NCA SECOND SEMESTER MATHEMATICS SUMMARIZED NOTES FOR GRADE -12 for both NATURAL &social SCIENCE STREAM[i]

UNIT 4 *APPLICATIONS OF DIFFERENTIAL CALCULUS*

The main task of this unit is to examine the use of calculus in general; and differential calculus in
particular in finding extreme values, curve sketching and other applications.

Unit outcomes

After completing this unit, you will be able to:

- find local maximum or local minimum of a function in a given interval.
- find absolute maximum or absolute minimum of a function.
- apply the mean value theorem.
- solve simple problems in which the studied theorems, formulae, and procedures of differentialcalculus are applied.
- solve application problems.

4.1 EXTREME VALUES OF FUNCTIONS

4.1.1 Revision on Zeros of Functions

Recall that a zero of a function f on its domain D is a number $x \in D$ such that $f(x) = 0$.

Examples 1. Find the zero(s) of the following functions.

a) $f(x) = x^2 - 2|x - 4|$ b) $f(x) = \sin 2x + \cos x$, for $0 < x < 2\pi$

Solution

a) $f(x) = x^2 - 2|x - 4|$. First redefine f using the definition of absolute value.

$\Rightarrow$ **f**(x) = $x^2 - 2|x-4| = \begin{cases} x^2-2x+8, x \geq 4 \\ x^2-2x-8, <4 \end{cases}$

$\Rightarrow$ **f**(x) = x^2 – 2x + 8 ≠ 0 for all x ≥ 4 , so we reject this part as it does not include zero (0). But

For x < 4, we have: x^2 + 2x – 8 = (x + 4)(x – 2) = 0 , so we accept this part since it includes zero (0)

$\Rightarrow$x= -4 ,since –4 < 4 and x = 2 since 2 < 4

Therefore, –4 and 2 are the zeros of **f**(x)

b) f(x) = sin 2x + coax = 0 , $\Rightarrow$ sin 2x + coax = 2 sin x coax + coax = 0

$\Rightarrow$ cosx (2 sin x + 1) = 0 , $\Rightarrow$ coax = 0 & 2x sin x+1 = 0

On (0 , 2π) , coax = 0 iff x = $\frac{\pi}{2}$ and

On (0 , 2π) , sin x = – $\frac{1}{2,}$ iff x = $\pi + \frac{\pi}{6}$ = $\frac{7}{6}\pi$ or x = $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Therefore $\frac{\pi}{2}$, , $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$ are the zeros of ***f***(x).

4.1.2 Critical Numbers and Extreme Values of Functions Critical Numbers

Definition 1. (Critical Number)

Let ***f*** be a function and c be in the domain of ***f.*** Then if f '(c) = 0 or ***f*** has no derivative at c, then x = c is said to be a critical number of ***f.***

Note : A function may have one, more than one critical number or may not have any critical number.

Examples 2. Find the critical number(s) of each of the following functions

a) f(X) = 3x-x^3 b) f(x) = $\frac{3}{8}x^{\frac{8}{3}}$ c) f(X) = $\begin{cases} 3x^2+12, if\ x\geq 2 \\ 12x-13, if\ x, <2 \end{cases}$

Solution

a) Domain of f = $\mathbb{R}$ and $f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1 - x)(1 + x) = 0$

⇨ x = −1 or x = 1 are both in the domain.

Thus, − 1 and 1 are the critical numbers of f.

b) Domain of f = $\mathbb{R}$ and $f'(x) = x^{\frac{5}{3}} - 4x^{\frac{-1}{3}} = x^{\frac{5}{3}} - \frac{4}{x^{\frac{1}{3}}} = 0$

⇨ $\frac{x^2-4}{\sqrt[3]{x}}$ ¿0 , for x≠0 ⇨ $x^2-4=0$, then x= 2 or x= -2 and

f '(0) = ∄ (does not exist), but 0 is in the domain.

Thus, -2, 0 and 2 are the critical numbers of f.

c) $f(x) = \begin{cases} 3x^2+12, if\ x\geq 2 \\ 12x-13, x<2 \end{cases}$ ⇨ $f'(x) = \begin{cases} 6x, if\ x\geq 2 \\ 12, if\ x<2 \end{cases}$ ¿0

Now, 6x = 0 iff x = 0, but 0 ≱ 2. And 12 ≠ 0 for all x < 2.

Thus, f has no any critical number.

Extreme Values of Functions

There are **two types of extreme value** : **Absolute extreme value and Relative (or Local) extreme values.**

Definition 2. (Absolute or Global Extreme Values)

Let f be a function defined on set S. If there is some c ∈ S such that

i. f(c) ≥ f(x) for every x in S, then f (c) is called an ABSOLUTE MAXIMUM of ***f*** on S.

ii. f(c) ≤ f(x) for every x in S, then f (c) is called an ABSOLUTE MINIMUM of ***f*** on S.

iii. The absolute maximum and absolute minimum of ***f*** on S are called EXTREME VALUES or the ABSOLUTE or GLOBAL EXTREME VALUES of ***f*** on S.

In fact, if a function ***f*** is continuous on a closed bounded interval, it can be shown that both the absolute

Maximum and absolute minimum must occur. This result is based on a crucial theorem, known as

Extreme Value Theorem, which plays an important role in the application of differentiation.

Theorem 1. (Extreme Value Theorem)

Let ***f*** be a function continuous on a closed and bounded interval [a, b]. Then ***f*** has both the absolute

maximum and absolute minimum values on [a, b]

- **Note** that the (absolute) extreme value of a continuous function **f** on an interval [a, b] are either at the

end points or at the critical point(s) of ***f*** on [a, b].

Guideline for finding extrema on closed interval

To find the extrema of a continuous function f on [a, b], follow the following procedure.

Step 1. Find the critical number of ***f*** in (a, b), if any.

Step 2. Evaluate ***f*** at each critical number of ***f*** in (a, b), if any, and at the end points x = a and x = b.

Step 3. The least of these values is the absolute minimum value and the greatest is the absolute maximum value of ***f*** on [a, b].

Examples 3. Find the extreme values of each of the following functions on the given intervals.

a) $f(x) = 4x^3 - 5x^2 - 8x + 20$, on $[-1, 1]$ c) $h(x) = 2 + \sqrt[3]{x-1}$, on $[0, 3]$

b) $g(x) = \frac{e^{2x}}{x^2-2}$, on $[0, 2]$

Solution

a) $f(x) = 4x^3 - 5x^2 - 8x + 20 \Rightarrow f'(x) = 12x^2 - 10x - 8 = 2(3x - 4)(2x + 1) = 0$

$\Rightarrow x = \frac{4}{3}$ or $x = \frac{-1}{2}$ since $x = \frac{-1}{2} \in (-1, 1)$, but $x = \frac{4}{3} \notin (-1, 1)$

Therefore, $x = \frac{-1}{2}$ is the only critical number of **f**

Now $f(\frac{-1}{2}) = 22\frac{1}{4}$

$f(-1) = 4(-1¿^3 - 5(-1¿¿^2 - 8(-1) + 20 = 19$

$f(1) = 4(1¿^3 - 5(1¿^2 - 8(1) + 20 = 11$

Here, $11 < 19 < 22\frac{1}{4}$

Therefore, the absolute minimum value is 11 and the absolute maximum value is $22\frac{1}{4}$

Moreover ,f attain its absolute minimum at the point (1 , f(1)) = (1 , 11) and its absolute maximum at the point $(-\frac{-1}{2}, f\left(\frac{-1}{2}\right)) = (-\frac{1}{2}, \frac{89}{4})$.Hence

- (1 ,11) is the absolute minimum point of f and
  $(\frac{-1}{2}, \frac{89}{4}¿$ is the absolute maximum point of f

b) $g(x) = \frac{e^{2x}}{x^2-2}$ on $[0, 2]$ $\Rightarrow$ $g'(x) = \frac{2e^{2x}(x^2-2)-2xe^{2x}}{¿¿} = 0$ $\Rightarrow$ $\frac{2e^{2x}(x^2-x-2)}{¿¿} = 0$

$\Rightarrow 2e^{2x}(x^2-x-2) = 0$ $\Rightarrow$ $2e^{2x}=0$ *or* $x^2-x-2=0$ *and* $g'(\pm\sqrt{2}¿$ ¿∄

But , $2e^{2x}>0$, since $2e^{2x}\neq 0$ $\forall x \in \mathbb{R}$. & for $x^2-x-2=0$, x = -1 or x = 2

And since $2 \notin (0, 2)$, $-1 \notin (0, 2)$, $-\sqrt{2} \notin (0, 2)$ and only $\sqrt{2} \in (0, 2)$

Therefore,x $=\sqrt{2}$ is the only critical number of g on $(0, 2)$. Hence, the extreme values of **g** occur at the end points

$\Rightarrow g(0) = \frac{e^{2(0)}}{0^2-2} = \frac{-1}{2}$ and $g(2) = \frac{e^{2(2)}}{2^2-2} = \frac{e^4}{2}$, since $\frac{e^4}{2} > \frac{-1}{2}$

Therefore, $\frac{\frac{-1}{2} \wedge e^4}{2}$ are the minimum and the maximum values of g on [0, 2],

respectively.

c) h(x) = 2 + $\sqrt[3]{x-1}$ on [0 , 3] $\Rightarrow$ h'(x)= $\frac{1}{3}(x-1¿¿^{\frac{-2}{3}}$ = 0 $\Rightarrow$ $\frac{1}{3\sqrt[3]{¿¿¿}}$ = 0 .but h'(1)=∄

Therefore ,x = 1 is a critical number of h. from numerator we have

1 = 0, which is impossible for all real number *x*.

Here, h is continuous on $1 \in [0, 3]$, but h(1) = ∄. Thus, 1 is the critical number of h

h(0) = 2 + $\sqrt[3]{-1}$ = 1., h(1) = 2 + $\sqrt[3]{1-1}$ = 2 and h(3) = 2 + $\sqrt[3]{3-1} = 2 + \sqrt[3]{2}$

Therefore, 1 and 2 + $\sqrt[3]{2}$ are the minimum and the maximum values of h on [0, 3], respectively.

Note: The continuity of the function f on [a, b] is the sufficient condition for the existence of extreme value on [a, b]. That is if f is not continuous on [a, b], then f may not have extreme value on [a, b].

Exercise 4.1.

1.Find the set of critical points of each of the given functions.

a) f(x) = $3x^4 - 4x^3 - 12x^2 + 5$ c) k(x) = $x - 3\sqrt[3]{x}$

b) g(x) = $\frac{x^2}{x^2-9}$ d) (x) = $\sin x + \sin^2 x$, on $(-\pi, \pi)$

2. Find the absolute extreme values of each of the given functions on the given intervals.

a) f(x) = $25 - 5x^4 - 10x^3$, on $[-2, 1]$ c) f(x) = $x^2 e^{-x}$, on $[-1, 3]$

b) f(x) = $|x^2 - x|$, on $[0, 2]$ d) f(x) = $2 + \sqrt{x-1}$ on $[2, 5]$

Increasing and Decreasing (Monotonic) Functions

Note: i) f is strictly increasing. ⇨f is increasing

ii) f is strictly decreasing. ⇨ f is decreasing, but the converses are not true.

Theorem 2. (Increasing and Decreasing Test)

Suppose that f is continuous on an interval I and differentiable in the interior of

i. If f '(x) = 0 for all x in the interior of I, then f is increasing on I.

ii. If f '(x) = 0 for all x in the interior of I, then f is decreasing on I.

iii. If f '(x) > 0 and f '(x) = 0 only for finite number of points on I, then f is strictly increasing on I.

iv. If f '(x) < 0 and f '(x) = 0 only for finite number of points on I, then f is strictly decreasing on I.

Note: To determine the interval on which a function is increasing or decreasing, it will be easier to use

sign chart for its derivative f ' based on the critical point of

Examples 5. Find the interval(s) on which each of the following functions is increasing and on which it is decreasing.

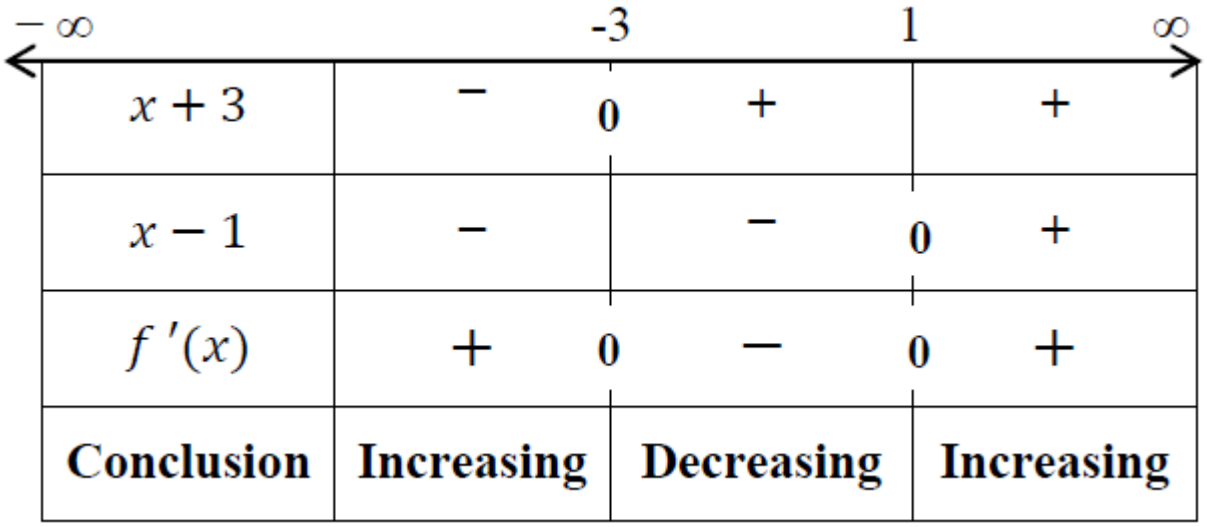
a) f(x) = $x^3 + 3x^2 - 9x + 7$ b) g(x) = $2 + 3^{(1-2x)}$ c) h(x) = $x\sqrt{x^2+1}$

solution :You are going to find intervals in which $f'(x)$ is positive or negative. Use sign charts for this purpose,

as follows:

a) f '(x) = 3(x+3)(x-1) =0 ⇨ x = -3 & x = 1 are critical numbers of f

using the sign chart we can determine the intervals of concavity as follows



Therefore, f is strictly increasing on the intervals $(-\infty , -3]$ and $[1 , \infty)$ .But f is not increasing$(-\infty , -3] \cup [1 , \infty)$,and f is decreasing [-3,1].

