

RISING ★ STAR ACADEMY

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Linear Algebra 1

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1.1 Matrices and their properties

Def. Transpose of a matrix : If A is any matrix of order $m \times n$, then a matrix obtained from A by inter changing its rows and columns is called the transpose of the matrix A and is denoted by A' or A^T which is of order $n \times m$.

The operation of interchanging rows with columns is called transposition. Symbolically if $A = [a_{ij}]_{m \times n}$ then $A' = [a_{ji}]_{n \times m}$ i.e., (i, j) th element of $A = (j, i)$ th element of A' .

Properties of transpose : If A' and B' denote transpose of A and B respectively then

- (i) $(A')' = A$
- (ii) $(A + B)' = A' + B'$, where A and B are conformable for addition.
- (iii) $(kA)' = kA'$, where k is any number, real or complex.
- (iv) $(AB)' = B'A'$, where A and B are conformable for multiplication.
- (v) $(A^n)' = (A')^n$, A being a square matrix and n is a positive integer.

Def. Conjugate of a matrix : If A is any matrix of order $m \times n$ whose elements are complex numbers, then a matrix obtained from A by replacing each of its elements by their corresponding complex conjugate is called the conjugate of A and is denoted by $\bar{A}$ where $\bar{A}$ is also of same order $m \times n$. Symbolically, if $A = [a_{ij}]_{m \times n}$ then $\bar{A} = [\bar{a}_{ij}]_{m \times n}$ where $\bar{a}_{ij}$ is complex conjugate of a_{ij} .

Note : If the elements of the matrix A are all real numbers then $\bar{A} = A$.

Properties of conjugate : If $\bar{A}$ and $\bar{B}$ denote the conjugate of A and B respectively then

- (i) $\overline{(\bar{A})} = A$
- (ii) $\overline{(A + B)} = \bar{A} + \bar{B}$, where A and B are conformable for addition.
- (iii) $\overline{(kA)} = \bar{k} \bar{A}$, where k is any complex number.
- (iv) $\overline{(AB)} = \bar{A} \bar{B}$, where A and B are conformable for multiplication.
- (v) $\overline{(A^n)} = (\bar{A})^n$, where A is a square matrix and n is a positive integer.

Def. Transposed conjugate of a matrix : If A is any matrix of order $m \times n$, then the transpose of the conjugate of A is called transposed conjugate of A and is denoted by A^θ or A^* . Symbolically if

$$A = [a_{ij}]_{m \times n}, \text{ then } \bar{A} = [\bar{a}_{ij}]_{m \times n} \text{ and } A^* = (\bar{A})' = [\bar{a}_{ji}]_{n \times m}.$$

Properties of transposed conjugate : If A^* and B^* denote the transposed conjugate of A and B respectively, then

$$(i) (A^*)^* = A$$

$$(ii) (A + B)^* = A^* + B^*, \text{ where } A \text{ and } B \text{ are conformable for addition.}$$

$$(iii) (kA)^* = \bar{k} A^*, \text{ where } k \text{ is any complex number.}$$

$$(iv) (AB)^* = B^* A^*, \text{ where } A \text{ and } B \text{ are conformable for addition.}$$

$$(v) (A^n)^* = (A^*)^n, \text{ where } A \text{ is any square matrix and } n \text{ is a +ve integer.}$$

Note : If k is any real number then $(kA)^* = k A^*$ because in this case $\bar{k} = k$

Def. Singular and Non-singular Matrix : A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$

Def. Adjoint of a square Matrix : Let $A = [a_{ij}]_{n \times n}$ be the square matrix of order n and A_{ij} be the corresponding cofactor of a_{ij} in $|A|$, then the matrix

$$B = [A_{ij}]' = [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

is called the adjoint of A and is denoted by $\text{adj. } A$. Thus the adjoint of a matrix is the transpose of the matrix formed by the co-factors of A .

Def. Inverse of a square matrix : Let A be a square matrix of order n . If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix A is said to be invertible and the matrix B is called inverse of the matrix A and is denoted by A^{-1} .

Note : (i) If the matrix B is inverse of A then A is the inverse of B .

(ii) For the products AB and BA both to be defined and be equal it is necessary that A and B are both square matrices of the same order. Thus non-square matrices cannot possess inverse.

The following theorem shows that if a matrix possesses an inverse, then the inverse must be unique.

Results :

1. If A be any n -square matrix, then $A(\text{adj. } A) = (\text{adj. } A)A = |A| I_n$, where I_n is the unit matrix of order n .

