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Synchronization in Networks of Nonlinear Circuits

Essential Topics with MatLab code



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Preface

In the past two decades, a great interest has been devoted to the study of complex networks and phenomena appearing in them, as the properties of many real natural, social, and technological systems are intrinsically determined by the structure of interconnections characterizing them. This book focuses on synchronization, a particular behavior observed in complex networks made of coupled nonlinear circuits and systems. The subject of our study is mathematically described by a set of ordinary differential equations that may represent the dimensionless equations of a large class of continuous-time dynamical systems, such as electronic circuits, stations of a power grid, mechanical, chemical systems, biological oscillators, and many others. These systems, when coupled together, may display a collective state where all of them follow the same *synchronized* behavior. Beyond this, synchronization in complex networks may appear with a rich repertoire of distinct forms, including the onset of distinct clusters of synchronized units and the coexistence of coherent and incoherent regions.

The aim of this book is to provide a concise reference for the study of these different forms of synchronization in complex networks of nonlinear dynamical systems. On the contrary, this book does not aim to exhaustively cover many of the topics, such as synchronization in chaotic circuits, synchronization of Kuramoto oscillators, structure of complex networks, that are related to the main subject, but would require much more detailed discussion than that possible in this brief. The interested reader is referred to the many excellent books and reviews covering these topics.

The book is intended for young researchers and students in the field and, for this reason, we have decided to illustrate only the basic ideas and concepts, through examples and simple MATLAB[®] codes to reproduce the results discussed. The book does not exhaustively cover the subject but should help the reader to have an overview on it and practical examples and codes to begin with. An important issue related to synchronization is its control, and for this reason, our considerations on the analysis of the distinct forms of synchronization are followed by the illustration of the techniques for controlling them.

The book is organized as follows. Chapter 1 provides a short introduction on synchronization and complex networks. Chapter 2 discusses the state where all the units of the networks are synchronized, while the other forms of synchronization are dealt with in the next chapters. In particular, Chap. 3 focuses on chimera states, Chap. 4 on cluster synchronization and Chap. 5 on remote and relay synchronization. In Chap. 6, examples of global synchronization and chimera states in time-varying networks are illustrated. Chapter 7 gives some conclusive remarks.

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Chapter 1

Introduction

In this chapter, introductory concepts on synchronization and complex networks are given. We provide basic notions on synchronization of pairs of nonlinear systems and on complex networks, and define the differential equations representing the dynamics of a nonlinear system. These are the ingredients needed for the study of synchronization in complex networks that will be dealt with in the next chapters.

1.1 Synchronization

The term synchronization refers to the simultaneous occurrence of an event in two (or more) systems that interact in some way [30, 35]. Even more, two systems may work in such a way that all their activity takes place at the same time. Consider, for instance, two nonlinear oscillators and a physical quantity (e.g., a voltage signal) characterizing their behavior. If these electronic circuits are independent of each other, then in general the two signals have an uncorrelated behavior with peaks in their waveform occurring at different times and rates. Now suppose that these systems interact through a simple circuitry coupling them, i.e., a resistor placed between two capacitors, one belonging to one circuit and the other one to the second oscillator. Very often this form of interaction is able to induce synchronization in the electronic circuits, either with the two signals displaying exactly the same waveform or with two different trends with peaks occurring at the same time. The two situations represent two instances of the phenomenon of synchronization.

There are many natural, social, and technological systems that display synchronization [14, 20, 24, 35, 36]. Crickets singing at unison, fireflies flashing with a common rhythm, or a human audience clapping at the same rate are a few examples encountered in nature, but synchronization is also ubiquitous in technological systems. The electric generators are connected to form the so-called power grid where all the units operate at the same frequency, i.e., 50 Hz in Europe and 60 Hz in U.S.;

many electronic devices operate with a clock signal devoted to synchronize all their activities so that they occur in synchrony with a common reference time; and so on. The phenomenon of synchronization is so common that one naturally asks the question of what is the reason of this ubiquity. Two examples can help us to provide some insights to this question. Consider again the power grid; in this case synchronization represents the normal operating condition of the system, and failure of it results in malfunctioning, faults, and blackouts at different scales [22]. Instead, in fireflies the flashing is a sexual attraction signal that becomes stronger when produced by many animals operating at unison [35]. Thus, in the first example, synchronization guarantees a proper behavior of the whole system, and in the second case it allows the system to operate in a better way.

This so common occurrence of the phenomenon has attracted a lot of interest and many studies have been carried out elucidating the way in which it manifests, the mechanisms leading to it, the mathematical models able to explain it and many other aspects of the phenomenon. Reviewing all these issues is beyond the scope of this book, and the reader is addressed to the many excellent books and reviews on the topic for a deep study on synchronization (see for instance [2, 3, 6, 30, 35]). On the contrary, this book will focus on a specific framework, synchronization in networks of dynamical units, to provide an introduction to the subject with worked examples. The idea is to illustrate a few basic concepts with the help of examples that the reader can easily reproduce in her/his own computer. For this reason, the theoretical information is limited to few essential results (and probably many fundamental ones are not discussed, only for brevity) which are however illustrated by the mathematical model and the corresponding code to numerically simulate it. To this aim, the authors have decided to use MATLAB®; two reasons have driven this choice, the diffusion of the program and the easy of generalization of the code.

In this book, we study synchronization in networks formed by many continuous-time nonlinear systems coupled each other, but first we introduce a few notions on synchronization of two systems. In fact, synchronization has been historically introduced by investigating first this scenario and then extending it to the case of more than two coupled units. In addition, this helps to simplify the illustration of phenomena which also appear in the complex network scenario.

The first report on synchronization is historically attributed to Christiaan Huygens who studied the motion of two pendulum clocks anchored to the same beam supported by two chairs [30]. He noticed that, after about half an hour from the beginning of oscillations, the two pendula were swinging with the same rhythm and opposite phase, i.e., when one reached the top point of its trajectory on one side, the other was at the top of the other side. He also noticed that the same behavior occurred for different initial conditions of the two pendula. What made possible this behavior was the interaction between the two pendula, that occurs through the motion of the wooden beam. Remarkably, synchronization did not require a strong interaction, but a weak interaction was enough to elicit a synchronous behavior. This conclusion does not hold only for the Huygens' experiment but, on the contrary, is much more general and constitutes one of the distinguishing traits of synchronization.

A modern version of the Huygens' experiment has been recently proposed in a paper by Ramirez and colleagues [31], who realized a new setup with two monumental clocks ad hoc designed and built, modeled it and performed an accurate theoretical and experimental study (in addition, they made available the setup to the large public by installing it in a clocks museum). They observed a synchronous (sympathetic, according to the term used by Huygens) motion of the two pendula, oscillating with the same rate and direction. In agreement with their model, they have found that the difference with the Huygens' finding (opposite vs. same phase) is due to the different damping in the two setups. Ramirez and colleagues also noticed that the frequency of oscillation, during synchronized motion, decreases and the clocks become inaccurate.

Another interesting experiment demonstrating that synchronization requires a weak interaction is the one discussed by James Pantaleone in his work [27]. The experiment consists of two metronomes, that are inverted pendula, placed on a moving base. Pantaleone demonstrated with experimental measurements, corroborated by the analysis of a mathematical model of the setup, that it is the small motion of the base (due to the oscillations of the pendula) at the origin of synchronization. He observed in-phase synchronization and antiphase for a limited set of experimental conditions. The experiment, with two or more units, is particularly easy to reproduce and many videos made by scholars or enthusiasts can be found on internet.

A significant contribution to the development of the study of synchronization came from chaos theory [29, 32, 37]. Due to their sensitivity to initial conditions, two chaotic systems evolve following trajectories which diverge each other in time, even if the two systems are identical, as it is impossible to avoid even a small difference in their starting point. However, it was realized that certain applications (such as the transmission of signals through chaotic carriers) require the synchronization between two chaotic systems (in the mentioned case, the synchronization between the chaotic circuit acting as transmitter and the one serving as receiver). This led to a series of techniques [9] based on system decomposition, linear feedback or inverse system, for achieving synchronization in a unidirectional way, i.e., from a master/driver circuit to a slave/response circuit. Synchronization of chaotic systems is made also possible by mutual (or bidirectional) interactions, that is, when the influence is reciprocal without a preferred direction. The resistor placed between the two capacitors of two oscillatory circuits is an example of an element allowing a reciprocal interaction as the current may flow in both directions. In particular, this is an example of *diffusive* coupling, that occurs when the coupling term is proportional to the differences between the state variables.

Synchronization of chaotic systems is also particularly interesting because it manifests in a rich variety of different forms. In fact, two chaotic systems may display different levels of synchronization as a function of the strength of their interaction. It may be found, for instance, that for a weak coupling two signals generated by two chaotic systems have poorly correlated amplitudes but the peaks of them occur simultaneously. Other forms of synchronization, even difficult to reveal, may also appear.

All these results, including the studies on periodic oscillators and those on chaotic systems, have led to the following definition of synchronization [6]: synchronization is the process through which two or more dynamical systems (that can be identical or not) coupled through some form of interaction or driven by a common forcing adjust a particular property of their motion.

Let us now discuss synchronization in the context of nonlinear dynamical systems, which is the focus of the book. Here, we consider the following system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (1.1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of state variables and $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the system dynamics. The dot indicates derivative with respect to time, i.e., $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$.

Equation (1.1) represents a generic model of a nonlinear system. For instance, it can represent the equations governing an electronic circuit with resistors, capacitors, inductors, and some nonlinear component, such as operational amplifiers working in their full operating range, or, equivalently, it may constitute the governing equation of a mechanical system, of a station of a power grid, of a chemical reactor, of a biological oscillator, and so on. For each specific example, the equation is derived using the physical principles of the field to which it belongs, but then it can be made dimensionless and be applied to a wide body of physical systems, thus yielding more universal conclusions.

In most cases, we will consider system (1.1) with chaotic dynamics, as this represents the more general and rich case, but for some specific topic we will restrict the analysis to periodic systems.

Consider now two nonlinear systems of the type (1.1) and suppose that they interact with each other in some way:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x}_2) + \mathbf{g}_2(\mathbf{x}_1, \mathbf{x}_2) \end{aligned} \quad (1.2)$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ have the same meaning of $\mathbf{f}$ in Eq. (1.1), while $\mathbf{g}_1, \mathbf{g}_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represent the way in which the two systems interact. As mentioned above, there exist different forms of synchronization between two nonlinear systems. We first consider *complete synchronization*, that represents the strongest synchronization condition. Two systems are in complete synchronization if their state variables asymptotically follow the same trajectory; equivalently, the two systems are said to be completely synchronized if

$$\lim_{t \rightarrow +\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = 0 \quad (1.3)$$

that is, if all the state variables of the two systems asymptotically converge to the same trajectory. Here, $\|\cdot\|$ denotes the Euclidean norm of a vector. Generally, Eq. (1.3)

is verified for a set of initial conditions that defines the *basin of attraction* of the synchronous state.

In complete synchronization, the two systems asymptotically follow exactly the same trajectory. In other forms of (weaker) synchronization, the agreement is not on the whole trajectory but on a property of it. An example is phase synchronization, where the two nonlinear systems asymptotically tend to two different trajectories but with the same phase. More precisely, phase synchronization between two systems occurs when their phases, $\varphi_1(t)$ and $\varphi_2(t)$, are such that:

$$|\varphi_1(t) - \varphi_2(t)| < \text{const}, \quad \forall t > t_0 \quad (1.4)$$

that is, the phase difference remains bounded for any time $t > t_0$.

For periodic systems, the phase represents the variable that accounts for motion along the limit cycle. For chaotic systems, the definition of a phase variable is non-trivial and various methods exist [30]. In the simplest case, when a reference point around which the trajectory of the chaotic system rotates can be identified, the phase can be defined as $\varphi(t) = \arctan \frac{x_2(t)}{x_1(t)}$. Here, $x_1(t)$ and $x_2(t)$ are the coordinates of a reference system such that the rotation reference point is the origin and the attractor projected on the plane $x_1 - x_2$ evidences the rotations around it. This approach can be used, for instance, in the Rössler system in the regime of the band attractor [30]. More in general, the phase of a chaotic system is defined by using appropriate mathematical tools such as the Hilbert transform or the Poincaré sections. From (1.4), it follows that the system frequencies $\omega_i = \dot{\varphi}_i$ are also locked, thus phase synchronization implies frequency locking.

Another form of synchronization is *lag synchronization* [33], which occurs when two systems have nearly identical oscillations, but one lags in time with respect to the other.

The most general, and less restrictive, form of synchronization between two nonlinear systems is called *generalized synchronization* [6]. Suppose to consider unidirectional interaction, i.e., the variables $\mathbf{x}_1(t)$ influence the dynamics of variables $\mathbf{x}_2(t)$, but not vice versa ($\mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2) = 0$). The systems are in generalized synchronization if there exists a mapping $\psi : \mathbf{x}_1 \rightarrow \mathbf{x}_2$, that can be asymptotically established transforming the trajectory of the driver system into that of the response system, $\mathbf{x}_2(t) = \psi(\mathbf{x}_1(t))$. The existence of this mapping indicates that the evolution of system 2 depends on system 1, or, equivalently, that the evolution of system 2 can be predicted from that of system 1. For mutually interacting oscillators, a functional relation cannot be explicitly written, but an implicit functional relation of the type $\psi(\mathbf{x}_1(t), \mathbf{x}_2(t)) = 0$ must be used [21]. Generalized synchronization can be very difficult to detect and appropriate methods and indicators have to be used [1, 28].

Quite interestingly, two systems may transit from one form of synchronization to another, typically when the strength of the interaction changes. Very often as the coupling strength increases from small to large values, first some forms of weak synchronization appear such, as for example, generalized or phase synchronization, and, then, some stronger one such as complete synchronization.

The scenario is even richer when more than two units are coupled, but, before discussing it, we need to introduce in the next section some basic notions on complex networks.

1.2 Complex Networks

Many natural and artificial complex systems have a structure characterized by a network of interactions [15, 19, 25]. In flocks of birds, for instance, each individual coordinates its motion by exchanging visual information on the relative speed with few other agents in his proximity neighborhood; still this local mechanism is capable of generating a surprisingly efficient collective behavior [18]. In social systems, interactions occur through acquaintances, friendships, coworker relationships, and are at the basis of opinion formation, group generation, segregation and many other collective behaviors. Many technological systems (internet, teams of mobile robots, smart grids, domotic systems) have now evolved into a complex architecture of many devices communicating and exchanging information each other. All these systems, and many others, have the prerogative that a collective, complex behavior, that is more than the sum of the single parts, emerges as the result of the interactions. The understanding of these systems and their behavior as well as the strategies for their control passes through the understanding of their underlying architecture. For this reason, complex networks constitute today a research field of growing interest. Many books [15, 19, 25] and reviews [7, 17, 26] provide excellent introductions to the topic. The discussion here is limited to few concepts useful in the rest of the book.

A network is formally described by a graph, that is an ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a set of vertices (or nodes) $\mathcal{V} = \{v_1, \dots, v_N\}$ and a set of edges (or links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The elements of $\mathcal{V}$ are ordered pairs of distinct vertices. If, for all $(v_i, v_j) \in \mathcal{E}$, the edge $(v_j, v_i) \in \mathcal{E}$, the graph is undirected, otherwise the graph is directed. In the latter case, the edge (v_i, v_j) represents an interaction from node v_j to v_i , while for an undirected graph, it represents a mutual interaction between the two vertices. In the following, the vertices will be indicated equivalently as v_i or i . If v_i is one of the extreme of a link of $\mathcal{E}$, the link is said incident on it.

Graphs may be described by two matrices, the adjacency and the Laplacian matrix. The adjacency matrix A of an undirected graph is defined in this way: if there is a link between nodes i and j with $i \neq j$, i.e., if $(v_i, v_j) \in \mathcal{E}$, then $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$. Here, we consider simple graphs, for which the diagonal terms are assumed to be $a_{ii} = 0$ for all $i = 1, \dots, N$, that denotes the absence of self-loops. For a directed graph, instead, the direction of interaction is important and is taken into account in the adjacency matrix by defining $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ji} = 1$ if $(v_j, v_i) \in \mathcal{E}$. Thus, in general, for a directed graph $a_{ij} \neq a_{ji}$ and the adjacency matrix is not symmetric.

For an undirected graph, the degree d_i is defined as the number of connections incident on node i : $d_i = \sum_{j=1, j \neq i}^N a_{ij} = \sum_{j=1, j \neq i}^N a_{ji}$ for $i = 1, 2, \dots, N$. For a directed

graph, instead, in- and out-degree may be defined considering the links from or to the node: $d_i^{out} = \sum_{j=1, j \neq i}^N a_{ij}$ and $d_i^{in} = \sum_{j=1, j \neq i}^N a_{ji}$.

Given a graph $\mathcal{G}$, the Laplacian matrix G is defined as $G_{ij} = -a_{ij}$ if $i \neq j$, and $G_{ii} = d_i^{out}$. The Laplacian G and adjacency matrix A are both $N \times N$ matrices; they are linked by the relation $G = D - A$, where D is a diagonal matrix containing the node out-degrees d_i^{out} , i.e., $D = \text{diag}\{d_1^{out}, d_2^{out}, \dots, d_N^{out}\}$.

If a real number, the weight, is associated to each edge of the graph, then the graph is weighted. In this case, the graph is described by the set of vertices, the set of edges, and the set of weights associated to them. These graphs may also be represented by an adjacency matrix with coefficient a_{ij} set equal to the weight of the edge (v_i, v_j) .

In our book, we will focus on a specific class of networks, those where the nodes are dynamical continuous-time systems. Such networks are defined by a set of ordinary differential equations, a model that will be introduced in the next chapter. Each of these networks is fully specified when two elements are described: the isolated dynamics of the nodes (also called, in this context, units) that is given by a model of the type in Eq. (1.1) and the way in which these units are coupled, that is, the structure of interactions, that can be modeled with the adjacency or the Laplacian matrix. To illustrate the phenomenon of synchronization in these networks, we will often consider some artificial models. Here, we discuss three of them, once again referring to the abovementioned textbooks for a detailed discussion on these and many other models. The description of each model is followed by a MATLAB[®] code to implement it.

Nowadays, many softwares and libraries are available for complex networks (e.g., the software for the analysis and visualization of large networks, Pajek [23]; the Python package NetworkX [13]; the Open Graph Viz Platform Gephi [5, 16]; the Multilayer Analysis and Visualization Platform MuxViz [10, 11]) and the same applies for MATLAB[®] routines (e.g., the MATLAB[®] tools for network analysis available in [12]). These also include routines for the generation of the models we are here discussing. However, for better exemplification and clarity, we have included simple MATLAB[®] scripts for the generation of these models.

We start discussing the Erdős-Rényi (ER) model for random graphs. The term actually indicates two different variants, with close statistical properties, to produce a graph with randomly connected pairs of nodes. The first one assigns a number L of edges at random between the N nodes of the graph, while, in the second one, each pair of nodes has a fixed probability p of being connected by a link. We will focus on this latter.

To define the model in MATLAB[®], we use the adjacency matrix. Fixed p as the probability for the existence of a link between each possible pair, we generate a matrix of $N \times N$ random numbers drawn with uniform distribution in the interval $[0,1]$. The existence of a link is established by checking if, for each pair, this random number is less than p . Finally, we take the upper triangular part of the obtained matrix, sum with its transpose to obtain a symmetric matrix, and assign zero to all the elements of the diagonal. The code is reported in the following.

Code for the creation of an ER network

```
1 %code to generate an ER network
2
3 N=100; %number of nodes in the network
4 p=0.03; %probability for the existence of a link
5 A=(rand(N,N)<p);
6 A=triu(A)+triu(A)';
7 for i=1:N, A(i,i)=0; end
```

Interestingly, several properties of the ER model have a closed-form expression [19]. For instance, the degree distribution, namely the probability that a node has d edges, is known to be

$$p_d = \binom{N-1}{d} p^d (1-p)^{N-1-d} \quad (1.5)$$

and the average degree

$$\langle d \rangle = p(N-1) \quad (1.6)$$

The next two models represent two milestones in the development of complex networks: the Watt–Strogatz (WS) [38] and the Barabási–Albert (BA) [4] model. The first one is a rewiring model, that is, a model that starts from a regular topology and then produces a random graph as the result of a rewiring process. The second one is a growth model, where, starting from a nucleus of nodes connected with a regular topology, new nodes are added with an iterative process whose outcome is a random graph.

To introduce the WS model, we first need to define the P -neighbor ring. This is a graph where the nodes are arranged into a circle, with uniform distribution on it, labeled as $1, 2, \dots, N$ and then connected to their $2P$ closest nodes (also called the neighbors) into the circle (P in one direction and P in the other). The graph is regular as each node has exactly the same degree (equal to $2P$).

The WS model starts from a P -neighbor ring and then considers a stochastic rewiring of the links. A direction (clockwise or counterclockwise) is fixed and each node is sequentially inspected following this direction. For every node i , each of its links pointing toward a node in the fixed direction is kept with probability $1-p$ or rewired with probability p to another end vertex. The rewired link then will have one extreme in i and the other in a randomly selected node. Self-loops and multiple links are avoided so that the graph is simple.

The MATLAB® code to produce a WS network with N nodes, $2P$ neighbors, and probability of rewiring equal to p is reported in the following. In this case too, the adjacency matrix has been used to represent the graph. The first part of the code generates the P -neighbor ring. In the second part, the rewiring process is implemented. There, several command lines are devoted to the identification of those nodes to which, in case of rewiring, the link can be connected.

Code for the creation of a network with WS model

```
1 %code for WS network
2
3 N=100; %number of nodes in the network
4 P=3; %number of neighbors in the P-neigh ring
5
6 A=zeros(N);
7
8 %create the P-neigh ring
9 for i=1:P
10     A=A+diag(ones(N-i,1),i);
11     A=A+diag(ones(N-i,1),-i);
12     A=A+diag(ones(i,1),N-i);
13     A=A+diag(ones(i,1),-N+i);
14 end
15
16 %rewiring
17 p=0.1; %rewiring probability
18 for i=1:N
19     for ir=1:P
20         if rand(1)<p
21             allnodes=[1:N];
22             neighbors=find(A(i,:)==1);
23             nodestoexclude=[i neighbors];
24             possiblenewneighbors=allnodes;
25             for ii=1:length(nodestoexclude)
26                 possiblenewneighbors=possiblenewneighbors(find(
27                     possiblenewneighbors~=nodestoexclude(ii)));
28             end
29             lpn=length(possiblenewneighbors);
30             possiblenewneighborsshuffled=possiblenewneighbors(
31                 randperm(lpn));
32             j= mod(i+ir,N)+(mod(i+ir,N)==0)*N;
33             A(i,j)=0;
34             A(j,i)=0;
35             j=possiblenewneighborsshuffled(1);
36             A(i,j)=1;
37             A(j,i)=1;
38         end
39     end
40 end
```

The WS model was introduced to account for a property commonly found in real networks but not reproduced by the ER model or by P -neighbor rings [38]. This is the small-world property that everyone of us has probably experienced in our life. No matter the size of the city where we live, the distance with any other citizen is shorter than what we can think so that we are often surprised to discover a common acquaintance with a person we have just met for the first time. The same holds for many real networks where the distance between two nodes (which can be for instance measured as the shortest path between them) grows linearly with the logarithm of N . At the same time, the structure of connections is such that there is a significant probability that the neighbors of a node are also connected. ER models do not reproduce this feature, while, in P -neighbor rings, the average distance scales linearly with N . On the contrary, WS networks, generated even with a small rewiring probability p , display the small-world property.

The last model we discuss is the BA model. As we have mentioned, BA networks are created with an iterative growth process [4]. The process starts with a set of n_0 nodes completely linked each other (all-to-all connections). At each iteration, a new

node is added and it is linked with m already existing nodes ($m \leq n_0$). These nodes are selected in a random way with a probability which is proportional to the degree of the node to be connected. Formally, node i is selected as end node of one of the new m edges with probability:

$$\Pi_i = \frac{d_i}{\sum_j d_j} \quad (1.7)$$

where the sum in the denominator is over all the already existing nodes before the addition of the new one. The process terminates after $N - n_0$ iterations, when a number N of nodes is reached.

The following MATLAB® code may be used to generate a BA network. In this case, for convenience, we have used the list of edges to represent the network, where each link is considered to be bidirectional. In the first part of the code, the links of the n_0 initial nodes are set (all the possible links are considered). In the second part, the iterative growth process is implemented.

Code for the creation of a BA network

```

1  %code to generate a BA network
2
3  N=100; %number of nodes in the network
4  m=2; %number of links established for each new node
5  n0=3; %initial number of nodes
6
7  EdgeList=[];
8
9  %create a complete graph of n0 nodes
10 for i=1:n0
11     for j=i+1:n0
12         EdgeList=[EdgeList; i j];
13     end
14 end
15
16 %grow the other N-n0 nodes
17 for t=1:(N-n0)
18     j=n0+t;
19
20     %calculate degrees
21     degree=zeros(j-1,1);
22     for id=1:j-1
23         degree(id)=sum(EdgeList(:,1)==id)+sum(EdgeList(:,2)==id);
24     end
25
26     %calculate attachment probabilities
27     nlinks=sum(degree);
28     attprob=cumsum(degree)/nlinks;
29
30     %add new links
31     newneighbors=[];
32     while length(newneighbors)<m
33         r=rand;
34         i=min(find(attprob>r));
35         newneighbors=[newneighbors i];
36         newneighbors=unique(newneighbors);
37     end
38     for ir=1:m
39         EdgeList=[EdgeList; newneighbors(ir) j];
40     end
41
42 end

```

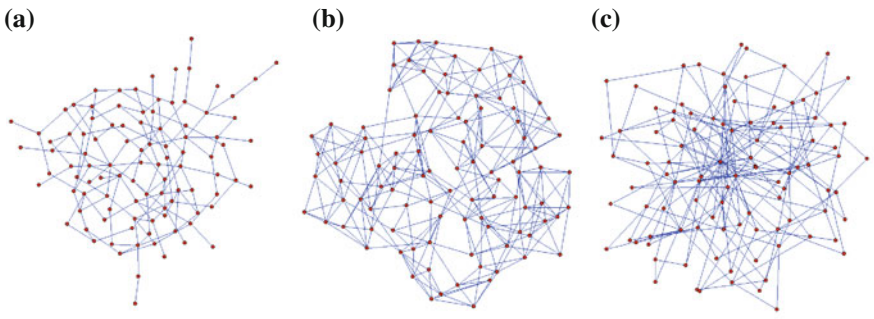


Fig. 1.1 **a** An ER network ($N = 100$, $p = 0.03$). **b** A WS network ($N = 100$, $P = 3$, $p = 0.1$). **c** A BA network ($N = 100$, $n_0 = 3$, $m = 2$)

The BA model reproduces a feature not present in ER and WS networks. Here, through the preferential attachment of Eq.(1.7), nodes with high degrees tend to acquire more and more links as the network grows. This richer-get-richer mechanism yields a power law degree distribution with no characteristic scale. For this reason, these networks are commonly referred to as scale-free (SF) networks.

Examples of the networks generated by the three models discussed are shown in Fig. 1.1. They have been generated by the MATLAB[®] scripts described above and then drawn with Pajek.

The two representations, through the adjacency matrix and the list of edges, that we have used in the MATLAB[®] codes are equivalent. To pass from one representation to the other, the following codes may be used:

```

1 %from EdgeList to adjacency matrix representation
2
3 N=100;
4 A=zeros(N);
5 [Lr Lc]=size(EdgeList);
6 for i=1:Lr
7     A(EdgeList(i,1),EdgeList(i,2))=1;
8     A(EdgeList(i,2),EdgeList(i,1))=1;
9 end

```

and

```

1 %from adjacency matrix to edge list representation
2
3 N=size(A,2);
4 EdgeList=[];
5 for i=1:N
6     for j=i+1:N
7         if A(i,j)~=1,
8             EdgeList=[EdgeList; i j];
9         end
10    end
11 end

```

1.3 The Dynamics of the Units

As the aim of the book is the study of synchronization, the dynamics of the units, namely Eq. (1.1), is assumed to be oscillatory, either periodic or chaotic. Periodic oscillations are associated to the presence of a stable limit cycle; if the system starts from a point on this limit cycle it will return to the starting point exactly after one period of time. Chaotic oscillations, instead, are irregular and a characteristic period cannot be defined. They are, in fact, long-term unpredictable and in the state-space form a dense set of points. For an in-depth discussion on nonlinear dynamics and chaos, the reader is referred to the textbooks on the topic (see, for instance, [8, 34, 37]).

The function $\mathbf{f}$ appearing in Eq. (1.1), usually, is assumed to satisfy the Lipschitz condition, so that, given an initial condition $\mathbf{x}(0) = \mathbf{x}_0$, there exists a unique solution starting from that initial point and satisfying Eq. (1.1). This solution is generally difficult or impossible to find in analytic way, and we often resort to the use of numerical methods.

Given the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1.8)$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, the solution is computed at discrete times $t_k = k \Delta t$ where $k = 1, 2, \dots$ and Δt is the integration fixed step size. More in detail, denoted $\mathbf{x}(t_k)$ as $\mathbf{x}_k$, in the fourth-order Runge–Kutta method, the solution is iteratively calculated, so that sample $\mathbf{x}_{k+1}$ is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{1}{6} \Delta t (\mathbf{K}_0 + 2\mathbf{K}_1 + 2\mathbf{K}_2 + \mathbf{K}_3) \quad (1.9)$$

where

$$\begin{aligned} \mathbf{K}_0 &= \mathbf{f}(\mathbf{x}_k, t_k) \\ \mathbf{K}_1 &= \mathbf{f}(\mathbf{x}_k + \frac{\Delta t}{2} \mathbf{K}_0, t_k + \frac{\Delta t}{2}) \\ \mathbf{K}_2 &= \mathbf{f}(\mathbf{x}_k + \frac{\Delta t}{2} \mathbf{K}_1, t_k + \frac{\Delta t}{2}) \\ \mathbf{K}_3 &= \mathbf{f}(\mathbf{x}_k + \Delta t \mathbf{K}_2, t_k + \Delta t) \end{aligned} \quad (1.10)$$

In the Euler method, instead, the solution is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{f}(\mathbf{x}_k, t_k) \quad (1.11)$$

The Euler method is faster but less accurate than the Runge–Kutta method. In the following, we will use the Euler method to integrate the equations of the network of coupled dynamical systems with the precaution to check the results for consistency with a smaller integration step or with the Runge–Kutta method.