Note: f is monotonic on intervals I_1 and I_2 has no guarantee for f to be monotonic on $I_1 \cup I_2$

a) $f(x) = 2 + 3^{(1-2x)} \Rightarrow f'(x) = (-2)(\ln 3)(3^{(1-2x)}) \neq 0 , \forall x \in \mathbb{R}$. WHY?

Now, since $2 < 0$, $\ln 3 > 0$ and $3^{(1-2x)}) > 0$, $\forall x \in \mathbb{R}$,

we have $f'(x) = (-2)(\ln 3)(3^{(1-2x)}) < 0, \forall x \in \mathbb{R}$.

Therefore, ***f*** (strictly) decreasing function on $\mathbb{R}$

c) $f(x) = x\sqrt{x^2+1} \Rightarrow f'(x) = \sqrt{x^2+1} + \frac{x(2x)}{2\sqrt{x^2+1}} = \frac{2(x^2+1)+2(x^2)}{2\sqrt{x^2+1}} = \frac{2x^2+!}{2\sqrt{x^2+1}} = 0$

$\Rightarrow 2x^2+1=0$. But $2x^2+1 > 0$,for all real number x.

Therefore,f strictly increasing function on its entire domain $\mathbb{R}$.

Exercise 4.2. Find intervals in which f is strictly increasing or strictly decreasing.

a. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

b. $f(x) = 2\sin x+\cos 2x$,on $[0 , \pi]$

c. f(x) = $\sqrt{16-x^2}$ d. f(x) = $\frac{x^2+1}{x-1}$

e. f(x) = xe^{x-4} f. f(x)= x − 2|x − 3|

Local (or Relative) Extreme Values of a Function on its Entire Domain

Recall that the local extreme values of a function f correspond to the peak (local maximum) and valley

(local minimum) on the graph of f. (See definition 4.2 and figure 4.2 on your text book : page 166)

Theorem 3. (First Derivative Test for Local Extreme Values of a Function)

Suppose that f be a continuous on some interval I and let x = c ∈ I be the critical number of f. Then

i. If f' changes its sign from positive to negative at c, then f has local maximum value at c.

(Here, point (c , f (c)) is the local max. point and y = f (c) is the local maximum value of f).

ii. If f' changes its sign from negative to positive at c, then f has local minimum value at c.

(Here, point (c , f (c)) is the local min. point and y = f (c) is the local minimum value of f).

iii. If f' doesn't change its sign at c, then has no any local extreme value at c.

We can relate monotonicity and local extreme value as follow:

Let f be decreasing on [a , c] and increasing on [c, b]. Then f has local minimum at x = c.

Let f be increasing on [a , c] and decreasing on [c , b]. Then f has local maximum at x = c.

Examples 6. Find the local maximum and minimum values of each of the following functions.

a) f(x) = $x^3 + 3x^2 + 3x + 4$ c) f(x) = $\begin{cases} x^2-4x+3, x\geq 1 \\ x^3-3x^2-9x+11, x<1 \end{cases}$

b) f(x) =(x^2-4 ¿¿$^{\frac{2}{3}}$

Solution

a) f'(x)= $3x^2+6x+3=0$ iff $3x(x+1)+3(x+1)=0$ iff $x = -1$

Therefore,$x=-1$ is the critical numbers of f.

	- 1	
$x+1$	− 0	+
$f'(x) = 3(x+1)^2$	+	+
Conclusion	**Increasing** 0	**Increasing**

From the sign chart we can see that, there is no change of sign of f' throughout the entire domain.

That is **f is strictly increasing** function.

Therefore, by First Derivative Test, f has no any local extreme val

.

b) $f(x) = (x^2-4)^{\frac{2}{3}} = \sqrt[3]{(x^2-4)^2}$. Notice that f is continuous on the entire domain of $\mathbb{R}$.

$\Rightarrow \quad f'(x) = \frac{2}{3}(x^2-4)^{-\frac{1}{3}} = \frac{2(2x)}{3\sqrt[3]{x^2-4}} = \frac{4x}{3\sqrt[3]{x^2-4}} = \frac{4x}{3\sqrt[3]{(x-2)(x+2)}}$

$\Rightarrow \quad f'(x) = 0$ when $x = 0 \in \mathbb{R}$, and $f'(x)$ doesn't exist when $x = -2$ or $2 \in \mathbb{R}$.

$\Rightarrow \quad -2, 0$ *and* 2 are the critical numbers of f.

		-2		0		2	
$4x$	$-$		$-$	0	$+$		$+$
$3\sqrt[3]{x^2-4}$	$+$	0	$-$		$-$	0	$+$
$f'(x)$	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$+$

By First Derivative Test,

- f has **relative maximum** at $x = 0$, and hence
  $f(0) = (0^2-4)^{\frac{2}{3}} = 2\sqrt[3]{2}$ is the **relative maximum value** of f.
- f has **relative minimum** at $x = -2$ and $x = 2$, and hence
  $f(-2) = f(2) = ((\pm 2)^2-4)^{\frac{2}{3}} = 0$ is the relative minimum value of f.
  In this example, we can see that f attains its minimum value twice at $x = -2$ and $x = 2$.

c) $f(x) = \begin{cases} x^2-4x+3 \; for \; x \geq 1 \\ x^3-3x^2-9x+11 \; for \; x < 1 \end{cases} \Rightarrow \quad f'(x) = \begin{cases} 2x-4 \quad for \quad x > 1 \\ \nexists \quad for \quad x = 1 \\ 3x^2-6x-9 \; for \; x < 1 \end{cases}$

Here we can observe that f is continuous at $x = 1$, but f is not differentiable at $x = 1$.
On $(1, \infty)$, $f'(x) = 2x-4 = 2(x-2) = 0$ iff $x = 2 \in (1, \infty)$.
On $(-\infty, 1)$, $f'(x) = 3x^2-6x-9 = 3(x-3)(x+1) = 0$ iff $x = 3$ *or* $x = -1$.
Here, $x = -1 \in (-\infty, 1)$ but $x = 3 \notin (-\infty, 1)$.

- -1, 1 and 2 are the critical points of f.

		-1		1		2	
$2x-4$ $= 2(x-2)$				$-$		0	$+$
$x-3$	$-$		$-$				
$x+1$	$-$	0	$+$				
$3x^2-6x-9$ $= (x-3)(x+1)$	$+$		$-$				
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$

Examples 7. Using Second Derivative Test, find the local maximum and minimum values of each of the following functions.

a) $f(x) = xe^x + 1$ b) $f(x) = \frac{9}{10}x^{\frac{5}{3}}$

Solution

a) $f(x) = xe^x + 1 \Rightarrow f'(x) = e^x + xe^x = e^x(1+x) = 0 \quad \Rightarrow \quad x = -1$

$\Rightarrow \quad f''(x) = 2e^x + xe^x = (2+x)e^x$

- $f''(-1) = (2-1)e^{-1} = \frac{1}{e} > 0.$

Therefore, by Second Derivative Test, f has local minimum at $x = -1$, and $f(-1) = (-1)e^{-1} + 1 = 1 - \frac{1}{e}$ is the local minimum value of f .

b) $f(x) = \frac{9}{10}x^{\frac{5}{3}} \quad \Rightarrow f'(x) = \left(\frac{9}{10}\right)\left(\frac{5}{3}\right)x^{\frac{2}{3}} = -\frac{3}{2}x^{\frac{2}{3}} = 0 \quad \Rightarrow \quad x = 0$

$\Rightarrow \quad f''(x) = -\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{-\frac{1}{3}} = -\frac{1}{\sqrt[3]{x}}$. But $f''(0) = \nexists$

Therefore, the Second Derivative Test fails to apply.

Concavity and inflection points

Definition 4.3

Let f be a function differentiable at c and let L be the line tangent to the graph of f at. $(c\,,\ f(c))$

i. The graph of f is said to be **CONCAVE UPWARD** at $(c\,,\ f(c))$ if there is an open interval I about c such that $\forall x \in I$ and $x \neq c$, the point $(x\,,\ f(x))$ lies above L.

ii. The graph of f is said to be **CONCAVE DOWNWARD** at $(c\,,\ f(c))$ if there is an open interval I about c such that $\forall x \in I$ and $x \neq c$, the point $(x\,,\ f(x))$ lies below L.

Theorem 4. (Concavity Test)

Let f be a function twice differentiable on an interval I.

i. If $f''(x) > 0$ for all x in I, then the graph of f is **concave upward** on I.

ii. If $f''(x) < 0$ for all x in I, then the graph of f is **concave downward** on I.

Examples 8. Discus the behaviour of the curves of the following functions w.r.t local maximum and local minimum, concavity and points of inflection.

a) $f(x) = 3x^4 - 4x^3$ b) $f(x) = \frac{x^2+1}{x}$

		0		1	
$12x^2$	+	0	+	|	+
$x-1$	−	|	−	0	+
$f'(x)$	−	0	−	0	+

Solution

a) **Local extreme values :** $f'(x) = 12x^3 - 12x^2 = 0$

$\Rightarrow f'(x) = 12x^2(x-1) = 0 \Rightarrow x = 0 \text{ or } x = 1$

$\Rightarrow 0 \text{ and } 1$ are the critical point of f.

Then sketch the sign chart for f'.

Hence, from the sign chart, by 1st Derivative Test, $f(1) = 3(1)^4 - 4(1)^3 = -1$ is the local minimum value of f.

But f has no local maximum value.

Concavity and points of inflection : $f''(x) = (12x^3 - 12x^2)' = 36x^2 - 24x = 12x(3x-2) = 0$

$\Rightarrow 0 \text{ and } \frac{2}{3}$ are the critical point of f'.

Then sketch the sign chart for f''.

		0		$\frac{2}{3}$	
$12x$	−	0	+	|	+
$3x-2$	−	|	−	0	+
$f'(x)$	+	0	−	0	+

Hence, from the sign chart, by 2nd Derivative Test,

The graph of f is **concave upward** on $(-\infty, 0)$ and on $\left(\frac{2}{3}, \infty\right)$; and f is **concave downward** on $(0, \frac{2}{3})$.

Concavity of f changes at $x = 0$ and $x = \frac{2}{3}$.

Therefore, the inflection points of f are $(0, f(0)) = (0, 0)$ and $\left(\frac{2}{3}, f\left(\frac{2}{3}\right)\right) = (\frac{2}{3}, -\frac{16}{27})$

b) $f'(x) = \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = -1 \text{ or } x = 1 \Rightarrow$ The critical numbers are -1 & 1.

$f''(x) = \frac{2x(x^2) - 2x(x^2-1)}{(x^2)^2} = \frac{2}{x^3} \neq 0$ for all real number $x \neq 0$. $\Rightarrow f'$ has no critical number.

Then, sketch the sign charts for f' and f''.

		-1		0		1	
x^2-1	+	0	−	|	−	0	+
x^2	+	|	+	0	+	|	+
$f'(x)$	+	0	−	∄	−	0	+

		0	
2	+	|	+
x^3	−	0	+
$f''(x)$	−	∄	+

- ✓ Hence, from the first sign chart, by 1st Derivative Test, f has local maximum value at the point $(-1,\ f(-1)) = (-1, -2)$ and local minimum value at the point $(1,\ f(1)) = (1,\ 2)$
- ✓ The graph of f is **concave upward** on the intervals $(-\infty,\ 0)$ and on $\left(\frac{2}{3},\ \infty\right)$; and f is **concave downward** on the interval on $(0,\ \frac{2}{3})$.
- ✓ Concavity of f changes at $= 0$, but $0 \notin$ domain of f. Therefore, the graph of f has no inflection point.

Exercise 4.4. Find the local extreme values, the intervals on which the graphs concave upward and concave downward, and the inflection point of each of the following functions.

1. $f(x) = x^4 - 6x^2 + 8x + 10$
2. $f(x) = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$
3. $f(x) = \frac{x}{x^2 + 1}$
4. $f(x) = 2x - \tan x,\ on\ \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$

Comprehensive Curve Sketching

Now, you are ready to develop a procedure for curve sketching. To sketch the graph of a given function, say f, you should follow the following procedure.

Step 1. Determine the domain of the function f.
Step 2. Check whether the function is odd or even. This will help to understand the shape of the graph.
Step 3. Determine the x- and y-intercepts of f.
Step 4. Determine the asymptotes of f, if any.
Step 5. Determine the intervals of monotonicity and the local extreme values f.
Step 6. Determine the intervals of concavity and the inflection point of f.
Step 7. Lastly, using the information obtained in the above six steps sketch the graph of f.

Examples 9: Discuss the behaviors and sketch the graph of the function $f(x) = x^4 - 2x^2$. of the following functions.

