



QUESTIONS WITH SOLUTIONS

CSIR UGC NET DECEMBER- 17

MATHEMATICAL SCIENCES

PART - B

21. A(21), B(26), C(32)

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}_{\geq 0}$ denote the set $\{0, 1, 2, 3, \dots\}$. Consider the map

$f: \mathbb{Z}_{\geq 0} \times \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$f(m, n) = 2^m \cdot (2n + 1)$. Then the map f is

1. onto (surjective) but not one-one (injective)
2. one-one (injective) but not onto (surjective)
3. both one-one and onto
4. neither one-one nor onto

Sol. (2) Correct answer (2)

$f: \mathbb{Z}_{\geq 0} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$f(m, n) = 2^m (2n + 1)$

For one-one:

$f(m_1, n_1) = f(m_2, n_2)$

$$\Rightarrow 2^{m_1} (2n_1 + 1) = 2^{m_2} (2n_2 + 1)$$

$$\Rightarrow m_1 = m_2, n_1 = n_2$$

Clearly one-one

'f' is not onto

As there does not exist pre image of 0

22. A(22), B(25), C(31)

Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers satisfying $a_1 \geq 1$ and $a_{n+1} \geq a_n + 1$ for all $n \geq 1$. Then which of the following is necessarily true?

1. The series $\sum_{n=1}^{\infty} \frac{1}{a_n^2}$ diverges
2. The sequence $\{a_n\}_{n \geq 1}$ is bounded
3. The series $\sum_{n=1}^{\infty} \frac{1}{a_n^3}$ converges
4. The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges

Sol. (3) Correct answer (3)

Take $\langle a_n \rangle = \langle n \rangle$

Clearly, (1), (2) and (4) are rule out

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

(2) $\{1, 2, 3, 4, \dots\}$ is not bounded

(4) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Clearly from given condition $a_n \geq n \forall n \geq 1$

$\Rightarrow a_n^2 \leq \frac{1}{n^2}$ as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

$\therefore$ by comparison test $\sum_{n=1}^{\infty} \frac{1}{a_n^2}$ is convergent

So, (3) is correct answer

23. A(23), B(24), C(30)

Let D be a subset of the real line. Consider the assertion: "Every infinite sequence in D has a subsequence which converges in D ". This assertion is true if

1. $D = [0, \infty)$
2. $D = [0, 1] \cup [3, 4]$
3. $D = [-1, 1) \cup (1, 2]$
4. $D = (-1, 1)$

Sol. (2) Correct answer (2)

Every infinite sequence in D has a subsequence which converges in D . This is true only if D is compact set. Compact set is given only in option (2) that is $[0, 1] \cup [3, 4]$

24. A(24), B(23), C(29)

Let $f: (0, \infty) \rightarrow \mathbb{R}$ be uniformly continuous. Then

1. $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$ exist
2. $\lim_{x \rightarrow 0^+} f(x)$ exists but $\lim_{x \rightarrow \infty} f(x)$ need not exist
3. $\lim_{x \rightarrow 0^+} f(x)$ need not exist but $\lim_{x \rightarrow \infty} f(x)$ exists
4. neither $\lim_{x \rightarrow 0^+} f(x)$ nor $\lim_{x \rightarrow \infty} f(x)$ need exist

Sol. (2) Correct answer (2)

$f: (0, \infty) \rightarrow \mathbb{R}$ be uniformly continuous

Take $f(x) = x$

Clearly $f(x)$ is uniformly continuous in $(0, \infty)$

But $\lim_{x \rightarrow \infty} f(x)$ does not exist

So (1) and (3) is rule out

Result: if $f: (a, b) \rightarrow \mathbb{R}$ is continuous then f is uniformly continuous on (a, b) iff $\lim_{x \rightarrow a^+} f(x)$ and

$\lim_{x \rightarrow b^-} f(x)$ exists, where a and b are finite

So in this question $\lim_{x \rightarrow 0^+} f(x)$ must exist

So (2) is correct and (4) is incorrect.



$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which among the following are true?

1. there is exactly one solution $y=y(x)$ with $y(0) = y'(0) = 1$ and $y\left(\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$
2. there is exactly one solution $y=y(x)$ with $y(0) = 1, y'(0) = -1$ and $y\left(-\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$
3. any solution $y=y(x)$ satisfies $y''(0)=y(0)$
4. If y_1 and y_2 are any two solutions then $(ax+b)y_1=(cx+d)y_2$ for some $a, b, c, d \in \mathbb{R}$

Sol. (1,2,3,4) Given differential equation is

$y'' - 2 \tan x \cdot y' - y = 0 \quad \dots (1)$
 Comparing (1) with $y'' + Py' + Qy = R$
 $P = -2 \tan x, \quad Q = -1, \quad R = 0$
 To remove the first derivative from (1) we choose

$$u = e^{-\frac{1}{2} \int P dx} = e^{-\frac{1}{2} \int -2 \tan x} = \sec x$$

$$\Rightarrow u = \sec x$$

Let the General Solution is $y(x) = \sec x \cdot v(x)$

Substitute the values of $y(x), y'(x)$ and $y''(x)$ in (1) we get

$$v''(x) = 0 \Rightarrow v(x) = ax + b$$

$$\therefore y(x) = (ax + b) \sec x$$

$$\therefore \text{For option (1); } y(0) = y'(0) = 1$$

$$\Rightarrow a = 1, b = 1$$

$$\therefore y(x) = (x + 1) \sec x$$

$$\Rightarrow y\left(\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$$

$\Rightarrow$ Option (1) is correct.

$$\text{For option (2); } y(0) = 1, y'(0) = -1$$

$$\Rightarrow a = -1, b = 1$$

$$\Rightarrow y(x) = (1 - x) \sec x$$

$$\Rightarrow y\left(-\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$$

$\therefore$ Option (2) is also correct.

$$\text{For option (3); } y''(0) = b \text{ and } y(0) = b$$

$$\Rightarrow y''(0) = y(0)$$

$\therefore$ option (3) is also correct

For option (4):

$$\text{Let } y_1 = (c_1 x + c_2) \sec x$$

$$\text{and } y_2 = (d_1 x + d_2) \sec x$$

$$\therefore (ax + b) y_1 = (cx + d) y_2$$

$$\Rightarrow (ax + b) (c_1 x + c_2) = (cx + d) (d_1 x + d_2)$$

$$\text{if } c_1 = c, c_2 = d, d_1 = a, d_2 = b$$

then equality holds

$\therefore$ Option (4) is also correct

Correct options are (1), (2), (3), (4)

94.

A(94), B(93), C(99)

Consider the Lagrange equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z. \quad \text{Then the}$$

general solution of the given equation is

$$1. \quad F\left(\frac{x}{z}, \frac{x-y}{z}\right) = 0 \text{ for an arbitrary}$$

differentiable function F

$$2. \quad F\left(\frac{x-y}{z}, \frac{1}{x} - \frac{1}{y}\right) = 0 \text{ for an}$$

arbitrary differentiable function F

$$3. \quad z = f\left(\frac{1}{x} - \frac{1}{y}\right) = 0 \text{ for an arbitrary}$$

differentiable function f

$$4. \quad z = xyf\left(\frac{1}{x} - \frac{1}{y}\right) = 0 \text{ for an arbitrary}$$

differentiable function f

Sol. (1,2) Given equation is

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z$$

By Lagrange linear equation

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\therefore \frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx - dy}{x - y} = \frac{dz}{z}$$

$$\Rightarrow \log(x - y) = \log z + \log c_1 \Rightarrow \frac{x - y}{z} = c_1$$

$$\text{Also } \frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\Rightarrow \frac{-1}{x} = \frac{-1}{y} + c_2 \Rightarrow \frac{1}{x} - \frac{1}{y} = c_2$$

$\therefore$ General Solution is $F(c_1, c_2) = 0$

or $c_1 = f(c_2)$ or $c_2 = f(c_1)$

$$\therefore F\left(\frac{x-y}{z}, \frac{1}{x} - \frac{1}{y}\right) = 0$$

$$\therefore x - y = zc_1$$



Sol.(4) $f(z) = e^z + e^{-z}$

Now $f(z) = 0 \Rightarrow e^z + e^{-z} = 0$

$\Rightarrow e^z + \frac{1}{e^z} = 0 \Rightarrow e^{2z} = -1 = e^{(2n+1)\pi i}$

$\Rightarrow z = \left(n + \frac{1}{2}\right)\pi i, n \in \mathbb{Z}$

$\Rightarrow$ Thus, $f(z) = e^z + e^{-z}$ has infinitely many zeros

34. A(34), B(35), C(39)

Let f be a holomorphic function in the open unit disc such that $\lim_{z \rightarrow 1} f(z)$ does not exist. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f about $z = 0$ and let R be its radius of convergence. Then

1. $R = 0$
2. $0 < R < 1$
3. $R = 1$
4. $R > 1$

Sol.(3) Take $f(z) = \frac{1}{1-z}$

Taylor series expansion of ' f ' about ' $z=0$ ' is
 $f(z) = (1-z)^{-1} = 1 + z + z^2 + z^3 + \dots$

