sqrt{x^2+16}}dx$.

Solution: The denominator of the integrand has an expression of the form $\sqrt{a^2+x^2}$ with a=4. Hence, using table 1.1, we make the substitution

$$x = 4\tan\theta, dx = 4\sec^2\theta d\theta$$

Consequently

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sqrt{1+\tan^2\theta} = 4\sqrt{\sec^2\theta} = 4\sec\theta$$

And

$$\int \frac{1}{x^2\sqrt{x^2+16}}dx = \int \frac{1}{16\tan^2\theta(4\sec\theta)}4\sec^2\theta d\theta$$
$$= \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$
$$= \frac{-1}{16\sin\theta}$$

To give the answer in terms of x, we use the triangle in Fig 1.2, with $x=4\tan\theta$ .We then

find that

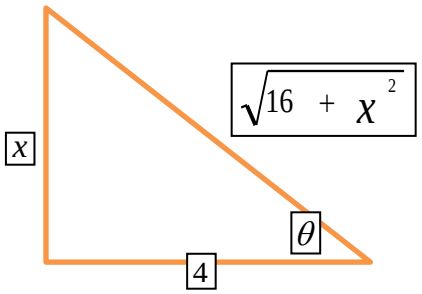


Fig 5.2

Hence, $\sin\theta = \dfrac{x}{\sqrt{16+x^2}}$

Therefore

$$\int \frac{1}{x^2\sqrt{x^2+16}}dx = -\frac{1}{16\sin\theta} = \frac{-\sqrt{16+x^2}}{16x} + C$$

Example 3 Evaluate $\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}dx.$

Solution: The domain of the integrand consists of $(-\infty,3]\ and\ [3,\infty)$ but since the interval over which we must integrate is [-6,-3], we seek an antiderivative whose domain is contained in $(-\infty,3]$ .Since $\sqrt{x^2-9} = \sqrt{x^2 = 3^2}$

$$x = 3\sec\theta, so that\ dx = \sec\theta\tan\theta d\theta$$

And notice that $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$ .For the limits of integration we observe that

$$x = -6, then\ \theta = \sec^{-1}(-2) = \frac{4\pi}{3}\ and\ if\ x = -3, then\ \theta = \pi$$

Therefore

$$\int_{-6}^{-3}\frac{\sqrt{x^2-9}}{x}dx=\int_{\frac{4\pi}{3}}^{\pi}\frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}9(3\sec\theta\tan\theta)d\theta$$

$$=\int_{\frac{4\pi}{3}}^{\pi}\frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)d\theta=3\int_{\frac{4\pi}{3}}^{\pi}\tan^2\theta d\theta$$

$$=3\int_{\frac{4\pi}{3}}^{\pi s}(\sec^2\theta-1)d\theta$$

$$=\left[3\tan\theta-\theta\right]_{4\pi/3}^{\pi}$$

$$=\pi-3\sqrt{3}$$

Integrals containing $\sqrt{bx^2+cx+d}$

By completing the square in bx^2+cx+d we can express $\sqrt{bx^2+cx+d}$ in terms of $\sqrt{a^2+x^2}, and\ \sqrt{x^2-a^2}\ or \sqrt{a^2-x^2}$ for suitable a > 0. Then a trigonometric substitution eliminates the square root as before.

Example 5 Evaluate $\int\frac{1}{\sqrt{x^2+8x+25}}dx.$

Solution: We complete the square for the quadratic expression as follows:

$$x^2+8x+25=(x^2+8x+16)+25-16$$

$$=(x+4)^2+9$$

Thus,

$$\int\frac{1}{\sqrt{x^2+8x+25}}dx=\int\frac{1}{\sqrt{(x+4)^2+9}}dx$$

If we make the trigonometric substitution

$$x+4=3\tan\theta, dx=3\sec^2\theta d\theta$$

Then

$$\sqrt{(x+4)^2+9}=\sqrt{9\tan^2\theta+9}=3\sqrt{\tan^2+1}=3\sec\theta$$

And

$$\int\frac{1}{\sqrt{x^2+8x+25}}=\int\frac{3\sec^2\theta}{3\sec\theta}d\theta$$
$$=\int\sec\theta d\theta$$
$$=\ln\left|\sec\theta+\tan\theta\right|+C$$

Using our formulas for $\tan\theta$ *and* $\sec\theta$,we conclude that

$$\int\frac{1}{\sqrt{x^2+8x+25}}=\ln\left|\frac{\sqrt{x^2+8x+25}}{3}+\frac{x+4}{3}\right|+C$$

5.3 Improper Integrals

The definite integral $\int_a^b f(x)dx$ has meaning only when f is continuous on $[a,b]$ consequently bounded on $[a,b]$.

Definition: We say f is bounded on an interval I if there is a constant M such that $|f(x)|\le M$ for all x in I.

In this section, we shall extend the definition of the definite integral when either the integrand or the inerval of integraion is unbounded. Such integrals are called **improper integrals.**

Integrals Over Unbounded Intervals

Definition: If f is continuous on $[a,\infty)$, then the improper integral $\int_a^\infty f(x)dx$ **converges** if

$\lim_{t\to\infty}\int_a^t f(x)dx$ exists. In that case

$$\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If the limit does not exist, the improper integral **diverges.**

Again if f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_{t}^{a} f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Provided the limit exists.