Solutions $f(x) = x^4 - 2x^2$

- Domain = $\mathbb{R}$
- $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$. Hence, f is even function.
  $\Rightarrow$ The graph of f is symmetric w.r.t the y – axis.
- $f(x) = x^4 - 2x^2 = x^2(x^2 - 2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$
  $\Rightarrow -\sqrt{2},\ 0$ and $\sqrt{2}$ are the x-intercepts of f.
  $y = f(0) = 0$ is the y – intercept of f.
- Since f is polynomial, its graph has no any asymptote.
- $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0 \Rightarrow x = -1,\ 0\ or\ 1$
  $\Rightarrow x = -1,\ 0$ and1 are the critical numbers of f.
- $f''(x) = 12x^2 - 4 = 4(3x^2 - 1) = 4(\sqrt{3}x - 1)(\sqrt{3}x + 1) = 0 \Rightarrow x = -\frac{\sqrt{3}}{3}\ or\ \frac{\sqrt{3}}{3}$
  $\Rightarrow -\frac{\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3}$ are the critical numbers of f.

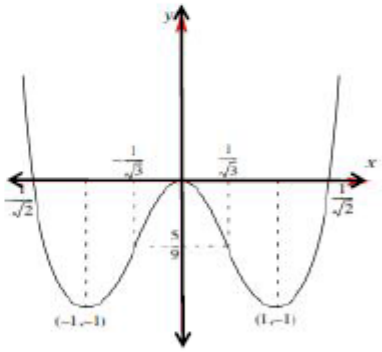
Sign chart for $f'(x)$

		-1		0		1	
$4x$	$-$		$-$	0	$+$		$+$
x^2-1	$+$	0	$-$		$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$

Sign chart for $f''(x)$

		$-\frac{\sqrt{3}}{3}$		$\frac{\sqrt{3}}{3}$	
4	$+$	$+$		$+$	$+$
$\sqrt{3}x-1$	$-$		$-$	0	$+$
$\sqrt{3}x+1$	$-$	0	$+$		$+$
$f''(x)$	$+$	0	$-$	0	$+$

- Hence, from the **first sign chart**, f is **monotonically decreasing** on the intervals $(-\infty, -1)$ and on $(0, 1)$. f is **monotonically increasing** on the intervals $(-1, 0)$ and on $(1, \infty)$.
- Again, from the **first sign chart**, by **1st Derivative Test**, f has local maximum value at the point $(-1, f(-1)) = (-1, -1)$ and local minimum value at the point $(1, f(1)) = (1, -1)$
- From the **second sign chart**, the graph of f is **concave upward** on the intervals $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and on $\left(\frac{\sqrt{3}}{3}, \infty\right)$; and f is **concave downward** on the interval on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.
- This implies that $\left(-\frac{\sqrt{3}}{3}, f\left(-\frac{\sqrt{3}}{3}\right)\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, f\left(\frac{\sqrt{3}}{3}\right)\right) = \left(\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ are the inflection points of the graphs of f.
- Therefore, the graph of f can be sketched as follow.



Exercise 4.5: Discuss the behaviors and sketch the graph of the following function.

a) $f(x) = x^3 - 12x$

b) $f(x) = 2 + \frac{x}{x^2-1}$

c) $f(x) = \frac{1}{2^x - 1}$

d) $f(x) = \begin{cases} x^2 - 6x + 8 & if \ 1 \le x < 4 \\ x + 2 & if \ -5 \le x < 1 \end{cases}$

e) $f(x) = e^{-x}\cos x \ \ on \ \left[0, \frac{5\pi}{2}\right]$

4.2 MINIMIZATION AND MAXIMIZATION PROBLEMS (OPTIMIZATION PROBLEMS)

The techniques and theories that have been developed in previous sections and chapter have important application to real life problems in which the maximum or minimum value of quantity is required. Problems that require the use of this theory can be found in many real life situations such as manufacturers wanting to minimize their costs, designers wanting to maximize the available space to be used A businessperson wants to minimize costs and maximize profits, etc. Such types of problem sometimes known as **OPTIMIZATION (MINIMIZATION AND MAXIMIZATION) PROBLEMS**

Guideline to solve optimization problems

Step 1. Understand the problem

Step 2. Introduce notation : Assign a symbol to the quantity that is to be maximized or minimized and select symbols for the unknowns.

Step 3. Draw a diagram (if necessary), and label the given and the required quantities on the diagram.

Step 4. Write a mathematical equation that describes the problem. Determine the domain.

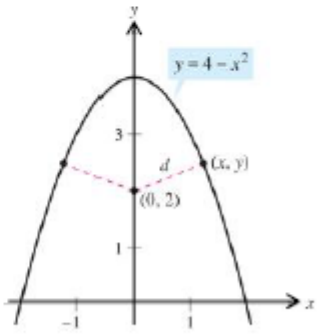
Step 5. Express the quantity which is going to be optimized in terms of the unknowns.

Step 6. Use the methods of solving maximization and minimization problems to get the quantity optimized.

Note: The maximum or minimum value does not always occur when the first derivative is zero. It is essential to also examine the values of the function at the endpoint(s) of the domain for global maxima and minima.

Examples 10. Find a point on the graph of $f(x) = 4 - x^2$ which is closest to the point (0, 2).

Solution: The graph of $f(x) = 4 - x^2$ shows that there are two points on the graph which have equal distance from (0, 2).



To optimize distance

Let (x, y) be a point on the graph of f which has least distance from the point (0, 2). Then by distance formula

$$d = \sqrt{(x-0)^2 + (y-2)^2} = d(x, y)$$

$\Rightarrow d(x, y) = \sqrt{x^2 + (y-2)^2}$ and domain of

Since $y = f(x) = 4 - x^2$, we have

$\Rightarrow d(x) = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$

The domain of $d(x) = \mathbb{R}$ and there is no end point on the domain. Because of d is smallest when $x^4 - 3x^2 + 4$ is smallest, you need only to find the critical numbers of $g(x) = x^4 - 3x^2 + 4$.

$\Rightarrow \quad g'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0 \Rightarrow x = 0, -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}} \in \mathbb{R}$ are critical numbers of f.

$\Rightarrow \quad g(0) = (0)^4 - 3(0)^2 + 4 = 4$, and

$$g\left(-\sqrt{3/2}\right) = \left(-\sqrt{3/2}\right)^4 - 3\left(-\sqrt{3/2}\right)^2 + 4 = \frac{7}{4} = g\left(\sqrt{3/2}\right) < g(0)$$

The relative minimum g occurs when $x = -\sqrt{3/2}$ *or* $x = \sqrt{3/2}$

Therefore, points $\left(-\sqrt{3/2}\ ,\ f\left(-\sqrt{3/2}\right)\right) = \left(-\sqrt{3/2}\ ,\ 5/2\right)$ and

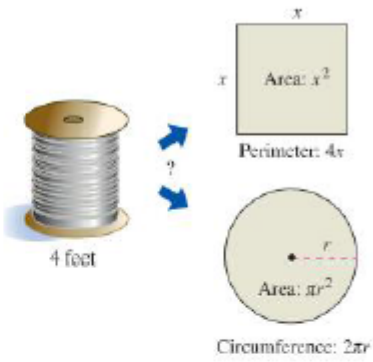
$\left(\sqrt{3/2}\ ,\ f\left(\sqrt{3/2}\right)\right) = \left(\sqrt{3/2}\ ,\ 5/2\right)$ are the closest points on the graph of f fro (0, 2).

Example 11. Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution Let L = length of the wire = 4 ft.

r = radius of the circle and

x = the length of a side of the square.



To optimize total area A

The total area A is given by

$A = a(square) + a(circle)$

$A = x^2 + \pi r^2 = A(x, r)$

This is the primary equation.

$L = P(square) + C(circle)$

$4 = 4x + 2\pi r$

Solving for r, we obtain

$r = 2(1 - x)/\pi$

Substitute this in A gives

$A = x^2 + \pi[2(1 - x)/\pi]^2 = \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4] = A(x)$

Based on the perimeter of the square, the feasible domain is $0 \le x \le 1$.

Then $\frac{dA}{dx} = \frac{2(\pi + 4)x - 8}{\pi} = 0$ iff $x = \frac{4}{\pi+4} \in (0\ ,\ 1)$

Hence, $x = \frac{4}{\pi+4}$ is the critical point of A(x).

Since there us a closed interval domain, we use Extreme Value Theorem.

$A(0) = \frac{1}{\pi}[(\pi + 4)(0)^2 - 8(0) + 4] = \frac{4}{\pi}$ ft^2.

$A\left(\frac{4}{\pi+4}\right) = \frac{1}{\pi}\left[(\pi + 4)(\frac{4}{\pi+4})^2 - 8(\frac{4}{\pi+4}) + 4\right] = \frac{4}{\pi+4}$ ft^2.

$A(1) = \frac{1}{\pi}[(\pi + 4)(1)^2 - 8(1) + 4] = 1$ ft^2.

Hence, the maximum area occurs when $x = 0$, that is the whole wire is used to make the circle.

The maximum area so obtained is $A_{max} = A(0) = \frac{4}{\pi}$ ft^2

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Exercise 4.6:

1. The height 's' of an object in free fall on the moon is approximately given by $s = s_0 + u_0 t - \frac{27}{10}t^2$, where '$s$' is measured in feet and 't' in seconds.

   a) What is the acceleration due to "gravity" on the moon?

   b) If an object is thrown upward from the surface of the moon with an initial velocity of 54 feet per second, what is the maximum height it will reach, and when will it reach that height?

2. An open box is to be constructed with a square base and is required to have a volume of 48 cubic inches. The bottom of the box costs 3 cents per square inch, whereas the sides cost 2 cents per square inch. Find the dimensions that will minimize the cost of the box.
3. A farmer must fence in a rectangular field with one side along a stream; no fence is needed on that side. If the area must be 1800 square meters and the fencing cost Br. 2 per meter, what dimensions will minimize the cost?
4. A thin-walled cone-shaped cup is to hold 36π cu. in. of water when full. What dimensions will minimize the amount of material needed for the cup?
5. Find the dimensions of the closed cylindrical can that will have a capacity of k volume units and used the minimum amount of material.

4.3 RATE OF CHANGE

There are many real-life applications of rates of change such as velocity, acceleration, population growth rates, unemployment rates, production rates, etc.. You can investigate the rate of change of one variable with respect to any other related variable.

When determining the rate of change of one variable with respect to another, you must be careful to distinguish between average and instantaneous rates of change.

- The **average rate of change** of a function f over the interval $[a, b]$ is the slope of a secant line is determined by two points $(a, f(a))$ and $(b, f(b))$ which is given by

$$\textbf{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- The **instantaneous rate of change** of a function f at the point $x = c$ is the slope of a tangent line to the graph of f at the point $(c, f(c))$ which is given by

$$\textbf{instantaneous rate of change} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

Related Rates

- In this section, you will study problems involving variables that are changing with respect to time. If two or more such variables are related to each other, then their rates of change with respect to time are also related.
- If two or more quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other quantities must be changing.
- Let x and y be variables which are both changing with time t and let $y = f(x)$.

  $\Rightarrow$ The rate of change of y is $\frac{dy}{dt}$ and the rate of change of x is $\frac{dx}{dt}$.

  Then by Chain Rule, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ related as

  $$y = f(x) \Rightarrow y(t) = f(x(t))$$

  Differentiating both sides w.r.t x gives

  $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For instance, suppose variables x and y be related by $y = 2x^3$

$\Rightarrow \quad y(t) = 2(x(t))^3$

$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = 6x^2\frac{dx}{dt}$

Examples 11. The volume of a cube increases at a constant rate of 10 cm^3/sec. Find the rate of change in its total surface area at the instant when its sides are 20 cm. long.

Solution : Let x be the lengths of the sides of the cube. Then the total surface area A and the volume V of the cube are

$A = 6x^2$ and $V = x^3$

$\Rightarrow \quad \frac{dA}{dt} = 12x\frac{dx}{dt}$ and $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

At the instant when $x = 20\ cm.$, $\frac{dV}{dt} = 3(20)^2\frac{dx}{dt} = 1200\frac{dx}{dt}$

$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{1200}\frac{dV}{dt} = \frac{1}{1200}(10) = \frac{1}{120}cm/s$

Thus, $\frac{dA}{dt} = 12x\frac{dx}{dt} = 12\left(\frac{1}{120}\right)(20) = 2\ cm^2/s$

Therefore, the surface area is increasing at the rate of $2cm^2s^{-1}$

Examples 12. The radius r of a right circular cone is increasing at a rate of 2 cm / min. The height h of the cone is related to the radius by $h = 3r$. Find the rate of change of the volume when $r = 3$cm.

Solution : The volume V of the cone is

$V = \frac{1}{3}\pi r^2 h$ and it is given that $h = 3r$

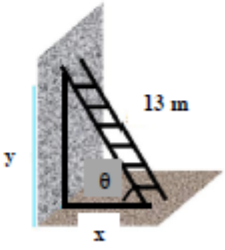
Hence, V can be written as

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 3r^2\frac{dr}{dt} = 3r^2 2cm/min \ = 6r^2 cm/min$

Then, at the instant r = 3 cm., the rate of change of the volume becomes

$\frac{dV}{dt} = 6(3cm)^2 cm/min \ = 54cm^3/min$

Examples 13. A ladder 13m long is leaning against a vertical wall. If the foot of the ladder slides away from the wall at a constant rate of 2 m / s, how fast is the angle formed between the ladder and the flat ground changing (rad. / s) at the instant when the top of the ladder is 12 m. above the ground?



Solution Let θ be the angle between the ladder and the flat ground.