$\therefore \sum_{n=0}^{\infty} z^n$ Clearly $R = 1$

35. A(35), B(34), C(38)

Let C be the circle of radius 2 with centre at the origin in the complex plane, oriented in the anti-clockwise direction. Then the integral

$\oint_C \frac{dz}{(z-1)^2}$ is equal to

1. $\frac{1}{2\pi i}$
2. $2\pi i$
3. 1
4. 0

Sol.(4) let $I = \oint_C \frac{dz}{(z-1)^2}$, where $C : |z| = 2$

$= \frac{1}{2\pi i} \times \left\{ \frac{d^2}{dz^2} (1) \right\}_{z=1} = 0$

$\therefore$ If $f(z)$ is analytic inside and on a closed contour ' c ' and ' a ' is any point inside c , then

$\oint_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{n!}{2\pi i} [f^{(n)}(a)]$

37. A(37), B(40), C(36)

Let $f: \mathbb{Z} \rightarrow (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ be the function
 $f(n) = (n \bmod 4, n \bmod 6)$. Then

1. $(0 \bmod 4, 3 \bmod 6)$ is in the image of f
2. $(a \bmod 4, b \bmod 6)$ is in the image of f , for all even integers a and b
3. image of f has exactly 6 elements
4. kernel of $f = 24\mathbb{Z}$

Sol (2): Given $f: \mathbb{Z} \rightarrow (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$

Defined by $f(n) = (n \bmod 4, n \bmod 6)$

$\Rightarrow f(0) = (0, 0)$

$f(1) = (1, 1)$,

$f(2) = (2, 2)$,

$f(3) = (3, 3)$,

$f(4) = (0, 4)$,

$f(5) = (1, 5)$,

$f(6) = (2, 0)$,

$f(7) = (3, 1)$,

$f(8) = (0, 2)$,

$f(9) = (1, 3)$,

$f(10) = (2, 4)$,

$f(11) = (3, 5)$,

$f(12) = (0, 0)$.

clearly, $(0, 3) \notin \text{Image of } f$

$\therefore$ option (1) is incorrect

Image of f has exactly 12 elements.

Thus, option (3) and (4) are incorrect

As all other options are rule out

$\therefore$ option (2) is correct answer

39. A(39), B(38), C(34)

Let R be a subring of $\mathbb{Q}$ containing 1. Then which of the following is necessarily true?

1. R is a principal ideal domain (PID)
2. R contains infinitely many prime ideals
3. R contains a prime ideal which is not a maximal ideal
4. for every maximal ideal m in R , the residue field R/m is finite

Sol.(1) Take $R = \mathbb{Q}$ is subring of $\mathbb{Q}$ containing 1

For option (2):

$\mathbb{Q}$ has only two Ideals zero Ideal and $\mathbb{Q}$

$\Rightarrow \mathbb{Q}$ will have finitely many prime Ideals

So, option (2) is incorrect

For option (3):

$\mathbb{Q}$ does not contain any prime ideal which is not maximal ideal

So, option (3) is incorrect

For option (4):

$I = \{0\}$ is Maximal Ideal of $\mathbb{Q}$

But $\frac{\mathbb{Q}}{\{0\}} \cong \mathbb{Q}$ is not finite

So, option (4) is not correct

For option (1)

Option (1) is true (as options (2), (3) and (4) are incorrect)

NOTE:- A subring of the field of fraction containing unity of a P.I.D. is P.I.D. as well.



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$f(m, n) = 2^m (2n + 1)$

For one-one:

$f(m_1, n_1) = f(m_2, n_2)$

$$\Rightarrow 2^{m_1} (2n_1 + 1) = 2^{m_2} (2n_2 + 1)$$

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Clearly one-one

'f' is not onto

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Take $\langle a_n \rangle = \langle n \rangle$

Clearly, (1), (2) and (4) are rule out

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$\therefore$ by comparison test $\sum_{n=1}^{\infty} \frac{1}{a_n^2}$ is convergent

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23. A(23), B(24), C(30)

Let D be a subset of the real line. Consider the assertion: "Every infinite sequence in D has a subsequence which converges in D ". This assertion is true if

1. $D = [0, \infty)$
2. $D = [0, 1] \cup [3, 4]$
3. $D = [-1, 1) \cup (1, 2]$
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24. A(24), B(23), C(29)

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2. $\lim_{x \rightarrow 0^+} f(x)$ exists but $\lim_{x \rightarrow \infty} f(x)$ need not exist
3. $\lim_{x \rightarrow 0^+} f(x)$ need not exist but $\lim_{x \rightarrow \infty} f(x)$ exists
4. neither $\lim_{x \rightarrow 0^+} f(x)$ nor $\lim_{x \rightarrow \infty} f(x)$ need exist

Sol. (2) Correct answer (2)

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Take $f(x) = x$

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But $\lim_{x \rightarrow \infty} f(x)$ does not exist

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Result: if $f: (a, b) \rightarrow \mathbb{R}$ is continuous then f is uniformly continuous on (a, b) iff $\lim_{x \rightarrow a^+} f(x)$ and

$\lim_{x \rightarrow b^-} f(x)$ exists, where a and b are finite

So in this question $\lim_{x \rightarrow 0^+} f(x)$ must exist

So (2) is correct and (4) is incorrect.



$$\therefore \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} = \frac{-zc_1}{xy}$$

$$\therefore \frac{1}{x} - \frac{1}{y} = c_2 \text{ can be written as } \frac{z}{xy} = c_2' \text{ or}$$

$$\frac{xy}{z} = c_2'$$

$\Rightarrow$ General Solution can be written as

$$F\left(\frac{x-y}{z}, \frac{xy}{z}\right) = 0$$

Now for option (3), if $z = f\left(\frac{1}{x} - \frac{1}{y}\right)$

$$\text{then } p = f'\left(\frac{1}{x} - \frac{1}{y}\right)\left(-\frac{1}{x^2}\right)$$

$$q = f'\left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{1}{y^2}\right)$$

$\therefore$ By given equation $x^2p + y^2q = (x+y)z$

$$\Rightarrow x^2 f'\left(\frac{1}{x} - \frac{1}{y}\right)\left(-\frac{1}{x^2}\right) + y^2 f'\left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{1}{y^2}\right)$$

$$= (x+y)z$$

$$\Rightarrow -f'\left(\frac{1}{x} - \frac{1}{y}\right) + f'\left(\frac{1}{x} - \frac{1}{y}\right) = (x+y)z$$

$$\Rightarrow 0 = (x+y)z$$

which is not true

$\therefore$ Options (1), (2), (4) are correct.

95. **A(95), B(92), C(98)**

Consider the second order PDE

$$8\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0. \text{ Then which of}$$

the following are correct ?

1. the equation is elliptic
2. the equation is hyperbolic
3. the general solution is

$$z = f\left(y - \frac{x}{2}\right) + g\left(y + \frac{3x}{4}\right),$$

for arbitrary differentiable functions f and g

4. the general solution is

$$z = f\left(y + \frac{x}{2}\right) + g\left(y - \frac{3x}{4}\right),$$

for arbitrary differentiable functions f and g

Sol. (2,3) Given Equation is

$$8\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$$

$$\therefore S^2 - 4RT = (-2)^2 - 4(8)(-3)$$

$$= 4 + 96 = 100 > 0$$

$\therefore$ Given equation is hyperbolic

Now for General solution put $D = m$ and $D' = 1$ in

$$(8D^2 - 2DD' - 3D'^2)z = 0$$

$$\Rightarrow (8m^2 - 2m - 3)z = 0$$

$$\Rightarrow 8m^2 - 2m - 3 = 0$$

$$\Rightarrow 8m^2 - 6m + 4m - 3 = 0$$

$$\Rightarrow 2m(4m - 3) + 1(4m - 3) = 0$$

$$\Rightarrow m = \frac{3}{4}, -\frac{1}{2}$$

$\therefore$ General Solution is

$$Z = f_1\left(y + \frac{3}{4}x\right) + f_2\left(y - \frac{1}{2}x\right)$$

$\therefore$ options (2) and (3) are correct.

98. **A(98), B(101), C(95)**

Let $X = \{u \in C^1[0,1] \mid u(0)=0\}$ and let $I: X \rightarrow \mathbb{R}$ be defined as

$$I(u) = \int_0^1 ((u'(t))^2 - (u(t))^2) dt.$$

Which of the following are correct ?