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Chapter 2

Global Synchronization

In this chapter, we discuss the most studied form of synchronization in a network, where all the units asymptotically converge toward the same trajectory. After describing the phenomenon and the measures to characterize it, we illustrate the methods for the study of the stability and present some results on how to control it.

2.1 Global Synchronization: Definition and Some Examples

The term “global synchronization” indicates that the property of synchronization applies to *all* the units of the network. This holds independently of the specific type of synchronization that is occurring; in other words, if we are considering phase synchronization, then all the units share the same phase, while, if we are considering complete synchronization, then all the units follow the same trajectory, and so on. Whenever otherwise specified in the following we will always refer to as global synchronization, omitting to specify that it is complete. We also note that this type of synchronization has been the first and most investigated form in complex networks and many studies refer to it simply as network synchronization (see for instance [1]). Following such studies, when the context is clear, we will simply indicate it as synchronization.

The phenomenon of global synchronization can be observed in undirected and directed networks of linearly and nonlinearly coupled dynamical systems and in coupled maps. Focusing on synchronization in networks of nonlinear circuits and systems, we illustrate the phenomenon by considering an ensemble of identical chaotic units. Global synchronization refers to the condition in which the trajectories of all the chaotic systems of the network are perfectly overlapped thanks to the coupling existing among them.

We start introducing the equations of an unweighted, undirected network of identical, linearly coupled units:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N g_{ij} \mathbf{H} \mathbf{x}_j \quad (2.1)$$

where $i = 1, \dots, N$, $\mathbf{x}_i \in \mathbb{R}^n$ is the state vector of the i -th oscillator (n is the system order), $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the dynamics of each uncoupled unit, g_{ij} are the elements of the Laplacian matrix as defined in the previous chapter, that is, $g_{ji} = g_{ij} = -1$ if i and j are connected by a link, while $g_{ji} = g_{ij} = 0$ otherwise, and $g_{ii} = -\sum_{j=1}^N g_{ij}$; $\mathbf{H} \in \mathbb{R}^{n \times n}$ is a constant matrix representing the inner coupling, that specifies the state variables through which node j is coupled to node i , and σ is the coupling coefficient or coupling strength.

Equations (2.1) can be equivalently rewritten as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^N a_{ij} \mathbf{H}(\mathbf{x}_j - \mathbf{x}_i) \quad (2.2)$$

where the coupling is expressed through the adjacency matrix $\mathbf{A} = \{a_{ij}\}$. We recall that the coefficients of $\mathbf{A}$ for unweighted and undirected networks are defined such as $a_{ij} = 0$ if nodes i and j are not connected, and $a_{ij} = a_{ji} = 1$ if i and j are linked, and $a_{ii} = 0$. Equations (2.2) emphasize that the coupling is diffusive, that is, the interaction is mutual and based on the difference of the value of the state variables.

The existence of synchronization, that is complete and global, in the network can be monitored by considering the following synchronization error:

$$e(t) = \left(\frac{1}{N(N-1)} \sum_{i,j} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|^2 \right)^{1/2} \quad (2.3)$$

built by considering the Euclidean norm between the state vectors of two units, i.e., $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| = \sqrt{(x_{1,i} - x_{1,j})^2 + (x_{2,i} - x_{2,j})^2 + \dots + (x_{n,i} - x_{n,j})^2}$, where $x_{1,i}, x_{2,i}, \dots, x_{n,i}$ are the state variables of the i -th node (i.e., $\mathbf{x}_i = [x_{1,i}, x_{2,i}, \dots, x_{n,i}]^T$), and averaging over all the possible pairs of nodes. Other vector norms may also adopted for the purpose, but their use is beyond the scope of our discussion.

The quantity in (2.3) becomes zero if and only if all the state vectors are equal each other, therefore, indicating synchronization in the network. A nonzero value of this parameter indicates that the network is not fully synchronized. From a practical perspective, small values of this parameter may indicate that the system is close to synchronization.

The synchronization error can also be expressed as

$$e(t) = \left(\frac{2}{N-1} \sum_i \|\mathbf{x}_i(t) - \bar{\mathbf{x}}(t)\|^2 \right)^{1/2} \quad (2.4)$$

where

$$\bar{\mathbf{x}}(t) = \frac{1}{N} \sum_j \mathbf{x}_j(t) \quad (2.5)$$

is the average trajectory at each time. Usually, in the study of synchronization one may be interested to look at the whole evolution of the synchronization error or to its time average. In the latter case, the time average, denoted as $\langle e(t) \rangle_T$, is performed on a sufficiently long time window T , after discarding the transient part of the dynamic evolution.

Let us consider a specific instance of network (2.1). For the node dynamics, that is, $\mathbf{f}$ in Eq. (2.1), let us select the Rössler equations [25]:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i \\ \dot{y}_i &= x_i + ay_i \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \quad (2.6)$$

where a , b , and c are the parameters of the isolated dynamics and the components of the state vector have been indicated as x_i , y_i , and z_i , i.e., $\mathbf{x}_i = [x_i, y_i, z_i]^T$. The system parameters are, here, fixed as $a = b = 0.2$ and $c = 9$, so that the uncoupled oscillator has chaotic dynamics.

Let us now specify the coupling. To fix it, we need to select a given network and the inner coupling, that is, the matrix H . Let us consider $N = 10$ identical Rössler oscillators arranged as in Fig. 2.1 and coupled with the second variable of node j acting on the dynamics of the second variable of node i . This results in the following equations:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i \\ \dot{y}_i &= x_i + ay_i - \sigma \sum_{j=1}^N g_{ij} y_j \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \quad (2.7)$$

which, therefore, correspond to Eq. (2.1) with $\mathbf{f}$ as in Eq. (2.6) and $H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$.

We now simulate system in Eq. (2.7) for $\sigma = 0.75$ and report the synchronization error (2.4) in Fig. 2.2a. The error is not converging toward zero, indicating that the network does not reach synchronization. However, if the coupling strength is increased as in Fig. 2.2b, where it has been fixed to $\sigma = 1.2$, then a different behavior is observed: after a transient, the network reaches synchronization and then never exits from this state. We conclude that the existence of synchronization depends on the value of σ . The next case study, however, demonstrates that it does not depend uniquely on σ , but also the topology and the coupling way are very important.

Consider again the network of Fig. 2.1, but a different inner coupling matrix $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$, corresponding to the following equations:

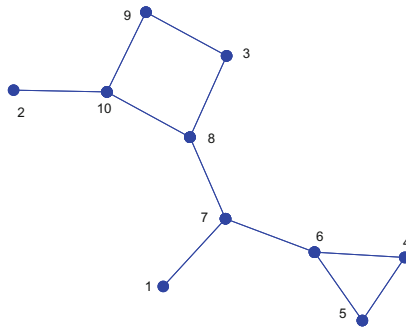


Fig. 2.1 A network of $N = 10$ Rössler oscillators

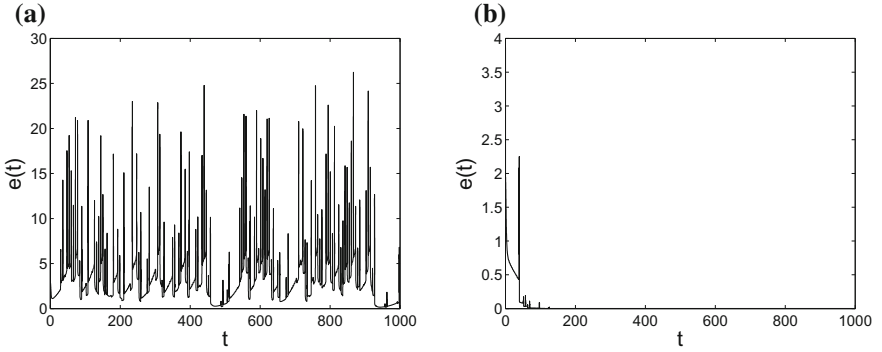


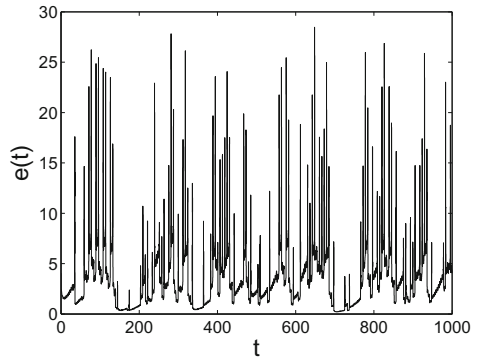
Fig. 2.2 Synchronization error of the network as in Eq. (2.7) for two different values of the coupling coefficient σ : **a** $\sigma = 0.75$, for which the network does not synchronize; **b** $\sigma = 1.2$, for which the network synchronizes

$$\begin{aligned}
 \dot{x}_i &= -y_i - z_i - \sigma \sum_{j=1}^N g_{ij} x_j \\
 \dot{y}_i &= x_i + a y_i \\
 \dot{z}_i &= b + z_i (x_i - c)
 \end{aligned} \tag{2.8}$$

Let us now fix again the coupling coefficient to $\sigma = 0.75$. Once again we find that for this value of the coupling the network does not synchronize (Fig. 2.3). However, this time, if the coupling coefficient is increased, synchronization is never achieved. This is a profound difference with the previous example, where, on the contrary, increasing the coupling yields to synchronization. The reasons of this difference rely on in the way in which coupling is realized and are made clear by the use of a mathematical tool known as the master stability function (MSF) and discussed in the next section.

We conclude the paragraph by illustrating the MATLAB[®] code used to simulate the network of Rössler nodes. To fix the ideas, in particular, let us consider the numer-

Fig. 2.3 Synchronization error of the network as in Eq. (2.8) for $\sigma = 0.75$



ical simulations of Eq. (2.7), with the nodes coupled through the second variable, and with $\sigma = 1.2$.

Code for numerical simulation of a network of Rössler systems

```

1 % numerical integration of a network of coupled Rossler systems
2
3 % network definition
4 N =10;
5 A= zeros(N,N);
6 links =[1 7; 2 10; 3 8; 3 9; 4 5; 4 6; 5 6; 6 7; 7 8; 8 10; 9 10];
7 for l=1:length(links)
8     A(links(l,1),links(l,2))=1;
9     A(links(l,2),links(l,1))=1;
10 end
11
12 % network Laplacian
13 degree = sum(A,2);
14 Gmatrix = diag(degree)-A;
15 dt =0.001;
16 steps =1000/ dt;
17
18 % parameters of the units
19 ar =0.2;
20 br =0.2;
21 cr =9;
22
23 % coupling
24 sigma =1.2;
25
26 % initialization
27 x= zeros(steps,N);
28 y= zeros(steps,N);
29 z= zeros(steps,N);
30
31 % initial conditions for the oscillators
32 xold =(30*rand(N,1)-15)/5;
33 yold =(30*rand(N,1)-15)/5;
34 zold =(40*rand(N,1)-5)/5;
35
36 for t=1: steps
37     coupling =-Gmatrix*yold ;
38
39     dxdt=-yold-zold;
40     dydt=xold+ar*yold+sigma*coupling ;
41     dzdt=br+zold.*(xold-cr);
42

```

```

43     xnew=xold+dt*dxdt;
44     ynew=yold+dt*dydt;
45     znew=zold+dt*dzdt;
46
47     xold=xnew;
48     yold=ynew;
49     zold=znew;
50     x(t,:)= xnew;
51     y(t,:)= ynew;
52     z(t,:)= znew;
53 end
54
55 %error evaluation
56 error=0;
57 for i=1:N
58     for j=1:N
59         Ee=(x(:,i)-x(:,j)).^2+(y(:,i)-y(:,j)).^2+(z(:,i)-z(:,j)).^2;
60         error=error+Ee;
61     end
62 end
63 error=sqrt(error/N/(N-1));
64
65 figure, plot((1:steps)*dt,error,'k '), xlabel('t'), ylabel('e(t)')

```

2.2 Stability of Global Synchronization Through the Master Stability Function

MSF is a technique introduced by Pecora and Carroll [23] to study the stability of the synchronous state, that is, the state in which all the units follow the same trajectory, here indicated as $\mathbf{s}(t)$. The synchronous state for global synchronization is, thus, expressed by the condition $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t) = \mathbf{s}(t)$. Although the technique applies to more general cases, we illustrate it by considering networks that are undirected and unweighted. However, we note that several extensions of the basic approach presented here have been proposed in the literature, for instance, to deal with the case of uncertainties in the node dynamics (nonidentical nodes, [28]) or of time-variant links (for which the coupling is represented by a time-ordered sequence of adjacency matrices, one for each time instant, [12, 27]), so that the state of the art of the conditions where the MSF approach may be applied is larger than that here discussed.

The reference model to introduce the MSF approach is slightly more general than Eqs. (2.1):

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_j g_{ij} \mathbf{h}(\mathbf{x}_j) \quad (2.9)$$

Here, in fact, the coupling, expressed by the coupling function $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, is nonlinear. Eqs. (2.9) reduce to Eqs. (2.1) for $\mathbf{h}(\mathbf{x}_j) = \mathbf{H}\mathbf{x}_j$.

As the Laplacian is a zero-row sum matrix, the network admits an invariant synchronization manifold; a flow starting on it remains there for any time. When all

the network nodes evolve along the same trajectory, the network is governed by the uncoupled dynamics at each identical node as the coupling terms vanish. In fact, the MSF approach derives the conditions under which N identical oscillators can be synchronized, under an arbitrary network configuration provided that it admits an invariant synchronization manifold. Diffusive coupling automatically satisfies this property. So, $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t) = \mathbf{s}(t)$ where $\mathbf{s}(t)$ such that $\dot{\mathbf{s}}(t) = \mathbf{F}(\mathbf{s}(t))$, i.e., the *synchronization manifold*, is a solution of Eqs. (2.9). To understand whether it is stable or not, the dynamics of the network is linearized around it. For convenience Eqs. (2.9) are rewritten in compact form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - \sigma \mathbf{G} \otimes \mathbf{I}_n \cdot \mathbf{H}(\mathbf{x}) \quad (2.10)$$

where $\mathbf{x} = [\mathbf{x}_1^T \ \mathbf{x}_2^T \ \dots \ \mathbf{x}_N^T]^T$ is the stack vector of all node states, and

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \mathbf{f}(\mathbf{x}_1) \\ \mathbf{f}(\mathbf{x}_2) \\ \vdots \\ \mathbf{f}(\mathbf{x}_N) \end{bmatrix} \quad (2.11)$$

describes the local dynamics of the network, and

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_1) \\ \mathbf{h}(\mathbf{x}_2) \\ \vdots \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \quad (2.12)$$

is the stack vector of the coupling function. Consider now new variables, for each node, defined as $\eta_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$, and the corresponding stack vector $\eta = [\eta_1^T \ \eta_2^T \ \dots \ \eta_N^T]^T$, and linearize Eqs. (2.10) around the synchronous state, thus obtaining the dynamics of η :

$$\dot{\eta} = (\mathbf{I}_N \otimes \mathbf{Df})\eta - \sigma (\mathbf{G} \otimes \mathbf{Dh})\eta \quad (2.13)$$

where $\mathbf{Df} = \left. \frac{\partial \mathbf{f}(\mathbf{x}_i)}{\partial \mathbf{x}_i} \right|_{\mathbf{x}_i=\mathbf{s}(t)}$ and $\mathbf{Dh} = \left. \frac{\partial \mathbf{h}(\mathbf{x}_i)}{\partial \mathbf{x}_i} \right|_{\mathbf{x}_i=\mathbf{s}(t)}$. Note that Eqs. (2.13) represent a linear time-varying system (of order nN) as the matrices $\mathbf{Df}$ and $\mathbf{Dh}$ are time dependent.

The next step is to define properly transformed variables leading to a system of equations that are decoupled. This can be done taking into account that, as the graph is undirected, the Laplacian matrix $\mathbf{G}$ is symmetric, so is diagonalizable and there exists a non-singular matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a diagonal matrix containing the eigenvalues of $\mathbf{G}$. Such eigenvalues are all real since the matrix is symmetric. Moreover, they are all nonnegative as $\mathbf{G}$ is positive semi-definite. Finally,

one is always zero as the matrix is zero-row sum. It follows that the eigenvalues of G can be ordered as $0 = \gamma_1 < \gamma_2 \leq \dots \leq \gamma_N$.

With the help of T , the following transformed variables are defined:

$$\xi = (T^{-1} \otimes I_n)\eta \quad (2.14)$$

that, applied to Eqs. (2.13), give a series of decoupled blocks:

$$\dot{\xi}_i = [Df - \sigma \gamma_i Dh]\xi_i \quad (2.15)$$

with $i = 1, \dots, N$. Note that the Jacobians Df and Dh are the same for each block; the blocks differ from each other only for the term $\sigma \gamma_i$. The first block corresponds to $\gamma_1 = 0$ and represents linearized motion along the trajectory $s(t)$. If, for instance, the uncoupled dynamics is chaotic, this block describes the divergence of two nearby trajectories in the chaotic attractor and thus is associated with a positive Lyapunov exponent. The other blocks ($i = 2, \dots, N$) describe the motion transverse to the synchronous solution $s(t)$. Therefore, their study allows to assess if this solution is stable or not. Stability, in fact, requires that all the transverse modes damp out, a condition that can be verified by checking the Lyapunov exponents (called transverse) associated with these blocks.

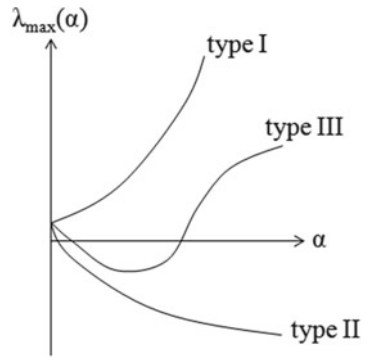
In practice, repeating the calculation for each γ_i is not necessary, but one can study Eqs. (2.15) as a function of a generic real eigenvalue α :

$$\dot{\zeta} = [Df - \alpha Dh]\zeta \quad (2.16)$$

This is the master stability equation (MSE), from which the maximum (conditional) Lyapunov exponent λ_{max} is studied as a function of α , thus obtaining the MSF, i.e., $\lambda_{max} = \lambda_{max}(\alpha)$. Once the MSF is computed, the stability of the synchronization manifold in a given network can be evaluated by computing the eigenvalues γ_i (with $i = 2, \dots, N$) of the matrix G and studying the sign of λ_{max} at the points $\alpha = \sigma \gamma_i$. If all eigenmodes with $i = 2, \dots, N$ are stable, then the synchronous state is stable at the given coupling strength. In such a way, the study of synchronization stability in a network is carried out in two steps: a first step, which is made independent on the specific topology, where the MSF is derived (note that this depends on the node dynamics and the coupling function), and a second step, where the MSF is evaluated at specific points that are determined by the coupling coefficient and the eigenvalues of the Laplacian of the network under study.

The functional dependence of λ_{max} on α gives rise to three different cases [1], shown in Fig. 2.4. The first case (type I) is such that λ_{max} is positive $\forall \alpha$ and, thus, the network nodes cannot be never synchronized. In the second case (type II) λ_{max} assumes negative values above a threshold value, and one always gets a stable synchronous state for high enough coupling strength. In the third case (type III) λ_{max} is negative only in a closed interval of values of α , that is, network nodes can be synchronized only if $\sigma \gamma_i$ for $i = 2, \dots, N$ lie in this interval.

Fig. 2.4 Different classes of MSFs for coupled chaotic systems



This classification leads to two different criteria for synchronizability. For type II MSF systems, there exists a positive number α_1 such that $\lambda_{\max}(\alpha) < 0$ for $\alpha > \alpha_1$, so synchronization requires that $\sigma > \alpha_1/\gamma_2$, that is, the coupling has to be greater than a threshold value given by $\sigma_c = \alpha_1/\gamma_2$.

For type III MSF systems, instead, there exist two positive numbers α_1 and α_2 such that $\lambda_{\max}(\alpha) < 0$ for $\alpha \in [\alpha_1, \alpha_2]$; hence, synchronization requires that $\sigma > \alpha_1/\gamma_2$ and $\sigma < \alpha_2/\gamma_N$. Therefore, for type III MSF systems, the coupling is bounded by two threshold values, $\sigma_{c1} = \alpha_1/\gamma_2$ and $\sigma_{c2} = \alpha_2/\gamma_N$. From this, one derives that γ_N/γ_2 has to satisfy the condition

$$\gamma_N/\gamma_2 < \alpha_2/\alpha_1 \quad (2.17)$$

that highlights as, for type III systems, there can be networks that are never synchronizable.

The MSF method, here only presented for unweighted and undirected networks, can be applied to arbitrary either weighted or unweighted topologies if the MSE is studied as function of a generic (complex) eigenvalue $\alpha + i\beta$ (with $i = \sqrt{-1}$). In addition, we have here considered only diagonalizable networks (an assumption always valid for undirected networks, where the Laplacian is symmetric) of identical units and with links not evolving in time. However, the method has been also generalized to non-diagonalizable networks [21], to scenarios where the nodes are not identical [28] and to topologies with time-varying links [12, 27]. An application to the case of time-varying links is dealt with in Chap. 6.

The MSF approach gives a necessary local condition (all Lyapunov exponents transverse to the synchronization manifold are negative) for the stability of complete synchronization. However, its use for chaotic systems is consolidated [1] as, in a very large number of cases, numerical simulations have proved that this condition reveals to be also sufficient. On the other hand, it should be noticed that there exist other approaches, mainly based on the use of Lyapunov functions, to prove the stability of the synchronous state [4, 10, 11, 16]. These approaches ultimately resort to the same criterion, seen for type II MSF systems, on the eigenvalues of the network Laplacian,

i.e., that synchronizability is expressed in terms of the first nonzero eigenvalue of the Laplacian for a class of systems, but sometimes may offer more conservative bounds for the values of the coupling coefficient leading to stable synchronization. We will discuss some of these methods in Sect. 2.4.

The MSF approach and, in particular, the classification into three different classes of MSFs explain the behavior observed for the examples discussed in the previous section. As mentioned before, the first step of the MSF method is the study of the variational equation. This step can be carried out by using standard routines for the calculation of the Lyapunov exponents (see, for instance, [9]) or papers reporting the behavior of the MSF for selected examples of nonlinear oscillators. In particular, for the Rössler systems one finds [13] that coupling as in Eq. (2.7) yields a type II MSF with $\alpha_1 = 0.157$. Therefore, in this case, one expects that synchronization is obtained if the coupling strength is large enough. This is exactly what was observed in the previous section, where we noticed that for $\sigma = 0.75$ the network of Fig. 2.1 was not synchronized, while for $\sigma = 1.2$ the network was synchronized; for this network, in fact, the threshold is $\sigma_c = \alpha_1/\gamma_2 = 0.8147$.

Let us now consider the second example of coupling, represented by Eq. (2.8). In this case, the MSF is of type III with $\alpha_1 \simeq 0.186$ and $\alpha_2 \simeq 4.614$ [13]. To check if a given network is synchronizable, we have, thus, to check the condition (2.17). For this network $\gamma_2 = 0.1927$ and $\gamma_N = 5.0672$. Therefore, since $\gamma_N/\gamma_2 = 26.2959 > \alpha_2/\alpha_1 = 24.8065$, synchronization cannot be obtained for any value of the coupling coefficient.

2.3 Basin Stability and Synchronization Time

The analysis based on the MSF, relying on linearization of the dynamics, essentially provides local information on the stability of synchronization. In fact, the result applies for perturbations around the synchronous state. However, in nonlinear systems it is quite common to observe multistability and it may occur that some states, although stable, can be observed very rarely compared to others. In general, the stability against non-small perturbations should also be quantified as these perturbations can move the system state from one attractor to another.

The basin of attraction is a concept that helps to clarify these issues. The basin of attraction of a state represents the set of initial conditions (points in the state space) from which the system converges to this state. A measure of the volume associated to the basin of attraction of a given state would provide an indication of its robustness to non-small perturbations. For high-dimensional nonlinear systems, as it is the case for networks of coupled oscillators, where the number of state variables is nN , measuring the volume of a basin of attraction is not a simple task. Recently, in [20] a probabilistic approach was proposed to quantify the volume of the basin of attraction of the synchronous state and, therefore, to complement the result of the local analysis of stability based on MSF. This measure, indicated as basin stability, *BS*, is obtained by performing a large number of numerical simulations of a network

of coupled oscillators starting from initial conditions in a finite subset of the state space and evaluating the fraction of simulations that converge to the synchronous state (rigorously this gives information restricted to the finite subset investigated).

Quite interestingly, the analysis of the basin stability can reveal important features of synchronization in real networks. In [20], in fact, a fascinating result is illustrated. We have already noted that synchronization is crucial for the proper operation of many real-world networks with small-world property, so one could expect that their structure is optimal for synchronization. In Chap. 1, we have seen that the WS model interpolates from regular to random networks as a function of the rewiring probability, including in-between networks with small-world property. Therefore, studying synchronization in the WS model, one expects to find an optimum in correspondence of these networks. However, one finds that synchronizability as measured by the ratio between the largest and the first nonzero eigenvalue of the Laplacian (see condition (2.17)) is monotonically decreasing, thus suggesting that it is greater in random networks than in regular ones. Instead, the analysis of the basin stability shows a non-monotonic curve, peaked in correspondence of the values of rewiring probability yielding small-world networks. This posits that real-world networks with small-world property have a structure with good performance in terms of the eigenvalue ratio as well as the robustness of synchronization.

Another quantity which is relevant in a dynamical system is the time needed to reach the steady state, i.e., in our context, the synchronization time. This quantity, indicated as t_s , is defined as the time required for the synchronization error to reach and remain within given bounds around zero. In what follows, we consider a network of Rössler oscillators obtained by adding a further link (3, 4) to the topology of Fig. 2.1 so that the Laplacian eigenvalues satisfy condition (2.17) (the new network has $\gamma_2 = 0.4024$, $\gamma_N = 5.2112$, and so $\gamma_N/\gamma_2 = 12.9503$), and show how to calculate the basin stability and the synchronization time. The MATLAB® code to do this is reported below, and the results are illustrated in Fig. 2.5. Both measures indicate that synchronization is attained in an interval of values of σ , in accordance with the MSF analysis as the system is type III. However, they also indicate that the characteristics are not flat in this region. The volume of the basin of attraction of the synchronous state first grows with σ and then decreases, suggesting an optimum value of it. The same is observed for the curve of t_s , which shows an optimum value of σ corresponding to a minimum synchronization time.

Code for basin stability and synchronization time

```

1 %Basin stability and synchronization time
2
3 %integration parameters
4 dt=0.001;
5 steps=1000000;
6
7 %parameters
8 ar=0.2;
9 br=0.2;
10 cr=9;
11
12 %number of initial conditions and values of the coupling strength
13 NIt=100;
14 sigmaVec=[0:0.02:1];

```

```

15
16
17 %Matrix of the mean error
18 ME=100*ones(length(sigmaVec),NIt);
19
20 %initialization
21 BS=zeros(length(sigmaVec),NIt);
22 ts=steps*dt*ones(length(sigmaVec),NIt);
23
24 %network
25 N=10;
26 A=zeros(N,N);
27 links=[1 7; 2 10; 3 8; 3 9; 4 5; 4 6; 5 6; 6 7; 7 8; 8 10; 9 10;
        3 4];
28
29 for l=1:length(links)
30     A(links(l,1),links(l,2))=1;
31     A(links(l,2),links(l,1))=1;
32 end
33
34 % network Laplacian
35 degree=sum(A,2);
36 Gmatrix=diag(degree)-A;
37
38 for ItIndex=1:NIt
39
40     for SigmaInd=1:length(sigmaVec)
41
42         sigma=sigmaVec(SigmaInd)
43
44         %initial conditions for Roessler oscillators
45         xold=(30*rand(N,1)-15);
46         yold=(30*rand(N,1)-15);
47         zold=(40*rand(N,1)-5);
48         error=zeros(steps,1);
49
50         for t=1:steps
51             coupling=-Gmatrix*xold;
52             dxdt=-yold-zold+sigma*coupling;
53             dydt=xold+ar*yold;
54             dzdt=br+zold.*(xold-cr);
55
56             xnew=xold+dt*dxdt;
57             ynew=yold+dt*dydt;
58             znew=zold+dt*dzdt;
59
60             xold=xnew;
61             yold=ynew;
62             zold=znew;
63
64             x(t,:)=xnew;
65             y(t,:)=ynew;
66             z(t,:)=znew;
67
68         end
69
70         error=0;
71         for i=1:N
72             for j=1:N
73                 Ee=((x(:,i))-x(:,j)).^2+(y(:,i))-y(:,j)).^2+(z(:,i))-z(:,j)).^2);
74                 error=error+Ee;
75             end
76         end
77
78     error=sqrt(error/N/(N-1));
79

```

```

80 % last value of the mean error
81 meanErrorInf=mean(error(end/4:end));
82 %BS
83 if erroremedioinf<0.001,
84     BS(SigmaInd,ItIndex)=1;
85     %Evaluation of ts for the synchronized system
86     %if the system does not synchronize ts=steps
87     S = stepinfo([200; error],1:steps+1,0,'
        SettlingTimeThreshold',0.0002);
88     ts(SigmaInd,ItIndex) = (S.SettlingTime-1)*dt;
89 end
90
91 if ~isnan(meanErrorInf),
92     ME(SigmaInd,ItIndex)=meanErrorInf;
93 end
94 end
95 end

```

2.4 Some Criteria for Global Stability of Synchronization

We have seen in Sect. 2.2 that the MSF approach to the problem of stability is based on linearization around the synchronization manifold and evaluation of the transverse Lyapunov exponents. This section offers a limited set of examples of criteria referring to global conditions for synchronization, that is, conditions that hold for any initial condition and not only around the synchronous state. These criteria mainly derive from the application of proper Lyapunov functions or contraction theory conditions, in most cases enabled by some more restrictive hypotheses on the dynamics of the units, on the coupling mechanisms and ultimately on the network topology and coupling.