Example 1: Determine whether the integral converges or diverges, and if it converges find its value.

a. $\int_{0}^{\infty}\frac{1}{(x+1)^2}dx$ b. $\int_{-\infty}^{0}\frac{1}{1-x}dx$

Solution: (a) Following the discussion above and equation (1) we have

$$\int_{0}^{\infty}\frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\int_{0}^{t}\frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\left[-\frac{1}{x+1}\right]_{0}^{t}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{t+1}+\frac{1}{0+1}\right] = 0+1=1.$$

Thus, the improper integral converges and has the value 1.

(b) Following the discussion above and equation (2) we have

$$\int_{-\infty}^{0}\frac{1}{1-x}dx = \lim_{t\to-\infty}\int_{t}^{0}\frac{1}{1-x}dx = \lim_{t\to-\infty}\left[-\ln(1-x)\right]_{t}^{0}$$

$$= \lim_{t\to-\infty}\left[-\ln(1-t)+\ln(1-0)\right] = -\infty+0=-\infty.$$

Thus, since the limit does not exist, the improper integral diverges.

Example 2: Determine whether the integral $\int_{-\infty}^{1} e^{x}dx$ converges or diverges, and if it converges find its value.

Solution: As in **example 1**,

$$\int_{-\infty}^{1} e^{x}dx = \lim_{t\to-\infty}\int_{t}^{1} e^{x}dx = \lim_{t\to-\infty}\left[e^{x}\right]_{t}^{1}$$

$$= \lim_{t\to-\infty}\left[e^{1}-e^{t}\right] = e-0=e.$$

Thus, the improper integral converges and has the value e.

Finally, for integrals over the range $(-\infty,\infty)$, we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Provided both of the improper integrals on the right converge.

If either of the integrals on the right in (3) diverges, then $\int_{-\infty}^{\infty} f(x)dx$ is said to **diverge**. It can be shown that (3) does not depend on the choice of the real number a.

Example 3: Determine whether the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ converges or diverges, and if it converges find its value.

Solution: Using (3), with $a=0$, we have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx.$$

Next, applying (2) we get

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}dx + \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}dx = \lim_{t\to-\infty}(\arctan x)_{t}^{0} + \lim_{t\to\infty}[\arctan x]_{0}^{t}$$

$$= \lim_{t\to-\infty}[\arctan 0 - \arctan t] + \lim_{t\to\infty}[\arctan t - \arctan o]$$

$$= \left[0-(_-\frac{\pi}{2})\right] + \left[\frac{\pi}{2}-0\right] = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

Thus, the given improper integral converges and has the value π.

Integrals with Unbounded Integrands

We now consider a function f that is continuous at every point in $(a,b]$ and unbounded near a. By assumption f is continuous on the interval $(t,b]$ for t in (a,b), so that $\int_{t}^{b} f(x)dx$ is defined for such t. If the one-sided limit

$$\lim_{t\to a^{+}}\int_{t}^{b} f(x)dx$$

exists, then we define $\int_a^b f(x)dx$ to be the limit. This idea leads us the following definitions:

Definition:

i. If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Provided the limit exists.

ii. If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^a f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Provided the limit exists.

As the preceding section, the integrals defined in (4) and (5) are reffered to as improper integrals and they converge if the limit exists. The limits are called the values of the improper integrals. If the limits does not exist, the improper integral diverge.

Example 4: Evaluate $\int_1^2 \frac{1}{\sqrt{2-x}}dx$.

Solution: Since the integrand has an infinite discontinuity at $x=2$, we apply (4) and have

$$\int_1^2 \frac{1}{\sqrt{2-x}}dx = \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt{2-x}}dx = \lim_{t\to 2^-}\left[-2\sqrt{2-x}\right]_1^t$$

$$= \lim_{t\to 2^-}\left[-2\sqrt{2-t}-(-2)\sqrt{2-1}\right] = 2$$

Thus, the given improper integral converges and has the value 2.

Example 5: Determine whether the improper integral $\int_0^1 \frac{1}{x}dx$ converges or diverges.

Solution: The integrand is unbounded near 0. Applying (5) gives us

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \lim_{t\to 0^+}\left[\ln 1 - \ln t\right] = \infty.$$

Consequently the given improper integral diverges, since the limit does not exist.

We give the definition of another improper integral as follows:

Definition: If f has a discontinuity at a number c in the open interval (a,b) but continuous elsewhere on $[a,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Provided both of the improper integrals on the right converge. If both converge, then the value of the improper integral $\int_a^b f(x)dx$ is the sum of the two values.

Example 6: Determine whether the improper integral $\int_0^4 \frac{1}{(x-3)^2}dx$ converges or diverges.

Solution: The integrand is undefined at $x=3$. Since this number is in the interval $(0,4)$, we use (6), with $c=3$:

$$\int_0^4 \frac{1}{(x-3)^2}dx = \int_0^3 \frac{1}{(x-3)^2}dx + \int_3^4 \frac{1}{(x-3)^2}dx.$$

For the integral on the left to converge, both integrals on the right must converge.