1. I is bounded below
2. I is not bounded below
3. I attains its infimum
4. I does not attain its infimum

Sol. (1,3) Given functional is

$$I(u) = \int_0^1 [(u'(t))^2 - (u(t))^2] dt$$

$\therefore$ By Euler-Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{d}{dt} \left(\frac{\partial f}{\partial u'} \right) = 0$$

$$\Rightarrow -2u(t) - \frac{d}{dt} (2u'(t)) = 0$$

$$\Rightarrow u''(t) + u(t) = 0$$

$$\Rightarrow u(t) = c_1 \cos t + c_2 \sin t$$

$$\text{for } u(0) = 0 \Rightarrow c_1 = 0 \Rightarrow u(t) = c_2 \sin t$$

For $u(1)$; use Transversality condition,

$$\text{i.e., } \left. \frac{\partial f}{\partial u'} \right|_{t=1} = 0$$

$$\Rightarrow 2u'(t)|_{t=1} = 0 \Rightarrow 2c_2 \cos t|_{t=1} = 0 \Rightarrow c_2 = 0$$

$$\therefore u(t) = 0$$

$$\Rightarrow I(y) = 0$$

$\therefore$ options (1) and (3) are correct.



44. A(44), B(41), C(45)

Let $u(x, t)$ be the solution of the initial value problem

$$u_{tt} - u_{xx} = 0, u(x, 0) = x^3, u_t(x, 0) = \sin x$$

Then $u(\pi, \pi)$ is

1. $4\pi^3$ 2. π^3 3. 0 4. 4

Sol. (1) Given differential equation is $u_{tt} = u_{xx}$

$$u(x, 0) = x^3$$

$$u_t(x, 0) = \sin x$$

By D'Alembert Solution:-

$$u(x, t) = \frac{1}{2} [(x+t)^3 + (x-t)^3] + \frac{1}{2} \int_{x-t}^{x+t} \sin S \cdot dS$$

$$u(x, t) = \frac{1}{2} [(x+t)^3 + (x-t)^3] + \frac{1}{2} [\cos(x-t) - \cos(x+t)]$$

Now

$$u(\pi, \pi) = \frac{1}{2} [(\pi+\pi)^3 + (\pi-\pi)^3] + \frac{1}{2} [\cos 0 - \cos 2\pi]$$

$$u(\pi, \pi) = 4\pi^3$$

$\therefore$ Option (1) is correct.

45. A(45), B(48), C(44)

The iterative method $x_{n+1} = g(x_n)$ for the solution of $x^2 - x - 2 = 0$ converges quadratically in a neighbourhood of the root $x = 2$ if $g(x)$ equals

1. $x^2 - 2$ 2. $(x-2)^2 - 6$
 3. $1 + \frac{2}{x}$ 4. $\frac{x^2 + 2}{2x - 1}$

Sol.(3) Given equation is $x^2 - x - 2 = 0$

Now for iterative method $x_{n+1} = g(x_n)$ for the solution of $x^2 - x - 2 = 0$ converges in neighbourhood of $x=2$ if $x = g(x)$ for $x^2 - x - 2 = 0$ and $|g'(x)| < 1$ for all x lying in the neighbourhood of $x = 2$.

$$\therefore x^2 - x - 2 = 0$$

$$\Rightarrow x^2 - x - 2 \Rightarrow x(x-1) = 2$$

$$\Rightarrow x = 1 + \frac{2}{x}$$

$$\Rightarrow g(x) = 1 + \frac{2}{x} \text{ and } g'(x) = \frac{-2}{x^2}$$

Also $|g'(x)| < 1 \forall x$ lying in neighbourhood of $x = 2$

$\therefore$ option (3) is correct

46. A(46), B(47), C(43)

Let $X = \{u \in C^1[0, 1] \mid u(0) = u(1) = 0\}$ and define $J: X \rightarrow \mathbb{R}$ by $J(u) = \int_0^1 e^{-u(x)^2} dx$. Then

1. J does not attain its infimum
 2. J attains its infimum at a unique $u \in X$

3. J attains its infimum at exactly two elements $u \in X$

4. J attains its infimum at infinitely many $u \in X$

Sol.(2) Given functional is $J(u) = \int_0^1 e^{-u(x)^2} dx$

By Euler - Lagrange Equation

$$\frac{\partial f}{\partial u} - \frac{d}{dx} \left(\frac{\partial f}{\partial u'} \right) = 0$$

$$\Rightarrow \frac{d}{dx} \left(e^{-u(x)^2} \cdot (-2u'(x)) \right) = 0$$

$$\Rightarrow u''(x) \cdot e^{-u(x)^2} + u'(x) e^{-u(x)^2} [-2u'(x)] = 0$$

$$\Rightarrow e^{-u(x)^2} u''(x) (1 - 2(u'(x))^2) = 0$$

$$\Rightarrow u''(x) = 0 \Rightarrow u(x) = ax + b$$

$$\text{For } u(0) = 0 = u(1) \Rightarrow u(x) = 0$$

$$J(u) = \int_0^1 e^0 dx = 1$$

$$\Rightarrow J(u) = 1 \text{ (Constant)}$$

$\therefore J$ attains its minimum at a unique $u \in X$ is $u(x) = 0$

$\therefore$ Option (2) is correct

47. A(47), B(46), C(42)

Let $u(x, t)$ be a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ in a rectangle } [0, \pi] \times [0, T]$$

subject to the boundary conditions $u(0, t) = u(\pi, t) = 0$, $0 \leq t \leq T$ and the initial condition $u(x, 0) = \phi(x)$, $0 \leq x \leq \pi$. If $f(x) = u(x, T)$, then which of the following is true for a suitable kernel $k(x, y)$?

1. $\int_0^\pi k(x, y) \phi(y) dy = f(x), 0 \leq x \leq \pi$

2. $\phi(x) + \int_0^\pi k(x, y) \phi(y) dy = f(x), 0 \leq x \leq \pi$

3. $\int_0^\pi k(x, y) \phi(y) dy = f(x), 0 \leq x \leq \pi$

4. $\phi(x) + \int_0^\pi k(x, y) \phi(y) dy = f(x), 0 \leq x \leq \pi$

Sol.(1) Given equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$u(0, t) = u(\pi, t) = 0$$

$$u(x, 0) = \phi(x)$$

$$u(x, T) = f(x)$$



Take sequence

$$a_{m_1} \quad a_{m_2} \quad a_{m_3} \dots$$

$$0 \quad 1 \quad 0$$

$$1 \quad 0 \quad 1$$

$$0 \quad 1 \quad 0$$

$$1 \quad 0 \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\liminf_{n \rightarrow \infty} a_{m_n} = \{0, 0, 0, \dots\}$$

$$R = \limsup_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} a_{m_k} = \limsup_{n \rightarrow \infty} \{0, 0, 0, \dots\} = 0$$

$$\text{Also, } \limsup_{n \rightarrow \infty} a_{m_n} = \{1, 1, 1, \dots\}$$

$$\therefore Q = \liminf_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} a_{m_k}$$

$$= \liminf_{n \rightarrow \infty} \{1, 1, 1, \dots\} = 1$$

Clearly $R < Q$

$\therefore$ (2) is incorrect

66. A(66), B(61), C(64)

$$\text{Evaluate } \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{n}{k^2 + n^2}$$

$$1. \frac{\pi}{2} \quad 2. \pi \quad 3. \frac{\pi}{8} \quad 4. \frac{\pi}{4}$$

$$\text{Sol. (4)} \quad \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{n}{k^2 + n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \frac{1}{\left(\frac{k}{n}\right)^2 + 1}$$

$$\text{Result } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \frac{1}{\left(\frac{k}{n}\right)^2 + 1} = \int_0^1 \frac{1}{x^2 + 1} dx$$

$$= \tan^{-1} x \Big|_0^1 = \frac{\pi}{4}$$

Correct answer (4)

74. A(74), B(77), C(74)

$$\text{Let } M = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and the} \right.$$

eigenvalues of A are in $\mathbb{Q}$. Then

1. M is empty

$$2. M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

3. If $A \in M$ then the eigenvalues of A are in $\mathbb{Z}$

4. If $A, B \in M$ are such that $AB=I$ then $\det A \in \{1, -1\}$

Sol. (3,4)

$$\text{Given } M = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right.$$

and the eigen values of A are in $\mathbb{Q}$

For option (1):

$$\text{Clearly, } A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in M$$

$\therefore$ M is non-empty

For option (2):

Clearly,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

$$\text{But } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \notin \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

and eigen values of A are in $\mathbb{Q}$

$\therefore$ option (2) is incorrect

For option (3):

If $A \in M$, then eigen values of A are in $\mathbb{Z}$

$\therefore$ if $\alpha_1, \alpha_2 \in \mathbb{Q} \cap \mathbb{Z}$ are eigenvalues of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M, \text{ then } a, b, c, d \in \mathbb{Z} \text{ is not}$$

possible which is contradiction]

$\therefore$ option (3) is correct.

For option (4) : Let $A, B \in M$ such that $AB=I$

$$\Rightarrow |AB| = 1 \Rightarrow |A| \cdot |B| = 1 \Rightarrow |A|, |B| \in \{-1, 1\}$$

($\because$ |A| is an integral value $\forall A \in M$)

Thus, option (4) is correct.

75. A(75), B(76), C(73)

Let A be a 3×3 matrix with real entries. Identify the correct statements.