Rate of change of x = $\frac{dx}{dt} = 2m/s$

To find $\frac{d\theta}{dt}$

From trigonometric identity $\cos\theta = \frac{x}{r} = \frac{x}{13}$

- $\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{x}{13}\right)$
- $(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$

We know that $\sin\theta = \frac{y}{r} = \frac{y}{13}$. At the instant y = 12 m., we have

$\sin\theta = \frac{y}{13} = \frac{12}{13}$

- $\frac{d\theta}{dt} = -\frac{1}{13}.\frac{13}{12}.2 = -\frac{1}{6}$ rad/s
- Therefore, the angle between the ladder and the flat ground decreases at the rate of $\frac{1}{6}$ rad/s.

Implicit Differentiation

Examples 12. If $x^3 + y^3 = 10$, then find $\frac{dy}{dx}$ and $\frac{dx}{dy}$, assuming that y is differentiable w.r.t x and x is differentiable w.r.t y

Solution $x^3 + y^3 = 10$

i) Differentiate both sides w.r.t x, and the solve for $\frac{dy}{dx}$

$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(10)$

$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 0 \Rightarrow 3x^2\frac{dx}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^2$

ii) Differentiate both sides w.r.t y, and the solve for $\frac{dx}{dy}$

$\frac{d}{dy}(x^3 + y^3) = \frac{d}{dy}(10)$

$\Rightarrow \frac{d}{dy}(x^3) + \frac{d}{dy}(y^3) = 0 \Rightarrow 3x^2\frac{dx}{dy} + 3y^2\frac{dy}{dy} = 0 \Rightarrow 3x^2\frac{dx}{dy} + 3y^2 = 0$

$\Rightarrow \frac{dx}{dy} = -\left(\frac{y}{x}\right)^2$

Exercise 4.7

1. Let $s = t^3 - 9t^2 + 24t$ describe the position s at time t of an object moving on a straight line.
   a) Find the velocity and acceleration.
   b) Determine when the velocity is positive and when it is negative.
   c) Determine when the acceleration is positive and when it is negative.
2. An open box is to be constructed with a square base and is required to have a volume of 48 cubic inches. The bottom of the box costs 3 cents per square inch, whereas the sides cost 2 cents per square inch. Find the dimensions that will minimize the cost of the box.
3. 4fts. of wire is to be used to form a square and a circle. How much of the wire should be used for the square and should be used for the circle to enclose the maximum total area?
4. The length of a rectangle having a constant area of 800 sq. mm is increasing at the rate of 4 mm/sec.
   a) What is the width of the rectangle at the moment when the width is decreasing at the rate of 0.5mm/sec?
   b) How fast is the diagonal of the rectangle changing when the width is 20 mm?
5. A cube of ice is melting. The side s of the cube is decreasing at the constant rate of 2 inches per minute. How fast is the volume V decreasing?

UNIT 5 *INTRODUCTION*

TO INTEGRAL

CALCULUS

Integral Calculus is the process of finding the function itself when its derivative is known.

Unit Outcomes

After completing this unit, you will be able to:

• *understand the concept of definite integrals.*

• *integrate different polynomial functions, simple trigonometric functions, exponential and*

logarithmic functions.

• *use the various techniques of integration to evaluate a given integral.*

• *use the fundamental theorem of calculus for computing definite integrals.*

• *apply the knowledge of integral calculus to solve real life mathematical problems*

5.1 INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

5.1.1 The Concept of Indefinite Integral

Definition 5.1 (Anti derivative)

The process of finding ( ) from its derivative $f'(x)$ is said to be anti-differentiation or integration.

$f(x)$ is said to be the anti-derivative of $f'(x)$.

Integration is the reverse operation of differentiation.

Definition 5.2

The set of all anti derivatives of a function $f(x)$ is called the *INDEFINITE INTEGRAL* of $f(x)$. The

indefinite integral of $f(x)$ is denoted by $\int f(x)\,dx$, read as "the integral of $f(x)$ with respect to x".

- The symbol ∫ is said to be the integral sign.
- The function $f(x)$ is said to be the integrand of the integral.
- dx denotes that the variable of integration is x.
- If a function has an integral, then it is said to be integrable.

If F '(x) = f(x), then ∫ f(x) dx = F(x) + c, where c is constant, and known as the constant of integration.

For instance, if f (x) = x , then a function whose derivative is f(x) is F(x) = $\frac{x^2}{2}$

f . Hence, F is an antiderivative of f(x) and we say that f(x) = $\frac{x^2}{2}$

F is the integral of f(x). Moreover, F'(x) = $\frac{x^2}{2}+c$

$F_2(x) = \frac{x^2}{2} - 3$, $F_3(x) = \frac{x^2}{2} + \sqrt{2}$, etc., are also the antiderivatives of $f(x)$. In general, for any constant c, $F(x) = \frac{x^2}{2} + c$ is the set of all antiderivatives of $f(x)$. This set of antiderivatives of f if denoted by

$$F(x) = \int x\,dx = \frac{x^2}{2} + c$$

which is called the **indefinite integral** of $f(x) = x$.

Example 1

a) $G(x) = \frac{x^3}{3}$ is an anti-derivative of $f(x) = x^2$, since $G'(x) = \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int x^2\,dx = \frac{x^3}{3} + c$$, for any constant c.

b) $G(x) = x^4$ is an anti-derivative of $f(x) = 4x^3$, since $G'(x) = \frac{d}{dx}(x^4) = 4x^3 = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int 4x^3\,dx = x^4 + c$$, for any constant c.

c) $G(x) = \sin x$ is an anti-derivative of $f(x) = \cos x$, since $G'(x) = (\sin x)' = \cos x$.
Thus, the indefinite integral of f is given by
$$F(x) = \int \cos x\,dx = \sin x + c$$, for any constant c.

d) $G(x) = e^x$ is an anti-derivative of $f(x) = e^x$, since $G'(x) = \frac{d}{dx}(e^x) = e^x = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int e^x\,dx = e^x + c$$, for any constant c.

e) $G(x) = \ln x$ is an anti-derivative of $f(x) = \frac{1}{x}$, since $G'(x) = (\ln x)' = \frac{1}{x} = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int \frac{1}{x}\,dx = \ln x + c$$, for any constant c.

Note : If $F(x)$ and $G(x)$ are anti-derivatives of f, then they differ only by a constant.
$F(x) = G(x) + c$, where c is an arbitrary constant.

Remark: From the definition of indefinite integral and anti-differentiation, we have the following:

i) $\int \frac{d}{dx} f(x)\,dx = \int f'(x)\,dx = f(x) + c$, where c is arbitrary constant.
For instance, $\int \frac{d}{dx}(\sin x)\,dx = \int \cos x\,dx = \sin x + c$, where c is arbitrary constant.
Also $\int \frac{d}{dx}(x^4)\,dx = x^4 + c$, where c is arbitrary constant.

ii) $\frac{d}{dx}\int f(x)\,dx = f(x)$
For instance, $\frac{d}{dx}\int \sin x\,dx = \frac{d}{dx}(-\cos x + c) = \frac{d}{dx}(-\cos x) + \frac{d}{dx}(c) = \sin x + 0 = \sin x$
Also $\frac{d}{dx}\int x^4\,dx = x^4$

$F_2(x) = \frac{x^2}{2} - 3$, $F_3(x) = \frac{x^2}{2} + \sqrt{2}$, etc., are also the antiderivatives of $f(x)$. In general, for any constant c, $F(x) = \frac{x^2}{2} + c$ is the set of all antiderivatives of $f(x)$. This set of antiderivatives of f if denoted by

$$F(x) = \int x\,dx = \frac{x^2}{2} + c$$

which is called the **indefinite integral** of $f(x) = x$.

Example 1

a) $G(x) = \frac{x^3}{3}$ is an anti-derivative of $f(x) = x^2$, since $G'(x) = \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 = f(x)$.
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b) $G(x) = x^4$ is an anti-derivative of $f(x) = 4x^3$, since $G'(x) = \frac{d}{dx}(x^4) = 4x^3 = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int 4x^3\,dx = x^4 + c$$, for any constant c.

c) $G(x) = \sin x$ is an anti-derivative of $f(x) = \cos x$, since $G'(x) = (\sin x)' = \cos x$.
Thus, the indefinite integral of f is given by
$$F(x) = \int \cos x\,dx = \sin x + c$$, for any constant c.

d) $G(x) = e^x$ is an anti-derivative of $f(x) = e^x$, since $G'(x) = \frac{d}{dx}(e^x) = e^x = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int e^x\,dx = e^x + c$$, for any constant c.

e) $G(x) = \ln x$ is an anti-derivative of $f(x) = \frac{1}{x}$, since $G'(x) = (\ln x)' = \frac{1}{x} = f(x)$.
Thus, the indefinite integral of f is given by
$$F(x) = \int \frac{1}{x}\,dx = \ln x + c$$, for any constant c.

Note : If $F(x)$ and $G(x)$ are anti-derivatives of f, then they differ only by a constant.
$F(x) = G(x) + c$, where c is an arbitrary constant.

Remark: From the definition of indefinite integral and anti-differentiation, we have the following:

i) $\int \frac{d}{dx} f(x)\,dx = \int f'(x)\,dx = f(x) + c$, where c is arbitrary constant.
For instance, $\int \frac{d}{dx}(\sin x)\,dx = \int \cos x\,dx = \sin x + c$, where c is arbitrary constant.
Also $\int \frac{d}{dx}(x^4)\,dx = x^4 + c$, where c is arbitrary constant.

ii) $\frac{d}{dx}\int f(x)\,dx = f(x)$
For instance, $\frac{d}{dx}\int \sin x\,dx = \frac{d}{dx}(-\cos x + c) = \frac{d}{dx}(-\cos x) + \frac{d}{dx}(c) = \sin x + 0 = \sin x$
Also $\frac{d}{dx}\int x^4\,dx = x^4$

Integration (Indefinite Integral) of some simple functions

For an arbitrary constant c,

a) $\int 0\,dx = c$

b) $\int dx = \int 1\,dx = x + c$

c) $\int k\,dx = kx + c$, for any constant k.

d) $\int x\,dx = \frac{x^2}{2} + c$

e) $\int x^2\,dx = \frac{x^3}{3} + c$

f) $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + c$

Power Rule

Thus, for any real number n, $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & if\ n \neq -1 \\ \ln|x| + c & if\ n = -1 \end{cases}$

g) $\int \sqrt{x}\,dx = \int (x)^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

h) $\int \sin x\,dx = -\cos x + c$

i) $\int \cos x\,dx = \sin x + c$

j) $\int sec^2 x\,dx = \tan x + c$

k) $\int csc^2 x\,dx = -\cot x + c$

l) $\int \sec x \tan x\,dx = \sec x + c$

m) $\int \csc x \cot x\,dx = -\csc x + c$

The above indefinite integrals are obtained directly from the differentiations of basic functions.

Examples 2. Evaluate the following indefinite integrals.

a) $\int x^5\,dx$

b) $\int x^{-0.3}\,dx$

c) $\int \sqrt[3]{x^5}\,dx$

d) $\int (x^{-7})^{-\frac{2}{3}}\,dx$

e) $\int \frac{x^2}{\sqrt[5]{x^3}}\,dx$

f) $\int \frac{\sqrt[5]{x^3}}{x^{\frac{8}{5}}}\,dx$

Solution : For an arbitrary constant c;

a) $\int x^5\,dx = \frac{x^6}{6} + c$

b) $\int x^{-0.3}\,dx = \frac{x^{-0.3+1}}{-0.3+1} + c = \frac{x^{0.7}}{0.7} = \frac{10}{7}x^{0.7}$

c) $\int \sqrt[3]{x^5}\,dx = \int x^{\frac{5}{3}}\,dx = \frac{3}{8}x^{\frac{8}{3}} + c$

d) $\int (x^{-7})^{-\frac{2}{3}}\,dx = \int x^{\frac{14}{3}}dx = \frac{3}{17}x^{\frac{17}{3}} + c$

e) $\int \frac{x^2}{\sqrt[5]{x^3}}\,dx = \int x^{\frac{7}{5}}\,dx = \frac{5}{12}x^{\frac{12}{5}} + c$

f) $\int \frac{\sqrt[5]{x^3}}{x^{\frac{8}{5}}}\,dx = \int x^{\frac{3}{5}-\frac{8}{5}}\,dx = \int x^{-1}dx = \ln|x| + c$

Integration (Indefinite Integral) of some simple functions

For an arbitrary constant c,

a) $\int 0\,dx = c$

b) $\int dx = \int 1\,dx = x + c$

c) $\int k\,dx = kx + c$, for any constant k.

d) $\int x\,dx = \frac{x^2}{2} + c$

e) $\int x^2\,dx = \frac{x^3}{3} + c$

f) $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + c$

Power Rule

Thus, for any real number n, $\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & if\ n \neq -1 \\ \ln|x| + c & if\ n = -1 \end{cases}$

g) $\int \sqrt{x}\,dx = \int (x)^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

h) $\int \sin x\,dx = -\cos x + c$

i) $\int \cos x\,dx = \sin x + c$

j) $\int sec^2 x\,dx = \tan x + c$

k) $\int csc^2 x\,dx = -\cot x + c$

l) $\int \sec x \tan x\,dx = \sec x + c$

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The above indefinite integrals are obtained directly from the differentiations of basic functions.