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2. If A is a non-singular matrix of order n , then $|\text{adj. } A| = |A|^{n-1}$.
3. Adjoint of a non-singular matrix is non-singular.
4. If A is a non-singular matrix of order n , then $\text{adj.}(\text{adj. } A) = |A|^{n-2}A$.
5. If A and B are two square matrices of same order then $\text{adj.}(AB) = (\text{adj. } B)(\text{adj. } A)$.
6. If A is a square matrix then $\text{adj. } A' = (\text{adj. } A)'$, where A' is the transpose of matrix A .
7. Inverse of a square matrix, if it exists, is unique.
8. A square matrix A is invertible if and only if A is non singular i.e. $|A| \neq 0$.
9. If A is a non singular matrix then $A^{-1} = \frac{\text{adj. } A}{|A|}$.
10. A singular matrix cannot have an inverse.
11. Reversal law for inverse : If A and B be non singular matrices then $(AB)^{-1} = B^{-1}A^{-1}$.
12. Generalization of Reversal law for inverse : If $A_1, A_2, \dots, A_n$ be non singular matrices of same order then $(A_1 A_2 \dots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \dots A_2^{-1} A_1^{-1}$.
13. If A is non singular matrix then $(A^k)^{-1} = (A^{-1})^k$, where k is any positive integer.
14. If A is a non-singular matrix then $|A^{-1}| = \frac{1}{|A|}$.
15. If A is a non-singular matrix then $(A')^{-1} = (A^{-1})'$.
16. If A is a non-singular matrix then $(\text{adj. } A)^{-1} = \text{adj. } A^{-1}$.
17. The reversal law for multiplication holds for adjoint, inverse, transpose and transposed conjugate but it does not hold for conjugate.

Def. Diagonal matrix : A matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0$ for $\forall i \neq j$.

In other words, a square matrix is a diagonal matrix if its non-diagonal elements are zero.

Remark : An n -rowed diagonal matrix whose diagonal elements are $d_1, d_2, \dots, d_n$ can be represented as $\text{diag.}(d_1, d_2, \dots, d_n)$.

Def. Scalar matrix : A square matrix is called a scalar matrix if A is a diagonal matrix and all diagonal elements are equal.

Mathematically $A = [a_{ij}]_{n \times n}$ is a scalar matrix if

- (i) $a_{ij} = 0$ for all $i \neq j$ (ii) $a_{ii} = k$, where k is any number

Def. Upper triangular matrix : A matrix is an upper triangular matrix if all elements below the principal diagonal are zero. Mathematically $A = [a_{ij}]_{n \times n}$ is a upper triangular matrix if $a_{ij} = 0$ for all $i > j$.

Def. Lower triangular matrix : A matrix is called lower triangular matrix if all elements above the principal diagonal are zero. Mathematically $A = [a_{ij}]_{n \times n}$ is a upper triangular matrix if $a_{ij} = 0$ for all $i < j$.

Def. Triangular matrix : A matrix which is either upper triangular or lower triangular is called a triangular matrix.

Remark : Every diagonal matrix is both upper and lower triangular.

Def. Super upper triangular matrix : A matrix is said to be super upper triangular matrix if all elements below and on the principal diagonal are zero. Mathematically $A = [a_{ij}]_{n \times n}$ is a super upper triangular matrix if $a_{ij} = 0$ for all $i \geq j$.

Def. Super lower triangular matrix : A matrix is said to be super lower triangular matrix if all elements above and on the principal diagonal are zero. Mathematically $A = [a_{ij}]_{n \times n}$ is a super lower triangular matrix if $a_{ij} = 0$ for all $i \leq j$.

Def. Backward diagonal matrix : A matrix $A = [a_{ij}]_{n \times n}$ is called a backward diagonal matrix if $a_{ij} = 0$ if $i + j \neq n + 1$.

Def. Backward scalar matrix : A matrix $A = [a_{ij}]_{n \times n}$ is called a backward scalar matrix if

$$a_{ij} = \begin{cases} k & \text{if } i + j = n + 1 \\ 0 & \text{if } i + j \neq n + 1 \end{cases}$$

Def. Symmetric matrix : A square matrix A is said to be symmetric if $A' = A$.

or

A square matrix $A = [a_{ij}]$ said to be symmetric if $a_{ij} = a_{ji}$ for all i and j .

e.g. the matrices are $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & h & c \end{bmatrix}$, $\begin{bmatrix} 1 & 1-i & 2+i \\ 1-i & 2 & 3+2i \\ 2+i & 3+2i & 3 \end{bmatrix}$ are symmetric matrices.