The hypotheses made on the unit dynamics, formalized in terms of mathematical conditions on the vector field $\mathbf{f}$, are quite important as they, in practice, define the class of systems to which the approach may be applied. The most common ones

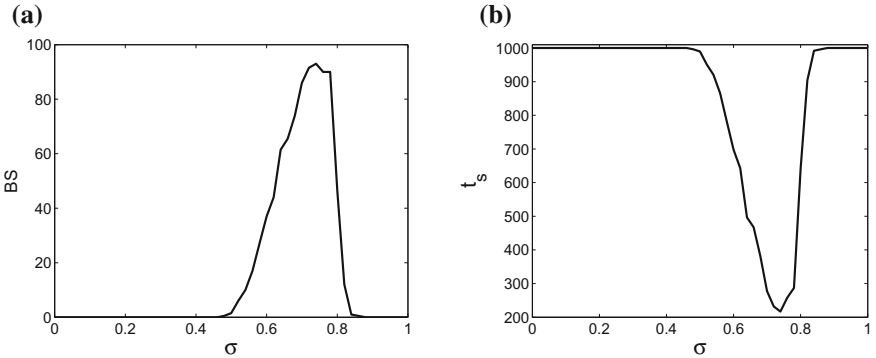


Fig. 2.5 **a** Basin stability BS and **b** synchronization time t_s versus the coupling coefficient σ for the network as in Eqs. (2.7) with type III MSF

are based on the Lipschitz condition or on the QUAD assumption. On the contrary, the approaches based on contraction theory rely on understanding which conditions guarantee the convergence of all solutions among each other. These three important concepts are actually linked as discussed in [7], to which we refer for a more in deep study on this topic.

As mentioned in Sect. 2.2, there is a vast literature on the subject and we will simply show some illustrative examples, whereas a comprehensive list of these results goes beyond the purpose of this book. We will restrict the discussion to undirected networks of linearly coupled nodes, as in Eqs. (2.1).

A first criterion is reported in [17] for networks that are completely connected, that is, for any pair of nodes there exists a path connecting them. For such networks the Laplacian matrix is irreducible. In this criterion it is assumed that the inner coupling matrix is diagonal, that is, $H = \text{diag}\{h_1, h_2, \dots, h_n\}$. In addition, an hypothesis is done on the dynamics of the units. To formulate it we need the concept of a QUAD function.

Given a $n \times n$ diagonal matrix Δ and a real positive constant ω , we say that a function $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is $\text{QUAD}(\Delta, \omega)$ if and only if 1

$$(\mathbf{x} - \mathbf{y})^T [\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)] - (\mathbf{x} - \mathbf{y})^T \Delta (\mathbf{x} - \mathbf{y}) \leq -\omega (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \quad (2.18)$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. So a dynamical system is said QUAD, or to satisfy the QUAD condition, if $\mathbf{f}$ is QUAD for certain values of Δ and ω .

We are now ready to discuss the criterion reported in [17] (Corollary 3). Suppose that the Laplacian matrix G of the network is symmetric and irreducible, let $0 = \gamma_1 < \gamma_2 \leq \dots \gamma_N$ be the eigenvalues of G , $\Delta = \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$ a diagonal matrix and ω a positive constant such that $\mathbf{f}$ is $\text{QUAD}(\Delta, \omega)$. If $\delta_i - h_i \gamma_k < 0$, for $i = 1, 2, \dots, n$ and $k = 2, \dots, N$, then the synchronization manifold is globally exponentially stable.

We now consider a peculiar form of linearly coupled dynamical systems, i.e., the one in which all the variables are used for coupling, that is, $H = I$. This condition enables a lot of analytical results proving the global stability of synchronization.

First, let us recall the definition of a Lipschitz function. Given $\alpha > 0$, we say that a function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz if $\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| < \alpha \|\mathbf{y} - \mathbf{x}\|$.

Note that the Lipschitz condition and the QUAD one are linked by this result (Theorem 3 of [7]): if $\mathbf{f}$ is globally Lipschitz with $\alpha > 0$, then it is $\text{QUAD}(\Delta, \omega)$ with $\Delta - \omega I_n \geq \alpha I_n$.

A sufficient condition for the global synchronization is given by the following result (Theorem 4 of [7]). If the network is connected, the vector field of the uncoupled nodes is Lipschitz with Lipschitz constant $\alpha > 0$, and the coupling coefficient satisfies

$$\sigma > \max\{\alpha, \alpha/\gamma_2\} \quad (2.19)$$

then the network synchronizes.

The next result (Theorem 5 of [7]) holds for more general coupling functions, but, as it can be observed, does not provide a condition on σ , but rather requires that each single value is checked for, to see if it satisfies the assumption of the theorem.

Consider a network whose node dynamics is given by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sigma \sum_{j=1}^N a_{ij}(\mathbf{h}(\mathbf{x}_j) - \mathbf{h}(\mathbf{x}_i)) \quad (2.20)$$

where $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth nonlinear function of the system state. If the network is connected, the coupling function $\mathbf{h}$ is strictly increasing, and the function $\mathbf{f}(\mathbf{x}_i, t) - \sigma d_i \mathbf{h}(\mathbf{x}_i)$, with d_i the degree of node i , is QUAD(Δ, ω) with $\Delta - \omega I_n \leq c I_n$ for some $c < 0$, then the network synchronizes.

We next consider a network whose dynamics is given by:

$$\dot{\mathbf{x}}_i = \mathbf{A}_c \mathbf{x}_i + \mathbf{f}(\mathbf{x}_i, t) - \sigma \sum_{j=1}^N g_{ij} \mathbf{x}_j \quad (2.21)$$

where we note that the coupling acts on all node variables, that is, $\mathbf{H}$ does not appear as it has been fixed equal to the identity matrix, $\mathbf{H} = \mathbf{I}$.

For this case, there are several interesting results available in literature. We here report a criterion (Theorem 2 of [15]) for global stability (as the other results previously mentioned). Given the network as in Eq. (2.21) and indicated with μ_i the eigenvalues of $\mathbf{A}_c + \mathbf{A}_c^T$, suppose that there exist nonnegative constants c_{ij} , $i, j = 1, 2, \dots, N$, such that

$$\|\mathbf{f}(\mathbf{x}_i, t) - \mathbf{f}(\mathbf{x}_j, t)\| \leq c_{ij} \|\mathbf{x}_i - \mathbf{x}_j\| \quad (2.22)$$

and that the second largest eigenvalue of the Laplacian matrix $\mathbf{G}$ satisfies

$$\sigma \gamma_2 \geq \frac{1}{2} \left(\eta + \max_{1 \leq i \leq n} \{\mu_i\} \right) \quad (2.23)$$

where $\eta > \max\left\{\frac{4(N-1)}{N} \sum_{i,j=1, i \neq j}^N c_{ij}, \max_{1 \leq i \leq n} \{\mu_i\}\right\}$, then the network synchronizes.

Finally, we report a criterion which is local, but particularly interesting as it is quite easy to check (Theorem 1 of [14]). Given a network of N linearly and diffusively coupled identical nodes

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^N a_{ij}(\mathbf{x}_j - \mathbf{x}_i) \quad (2.24)$$

consider the Lyapunov exponents of each individual n -dimensional dynamical node, and denote with ρ_{max} the maximum Lyapunov exponent ($\rho_{max} > 0$ in the case of chaotic units). If $\mathbf{f}$ is Lipschitz and

$$\sigma > \frac{\rho_{max}}{\gamma_2} \quad (2.25)$$

then the synchronized state, $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t) = \mathbf{s}(t)$ is exponentially stable.

We now apply two of the criteria previously described, and show their suitability to predict the stability of global synchronization in a network of Chua's oscillators. The Chua's oscillator is a particularly important circuit as it has been the first to be intentionally designed to be chaotic by its inventor Leon O. Chua, as well as for many dynamical features it embeds (for detailed discussion on the circuit see for instance [2, 19]). Here, we consider the dimensionless equation describing the circuit dynamics:

$$\begin{aligned} \dot{x}_1 &= \alpha_C(x_2 - x_1 + g(x_1)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta_C x_2 - \gamma_C x_3 \end{aligned} \quad (2.26)$$

where $g(x_1)$ is a piecewise linear function defined as $g(x_1) = -bx_1 + 0.5(b - a)(|x_1 + 1| - |x_1 - 1|)$ with $a = -1.27$ and $b = -0.68$. The other parameters of the Chua's oscillator are chosen so that the system is chaotic, i.e., $\alpha_C = 10$, $\beta_C = 15$ and $\gamma_C = 0.0385$.

We study a network of these oscillators with topology as in Fig. 2.1. For this network, $\gamma_2 = 0.1927$. In addition, $\rho_{max} \simeq 0.37$ and the Lipschitz constant of the vector field of the Chua's equations is $\alpha = 20.4$.¹

The sufficient condition for global synchronization is determined by Eq. (2.19), that is, $\sigma > \max\{\alpha, \alpha/\gamma_2\}$. We get $\sigma > \max\{20.4, 20.4/0.1927\} = 105$. For the local criterion, we have to check condition (2.25). In this case, we get $\sigma > \frac{0.37}{0.1927} = 1.92$. Clearly, the threshold predicted by the global criterion is much larger than the local one.

Figure 2.6 shows the behavior of the first state variable of the Chua's oscillators circuits (Fig. 2.6a, c), and the synchronization error evaluated as in Eq. (2.3) (Fig. 2.6b, d), for two values of the coupling strength σ selected according to the criteria mentioned. In both the cases, the network synchronizes with the main difference evident in the evolution of the synchronization error. In the first case, the error tends to zero much faster than in the second case, being the threshold for global synchronization more conservative.

The MATLAB[®] code to simulate the network of Chua's oscillators and obtain panels (c) and (d) of Fig. 2.6 is the following.

¹Note that this constant refers to the whole vector field and, so, differs from that of the scalar nonlinearity appearing in it.

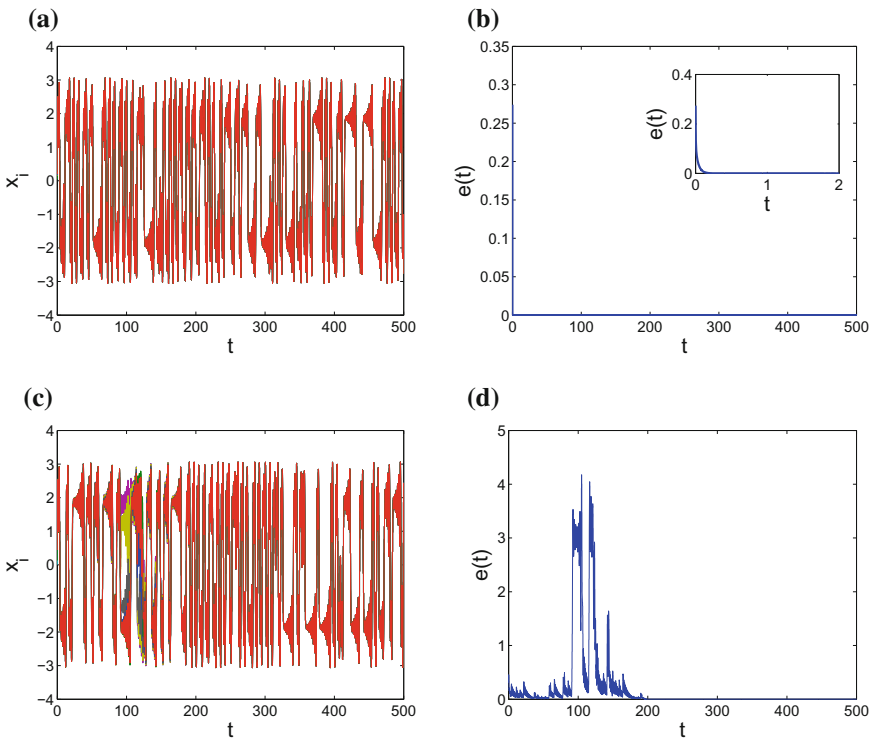


Fig. 2.6 Behavior of state variable x_i in a network of Chua's oscillators, **a** and **c**, and of the synchronization error, **b** and **d**. The topology is as in Fig. 2.1, the units dynamics as in Eqs. (2.26), the inner coupling $H = I$. The coupling coefficient has been fixed as $\sigma = 106$ in **a–b** to satisfy the global condition of Eq. (2.19), and as $\sigma = 1.91$ in **c–d** to satisfy condition (2.25)

Code for integration of the network of Chua's oscillators

```

1 %Numerical integration of a network of Chua's oscillators
2
3 %Network
4 N=10;
5 A= zeros(N,N);
6 links=[1 7; 2 10; 3 8; 3 9; 4 5; 4 6; 5 6; 6 7; 7 8; 8 10; 9 10];
7 for l=1:length(links)
8     A(links(l,1),links(l,2))=1;
9     A(links(l,2),links(l,1))=1;
10 end
11
12 G=diag(sum(A))-A;
13 lg=eig(G);
14
15 %integration parameters
16 dt=0.001;
17 steps=1000/dt;
18
19 %parameters
20 ac=-1.27;
21 bc=-0.68;

```

```

22 gamma=0.0385;
23 alpha=10;
24 beta=15;
25
26 sigma=1.91;
27
28 %initialization
29 x=zeros(steps,N);
30 y=zeros(steps,N);
31 z=zeros(steps,N);
32
33 %initial conditions for Chua's oscillators
34 xold=0.5*rand(N,1);
35 yold=0.5*rand(N,1);
36 zold=0.5*rand(N,1);
37
38 for t=1:steps
39
40     %%coupling
41     couplingx=G*xold;
42     couplingy=G*yold;
43     couplingz=G*zold;
44
45     %%dynamics
46     fc=-bc*xold+(-ac+bc)/2*(abs(xold+1)-abs(xold-1));
47     dxdt=alpha*(yold-xold+fc)-sigma*couplingx;
48     dydt=xold-yold+zold-sigma*couplingy;
49     dzdt=-beta*yold-gamma*zold-sigma*couplingz;
50
51     xnew=xold+dt*dxdt;
52     ynew=yold+dt*dydt;
53     znew=zold+dt*dzdt;
54
55     xold=xnew;
56     yold=ynew;
57     zold=znew;
58
59     x(t,:)=xnew;
60     y(t,:)=ynew;
61     z(t,:)=znew;
62
63 end
64
65 %%plot of the x-state variables
66 figure,plot([1:steps]*dt,x),xlabel('t'),ylabel('x_i')
67
68
69 %%Global Error
70 E=0;
71 for i=1:N
72     for j=1:N
73         Ee=((x(:,i))-x(:,j)).^2+(y(:,i))-y(:,j)).^2+(z(:,i))-z(:,j)).^2));
74         E=E+Ee;
75     end
76 end
77
78 E=sqrt(E/N/(N-1));
79
80 %%plot of the synchronization Error
81 figure,plot([1:steps]*dt,E)

```

2.5 Pinning Control

In the previous sections, we have observed that there are networks which cannot reach synchronization either because the coupling value is not set at the proper value or because the units have a type III MSF and the eigenvalues of the network Laplacian do not satisfy condition (2.17). Similar scenarios motivate the problem of *controlling synchronization*.

Even for networks reaching synchronization, control may be very important. In fact, in synchronization the units follow a trajectory which is the result of the system self-organization, but, if one aims at steering them toward a specific trajectory, a form of control has to be included.

A possible solution to this problem would be to add a feedback control input to each node of the network. However, this solution is clearly difficult or impossible to apply when there is a large number of units and is possibly inefficient even when the number of nodes is low.

The idea underlying *pinning control* originated from these considerations: the control action is exerted only on a small fraction of the network nodes, still under proper conditions this suffices to steer all the network nodes towards the desired trajectory. More specifically, an additional node, called the pinner, is included in the network and connected only to a subset of nodes, called the pinned nodes. This node influences the pinned nodes but is not influenced by them, so that its (autonomous) evolution determines the desired trajectory. This dynamics can be selected to be an equilibrium point, a limit cycle or a trajectory of a chaotic attractor. In the first two cases, the whole network will converge to the equilibrium point or periodic solution with suppression of chaos, while, in the latter case, the action of the pinning control is to induce synchronization in all the network, preserving chaos and regulating the network behavior towards its trajectory. The technique is also widely used in consensus problems [3, 22], where, typically, the node dynamics is simpler than that considered in this chapter, e.g., it could be a first-order linear system or integrator.

We now focus on the application of pinning control to the problem of steering the network towards a common chaotic solution. In this case, the dynamics of the pinner node is described by

$$\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s}) \quad (2.27)$$

where $\mathbf{s}(t)$ is the desired trajectory (in the case of synchronization control it satisfies exactly the same equations of the other nodes, while for stabilization of an equilibrium point or periodic solution it may be required changing a system parameter).

The dynamics of the other nodes includes a control term $\mathbf{u}_i$ as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N g_{ji} \mathbf{H} \mathbf{x}_j + \mathbf{u}_i \quad (2.28)$$

Following the main approach to the topic, the control action is set as a state feedback law with respect to the reference solution $\mathbf{s}(t)$, that is, $\mathbf{u}_i = -\delta_i \kappa (\mathbf{x}_i - \mathbf{s})$, where κ is the gain of the control and $\delta_i = 1$ if node i is pinned or $\delta_i = 0$ otherwise. The strength of the control κ and the constants δ_i are design parameters. In particular, selecting δ_i means determining how many ($n_p = \sum_i \delta_i$) and which nodes are connected to the extra node, an issue particularly important in pinning control.

Entering in all the details of a very rich technical literature on the problem goes beyond the aim of this discussion which is limited to few results. A first one relies on the use of MSF to establish if the pinning control settings (i.e., the choice of κ and δ_i) yields synchronization. To this purpose, an extended network of $N + 1$ dynamical systems is considered

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N \tilde{g}_{ji} \mathbf{H} \mathbf{x}_j \quad (2.29)$$

for $i = 1, 2, \dots, N + 1$, where $\mathbf{x}_{N+1} = \mathbf{s}(t)$ defines the extra node (the dynamics of the pinner). $\tilde{g}_{ji}$ are the elements of a new $(N + 1) \times (N + 1)$ square matrix $\tilde{G}$ related to the elements of the Laplacian matrix of the original graph and to the control parameters by this relationship:

$$\tilde{G} = \begin{pmatrix} g_{11} + \delta_1 \kappa & g_{12} & \dots & g_{1N} & -\delta_1 \kappa \\ g_{21} & g_{22} + \delta_2 \kappa & \dots & g_{2N} & -\delta_2 \kappa \\ \vdots & & \ddots & & \vdots \\ g_{N1} & g_{N2} & \dots & g_{NN} + \delta_N \kappa & -\delta_N \kappa \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (2.30)$$

Note that this matrix is zero-row sum; this means that system (2.29) admits the global synchronization manifold. The goal of pinning control, that can be formally expressed by

$$\lim_{t \rightarrow +\infty} \|\mathbf{x}_i(t) - \mathbf{s}(t)\| = 0 \quad (2.31)$$

for $i = 1, \dots, N$, is equivalent to synchronization in the augmented system (2.29)

$$\lim_{t \rightarrow +\infty} \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| = 0 \quad (2.32)$$

for $i, j = 1, \dots, N + 1$. Therefore, the system in Eqs. (2.28) follows the trajectory $\mathbf{s}$ if the system in Eqs. (2.29) synchronizes. In particular, one can use the MSF approach to check if a given setting of the control parameters yields synchronization or not. One can also fix some of the parameters (e.g., σ and δ_i), to find the values of the control gain κ , if they exist, leading to synchronization.

Suppose to consider again the network in Fig. 2.1 and the coupling as in Eq. (2.8). We have already observed that for $\sigma = 0.75$ the network does not reach synchronization. Now suppose to apply pinning control as in Eq. (2.28) with $\kappa = 2$ to the first four nodes of the network, i.e., $\delta = [\delta_1 \delta_2 \dots \delta_{10}]^T = [1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$. In correspondence of these values, the first nonzero and the largest eigenvalue of $\tilde{G}$ are $\tilde{\gamma}_2 = 0.4845$ and $\tilde{\gamma}_{N+1} = 5.4569$, that satisfy condition (2.17). Numerical simulations, obtained with the following code, show that with pinning control the network reaches synchronization (Fig. 2.7).

Code for pinning control of a network of Rössler systems

```

1 %pinning control of a network of Rossler systems
2
3 %network
4 N=10;
5 A=zeros(N,N);
6 links=[1 7; 2 10; 3 8; 3 9; 4 5; 4 6; 5 6; 6 7; 7 8; 8 10; 9
       10];
7
8 for l=1:length(links)
9     A(links(l,1),links(l,2))=1;
10    A(links(l,2),links(l,1))=1;
11 end
12
13 % network Laplacian
14 degree=sum(A,2);
15 Gmatrix=diag(degree)-A;
16
17 %integration parameters
18 dt=0.001;
19 steps=1000/dt;
20
21 %parameters of the control
22 delta=[1 1 1 1 0 0 0 0 0 0]';
23 kappa=2;
24 c=diag(delta*kappa);
25 Gtilde=[Gmatrix+c -delta*kappa;zeros(1,N) 0];
26
27 % parameters of the units
28 ar=0.2;
29 br=0.2;
30 cr=9;
31 % coupling
32 sigma=0.75;
33
34 % initialization
35 x=zeros(steps,N+1);
36 y=zeros(steps,N+1);
37 z=zeros(steps,N+1);
38
39 % initial conditions for the oscillators
40 xold=(30*rand(N+1,1)-15)/5;
41 yold=(30*rand(N+1,1)-15)/5;
42 zold=(40*rand(N+1,1)-5)/5;
43
44 for t=1: steps
45     coupling=- Gtilde * xold ;
46
47     dxdt=-yold-zold+sigma*coupling;
48     dydt=xold+ar*yold;
49     dzdt=br+zold.*(xold-cr);
50
51     xnew=xold+dt*dxdt;
52     ynew=yold+dt*dydt;

```

```

53     znew=zold+dt*dzdt;
54
55     xold=xnew;
56     yold=ynew;
57     zold=znew;
58
59     x(t,:)=xnew;
60     y(t,:)=ynew;
61     z(t,:)=znew;
62 end
63
64 error=0;
65 for i=1:N+1
66     for j=1:N+1
67         Ee=(x(:,i)-x(:,j)).^2+(y(:,i)-y(:,j)).^2+(z(:,i)-z(:,j)).^2;
68         error=error+Ee;
69     end
70 end
71
72 error=sqrt(error/N/(N+1));
73
74 figure, plot([1:steps]*dt,error,'k'), xlabel('t'), ylabel('e(t)')
75 figure, plot([1:steps]*dt,x), xlabel('t'), ylabel('x_i(t)')

```

The method presented, based on the MSF, gives local necessary conditions on the synchronization stability, so that one has to resort to numerical simulations to verify the sufficiency part. Several sufficient conditions for global stability, generally relying on more restrictive hypotheses on the dynamics and type of coupling, have been derived; some examples are the case of identical coupling (i.e., $\mathbf{h}(\mathbf{x}_i) = \mathbf{x}_i$) discussed in [24] and that of linear coupling for any number of pinned nodes [26, 30] or for a single controller [5].

Ultimately, these criteria aim at linking the conditions for the stability of synchronization under pinning control to some topological properties of the nodes, therefore providing simple methods to select the nodes based on the inspection of such prop-

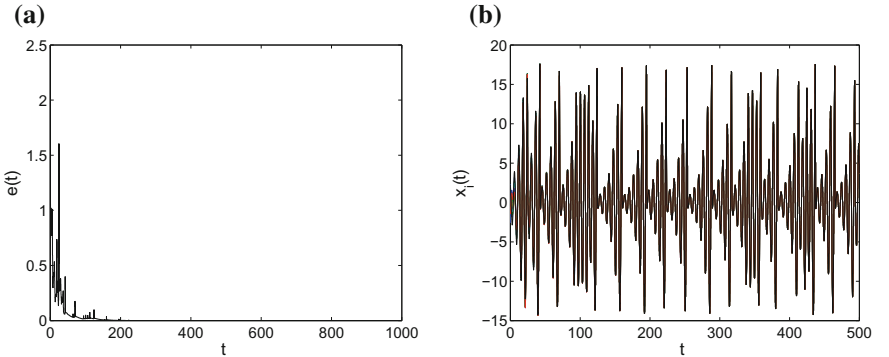


Fig. 2.7 Pinning control of the network in Fig. 2.1 and coupling as in Eq. (2.8) for $\sigma = 0.75$ and $\delta = [1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0]^T$: **a** synchronization error $e(t)$ of the augmented system (2.29); **b** evolution of the state variables $x_i(t)$ $i = 1, \dots, N$ ($x_i(t)$ is the first state variable of oscillator i)

erties. This leads to the fundamental question in pinning control for establishing a relationship between the structure of the network and the nodes to be pinned. Early works [29] comparing random choices with degree-based selective approaches showed that the latter is a preferable strategy. More recently, it has been shown that, under appropriate conditions on the node dynamics, if the directed network admits a spanning tree, then the control can be achieved by only pinning the root node [5]. If the network does not admit a spanning tree, then at least one node in each root strongly connected component has to be pinned [18].

These works identify the minimum number of nodes to control the network; however, it should be pointed out that, in some cases, it could be convenient to apply the control action to a higher number of nodes as this generally reduces the gain required by the controller. Equivalently, the optimization of the number and locations of the pinned nodes has to be accomplished by taking also into account constraints on the feasibility of the control gain. Further methods for the selection of the nodes to pin, which additionally may work at a fixed value of the control gain, thus also allowing to select the pinned nodes for a specific value of this parameter, are based on adaptive rules to determine the values of the parameters δ_i [6].

Finally, we mention that new trends in pinning control also include another relevant problem recently investigated in [8]. The authors consider the case in which the number of nodes to pin is limited, so that not all the network nodes may be controlled to the desired trajectory, and develop methods to select the pinned nodes in order to maximize the number of units controlled. This problem is referred to as the partial pinning control problem.

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Chapter 3

Chimera States

In this chapter we study chimera states, a regime where the units of a symmetric network split into groups with coherent and incoherent behaviors, thus breaking the symmetry of the structure. We discuss illustrative examples, control techniques, and experimental observations of these states, with particular focus on those performed on networks of electronic circuits.

3.1 The Discovery of Chimera States

So far, we have discussed a state where all the units of the networks are synchronized to each other. However, other more intricate forms of synchronization exist: chimera states represent one of them. These states are characterized by the coexistence of a coherent domain, that is formed by synchronous oscillators, and an incoherent one, where, on the contrary, the units are not synchronized, but behave in an incoherent way. Quite surprisingly, these states appear in networks of identical oscillators, thus representing a solution breaking the symmetry of the structure.

An example can help to better introduce the chimera states. We refer to the case study presented by Kuramoto and Battogtokh in their paper [12]. They studied a simple structure of phase oscillators, that is, oscillators that can be described by a single state variable representing motion along the system limit cycle. This assumption is usually made in systems where the attraction to the limit cycle is strong compared to the coupling as it simplifies the analysis; for nonlinear circuits this clearly represents an abstraction, which can be, however, useful for a first glimpse at the system behavior (in fact, the counterpart of the phase variable, that is, the amplitude, can have an important role as shown in [8]). The structure considered by Kuramoto and Battogtokh was a circular one with each oscillator connected to its first P neighbors at each side (the P -neighbor ring discussed in Chap. 1). This structure is also referred to as a ring of nonlocally coupled nodes. Contrary to expectations which

could lead one to imagine that all the units of this structure either synchronize in phase or drift in an incoherent way, they observed that there were oscillators synchronizing each other while others incoherently drifting. So, they discovered that oscillators that have identical parameters (in particular, identical natural frequency) and that are coupled in the same way can exhibit a behavior that is different from one another. This dynamical regime was observed for some initial conditions, while for others global synchronization was found, thus showing the possibility of distinct dynamical patterns of synchronization for the same structure.

The paper by Kuramoto and Battogtokh is considered the first reporting this type of phenomenon, even if the name chimera state was introduced later in the work of Abrams and Strogatz [1] with the aim of remarking that, like the beast from Greek mythology, these states are made of incongruous elements. Later on, it was recognized that similar states were also observed in earlier works [29]. A series of studies have now pointed out that the phenomenon can be observed for phase oscillators [1, 2, 12, 15, 18, 31], as well as for other individual dynamics, e.g., neuron-like systems [7, 26, 33], chemical units [34], chaotic oscillators [19, 20], superconducting quantum interference device (SQUID) oscillators [10], and others.

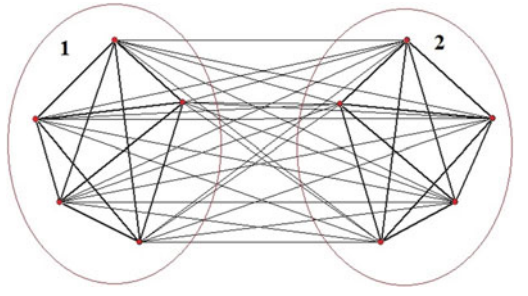
Initially, the appearance of chimera states was supposed to be strictly connected to the nonlocal character of the coupling of the structure, but it was then clarified that they can be induced in a variety of structures: one-dimensional [1, 12] as well as two-dimensional systems [15, 18, 31]; structures with global coupling or with not negligible amplitude dynamics [21, 27, 29, 30, 33, 36]; structures with time-varying connections [5], and so on. Besides numerical evidence, chimera states were also experimentally observed: coupled chemical oscillators [34], coupled map lattices [9], mechanical oscillators [16], populations of coupled electronic neurons [7] are a few examples of a plenty of different scenarios.

3.2 Chimera States in a Network of Two Interacting Populations

The aim of this section is to illustrate chimera states through an example. To simplify the discussion, we consider phase oscillators. As mentioned above, they represent the reduction to a single-variable dynamics (the phase) of oscillators whose motion is confined to a limit cycle as the result of a strong attraction to it, compared to the coupling with the other oscillators, that, on the contrary, is weak. These oscillators are, therefore, described by an equation of the type $\dot{\theta}_i = \omega_i$, where θ_i is the phase variable (an angle) and ω_i the natural oscillation frequency, i.e., the oscillation frequency of the i th oscillator when uncoupled from the others.