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Solution : For an arbitrary constant c;

a) $\int x^5\,dx = \frac{x^6}{6} + c$

b) $\int x^{-0.3}\,dx = \frac{x^{-0.3+1}}{-0.3+1} + c = \frac{x^{0.7}}{0.7} = \frac{10}{7}x^{0.7}$

c) $\int \sqrt[3]{x^5}\,dx = \int x^{\frac{5}{3}}\,dx = \frac{3}{8}x^{\frac{8}{3}} + c$

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f) $\int \frac{\sqrt[5]{x^3}}{x^{\frac{8}{5}}}\,dx = \int x^{\frac{3}{5}-\frac{8}{5}}\,dx = \int x^{-1}dx = \ln|x| + c$

5.1.2 Properties of the Indefinite Integral

Properties of the Indefinite Integral

For two integrable functions f and g,

1. $\int kf(x)dx = k\int f(x)dx$, for any constant k
2. $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
3. $\int[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

Remark : For a polynomial function of degree n given by

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots \ldots\ldots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$,

$$\int f(x)dx = \int[a_n x^n + a_{n-1}x^{n-1} + \cdots \ldots\ldots + a_2x^2 + a_1x + a_0]dx$$

$$= \int a_n x^n dx + \int a_{n-1}x^{n-1}dx + \ldots\ldots\ldots + \int a_1 x dx + \int a_0 dx$$

$$= a_n\frac{x^{n+1}}{n+1} + a_{n-1}\frac{x^n}{n} + \cdots \ldots\ldots + a_1\frac{x^2}{2} + a_0x$$

which is also polynomial function of degree $n+1$.

Examples 2. Evaluate the following indefinite integrals.

a) $\int[\pi x^3 - 5x^2 + x - 3]dx$

b) $\int(2t + 3sec^2 t)dt$

c) $\int\left(\frac{1}{2\sqrt{x}} - \frac{1}{sin x}\right)dx$

d) $\int\frac{(1-2y^2)^2}{\sqrt{y^3}}dy$

e) $\int\left(1 - 3z^2\sqrt[3]{z^2}\right)dz$

Solution

a) $\int[\pi x^3 - 5x^2 + x - 3]dx = \int \pi x^3 dx - \int 5x^2 dx + \int x\,dx - \int 3dx$

$= \pi\int x^3 dx - 5\int x^2 dx + \int x\,dx - 3\int dx = \pi\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - 3x + c$, c is a constant.

b) $\int(2t + 3sec^2 t)dt = 2\int t\,dt + 3\int sec^2 t\,dt = 2\left(\frac{x^2}{2}\right) + 3\tan t + c$, c is a constant.

c) $\int\left(\frac{1}{2\sqrt{x}} - \frac{1}{sin x}\right)dx = \int\left(\frac{1}{2}x^{-\frac{1}{2}} - csc^2 x\right)dx = \sqrt{x} + \cot x + c$, c is a constant.

d) $\int\frac{(1-2y^2)^2}{\sqrt{y^3}}dy = \int(1 - 4y^2 + 4y^4)y^{-\frac{3}{2}}\,dy = \int\left(y^{-\frac{3}{2}} - 4y^{\frac{1}{2}} + 4y^{\frac{5}{2}}\right)dy$

$= -2y^{-\frac{1}{2}} - 4.\frac{2}{3}y^{\frac{3}{2}} + 4.\frac{2}{7}y^{\frac{5}{2}} + c = -2y^{-\frac{1}{2}} - \frac{8}{3}y^{\frac{3}{2}} + \frac{8}{7}y^{\frac{5}{2}} + c,\ c \in \mathbb{R}$

e) $\int\left(1 - 3z^2\sqrt[3]{z^2}\right)dz = \int\left(1 - 3z^{\frac{8}{3}}\right)dz = z - 3.\frac{3}{11}z^{\frac{11}{3}} + c = z - \frac{9}{11}z^{\frac{11}{3}} + c$, $c \in \mathbb{R}$

Examples 3. For $a > 0$, $a \neq 1$ *and any re alnumbe r* x, show that

$$\int a^x dx = \frac{a^x}{\ln a} + c$$, where c is arbitrary constant.

Solution We know that $(a^x)' = a^x \ln a \Rightarrow a^x = \frac{1}{\ln a}(a^x)' = \left(\frac{a^x}{\ln a}\right)'$, since $\frac{1}{\ln a}$ is constant.

Then by definition of indefinite integral , we have

$$\int a^x dx = \frac{a^x}{\ln a} + c$$, where c is arbitrary constant.

Thus, the indefinite integral of expnonential function is given as follow:

The Indefinite Integral of Expnonential Function

For $a > 0$, $a \neq 1$ *and any real number* x, show that

$$\int a^x dx = \frac{a^x}{\ln a} + c, \quad \text{where c is arbitrary constant.}$$

- If $a = e$, then

$$\int e^x dx = \frac{e^x}{\ln e} + c = e^x + c, \quad \text{where c is arbitrary constant.}$$

Examples 4. Evaluate the following indefinite integrals.

a) $\int 2^x \, dx$ b) $\int \frac{3(2^x)-2}{\sqrt{5^x}} dx$

Solution

a) $\int 2^x \, dx = \frac{2^x}{\ln 2} + c$, where c is arbitrary constant.

b) $\int \frac{3(2^x)-2}{\sqrt{5^x}} dx = \int \frac{3(2^x)-2}{\sqrt{5}^x} dx = \int \left[3\left(\frac{2}{\sqrt{5}}\right)^x - 2\left(\frac{1}{\sqrt{5}}\right)^x\right] dx = 3\frac{\left(\frac{2}{\sqrt{5}}\right)^x}{\ln\left(\frac{2}{\sqrt{5}}\right)} - 2\frac{\left(\frac{1}{\sqrt{5}}\right)^x}{\ln\left(\frac{1}{\sqrt{5}}\right)} + c,$

$$= 3\left(\frac{2}{\sqrt{5}}\right)^x \left[\ln\left(\frac{2}{\sqrt{5}}\right)\right]^{-1} - 2\left(\frac{1}{\sqrt{5}}\right)^x \left[\ln\left(\frac{1}{\sqrt{5}}\right)\right]^{-1} + c, \text{ for arb. constant.}$$

Exercise 5.1 Evaluate the following indefinite integrals.

a) $\int 3^{-x} \, dx$ d) $\int \frac{2^x - 3^x}{6^x} dx$ g) $\int \frac{y^2 - 4y + 10}{y^2 \sqrt[3]{\sqrt{y}}} dy$

b) $\int (1 - s^2)^2 (3s + s^3) \, ds$ e) $\int (2e^{2x} - e^{-3x}) \, dx$ h) $\int \left(2t + \frac{1}{t}\right)^2 dt$

c) $\int (x^2 - 5^{2x}) \, dx$ f) $\int \frac{\cos x}{1 - \cos^2 x} dx$ i) $\int e^{7-2x} dx$

5.2 TECHNIQUES OF INTEGRATION

There are different techniques of integrations. The most commonly used methods are : **substitution, partial fractions,** and **integration by parts**.

5.2.1 Integration by Substitution

Integration by substitution is a counter part to the chain rule of differentiation. It is a method of finding integrals by changing variables. In integration by substitution, we may

- Use **pattern recognition** to evaluate indefinite integral.
- Use **change of variable** to evaluate indefinite integral.
- Use **general power rule of integration** to evaluate indefinite integral.

Rule of Integration by Substitution

Let f and g be functions such that fog and g' are continuous on an interval I. If F is an anti-derivative of f on I, then

$$\int f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + c, \text{ for an arbitrary c.}$$

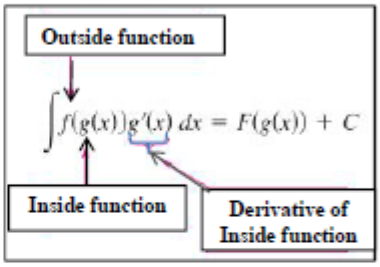
Change of variable

In the above rule, letting $u = g(x)$ gives $du = g'(x)dx$. Then the rule can be written as

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du = F(u) + c$$

$$\Rightarrow \quad \int f\big(g(x)\big)g'(x)dx = (g(x)) + c\,, \quad \text{for an arbitrary c}$$

- To apply Rule of Integration by Substitution given above, you must recognize the presence of $f(g(x))$, and $g'(x)$ in the integrand.
- The composite function $f(g(x))$ in the integrand has **the outside function f the inside function g.**



Examples 5. Evaluate the following indefinite integrals.

(pattern recognition and change of variable)

a) $\int 2x(x^2+1)\,dx$ c) $\int \sqrt[3]{(1-x)^4}\,dx$

b) $\int xe^{x^2}dx$ d) $\int sin^4x\ \cos x\,dx$

Solution

a) Let $u = x^2 + 1 \Rightarrow du = 2xdx$. Then by Substitution Rule

$\int 2x(x^2+1)\,dx = \int (x^2+1)\,2xdx = \int u\,du = \frac{u^2}{2} + c = \frac{(x^2+1)^2}{2} + c$

b) Let $u = x^2 \Rightarrow du = 2xdx \Rightarrow xdx = \frac{1}{2}du$. Then by Substitution Rule

$\int xe^{x^2}dx = \int e^{x^2}xdx = \int \frac{1}{2}e^u du = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$

c) Let $u = 1 - x \Rightarrow du = -dx \Rightarrow dx = -du$. Then by Substitution Rule

$\int \sqrt[3]{(1-x)^4}\,dx = \int -1.\sqrt[3]{u^4}\,du = -\int u^{\frac{4}{3}}du = \frac{3}{7}u^{\frac{7}{3}} + c = \frac{3}{7}(1-x)^{\frac{7}{3}} + c$

d) Let $u = \sin x \Rightarrow du = \cos x\,dx$. Then by Substitution Rule

$\int sin^4x\ \cos x\,dx = \int u^4 du = \frac{u^5}{5} + c = \frac{sin^5x}{5} + c$,

Note : If F is an antiderivative of f and $a \neq 0$ is constant, then by change of variable

$$\int f(ax)\,dx = \frac{1}{a}F(ax) + c$$

For instance,

i) $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

ii) $\int e^{\sqrt{3}x}\,dx = -\frac{1}{\sqrt{3}}e^{\sqrt{3}x} + c$

iii) $\int \cos(5x)\,dx = \frac{1}{5}\cos\,(5x) + c$

Other version of change of variable

Examples 6. Evaluate the following indefinite integrals.

a) $\int x\sqrt{x+1}\,dx$ b) $\int \frac{x^2}{2x-1}\,dx$

Solution From the nature of the integrands, the above rule of integration may not helpful to evaluate these integrals. The reason is that the integrands have no the form $f(g(x))g'(x)$. But still you can use change of variable to evaluate them.

a) Let $u = x+1 \Rightarrow du = dx$ and $x = u-1$. Then

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int (u\sqrt{u} - \sqrt{u}\,)du$$
$$= \int (u\sqrt{u} - \sqrt{u}\,)du = = \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$$
$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

b) Let $u = 2x-1 \Rightarrow du = 2dx$ and $x = \frac{u+1}{2} \Rightarrow x^2 = \left(\frac{u+1}{2}\right)^2 = \frac{u^2+2u+1}{4}$. Then

$$\int \frac{x^2}{2x-1}\,dx = \int \frac{u^2+2u+1}{4u}du = \int \left(\frac{u}{4} + \frac{1}{2} + \frac{1}{4}u^{-1}\right)du = \frac{u^2}{8} + \frac{u}{2} + \frac{\ln|u|}{4}$$
$$= \frac{(2x-1)^2}{8} + \frac{2x-1}{2} + \frac{\ln|2x-1|}{4} + c$$

Other version of Integration by Substitution

i) $\int \frac{f'(x)}{f(x)}\,dx$; letting $u = f(x) \Rightarrow du = f'(x)dx$. Then the given integral can be written as $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u}$, then evaluate the resulting integral.

ii) $\int f(x)f'(x)\,dx$; letting $u = f(x) \Rightarrow du = f'(x)dx$. Then the given integral can be written as $\int f(x)f'(x)\,dx = \int u\,du$, then evaluate the resulting integral.

Examples 7. Evaluate the following indefinite integrals.

a) $\int \frac{1}{x\ln x}\,dx$ b) $\int \sin x \cos x\,dx$

Solution

a) Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$. Then

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}du = \ln|u| = \ln|\ln x| + c$$

b) Let $u = \sin x \Rightarrow du = \cos x\,dx$. Then

$$\int sinx\,cosx\,dx = \int u\,du = \frac{u^2}{2} = \frac{1}{2}sin^2x + c$$

Indefinite Integral of $f(x) = tanx$ and $f(x) = cotx$

i) $\int tanx\,dx = \int \frac{sinx}{cosx}dx$; letting $u = cos\,x$ then $du = -\sin x\,dx$

$= \int -\frac{du}{u} = -ln|u| + c = -ln|\cos x| + c$

ii) $\int \cot x\,dx = \int \frac{\cos x}{sinx}dx$; letting $u = \sin x$. Then $du = \cos x\,dx$

$= \int \frac{du}{u} = ln|u| + c = ln|\sin x| + c$

The Indefinite Integral of $f(x) = tanx$ and $f(x) = cotx$

1) $\int tanx\,dx = -ln|cosx| + c$
2) $\int cotx\,dx = ln|sinx| + c$

For an arbitrary constant c.

Note that also we can use Integration by substitution to show :

The Indefinite Integral of $f(x) = secx$ and $f(x) = cscx$

1. $\int secx\,dx = ln|secx + tanx\,cosx| + c$
2. $\int cscx\,dx = -ln|cscx + cotx| + c$

For an arbitrary constant c.

Exercise 5.2 Evaluate the following indefinite integrals.

1. $\int 5x^2\cos(1-2x^3)\,dx$
2. $\int \sqrt{x}\sqrt{1+x\sqrt{x}}\,dx$
3. $\int \frac{4x-5}{\sqrt{2x^2-5x+1}}\,dx$
4. $\int 5^{(2-3x)}dx$
5. $\int x\sqrt{x+3}\,dx$
6. $\int tanx\,sec^2 x\,dx$
7. $\int \sin^2 x\,dx$
8. $\int \cos^2 x\,dx$

5.2.2 Integration by Partial Fractions

In this sub-unit, we shall use decomposition of a rational expression into partial fractions along with the method of substitution in order to evaluate the indefinite integrals of some rational expressions.