Def. Skew-symmetric matrix : A square matrix A is said to be skew-symmetric if $A' = -A$

or

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j .

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e.g. the matrices $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i & 2 \\ -i & 0 & 1-i \\ -2 & -1+i & 0 \end{bmatrix}$ are skew-symmetric.

Def. Hermitian matrix : A square A is said to be hermitian if $A^* = A$

or

A square matrix $A = [a_{ij}]$ is said to be Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j .

e.g. The matrices $\begin{bmatrix} 1 & 3-i \\ 3+i & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2+i & 1-2i \\ 2-i & 2 & 3i \\ 1+2i & -3i & 3 \end{bmatrix}$ are hermitian matrices.

Def. Skew-hermitian matrix : A square matrix A is said to be skew-hermitian if $A^* = -A$

or

A square matrix $A = [a_{ij}]$ is said to be skew-hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j .

e.g. The matrices $\begin{bmatrix} 0 & 3+5i \\ -3+5i & 2i \end{bmatrix}$, $\begin{bmatrix} 3i & 1+i & 2-3i \\ -1+i & 0 & 5+i \\ -2-3i & -5+i & -4i \end{bmatrix}$ are skew-hermitian matrices.

Results A :

1. The diagonal elements of a symmetric matrix are arbitrary.
2. The diagonal elements of a skew-symmetric matrix are all zero.
3. The diagonal element of a hermitian matrix are purely real.
4. The diagonal elements of a skew-hermitian matrix are either zero or purely imaginary.

Results B :

1. If A is a square matrix, then

- | | |
|------------------------------|----------------------------------|
| (i) $A + A'$ is symmetric | (ii) $A - A'$ is skew-symmetric |
| (iii) $A + A^*$ is hermitian | (iv) $A - A^*$ is skew-hermitian |

2. If A is any square matrix, then

- | | |
|--|---|
| (i) AA' and $A'A$ are both symmetric | (ii) AA^* and A^*A are both hermitian |
|--|---|

3. If A is symmetric then kA is symmetric, where k is any number real or complex.

4. If A is skew-symmetric then kA is skew-symmetric, where k is any number real or complex.

5. If A is hermitian then kA is hermitian, where k is any real number.
6. If A is skew-hermitian then kA is skew-hermitian, where k is any real number.
7. If A is hermitian then iA is skew-hermitian.
8. If A is skew-hermitian then iA is hermitian.
9. If A and B are symmetric matrices of same order, then
 - (i) $A + B$, $A - B$, $AB + BA$ are symmetric
 - (ii) $AB - BA$ is skew-symmetric
 - (iii) BAB , ABA are symmetric.
10. If A and B are skew-symmetric matrices of same order, then
 - (i) $A + B$, $A - B$, $AB - BA$ are skew-symmetric
 - (ii) $AB + BA$ is symmetric
 - (iii) BAB , ABA are skew-symmetric.
11. If A and B are hermitian matrices of same order, then
 - (i) $A + B$, $A - B$, $AB + BA$ are hermitian
 - (ii) $AB - BA$ is skew-hermitian
 - (iii) BAB , ABA are hermitian.
12. If A and B are skew-hermitian matrices of same order, then
 - (i) $A + B$, $A - B$, $AB - BA$ are skew-hermitian
 - (ii) $AB + BA$ is hermitian
 - (iii) BAB , ABA are skew-hermitian
13. If A and B are symmetric matrices of same order, then prove that AB is symmetric iff $AB = BA$.
14. If A and B are hermitian matrices of same order, then prove that AB is hermitian if and only if $AB = BA$.
15. If A and B are square matrices of same order then
 - (i) $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.
 - (ii) B^*AB is hermitian or skew-hermitian according as A is hermitian or skew-hermitian.
16. If A is a symmetric matrix then A^n is also symmetric for all positive integers n .
17. If A is a skew-symmetric matrix then A^n is symmetric if n is a positive even integer and A^n is skew-symmetric if n is positive odd integer.
18. If A is a hermitian matrix then A^n is also hermitian for all positive integers n .
19. If A is a skew-hermitian matrix then A^n is hermitian or skew-hermitian according as n is a positive even integer or an odd integer.
20. If A is a non-singular and symmetric matrix then $\text{adj. } A$ is also symmetric.
21. If A is a skew-symmetric matrix of order n , then $\text{adj. } A$ is symmetric or skew-symmetric according as n is odd or even.
22. Determinant of a skew-symmetric matrix of odd order is always zero.
23. Determinant a skew-symmetric matrix of even order with integral entries is always a perfect square.