One of the simplest systems able to show chimera states [1] is formed by two interacting populations of such phase oscillators. In particular, the oscillators are considered identical ($\omega_i = \omega \forall i$) and labeled with two indices, σ and σ' , indicating

Fig. 3.1 Model used to illustrate chimera states: a network of two interacting populations, fully connected each other. Gray (black) links have weight ν (μ)



the population to which they belong, each formed by the same number ($N_\sigma = N/2$ with $\sigma = 1, 2$) of units.¹ The equations of the system read:

$$\dot{\theta}_i^\sigma = \omega + \sum_{\sigma'=1}^2 \frac{1}{N_{\sigma'}} \sum_{j=1}^{N_{\sigma'}} K^{\sigma\sigma'} \sin(\theta_j^{\sigma'} - \theta_i^\sigma - \alpha) \quad (3.1)$$

where θ_i^σ is the phase of oscillator i in population σ , ω is the natural frequency, identical for all the units, α is the phase lag, and $K^{\sigma\sigma'}$ are the couplings between the populations σ and σ' or within a single population if $\sigma' = \sigma$. In particular, in accordance with [1], we set $K^{11} = K^{22} = \mu > 0$, $K^{12} = K^{21} = \nu > 0$ and consider $\mu > \nu$. This means that the systems are formed by two clusters, where the coupling inside each cluster is with all the other individuals and with the same weight, μ , while the coupling with the elements of the other populations (all to all connections also in this case) is weaker (weight ν). By rescaling time, the two coupling values may be chosen to satisfy the condition $\mu + \nu = 1$. The network is shown in Fig. 3.1.

To monitor the coherence of each population the following order parameter is defined:

$$r_\sigma(t) = \left| \left\langle e^{i\theta_i(t)} \right\rangle_\sigma \right| \quad (3.2)$$

with $\sigma = 1, 2$ and $\iota = \sqrt{-1}$. $\langle \cdot \rangle_\sigma$ denotes the average over all elements in population σ .

As a function of the values of the intra- and inter-population couplings μ and ν and parameter α , the system displays distinct dynamical regimes, including chimera states of different nature. A detailed discussion of these behaviors is given in [2], here we briefly report on one of them, called *long-period breather chimera*. This state (see also Fig. 3.2, obtained for $A = \mu - \nu = 0.35$ ($\mu = 0.675$, $\nu = 0.325$), and $\alpha = \pi/2 - 0.1$) is characterized by a population that is synchronized while the other has an order parameter with large fluctuations, indicating that the level of coherence in this population is not constant (the units of this population do not reach

¹In this section, to keep notation coherent with the original papers on chimera states, the coupling coefficient is indicated with K , whereas σ is an index labeling oscillator populations.

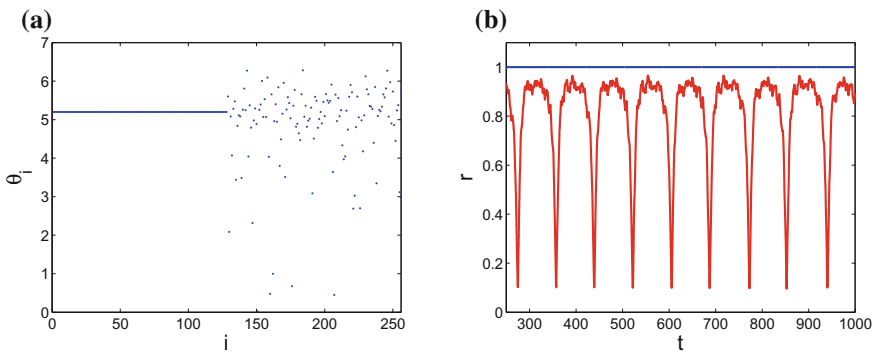


Fig. 3.2 **a** Snapshot of the phase variables θ_i of the chimera state in system (3.1): oscillators in the first group ($i \in N_1 = \{1, \dots, 128\}$) are synchronized, while the remaining units ($i \in N_2 = \{129, \dots, 256\}$) are not. **b** Order parameter r_σ versus time: r_1 (blue line) is constant, while r_2 (red line) has fluctuations, indicating that the behavior is a long-term breathing chimera state. The network consists of $N_1 = N_2 = \frac{N}{2} = 128$ phase oscillators

synchronization). Despite the perfect symmetry of the structure, the chimera state solution is not symmetric. In particular, in Fig. 3.2 the first population is synchronized and the second not, but clearly the opposite may also be observed with the first population behaving in an incoherent way and the second in the synchronous state. The difference is due to the initial conditions which may lead the system to one or the other chimera state.

For the same set of parameters, another regime also exists: global synchronization may be also observed. We also note that other chimera states are possible for other values of the parameters: the stable chimera, where the incoherent population has a constant order parameter, and the breathing chimera, similar to the long-period breather but with smaller fluctuations in the order parameter of the incoherent population.

The system behavior may be numerically investigated with the following MATLAB® code.

Code for Chimera States in Kuramoto Oscillators

The code below reports the construction of the adjacency matrix A for the two interconnected populations of Kuramoto oscillators and the integration of the network equations with parameters to obtain the chimera state of Fig. 3.2. It is sufficient to change μ and ν to generate the other types of chimera states (stable chimera for $\mu = 0.6$ and $\nu = 0.4$; breathing chimera for $\mu = 0.64$ and $\nu = 0.36$).

The code also plots the two order parameters $r_\sigma(t)$ (in the code they are $R1$ and $R2$) and the snapshot of the phase variables θ_i in the last time instant. As discussed in this chapter, chimera states depend on initial conditions, for this reason, while simulating the systems one has to pay attention to this issue. A criterion for the choice of the suitable set of initial conditions is to run simulations near the chimera states.

```

%chimera states in a structure of two coupled populations of
oscillators

%% integration parameters
dt=0.01;
max_step=1000/dt;

%% construction of the adjacency matrix
N=2*128;
miu=0.675;
ni=0.325;
A=[miu*ones(N/2) ni*ones(N/2);ni*ones(N/2) miu*ones(N/2)];
A=A-eye(N)*miu;

%% parameters
alpha=pi/2-0.1;
omega=ones(N,1);

%% initialization
x=zeros(N,max_step);
x(:,1)=2*pi*rand(N,1);
dxdt=zeros(N,1);
R=zeros(max_step,1);

for t=1:max_step

    %% main integration
    coup=zeros(N,1);
    for i=1:N
        coup(i)=sum(A(:,i).*sin(x(:,t)-x(i,t)-alpha));
        dxdt(i)=omega(i)+2/N*coup(i);
        x(i,t+1)=mod(x(i,t)+dt*(dxdt(i)),2*pi);
    end
end

%% order parameters
R1=abs(mean(exp(j*(x(1:N/2,:))),1));
R2=abs(mean(exp(j*(x(N/2+1:end,:))),1));

%% plots
figure,plot([1: max_step+1]*dt,R1),hold on,plot([1: max_step+1]*dt
,R2,'r')
figure,plot(x(:,end),'.')

```

3.2.1 Ott–Antonsen Reduction

System (3.1) offers the possibility of some analytical considerations. In particular, in [24] an approach to reduce the system (3.1) of N differential equations into a low-dimensional one is introduced. The reduction is possible only in the thermodynamic limit, $N \rightarrow \infty$. Under this hypothesis, the phase in each population σ can be expressed by a probability density function $f^\sigma(\theta, t)$ obeying to the continuity equation

$$\frac{\partial f^\sigma}{\partial t} + \frac{\partial (f^\sigma v^\sigma)}{\partial \theta} = 0 \quad (3.3)$$

where v^σ is the phase velocity of the oscillators in population σ . Expanding $f^\sigma(\theta, t)$ in Fourier series we get

$$f^\sigma(\theta, t) = \frac{1}{2\pi} \left[1 + \sum_{n=1}^{\infty} \{ [f_n(\theta, t)e^{i\theta}]^n + [f_n^*(\theta, t)e^{-i\theta}]^n \} \right] \quad (3.4)$$

We adopt the ansatz proposed in [24] $f_n(\theta, t) = a_\sigma(\theta, t)^n$, with $|a_\sigma(\theta, t)| \leq 1$ to avoid divergence of the series; this means that we use the same function a_σ in all the Fourier harmonics (that is, raised to n th power in the n harmonic).

We define a complex-order parameter

$$z_\sigma(t) = \sum_{\sigma'=1}^2 K^{\sigma\sigma'} \int e^{i\theta'} f^{\sigma'}(\theta', t) d\theta' \quad (3.5)$$

that, by using the definition of the density function f^σ , becomes

$$z_\sigma(t) = \sum_{\sigma'=1}^2 K^{\sigma\sigma'} a_{\sigma'}^*(t) \quad (3.6)$$

The phase velocity can be written as follows:

$$v^\sigma(\theta, t) = \omega + \frac{1}{2t} (z_\sigma e^{-i\alpha} e^{-i\theta} - z_\sigma^* e^{i\alpha} e^{i\theta}) \quad (3.7)$$

Substituting into the continuity equation, we find that the particular choice of f^σ is solution if

$$\frac{da_\sigma}{dt} + i\omega a_\sigma + \frac{1}{2} [a_\sigma^2 z_\sigma e^{-i\alpha} - z_\sigma^* e^{i\alpha}] = 0 \quad (3.8)$$

Introducing polar coordinates $a_\sigma = \rho_\sigma e^{-i\phi_\sigma}$, (ϕ_σ represents the center of the density function f^σ and ρ_σ its amplitude) we obtain two identical equations for each cluster, for $\sigma = 1$

$$\dot{\rho}_1 = \frac{1 - \rho_1^2}{2} [\mu \rho_1 \cos \alpha + \nu \rho_2 \cos(\phi_2 - \phi_1 - \alpha)] \quad (3.9)$$

$$\dot{\phi}_1 = \omega - \frac{1 + \rho_1^2}{2} [\mu \rho_1 \sin \alpha + \nu \rho_2 \sin(\phi_1 - \phi_2 + \alpha)] \quad (3.10)$$

and similarly for $\dot{\rho}_2$ and $\dot{\phi}_2$, by exchanging 1 with 2. By defining the phase difference $\psi = \phi_2 - \phi_1$ between the two populations, the dimensionality of the system reduces to three.

The chimera state reported in Fig. 3.2 corresponds to a stationary solution where one of the populations is synchronized, $\rho_\sigma = 1$, and the other is not $\rho_\sigma < 1$. If we suppose that population one is synchronized, $\rho_1 = 1$, and we can indicate $\rho_2 = r$.

The system can be fully represented by two equations

$$\dot{r} = \frac{1-r^2}{2}[\mu r \cos \alpha + \nu \cos(\psi - \alpha)] \quad (3.11)$$

$$\dot{\psi} = \frac{1+r^2}{2r}[\mu r \sin \alpha - \nu \sin(\psi - \alpha)] - \mu \sin \alpha - \nu r \sin(\psi + \alpha) \quad (3.12)$$

that can be analytically studied. A stable chimera state corresponds to a stable equilibrium point in this system, while a breathing chimera corresponds to a stable limit cycle. These two possible attractors, in any case, coexist with the stable equilibrium point $r = 1$ and $\psi = 0$, which indicates global synchronization in the network.

3.3 Chimera States in Nonlocally Coupled Rings

We now look back to the structure where the first evidence of chimera states by Kuramoto and Battogtokh was given and consider a ring of N nonlocally coupled oscillators, where each node is connected with a number, $2P$, of nearest neighbors (P for each side). In the limit cases, when $P = 1$ each node is connected only to its first neighbors and the system is a locally coupled structure, while when $P = N/2$ the structure is globally coupled. So, here we fix $1 < P < N/2$ to account for nonlocal coupling. The system equations are

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} \mathbf{H}(\mathbf{x}_j, \mathbf{x}_i) \quad (3.13)$$

where $\mathbf{x}_i = [x_{1,i} \ x_{2,i} \ \dots \ x_{n,i}]^T$ represents the state vector of the i th oscillator ($i = 1, \dots, N$), $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$, as usual, the uncoupled dynamics (identical over the network), σ the coupling strength, $2P$ the number of nearest neighbors to which each unit is coupled (P to the right and P to the left), $\mathbf{H}(\mathbf{x}_j, \mathbf{x}_i)$ is the coupling function. Boundary conditions are periodic so that all indices are modulo N .

In their study [12], Kuramoto and Battogtokh have considered phase oscillators. Their results have been then extended to other oscillators with more complex dynamics. Here, as in [22], we illustrate the case where the dynamics of the units is represented by the FitzHugh–Nagumo (FHN) oscillator with neuron-like dynamics as follows:

$$\varepsilon \dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} [b_{xx}(x_j - x_i) + b_{xy}(y_j - y_i)] \quad (3.14)$$

$$\dot{y}_i = x_i + a_i + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} [b_{yx}(x_j - x_i) + b_{yy}(y_j - y_i)]$$

where the components of the state vector of system i , with $i = 1, \dots, N$, have been indicated as x_i and y_i , i.e., $\mathbf{x}_i = [x_i \ y_i]^T$. In terms of reaction–diffusion equations, these variables are the activator and the inhibitor, respectively. They can also be interpreted as the voltage across the membrane and a generic variable to account for the dynamics of the ionic channels in the neuron. ε is the time scale ratio between the two variables and a_i is a threshold parameter indicating if the units are oscillatory ($|a_i| < 1$) or excitable ($|a_i| > 1$). Here, we assume that all the units are equal and in the oscillatory regime, i.e., $a_i = 0.5 \ \forall i$.

The coupling in Eq.(3.14) contains the direct and cross- coupling between the two variables x_i and y_i , modeled by a rotational coupling matrix

$$\mathbf{B} = \begin{pmatrix} b_{xx} & b_{xy} \\ b_{yx} & b_{yy} \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \quad (3.15)$$

that is, $\mathbf{H}(\mathbf{x}_j, \mathbf{x}_i) = \mathbf{B}(\mathbf{x}_j - \mathbf{x}_i)$.

Chimera states are characterized by one or more coherent domains coexisting with incoherent ones; in nonlocally coupled rings, the oscillators split into groups of adjacent nodes forming these domains, whose position in the structure is, however, not known a priori. Differently from the previous example, this makes impossible to define a priori the set of nodes over which the order parameter has to be averaged. Therefore, chimera states in nonlocally coupled rings are commonly characterized by a local order parameter, that is defined over a region of size ν and centered in i and measures the level of the spatial coherence of oscillators in that region:

$$R_i(t) = \left| \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} e^{i\theta_j} \right| \quad (3.16)$$

where θ_i is the phase variable of oscillator i , and $\mathcal{N}_i = \{j : |j - i| \leq \nu\}$. The phase variable θ_i of the units (that are limit cycle oscillators) is defined as $\theta_i = \arctan(y_i/x_i)$. The local-order parameter allows to assess if an oscillator belongs to the coherent (where it takes values close to one) or incoherent domain (where it has much lower values).

In Fig.3.3 the chimera state obtained in a ring of $N = 200$ nonlocally coupled FHN units for $P = 70$, $\sigma = 0.15$, and $\beta = \pi/2 - 0.1$ is illustrated. Figure 3.3a reports the local order parameter $R_i(t)$ as in Eq.(3.16), showing that the coherent segment is in the central region of the array. This is also evident from the snapshot of the phase variables y_i illustrated in Fig. 3.3b. The result is obtained with the following MATLAB® code.

Code for Chimera States in a Ring of Nonlocally Coupled FHN Units

```
%chimera states in a ring of non-locally coupled FHN units
```

```
%integration parameters
```

```
dt=0.01;  
steps=1000/dt;
```

```
%network
```

```
m=70;
```

```

N=200;
Ab=diag(ones(N-1,1),1) + diag(ones(N-1,1),-1);
Ab(end,1)=1;
Ab(1,end)=1;
A1=zeros(N);
for ii=2:m
    A1=A1+diag(ones(N-ii,1),ii)+diag(ones(N-ii,1),-ii)+diag(ones(
        ii,1),N-ii)+diag(ones(ii,1),-N+ii);
end
A=Ab+A1;

%initialization
x=zeros(N,steps+1);
y=zeros(N,steps+1);
Ri=zeros(N,steps+1);

%initial conditions
X(:,1)=2*rand(N,1)-1;
Y(:,1)=2*rand(N,1)-1;
x(:,1)=2*(X./sqrt(X.^2+Y.^2));
y(:,1)=2*(Y./sqrt(X.^2+Y.^2));

%parameters
alpha=pi/2-0.1;
eps=0.05;
sigma=0.15;
lambda=sigma/2/m;

%integration
for t=1:steps

    coupx=zeros(N,1);
    coupy=zeros(N,1);

    for ii=1:N
        coupx(ii)=sum(A(:,ii).*(cos(alpha)*(x(:,t)-x(ii,t))+sin(
            alpha)*(y(:,t)-y(ii,t))));
        coupy(ii)=sum(A(:,ii).*(-sin(alpha)*(x(:,t)-x(ii,t))+cos(
            alpha)*(y(:,t)-y(ii,t))));

        x(ii,t+1)=x(ii,t)+dt/eps*(x(ii,t)-1/3*x(ii,t)^3-y(ii,t)+
            lambda*coupx(ii));
        y(ii,t+1)=y(ii,t)+dt*(x(ii,t)+0.5+lambda*coupy(ii));
    end
end

%plot of the final snapshot
figure,plot(y(:,end),'.')

%local order parameter
niu=20;
theta2=atan2(y,x);

for ii=1:N
    if (ii<=niu)
        thetaN=theta2([N-abs(ii-niu):N,1:ii+niu],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    elseif (ii+niu<=N)
        thetaN=theta2([ii-niu:ii+niu],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    else
        thetaN=theta2([ii-niu:N,1:niu-abs(N-ii)],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    end
end
end

```

```
%space-time plot of local order parameter
figure, imagesc(1:N, dt:dt:steps*dt, Ri')
```

One of the issues related to chimera states in nonlocally coupled rings is whenever the state is stable or not, and only a few studies report an exhaustive analysis of this problem. Wolfrum and Omel’chenko in [35] show how in a small system of N ($N = 40$) nonlocally coupled phase oscillators, the chimera state collapses into a completely coherent behavior. The important finding is that the transient nature of the chimera states vanishes with the size of the system, that is, the transient time grows together with it. For another type of chimera states known as *chimera death* and characterized by the coexistence of coherent and incoherent inhomogeneous steady states, it was demonstrated that this state is transient in a region of the parameter space $\sigma - P$ [28].

3.4 Control of Chimera States

In the previous examples, we have seen that a common feature of chimera states is their dependence on the initial conditions, so that the spatial position of the coherent, or incoherent, domain is not fixed. In addition, we have also noted that in many cases chimera states coexist with other regimes, e.g., global synchronization, or are transient behaviors. These issues are at the origin of the problem of controlling the onset of a chimera state and the position of its domains. Several techniques have been developed to address this problem [3, 6, 11, 23, 32]. They have been often tailored to the specific structure considered. For instance, in [32] a system of nonlocally coupled phase oscillators is controlled by acting on a global parameter (the phase lag) and linking it to the difference between the actual order parameter and its reference value. Similarly, another feedback technique is used for controlling small size systems by asymmetrically modulating the coupling strength in order to stabilize the chimera state, avoiding its drift and collapse [23]. Introducing asymmetries in the coupling

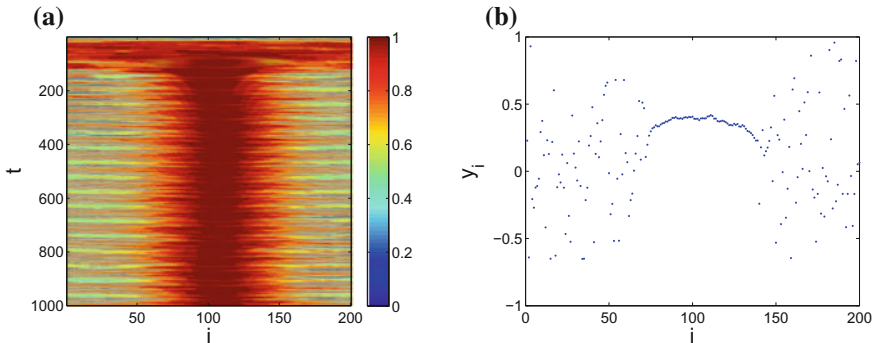


Fig. 3.3 **a** Local order parameter $R_i(t)$ as defined in Eq.(3.16) for a ring of $N = 200$ nonlocally coupled FHN units. **b** Snapshot of the phase variables y_i at $t = 1000$. The parameters are fixed as: $P = 70$, $\sigma = 0.15$, and $\beta = \pi/2 - 0.1$

kernel has been proved to be another suitable way to control the position of the coherent domain in nonlocally coupled phase oscillators [3]. A technique proposed for amplitude and phase oscillators, in particular FHN oscillators, is, instead, based on the exploitation of the local individual dynamics: in this case, control is performed by introducing inhomogeneities in proper positions of the ring so that local changes in the dynamics of some of the units yield the desired global behavior [11].

Here, we discuss in detail a method [6] based on the use of pinning control and able to induce a chimera state where the nodes belonging to one domain, either the coherent or the incoherent, are fixed by the control action. The dynamics of the nonlocally coupled units with pinning control is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \frac{\sigma}{2P} \sum_{j=i-P}^{i+P} \mathbf{H}(\mathbf{x}_j, \mathbf{x}_i) + \mathbf{u}_i \quad (3.17)$$

where $\mathbf{u}_i = [u_{1,i} \ u_{2,i} \ \dots \ u_{n,i}]^T$ are the controllers. They are applied to N_p ($N_p < N$) *pinned* nodes. Without loss of generality, we assume that these are the first N_p oscillators, that is, $\mathbf{u}_i = 0$ for $i = N_p + 1, \dots, N$. We distinguish between two different control objectives: (i) control of the position of the coherent domain, in short *coherent domain control* (CDC); and (ii) control of the position of the incoherent domain, in short *incoherent domain control* (IDC). For CDC, the controllers are given by

$$\mathbf{u}_i(t) = w\delta_i(\Theta(t) - \Theta(t - t_c))\mathbf{H}_c(\mathbf{x}_r, \mathbf{x}_i) \quad (3.18)$$

where w represents the strength of control, $\delta_i = 1$ if $i = 1, \dots, N_p$ while $\delta_i = 0$ otherwise, $\Theta(t)$ is the Heaviside step function and t_c is the period of application of the control, $\mathbf{x}_r$ is a reference trajectory generated by a further unit r , and $\mathbf{H}_c(\mathbf{x}_r, \mathbf{x}_i)$ the coupling function between the pinned units and the reference. The oscillator r acts as a leader in driving all the pinned oscillators toward a common (synchronized) trajectory which is slightly different from the uncoupled one in order to avoid to drive all the structure to the regime of global synchronization. For this reason, the reference node obeys to the same dynamical equations of the other network nodes but has different parameters.

In the case of IDC, we set the controllers as identically independently distributed random variables:

$$\mathbf{u}_i(t) = \delta_i(\Theta(t) - \Theta(t - t_c))\sqrt{2D}\mathbf{H}_n\xi_i(t) \quad (3.19)$$

where $\xi_i(t)$ represents additive Gaussian noise with zero mean, $\langle \xi_i(t) \rangle = 0$, and unit variance, $\langle \xi_i(t)\xi_i(t) \rangle = 1$. $D > 0$ is the noise intensity, and $\mathbf{H}_n \in \mathbb{R}^{n \times n}$ is a matrix of zero or one entries selecting to which variable the noise is applied.

It is assumed that only a portion of the structure is accessible and the target is to induce a coherent or an incoherent domain in that accessible region. Although the general objective of the control is similar for IDC and CDC techniques, inducing one type of behavior rather than the other requires a different strategy: one driving

the pinned units toward the synchronous state, and the other forcing them to behave in an incoherent way.

The parameters tuning the control are the period of application, t_c , after which the control is switched off, the fraction of pinned nodes, N_p/N , and the strength of the control, represented by the parameter w in Eq. (3.18) or the noise level D in Eq. (3.19). The CDC and IDC strategies are illustrated here for a fixed set of these parameters; readers interested to the discussion on the effect of the parameters are referred to Ref. [6]. In particular, we consider here $N = 200$ FHN units with $\sigma = 0.15$, $P = 70$, $\varepsilon = 0.05$, $a = 0.5$, and $\beta = 1.47$ so that the system supports a chimera state with one coherent domain coexisting with an incoherent one. Without control the position of the domains along the structure depends on the initial conditions, while, when the control is applied, it is determined by the sites of pinning control. In the case of CDC, the controllers $\mathbf{u}_i$ are given by Eq. (3.18) with

$$\mathbf{H}_c(\mathbf{x}_r, \mathbf{x}_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{x}_r - \mathbf{x}_i) \quad (3.20)$$

and $\mathbf{x}_r$ obeying to:

$$\begin{aligned} \varepsilon_r \dot{x}_r &= x_r - \frac{1}{3} x_r^3 - y_r \\ \dot{y}_r &= x_r + a_r \end{aligned} \quad (3.21)$$

where $a_r = 0.5$ and $\varepsilon_r = 0.06$, so that the reference trajectory is generated by a FHN system with slightly different parameters. In the case of IDC, we have considered Eq. (3.19) with $\mathbf{H}_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Figure 3.4 shows the results for $N_p = 100$ pinned nodes (the period of activation of the control is $t_c = 150$). Figure 3.4 clearly shows how, after switching off the control, a chimera state appears with the coherent (incoherent) domain located in the region of pinned nodes, $i = 1, \dots, 100N_p$. The initial conditions for both panel in Fig. 3.4 are the same. We observe that in the case of IDC the external noise significantly influences the system behavior. This result is different from [14] where chimera states are shown to be robust to the presence of noise as the noise intensity is several orders of magnitude larger.

The MATLAB[®] code to control chimera states in the ring of nonlocally coupled FHN units is reported in the following. Both strategies are implemented and may be selected commenting/uncommenting a few lines of the code. The simulation in the code starts from random initial conditions.

Code for Pinning Control of Chimera States

```
%pinning control of chimera states

%integration parameters
dt=0.01;
steps=1000/dt;

%network
m=75;
```

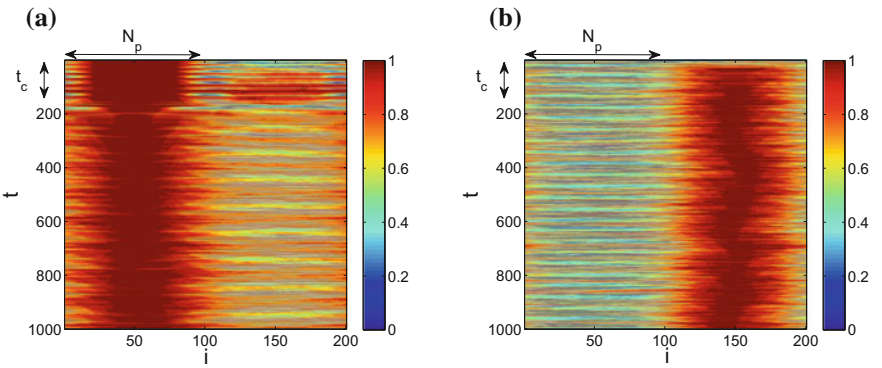


Fig. 3.4 Local order parameter $R_i(t)$ as in Eq.(3.16) with $\nu = 20$, for a ring of nonlocally coupled FHN. The system starts from the same initial condition of the simulation in Fig.3.3. **a** Results of CDC with $w = 0.25$, $\varepsilon_r = 0.06$, and $a_r = 0.5$; **b** Results of IDC with $D = 0.125$. In both the cases, we have fixed $N_p = N/2$ and $t_c = 150$. The other parameters are $N = 200$, $P = 70$, $\sigma = 0.15$, $\beta = 1.47$, $\varepsilon = 0.05$, and $a = 0.5$

```

N=200;
Ab=diag(ones(N-1,1),1) + diag(ones(N-1,1),-1);
Ab(end,1)=1;
Ab(1,end)=1;
A1=zeros(N);
for ii=2:m
    A1=A1+diag(ones(N-ii,1),ii)+diag(ones(N-ii,1),-ii)+diag(ones(
        ii,1),N-ii)+diag(ones(ii,1),-N+ii);
end
A=Ab+A1;

%initialization
x=zeros(N,steps+1);
y=zeros(N,steps+1);
theta2=zeros(N,steps+1);
Ri=zeros(N,steps+1);

%initial conditions
X(:,1)=2*rand(N,1)-1;
Y(:,1)=2*rand(N,1)-1;
x(:,1)=2*(X./sqrt(X.^2+Y.^2));
y(:,1)=2*(Y./sqrt(X.^2+Y.^2));
xr(1,1)=2;
yr(1,1)=0;

%parameters
alpha=pi/2-0.1;
eps=0.05;
eps2=0.06;
ar=0.5;
sigma=0.15;
lambda=sigma/2/m;

%CDC control
F=zeros(N,1);
F(1:100,1)=0.25;

%IDC control
% G=zeros(N,1);

```

```

% G(1:100,1)=0.5;

for t=1:steps

    if t>15000
        F=zeros(N,1);
        G=zeros(N,1);
    end

    coupx=zeros(N,1);
    coupy=zeros(N,1);

    for ii=1:N
        coupx(ii)=sum(A(:,ii).*(cos(alpha)*(x(:,t)-x(ii,t))+sin(
            alpha)*(y(:,t)-y(ii,t))));
        coupy(ii)=sum(A(:,ii).*(-sin(alpha)*(x(:,t)-x(ii,t))+cos(
            alpha)*(y(:,t)-y(ii,t))));

        %CDC control
        x(ii,t+1)=x(ii,t)+dt/eps*(x(ii,t)-1/3*x(ii,t)^3-y(ii,t)+
            lambda*coupx(ii)+F(ii)*(xr(1,t)-x(ii,t)));
        y(ii,t+1)=y(ii,t)+dt*(x(ii,t)+0.5+lambda*coupy(ii)+F(ii)*(
            yr(1,t)-y(ii,t)));

        %IDC control
        %x(ii,t+1)=x(ii,t)+dt/eps*(x(ii,t)-1/3*x(ii,t)^3-y(ii,t)+
            lambda*coupx(ii)+G(ii)*randn);
        %y(ii,t+1)=y(ii,t)+dt*(x(ii,t)+0.5+lambda*coupy(ii));
    end

    %ref node
    xr(1,t+1)=xr(1,t)+dt/eps2*(xr(1,t)-1/3*xr(1,t)^3-yr(1,t));
    yr(1,t+1)=yr(1,t)+dt*(xr(1,t)+ar);

end

%local order parameter
niu=20;
theta2=atan2(y,x);

for ii=1:N
    if (ii<=niu)
        thetaN=theta2([N-abs(ii-niu):N,1:ii+niu],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    elseif (ii+niu<=N)
        thetaN=theta2([ii-niu:ii+niu],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    else
        thetaN=theta2([ii-niu:N,1:niu-abs(N-ii)],:);
        Ri(ii,:)=abs(mean(exp(i*thetaN)));
    end
end

%space-time plot of local order parameter
figure,imagesc(1:N,dt:dt:steps*dt,Ri')

```

3.5 Experimental Observations of Chimera States

In the previous sections, we have seen as chimera states emerge in a system of interacting units, and which conditions are necessary for the onset of this intriguing phenomenon. The large number of theoretical studies shed light on the fact that

chimera states may appear in a wide variety of networks with different types of units and coupling topologies, resulting in a significant importance of the state in various fields. One of the most cited phenomena that is related to chimera states is the unihemispheric sleep in animals [25], where one half of the brain is sleeping while the other half stays awake. Important is also the similarity between neural chimera states and neural bump states, which has been proposed as a model for visual orientation tuning in the cortex [13].