Example 8: Decompose each of the following rational expressions into partial fractions

a) $f(x) = \frac{1}{x^2 - x}$

b) $f(x) = \frac{x+2}{x^3 - 3x^2}$

Solution :

a) $x^2 - x = x(x-1)$

Let $\frac{1}{x^2-x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, where A and B are constants to be determined.

$\Rightarrow \quad \frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$

$\Rightarrow \quad (A+B)x - A = 1 \quad \Rightarrow \quad A + B = 0$ and $-A = 1 \quad \Rightarrow \quad A = -1$ and $B = 1$

$\Rightarrow \quad -\frac{1}{x}$ *and* $\frac{1}{x-1}$ are the partial fractions of f.

The partial fraction decomposition of $f(x)$ is

$$f(x) = \frac{1}{x^2-x} = -\frac{1}{x} + \frac{1}{x-1} = \frac{1}{x-1} - \frac{1}{x}$$

b) $f(x) = \frac{x+2}{x^3-3x^2} = \frac{x+2}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{(A+C)x^2+(-3A+B)x-3B}{x^2(x-3)}$

$\Rightarrow \quad (A+C)x^2 + (-3A+B)x - 3B = x + 2 \quad \Rightarrow \quad A + C = 0\,,\ -3A + B = 1\,, -3B = 2$

$\Rightarrow \quad A = -\frac{5}{9}\,,\ B = -\frac{2}{3}\,,\ C = \frac{5}{9}$

Hence, $f(x) = \frac{x+2}{x^3-3x^2} = -\frac{5}{9x} - \frac{2B}{3x^2} + \frac{5}{9(x-1)}$ is the partial decomposition of f.

Note : Indefinite integral of some rational functions need not partial decomposition.

Example 8: Devaluate the following indefinite integrals of rational functions.

a) $\int \frac{1}{3-2x}\,dx$ b) $\int \frac{2x-1}{x+1}\,dx$ c) $\int \frac{x^3}{x-2}\,dx$

Solution: The above indefinite integrals need not partial decomposition .

a) Let $u = 3 - 2x \Rightarrow du = -2dx$. Then

$\int \frac{1}{3-2x}dx = \int -\frac{1}{2u}du = -\frac{1}{2}\int \frac{1}{u}du = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|3-2x| + c$

b) Let $u = x + 1 \Rightarrow du = dx$ and $x = u - 1 \Rightarrow \frac{2x-1}{x+1} = \frac{2(u-1)-1}{u} = \frac{2u-3}{u} = 2 - \frac{3}{u}$. Then

$\int \frac{2x-1}{x+1}\,dx = \int \left(2 - \frac{3}{u}\right)du = 2u - 3\ln|u| = 2(x+1) - 3\ln|x+1| + c$

c) The integrand $\frac{x^3}{x-2}$ is improper rational function. Write $\frac{x^3}{x-2}$ in the form of

$\Rightarrow \quad \frac{x^3}{x-2} = (x^2 + 4x + 4) + \frac{8}{x-2}$

$\Rightarrow \int \frac{x^3}{x-2}dx = \int \left(x^2 + 4x + 4 + \frac{8}{x-2}\right)dx = \frac{x^3}{3} + 2x^2 + 4x + 8\ln|x-2| + c$

Example 9: Devaluate the following indefinite integrals of rational functions.

a) $\int \frac{x+1}{x^2-x}\,dx$ b) $\int \frac{x^3}{x^2-9}\,dx$

Solution : The above indefinite integrals need not partial decomposition .

a) $x^2 - x = x(x-1)$

Let $\frac{x+1}{x^2-x} = \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, where A and B are constants to be determined.

$\Rightarrow \quad \frac{x+1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$

$\Rightarrow \quad (A+B)x - A = x + 1 \Rightarrow \ A + B = 1$ and $-A = 1 \quad \Rightarrow \quad A = -1$ and $B = 2$

$$\int \frac{x+1}{x^2-x}dx = \int\left(\frac{-1}{x}+\frac{2}{x-1}\right)dx = -\int\frac{1}{x}dx + 2\int\frac{1}{x-1}dx = -\ln|x| + 2\ln|x-1| + c$$

b) $\frac{x^3}{x^2-9} = x + \frac{9x}{x^2-9}$. Let $\frac{9x}{x^2-9} = \frac{9x}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow (A+B)x + 3(A-B) = 9x$

$$\Rightarrow \quad A = B = \frac{9}{2} \quad \Rightarrow \quad \frac{x^3}{x^2-9} = x + \frac{9x}{(x-3)(x+3)} = x + \frac{9/2}{x-3} + \frac{9/2}{x+3}$$

$$\int\frac{x^3}{x^2-9}dx = \int\left(x + \frac{9/2}{x-3} + \frac{9/2}{x+3}\right)dx = \frac{x^2}{2} + \frac{9}{2}(\ln|x-3| + \ln|x+3|) + c = \frac{x^2}{2} + \frac{9}{2}(\ln|x^2-3|)$$

Exercise 5.3 Evaluate the following indefinite integrals.

1. $\int\frac{2x-1}{x+1}dx$
2. $\int\frac{x+2}{x^2(x-3)}dx$
3. $\int\frac{x^2+1}{x^2-1}dx$
4. $\int\frac{\sqrt{x}}{1+\sqrt{x}}dx$
5. $\int\frac{dx}{4x^2+4x+1}$
6. $\int\frac{1}{e^x-e^{-x}}dx$
7. $\int\frac{x^3-1}{x^3+x}dx$

5.2.3 Integration by Parts

Integration by parts is a method which is a counter part of the **product rule of differentiation**. This technique can be applied to a wide variety of functions and is particularly useful for integrands involving products of **algebraic** and **transcendental functions**. For instance, Integration by parts well with integrals such as

$$\int x\ln x\,dx\,, \quad \int x^2e^x dx\,, \quad \int e^x\sin x\,dx\,, \quad \text{etc}$$

Rule of Integration by Parts

Let f and g be functions that have continuous derivatives. Then

$$\int \boldsymbol{f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}$$

For instance, consider the indefinite integral $\int x\sin x\,dx$.

Let $f(x) = x$ and $g'(x) = \sin x\,dx$

$\Rightarrow$ $f'(x) = dx$ and $g(x) = \int \sin x\,dx = -\cos x$

Then by the rule of integration by parts, we have

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\Rightarrow \quad \int x\sin x\,dx = x(-\cos x) - \int(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c,$$

Note : For computational purpose, integration by parts can be written in more convenient and simpler form as follow:

Let $u = f(x)$ and $dv = g'(x)dx$

$\Rightarrow$ $du = f'(x)dx$ and $v = g(x)$

Then rule of integration by parts can be written as

$$\int \boldsymbol{u\,dv = uv - \int v\,du}$$

Remark: In integration by part, we choose u for a function which is easily differentiable and dv for a function which is easily integrable.

Example 10: Evaluate the following indefinite integrals.

a) $\int xe^{2-3x}\,dx$ b) $\int e^{3x}\cos 2x\,dx$ c) $\int 1\text{n}x\,dx$

Solution : The above indefinite integrals need not partial decomposition .

a) $\int xe^{2-3x}\,dx$; Let $u = x$ and $dv = e^{2-3x}\,dx$

$\Rightarrow du = dx$ and $v = \int e^{2-3x}\,dx = e^2\int e^{-3x}\,dx = e^2\left(-\frac{1}{3}\right)e^{-3x} = -\frac{1}{3}e^{2-3x}$

$\Rightarrow \int xe^{2-3x}\,dx = uv - \int v\,du = -\frac{1}{3}xe^{2-3x} - \int -\frac{1}{3}e^{2-3x}\,dx = \frac{1}{3}xe^{2-3x} + \frac{1}{3}\int e^{2-3x}\,dx$

$= -\frac{1}{3}xe^{2-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{2-3x}\right) = -\frac{1}{3}xe^{2-3x} - \frac{1}{9}e^{2-3x} + c$

b) $\int e^{3x}\cos 2x\,dx$; Let $u = e^{3x}$ and $dv = \cos 2x\,dx$

$\Rightarrow \quad du = 3e^{3x}dx$ and $v = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$

$\Rightarrow \int e^{3x}\cos 2x\,dx = uv - \int v\,du = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\,dx$ --------- (*)

Repeat integration by part for $\int e^{3x}\sin 2x\,dx$:

Let $u = e^{3x}$ and $dv = \sin 2x\,dx$

$\Rightarrow du = 3e^{3x}dx$ and $v = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$\Rightarrow \int e^{3x}\sin 2x\,dx = uv - \int v\,du = -\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x\,dx$ --------- (**)

Now substitute (**) in (*) gives

$\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\,dx$

$= \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left[-\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x\,dx\right]$

$= \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \frac{9}{4}\int e^{3x}\cos 2x\,dx$

Then, collect like terms gives

$\int e^{3x}\cos 2x\,dx + \frac{9}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$

$\Rightarrow \quad \frac{13}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$

Therefore, $\int e^{3x}\cos 2x\,dx = \frac{4}{13}\left(\frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x\right) + c$

$= \frac{2}{13}e^{3x}\left(\sin 2x + \frac{3}{2}\cos 2x\right) + c$

c) $\int 1\text{n}x\,dx = \int 1.1\text{n}x\,dx$

Let $u = 1\text{n}x$ and $dv = 1dx$

$\Rightarrow du = \frac{1}{x}\,dx$ and $v = \int 1\,dx = x$

$\Rightarrow \int 1\text{n}x\,dx = uv - \int v\,du = x\,ln\,x - \int x.\frac{1}{x}\,dx = x\,ln\,x - \int dx = x\,ln\,x - x + c$

Therefore,

$$\int 1\text{n}x\,dx = x\,l\,nx - x + c$$

Indefinite Integral of Logarithm Functions

1. $\int \ln x\,dx = x\ln x - x + c$
2. For $a > 0$ and $a \neq 1$,

$$\int \log_a x\,dx = \int \frac{\ln x}{\ln a}dx = \frac{1}{\ln a}\int \ln x\,dx = \frac{1}{\ln a}(x\ln x - x) = x\log_a x - \frac{x}{\ln a} + c$$

Example 11: Evaluate the following indefinite integrals.

a) $\int \log(1-3x)\,dx$ c) $\int \ln^2 x\,dx$

b) $\int (2x+1)\ln 5x\,dx$ d) $\int x^{-3}\ln x\,dx$

Solution : The above indefinite integrals need not partial decomposition .

a) Let $u = 1-3x \Rightarrow du = -3dx$

$$\int \log(1-3x)\,dx = -\frac{1}{3}\int \log u\,du = -\frac{1}{3}\int \log u\,du = -\frac{u}{3\ln 10}(\ln u - 1)$$
$$= -\frac{1-3x}{3\ln 10}[\ln(1-3x) - 1] + c$$

b) Let $u = 2-3x$ and $dv = e^{2-3x}\,dx$

$$\Rightarrow du = dx \text{ and } v = \int e^{2-3x}\,dx = e^2\int e^{-3x}\,dx = e^2\left(-\frac{1}{3}\right)e^{-3x} = -\frac{1}{3}e^{2-3x}$$
$$\Rightarrow \int xe^{2-3x}\,dx = uv - \int v\,du = -\frac{1}{3}xe^{2-3x} - \int -\frac{1}{3}e^{2-3x}\,dx = \frac{1}{3}xe^{2-3x} + \frac{1}{3}\int e^{2-3x}\,dx$$
$$= -\frac{1}{3}xe^{2-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{2-3x}\right) = -\frac{1}{3}xe^{2-3x} - \frac{1}{9}e^{2-3x} + c$$

c) Let $u = \ln^2 x$ and $dv = 1\,dx \Rightarrow du = 2\frac{\ln x}{x}dx$ and $v = \int 1\,dx = x$

$$\Rightarrow \int \ln^2 x\,dx = uv - \int v\,du = x\ln^2 x - \int 2x\frac{\ln x}{x}dx = x\ln^2 x - 2\int \ln x\,dx$$
$$\Rightarrow \int \ln^2 x\,dx = x\ln^2 x - 2(x\ln x - x) + c$$

d) Let $u = \ln x$ and $dv = x^{-3}dx$

$$\Rightarrow du = \frac{1}{x}dx \text{ and } v = \int x^{-3}\,dx = \frac{x^{-2}}{-2}$$
$$\Rightarrow \int x^{-3}\ln x\,dx = uv - \int v\,du = \frac{x^{-2}}{-2}\ln x - \int \frac{1}{x}\frac{x^{-2}}{-2}dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\int x^{-3}dx$$
$$\Rightarrow \int x^{-3}\ln x\,dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\frac{x^{-2}}{-2} = -\frac{1}{4x^2}(2\ln x + 1) + c$$

Exercise 5.3 Evaluate the following indefinite integrals.

1. $\int e^{3x}\sin(2x-1)\,dx$
2. $\int x^3\ln^2 x\,dx$
3. $\int x^{-1}e^{3x}dx$
4. $\int (x^2+3x)\sin 2x\,dx$
5) $\int \sec^2 x\,dx$

5.3 DEFINITE INTEGRALS, AREA AND THE FUNDAMENTAL THEOREM OF CALCULUS

5.3.1 Definite Integrals

- Unlike the previous sub-units where the indefinite integral of a function resulted in a new function, when finding the **definite integral** we produce a **numerical value**.

- Definite integrals are important because they can be used to find different types of measures such as areas, volumes, lengths and so on.