1. A is necessarily diagonalizable over $\mathbb{R}$

2. If A has distinct real eigenvalues then

it is diagonalizable over $\mathbb{R}$

3. If A has distinct eigenvalues then it is

diagonalizable over $\mathbb{C}$

4. If all eigenvalues of A are non-zero

then it is diagonalizable over $\mathbb{C}$

Sol. (2,3) correct answer (2), (3)

$$\text{For option (1): } A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not}$$

diagonalizable



$$A_3 = \left[\frac{999}{3} \right] - \left[\frac{999}{3^2} \right] + \left[\frac{999}{3^3} \right] - \left[\frac{999}{3^4} \right]$$

$$+ \left[\frac{999}{3^5} \right] - \left[\frac{999}{3^6} \right]$$

$$= 333 - 111 + 37 - 12 + 4 - 1$$

$$A_3 = 250$$

So, option (1) is correct

For option (2):

$$A_5 = \left[\frac{999}{5} \right] - \left[\frac{999}{5^2} \right] + \left[\frac{999}{5^3} \right] - \left[\frac{999}{5^4} \right]$$

$$= 199 - 39 + 7 - 1 = 166$$

So, option (2) is incorrect

For option (3):

$$A_7 = \left[\frac{999}{7} \right] - \left[\frac{999}{7^2} \right] + \left[\frac{999}{7^3} \right]$$

$$= 142 - 20 + 2 = 124$$

So, option (3) is correct

For option (4):

$$A_{11} = \left[\frac{999}{11} \right] - \left[\frac{999}{11^2} \right]$$

So, option (4) is correct

91. A(91), B(96), C(102)

Consider a system of first order differential

$$\text{equations } \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) + y(t) \\ -y(t) \end{bmatrix}$$

The solution space is spanned by

1. $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$ and $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cosh t \\ e^{-t} \end{bmatrix}$

3. $\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$ and $\begin{bmatrix} \sinh t \\ e^{-t} \end{bmatrix}$

4. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} e^t - \frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix}$

Sol.(3,4) Given system of equations is

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t) + y(t) \\ -y(t) \end{bmatrix}$$

$$\Rightarrow x'(t) = x(t) + y(t) \quad \dots(1)$$

$$y'(t) = -y(t) \quad \dots(2)$$

From (2), $y'(t) + y(t) = 0$

$$\Rightarrow (D + 1)y(t) = 0 \Rightarrow y(t) = c e^{-t}$$

Now from (1) $x'(t) - x(t) = c e^{-t}$

$$\Rightarrow x(t) = c_1 e^t + \left(\frac{-1}{2} \right) c e^{-t}$$

$$\therefore \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \end{bmatrix} c_1 + \left[\frac{-1}{2} \frac{e^{-t}}{e^{-t}} \right] c$$

$\therefore$ Solution space is spanned by $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and

$$\begin{bmatrix} \frac{e^{-t}}{2} \\ e^{-t} \end{bmatrix}$$

$\therefore$ Option (1) is incorrect

For option (2): $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cosh t \\ e^{-t} \end{bmatrix}$

Can't span solution space

$\therefore \begin{bmatrix} \cosh t \\ e^{-t} \end{bmatrix}$ doesn't satisfy given

differential equation.

$\therefore$ Option (2) is incorrect

For option (3):

$$\therefore \begin{bmatrix} \sinh t \\ e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^t - e^{-t}}{2} \\ e^{-t} \end{bmatrix} \text{ and } \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$$

is again a linear combination of

$$\begin{bmatrix} e^t \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix} \text{ and also } \begin{bmatrix} \sinh t \\ e^{-t} \end{bmatrix} \&$$

$$\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \text{ satisfy given system}$$

$\therefore$ option (3) is also correct

Also, $\begin{bmatrix} e^t - \frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix}$ is a linear combination

$$\text{of } \begin{bmatrix} e^t \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix}$$

$\therefore$ options (3), (4) are correct.

92. A(92), B(95), C(101)

Consider the differential equation

$$\frac{d^2 y}{dx^2} - 2 \tan x \frac{dy}{dx} - y = 0 \text{ defined on}$$



QUESTIONS WITH SOLUTIONS

CSIR UGC NET DECEMBER- 17

MATHEMATICAL SCIENCES

PART - C

61. A(61), B(66), C(69)
 Which of the following are convergent ?

1. $\sum_{n=1}^{\infty} n^2 2^{-n}$ 2. $\sum_{n=1}^{\infty} n^{-2} 2^n$
 3. $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ 4. $\sum_{n=1}^{\infty} \frac{1}{n \log(1+1/n)}$

Sol. (1) Correct answer (1)

(1) $\sum_{n=1}^{\infty} n^2 2^{-n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

We have $\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n^2}{2^n} \right)^{\frac{1}{n}}$

$\lim_{n \rightarrow \infty} \frac{(n^2)^{\frac{1}{n}}}{2} = \frac{1}{2} < 1$

So, by root test, series is convergent

(2) $\sum_{n=1}^{\infty} n^{-2} 2^n = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$

By Root-test

$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = 2 > 1$

By root - test, Series is divergent

(3) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Use integral - test

Let $I = \int_2^{\infty} \frac{1}{x \log x} dx$

(By changing of variables and substitution)

$I = \int_{\log 2}^{\infty} \frac{1}{y} dy$

By p-test $\int_{\log 2}^{\infty} \frac{1}{y} dy$

is divergent

(4) $\sum_{n=1}^{\infty} \frac{1}{n \log \left(1 + \frac{1}{n} \right)}$

$a_n = \frac{1}{n \log \left(1 + \frac{1}{n} \right)}$

$= \lim_{n \rightarrow \infty} \frac{1}{n \left[\frac{1}{n} - \frac{1}{2n^2} + \dots \right]} = 1$

$\lim_{n \rightarrow \infty} a_n \neq 0$

$\Rightarrow$ Series is not convergent

62. A(62), B(65), C(68)

Let $a_{mn}, m \geq 1, n \geq 1$ be a double array of real numbers. Define

$P = \liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} a_{mn},$

$Q = \liminf_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_{mn},$

$R = \limsup_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} a_{mn},$

$S = \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_{mn}$

Which of the following statements are necessarily true ?

1. $P \leq Q$ 2. $Q \leq R$ 3. $R \leq S$ 4. $P \leq S$

Sol. (1,3,4) correct answer (1), (3) and (4)

Given sequence of real numbers is

$\{a_{m1}, a_{m2}, a_{m3}, \dots\}$ i.e.,

$\{a_{11}, a_{12}, a_{13}, \dots,$

$a_{21}, a_{22}, a_{23}, \dots,$

$a_{31}, a_{32}, a_{33}, \dots\}$

$\liminf_{m \rightarrow \infty} a_{mn} = \left\{ \liminf_{m \rightarrow \infty} a_{m1}, \liminf_{m \rightarrow \infty} a_{m2}, \dots \right\}$

$\limsup_{m \rightarrow \infty} a_{mn} = \left\{ \limsup_{m \rightarrow \infty} a_{m1}, \limsup_{m \rightarrow \infty} a_{m2}, \dots \right\}$

Clearly $\liminf_{m \rightarrow \infty} a_{mn} \leq \limsup_{m \rightarrow \infty} a_{mn} \quad \forall i$

Implies $\liminf_{m \rightarrow \infty} a_{mn} \leq \limsup_{m \rightarrow \infty} a_{mn} \quad \forall n \in \mathbb{N}$

that implies

$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} a_{mn} \leq \liminf_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_{mn}$

$\Rightarrow P \leq Q$

$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} a_{mn} \leq \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_{mn}$

$\Rightarrow P \leq S$

Also $\liminf_{m \rightarrow \infty} a_{mn} \leq \limsup_{m \rightarrow \infty} a_{mn}$

$\Rightarrow \limsup_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} a_{mn} \leq \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} a_{mn}$

$\Rightarrow R \leq S$

So (1), (3), (4) is correct

(2) may not hold



∴ Solution of Heat Equation can be written in the form of Integral by using Green's function. As if $G(x, y, t)$ be the Green's function of given differential equation then solution is

$$u(x, t) = \int_0^x G(x, y, t) \phi(y) dy$$

∴ at $t = T$

$$u(x, T) = \int_0^x G(x, y, T) \phi(y) dy$$

$$\Rightarrow f(x) = \int_0^x k(x, y) \phi(y) dy$$

where $k(x, y) = G(x, y, T)$

∴ Option (1) is correct

49. A(49), B(54), C(60)

There are five empty boxes. Balls are placed independently one after another in randomly selected boxes. The probability that the fourth ball is the first to be placed in an occupied box equals

1. $\frac{4}{5} \left(\frac{3}{5}\right)^2$ 2. $\left(\frac{3}{5}\right)^3$ 3. $\left(\frac{3}{5}\right)^2$ 4. $\left(\frac{4}{5}\right) \left(\frac{3}{5}\right)$