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24. Let $A = \begin{bmatrix} 0 & \dots & 0 & d_1 \\ 0 & \dots & d_2 & 0 \\ \dots & \dots & \dots & \dots \\ d_n & \dots & 0 & 0 \end{bmatrix}$ be a backward diagonal matrix then

$$(i) \det(A) = (-1)^{\left[\frac{n}{2}\right]} d_1 \cdot d_2 \cdots d_n$$

$$(ii) \text{trace}(A) = \begin{cases} 0 & \text{if } n \text{ is even} \\ d_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Results C :

1. Every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.
2. Every square matrix is uniquely expressible as the sum of a hermitian matrix and a skew-hermitian matrix.
3. Every square matrix A can be uniquely expressed as $A = P + iQ$ where P and Q are hermitian matrices.
4. Every hermitian matrix A can be written as $A = B + iC$ where B is real symmetric and C is real skew-symmetric.

Exercise 1.1

1. (i) Compute the adjoint of the matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ and verify that $(\text{adj. } A)A = |A|I$.

(ii) For the matrix $\begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $(\text{adj. } A)A = |A|I$.

2. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, verify that (i) $(\text{adj. } A)^{-1} = \text{adj. } A^{-1}$ (ii) $(A^{-1})^{-1} = A$

3. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

4. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $i^2 = -1$ verify that (i) $(AB)' = B'A'$ (ii) $(AB)^* = B^*A^*$

5. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$ verify that $(A^2)' = (A')^2$

6. If $A = \begin{bmatrix} 2+3i & i \\ 6+5i & 0 \end{bmatrix}$, $B = \begin{bmatrix} i & 2i+1 \\ 2-i & -i \end{bmatrix}$, verify that (i) $\overline{AB} = \bar{A} \cdot \bar{B}$ (ii) $(AB)^* = B^* A^*$

7. Express the matrix $\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as a sum of a symmetric and skew-symmetric matrices.

8. Express $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

9. Express the following matrices as a sum of hermitian and skew-hermitian matrices.

(i) $\begin{bmatrix} 2 & 1+i & 2+3i \\ 2-i & 1+2i & -i \\ 2-4i & 3-2i & 4+5i \end{bmatrix}$

(ii) $\begin{bmatrix} 7-4i & 2 & 3+5i \\ 3i & 6i & 1-4i \\ -3i & 2-i & 3 \end{bmatrix}$

10. Show that the following matrices are orthogonal and hence find their inverse.

(i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(iii) $\frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Answers

7. $\begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ -\frac{5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -1 & 4 & \frac{7}{2} \\ -1 & 1 & 4 & -\frac{3}{2} \\ 4 & 4 & 7 & -\frac{1}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & -\frac{1}{2} \\ -1 & 0 & 2 & \frac{5}{2} \\ -1 & -2 & 0 & \frac{3}{2} \\ \frac{1}{2} & -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

9. (i) $\frac{1}{2} \begin{bmatrix} 4 & 3+2i & 4+7i \\ 3-2i & 2 & 3+i \\ 4-7i & 3-i & 8 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & -1 & -i \\ 1 & 4i & -3-3i \\ -i & 3-3i & 10i \end{bmatrix}$

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$$(ii) \quad \frac{1}{2} \begin{bmatrix} 14 & 2-3i & 3+8i \\ 2+3i & 0 & 3-3i \\ 3-8i & 3+3i & 6 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -8i & 2+3i & 3+2i \\ -2+3i & 12i & -1-5i \\ -3+2i & 1-5i & 0 \end{bmatrix}$$

10. (i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$$(ii) \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\text{(iii)} \quad \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$(iv) \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

1.2 Orthogonal and Unitary matrices

Def. Orthogonal Matrix : A square matrix A is said to be orthogonal if $AA' = A'A = I$. It is clear from the definition that if the matrix A is orthogonal then $A^{-1} = A'$.

e.g.

If $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, then $A' = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

$$\therefore AA' = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $A' A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence A is an orthogonal matrix.

Remark: An orthogonal matrix is always non-singular.

Properties of orthogonal Matrix :

1. The determinant of an orthogonal matrix is ± 1 .

OR

The absolute value of determinant of an orthogonal matrix is unity.