Definition 5.3 (Fundamental Theorem of Calculus)

Let f be f function continuous on the interval $[a, b]$ and F be the antiderivative of f. Then the **DEFINITE INTEGRAL** f on $[a, b]$ is the value of the integral $\int_a^b f(x)dx$ which is given by

$$\int_a^b \boldsymbol{f(x)dx} = (\boldsymbol{F(x)})|_a^b = \boldsymbol{F(b) - F(a)}$$

which is read as "the (definite) integral of $f(x)$ w.r.t x from a to b".

Note : The constant of integration c is omitted, because it would be cancel itself out upon carrying out the subtraction :

$$\int_a^b \boldsymbol{f(x)dx} = (\boldsymbol{F(x) + c})|_a^b = \boldsymbol{[F(b) + c] - [F(a) + c] = F(b) - F(a)}$$

In the expression $\int_a^b \boldsymbol{f(x)dx}$,

- x is called the **variable of integration**, and
- a and b are called the **lower limit** and **upper limit**, respectively.

Properties of Definite Integrals

Let f and g be two integrable functions on $[a, b]$ and k be constant. Then

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any c where $a < c < b$

Example 12: Evaluate the following definite integrals.

a) $\int_1^4 (\frac{1}{x} + 2\sqrt{x})dx$ b) $\int_{-4}^0 \frac{x}{\sqrt{x^2+9}}\,dx$

Solution

a) $F(x) = \int \left(\frac{1}{x} + 2\sqrt{x}\right)dx = \ln x + \frac{4}{3}\sqrt{x^3} + c$. Then

$\int_1^4 (\frac{1}{x} + 2\sqrt{x})dx = \left(\ln x + \frac{4}{3}\sqrt{x^3}\right)\Big|_1^4 = \left(\ln 4 + \frac{4}{3}\sqrt{4^3}\right) - \left(\ln 1 + \frac{4}{3}\sqrt{1^3}\right) = \ln 4 + \frac{28}{3}$

b) Let $u = x^2 + 9 \Rightarrow \quad du = 2xdx \Rightarrow xdx = \frac{du}{2}$

$\int \left(\frac{x}{\sqrt{x^2+9}}\right)dx = \int \frac{1}{2\sqrt{u}}du = \sqrt{u} = \sqrt{x^2+9}$

$\Rightarrow \int_{-4}^0 \frac{x}{\sqrt{x^2+9}}\,dx = \sqrt{x^2+9}\Big|_{-4}^0 = \sqrt{(0)^2 + 9} - \sqrt{(-4)^2 + 9} = 3 - 5 = -2$

Notice that in (b), we use change of variable. But we do not change the limit of integrations. This is because we express the antiderivative F interms of the original variable x.
But it is possible to change the limit of integration to the new variable when we use change of variable in evaluation of the integral.

Change of variable in Definite Integral

$$\int_a^b f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + c, \text{ for an arbitrary c.}$$

letting $u = g(x)$ gives $du = g'(x)dx$.

$$x = a \Rightarrow u = g(a) \text{ and } x = b \Rightarrow u = g(b)$$

Then the rule can be written as

$$\int_a^b f\big(g(x)\big)g'(x)dx = \int_a^b f(u)du = F(u)\Big|_{g(a)}^{g(b)}$$

Example 13: Evaluate the following definite integrals.

a) $\int_0^{\frac{\pi}{3}} \tan x \sec^2 x\, dx$ b) $\int_{\ln 2}^{\ln 5} \frac{1}{e^x - e^{-x}}\, dx$

Solution

a) Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$

$x = 0 \Rightarrow u = g(0) = \tan(0) = 0$ and $x = \frac{\pi}{3} \Rightarrow u = g\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$$\Rightarrow \int_0^{\frac{\pi}{3}} \tan x \sec^2 x\, dx\, dx = \int_0^{\sqrt{3}} u\, du = \frac{u^2}{2}\Big|_0^{\sqrt{3}} = \frac{\sqrt{3}^2}{2} - \frac{0^2}{2} = \frac{3}{2} - 0 = \frac{3}{2}$$

b) $\int_{\ln 2}^{\ln 5} \frac{1}{e^x - e^{-x}}\, dx = \int_{\ln 2}^{\ln 5} \frac{e^x}{e^{2x} - 1}\, dx$. $u = e^x \Rightarrow du = e^x dx$

$x = \ln 2 \Rightarrow u = e^{\ln 2} = 2$ and $x = \ln 5 \Rightarrow u = e^{\ln 5} = 5$

$$\Rightarrow \int_{\ln 2}^{\ln 5} \frac{1}{e^x - e^{-x}}\, dx = \int_{\ln 2}^{\ln 5} \frac{e^x}{e^{2x} - 1}\, dx = \int_2^5 \frac{du}{u^2 - 1} = \frac{1}{2}\int_2^5 \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$\Rightarrow \int_{\ln 2}^{\ln 5} \frac{1}{e^x - e^{-x}}\, dx = \int_2^5 \frac{du}{u^2 - 1} = \frac{1}{2}(\ln|u-1| - \ln|u+1|)\Big|_2^5 = \frac{1}{2}[(\ln 4 - \ln 6) - (\ln 1 - \ln 3)]$$

$$= \frac{1}{2}[\ln 4 - \ln 6 + \ln 3] = \frac{1}{2}\ln\frac{4 \times 3}{6} = \frac{1}{2}\ln 2 = \ln\sqrt{2}$$

Exercise 5.4 Evaluate the following definite integrals.

a) $\int_1^3 \sqrt{x}(1 - \frac{1}{x})\, dx$

b) $\int_{-1}^3 \frac{2x}{x+3}\, dx$

c) $\int_{-\frac{\pi}{2}}^{\frac{4\pi}{3}} |\sin x|\, dx$

d) $\int_{-1}^3 \frac{\cos x}{\sqrt{1+\sin x}}\, dx$

e) $\int_1^3 \frac{x^3+1}{x^2(x-4)}\, dx$

f) $\int_e^{e^2} \left(\frac{\ln x}{x^2}\right) dx$

g) $\int_{-1}^3 (e^x + e^{-x})^2\, dx$

h) $\int_0^1 \frac{2^{2x-1} - 5^{2x+1}}{10^x}\, dx$

5.4 APPLICATIONS OF INTEGRAL CALCULUS

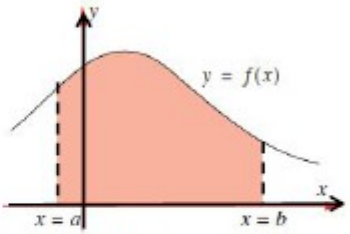
- **Mathematical application** : To calculate area of a region bounded by curves of continuous functions defined on a closed interval $[a\ ,\ b]$ and the volume of a solid of revolution.
- **Physical applications** : To calculate work done by a variable force along a straight line, acceleration, velocity and displacement.

5.4.1 The Area Between Two Curves

1. Area between a curve and the x-axis

If $f(x)$ is a **positive** and continuous function on the interval $[a, b]$, then the **area A** bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

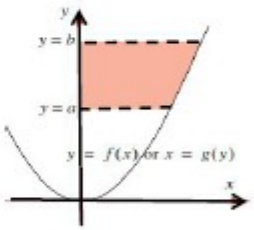
$$A = \int_a^b f(x)\, dx$$



2. Area between a curve and the y-axis

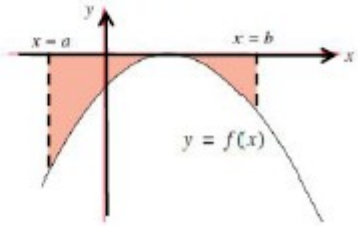
If $f(y)$ is a **positive** and continuous function on the interval $[a, b]$, then the **area A** bounded by the graph of $x = f(y)$, the y-axis, and the horizontal lines y= a and $y = b$ is given by

$$A = \int_a^b f(y)\, dy$$



Remark : If $f(x)$ is a **negative** and continuous function over the interval $[a, b]$, then the **area A** bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines x = a and x = b is given by

$$A = \left| \int_a^b f(x)\, dx \right|$$



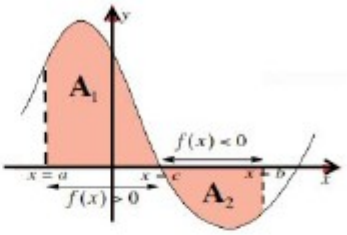
Note: It is possible for a continuous function $y = f(x)$ to alternate between negative and positive values over the interval $[a\ ,\ b]$. In this case there is atleast a number $c \in (a\ ,b)$ where the graph of $y = f(x)$ crosses the x-axis. For instance, as shown in the figure below,

$$A = \int_a^b f(x)\,dx = A_1 + A_2$$

$$\Rightarrow \quad A = \int_a^c f(x)\,dx + \left(-\int_c^b f(x)\,dx\right)$$

But instead, we can write A, in general, as

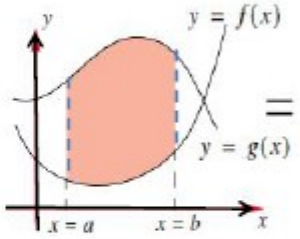
$$\Rightarrow \quad A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$$



3. The Area Between Two Curves

Suppose f and g are two continuous functions on $[a, b]$ with $f(x) \geq g(x)$ on $[a, b]$. Then the area A bounded by the curves of $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$



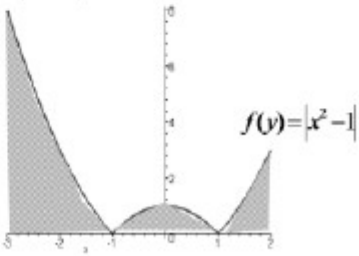
Examples 14. Find the area of the region bounded by the graph of the function $f(x) = |x^2 - 1|$ and the x-axis between the lines $x = -3$ *and* $x = 2$

Solution

Notice that the values of $f(x) = |x^2 - 1|$ to alternate between negative and positive values over the interval $[-3, 2]$.

$$x^2 - 1 = 0 \ iff \ x = -1 \ or \ x = 1$$

$$f(x) = |x^2 - 1| = \begin{cases} \geq 0 \ on \ [-3, -1] \\ \leq 0 \ on \ [-1, 1] \\ \geq 0 \ on \ [1, 2] \end{cases}$$



Hence, $A = A_1 + A_2 + A_3$

$$A = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$$

$$A = \int_{-3}^{-1} (x^2 - 1)\,dx + \int_{-1}^{1} (1 - x^2)\,dx + \int_{1}^{2} (x^2 - 1)\,dx$$

$$= \left(\frac{x^3}{3} - x\right)_{-3}^{-1} + \left(x - \frac{x^3}{3}\right)_{-1}^{1} + \left(\frac{x^3}{3} - x\right)_{1}^{2} = \frac{20}{3} + \frac{4}{3} + \frac{4}{3} = \frac{28}{3}\ sq.u$$

Examples 15. Find the area of the region bounded by the graph of the function $y = x^2 + x - 2$ and $y = x + 2$

Solution

$x^2 + x - 2 = x + 2$

$\Rightarrow \quad x^2 - 4 = 0$

$\Rightarrow \quad x = -2 \text{ or } x = 2$

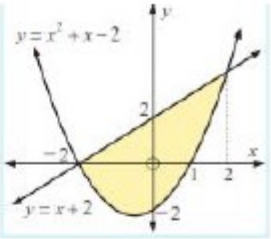
The two functions intersect at $x = -2$ and $x = 2$.

On $[-2, 2]$, $x + 2 \geq x^2 + x - 2$

$\Rightarrow$ Then the area of the region is

$$A = \int_{-2}^{2}[f(x) - g(x)]\,dx = \int_{-2}^{2}[(x-2) - (x^2 + x - 2)]\,dx$$

$$= \int_{-2}^{2}[4 - x^2]\,dx = \frac{32}{3} \text{ sq. u}$$



Examples 15. Find the area of the region bounded by the graph of the function $x = 4 - y^2$ and $x = y - 2$

Solution

$4 - y^2 = y - 2$

$\Rightarrow \quad y^2 + y - 6 = (y + 3)(y - 2) = 0$

$\Rightarrow \quad y = -3 \text{ or } y = 2$

The two functions intersect at $y = -3$ and $y = 2$.

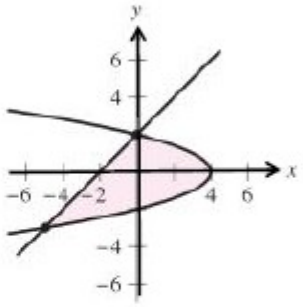
$y - 2 \leq 4 - y^2$ for $-3 \leq y \leq 2$

$\Rightarrow$ Then the area of the region is

$$A = \int_{-3}^{2}[(4 - y^2) - (y - 2)]dy$$

$$= \int_{-3}^{2}[(6 - y - y^2)]dy$$

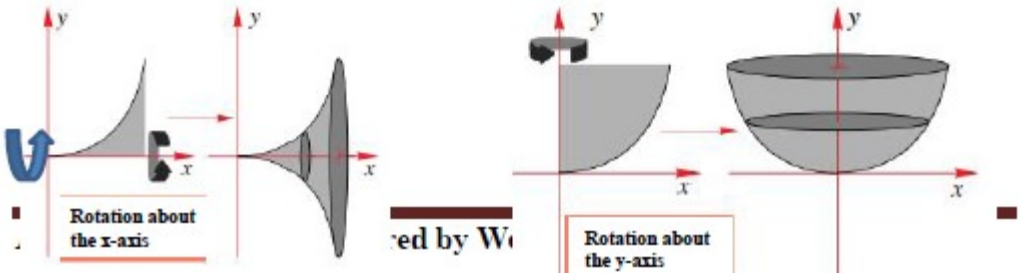
$$= \left(6y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-3}^{2} = \frac{32}{3} + \frac{27}{2} = \frac{145}{6}\ sq.u$$



5.4.2 Volume of Revolution

The volume of a solid of revolution

- Another important application of definite integral is its use in finding the volume of a three dimensional solid generated by revolving a plane region about a line.
- If a region is revolving about a line, the resulting solid is called a **SOLID OF REVOLUTION** and the **AXIS OF ROTATION** is called axis of revolution.
- The volume of a solid of revolution is said to be a **VOLUME OF REVOLUTION.**
- In this section we consider only two axis of rotation : the **x-axis** and the **y-axis.**



Rule 1 : Disk Method

1. Let $f(x)$ be a function continuous on $[a\ ,\ b]$. The volume V of the solid of revolution obtained by revolving the plane region enclosed by the graph of $y = f(x)$, the x-axis and the lines $x = a\ \ and\ \ x = b$ is given by
$$V = \pi \int_a^b [f(x)]^2 dx$$
2. Let $f(y)$ be a function continuous on $[c\ ,\ d]$. The volume V of the solid of revolution obtained by revolving the plane region enclosed by the graph of $x = f(y)$, the y-axis and the lines $y = c\ \ and\ \ y = d$ is given by
$$V = \pi \int_c^d [f(y)]^2 dy$$

This method is sometimes known as **DISK METHOD**.