Sol.(1) 1st ball can be placed in any one of the five boxes

∴ Number of choices for 1st ball = 5

2nd ball can be placed in any one of the remaining 4 boxes

∴ Number of choices for 2nd ball = 4

Number of choices for 3rd ball = 3

Now, 4th ball is to be placed in any one of the occupied boxes

∴ Number of choices for 4th ball = 3

$$\therefore \text{Required probability} = \frac{5 \times 4 \times 3 \times 3}{5^4} = \frac{4}{5} \left(\frac{3}{5}\right)^2$$

59. A(59), B(56), C(50)

Let X and Y be independent exponential random variables. If $E[X] = 1$ and $E[Y] = \frac{1}{2}$ then

$P(X > 2Y | X > Y)$ is

1. $\frac{1}{2}$ 2. $\frac{1}{3}$ 3. $\frac{2}{3}$ 4. $\frac{3}{4}$

Sol.(4)



Since X and Y are exponential random variables with mean = 1 and 1/2 respectively. i.e. $E(X) = 1$ and $E(Y) = 1/2$

∴ marginal p.d.f of X and Y are given by

$$f_X(x) = e^{-x}, x > 0, f_Y(y) = 2e^{-2y}, y > 0$$

Since X and Y are independent

$$\therefore f_{XY}(x, y) = f_X(x) \cdot f_Y(y) = e^{-x} \cdot 2e^{-2y}, x > 0, y > 0$$

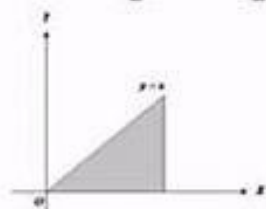
$$\begin{aligned} \text{Now, } P(X > 2Y | X > Y) &= \frac{P[(X > 2Y) \cap (X > Y)]}{P(X > Y)} \\ &= \frac{P(X > 2Y)}{P(X > Y)} \quad \dots (1) \end{aligned}$$

$$\text{Now, } P(X > 2Y) = \int_0^{\infty} \int_0^{\frac{x}{2}} e^{-x} \cdot 2e^{-2y} dy dx$$

$$= 2 \int_0^{\infty} e^{-x} \left[\frac{e^{-2y}}{-2} \right]_0^{\frac{x}{2}} dx = \int_0^{\infty} e^{-x} [-e^{-x} + 1] dx$$

$$= \int_0^{\infty} (e^{-x} - e^{-2x}) dx = \left[\frac{e^{-x}}{(-1)} - \frac{e^{-2x}}{(-2)} \right]_0^{\infty}$$

$$= (0 + 1) + \frac{1}{2} (0 - 1) = \frac{1}{2}$$



$$\text{and } P(X > Y) = \int_0^{\infty} \int_0^x e^{-x} \cdot 2e^{-2y} dy dx$$

$$= 2 \int_0^{\infty} e^{-x} \left[\frac{e^{-2y}}{-2} \right]_0^x dx$$

$$= \int_0^{\infty} (e^{-x} - e^{-3x}) dx = \left[\frac{e^{-x}}{(-1)} - \frac{e^{-3x}}{(-3x)} \right]_0^{\infty}$$

$$= (0 + 1) + \frac{1}{3} (0 - 1) = \frac{2}{3}$$

$$\text{Put in (1), } P(X > 2Y | X > Y) = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

∴ option (4) is correct.



31. A(31), B(28), C(22)

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ \beta \end{bmatrix}$. Then the

system $AX = b$ over the real numbers has

1. no solution whenever $\beta \neq 7$.
2. an infinite number of solutions whenever $\alpha \neq 2$.
3. an infinite number of solutions if $\alpha = 2$ and $\beta \neq 7$
4. a unique solution if $\alpha \neq 2$

Sol.(4) $[A;b] = \begin{bmatrix} 1 & -1 & 1 & : & 1 \\ 1 & 1 & 1 & : & 3 \\ 2 & 3 & \alpha & : & \beta \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & -1 & 1 & : & 1 \\ 0 & 2 & 0 & : & 2 \\ 0 & 5 & \alpha-2 & : & \beta-2 \end{bmatrix}$

Thus given system has unique solution if $\alpha-2 \neq 0$ i.e. $\alpha \neq 2$

Clearly, option (4) is correct.

32. A(32), B(27), C(21)

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \in M_2(\mathbb{R})$ and $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$

be the bilinear map defined by $\phi(v, w) = v^T A w$. Choose the correct statement from below:

1. $\phi(v, w) = \phi(w, v)$ for all $v, w \in \mathbb{R}^2$
2. there exists nonzero $v \in \mathbb{R}^2$ such that $\phi(v, w) = 0$ for all $w \in \mathbb{R}^2$
3. there exists a 2×2 symmetric matrix B such that $\phi(v, v) = v^T B v$ for all $v \in \mathbb{R}^2$
4. the map $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$\psi \left(\begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} \right) = \phi \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)$ is linear

Sol.(3) Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Let $v = (v_1, v_2)^T$, $w = (w_1, w_2)^T$

For option(1): $\phi(v, w) = v^T A w$

$= [v_1 \ v_2] \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$= [v_1 + 4v_2 \ 2v_1 + 3v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$= v_1 w_1 + 4v_2 w_1 + 2v_1 w_2 + 3v_2 w_2$

& $\phi(w, v) = w^T A v$

$= w_1 v_1 + 4w_2 v_1 + 2w_1 v_2 + 3w_2 v_2$

Clearly, $\phi(v, w) \neq \phi(w, v)$

$\therefore$ option(1) is not correct.

For option(2):

$\phi(v, w) = v_1(w_1 + 4w_2) + v_2(2w_1 + 3w_2)$

$\phi(v, w) = 0$

$\Rightarrow v_1(w_1 + 4w_2) + v_2(2w_1 + 3w_2) = 0$

Since $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = -5 \neq 0$

$\therefore v_1(w_1 + 4w_2) + v_2(2w_1 + 3w_2) = 0$

$\Rightarrow v_1 = 0, v_2 = 0$

Option (2) is incorrect.

For option(3):

for $v = (v_1, v_2)^T$

$\phi(v, v) = v^T B v = v_1^2 + 6v_1 v_2 + 3v_2^2$

$= [v_1 \ v_2] \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Thus, $\exists$ symmetric matrix $B = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}$ such

that $\phi(v, u) = v^T B u$

$\therefore$ option (3) is correct.

For option (4): it can be easily verified that

$\phi \left(\alpha \begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} \right) \neq \alpha \phi \left(\begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix} \right)$

$\therefore \psi : \mathbb{R}^4 \rightarrow \mathbb{R}$ is not linear

$\therefore$ option (4) is incorrect.

33. A(33), B(36), C(40)

The function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$f(z) = e^z + e^{-z}$ has

1. finitely many zeros
2. no zeros
3. only real zeros
4. has infinitely many zeros



$\Rightarrow \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \frac{1}{\alpha_4}$ are eigen values of A^{-1}

$\Rightarrow \frac{1}{\alpha_1^2}, \frac{1}{\alpha_2^2}, \frac{1}{\alpha_3^2}, \frac{1}{\alpha_4^2}$ are eigen values of A^{-2}

Clearly, A^{-2} is positive definite as $\frac{1}{\alpha_i^2} > 0$

$\forall i=1,2,3,4$

$\therefore$ Option (3) is correct

For option(4):

$$\text{For } A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\text{Exp}(pA)$ can't be positive definite for any positive integer p

$\therefore$ option (4) is incorrect.

79. A(79), B(84), C(90)

Let f be an entire function. Consider $A = \{z \in \mathbb{C} \mid f^{(n)}(z) = 0 \text{ for some positive integer } n\}$. Then

1. if $A = \mathbb{C}$, then f is a polynomial
2. if $A = \mathbb{C}$, then f is a constant function
3. if A is uncountable, then f is a polynomial
4. If A is uncountable, then f is a constant function

Sol. (1,3)

For $f(z) = z$

$f^{(n)}(z) = 0 \forall z \in \mathbb{C}$ i.e. $A = \mathbb{C}$ for $f(z) = z$

Thus options (2) and (4) are incorrect.

Further, if A is uncountable, i.e., $f^{(n)}(z) = 0$ at uncountably many points for some positive integer n

$\Rightarrow f^{(n)}(z) = 0 \forall z \in \mathbb{C}$

[because zeros of a non-constant entire function are countable]

$\Rightarrow A = \mathbb{C}$

Thus if option (3) is correct, then (1) is also correct and vice versa

80. A(80), B(83), C(89)

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function and let u be the real part of f and v the imaginary part of f . Then, for

$x, y \in \mathbb{R}$, $|f'(x+iy)|^2$ is equal to

1. $u_x^2 + u_y^2$
2. $u_x^2 + v_x^2$
3. $v_x^2 + u_y^2$
4. $v_x^2 + v_y^2$

Sol. (1,2,3,4)

Let $f(z) = u+iv$

As $f(z)$ is analytic

$\therefore u_x = v_y, u_y = -v_x$

Thus $f'(z) = u_x + iv_x$... (1)

or

$f'(z) = u_y + iv_y$... (2)

also, $v_x = -u_y \Rightarrow f'(z) = u_x - iv_y$

And $u_x = v_y$

$\Rightarrow f'(z) = v_y + iv_x$... (3)

and $u_x = v_y$

$\Rightarrow f'(z) = v_y + iv_x$... (4)

By(1), $|f'(z)|^2 = u_x^2 + v_x^2$

By(2), $|f'(z)|^2 = u_y^2 + v_y^2$

By(3), $|f'(z)|^2 = u_x^2 + u_y^2$

By(4), $|f'(z)|^2 = u_x^2 + v_y^2$

Thus options (1),(2),(3) and (4) are correct.