Many efforts have been devoted to the experimental observation of this phenomenon in artificial setups. In 2012, two independent groups reported evidence of chimera states in laboratory. The first paper [34] studies a population of N coupled Belousov–Zhabotinsky (BZ) oscillators divided into two clusters; following the scheme proposed in [1] and illustrated in Sect. 3.2, the units are strongly coupled with the members of each group and weakly coupled with the members of the other group; the coupling is controlled by a computer and the particles are monitored using a CCD camera. The second experiment shows chimera states in a coupled map lattice made by a liquid-crystal spatial light modulator [9]. Also, in this case, the setup includes a computer to realize the coupling and the feedback.

The first totally analog experiment, realized without the need of a computer, was proposed by Martens and colleagues [16]. This experiment also reproduces the theoretical model of the two populations discussed by Strogatz and Abrams [1]. In the setup, schematically shown in Fig. 3.5, N identical metronomes with a nominal beating frequency f were placed on two swings connected by a spring. There, the strong coupling is obtained through the vibrations of the swing, while the spring realizes the weak coupling. The chimera states are observed for given values of the spring constant. A fascinating video [17] shows the synchronous beating of the metronomes in one plate and the asynchronous one in the other plate.

An experiment realized with electronic circuits (populations of coupled electronic neurons [7]) is now discussed. The core of the experimental setup is an electronic circuit implementing the dynamics of a FHN neuron designed and realized with off-the-shelf components. The experiment has been carried out on $N = 10$ circuits

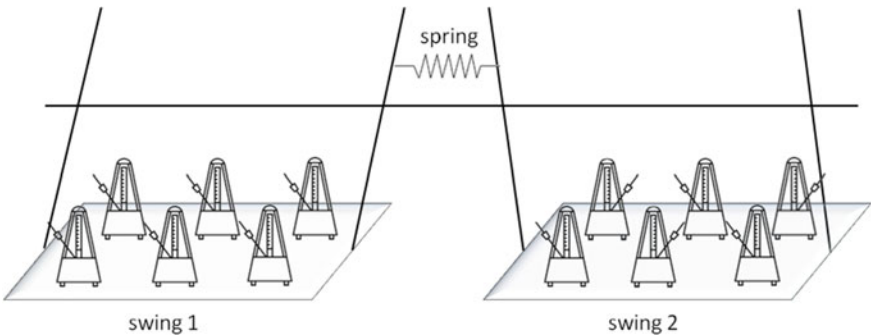


Fig. 3.5 Schematic of the experimental setup used in [16] to observe chimera states. The metronomes placed on swing 1 are synchronized while the metronomes in the second swing are desynchronized

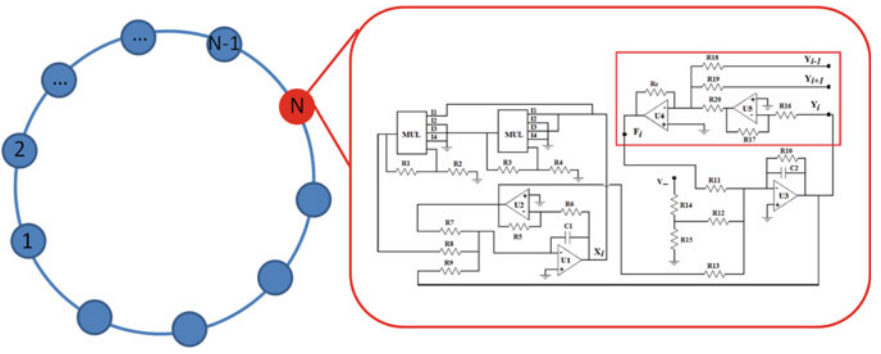


Fig. 3.6 The experimental setup consists of a chain (either open or closed) of $N = 10$ coupled FHN oscillators. Each node represents an analog circuit; the details for the realization of the circuits are reported in [7]

coupled either in an open chain or in a ring configuration, where each circuit is connected with its nearest neighbors, as schematically reported in Fig. 3.6.

The dimensionless equations describing the FHN circuit neuron are

$$\begin{aligned}\dot{x}_i &= x_i - y_i - ax_i^3 \\ \dot{y}_i &= \varepsilon(x_i - by_i + c + f_i)\end{aligned}\quad (3.22)$$

where x_i and y_i are the state variables of neuron i , with $i = 1, \dots, N$, and a, b, c are parameters governing the single-neuron dynamics. The term f_i is the coupling of neuron i with neurons $i - 1$ and $i + 1$, and is given by

$$f_i = \sigma(y_{i-1} + y_{i+1} - 2y_i) \quad i = 1, \dots, N \quad (3.23)$$

for the closed chain (ring configuration) and by

$$f_i = \begin{cases} \sigma(y_{i+1} - y_i) & i = 1 \\ \sigma(y_{i-1} + y_{i+1} - 2y_i) & i = 2, \dots, N - 1 \\ \sigma(y_{i-1} - y_i) & i = N \end{cases} \quad (3.24)$$

for the open chain. The parameter σ represents the coupling coefficient.

In the electronic implementation of Eq.(3.22) each state variable is associated with the voltage across a capacitor and operational amplifiers and analog multipliers are used to implement the mathematical operations in the model. The design follows a standard approach for nonlinear circuit implementations [4].

Several dynamical regimes have been experimentally identified in the system: a disordered state where all neurons oscillate independently and with their own phase; a regime of phase synchronization of all the FHN units, where all frequencies are locked and the phase differences between units remain constant in time; a regime of spatially patterned oscillation death, that is a regime, where all the neurons reach a steady state, which, however, varies from neuron to neuron forming an inhomogeneous pattern;

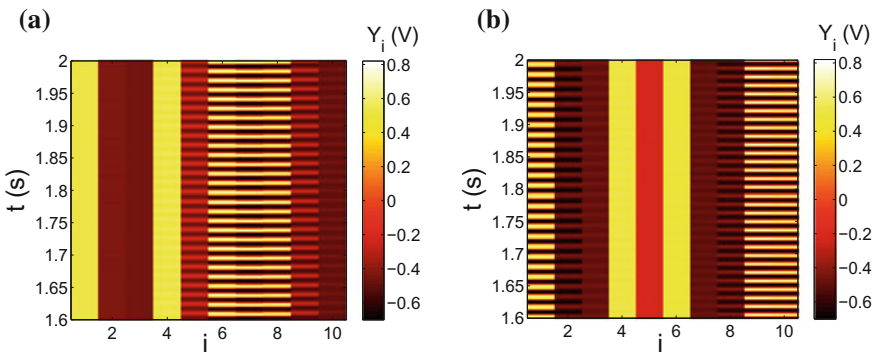


Fig. 3.7 Two chimera states obtained in the experimental setup in Fig. 3.6 for the two configurations studied: **a** ring with $\sigma = 0.575$ and **b** open chain for $\sigma = 0.7$

and a chimera state, where coherent oscillations in a part of the ring (or in part of the open chain) coexist with a spatially patterned stationary state for a subset of the neurons. As in many other systems showing chimera states, the appearance of a given dynamical regime is function of the coupling. One important finding is that, for the same coupling strength but different initial conditions, more than one dynamical states are simultaneously possible and stable, and different patterns of the same type have been found. Figure 3.7 illustrates examples of chimera states obtained in the two configurations, closed (Fig. 3.7a) and open (Fig. 3.7b) chain.

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Chapter 4

Cluster Synchronization

In this chapter, synchronization of a network into groups with distinct behavior is discussed. Both networks with heterogeneous units and networks with identical node dynamics are considered, and the peculiarities of this form of synchronization in the two cases are illustrated.

4.1 Introduction

In Chaps. 2 and 3 we have seen that the units of a network can organize themselves (or be controlled to) in the collective state of global synchronization or of chimera states. There is another important collective phenomenon occurring in ensembles of coupled oscillators that is called *cluster synchronization*. While in global synchronization all the units follow the same trajectory regardless of their initial conditions, in cluster synchronization the network splits into different groups of nodes where the units within each cluster converge to the same behavior that is distinct for each group.

We can distinguish different scenarios of cluster synchronization: one where the network is made of identical units and one where the network is not homogeneous and some sources of mismatch are present. The mismatch in a network can be realized by units with slightly different parameters, or by a coupling function promoting the intra-cluster coupling while weakening the inter-cluster one. In the first scenario, the emergence of clusters in a network only depends on the topology, and, in particular, it has been demonstrated that clustering reflects network symmetries. Instead, in the second case the clusters are determined by the weights of the links and, possibly, can be ad hoc defined by adjusting them, or emerge as the result of the distribution of some parameters of the oscillators (e.g., the natural frequency).

In this chapter, we aim to show the characteristics of cluster synchronization and the emergence of patterns of groups of synchronized nodes in networks of either

heterogeneous or homogeneous units. We will first consider the case of heterogeneous units and then that of identical nodes, where we will discuss a recently highlighted important ingredient, that is the role of symmetries for group formation.

4.2 Cluster Synchronization in Networks of Heterogeneous Units

In many real-world networks, a common scenario is represented by the situation in which global synchronization is not possible, because the units are not identical or because the coupling is not strong enough. In particular, in the regime of weak coupling it is common to observe that the strength of interaction is not enough to yield global synchronization but it still allows some form of organization in the units interacting to create patterns of synchronized nodes following the connection topology. This is evident if one studies the so-called path to synchronization, from small to large coupling, as discussed in [5]. Here, we do this in SF and ER networks of periodic oscillators, with a particular focus on the cluster formation in the regime of weak coupling.

We consider a network of N coupled Stuart–Landau (SL) oscillators [13]. The SL model [2, 11] is important because it allows one to bridge the gap between the simplicity of the Kuramoto phase model and the completeness of the amplitude-phase framework. In particular, the SL model may describe, by means of a single parameter, both the behavior close to the Hopf bifurcation, which represents an instance where the attraction to the limit cycle is weak, and the case where, on the contrary, the motion is confined to the limit cycle and the system behaves as a pure phase oscillator.

Each SL oscillator is characterized by two variables, x_i and y_i , whose dynamical evolution is described by

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \begin{pmatrix} \alpha - x_i^2 - y_i^2 & -\omega_i \\ \omega_i & \alpha - x_i^2 - y_i^2 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad (4.1)$$

where $\sqrt{\alpha}$ and ω_i are, respectively, the amplitude and the natural frequency of oscillator i , i.e., the frequency of oscillations when uncoupled from the rest of the network.

In complex coordinates, the system in Eq. (4.1) becomes

$$\dot{u}_i = (\alpha + i\omega_i - |u_i|^2)u_i \quad (4.2)$$

where u_i is a complex variable, $u_i = \rho_i e^{i\theta_i} = x_i + iy_i$, being $\rho_i = \sqrt{x_i^2 + y_i^2}$ the amplitude of oscillator i and $\theta_i = \arctan(y_i/x_i)$ its phase.

The Stuart–Landau oscillator is quite interesting as it represents the normal form of an oscillator subject to a Hopf bifurcation. For an isolated unit, as in Eq. (4.2), one can easily check that $u_i(t) = \sqrt{\alpha} e^{i(\omega_i t + \theta_0)}$ is a solution. This means that in the absence of coupling for $\alpha > 0$ the systems oscillates with constant amplitude

$\rho = \sqrt{x_i(t) + y_i(t)} = \sqrt{\alpha}$ and phase given by $\theta(t) = \tan^{-1} \frac{y_i(t)}{x_i(t)} = \omega_i t + \theta_0$, where θ_0 is the initial phase. The parameter α is thus the bifurcation parameter, with the critical point at $\alpha = 0$, where the eigenvalues of the Jacobian around the origin (i.e., the equilibrium point) cross the imaginary axis.

The dynamics of node i of a network of Stuart–Landau oscillators is written as

$$\dot{u}_i = (\alpha + i\omega_i - |u_i|^2)u_i + \sigma \sum_{j=1}^N a_{ij}(u_j - u_i) \quad (4.3)$$

The strength of the coupling is controlled by σ , while $A = \{a_{ij}\}$ represents the adjacency matrix of the network. Phase synchronization in system (4.3) is monitored through the order parameter r . To define this parameter we consider the phase variable of each oscillator, θ_i , and measure the degree of synchronization of any (connected or not) pair of oscillators:

$$r_{ij} = |\langle e^{i[\theta_i(t) - \theta_j(t)]} \rangle_T| \quad (4.4)$$

where $\langle \cdot \rangle_T$ denotes averaging over a large enough time interval T . We consider two nodes as synchronized when $r_{ij} > \delta$, where δ is a constant threshold that we fix to $\delta = 0.8$. Nonetheless, the results are robust as other values of δ yield qualitatively the same outcomes.

The overall degree of synchronization is then measured by the parameter r , defined by averaging r_{ij} over all the possible pairs of nodes:

$$r = \frac{1}{N(N-1)} \sum_{i,j=1}^N r_{ij} \quad (4.5)$$

This parameter represents the degree of global synchronization in the whole network. Values of r close to 1 indicates that all the pairs of oscillators are phase synchronized, whereas $r = 0$ the absence of synchronization. Note that this parameter differs from that illustrated in Chap. 3 aimed at measuring when all the units have exactly the same phase, rather than a constant phase difference.

We study the dynamics in Eq. (4.3) in SF and ER networks with $N = 200$ nodes and first monitor the system behavior through the parameter r (reported in Fig. 4.1) as the coupling coefficient σ is increased. Both networks reach synchronization, but in ER networks the transition to the synchronized state occurs for a critical value of the coupling coefficient smaller than in SF networks. This is not the only difference in the process of synchronization of the networks considered: in fact, the route that the system follows to reach global coherence strongly depends on the characteristic features of the graphs of interactions. This important finding has been explored in [5] where it has been highlighted that, during the synchronization process, the network nodes organize and form synchronization patterns. To study them, the links of the network associated to pairs of synchronized nodes are monitored for growing values of the coupling coefficient. This is done by checking for which links of $\mathcal{E}$ it holds

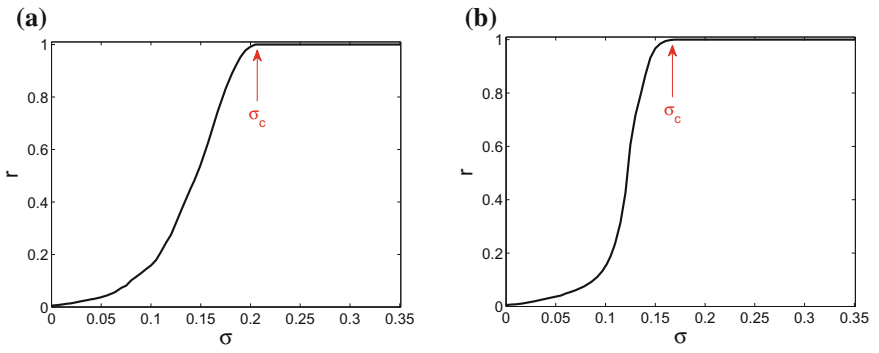


Fig. 4.1 Evolution of the order parameter r with respect to the coupling coefficient σ for **a** an SF network and **b** an ER network. The networks have $N = 200$ nodes and $\langle d \rangle = 4$. The red arrows indicate the critical value of the coupling strength for which the networks is globally synchronized. As expected the value σ_c on ER is smaller then in SF ($\sigma_c^{ER} = 0.17$, $\sigma_c^{SF} = 0.20$)

that $r_{ij} > \delta$. The connected components with such links represent the synchronized clusters.

Figure 4.2 shows the synchronized clusters for the two networks, illustrating the different path to synchronization of ER and SF networks. As demonstrated in [5], when the coupling is not strong enough to yield global synchronization, the network nodes arrange themselves into groups of synchronized units. In particular, in SF networks, due to degree heterogeneity, the units display hierarchical organization with respect to the collective oscillations, with the hubs acting as the leaders in the formation of the clusters which are the main responsible of the collective dynamics of the whole network.

The main difference between the two networks is represented by characteristic features of the topological structure; in SF networks there is a small portion of nodes with a large degree, the hubs, while most of the nodes have small degree, while in ER networks the degree distribution as in Eq. (1.5) is more homogeneous. This implies that in SF networks the most significant contribution to synchronization is given by the hubs, while in ER networks all the nodes contribute equally. In SF structures, even in the regime of weak coupling, hubs are able to adjust their behavior and form a cluster of synchronized nodes, and in most cases a giant component emerges. In ER topologies the weak coupling induces the formation of small clusters, often formed by a single pair of nodes. While increasing the coupling coefficient, the dominant cluster in SF networks recruits other nodes thus increasing its size; in ER networks, instead, since all the nodes contribute at the same time to synchronization, small clusters merge together and, for an intermediate value of the coupling, the network is almost globally synchronized.

Cluster synchronization may be also obtained as the results of adaptivity of the links. In [6], an adaptive law reinforcing the coupling between synchronized nodes in the presence of limited resources for the nodes to link with the other network units is

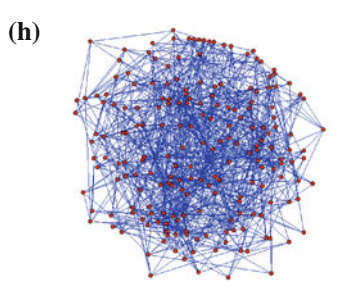
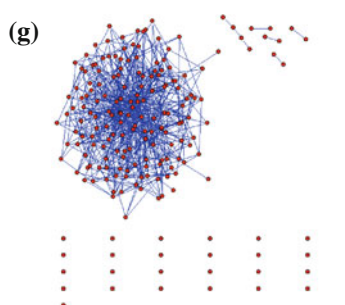
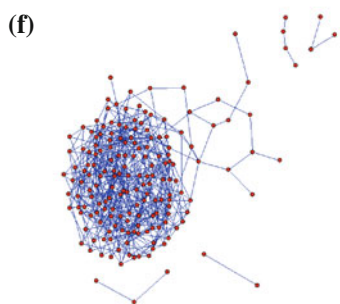
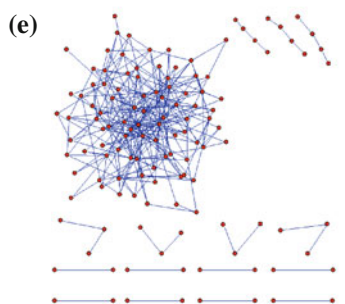
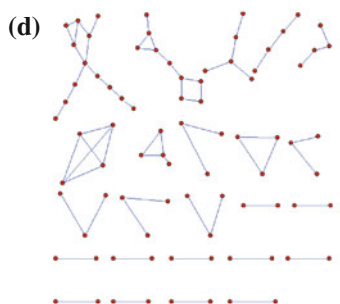
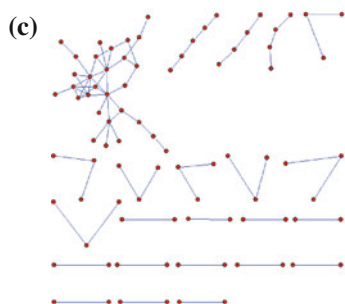
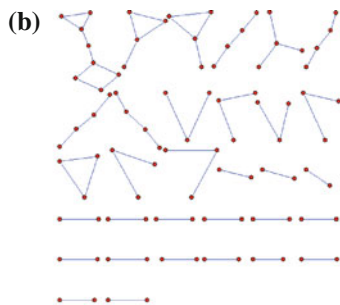
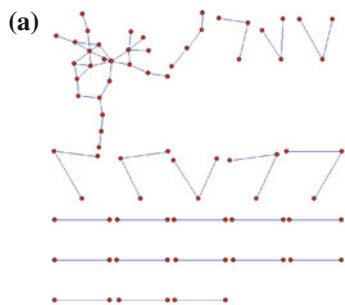


Fig. 4.2 (Continued)

◀ **Fig. 4.2** Synchronized clusters for different values of the coupling coefficient σ . In the left panels, we report the behavior of SF networks, while in the right panels that of ER networks. Panels **a** and **b** report the behavior of clusters when $\sigma = 0.085$, for this value in the SF network there exists a quite large cluster of synchronized nodes, while for the ER network there are different small synchronized clusters. In panels **c** and **d** $\sigma = 0.1$, for this value the cluster of the SF network is grown after having recruited other nodes, while in the ER network some small clusters merged. When $\sigma = 0.13$, the giant component of the SF network (panel **e**) includes many nodes, while in the ER network (panel **f**) most of the small clusters already existing merged together to form a large synchronized cluster. Finally, for $\sigma = 0.17$ in the SF network (panel **g**) the process of aggregation to the giant component is still active, while in the ER networks (panel **h**) all the networks are synchronized. This latter value of coupling confirms that in the ER network global synchronization is attained for value of the coupling smaller than in the SF network

considered. Such a competitive adaptive mechanism leads the network to evolve and form meso- and macro-scale structures. At a certain value of the coupling coefficient several distinct synchronized communities form, while for coupling exceeding the threshold for synchronization all the nodes coalesce into a single community.

The previous examples demonstrate that cluster synchronization results as the effect of weak coupling in the network and, in addition, suggest a strategy to control cluster synchronization by intentionally introducing mismatches in the node dynamics or using adaptive laws which reinforce a proper set of links in the network.

4.3 Cluster Synchronization in Networks of Identical Nodes

4.3.1 Networks with Symmetries

Recent studies [12, 15] have pointed out the existence of a link between the formation of synchronization patterns and network symmetries in structures with identical units. In a network of N identical coupled units, the onset of cluster synchronization is the result of the properties of the network topology. This important finding has paved the way to a new branch of synchronization studies, where the intrinsic properties of the network are mainly responsible for the synchronization behavior. It is worth to note that symmetry plays an important role in many dynamical models, suggesting, thus, that it is a more general feature in the study of complex networks.

For the purpose of illustration, consider first a network made of $N = 5$ identical nodes, connected in a star-like configuration, as reported in Fig.4.3a, where, as usually, each node represents a n -dimensional dynamical system governed by the following equations:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N g_{ij} \mathbf{H} \mathbf{x}_j \quad (4.6)$$

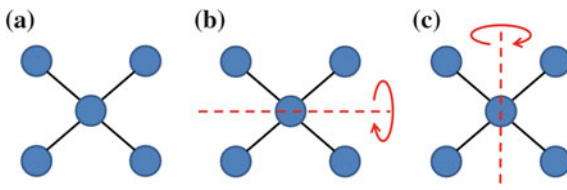


Fig. 4.3 Example of a simple network of five identical nodes **a** along with two types of symmetries: **b** mirroring around the horizontal axis and **c** mirroring around vertical axis

Before entering into the mathematical formalism to describe the symmetries of a network, let us look at Fig. 4.3a and note that, if we mirror the network around one of the two main axes, the horizontal (Fig. 4.3b) or the vertical one (Fig. 4.3c), the structure of the network remains the same, so that we say that mirroring around the horizontal axis (Fig. 4.3b) and around the vertical one (Fig. 4.3c) represent two symmetries of the network of Fig. 4.3a.

Symmetries in Fig. 4.3b, c leave the network unchanged. They can be viewed as permutations of some nodes of the network. These are also known as mirror/reflection symmetries, in fact they can be obtained by mirroring the object through one of the main axes. Other trivial symmetries can be recovered in the network, for example, by rotating the structure. In particular each node, except the central one, can be replaced by the node at the right (or at the left), or by considering rotation along one of the diagonal axes. These symmetries are important for the equations of motion: the identification of the groups of symmetries allow to assess which nodes (those within each group) may evolve into the same trajectory, while nodes in different groups form separate clusters. These concepts are more general and we now introduce them in a more formal context.

First, we briefly recall some properties of group theory [4] that are useful for the study of network symmetries, and more specifically for the study of synchronization in such networks.

We say that an object has a symmetry if a transformation/operation applied to the object leaves it unchanged. For graphs, symmetries are associated to automorphisms (i.e., permutations of the graph nodes) preserving the adjacency of nodes [7]. The symmetry operations of a graph form a mathematical group. Each element of the group is a symmetry of the graph and can be represented as a permutation matrix, R_g , i.e., a square matrix containing exactly one nonzero entry for each row and column, with $r_{ij} = 1$ if nodes i and j permute, and $r_{ij} = 0$ otherwise. The adjacency and the Laplacian matrix are invariant to graph symmetries which imply that $R_g A = A R_g$ and $R_g G = G R_g$. Furthermore, for any permutation matrix we have that $R_g R_g^{-1} = R_g^2 = I_N$.

Consider again the network in Fig. 4.3a, its adjacency matrix and the corresponding permutation matrices for the two symmetries in Fig. 4.3b, c, indicated as R_{g1} and R_{g2} , respectively, are given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}; R_{g1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; R_{g2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.7)$$

One can easily verify that the relation $R_g A = A R_g$ is satisfied for R_{g1} and R_{g2} .

In general, detection of graph symmetries is not as simple as in the illustrated example, but on the contrary is a challenging task. Except for graphs with a small number of nodes where trivial symmetries can be recognized by inspection, appropriate tools have to be used [12]. These methods are based on computational group theory, and through discrete algebra allow one to obtain the symmetry groups and the permutation orbits that, applied to network nodes, partition the graph into M distinct clusters.

As the Laplacian matrix is invariant under symmetry transformation, Eqs. (4.6) remain unchanged after the application of it: hence, we obtain that nodes within each group have the same equation of motion. Suppose that nodes within each group start from the same initial condition. Then, they will evolve following the same trajectory and the overall system will display at most M distinct synchronized motions. This demonstrates that the state in which the nodes of the same group, as identified by the network symmetries, are synchronized is a solution of Eqs. (4.6). It constitutes an invariant manifold, referred to as the synchronization manifold.

In the example of the network of Fig. 4.3a the two permutations divide the network nodes into two different clusters, one formed by the four outer nodes, and one with only the central node, so that a possible pattern of synchronization is the one where the four outer nodes are synchronized and the central one undergoes a different behavior, thus resulting into two distinct trajectories. However, ascertained the existence of a nontrivial synchronization manifold determined by the network symmetries, it is not enough to conclude that the network will evolve toward this state. Its stability has to be studied. This step is accomplished by applying a method that ultimately resorts to the same idea of the MSF discussed in Chap. 2. First, a proper transformation of the state variables decoupling the equations of motion is established and then the stability of the modes transverse to the synchronization manifold is studied.

The definition of permutation matrices R_g is important to determine a transformation matrix T that block-diagonalizes the adjacency and the Laplacian matrix. In the most general case, the derivation of both R_g and T is not trivial and the use of tools from algebraic group theory [12] is required. However, if one is able to construct the permutation matrix R_g , as done in Eq. (4.7) for the star-like network, the transformation matrix T can be easily derived. In fact, T is constructed by using the eigenvectors of R_g [8, 9]. One first determines T such that to diagonalize R_g , e.g., $T^{-1} R_g T = \text{diag}\{\lambda_1(R_g), \dots, \lambda_N(R_g)\}$, where $\lambda_1(R_g), \dots, \lambda_N(R_g)$ are the eigenvalues of R_g . The matrix T is then used to define transformed variables as, when applied to the Laplacian, it yields $G^* = T^{-1} G T$ where G^* is a block diagonal matrix.