Examples 15. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \sqrt{e^x + 1}$ and the x-axis between $x = 0$ and $x = \ln 2$ about the x-axis.

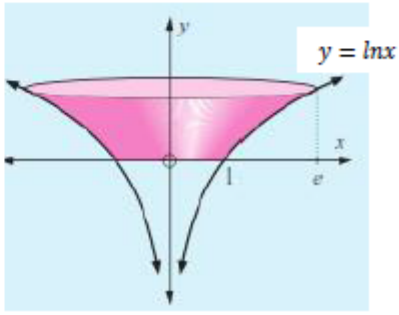
Solution : Clearly f is continuous on [0 , ln x]
$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_0^{\ln 2} [\sqrt{e^x + 1}]^2 dx$$
$$V = \pi \int_0^{\ln 2} (e^x + 1) dx = (e^x + 1)_0^{\ln 2} = (e^{\ln 2} + 1) - (e^0 + 1) = 2\ cu.u$$

Examples 16. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \sqrt{e^x + 1}$ and the x-axis between $x = 0$ and $x = \ln 2$ abType equation hereout the x-axis.

Solution : Clearly f is continuous on [0 , ln x]
$$V = \pi \int_a^b [f(x)]^2 dx = \pi \int_0^{\ln 2} [\sqrt{e^x + 1}]^2 dx$$
$$V = \pi \int_0^{\ln 2} (e^x + 1) dx = (e^x + 1)_0^{\ln 2} = (e^{\ln 2} + 1) - (e^0 + 1) = 2\ cu.u$$

Examples 17. Find the volume of the solid generated by revolving the region bounded by the graph of $f(x) = \ln x$, the lines $x = 1$ and $x = e$ about the y-axis.

Solution : Clearly f is continuous on [1, e].



First express x interms of y.

$y = f(x) = \ln x \ \Rightarrow\ x = e^y = g(y)$

$x = 1 \ \Rightarrow\ y = \ln 1 = 0 = c$ and

$x = e \ \Rightarrow\ y = \ln e = 1 = d$. Then apply the formula.

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_0^1 [e^y]^2 dy = \pi \int_0^1 e^{2y} dy$$

$$\Rightarrow\ V = \left(\tfrac{1}{2}\pi e^{2y}\right)_0^1 = \tfrac{1}{2}\pi e^2 - \tfrac{1}{2}\pi e^0 = \tfrac{1}{2}\pi(e^2 - 1)$$

Examples 18. Find the volume of the solid generated by revolving the region bounded by the graph of $x = 4y - y^2$, the y-axis from $y = -2$ to y = 2 about the y-axis.

Solution : Clearly $x = g(y) = 4y - y^2$ is continuous on for $-2 \le y \le 2$.

$$V = \pi \int_c^d [g(y)]^2 dy = \pi \int_{-2}^{2} [4y - y^2]^2 dy = \pi \int_0^1 [16y^2 - 8y^3 - y^4] dy$$

$$\Rightarrow V = \left(16\frac{y^3}{3} - 8\frac{y^4}{4} - \frac{y^5}{5}\right)_{-2}^{2} = \frac{256}{3} - \frac{64}{5} = \frac{1088}{15}$$

Rule 2 : Washer Method

Let $f(x)$ and $g(x)$ be two functions continuous on $[a\ ,\ b]$ with $(x) \ge g(x)\ \forall x \in [a, b]$. Then the volume V of the solid of revolution generated by revolving the plane region enclosed by the graphs of $y = f(x)$ and $y = g(x)$ over $a \le x \le b$ about the x-axis is given by

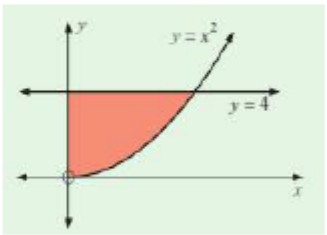
$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

This method is sometimes known as **WASHER METHOD**.

Examples 19. Find the volume of revolution generated by rotating the region bounded by the graphs of $y = 4$ *and* $y = x^2$, and the y-axis, in the first quadrant about the x-axis.

Solution : $4 = x^2 \Rightarrow x = -2$ *or* $x = 2$. But $x \ge 0$. Hence $0 \le x \le 2$.

On [0, 2], $4 \ge x^2$. Then the required volume is



$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

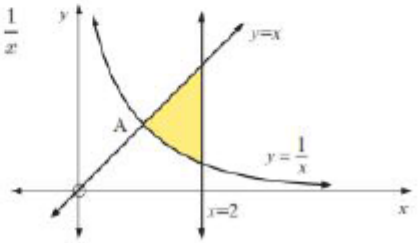
$$= \pi \int_0^2 ([4]^2 - [x^2]^2) dx$$

$$= \pi \int_0^2 (16 - x^4) dx$$

$$\Rightarrow \quad V = \left(16x - \frac{x^5}{5}\right)_0^2 = 32 - \frac{32}{5} = \frac{118}{5} \text{ cubic u.}$$

Examples 20. In the figure, the shaded region between the graphs of $y = x,\ y = \frac{1}{x}$ *and* $x = 2$.

Find the volume of revolution generated by rotating the region about the x-axis.



Solution : $x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = 1$, since $x > 0$.

The shaded region lies between x = 1 and x = 2, and on [1, 2], $x \ge \frac{1}{x}$. Hence

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx$$

$$= \pi \int_1^2 \left([x]^2 - [\tfrac{1}{x}]^2\right) dx$$

$$= \pi \int_1^2 (x^2 - x^{-2}) dx = \pi \left(\frac{x^3}{3} + x^{-1}\right)_1^2 = \frac{19}{6} - \frac{4}{3} = \frac{11}{6} \text{ cubic u.}$$

Chemistry Summary Notes

UNIT-4

ELECTROCHEMISTRY

Oxidation is loss of electrons, and reduction is gain of electrons.

A reduced substance is an oxidizing agent and an oxidized substance is a reducing agent.

Redox reactions are balanced, using the oxidation number change method or the ion- electron method.

Metals can conduct electricity because they have delocalized electrons, and electrolytes can also conduct electricity, due to the presence of free ions.

In the electrolytic cell, electrical energy is used to bring about a chemical change.

The cathode is the electrode at which reduction takes place and the anode is the electrode at which oxidation takes place.

In an electrolytic cell, the anode is positive and the cathode is negative. In Galvanic cells, the electrode polarity is reversed.

The preferential discharge of ions is affected by the nature of the electrodes, the position of the ions in the electrochemical series and the concentration of the ions in the electrolyte.

In the electrolysis of molten sodium chloride, sodium is discharged at the cathode, and chlorine is released at the anode. But, in the electrolysis of concentrated brine solution, hydrogen is discharged at the cathode, and chlorine is discharged at the anode.

In the electrolysis of dilute sodium chloride solution and dilute sulphuric acid solution, hydrogen gas discharges at the cathode and oxygen gas discharges at the anode.

The Faraday's Laws are used to calculate the amount of substance deposited or liberated at the cathode or anode during electrolysis.

Some of the applications of electrolysis are electroplating, electrorefining and extraction of metals and non-metals.

A Galvanic cell is a device in which chemical energy is transformed to electrical energy.

A salt bridge preserves the electroneutrality of a solution.

A cell notation allows us to describe a Galvanic cell without drawing a diagram.

The anode is written on the left and the cathode on the right.

Cell potential is the difference between the electrode potentials of the cathode and anode.

- **For spontaneous cell reactions, $\Delta G < 0$ and $E_{cell} > 0$.**
- **The Nernst equation enables us to calculate Ecell as a function of reactant and product concentrations in a redox reaction.**
- **A battery is a Galvanic cell that gives a direct current at a constant voltage.**

A battery can be either primary or secondary.

Corrosion is the deterioration of metals by electrochemical processes. It is a redox reaction.

Corrosion can be prevented by painting, electroplating and by way of cathodic protection.

UNIT-5

SOME ELEMENTS IN CHEMISTRY AND INDUSTRY

There are 113 elements in the periodic table. Of these, 92 elements occur in nature, and the others are man-made.

Minerals that are suitable for the extraction of elements are called ores.

All ores are minerals, but not all minerals are ores.

The twelve most abundant elements in the earth's crust, in decreasing order of abundance, are: oxygen, silicon, aluminium, iron, calcium, magnesium, sodium, potassium, titanium, hydrogen, phosphorus and manganese.

Metals found in the Earth's crust are in the form of oxides, carbonates, sulphides, halides, sulphates and phosphates.

Nutrients are limited in nature, and they are exchanged between living and nonliving things.

In the carbon cycle, atmospheric carbon dioxide is converted to glucose by green plants. Animals incorporate carbon by eating plants or other animals. Carbon returns to the physical environment during respiration, the burning of wood or fossil fuels, and the decomposition of plants and animals.

In the nitrogen cycle, atmospheric nitrogen is converted to nitrogen compounds by lightning, through the Haber process or by bacteria on root nodules of leguminous plants. Plants use these compounds to prepare their proteins and nucleic acids. Animals also prepare their proteins and nucleic acids through eating plants or other animals.

Nitrogen returns to the atmosphere during the decomposition of plants and animals through the action of denitrifying bacteria.

The phosphorus cycle differs from the nitrogen and carbon cycles primarily because phosphorus cannot exist in gaseous form in the atmosphere. Plants absorb phosphates, using their roots, and make DNA, and ATP. Animals get phosphates by eating plants or other animals. When plants and animals die, they decompose and return phosphates into the soil or water.

Metallurgy is the science and technology of extracting metals from their natural sources and preparing them for practical use. This process has three main steps: concentrating the ore, extracting the metal and purifying the crude metal.

Sodium is the 7th most abundant element in the Earth's crust and is extracted from molten sodium chloride using Downs cell.

Calcium is the 5th most abundant element in the Earth's crust and is extracted from molten calcium chloride. It is a major component in bones and teeth.

Tin is mainly extracted from cassiterite (SnO_2), using coke as reducing agent. It is used to make alloys.

Lead can be extracted from galena (PbS) by partially oxidizing galena and self-reduction. It is used to make lead /acid storage batteries.

Zinc is extracted from zinc blende (ZnS) by roasting, followed by reduction with carbon monoxide. Large amounts of zinc are used for coating iron.

Chromium is extracted from chromite ore ($FeCr_2O_4$): First it is oxidized to sodium chromate (Na_2CrO_4). Then

sodium chromate is converted to sodium dichromate by the addition of acids. Next the sodium dichromate is reduced to chromium (III) oxide, using carbon. Finally, Cr_2O_3 is reduced to chromium metal by aluminium. Chromium is used to make alloys.

Silicon is the 2nd most abundant element in the Earth's crust and is extracted from silica, using carbon as reducing agent. Silicon is used in the construction of transistors and microprocessors.

Ammonia is prepared by the Haber process and is mainly used to make fertilisers.

Nitric acid is prepared by the Ostwald process, and is used to make fertilizers, plastics and explosives.

Sulphuric acid is produced by the Contact process, and is mainly used to make fertilizers, detergents and paints.

Diammonium monohydrogen phosphate (DAP) is prepared by reacting anhydrous ammonia with phosphoric acid in two steps. DAP is mainly used as a fertilizer.

UNIT- 6

POLYMERS

Monomers are the building blocks of polymers.

Homopolymers are made from identical monomers, and copolymers are synthesized from more than one kind of monomer.

Based on their sources, polymers can be classified as synthetic or natural. Synthetic polymers are man-made, while natural polymers exist naturally.

Polymerization is the process of combining monomer units to form large molecules called polymers.

Polymerization can be effected through either addition polymerization or condensation polymerization.

Each of the following polymers are made by addition polymerization.

- **Polyethylene**
- **Polypropylene**
- **Polyvinyl chloride (PVC)**
- **Polymethyl methacrylate (PMMA)**
- **Teflon**
- **Polystyrene**
- **Natural rubber**
- **Neoprene**
- **Styrene-butadiene rubber and**
- **Butyl rubber**

- **Polymers that are synthesized by condensation polymerization are.**
  - **Nylons**
  - **Polyesters**
  - **Bakelite**
  - **Carbohydrates, and**
  - **Proteins.**
- **Thermoplastics soften on heating and can be molded into different shapes, and the process of heating and cooling can be repeated several times, but once thermosetting plastics are solidified they cannot be**

softened or remolded on heating.

Synthetic polymers pollute the environment during their preparations, use and disposal.

Monosaccharides are the building blocks of carbohydrates.

Proteins are made from the twenty naturally occurring amino acids.