81. A(81), B(82), C(88)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$, where $a_0, \dots, a_{n-1}$ are complex numbers and let $q(z) = 1 + a_{n-1}z + \dots + a_0z^n$. If $|p(z)| \leq 1$ for all z with $|z| \leq 1$, then

1. $|q(z)| \leq 1$ for all z with $|z| \leq 1$

2. $q(z)$ is a constant polynomial

3. $p(z) = z^n$ for all complex numbers z

4. $p(z)$ is a constant polynomial

Sol. (1,2,3)

Given $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$

such that $|p(z)| \leq 1 \forall z$ with $|z| \leq 1$

This is possible only if $p(z) = z^n$

Also $q(z) = 1 + a_{n-1}z + a_{n-2}z^2 + \dots + a_0z^n$

$$= z^n p\left(\frac{1}{z}\right) = z^n \times \frac{1}{z^n} = 1$$

$\Rightarrow q(z)$ is constant polynomial &

$|q(z)| \leq 1 \forall z$ with $|z| \leq 1$

Thus, options (1),(2),(3) are correct and (4) is incorrect.

82. A(82), B(81), C(87)

Let f be a non-constant entire function and let E be the image of f . Then



26. A(26), B(21), C(27)

Which of the following is necessarily true for a function $f: X \rightarrow Y$?

1. if f is injective, then there exists $g: Y \rightarrow X$ such that $f(g(y)) = y$ for all $y \in Y$
2. if f is surjective, then there exists $g: Y \rightarrow X$ such that $f(g(y)) = y$ for all $y \in Y$
3. if f is injective and Y is countable then X is finite
4. if f is surjective and X is uncountable then Y is countably infinite

Sol.(2) Correct answer (2)

For Option (1):

Let $X = \{0, 1\}$, $Y = \{0, 1, 2\}$

Let $f: X \rightarrow Y$ be defined as $f(x) = x$

Clearly f is injective but for any function $g: Y \rightarrow X$ there does not exist pre image of 2 under $f \circ g$.

For option (2):

As f is surjective

$\therefore f: X \rightarrow Y$ for any $y \in Y$ there exists atleast one $x \in X$ for which

$f(x) = y$

Define $g: Y \rightarrow X$ in such a way that $g(y) = x$

$f \circ g(y) = f(x) = y$

So, (2) is correct

For option (3):

Take $X = \mathbb{N}$, $Y = \mathbb{N}$

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(x) = x$, clearly f is injective and Y is countable but X is also countable

For option (4):

Take $X = \mathbb{R}$, $Y = \mathbb{R}$

Let $f: X \rightarrow Y$ defined by $f(x) = x$

clearly f is surjective and X is uncountable but Y is also uncountable

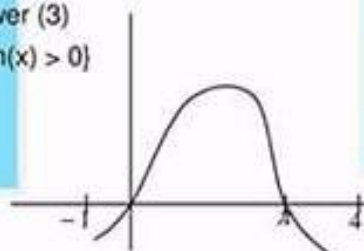
28. A(28), B(31), C(25)

Let $S = \{x \in [-1, 4] \mid \sin(x) > 0\}$. Which of the following is true?

1. $\inf(S) < 0$
2. $\sup(S)$ does not exist
3. $\sup(S) = \pi$
4. $\inf(S) = \pi/2$

Sol. (3) Correct answer (3)

$S = \{x \in [-1, 4] \mid \sin(x) > 0\}$



$S = [0, \pi]$

$\sup(S) = \pi$, $\inf(S) = 0$

Correct answer (3)

29. A(29), B(30), C(24)

Let A be a real symmetric matrix and $B = I + iA$, where $i^2 = -1$. Then

1. B is invertible if and only if A is invertible
2. all eigenvalues of B are necessarily real
3. $B - I$ is necessarily invertible
4. B is necessarily invertible

Sol. (4) Since A is real symmetric matrix

$\therefore$ Eigen values of A are purely real

Eigen values of $B = I + iA$ = Eigen values of $I + iA$ (Eigen values of A)

Clearly, Eigen values of B are real only if A has zero eigen values

Thus option (2) is incorrect

$B - I = iA \Rightarrow |B - I| = |iA| = |A|$

$B - I$ is invertible only if A is invertible

$\therefore$ option (3) is incorrect

Further B is singular iff '0' is eigen value of B

$\Rightarrow 1 + i\alpha = 0$ [where ' α ' is eigen value of A]

$\Rightarrow \alpha = \frac{-1}{i} = i$, which is not possible as ' α ' is

purely real

Thus, B is always non-singular and hence invertible.

Thus option (4) is correct and (1) is incorrect

30. A(30), B(29), C(23)

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Then the smallest positive

integer n such that $A^n = I$ is

1. 1
2. 2
3. 4
4. 6

Sol.(4) Given $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ The least positive integer for which $A^n = I$ is

$n = 6$



∴ option (1) is not correct
For option (4):

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Clearly, all eigenvalues of A are non-zero

But A is not diagonalizable

[∵ A.M. of '3' = 3 but G.M. of '3' = 1]

∴ option (4) is not correct

Result:

(i) If a matrix has distinct real eigenvalues then it is diagonalizable over $\mathbb{R}$

(ii) if A matrix has distinct eigenvalues then it is diagonalizable over $\mathbb{C}$

Thus options (2) and (3) are correct.

76. A(76), B(75), C(72)

Let V be the vector space over $\mathbb{C}$ of all polynomials in a variable x of degree at most 3. Let $D: V \rightarrow V$ be the linear operator given by differentiation with respect to x. Let A be the matrix of D with respect to some basis for V. Which of the following are true?

1. A is a nilpotent matrix

2. A is a diagonalizable matrix

3. The rank of A is 2

4. The Jordan canonical form of A is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol.(1,4)

Let $B = \{1, x, x^2, x^3\}$ be the standard basis of V

∴ $D(1) = 0$; $D(x) = 1$

$D(x^2) = 2x$; $D(x^3) = 3x^2$

∴ matrix of D w.r.t. basis B =

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly A is nilpotent

[∵ upper triangular matrix with diagonal entries '0' is nilpotent]

∴ option (1) is correct

Also, A nilpotent matrix is diagonalizable iff it is zero matrix

∴ option (2) is incorrect

Clearly, $\rho(A) = 3$

∴ option (3) is incorrect

Further, only eigenvalues of A is "0" with algebraic multiplicity = 4

And geometric multiplicity = $4 - \rho(A - 0 \cdot I)$
 $= 4 - 3 = 1$

∴ Jordan canonical form of A is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ option (4) is correct

77. A(77), B(74), C(71)

For every 4×4 real symmetric non-singular matrix A, there exists a positive integer p such that

1. $pI + A$ is positive definite

2. A^p is positive definite

3. A^{-p} is positive definite

4. $\exp(pA) - I$ is positive definite

Sol. (1,2,3)

Let A be a 4×4 real symmetric non-singular matrix and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be eigen values of A

Clearly $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non-zero and purely real.

[∵ Eigen values of a symmetric matrix are purely real]

For option(1):

Choose $p > |\alpha_i| \forall i = 1, 2, 3, 4$

∴ Eigen values of $(pI + A)$ = Eigen values of pI + eigen values of A

$\Rightarrow p + \alpha_1, p + \alpha_2, p + \alpha_3, p + \alpha_4$ are eigen values of $pI + A$

Clearly, $p + \alpha_i > 0 \forall i = 1, 2, 3, 4$

Thus, $pI + A$ is positive definite

For option (2): As $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are eigen values of A

$\Rightarrow \alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$ are eigen values of A^2

Clearly $\alpha_i^2 > 0 \forall i = 1, 2, 3, 4$

Thus, A^2 is positive definite

∴ option (2) is correct.

For option(3): As $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are eigen values of A



100. A(100), B(99), C(93)

Consider a non-zero, real-valued polynomial function $p(x) = a_0 + a_1x + a_2x^2$ of degree at most 2. Let $y=y(x)$ be a solution of the integral equation

$$y = p(x) + \int_0^x y(t) \sin(x-t) dt$$

Which of the following statements are necessarily correct?