More in details, let us indicate with $l = 1, \dots, M$ the clusters identified in the network, with $s_1, s_2, \dots, s_M$ the synchronized motions associated with cluster

$l = 1, \dots, M$, and V_l the set of nodes belonging to cluster l . Stability is studied by linearizing the system around this solution. Let us rewrite the state variables as $\mathbf{x}_i = \mathbf{s}_l + \delta \mathbf{x}_i$ ($i \in V_l$), where $\delta \mathbf{x}_i$ represents the perturbations of the i th node from the synchronous state $\mathbf{s}_l$. The variational equations read

$$\delta \dot{\mathbf{x}}_i = \mathbf{Df}(\mathbf{s}_l) \delta \mathbf{x}_i - \sigma \sum_{j \in V_l} g_{ij} \mathbf{Dh}(\mathbf{s}_l) \delta \mathbf{x}_j \quad (4.8)$$

where $\mathbf{Df}$ is the Jacobian of $\mathbf{f}$ evaluated at the synchronous solution (for each cluster l , it depends on the trajectory $\mathbf{s}_l$ associated to it), and $\mathbf{Dh}$ the Jacobian of the coupling function $\mathbf{h}$. We define $\Delta \mathbf{X}_l = [\Delta \mathbf{x}_j^T, \Delta \mathbf{x}_{j+1}^T, \dots, \Delta \mathbf{x}_{j+n_l}^T]^T$ the stack vector of the perturbations associated to cluster l (n_l is the number of units in cluster l , and $j = 1, \dots, M$), and $\Delta \mathbf{X} = [\Delta \mathbf{X}_1^T, \Delta \mathbf{X}_2^T, \dots, \Delta \mathbf{X}_M^T]^T$ the stack vector of perturbations for the entire network, so that Eqs. (4.8) can be rewritten as

$$\Delta \dot{\mathbf{X}} = \left[\sum_{l=1}^M \mathbf{E}^l \otimes \mathbf{Df}(\mathbf{s}_l) - \sigma \mathbf{G} \sum_{l=1}^M \mathbf{E}^l \otimes \mathbf{Dh}(\mathbf{s}_l) \right] \Delta \mathbf{X} \quad (4.9)$$

where the matrix $\mathbf{E}^l$ is a diagonal matrix that indicates the nodes belonging to cluster l . Its diagonal elements are given by $E_{ii}^l = 1$ if node $i \in V_l$, $E_{ii}^l = 0$ otherwise. By considering new transformed variables defined by $\Delta \mathbf{Y} = \mathbf{T}^{-1} \otimes \mathbf{I}_n \Delta \mathbf{X}$, Eq. (4.9) become

$$\Delta \dot{\mathbf{Y}} = \left[\sum_{l=1}^M \mathbf{E}^l \otimes \mathbf{Df}(\mathbf{s}_l) - \sigma \mathbf{G}^* \sum_{l=1}^M \mathbf{E}^l \otimes \mathbf{Dh}(\mathbf{s}_l) \right] \Delta \mathbf{Y} \quad (4.10)$$

where the Laplacian matrix $\mathbf{G}^*$ is block-diagonal. More in details, the matrix $\mathbf{G}^*$ has a structure with two main diagonal blocks, say $\mathbf{B}$ and $\mathbf{D}$: the first one, the upper left block $\mathbf{B}$ is a $(N - M) \times (N - M)$ matrix given by $\mathbf{B} = \oplus_{l=1}^M \mathbf{B}_l$. Each block $\mathbf{B}_l$ is a $(n_l - 1) \times (n_l - 1)$ matrix characterizing the motion transverse to the synchronous manifold $\mathbf{s}_l$. The remaining diagonal block is constituted by the matrix $\mathbf{D}$, of size $M \times M$, that represents the dynamics within the synchronization manifold, thus not affecting the stability of the cluster synchronous state. The transformation $\Delta \mathbf{Y} = \mathbf{T}^{-1} \otimes \mathbf{I}_n \Delta \mathbf{X}$ decouples the transverse motions from that within the synchronization manifold in such a way that the stability of each cluster can be studied separately by looking at the l th block:

$$\Delta \dot{\mathbf{Y}}_l = [\mathbf{I}_{n_l-1} \otimes \mathbf{Df}(\mathbf{s}_l) - \sigma \mathbf{B}_l \otimes \mathbf{Dh}(\mathbf{s}_l)] \Delta \mathbf{Y}_l \quad (4.11)$$

The stability of Eq. (4.11) is studied by calculating the Lyapunov exponents as in the MSF approach discussed in Chap. 2.

We now show with an example the onset of cluster synchronization in a network of six identical nodes as reported in Fig. 4.4a. Figure 4.4b, c illustrates the expected

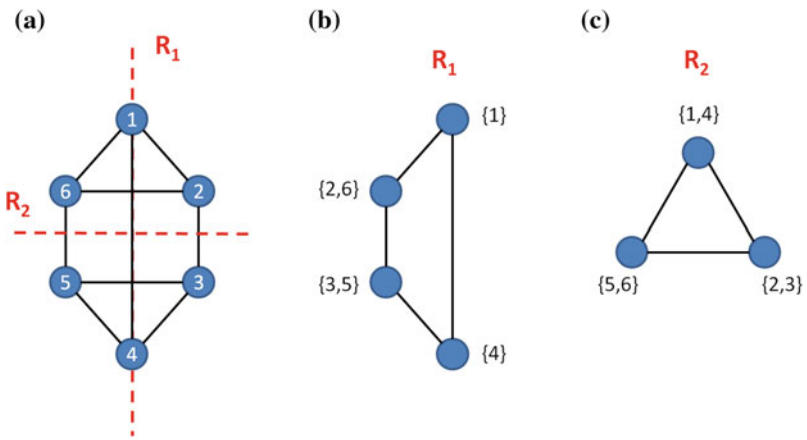


Fig. 4.4 Example of a simple network of six identical nodes **a** along with two types of symmetries: **b** mirroring around the vertical axis and **c** mirroring around horizontal axis

clusters deriving from two distinct rotations of the nodes, R_1 around the vertical axis, and R_2 around the horizontal one.

By using the permutation matrix associated to R_1 and R_2 , we can study the stability of cluster synchronization for the network in Fig. 4.4. The Laplacian and the permutation matrices are given by

$$G = \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{bmatrix}; R_{g1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; R_{g2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We derive the transformation matrix T associated to R_{g1} and R_{g2}

$$T_{g1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix};$$

and

$$T_{g2} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 1 \end{bmatrix}$$

Applying T_{g1} or T_{g2} to the Laplacian of the network, we get

$$G_1^* = T_{g1}^T G T_{g1} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -\sqrt{2} & 0 \\ 0 & 0 & -1 & 3 & 0 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & 0 & 2 & -1 \\ 0 & 0 & 0 & -\sqrt{2} & -1 & 2 \end{bmatrix}$$

and

$$G_2^* = T_{g2}^T G T_{g2} = \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 1 & 4 & -1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (4.12)$$

The two matrices are now block diagonalized, in particular the first one $G_1^* = T_{g1}^T G T_{g1}$ is divided into two blocks, the top block of size 2×2 and the bottom one that is 4×4 . This latter size identifies the number of possible clusters in the network, as reported in Fig. 4.4b. In the second transformation, $G_2^* = T_{g2}^T G T_{g2}$, the two blocks have dimension 3×3 , thus identifying the three clusters in Fig. 4.4c.

Let us consider the network in Fig. 4.4a, where each node is described by Chua's equations (2.26) with coupling matrix $H = [0 \ 1 \ 0]^T [0 \ 1 \ 0]$, and coupling coefficient $\sigma = 0.6$. The analysis of the stability can be carried out in an approximated way by assuming that the MSF is weakly affected by the specific trajectory considered. The node dynamics is characterized by a type II MSF with $\alpha_1 = 1.6$, so that the cluster synchronization state associated to symmetry R_1 is expected to be stable as the eigenvalues of the top block of G_1^* are $\gamma_3 = 3$ and $\gamma_5 = 5$ and for both of them $\sigma\gamma_h > \alpha_1$ ($h = \{3, 5\}$), while the state associated to R_2 is unstable as the eigenvalues of the top block of G_2^* are $\gamma_2 = 2$, $\gamma_5 = 5$ and $\gamma_6 = 5$ and one of them (γ_2) is not in the synchronization region as $\sigma\gamma_2 < \alpha_1$.

The results of the numerical simulations of the network are in agreement with these considerations. From Fig. 4.4b and the previous analysis, we expect four distinct clusters, $\{1\}$, $\{2, 6\}$, $\{3, 5\}$, $\{4\}$. In Fig. 4.5 we show the state variables $x_i(t)$ along with the overall synchronization error and those referring to the nontrivial clusters, i.e.,

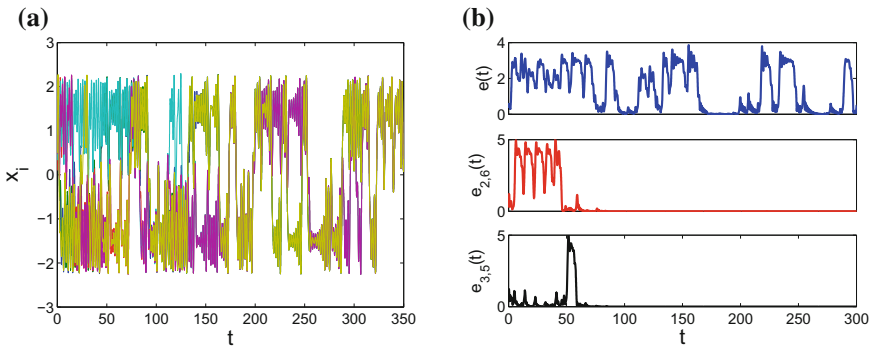


Fig. 4.5 Waveforms of the state variables x_i in a network of Chua's oscillators, (a), and synchronization error, (b). The topology is as in Fig. 4.4a, the unit dynamics as in Eq. (2.26)

$e_{2,6}(t) = \|\mathbf{x}_2(t) - \mathbf{x}_6(t)\|$ and $e_{3,5}(t) = \|\mathbf{x}_3(t) - \mathbf{x}_5(t)\|$, measuring synchronization in cluster $\{2, 6\}$ and in $\{3, 5\}$. Figure 4.5a illustrates that, after a transient, the network nodes organize in clusters. In particular, Fig. 4.5b shows that the whole network is not synchronized (top panel), while the nodes belonging to the two new main clusters $\{2, 6\}$ and $\{3, 5\}$ are synchronized (middle and bottom panel of Fig. 4.5b). In this case, we observe that some clusters have merged together. Figure 4.5 shows in fact that the state variables are grouped into only two clusters, formed by nodes $\{1, 2, 6\}$ and $\{3, 4, 5\}$. Cluster $\{1\}$ has merged with cluster $\{2, 6\}$, and cluster $\{4\}$ with cluster $\{3, 5\}$. The MATLAB[®] code for the simulation is the same used in Sect. 2.4 after changing the inner coupling matrix and the network topology.

4.3.2 Networks with External Equitable Partitions

Cluster synchronization also appear in networks whose nodes can be partitioned into groups according to the so-called *external equitable partition* (EEP). EEP is the extension and more relaxed version of the equitable partition (EP) [3]. We first briefly recall some definitions on the topic and then we provide examples on the use of this approach to study cluster synchronization.

Given a graph $\mathcal{G}$ and the set of vertices associated to it $\mathcal{V} = \mathcal{V}(\mathcal{G})$, EP groups the graph into s distinct cells, $C_1, C_2, \dots, C_s$. The set of the cells is called the partition and it is defined as $\pi = \{C_1, C_2, \dots, C_s\}$. The operation of partitioning groups the nodes into cells, thus creating a map $\pi : \mathcal{V} \rightarrow \{C_1, C_2, \dots, C_s\}$. A partition π is said equitable if, given two cells C_i and C_j , there exists a constant b_{ij} such that each vertex $v_i \in C_i$ has exactly b_{ij} neighbors in C_j ; this also holds when $i = j$. EP, thus, requires that nodes inside a cell of an EP have the same out-degree pattern with respect to any other cell (including internal links). Given an equitable partition π ,

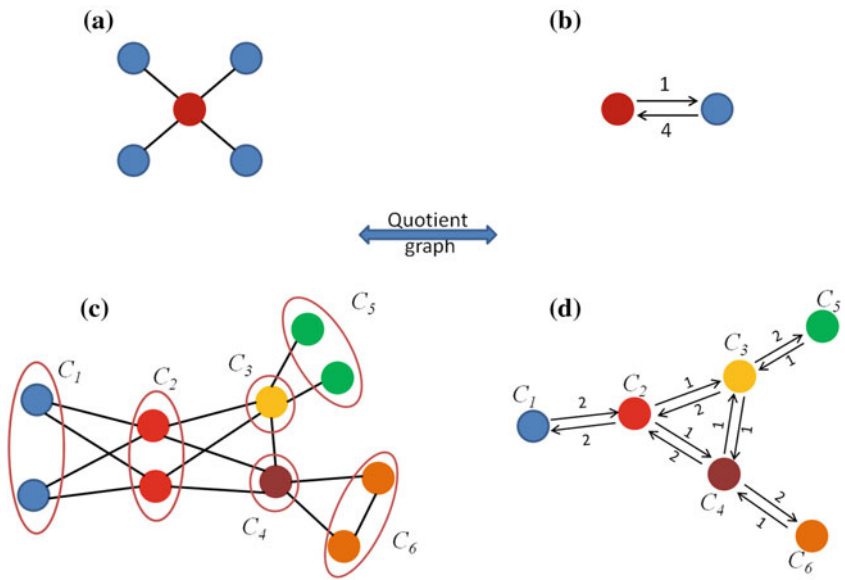


Fig. 4.6 Two examples of graphs along with their resulting quotient graphs. The top graph is a star-like configuration with a central node connected to four peripheral units (a), in this case the EEP consists of two cells, indicated with different colors (b). The bottom graph [1] is a network with ten identical nodes (c) and correspondingly has an EEP with six cells, again identified with different colors (d)

the quotient graph of $\mathcal{G}$ over π , denoted by $\mathcal{G}/\pi$, is the directed graph with vertices $C_1, C_2, \dots, C_s$ and b_{ij} arcs from C_i to C_j . For EPs, the quotient graph is regular.

The concept of external equitable partition relaxes the previous definition. Here, it is not important that the subgraph induced by the partition is regular, and thus nodes within a cell do not necessarily have the same numbers of neighbors. While the cells of an EP have the same out-degree pattern with respect to every cell; in EEPs, this holds only for the number of connections between distinct cells. The quotient graph resulting from the EEP is a weighted graph, where each cell constitutes a node and the weights between nodes are the out-degrees between the cells. Figure 4.6 illustrates two graphs and the related quotient graphs. Note that the quotient graph of the bottom network is an EEP, but it is not minimal, as it can be further reduced to a graph with only four nodes [1].

Given an equitable partition π , with cells $C_1, C_2, \dots, C_s$, we define the characteristic matrix P as the $N \times s$ matrix with $p_{ij} = 1$ if node i belongs to cell C_j , and $p_{ij} = 0$ otherwise. The characteristic matrix P has an important property: the product matrix $P^T P$ is diagonal with the j th element equal to the number of vertices in the cell C_j . Since the diagonal terms are nonzero, $P^T P$ is invertible.

Consider the star-like graph of Fig. 4.6, it can be partitioned in two cells: $C_1 = \{1\}$ and $C_2 = \{2, 3, 4, 5\}$. The corresponding characteristic matrix reads

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad (4.13)$$

By direct calculation one obtains:

$$\mathbf{P}^T \mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad (4.14)$$

which satisfies the above-mentioned properties.

Let us indicate the Laplacian matrix of the quotient graph as $\mathbf{G}^\pi$, we have that $\mathbf{G}\mathbf{P} = \mathbf{P}\mathbf{G}^\pi$. Hence, $\mathbf{G}^\pi = (\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T \mathbf{G} \mathbf{P}$. If π is an equitable partition of a graph $\mathcal{G}$, then the eigenvalues of the $\mathbf{G}^\pi$ are a subset of the eigenvalues of $\mathbf{G}$, $\lambda(\mathbf{G}^\pi) \subseteq \lambda(\mathbf{G})$; the same holds for the eigenvectors, $\mathbf{v}^\pi$ is an eigenvector for $\mathbf{G}^\pi$ if $\mathbf{v}$ ($\mathbf{v} = \mathbf{P}\mathbf{v}^\pi$) is an eigenvector of $\mathbf{G}$ with the same eigenvalue.

The EEP π of a network determines the cluster synchronization manifold. Each cell of the partition is in fact associated with a distinct motion (at most, as we will see below, two or more cells may synchronize and the synchronous state may lie in a space of dimension lower than that of the synchronization manifold). The cells essentially correspond to the synchronized clusters that may form. Similarly to symmetries, in fact, EEP yields the invariance of the cluster synchronization motion [10, 14].

We indicate with $\mathbf{V}^\pi = [\mathbf{v}_1^\pi, \mathbf{v}_2^\pi, \dots, \mathbf{v}_s^\pi]$ the eigenvectors of $\mathbf{G}^\pi$, with $\mathbf{V}_s$ the eigenvectors of the Laplacian of the original graph defining the cluster synchronization manifold with respect to the partition π , that is $\mathbf{V}_s = \mathbf{P}\mathbf{V}^\pi$, and with $\mathbf{V}_\perp$ those transverse to $\mathbf{V}_s$ which may eventually drive the system out of the cluster synchronization state. The eigenvectors of the Laplacian of the original graph are thus $\mathbf{V} = [\mathbf{V}_s, \mathbf{V}_\perp]$. As in the case of cluster synchronization for networks with symmetries, to study the stability of the synchronization manifold, we write the variational equations of the system

$$\Delta \dot{\mathbf{X}} = \left[\sum_{l=1}^s \mathbf{E}^l \otimes \mathbf{D}\mathbf{F}(\mathbf{s}_l) - \sigma \sum_{l=1}^s \mathbf{G}\mathbf{E}^l \otimes \mathbf{D}\mathbf{H}(\mathbf{s}_l) \right] \Delta \mathbf{X} \quad (4.15)$$

where the matrices $\mathbf{E}^l$ ($l = 1, \dots, s$) have the same meaning as those appearing in cluster synchronization for networks with symmetries, now denoting the belonging of a node to a given cell/cluster l .

Clearly, the matrix $\mathbf{G}$ can be diagonalized through its eigenvectors, i.e., $\mathbf{V}^T \mathbf{G} \mathbf{V} = \mathbf{\Gamma}$, with $\mathbf{\Gamma} = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_N\}$, so that we can consider the new coordinates defined by the transformation $\Delta \mathbf{Y} = (\mathbf{V}^T \otimes \mathbf{I}_n) \Delta \mathbf{X}$. With this transformation of variables, Eq. (4.15) become

$$\begin{aligned}\Delta \dot{\mathbf{Y}} &= (V^T \otimes \mathbf{I}_n) \left[\sum_{l=1}^s \mathbf{E}^l \otimes D\mathbf{F}(\mathbf{s}_l) - \sigma \sum_{l=1}^s \mathbf{G}\mathbf{E}^l \otimes D\mathbf{H}(\mathbf{s}_l) \right] (V \otimes \mathbf{I}_n) \Delta \mathbf{Y} \\ &= \left[\sum_{l=1}^s V^T \mathbf{E}^l V \otimes D\mathbf{F}(\mathbf{s}_l) - \sigma \sum_{l=1}^s V^T \mathbf{G}\mathbf{E}^l V \otimes D\mathbf{H}(\mathbf{s}_l) \right] \Delta \mathbf{Y}\end{aligned}\quad (4.16)$$

where $V^T \mathbf{G}\mathbf{E}^l V = V^T (V \Gamma V^T) \mathbf{E}^l V = \Gamma V^T \mathbf{E}^l V$.

The matrix $V^T \mathbf{E}^l V$ is constituted by four blocks

$$[V_s, V_\perp]^T \mathbf{E}^l [V_s, V_\perp] = \begin{bmatrix} V_s^T \mathbf{E}^l V_s & V_s^T \mathbf{E}^l V_\perp \\ V_\perp^T \mathbf{E}^l V_s & V_\perp^T \mathbf{E}^l V_\perp \end{bmatrix} \quad (4.17)$$

As the off-diagonal blocks are null ($V_\perp^T V_s = 0$), we have that

$$[V_s, V_\perp]^T \mathbf{E}^l [V_s, V_\perp] = \begin{bmatrix} V_s^T \mathbf{E}^l V_s & 0 \\ 0 & V_\perp^T \mathbf{E}^l V_\perp \end{bmatrix} \quad (4.18)$$

which show that the transverse modes are decoupled. Therefore, the dynamics of the transverse modes can be studied from Eq. (4.16). If these modes damp out, then the cluster synchronization state is stable. Such study does not specify the behavior within the cluster synchronization manifold, where further damping may occur, thus resulting in a number of clusters lower than the number of cells. Two or more cells, in fact, may merge and follow the same shared trajectory, so that the resulting synchronization manifold is lower, as it may occur for cluster synchronization in networks with symmetries.

To illustrate cluster synchronization in networks with EEPs, we consider a network with topology as in the bottom graph of Fig. 4.6 and units with Rössler dynamics with parameter $a = 0.2$, $b = 0.2$ and $c = 9$ and linear coupling with $\mathbf{H} = [0 \ 1 \ 0]^T [0 \ 1 \ 0]$. We select the coupling coefficient equal to $\sigma = 0.2$. To monitor cluster synchronization with respect to the cells defined by the EEP, we consider the global synchronization error defined as in Eq. (2.3) and that associated to the cluster C_l , $E_{C_l}(t) = \left(\frac{1}{N_{C_l}(N_{C_l}-1)} \sum_{i,j \in C_l} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \right)^{1/2}$. In Fig. 4.7 we report the global synchronization error and that of non-trivial clusters, showing that all the units of each cell are synchronized.

In addition, the quotient graph can be used to describe the dynamics of the original network if the initial conditions are set to be equal for each unit of the cells. This property is exploited to calculate $\mathbf{s}_l(t)$ for $l = 1, \dots, s$ needed for the stability analysis.

4.4 Cluster Synchronization and Chimera States

Cluster synchronization has some similarities with chimera states, but with a profound difference. In both forms of synchronization, units with incoherent behavior can be observed. In chimera states they belong to a region of the system, while in

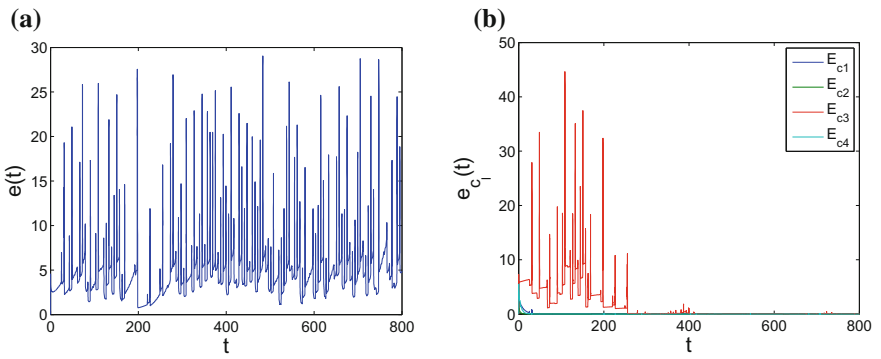


Fig. 4.7 Synchronization error for the bottom graph of Fig.4.6. The nodes are Rössler chaotic systems with parameters $a = 0.2$, $b = 0.2$ and $c = 9$, the coupling coefficient is $\sigma = 0.2$. **a** Global synchronization error; **b** synchronization error of each nontrivial cluster C_l of the quotient graph

cluster synchronization they derive from clusters with a single element. However, in chimera states the solution breaks the symmetry of the structure, while in cluster synchronization it is the symmetry that makes possible the clustering of the nodes in the same group. In other words, the synchronous state is a solution which reflects the symmetries in cluster synchronization and breaks the symmetry in chimera states.

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Chapter 5

Remote and Relay Synchronization

In this chapter, two related phenomena, remote and relay synchronization, are dealt with. They are first illustrated in simple motifs (chains and star-like configurations) and, then, in arbitrary random networks of heterogeneous oscillators. The case of networks with identical node dynamics in the presence of symmetries is also briefly illustrated.

5.1 Introduction

In this chapter, we discuss remote and relay synchronization, two issues that, despite having been introduced in different contexts and displaying their own unique features, reveal a common mechanism in the synchronization of complex networks: two nodes may synchronize not only by the effect of a direct interaction, but also thanks to the action of a further node which relays the information but does not synchronize with them. An example of this behavior is found in a star configuration of periodic oscillators where the central node is mismatched with the other (peripheral) nodes [2]; here, it is observed that there are values of the coupling such that the peripheral nodes are phase synchronized while the central hub is not. We will discuss several other examples of similar behaviors in this chapter, but first note that previous studies have focused on the specific features of the phenomena reported rather than on their common characteristics; we try to provide here a focus perspective on their similarities rather than on their differences.

We first consider synchronization by dynamical relying or, shortly, relay synchronization. An example of relay synchronization is reported in [4]: three nodes are connected in a chain configuration, so that there is no link between the two end points, i.e., node 1 and node 3; however, it turns out that these two nodes may synchronize with zero lag whereas node 2 lags behind the two. The units are either laser or units with neuron-like dynamics while the coupling is considered with delay. The same finding is also observed when populations of neurons are coupled in a delay chain

configuration [15] or when the chain consists of more than three units [1]. In the same work [1], two other important issues are demonstrated: (i) dynamical relaying also occurs where there is no delay, but the inner and outer oscillators have different parameters; (ii) dynamical relaying may yield an enhancement of synchronization, in the sense that the threshold for synchronization that would have been observed in absence of the dynamical relay is greater than in the presence of it. In these works, the oscillators are chaotic.

The term remote synchronization was, instead, introduced in [2]. The configuration considered is a star, but, unlike in relay synchronization, the oscillators are periodic. As mentioned, the central unit is mismatched with the outer ones, as its natural frequency is either larger or smaller than the others. The phenomenon was then demonstrated to commonly occur between the nodes of arbitrary random network topologies [5].

The same term, remote synchronization, was also used to indicate another phenomenon appearing in complex networks in the presence of symmetries: the onset of pairs of nodes at distant locations in the graph with exactly the same phase [12]. In this case, the units, which are identical (in particular, they are phase oscillators with a nonzero frustration parameter), may display patterns of synchronization where the units group into clusters having the same phase, but not linked each other. The other groups have the same frequency, but different phase lags.

All these examples highlight as there can be many scenarios, differing for the structure of the interactions and for the dynamics of the units, where a pair of nodes synchronizes by the action of a node relaying the information but synchronizing in a weaker form or not synchronizing at all with them. All these examples can be viewed as instances of the same phenomenon if the following definition is adopted: two nodes of a network are remotely synchronized when they are not directly connected and they are synchronized to each other, but not (or in a weaker form) with the nodes of any path connecting them.

This definition encompasses the abovementioned examples as it includes: identical synchronization among the remotely synchronized nodes and phase lag synchronization with the dynamical relay in phase oscillators with frustration in the presence of network symmetries [12]; phase synchronization and no synchronization in periodic oscillators in the presence of mismatches [2, 5]; complete synchronization and generalized synchronization in networks of chaotic oscillators in the presence of mismatches [7, 9]; higher levels of synchronization in nonadjacent nodes rather than in neighboring ones in rings of unidirectionally coupled nonlinear electronic circuits [11].

In the following we will keep separate the two names, remote and relay, to maintain the same notation of the original references, while trying to emphasize the similarities between them.

5.2 Remote Synchronization in a Star Network of Nonidentical Stuart–Landau Oscillators

The first example that we illustrate in some detail is remote synchronization in a star-like configuration of Stuart–Landau oscillators [2]. Labeling the central node as node 1 and the other nodes as $2, \dots, N$, the star-like configuration results in the following Laplacian matrix:

$$G = \begin{pmatrix} (N-1) & -1 & -1 & \dots & -1 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix} \quad (5.1)$$

As usual, this matrix sets the connections between the network nodes, which are described by the following equations:

$$\begin{aligned} \dot{x}_i &= \alpha x_i - \omega_i y_i - x_i(x_i^2 + y_i^2) - \frac{\sigma}{d_i} \sum_{j=1}^N g_{ij} x_j \\ \dot{y}_i &= \omega_i x_i + \alpha y_i - y_i(x_i^2 + y_i^2) - \frac{\sigma}{d_i} \sum_{j=1}^N g_{ij} y_j \end{aligned} \quad (5.2)$$

where x_i and y_i represent the state variables of the oscillator, α is the bifurcation parameter, ω_i is the natural frequency of the oscillator, that is, the frequency of oscillations when the unit is not coupled to the others, and the other symbols have the usual meaning (σ is the coupling coefficient, d_i the degree of node i). Notice here that the coupling coefficient is divided by the node degree, thus resulting in an effective network of interactions that is weighted.

The equations of the network of Stuart–Landau oscillators can also be restated in complex form, by considering the complex variable $u_i = x_i + iy_i$:

$$\dot{u}_i = (\alpha + i\omega_i - |u_i|^2)u_i - \frac{\sigma}{d_i} \sum_{j=1}^N g_{ij} u_j \quad (5.3)$$

In Chap. 4, we mentioned that, in the uncoupled limit case ($\sigma = 0$), α represents the bifurcation parameter. Going back to the system of coupled units, e.g., $\sigma > 0$ in Eqs. (5.2) or (5.3), then another role of the parameter α emerges: the parameter controls the decay rate of the trajectory onto the attractor. This is particularly important in relation to the coupling: if the attraction to the limit cycle is strong compared to the coupling, then the oscillator can be approximated as a pure phase oscillator. It is interesting to note that significant differences can occur in the phase synchronization of networks of Stuart–Landau oscillators when this hypothesis is satisfied or not (for more details, the reader is referred to [8]).

To study remote synchronization in system (5.2), we need to monitor phase synchronization between each pair of nodes. To this aim, following [2], we can use the measure introduced in Chap. 4 for two generic units i and j and here repeated for convenience:

$$r_{ij} = |\langle e^{i[\theta_i(t) - \theta_j(t)]} \rangle_T| \quad (5.4)$$

where $\theta_i(t)$ is the phase of the oscillator i calculated as $\theta_i(t) = \arctan \frac{y_i(t)}{x_i(t)}$ or $\theta_i(t) = -t \ln \frac{|u_i|}{|u|}$.

As already mentioned, in this structure remote synchronization appears as a strong level of phase coherence among the peripheral nodes, while the phase variable of the central node is poorly correlated with the other ones. Proper measures to detect the occurrence of such condition are as follows:

$$r^{direct} = \frac{1}{N-1} \sum_{i=2}^N r_{1i} \quad (5.5)$$

measuring the coherence of the hub with the other nodes, and

$$r^{indirect} = \frac{2}{(N-1)(N-2)} \sum_{i=2, j>i}^N r_{ij} \quad (5.6)$$

measuring the coherence among the peripheral nodes.