Def. Proper and Improper orthogonal matrix : An orthogonal matrix is said to be proper or improper according as $|A| = 1$ or $|A| = -1$

2. If A is an orthogonal matrix with $|A| = 1$, then each element of A is equal to its cofactor in $|A|$.

Believe that you have it, and you will have it. - Latin Proverb

3. If A is a orthogonal matrix with $|A| = -1$, then each element of A is equal to negative of its cofactor in $|A|$.
4. If A is an orthogonal matrix, then A' is also orthogonal.
5. If A is an orthogonal matrix, then A^{-1} is also orthogonal.
6. The product of two orthogonal matrices of same order is again orthogonal.
7. Generalization : If $A_1, A_2, \dots, A_n$ be n orthogonal matrices of same order then their product $A_1 A_2 \dots A_n$ is also orthogonal.

Def. Orthonormal set : A set of vectors $\{v_1, v_2, \dots, v_n\}$ is said to be an orthonormal set if $\langle v_i, v_j \rangle = 0$ for $i \neq j$ and $\langle v_i, v_i \rangle = 1$ for all i . e.g. the set $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is an orthonormal set.

8. A matrix is orthogonal iff its rows form an orthonormal set.

Example : Show that the direction cosines of the three mutually perpendicular lines in rectangular co-ordinate system constitutes the rows of orthogonal matrix.

Solution : Let l_r, m_r, n_r ($r = 1, 2, 3$), be the direction cosines of three mutually perpendicular lines.

Let $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ then $A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

Now, $AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$= \begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_2 l_1 + m_2 m_1 + n_2 n_1 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_3 l_1 + m_3 m_1 + n_3 n_1 & l_3 l_2 + m_3 m_2 + n_3 n_2 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ as } l_i l_j + m_i m_j + n_i n_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Similarly $A'A = I \Rightarrow A$ is orthogonal

Hence the direction cosines of three mutually perpendicular lines in rectangular co-ordinate system constitute the rows of orthogonal matrix.

Def. Unitary Matrix : A square matrix A is said to be unitary if $AA^* = A^*A = I$. It is clear from the definition that if a matrix A is unitary then $A^{-1} = A^*$.

e.g. $A = \frac{1}{2} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

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Linear Algebra 1

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For,
$$AA^* = \frac{1}{2} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly $A^*A = I$. Hence A is a unitary matrix.

Note : If each element of the matrix A is real, then $\bar{A} = A$

$$\therefore A^* = (\bar{A})' = A'$$

$$\therefore AA^* = A^*A = I \quad \Rightarrow \quad AA' = A'A = I$$

$\therefore$ A unitary matrix whose elements are all real, is an orthogonal matrix.

Remark : A unitary matrix is always a non-singular matrix.

Def. Normal Matrix : A square matrix A is said to be normal if $AA^* = A^*A$.

Properties of unitary matrix :

1. The transpose of a unitary matrix is unitary.
2. The conjugate of a unitary matrix is unitary.
3. The transposed conjugate of a unitary matrix is unitary.
4. Inverse of a unitary matrix is unitary.
5. The product of two unitary matrices of same order is again unitary.
6. Generalization : If $A_1, A_2, \dots, A_n$ be n unitary matrices of same order then their product $A_1 A_2 \dots A_n$ is also unitary.
7. The determinant of a unitary matrix has absolute value 1.

Exercise 1.2

1. Check whether the following matrices are proper orthogonal or improper orthogonal :

(i)
$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

(ii)
$$\begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$$

(iii)
$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

2. Prove that a matrix $A = [a]$ of order 1 is orthogonal if and only if $a = \pm 1$.

3. Show that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal iff $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2 + b^2 = 1$.

4. If A is an orthogonal matrix and if $B = AP$, where P is non-singular, then prove that PB^{-1} is orthogonal.

5. If A is symmetric and P is orthogonal, then show that $P^{-1}AP$ is symmetric.

6. If A and B commute, then $C'AC$ and $C'BC$ commute if C is orthogonal.
7. If A is skew-symmetric and $I + A$ is non-singular, then $B = (I - A)(I + A)^{-1}$ is orthogonal.
8. Show that the following matrices are unitary and hence find their inverse :

(i) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(iii) $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{3}} & \frac{3+i}{2\sqrt{15}} \\ -\frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{4+3i}{2\sqrt{15}} \\ \frac{1}{2} & -\frac{i}{\sqrt{3}} & \frac{5i}{2\sqrt{15}} \end{bmatrix}$