1. $y(x)$ is a polynomial function of degree ≤ 2
2. $y(x)$ is a polynomial function of degree ≤ 4
3. if $a_1 \neq 0$ and $a_0 + 2a_2 = 0$, then $y'(0) = 0$
4. If $a_1 \neq 0$ and $a_0 + 2a_2 = 0$, then $y''(0) = 0$

Sol. (2,4) Given Integral Equation is

$$y = p(x) + \int_0^x \sin(x-t) y(t) dt$$

where $p(x) = a_0 + a_1x + a_2x^2$

Apply Laplace transformation on both sides ;

$$\Rightarrow \bar{y} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \left(\frac{1}{s^2+1} \right) \bar{y}$$

where $\bar{y} = L(y(x))$

$$\Rightarrow \bar{y} \left[1 - \frac{1}{s^2+1} \right] = \frac{a_0 s^2 + a_1 s + 2a_2}{s^3}$$

$$\Rightarrow \bar{y} = \left(\frac{a_0 s^2 + a_1 s + 2a_2}{s^3} \right) \times \frac{(s^2+1)}{s^2}$$

$$\Rightarrow \bar{y} = \frac{a_0 s^4 + a_1 s^3 + 2a_2 s^2 + a_0 s^2 + a_1 s + 2a_2}{s^5}$$

$$\Rightarrow \bar{y} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \frac{a_0}{s^3} + \frac{a_1}{s^4} + \frac{2a_2}{s^5}$$

Now taking Laplace Inverse on both sides:-

$$y(x) = a_0 + a_1 x + a_2 x^2 + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2} \right) x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$\therefore y(x)$ is polynomial of degree ≤ 4

$\therefore$ option (2) is correct

Now if $a_1 \neq 0$ and $a_0 + 2a_2 = 0$

$$\text{then } y(x) = a_0 + a_1 x + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$$\Rightarrow y'(x) = a_1 + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$$

$$\Rightarrow y'(0) = a_1 \neq 0$$

$$y''(x) = a_1 x + a_2 x^2$$

$$\Rightarrow y''(0) = 0$$

$\therefore$ options (2) and (4) are correct.

101. A(101), B(98), C(92)

Let ϕ be the solution of the integral equation

$$\frac{1}{2} \phi(x) - \int_0^1 e^{x-y} \phi(y) dy = x^2; 0 \leq x \leq 1. \text{ Then}$$

- | | |
|-----------------------------|-----------------------------|
| 1. $\phi(0) = 20e^{-1} - 8$ | 2. $\phi(0) = 20e^{-8}$ |
| 3. $\phi(1) = 22 - 8e$ | 4. $\phi(1) = 22 - 8e^{-1}$ |

Sol. (1,3) Given Integral Equation is

$$\frac{1}{2} \phi(x) - \int_0^1 e^{x-y} \phi(y) dy = x^2; 0 \leq x \leq 1$$

$$\Rightarrow \phi(x) = 2x^2 + 2 \int_0^1 e^{x-y} \phi(y) dy$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x \int_0^1 e^{-y} \phi(y) dy$$

$$\text{Consider } c = \int_0^1 e^{-y} \phi(y) dy$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x c$$

$$\text{Now } c = \int_0^1 e^{-y} (2y^2 + 2e^y c) dy$$

$$= \int_0^1 [2y^2 e^{-y} + 2c] dy$$

$$c = 2 \left[\left[y^2 \frac{e^{-y}}{(-1)} \right]_0^1 - \int_0^1 2y \frac{e^{-y}}{(-1)} dy \right] + [2cy]_0^1$$

$$c = 2 \left[\frac{e^{-1}}{(-1)} + 2 \int_0^1 y e^{-y} dy \right] + [2c]$$

$$c = -2e^{-1} + 4 \left[\frac{y e^{-y}}{(-1)} \right]_0^1 - 4 \left[e^{-y} \right]_0^1 + 2c$$

$$\Rightarrow c = -2e^{-1} - 4e^{-1} - 4e^{-1} + 4 + 2c$$

$$c = 10e^{-1} - 4$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x (10e^{-1} - 4)$$

$$\Rightarrow \phi(0) = 20e^{-1} - 8 \text{ and } \phi(1) = 22 - 8e$$

$\therefore$ options (1) and (3) are correct



100. A(100), B(99), C(93)

Consider a non-zero, real-valued polynomial function $p(x) = a_0 + a_1x + a_2x^2$ of degree at most 2. Let $y=y(x)$ be a solution of the integral equation

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2. $y(x)$ is a polynomial function of degree ≤ 4
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4. If $a_1 \neq 0$ and $a_0 + 2a_2 = 0$, then $y''(0) = 0$

Sol. (2,4) Given Integral Equation is

$$y = p(x) + \int_0^x \sin(x-t) y(t) dt$$

where $p(x) = a_0 + a_1x + a_2x^2$

Apply Laplace transformation on both sides ;

$$\Rightarrow \bar{y} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \left(\frac{1}{s^2+1} \right) \bar{y}$$

where $\bar{y} = L(y(x))$

$$\Rightarrow \bar{y} \left[1 - \frac{1}{s^2+1} \right] = \frac{a_0 s^2 + a_1 s + 2a_2}{s^3}$$

$$\Rightarrow \bar{y} = \left(\frac{a_0 s^2 + a_1 s + 2a_2}{s^3} \right) \times \frac{(s^2+1)}{s^2}$$

$$\Rightarrow \bar{y} = \frac{a_0 s^4 + a_1 s^3 + 2a_2 s^2 + a_0 s^2 + a_1 s + 2a_2}{s^5}$$

$$\Rightarrow \bar{y} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \frac{a_0}{s^3} + \frac{a_1}{s^4} + \frac{2a_2}{s^5}$$

Now taking Laplace Inverse on both sides:-

$$y(x) = a_0 + a_1 x + a_2 x^2 + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2} \right) x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$\therefore y(x)$ is polynomial of degree ≤ 4

$\therefore$ option (2) is correct

Now if $a_1 \neq 0$ and $a_0 + 2a_2 = 0$

$$\text{then } y(x) = a_0 + a_1 x + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4$$

$$\Rightarrow y'(x) = a_1 + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$$

$$\Rightarrow y'(0) = a_1 \neq 0$$

$$y''(x) = a_1 x + a_2 x^2$$

$$\Rightarrow y''(0) = 0$$

$\therefore$ options (2) and (4) are correct.

101. A(101), B(98), C(92)

Let ϕ be the solution of the integral equation

$$\frac{1}{2} \phi(x) - \int_0^1 e^{x-y} \phi(y) dy = x^2; 0 \leq x \leq 1. \text{ Then}$$

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Sol. (1,3) Given Integral Equation is

$$\frac{1}{2} \phi(x) - \int_0^1 e^{x-y} \phi(y) dy = x^2; 0 \leq x \leq 1$$

$$\Rightarrow \phi(x) = 2x^2 + 2 \int_0^1 e^{x-y} \phi(y) dy$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x \int_0^1 e^{-y} \phi(y) dy$$

$$\text{Consider } c = \int_0^1 e^{-y} \phi(y) dy$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x c$$

$$\text{Now } c = \int_0^1 e^{-y} (2y^2 + 2e^y c) dy$$

$$= \int_0^1 [2y^2 e^{-y} + 2c] dy$$

$$c = 2 \left[\left[y^2 \frac{e^{-y}}{(-1)} \right]_0^1 - \int_0^1 2y \frac{e^{-y}}{(-1)} dy \right] + [2cy]_0^1$$

$$c = 2 \left[\frac{e^{-1}}{(-1)} + 2 \int_0^1 y e^{-y} dy \right] + [2c]$$

$$c = -2e^{-1} + 4 \left[\frac{y e^{-y}}{(-1)} \right]_0^1 - 4 \left[e^{-y} \right]_0^1 + 2c$$

$$\Rightarrow c = -2e^{-1} - 4e^{-1} - 4e^{-1} + 4 + 2c$$

$$c = 10e^{-1} - 4$$

$$\Rightarrow \phi(x) = 2x^2 + 2e^x (10e^{-1} - 4)$$

$$\Rightarrow \phi(0) = 20e^{-1} - 8 \text{ and } \phi(1) = 22 - 8e$$

$\therefore$ options (1) and (3) are correct



1. E is an open set
2. $E \cap \{z: |z| < 1\}$ is empty
3. $E \cap \mathbb{R}$ is non-empty
4. E is a bounded set

Sol. (1,3)

since 'f' is non-constant entire function

$\therefore$ E = image of f is either $\mathbb{C}$ or $\mathbb{C} - \{a\} (a \in \mathbb{C})$

Clearly, E is open and unbounded set

Also, $E \cap \{z: |z| < 1\} \neq \emptyset$

$E \cap \mathbb{R} \neq \emptyset$

Thus, options (1) and (3) are correct and (2), (4) are incorrect.

84. A(84), B(79), C(85)

Let G be a finite abelian group and $a, b \in G$ with $\text{order}(a) = m$, $\text{order}(b) = n$. Which of the following are necessarily true?