However, since

$$\int_0^3 \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-} -\left[\frac{1}{x-3}\right]_0^t$$

$$= \lim_{t\to 3^-}\left[-\frac{1}{t-3} - \frac{1}{3}\right] = \infty.$$

Thus, the given improper integral diverges.

The other kind of improper integral is found if f is continuous in (a,b) and is unbounded near both a and b. We say that $\int_a^b f(x)dx$ **converges** if for some point c in (a,b) both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. Otherwise we say that the integral is divergent.

Example 7: Determine whether $\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}dx$ converges or diverges.

Solution: The integrand is unbounded near both the end points 0 *and* 1 and is coninuous on $(0,1)$. Consequently the integral is of the type under consideretion. If we let $c=\frac{3}{4}$, then we need to analyze the convergence of

$$\int_0^{3/4}\frac{1-2x}{\sqrt{x-x^2}}dx \text{ and } \int_{3/4}^{1}\frac{1-2x}{\sqrt{x-x^2}}dx$$

For $0<t<\frac{3}{4}$, we have

$$\int_0^{3/4}\frac{1-2x}{\sqrt{x-x^2}}dx=\lim_{t\to 0^+}\int_t^{3/4}\frac{1-2x}{\sqrt{x-x^2}}dx=\lim_{t\to 0^+}\left[2\sqrt{x-x^2}\right]_t^{3/4}$$

$$=\lim_{t\to 0^+}\left[2\left(\sqrt{\frac{3}{16}}-\sqrt{t-t^2}\right)\right]=\frac{\sqrt{3}}{2}.$$

A similar computation shows the second improper integral also converges and that

$$\int_{3/4}^{1}\frac{1-2x}{\sqrt{x-x^2}}dx=-\frac{\sqrt{3}}{2}.$$

Therefore the original integral converges, and

$$\int_0^{1}\frac{1-2x}{\sqrt{x-x^2}}dx=\int_0^{3/4}\frac{1-2x}{\sqrt{x-x^2}}dx+\int_{3/4}^{1}\frac{1-2x}{\sqrt{x-x^2}}dx=\frac{\sqrt{3}}{2}+\left(-\frac{\sqrt{3}}{2}\right)=0.$$

Activity 5.5

1. Determine whether the following improper integral converges or diverges, and if it converges find its value.

   a. $\int_0^{\infty}\frac{x}{1+x^2}dx$

   b. $\int_{-\infty}^{0}\frac{1}{(x+3)^2}dx$

   c. $\int_1^{\infty}\frac{1}{\sqrt{x^2-1}}dx$

   f. $\int_0^{\pi/2}\sec^2 x\,dx$

   g. $\int_{-2}^{0}\frac{1}{\sqrt{4-x^2}}dx$

   h. $\int_0^{\pi}\sec x\,dx$

d. $\int_{-\infty}^{\infty} xe^{-x^2}dx$

e. $\int_{0}^{9} \frac{1}{\sqrt{x}}dx$

i. $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}dx$

j. $\int_{0}^{1} \frac{3x^2-1}{x^3-x}dx$

Self Test Exercises

1. Evaluate the following indefinite integrals

a) $\int x^3 e^{2x}dx$

b) $\int \frac{x}{\sqrt{x^2+1}}dx$

c) $\int \cos^5 x \sin x dx$

d) $\int x^2 \sin^2 x dx$

e) $\int \frac{x}{\sqrt{x^2+6x+12}}dx$

f) $\int e^x \sqrt{1-e^{2x}}dx$

g) $\int (x+1)\sqrt{x^2+2x+2}dx$

h) $\int \frac{1}{x\sqrt{4x^2+9}}dx$

i) $\int \frac{x^2-1}{x^3+1}dx$

j) $\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}dx$

k) $\int \frac{x}{(a+bx)^2}dx$

l) $\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}dx$

m) $\int -\frac{1}{\sqrt{1-(4x+1)^2}}dx$

n) $\int (\ln x)^2 dx$

2. Evaluate the following definite integrals.

a) $\int_{1}^{e} \frac{1-\ln x}{x}dx$

b) $\int_{0}^{8} \frac{2x}{\sqrt{x^2+36}}dx$

c) $\int_{0}^{2/\sqrt{3}} \frac{1}{4+9x^2}dx$

d) $\int_{1}^{2} \sqrt{x}\ln x dx$

e) $\int_{2}^{4} x \operatorname{arcsec} x dx$

f) $\int_{3}^{6} \frac{\sqrt{x^2-9}}{x^2}dx$

3. Determine whether the following improper integral converges or diverges. Evaluate the integral if it converges.

a) $\int_0^{\infty} \frac{e^x}{1+e^x} dx$

b) $\int_0^{e} \ln x^2 dx$

c) $\int_0^{5} \frac{1}{\sqrt{25-x^2}} dx$

d) $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$

e) $\int_{-\infty}^{0} xe^{-4x} dx$

f) $\int_{-1}^{4} \frac{1}{\sqrt{|x|}} dx$