Let us now consider a concrete example, a star-like network with $N = 5$. In particular, as in [2], we fix the natural frequency of the central node as $\omega_1 = 2.5$ and those of the other nodes as $\omega_2 = 0.975$, $\omega_3 = 0.992$, $\omega_4 = 1.008$, $\omega_5 = 1.025$. This choice is arbitrary provided that an enough large mismatch is set between the central node and the other ones. A typical scenario of remote synchronization is illustrated

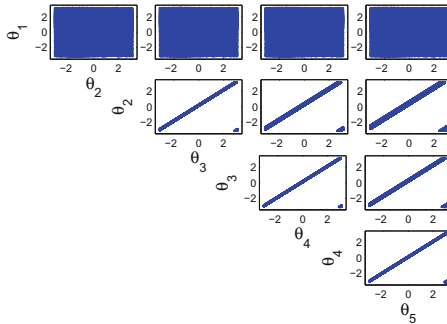


Fig. 5.1 Remote synchronization in a star-like network of $N = 5$ Stuart–Landau oscillators with the central unit mismatched with respect to the other ones for coupling coefficient equal to $\sigma = 0.6$. The phases of the oscillators are plotted against each other, showing that the peripheral nodes are phase synchronized with each other, while the central node is not

in Fig. 5.1, where one can observe that the peripheral nodes are phase synchronized with each other, while the central node is not.

The network is simulated with the following code.

Code for numerical simulation of a star-like network of Stuart–Landau systems

```
%Star network of SL oscillators
N=5;

dt=0.01;
steps=100000;

%initial conditions for SL oscillators
xold=2*rand(N,1)-1;
yold=2*rand(N,1)-1;

%parameters
a=1;
omega=[2.5; 0.975; 0.992; 1.008; 1.025];

x=zeros(steps,N);
y=zeros(steps,N);

A=[0 ones(1,N-1); ones(N-1,1) zeros(N-1)];
A=A./ (sum(A,2)*ones(1,N));
G=diag(sum(A,2))-A;

sigma=0.6;

for t=1:steps

    couplingx=-G*xold;
    couplingy=-G*yold;

    %integration
    dxdt=a*xold-omega.*yold-xold.*(xold.^2+yold.^2)+sigma*
        couplingx;
    dydt=omega.*xold+a*yold-yold.*(xold.^2+yold.^2)+sigma*
        couplingy;

    xnew=xold+dt*dxdt;
    ynew=yold+dt*dydt;

    xold=xnew;
    yold=ynew;

    x(t,:)=xnew;
    y(t,:)=ynew;

end

%phase of the units
phi=atan2(y,x);

figure(1)
subplot(441),plot(phi(end/2:end,1),phi(end/2:end,2),'.'),xlim
([-3.5,3.5]),ylim([-3.5,3.5]),ylabel('\theta_1'),xlabel('\theta_2')
subplot(442),plot(phi(end/2:end,1),phi(end/2:end,3),'.'),xlim
([-3.5,3.5]),ylim([-3.5,3.5])
subplot(443),plot(phi(end/2:end,1),phi(end/2:end,4),'.'),xlim
([-3.5,3.5]),ylim([-3.5,3.5])
subplot(444),plot(phi(end/2:end,1),phi(end/2:end,5),'.'),xlim
([-3.5,3.5]),ylim([-3.5,3.5])
```

```

subplot(446), plot(phi(end/2:end,2), phi(end/2:end,3), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5]), ylabel('\theta_2'), xlabel('\theta_3')
subplot(447), plot(phi(end/2:end,2), phi(end/2:end,4), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5])
subplot(448), plot(phi(end/2:end,2), phi(end/2:end,5), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5])

subplot(4,4,11), plot(phi(end/2:end,3), phi(end/2:end,4), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5]), ylabel('\theta_3'), xlabel('\theta_4')
subplot(4,4,12), plot(phi(end/2:end,3), phi(end/2:end,5), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5])

subplot(4,4,16), plot(phi(end/2:end,4), phi(end/2:end,5), '.'), xlim
([-3.5, 3.5]), ylim([-3.5, 3.5]), ylabel('\theta_4'), xlabel('\theta_5')

```

For the same network, we now study the evolution of the two order parameters r^{direct} and $r^{indirect}$ as the coupling coefficient σ is varied. The result is shown in Fig. 5.2. The curve of $r^{indirect}$ reaches 1 much before than r^{direct} , denoting a range of coupling values ($\sigma \in [0.55, 0.75]$), where the peripheral oscillators form a coherent cluster whereas the central unit oscillates with its own independent phase. For larger values ($\sigma > 0.75$), the network reaches global phase synchronization.

The integration of the network equations and the calculation of the order parameters is carried out with the following code.

Code for numerical simulation of a star-like network of Stuart–Landau systems with respect to increasing values of σ

```

%Calculation of the two order parameters rdirect and rindirect
N=5;

dt=0.01;
steps=500000;

%parameters
a=1;
omega=[2.5; 0.975; 0.992; 1.008; 1.025];

A=[0 ones(1,N-1); ones(N-1,1) zeros(N-1)];
A=A./ (sum(A,2)*ones(1,N));
G=diag(sum(A,2))-A;

sigmav=[0:0.05:1];

for sigmaI=1:length(sigmav)

    %initial conditions for SL oscillators
    xold=2*rand(N,1)-1;
    yold=2*rand(N,1)-1;

    %memorizzazione to check integration
    x=zeros(steps,N);
    y=zeros(steps,N);

    errormatrix=zeros(N,N);
    sigma=sigmav(sigmaI);

    for t=1:steps

        couplingx=-G*xold;

```

```

couplingy=-G*yold;

%%integration
dxdt=a*xold-omega.*yold-xold.*(xold.^2+yold.^2)+sigma*
    couplingx;
dydt=omega.*xold+a*yold-yold.*(xold.^2+yold.^2)+sigma*
    couplingy;

xnew=xold+dt*dxdt;
ynew=yold+dt*dydt;

xold=xnew;
yold=ynew;

x(t,:)=xnew;
y(t,:)=ynew;

end

%phase of the units
phi=atan2(y,x);

for ii=1:N
    for jj=ii+1:N
        errormatrix(ii,jj)=abs(mean(exp(i*(phi(end/2:end,ii)-
            phi(end/2:end,jj)))));
    end
end

rdirect(sigmaI)=mean(errormatrix(1,2:N));
rindirect(sigmaI)=sum(sum(triu(errormatrix(2:end,2:end),1))
    *2/(N-1)/(N-2));

end

figure,plot(sigmav,rdirect,sigmav,rindirect)

```

The behavior here illustrated for numerical simulations of Eqs. (5.3) is also found in experiments on electronic circuits [2]. These circuits are realized with off-the-shelf components (resistors, capacitors, operational amplifiers and analog multipliers) and

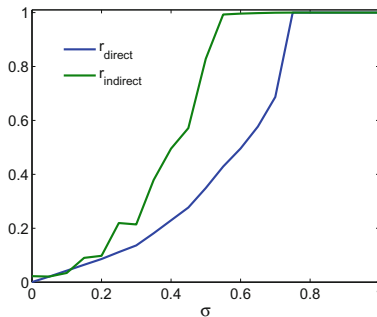


Fig. 5.2 Behavior of a star-like network of $N = 5$ Stuart–Landau oscillator with the central unit mismatched with respect to the other ones: evolution of the order parameters as the coupling coefficient σ is varied. Remote synchronization is characterized by high values of $r^{indirect}$ in correspondence of poor level of phase coherence of the peripheral nodes with the central one, as measured by r^{direct}

are designed to be governed by the same equations of the Stuart–Landau oscillator, thus representing the electronic analog of this mathematical model. Remote synchronization is found in accordance with the theoretical expectations, proving the robustness of the phenomenon to the presence of parameter tolerances, noise, and other unavoidable nonidealities present in an experimental setup.

5.3 Remote Synchronization in Random Networks of Nonidentical Stuart–Landau Oscillators

The example discussed in the previous section is based on a peculiar structure, the star-like motif, that permits to a priori identify which nodes can be remotely synchronized. However, in [5] it has been demonstrated that remote synchronization is a more general phenomenon, commonly found in random network topologies. In particular, the setup there considered is based on random topologies (ER and BA networks) of heterogeneous units. The heterogeneity is set in such a way that the distribution of the natural frequencies of the oscillators is bimodal with two peaks, one centered around ω_L and one around ω_H . Nodes with lower (higher) degrees are assigned natural frequencies centered around the first (second) peak to encompass a scenario extending the star-like network case.

The definition of proper measures to detect remote synchronization is a key aspect of the study of the phenomenon in these more general structures. Consider two nodes i and j that are phase synchronized (this can be detected by measuring r_{ij} and checking if it is higher than a threshold, i.e., $r_{ij} > \delta$). Three situations can occur: (1) nodes i and j are directly connected ($a_{ij} = a_{ji} = 1$); (2) there is a path of mutually synchronized nodes between them; or (3) none of the two previous cases is verified. Situations (1) and (2) are both examples of synchronization through *physical* links. The third case, instead, represents an instance of remote synchronization. Two nodes can be thus defined as remotely synchronized if they are synchronized ($r_{ij} > \delta$) and they are not connected by either a direct link or a path of synchronized nodes.

Once identified the nodes whose synchronization is ascribable to a remote mechanism, three measures can be defined for a systematic quantification of the impact of this mechanism in the synchronization of a complex network. The first one is the normalized number of RS nodes with respect to the total number of nodes N : $n_{RS} = N_{RS}/N$, where N_{RS} is the number of nodes that are in remote synchronization with at least another node. The second one is L_{RS} , that is the number of RS links (equivalently, the number of pairs of remotely synchronized nodes). The third one is the fraction of synchronized links due to a remote interaction and is defined starting from the following considerations. We take into account all synchronized pairs and partition them into those due to remote synchronization and those due a physical interaction, which can derive either from a direct link or a path of synchronized nodes. Hence, we introduce a parameter, η_{ij} which is equal to one if nodes i and j are connected by a physical link or a path of synchronized nodes, and zero otherwise.

Then, the following quantities are considered:

$$f_P = \frac{\sum_{i,j=1}^N \eta_{ij} \Theta(r_{ij} - \delta)}{\sum_{i,j=1}^N \Theta(r_{ij} - \delta)} \quad (5.7)$$

and

$$f_{RS} = \frac{\sum_{i,j=1}^N (1 - \eta_{ij}) \Theta(r_{ij} - \delta)}{\sum_{i,j=1}^N \Theta(r_{ij} - \delta)} \quad (5.8)$$

where $\Theta(x)$ is the Heaviside function. These parameters represent the fraction of synchronized links due to a physical or remote connection, respectively. Since they are not independent as $f_P + f_{RS} = 1$, monitoring f_{RS} is sufficient.

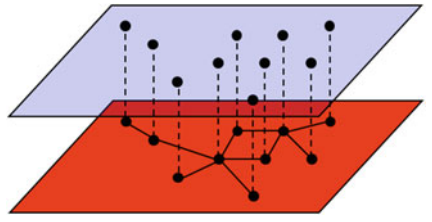
Numerical simulations carried out monitoring these measures permit to conclude that the phenomenon of remote synchronization commonly appears in random general topologies, in a region of the parameters with a significant frequency mismatch $\Delta\omega = \omega_H - \omega_L$ and a moderate coupling σ . For low values of the coupling parameter, nodes cannot synchronize (either in a physical or remote way), while, on the contrary, for large values of σ the network is globally synchronized. Instead, if the frequency mismatch is large and the coupling is not enough for global synchronization, then many nodes appear to be remotely synchronized, and correspondingly the parameters n_{RS} and L_{RS} assume high values.

The evolution of f_{RS} with σ is also interesting. Although in a manner that depends on the specific network model (ER or SF), this parameter first grows with σ and then decreases, as the network recruits physical links to get synchronized. In correspondence of this, the regions that are remotely synchronized merge into fewer components made of physically synchronized links.

From a practical point of view, to detect remotely synchronized links in a network, one has to first compute r_{ij} for all i and j and check if the value is higher than the threshold for synchronization. This computation may be possibly averaged over different runs (considering different values of natural frequencies of the oscillators) to better isolate the effects of the topology. Once this step is accomplished, one can calculate η_{ij} for $i, j = 1, \dots, N$, by checking if there exists or not a path of synchronized nodes between i and j . From η_{ij} , it is straightforward to compute n_{RS} , L_{RS} , and f_{RS} .

Another interesting finding related to remote synchronization is discussed in [6]. All the examples so far considered in this book have dealt with oscillators that are directly coupled through the interactions that are represented by the links of a network. However, there are physical systems, such as cellular populations communicating thanks to small molecules diffused in the medium [10], chemical oscillators interacting through a stirred solution [14] or pedestrian crowd walking in synchrony [13], where the interactions are not direct but occur through a shared environment. It turns out that oscillators, indirectly coupled by a medium that is distributed and dynamical, can also be modeled by networks if one resorts to use *multilayer* models [3]. These models account for the existence of multiple categories of interactions,

Fig. 5.3 Oscillators that interact through a distributed and dynamical medium are represented by a multilayer network consisting of two layers with one-to-one coupling between the layers



by representing each of them as one layer and allowing each node to have different kinds of interactions (i.e., the links in the various layers). For example, a social system is characterized by individuals which interact through different channels such as friendship, coworker relationships, online social networks, and many others: in a multilayer network, each of these channels is a layer with its own topological characteristics.

In the case of oscillators that are indirectly coupled through a distributed and dynamical medium [6], one layer (to fix the ideas the top one) is made of disconnected oscillators and the other one (the bottom one), modeling the medium, consists of oscillators with their own pattern of interactions (see Fig. 5.3). The different dynamics of the medium and the layer of oscillators is implemented by considering in both layers Stuart–Landau oscillators but including a frequency mismatch between the two layers.

Such a system is described by the following system of differential equations:

$$\begin{aligned}\dot{u}_j^x &= (a + i \cdot \omega_j^x - |u_j^x|^2) \cdot u_j^x + \sigma_{zx} \cdot (u_j^z - u_j^x) , \\ \dot{u}_j^z &= (a + i \cdot \omega_j^z - |u_j^z|^2) \cdot u_j^z + \sigma_{zx} \cdot (u_j^x - u_j^z) - \sigma_z \cdot \sum_{l=1}^N g_{jl} u_l^z ,\end{aligned}\quad (5.9)$$

where the state variables of the units of the top (bottom) layer are indicated as u_j^x (u_j^z), and σ_z and σ_{zx} represent the coupling strength between the nodes in the medium (i.e., the bottom layer) and the interlayer coupling coefficient, respectively.

In this system, intralayer coherence can be observed in the absence of interlayer coherence [6]. This is a form of remote synchronization as the units of the top layer are not directly connected by a physical link or by a path of synchronized nodes, but synchronize by the effect of the bottom layer. The regime is supported by arbitrary topologies in the bottom layer, provided that synchronization of the oscillators of the medium can be reached.

5.4 Relay Synchronization in Rössler Oscillators

As mentioned in the introduction, relay synchronization [1, 4, 15] has been reported in chains of chaotic oscillators under different assumptions: identical units with time-delay coupling and nonidentical units with instantaneous, diffusive coupling. As for the units, different examples have been used: Mackey–Glass systems, Rössler oscillators, Lorenz systems, neuron-like units, and laser models. In this section, we limit the discussion to a few considerations. We show an example of relay synchronization in a chain of Rössler oscillators and then discuss its appearance in complex networks.

The example we consider refers to a system formed by three Rössler units coupled into an open chain configuration [1]. The central node is mismatched with respect to the others. More in details, the system is governed by the following equations:

$$\begin{aligned}\dot{x}_i &= -\omega_i y_i - z_i(x_i^2 + y_i^2) - \sigma \sum_{j=1}^N g_{ij} x_j \\ \dot{y}_i &= \omega_i x_i + a y_i \\ \dot{z}_i &= b + z_i(x_i - c)\end{aligned}\tag{5.10}$$

with $G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. The parameters of the units are selected as $a = 0.2$, $b = 0.4$, $c = 7.5$, $\omega_1 = 1$, $\omega_2 = 1.1$, and $\omega_3 = 1$. With these parameters, all the units are chaotic when uncoupled.

To illustrate relay synchronization in this example, we consider the average synchronization error between node 1 and 3 and that between node 1 and 2, that is, $\langle e_{13} \rangle = \langle \|\mathbf{x}_1 - \mathbf{x}_3\| \rangle_T$ and $\langle e_{12} \rangle = \langle \|\mathbf{x}_1 - \mathbf{x}_2\| \rangle_T$. Relay synchronization manifests as the onset of complete synchronization between the nodes 1 and 3, i.e., $\langle e_{13} \rangle = 0$, while node 2 is not in complete synchronization with them, i.e., $\langle e_{12} \rangle \neq 0$. A weaker form of synchronization, instead, arises between the outer and the central node, that is lag synchronization. Figure 5.4 illustrates the values of these two errors when the coupling coefficient is varied, showing as at a critical point $\langle e_{13} \rangle$ becomes zero, while $\langle e_{12} \rangle$ is still different than zero.

The MATLAB[®] code to obtain these numerical results is the following.

Code for numerical simulation of a chain of three Rössler systems

```
%synchronization error in a chain of three Rossler units
N=3;

dt=0.01;
steps=500000;

%parameters
ar=0.2;
br=0.4;
cr=7.5;
omega=[1; 1+0.1; 1];

A=[0 1 0; 1 0 1; 0 1 0];
```

```

G=diag (sum(A,2)) -A;

sigmaV=[0:0.01:0.2];

for sigmaI=1:length(sigmaV)

    %initial conditions for Roessler oscillators
    xold=(30*rand(N,1)-15)/5;
    yold=(30*rand(N,1)-15)/5;
    zold=(40*rand(N,1)-5)/5;

    x=zeros(steps,N);
    y=zeros(steps,N);
    z=zeros(steps,N);

    e12=zeros(steps,1);
    e13=zeros(steps,1);

    sigma=sigmaV(sigmaI)

    for t=1:steps

        coupling=-G*xold;

        %%integration
        dxdt=-omega.*yold-zold+sigma*coupling;
        dydt=omega.*xold+ar*yold;
        dzdt=br+zold.*(xold-cr);

        xnew=xold+dt*dxdt;
        ynew=yold+dt*dydt;
        znew=zold+dt*dzdt;

        xold=xnew;
        yold=ynew;
        zold=znew;

        x(t,:)=xnew;
        y(t,:)=ynew;
        z(t,:)=znew;

        e12(t)=sqrt((x(t,1)-x(t,2))^2+(y(t,1)-y(t,2))^2+(z(t,1)-z(t,2))^2);
        e13(t)=sqrt((x(t,3)-x(t,1))^2+(y(t,3)-y(t,1))^2+(z(t,3)-z(t,1))^2);

    end

    error12(sigmaI)=mean(e12(end/2:end));
    error13(sigmaI)=mean(e13(end/2:end));

end

figure,plot(sigmaV,error12,sigmaV,error13)

```

There is an enhancing effect in synchronization associated to the presence of the dynamical relay (node 2), that can be illustrated by comparing the threshold for synchronization of two identical Rössler units when directly coupled and that when they are coupled through a mismatched relaying node. Figure 5.5 shows the results, demonstrating that the effect of the dynamical relay is to lower the synchronization threshold.

The results may be reproduced by using the following MATLAB[®] code.

Code to observe the enhancing synchronization effect in relay synchronization

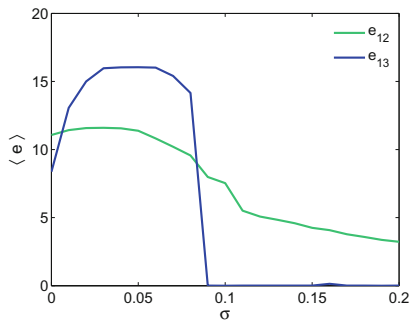


Fig. 5.4 Relay synchronization in a chain of three Rössler oscillators. For $\sigma > 0.09$, the outer nodes reach complete synchronization while they are not in complete synchronization with the central node. The blue curve is the average synchronization error between the outer nodes, i.e., $\langle e_{13} \rangle$, while the green curve is the average synchronization error between an outer and the central node, i.e., $\langle e_{12} \rangle$

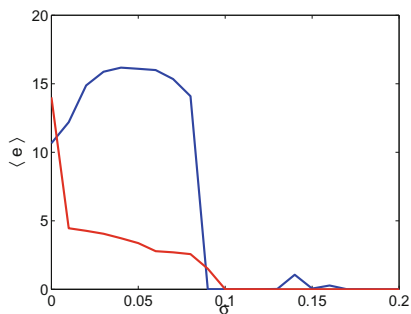


Fig. 5.5 Comparison between the synchronization error obtained for two directly coupled Rössler systems (red curve) and that obtained when the interaction is by dynamical relaying (blue curve). The enhancing effect due to dynamical relaying is apparent in the decrease of the coupling coefficient threshold for complete synchronization

`% Synchronization error for two Coupled Rossler systems with and without dynamical relaying`

`N=5;`

`dt=0.01;`
`steps=500000;`

`%parameters`

`ar=0.2;`
`br=0.4;`
`cr=7.5;`
`omega=[1; 1+0.1; 1; 1; 1];`

`A=[0 1 0 0 0;`
`1 0 1 0 0;`
`0 1 0 0 0;`
`0 0 0 0 1;`
`0 0 0 1 0];`

```

G=diag (sum(A,2)) -A;

sigmaV=[0:0.01:0.2];

for sigmaI=1:length(sigmaV)

    %initial conditions for Roessler oscillators
    xold=(30*rand(N,1)-15)/5;
    yold=(30*rand(N,1)-15)/5;
    zold=(40*rand(N,1)-5)/5;

    x=zeros(steps,N);
    y=zeros(steps,N);
    z=zeros(steps,N);

    e13=zeros(steps,1);
    e45=zeros(steps,1);

    sigma=sigmaV(sigmaI);

    for t=1:steps

        coupling=-G*xold;

        %%integration
        dxdt=-omega.*yold-zold+sigma*coupling;
        dydt=omega.*xold+ar*yold;
        dzdt=br+zold.*(xold-cr);

        xnew=xold+dt*dxdt;
        ynew=yold+dt*dydt;
        znew=zold+dt*dzdt;

        xold=xnew;
        yold=ynew;
        zold=znew;

        x(t,:)=xnew;
        y(t,:)=ynew;
        z(t,:)=znew;

        e13(t)=sqrt((x(t,3)-x(t,1))^2+(y(t,3)-y(t,1))^2+(z(t,3)-z(t,1))^2);
        e45(t)=sqrt((x(t,4)-x(t,5))^2+(y(t,4)-y(t,5))^2+(z(t,4)-z(t,5))^2);

    end

    error13(sigmaI)=mean(e13(end/2:end));
    error45(sigmaI)=mean(e45(end/2:end));

end

figure,plot(sigmaV,error13,sigmaV,error45)

```

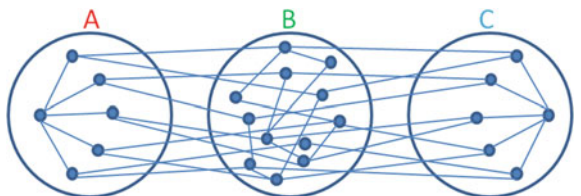
Evidence of relay synchronization in generic complex networks is given in [7], where a setup similar to that of Sect. 5.3 is considered: it consists of Rössler units coupled through ER or SF random networks with a mismatch introduced to a percentage of nodes, selected as those having the higher degrees. In such networks, many nodes synchronize as an effect of dynamical relaying. The work points out several other important facts:

1. As it occurs for the simple chain configuration, the underlying mechanism of dynamical relaying is generalized synchronization.
2. Relay synchronization appears in SF and ER networks with distinct traits. Since in SF networks many paths pass through few high-degree nodes (the hubs), these nodes can act as dynamical relays to synchronize many pairs of nodes. Hence, in SF, networks relay synchronization of many pairs of nodes is induced by the inhomogeneity of a few relay nodes. Instead, a higher fraction of inhomogeneous nodes is required in ER networks.
3. The phenomenon of remote synchronization in networks of chaotic units (that, following the main literature, we have referred to, so far, as relay synchronization) presents some differences with that in periodic units (Sect. 5.3). In the case of chaotic units, generalized synchronization is observed between the dynamical relays and the other nodes, while it is not for periodic units. In addition, in periodic units, remote synchronization anticipates and favors the onset of global synchronization observed for larger coupling, whereas, in chaotic units, global synchronization is prevented by the large mismatch present among the network nodes.
4. Relay synchronization is associated with an enhancing effect observed in generic topologies as well. The addition of mismatched nodes to randomly selected high-degree units of a generic pristine network in fact lowers its synchronization threshold.

5.5 Remote Synchronization in the Presence of Network Symmetries

Remote synchronization has been also found in networks with identical nodes. In this case, a key ingredient is the presence of graph symmetries [12]. Despite the fact that the result was obtained independently of the studies on cluster synchronization, there is a strong relationship between the two forms of synchronization. Consider, for instance, cluster synchronization in the network of Fig. 4.6c. It appears that the nodes in the cluster C_1 are remotely synchronized; in fact, they are synchronized each other, but not connected by a physical link or by a path of synchronized nodes, as cluster C_1 is not synchronized with cluster C_2 . The same considerations apply to

Fig. 5.6 Setup for incoherence-mediated remote synchronization as discussed in [16]



the nodes of C_2 and C_5 , while the nodes of C_6 are synchronized in the classical sense as there is a link connecting them.

In the example of Fig. 4.6c, remote synchronization is mediated by synchronized clusters. As reported in [16] it is also possible that mediation occurs by a subnetwork of nodes with incoherent behavior. Figure 5.6 shows the setup considered in [16]. The network is divided into three subnetworks; the subnetworks A and C are built in such a way that they are connected to the same nodes in subnetwork B. This construction creates a symmetry between these subnetworks. If subnetwork B does not have internal symmetries (equivalently, each cluster in it is formed by a single node), then remote synchronization mediated by incoherent nodes may appear. In particular, there could exist a state where the nodes of subnetwork A are synchronized with their counterpart in subnetwork C. To observe it, clearly this state should exist and be stable, so that stability has to be studied. To do this, the same technique illustrated for cluster synchronization in Chap. 4 can be applied.

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Chapter 6

Synchronization in Time-Varying Networks

In this chapter, we study synchronization in time-varying networks. These are structures modeling systems where the interactions among the nodes change in time. We analyze global synchronization in two relevant cases: when the time-dependence derives from motion of agents and when it is induced by a pattern of activity. Finally, we illustrate the onset of chimera states in a time-varying topology.

6.1 Introduction

In most of the examples discussed so far, the interactions have been assumed to occur along links that are fixed in time, thus each unit always interacts with the same set of nodes. This is clearly a simplifying hypothesis as thinking, for instance, to social systems it is clear that the acquaintances vary in time in their composition and strength. In this chapter we study synchronization in the presence of interconnections that are the result of a dynamical process leading pairs of network nodes to interact in some time window and to not interact at other times. To this aim, models accounting for time-varying links have to be adopted [2, 3, 9, 17, 18, 20, 22, 25]. These models differ for the assumptions of the dynamics of link evolution.

A first class of models is the one based on a structure of interconnections that are stochastically activated or deactivated: this network of switching links may coexist with another structure of static ones (that is, not evolving in time) as in [2, 7, 25] or constitute the whole set of possible interactions among the units as in [3, 17, 22]. The links can be switched on or off independently each other, as in the previously mentioned works, or simultaneously as in [9]. Synchronization in time-varying networks exhibits distinguishing features and this first class of models already shows one of these peculiar traits. In stochastically on-off switched links, synchronization is possible also when the switched interaction topologies do not support it in the static case [3].

Two further examples of classes of time-varying models are those deriving from proximity networks in mobile agents and those resulting from an activity pattern of the network nodes. In the first case, the units (also called in this context the agents) are assumed to move in a given space (for instance, planar) and interactions to be limited by proximity [8, 13–15]. We will discuss one of these models in Sect. 6.2.

The second class of models considers activity-driven interactions. These models are predicated upon the works [21, 24, 26] that introduce the activity-driven network model to describe the instantaneous dynamics of time-varying networks and will be discussed in Sect. 6.3.

The chapter is concluded with an example showing that time-varying networks can also support synchronization patterns different than the global one, and in particular chimera states.

6.2 Synchronization of Mobile Agents

In this section we introduce the model to study synchronization in time-varying networks and focus on the case study of mobile agents. In this context, the nodes are mobile agents, each one associated with an oscillator, interacting each other on the basis of a proximity network, that is, if they are at a distance lower than the *interaction radius*. The model may find applications to the study of clock synchronization in mobile robots [6] and in sensor networks with limited communication [29], task coordination in swarming animals, or synchronized bulk oscillations in populations of yeast cells [10–12].

To account for interactions varying in time, the node dynamics is written as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N g_{ij}(t) \mathbf{H} \mathbf{x}_j \quad (6.1)$$

where, as usual, $\mathbf{x}_i \in \mathbb{R}^n$ is the state vector of the i -th oscillator (with $i = 1, \dots, N$), $\mathbf{f}$ the uncoupled dynamics, $\mathbf{H} \in \mathbb{R}^{n \times n}$ the inner coupling matrix, and σ the coupling strength. The time-dependence of the links is incorporated in the coefficients $g_{ij}(t)$ that are the elements of a time-varying Laplacian matrix. We suppose that the interactions are mutual, so that at each time instant the Laplacian matrix is symmetric. The coefficients are, therefore, defined as $g_{ij}(t) = g_{ji}(t) = -1$ if i and j are connected by a link at time t , while $g_{ij}(t) = g_{ji}(t) = 0$ otherwise, and $g_{ii}(t) = -\sum_{j=1, j \neq i}^N g_{ij}(t)$.