1. $\text{order}(ab) = mn$
2. $\text{order}(ab) = \text{lcm}(m, n)$
3. there is an element of G whose order is $\text{lcm}(m, n)$
4. $\text{order}(ab) = \text{gcd}(m, n)$

Sol. (3)

Take $G = \{1, -1\}$ under multiplication

Let $a = -1, b = -1$

So, $O(a) = 2 = m, O(b) = 2 = n$

Then $ab = (-1) \cdot (-1) = 1$

$\Rightarrow O(ab) = O(1) = 1$

which is not equal to $m \cdot n, \text{gcd}(m, n), \text{lcm}(m, n)$

i.e. $O(ab) = 1 \neq 4 = 2 \cdot 2$

$1 \neq 2 = \text{gcd}(2, 2)$

$1 \neq 2 = \text{lcm}(2, 2)$

So, options (1), (2) and (4) are not correct

So, option (3) is correct

85. A(85), B(90), C(84)

Which of the following rings are principal ideal domains (PIDs)?

1. $\mathbb{Z}[X]/\langle X^2 + 1 \rangle$
2. $\mathbb{Z}[X]$
3. $\mathbb{C}[X, Y]$
4. $\mathbb{R}[X, Y]/\langle X^2 + 1, Y \rangle$

Sol. (1,4) For option (1):

$$\frac{\mathbb{Z}[X]}{\langle X^2 + 1 \rangle} \cong \mathbb{Z}[i]$$

$\mathbb{Z}[i]$ is an E.D.

$\therefore \exists$ a map $d: \mathbb{Z}[i] \setminus \{0\} \rightarrow \mathbb{Z}^+ \cup \{0\}$ defined as $d(a + ib) = a^2 + b^2$

such that it follows two conditions:

(1) $d((a + ib)(c + id)) \geq d(a + ib)$

$$\forall a + ib, c + id \in \mathbb{Z}[i]$$

$$(2) \forall a + ib \in \mathbb{Z}[i], c + id \in \mathbb{Z}[i] \setminus \{0\},$$

$$\exists q = q_1 + iq_2$$

$$r = r_1 + ir_2 \in \mathbb{Z}[i] \text{ such that}$$

$$a + ib = (c + id)q + r, \quad r = 0 \text{ or}$$

$$d(r) < d(c + id)$$

Every E.D. is P.I.D. $\Rightarrow \mathbb{Z}[i]$ is P.I.D.

So, option (1) is correct

For Option (2):

$\mathbb{Z}[X]$ is not a PID

$\therefore \exists$ an Ideal $I = \langle 2, X \rangle$ of $\mathbb{Z}[X]$ which is not generated by single element, i.e., it is not principle Ideal

$\Rightarrow \mathbb{Z}[X]$ is not a P.I.D.

So, option (2) is not correct.

For option (3):

If $\mathbb{C}[X, Y]$ is a P.I.D.

Then $\frac{\mathbb{C}[X, Y]}{\langle Y \rangle} \cong \mathbb{C}[X]$ is a field

($\because \frac{\text{P.I.D.}}{\langle \text{Irreducible Element} \rangle} \cong \text{field}$)

which is wrong (as polynomial ring can never be field)

So, option (3) is not correct

For Option (4):

$$\frac{\mathbb{R}[X, Y]}{\langle X^2 + 1, Y \rangle} \cong \frac{\mathbb{R}[X]}{\langle X^2 + 1 \rangle} \cong \mathbb{R}[i] = \mathbb{C}$$

Since $\mathbb{C}$ is field and every field is P.I.D.

$\Rightarrow \frac{\mathbb{R}[X, Y]}{\langle X^2 + 1, Y \rangle}$ is P.I.D.

So, option (4) is correct

86. A(86), B(89), C(83)

For any prime number p, let A_p be the set of integers $d \in \{1, 2, \dots, 999\}$ such that the power of p in the prime factorisation of d is odd. Then the cardinality of

1. A_2 is 250

2. A_5 is 160

3. A_3 is 124

4. A_{11} is 82

Sol. (1,3,4)

A_p = The number of multiples of 'p' from 1 to 999

- The number of multiples of p^2 from 1 to 999

+ The number of multiples of p^3 from 1 to 999

- The number of multiples of p^4 from 1 to 999 and so on

The number of multiples of 'K' from 1 to n

is given by $\left[\frac{n}{K} \right]$, where $[.]$ represents the

greatest integer function.

For option (1)



40. A(40), B(37), C(33)

Let A be a connected open subset of $\mathbb{R}^2$. The number of continuous surjective functions from $\bar{A}$ (the closure of A in $\mathbb{R}^2$) to $\mathbb{Q}$ is:

1. 1 **2. 0** 3. 2 4. not finite

Sol.(2) Result: if $f: X \rightarrow Y$ is continuous, then image of any connected subset of X is connected
Given: A is connected open subset of $\mathbb{R}^2$ But $\mathbb{Q}$ is not connected

$\therefore$ There is no continuous surjective function from A to $\mathbb{Q}$

$\therefore$ option (2) is correct

41. A(41), B(44), C(48)

Consider the differential equation

$$(x-1)y'' + xy' + \frac{1}{x}y = 0. \text{ Then}$$

1. $x = 1$ is the only singular point
2. $x = 0$ is the only singular point
3. both $x = 0$ and $x = 1$ are singular points
4. neither $x = 0$ nor $x = 1$ are singular points

Sol.(3) Given differential equation is

$$(x-1)y'' + xy' + \frac{1}{x}y = 0$$

$$\Rightarrow y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0 \quad \dots(1)$$

By comparing (1) with $y'' + P(x)y' + Q(x)y = 0$,

$$\text{we get } P(x) = \frac{x}{x-1}; \quad Q(x) = \frac{1}{x(x-1)}$$

Now for

$$x = 0; \lim_{x \rightarrow 0} (x-0)P(x) = \lim_{x \rightarrow 0} x \cdot \frac{x}{x-1} = 0$$

$$\lim_{x \rightarrow 0} (x-0)^2 Q(x) = \lim_{x \rightarrow 0} \frac{x^2}{x(x-1)} = 0$$

and for

$$x = 1; \lim_{x \rightarrow 1} (x-1)P(x) = \lim_{x \rightarrow 1} (x-1) \cdot \frac{x}{x-1} = 1$$

$$\lim_{x \rightarrow 1} (x-1)^2 Q(x) = \lim_{x \rightarrow 1} (x-1)^2 \cdot \frac{1}{x(x-1)} = 0$$

$\therefore$ Both 0 and 1 are Regular Singular Points

$\therefore$ Option (3) is correct

42. A(42), B(43), C(47)

Let D denote the unit disc given by $\{(x, y) \mid x^2 + y^2 \leq 1\}$ and let D^c be its complement in the plane. The partial differential equation

$$(x^2 - 1) \frac{\partial^2 u}{\partial x^2} + 2y \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0 \text{ is}$$

1. parabolic for all $(x, y) \in D^c$
2. hyperbolic for all $(x, y) \in D$
3. hyperbolic for all $(x, y) \in D^c$
4. parabolic for all $(x, y) \in D$

Sol.(3) Given Equation is

$$(x^2 - 1) \frac{\partial^2 u}{\partial x^2} + 2y \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

By comparing given equation with

$$R \frac{\partial^2 u}{\partial x^2} + S \frac{\partial^2 u}{\partial x \partial y} + T \frac{\partial^2 u}{\partial y^2} = 0; \text{ we get}$$

$$R = x^2 - 1; \quad S = 2y; \quad T = -1$$

$$\text{Now } S^2 - 4RT = 4y^2 - (x^2 - 1)(-1) \cdot 4$$

$$= 4(y^2 + x^2 - 1)$$

$$= 4(x^2 + y^2 - 1)$$

$$\therefore S^2 - 4RT > 0 \text{ if } x^2 + y^2 > 1$$

$$\therefore S^2 - 4RT > 0 \text{ if } (x, y) \in D^c$$

Hence given equation is hyperbolic if $(x, y) \in D^c$.

$\therefore$ Option (3) is correct

43. A(43), B(42), C(46)

The set of real numbers λ for which the

boundary value problem $\frac{d^2 y}{dx^2} + \lambda y = 0$,

$y(0) = 0, y(\pi) = 0$ has nontrivial solutions is

1. $(-\infty, 0)$
2. $\{\sqrt{n} \mid n \text{ is a positive integer}\}$
3. $\{n^2 \mid n \text{ is a positive integer}\}$

4. $\mathbb{R}$

Sol.(3) Given differential equation is

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

$\therefore$ This is a Sturm-Liouville Problem

$\therefore$ Eigenvalues of given problem are countable

$\Rightarrow$ Options (1) and (4) are incorrect.

Now if $\lambda = \alpha^2 (\alpha \neq 0)$

$$\therefore \frac{d^2 y}{dx^2} + \alpha^2 y = 0$$

$$\Rightarrow (D^2 + \alpha^2)y = 0 \Rightarrow y = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$\text{Now for } y(0) = 0 \Rightarrow C_1 = 0$$

$$\text{and } y(\pi) = 0 \Rightarrow C_2 \sin(\alpha\pi) = 0$$

$$\Rightarrow \text{if } C_2 \neq 0, \text{ then } \sin(\alpha\pi) = 0$$

$$\Rightarrow \alpha = n; \quad n \in \mathbb{N}$$

$$\Rightarrow \lambda = n^2$$

$\therefore$ option (3) is correct