The evolution of $g_{ij}(t)$ is determined by the motion of the agents. They are considered to lie in a planar space of size L and periodic boundary conditions. Each agent moves with velocity $\mathbf{v}_i(t)$ and direction of motion $\theta_i(t)$, i.e., $\mathbf{v}_i(t) = v e^{i\theta_i(t)}$. We assume the modulus of the agent velocity v to be constant in time and equal for all the agents. Let us indicate with $\mathbf{y}_i(t)$ the position of the i -th agent in the plane at time t . For its update we consider a random walker model as follows:

$$\begin{aligned} \mathbf{y}_i(t + \Delta t_M) &= \mathbf{y}_i(t) + \mathbf{v}_i(t) \Delta t_M \\ \theta_i(t + \Delta t_M) &= \theta_i(t + \Delta t_M), \end{aligned} \quad (6.2)$$

where $\eta_i(t)$ ($i = 1, \dots, N$) are N independent random variables chosen at each time with uniform probability in the interval $[-\pi, \pi]$. Δt_M is the motion integration step size that we here assume to be equal to the integration step size of Eqs. (6.1) (more generally, taking the two parameters independent each other enables the study of the different time scales on the system behavior, see [13, 15]). We further incorporate in the model the possibility of long-distance jumps by including a parameter p_j , representing the probability for an individual to jump into a completely random new position. Thus, the position of each agent is updated according to Eqs. (6.2) with probability $1 - p_j$, or to random coordinates in the plane with probability p_j . We will show below that the parameter p_j plays a fundamental role in the system behavior.

From the positions of the agents at each time instant the links of the time-varying matrix are calculated. In particular, indicated as r the interaction radius, we have that $g_{ij}(t) = 1$ if $\|\mathbf{y}_j(t) - \mathbf{y}_i(t)\| \leq r$, and $g_{ij}(t) = 0$, otherwise.

Some insight into the model is gained under the assumption that the switching among the possible topologies occurs at a fast time scale. This hypothesis is referred to as fast switching in the context of blinking networks [17, 28] and enables the study of the stability of the synchronization manifold of the time-varying network by inspecting the properties of its time average. Given a system of (statically) coupled oscillators described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N \bar{g}_{ij} \mathbf{H} \mathbf{x}_j \quad (6.3)$$

assume that it supports a stable synchronization manifold and that there exists a constant $\bar{T}$ such that $\frac{1}{\bar{T}} \int_t^{t+\bar{T}} G(\tau) d\tau = \bar{G}$, then, according to [28], there exists η^* such that for all fixed $0 < \eta < \eta^*$ the system of time-varying coupled oscillators described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^N g_{ij}(t/\eta) \mathbf{H} \mathbf{x}_j \quad (6.4)$$

also admits a stable synchronization manifold.

The network in Eqs. (6.3) is static and, therefore, its synchronization properties can be studied with the methods of Chap. 2, for instance with the MSF approach. Thus, to perform the study of the stability of synchronization of the time-varying network inherited by mobile agents, it suffices to calculate $\bar{G}$.

This calculation (detailed in [13]) gives $\bar{G} = \frac{\pi r^2 \rho}{N} \mathbf{G}_{\mathcal{K}}$, where $\rho = \frac{N}{L^2}$ is the agent density and $\mathbf{G}_{\mathcal{K}}$ the Laplacian network of a complete graph $\mathcal{K}$. The term $\frac{\pi r^2 \rho}{N}$ represents the probability, in the limit of fast switching, that two agents are neighbor; in fact, in this limit the agents at each time instant occupy random positions in the

space and the probability that a link exists is simply the fraction of the total area that is overlapped by the sensing region, that is, $\frac{\pi r^2}{L^2} = \frac{\pi r^2 \rho}{N}$.

Now considering a system with type III MSF, with thresholds α_1 and α_2 , and taking into account that the eigenvalues of the complete graph are $\gamma_i = N$ for $i = 2, \dots, N$, one can derive that

$$\alpha_1 < \sigma \pi r^2 \rho < \alpha_2 \quad (6.5)$$

and thus

$$\frac{\alpha_1}{\pi r^2 \sigma} < \rho < \frac{\alpha_2}{\pi r^2 \sigma} \quad (6.6)$$

which expresses the fact that for type III MSF systems synchronization is attained if the density is in the interval $[\frac{\alpha_1}{\pi r^2 \sigma}, \frac{\alpha_2}{\pi r^2 \sigma}]$.

With similar arguments, it is easy to derive that for type II MSF systems the condition is $\rho > \frac{\alpha_1}{\pi r^2 \sigma}$. The prediction of the model is particularly interesting as it is in agreement with what observed in some real systems, for instance in yeast cell populations where sustained oscillations emerge provided that the cell density is sufficiently high [10, 11].

The previous analysis suggests that the density is a fundamental parameter. The other model parameters, however, are also important as they determine the validity of the fast switching hypothesis. Both the agent velocity v and the jumping probability p_j , for instance, increase the likelihood of this assumption, in fact they both increase the average distance covered at each motion step, resulting in a faster switching among the possible interaction configurations.

We now report the MATLAB® code to simulate a network of mobile agents taking as reference dynamics the Rössler systems with $a = b = 0.2$, $c = 7$, and $H = [1 \ 0 \ 0]^T \cdot [1 \ 0 \ 0]$:

$$\begin{aligned} \dot{x}_i &= -y_i - z_i - \sigma \sum_{j=1}^N g_{ij}(t)x_j \\ \dot{y}_i &= x_i + ay_i \\ \dot{z}_i &= b + z_i(x_i - c) \end{aligned} \quad (6.7)$$

Suppose that the interaction radius is $r = 1$ and the coupling coefficient $\sigma = 10$, considering that $\alpha_1 \simeq 0.2$ and $\alpha_2 \simeq 4.8$, the theoretical calculations predict that the time-varying network synchronizes for $0.006 = \frac{\alpha_1}{\pi r^2 \sigma} < \rho < \frac{\alpha_2}{\pi r^2 \sigma} = 0.153$. Using the following code it can be checked that for $\rho = 0.08$, $v = 0.1$ and $p_j = 0$ (default parameters) synchronization is indeed achieved. Reducing the agent speed (e.g., $v = 0.01$ and $p_j = 0$) yields loss of synchronization, that can be recovered if the jumping probability p_j is increased (e.g., $v = 0.01$ and $p_j = 1$).

Code for numerical simulation of a network of mobile Rössler systems

```
1 %synchronization in mobile agents
2
3 %system motion parameters
4 rho=0.08;
5 v=0.1;
6 rquad=1;
7 N=10;
8 L=sqrt(N/rho);
9 pj=0; %jumping prob
10
11 %inicialization of positions
12 x=L*rand(N,1);
13 y=L*rand(N,1);
14
15 %coupling coefficient
16 sigma=10;
17
18 %integration parameters
19 dt=0.001;
20 stepsT=1000000;
21
22 %dynamic unit parameters
23 ar=0.2;
24 br=0.2;
25 cr=7;
26
27 %%%%memorization of the variables
28 % x=zeros(stepsT,N);
29 % y=zeros(stepsT,N);
30 % z=zeros(stepsT,N);
31
32 %initial conditions for Roessler oscillators
33 xold=(30*rand(N,1)-15)/5;
34 yold=(30*rand(N,1)-15)/5;
35 zold=(40*rand(N,1)-5)/5;
36
37 syncerror=zeros(stepsT,1);
38
39 for t=1:stepsT
40     t
41
42     %%%network update
43     %evaluate distances on a torus and find neighbors
44     X=x*ones(1,N);
45     M(:, :, 1)=abs(X-X');
46     M(:, :, 2)=L-abs(X-X');
47     dx=min(M, [], 3);
48     Y=y*ones(1,N);
49     M(:, :, 1)=abs(Y-Y');
50     M(:, :, 2)=L-abs(Y-Y');
51     dy=min(M, [], 3);
52     d=sqrt(dx.^2+dy.^2); %d matrix of distances
53     neighbors=(d<=rquad)-eye(N); %matrix of neighbors
54     G=diag(sum(neighbors,2))-neighbors;
55
56     %%%synchronization
57     %%%evaluation of the coupling
58     coupling=-G*xold;
59
60     %%%evaluation of the dynamics
61     dxdt=-yold-zold+sigma*coupling;
62     dydt=xold+ar*yold;
63     dzdt=br+zold.*(xold-cr);
64
65     xnew=xold+dt*dxdt;
```

```

66     ynew=yold+dt*dydt;
67     znew=zold+dt*dzdt;
68
69     xold=xnew;
70     yold=ynew;
71     zold=znew;
72
73     %%memorization to check integration
74     x(t,:)=xnew;
75     y(t,:)=ynew;
76     z(t,:)=znew;
77
78     %%evaluation of the sync error
79     Xn=xnew*ones(1,N);
80     Yn=ynew*ones(1,N);
81     Zn=znew*ones(1,N);
82     syncerror(t)=sqrt(sum(sum((Xn-Xn').^2+(Yn-Yn').^2+(Zn-Zn').^2)
83         )/N/(N-1));
84
85     %%update of the positions
86     teta=2*pi*rand(N,1);
87     m=(pj- rand(N,1))>0;    %m(i)=1 if agent i jumps, m(i)=0
88                             otherwise
89
90     x=(1-m).*(x+v*cos(teta))+L*m.*rand(N,1);
91     y=(1-m).*(y+v*sin(teta))+L*m.*rand(N,1);
92
93     x=mod(x,L);
94     y=mod(y,L);
95
96 end

```

As observed, the time scales at which the network dynamics evolves play an important role in the determination of the collective behavior of the populations of agents. This issue is in depth studied in [15, 16] for a population of Kuramoto oscillators.

Strategies to control networks of mobile agents are instead investigated in [14], where, in particular, spatial pinning control is used. The idea is that the control is applied to a fixed region of the space rather than to a fixed set of agents. In the work it is demonstrated that there exists suitable values of the parameters (size of the control region, density, velocity) such that the network can be fully controlled towards a desired trajectory.

Another interesting aspect connected to time-varying networks inherited by mobile agents is the influence of the type of motion that can be disordered, as in the above example, or ordered as in platoons. Depending on the system parameters, global synchronization may be induced by a coordinated motion or, at the opposite, by a disordered motion [8], thus resulting in a complex interplay between motion and synchronization.

6.3 Synchronization in Activity-Driven Networks

In activity-driven network models [24, 26], an activity potential function that represents the propensity of a node to contact the other ones is defined and used at each time to generate the pattern of interactions. The activity potential distribution can be

empirically measured, so that activity-driven networks have been applied to describe real datasets such as scientific collaboration networks [24], online social networks [24], and human interactions via mobile phone calls [21]. The structural properties of the real networks mentioned, such as the presence of hubs, are reproduced in the model assuming an heterogeneous activity pattern.

As for the proximity networks discussed in the previous section, also in this case, the time-varying network is represented by a time-varying Laplacian matrix $G(t) = \{g_{ij}(t)\}$. According to the model introduced in [24], each node is assigned a probability χ_i (that is fixed at the beginning and kept constant in time as it represents a characteristic of the node) to create new contacts, drawn by a probability distribution $F(\chi)$, arbitrarily chosen or derived from empirical data. Each τ integration steps the network is reset, all links are deleted and wiring is restarted with N disconnected nodes. With probability χ_i the node i becomes active and is linked with m other (randomly chosen) nodes. These links are maintained for a window of time equal to $\tau \Delta t$ (Δt is the integration step size), after which the process of link generation is iterated. Following [24], we adopt a power-law distribution with lower cut-off, i.e. $F(\chi) = \chi^\gamma$ with $\varepsilon \leq \chi \leq 1$.

The mechanism generating the links according to the node activity rate produces a sequence of graphs which are switched with a time scale equal to $\tau \Delta t$. The ratio between this time scale and that of the dynamics of the units is particularly important. As a proxy of it we consider the parameter τ , large values of this parameter represent the scenario where graphs vary slowly with respect to the dynamics of the units, whereas values close to one indicate graphs evolving with a fast time scale.

If the switching among the possible topologies occurs at a fast time scale, then the same approach discussed in the previous section may be applied. The only difference is the matrix $\bar{G}$, which is here calculated for an activity-driven network. To account for a fast switching, we consider $\tau = 1$. The matrix $\bar{G}$ is given by

$$\bar{G} = \frac{1}{L} \sum_{k=1}^L G(t_k) \quad (6.8)$$

where L are all the possible instantaneous configurations of the graph. For large $\bar{T}$, $\bar{g}_{ij}$ approaches $-p_{ij}$, where p_{ij} is the probability that node i and node j (with $i \neq j$) are connected at a given time. For activity-driven networks this probability is [27]:

$$p_{ij} = 1 - \left[\left(1 - \frac{\chi_i}{N} \right) \left(1 - \frac{\chi_j}{N} \right) \right]^{mN} \quad (6.9)$$

where χ_i and χ_j are the contact probabilities of node i and j . Therefore, we approximate $\bar{g}_{ij} \simeq -p_{ij}$, and, in addition, define $\bar{g}_{ii} = -\sum_j \bar{g}_{ij}$, so that the the matrix $\bar{G}$ represents the Laplacian of a fully connected, weighted, undirected network. The power-law distribution of χ_i yields a heterogeneous distribution of the coefficients of $\bar{G}$ (that represent the *weights* of the coupling between two generic nodes), whereas in

the case of mobile agents or uniform activity distributions we obtain homogeneous coefficients.

To illustrate an example of synchronization in the activity-driven model [4], we consider a network with $N = 200$ nodes, $m = 5$ and power-law distribution with exponent $\gamma = -2$, and cut-off $\varepsilon = 0.001$. The node dynamics is given by identical Rössler systems as in Eqs.(6.7) with the same parameters and coupling function. Eqs.(6.7) are integrated with a step size equal to $\Delta t = 0.001$ for a duration of time equal to $T = 10^4$. For the whole length of the simulation, we monitor the global synchronization error $e(t) = \sqrt{\frac{1}{N(N-1)} \sum_{i,j} \|\mathbf{x}_j - \mathbf{x}_i\|^2}$ (as in Eq. (2.3)) and then calculate the average of its last value $e(T)$ over M numerical simulations starting from different initial conditions $E = \frac{1}{M} \sum_{k=1}^M e_k(T)$. The initial conditions are randomly drawn from uniform distributions in the interval $x \in [-3, 3]$, $y \in [-3, 3]$, and $z \in [-1, 7]$, so that to start from a point in the basin of attraction of the uncoupled chaotic attractor.

The error E is plotted versus σ in Fig. 6.1 for several values of τ ($\tau = 1$, $\tau = 10$, $\tau = 30$, and $\tau = 100$). The case $\tau = 1$ is compared with the theoretical prediction under fast switching, obtained by simulating a static network with connectivity

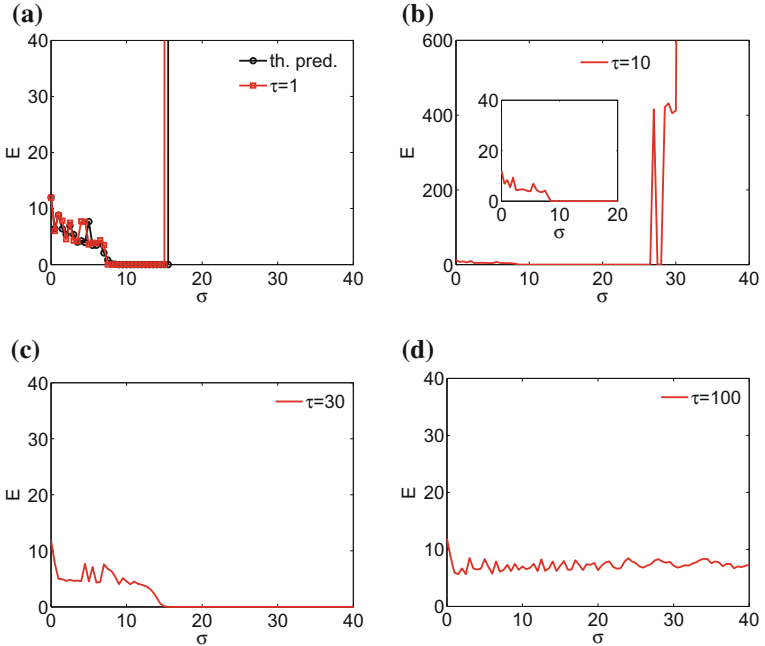


Fig. 6.1 Average synchronization error E with respect to the coupling coefficient σ for an activity-driven network of $N = 200$ Rössler units with $m = 5$ and power-law activity distribution with $\gamma = -2.1$. The parameter τ is: **a** $\tau = 1$; **b** $\tau = 10$; **c** $\tau = 30$; **d** $\tau = 100$. Results are averaged over 100 realizations starting from different initial conditions

given by $\bar{G}$. The numerical results for $\tau = 1$ are in agreement with the theoretical expectations, thus showing as the thresholds for synchronization when the network evolves with a fast dynamics can be accurately estimated.

The behavior for slower switching is also interesting. In fact, increasing τ , we note that the transition from synchronized to unsynchronized behavior is shifted to the right and eventually disappears. For example, for $\tau = 10$ we observe a larger region of values of σ yielding synchronization, and for $\tau = 30$ the region is unbounded. Similar results have been found in other time-varying topologies [5, 22], altogether showing that synchronization is generally promoted by a fast dynamics of the network evolution, but eventually even more favorable scenarios arise when the switching rates are slightly lower than the fast switching condition. If the switching is too slow, as for $\tau = 100$ in Fig. 6.1, the network never synchronizes.

We conclude the section by including the MATLAB® code for the simulation of a single value of σ .

Code for numerical simulation of a time-varying activity-driven network of Rössler systems

```

1  % synchronization in activity driven networks
2
3  % activity distribution
4  N=200;
5  gamma=-2.1; %gamma
6  xmin=0.001; %epsilon
7  xmax=1;
8  a = [(xmax^(gamma+1) - xmin^(gamma+1))*rand(N,1) + xmin^(gamma+1)
        ].^(1/(gamma+1));
9
10 %network parameter
11 m=5;
12 tau=10;
13 sigma=0.2;
14
15 %integration parameters
16 dt=0.001;
17 stepsT=1000000;
18
19 %dynamic unit parameters
20 ar=0.2;
21 br=0.2;
22 cr=7;
23
24 % %%memorization of the variables
25 % x=zeros(stepsT,N);
26 % y=zeros(stepsT,N);
27 % z=zeros(stepsT,N);
28
29 %initial conditions for Roessler oscillators
30 xold=(30*rand(N,1)-15)/5;
31 yold=(30*rand(N,1)-15)/5;
32 zold=(40*rand(N,1)-5)/5;
33
34 syncerror=zeros(stepsT,1);
35
36 for t=1:stepsT
37
38     if mod(t,tau)==1
39
40         %%generation of the links, no self-loops, no repeats
41         edgelist=[];

```

```

42     edgelistu=[];
43     activenodes=(rand(N,1)<(a));
44     indicesactivenodes=find(activenodes==1);
45
46     for i=1:length(indicesactivenodes)
47         nodei=indicesactivenodes(i);
48         V=randperm(N)';
49         vectorV=V(V~=nodei);
50         vectorV=vectorV(1:m);
51         edgelist=[edgelist; vectorV nodei*ones(m,1); nodei*ones(m
                    ,1) vectorV];
52         edgelistu=unique(edgelist,'rows');
53
54     end
55     %%if net update
56
57     %%%synchronization
58     %%%evaluation of the coupling
59     coupling=zeros(N,1);
60     [n1,m1]=size(edgelistu);
61     for indicel=1:n1
62         coupling(edgelistu(indicel,1))=coupling(edgelistu(indicel
                    ,1))+xold(edgelistu(indicel,2))-xold(edgelistu(indicel
                    ,1));
63
64     end
65
66     %%%evaluation of the dynamics
67     dxdt=-yold-zold+sigma*coupling;
68     dydt=xold+ar*yold;
69     dzdt=br+zold.*(xold-cr);
70
71     xnew=xold+dt*dxdt;
72     ynew=yold+dt*dydt;
73     znew=zold+dt*dzdt;
74
75     xold=xnew;
76     yold=ynew;
77     zold=znew;
78
79     % %%%memorization to check integration
80     x(t,:)=xnew;
81     y(t,:)=ynew;
82     z(t,:)=znew;
83
84     %%%evaluation of the sync error
85     Xn=xnew*ones(1,N);
86     Yn=ynew*ones(1,N);
87     Zn=znew*ones(1,N);
88     syncerror(t)=sqrt((sum((Xn-Xn')).^2+(Yn-Yn')).^2+(Zn-Zn')).^2
                    )/N/(N-1));
89 end

```

6.4 Chimera States in Time-Varying Networks

In Chap. 3 we have discussed the onset of chimera states with different coupling schemes and topologies, however, in all the examples discussed the coupling structure was considered to be static. In this chapter we have seen that the pattern of connectivity in real systems may depend on time. Having this in mind, we start from

the model of two interacting populations of phase oscillators as in Eqs. (3.1), and we hypothesize that the links between the two groups vary over time, while the connections within each group remain fixed [7]. We will show that the system can support stable and breathing chimeras and, in addition, the so-called *alternating* chimera state, where the two populations alternate each other in the high synchrony regime.

In the model the inter-population links are randomly switched on or off at fixed equally spaced time intervals of length τ . During each time interval, every possible connection between two nodes in different groups is turned on, with probability p_{switch} and weight equal to one, independently of the other links, and independently of whether or not it has been turned on during the previous time interval. This leads to an inter-population connectivity which is a time-varying matrix given by a random sequence of Erdős-Rényi graphs with average in-degree $p_{\text{switch}} N_\sigma$.

The system is described by¹:

$$\dot{\theta}_i^\sigma = \omega + \sum_{\sigma'=1}^2 \frac{1}{N_{\sigma'}} \sum_{j=1}^{N_{\sigma'}} K_{ij}^{\sigma\sigma'}(t) \sin(\theta_j^{\sigma'} - \theta_i^\sigma - \alpha), \quad (6.10)$$

where θ_i^σ is the phase of oscillator i in population σ , ω is the intrinsic frequency (equal for all the oscillators, fixed at $\omega = 1$), α is the phase lag, and $K_{ij}^{11}(t) = K_{ij}^{22}(t) = \mu > 0 \forall t$. $K^{12}(t) = [K^{21}(t)]^T$ are stochastic matrices whose elements are defined as $K_{ij}^{12}(t) = K_{ji}^{21}(t) = s_{ij}(q)$ for $(q-1)\tau < t < q\tau$ with:

$$s_{ij}(q) = \begin{cases} 1 & \text{with probability } p_{\text{switch}} \\ 0 & \text{with probability } 1 - p_{\text{switch}} \end{cases} \quad (6.11)$$

where $q \in \mathbb{N}^+$ defines the number of switching intervals, each of length τ .

To monitor coherence in each population we evaluate two separate Kuramoto order parameters as defined in Eq. (3.2). In Fig. 6.2 their evolution, for selected values of p_{switch} with τ fixed to $\tau = 0.1$, is reported. Stable chimeras (Fig. 6.2a) are characterized by a coherent population, showing synchronized oscillations and an order parameter close to one, and a desynchronized population characterized by an approximately constant phase coherence lower than one. For breathing chimeras (Fig. 6.2b) the phase coherence of the desynchronized population is not constant, but pulsates. Alternating chimera states (Fig. 6.2c) are characterized by alternating synchrony between the two populations. While one population is nearly synchronized, the other displays a pulsating order parameter; the oscillators in the desynchronized population may then gain synchrony at the expense of the oscillators in the other population which lose synchrony. The behavior is found to alternate with either regular or irregular periods as a function of the value of p_{switch} and τ .

¹Similarly to Sect. 3.2, also here the notation is kept coherent with that of the original papers on chimera states, so that the coupling coefficient is indicated with K , whereas σ is the index labeling oscillator populations.

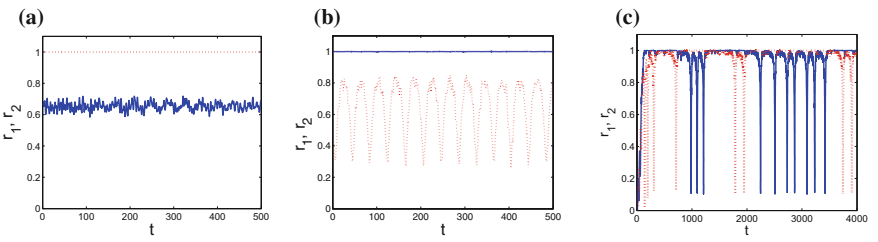


Fig. 6.2 Behavior of the two order parameters $r_1(t)$ (blue solid) and $r_2(t)$ (red dotted) in for $\tau = 0.1$ and three values of the switching probability p_{switch} : **a** $p_{switch} = 0.38$; **b** $p_{switch} = 0.33$; **c** $p_{switch} = 0.25$. The population size is $N = 200$, the coupling strength within each group is fixed to $\mu = 0.6$, the oscillators frequency $\omega = 1$, and the phase lag $\alpha = 1.5$

The MATLAB[®] code to numerically integrate the time-varying network is similar to the one discussed in Sect. 3.2 with the inclusion of commands to generate, at each time iteration, the time-varying adjacency matrix.

Commands to implement the time-varying network for the study of alternating chimera states

```

1  %%chimera states in a time-varying structure of two coupled
   populations of oscillators
2
3  %% integration parameters
4  dt=0.01;
5  max_step=1000/dt;
6
7  %% construction of the adjacency matrix
8  N=2*100;
9  miu=0.6;
10 ni=0.4;
11 A=[miu*ones(N/2) ni*ones(N/2); ni*ones(N/2) miu*ones(N/2)];
12 A=A-eye(N)*miu;
13
14 %% parameters
15 alpha=1.5;
16 omega=ones(N,1);
17
18 %% initialization
19 x=zeros(N,max_step);
20 x(:,1)=2*pi*rand(N,1);
21 dxdt=zeros(N,1);
22 R=zeros(max_step,1);
23
24 %% parameters for the switch
25 pswitch=0.25;
26 tau=100;
27
28 for t=1:max_step
29
30     if mod(t,tau)==1,
31         InterpopM=rand(N/2)<pswitch;
32         A=[miu*ones(N/2) InterpopM; InterpopM' miu*ones(N/2)];
33     end
34
35     %% main integration
36     coup=zeros(N,1);
37     for i=1:N

```

```

38     coup(i)=sum(A(:,i).*sin(x(:,t)-x(i,t)-alpha));
39     dxdt(i)=omega(i)+2/N*coup(i);
40     x(i,t+1)=mod(x(i,t)+dt*(dxdt(i)),2*pi);
41 end
42 end
43
44 %% order parameters
45 R1=abs(mean(exp(j*(x(1:100,:))),1));
46 R2=abs(mean(exp(j*(x(101:end,:))),1));
47
48 %% plots
49 figure,plot(R1),hold on,plot(R2,'r')
50 figure,plot(x(:,end),'.')

```

Also in the time-varying case the onset of chimera states depends on initial conditions, however, differently from the cases studied in Chap. 3 it does not require that the system starts from initial conditions close to the final state; the basin of attraction of the chimera state, in this case, is larger. Other studies have revealed that when the connectivity is fixed in time and the units are homogeneous only stable and breathing chimeras appear [1], while the onset of alternating chimeras has been identified in network with heterogeneity of the oscillators [23]. In networks of identical units, the time-dependence seems a key ingredient for alternating chimera states [7].

When the rate of connectivity changes is fast, compared to the oscillator dynamics, the network may be described by a low-dimensional system that is derived following an approach similar to that presented in Chap. 3. Here, in the fast switching hypothesis, we have to consider the time average value of the inter-population coupling coefficient, so that $\nu = p_{switch}$ in Eq. (3.11). The onset of alternating chimera states is due to the presence of fluctuations, which may be induced either by the finite size of the network or by large switching times. To model them, we may consider a stochastic term added to the averaged reduced system as follows:

$$\dot{\rho}_1 = \frac{1 - \rho_1^2}{2} [\mu \rho_1 \cos \alpha + p_{switch} \rho_2 \cos(-\psi - \alpha)] + \xi(t) \quad (6.12a)$$

$$\dot{\rho}_2 = \frac{1 - \rho_2^2}{2} [\mu \rho_2 \cos \alpha + p_{switch} \rho_1 \cos(\psi - \alpha)] - \xi(t) \quad (6.12b)$$

$$\begin{aligned} \dot{\psi} = & -\frac{1 + \rho_1^2}{2} \left[\mu \sin \alpha + p_{switch} \frac{\rho_2}{\rho_1} \sin(\psi + \alpha) \right] \\ & + \frac{1 + \rho_2^2}{2} \left[\mu \sin \alpha + p_{switch} \frac{\rho_1}{\rho_2} \sin(-\psi + \alpha) \right], \end{aligned} \quad (6.12c)$$

where $\xi(t)$ is a Gaussian white noise satisfying $\langle \xi(t) \xi(t') \rangle = D \delta(t - t')$ with noise intensity D .

We have numerically verified that a small level of noise in Eq. (6.12) leads to alternating chimera states analogous to those observed in the switching system. For instance, the alternating chimera state of Fig. 6.2c is also identified in the averaged reduced model (6.12) for $p_{switch} = 0.25$ and $D = 0.004$.

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Chapter 7

Conclusive Remarks

Many natural and man-made systems are complex as they are formed by units that are dynamical and interact with each other. Such systems are modeled by a set of ordinary differential equations, often written in a dimensionless form to capture the generality of the approach, where a complex network accounts for the interactions among the system units. Despite each unit has its own intrinsic dynamics, which can differ from that of the other system elements, the presence of the interactions may generate the onset of a collective synchronous status, where all the units agree to follow a common behavior.

The regime where all the units follow the same behavior is not the only one possible in complex networks and different forms of synchronization have been discovered and characterized under the framework of coupled dynamical systems, unveiling a more rich, and in some cases more close to the real world, scenario. To make an example, the complete synchronization of all the neurons in the brain is associated to pathological epileptic states, whereas in normal conditions, neurons separate into groups showing different synchronous activities.

In this book, we have discussed key ideas, examples, and control techniques on synchronization in complex networks of nonlinear dynamical systems. Starting from global synchronization, we have examined chimera states, cluster synchronization, relay and remote synchronization and then also considered the case of time-varying links. All these topics are the subject of an intense research that day by day shapes our understanding of them. Often the phenomena observed have been named with different terms with some degree of overlap between the definitions given and studied independently each other without exploring the common aspects of the phenomenon, so that a study of the similarities and the differences between them is still to be fully addressed. With this book, we hope to have provided a short introduction to the theme that could give to the reader the basic concepts and tools and further stimulate the interest toward the research in the field.