9. Show that the matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2 + b^2 + c^2 + d^2 = 1$.

10. Show that if A is hermitian and P is unitary, then $P^{-1}AP$ is hermitian.

11. If A is unitary and hermitian matrix, then show that A is involutory ($A^2 = I$).

12. If A is unitary and $B = AP$ where P is non-singular then PB^{-1} is unitary.

13. If A is skew hermitian and $(A - I)$ is non-singular then $(A + I)(A - I)^{-1}$ is unitary.

14. Show that the following matrices are unitary :

(i) $\begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$ (ii) $\begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -i & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{bmatrix}$

15. What do you notice in above matrices.

Answers

1. (i) Improper orthogonal. (ii) Proper orthogonal (iii) The matrix is not orthogonal.

8. (i) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (iii) $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$(iv) \begin{bmatrix} \frac{1-i}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{3-i}{2\sqrt{15}} & \frac{4-3i}{2\sqrt{15}} & -\frac{5i}{2\sqrt{15}} \end{bmatrix}$$

1.3 Idempotent, Involutory and Nilpotent matrices

Def. Idempotent matrix : A square matrix A is said to be idempotent matrix if $A^2 = A$. e.g. Zero matrix and identity matrix are always idempotent. The matrix $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$ is always idempotent.

Results :

1. Determinant of an idempotent matrix is always 0 or 1.
2. Trace of a $n \times n$ idempotent matrix always belongs to the set $\{0, 1, 2, \dots, n\}$.

Def. Involutory matrix : A square matrix A is said to be involutory matrix if $A^2 = I$. e.g. The identity

matrix is always involutory. The matrices $\begin{bmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \pm 1 \end{bmatrix}$ are always involutory.

Results :

1. Determinant of an involutory matrix is always 1 or -1.
2. Trace of a $n \times n$ involutory matrix always belongs to the set $\{-n, -(n-1), \dots, -1, 0, 1, 2, \dots, n\}$.

Def. Nilpotent matrix : A square matrix A is said to be nilpotent matrix if there exists a positive integer k such that $A^k = O$. e.g. Zero matrix is always nilpotent.

Def. Index of nilpotency : Let A be a nilpotent matrix then the smallest positive integer k for which $A^k = O$ is called index of nilpotency of A .

Results :

1. Determinant of a nilpotent matrix is always 0.
2. Trace of a nilpotent matrix is always 0.
3. All super lower triangular and super upper triangular matrices are nilpotent.

4. Index of nilpotency of a nilpotent matrix cannot exceed its size. In other words if A is a $n \times n$ nilpotent matrix with index of nilpotency k then $k \leq n$.
5. If A is a $n \times n$ nilpotent matrix then we must have $A^n = O$.
6. If A is a $n \times n$ matrix such that $A \neq 0, A^2 \neq 0, \dots, A^n \neq 0$ then A cannot be nilpotent.

Exercise 1.3

1. Show that the following matrices are nilpotent and also find their index of nilpotency.

$$(i) A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(ii) B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iii) C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iv) D = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(v) E = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vi) F = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vii) G = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(viii) H = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 3 & 6 & 9 \end{bmatrix}$$

2. Construct five different nilpotent matrices by taking hint from the matrix H in above question.

3. Show that the matrix $\begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}$, $b \neq 0$ is always involutory.

4. Show that the following matrices are idempotent :

$$(i) \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & 1 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. What do you notice in above matrices.

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Miscellaneous Exercise

1. Suppose A is invertible. Show that if $AB=AC$, then $B=C$. Give an example of a nonzero matrix A such that $AB=AC$ but $B \neq C$.

2. Find 2×2 invertible matrices A and B such that $A+B \neq 0$ and $A+B$ is not invertible.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Find A^n . (b) Find B^n .

4. Find all real triangular matrices A such that $A^2=B$, where (a) $B = \begin{bmatrix} 4 & 21 \\ 0 & 25 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 4 \\ 0 & -9 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix}$. Find all numbers k for which A is a root of the polynomial :

(a) $f(x) = x^2 - 7x + 10$

(b) $g(x) = x^2 - 25$

(c) $h(x) = x^2 - 4$

6. Let $B = \begin{bmatrix} 1 & 0 \\ 26 & 27 \end{bmatrix}$. Find a matrix A such that $A^3 = B$.

7. Let $B = \begin{bmatrix} 1 & 8 & 5 \\ 0 & 9 & 5 \\ 0 & 0 & 4 \end{bmatrix}$. Find a triangular matrix A with positive diagonal entries such that $A^2 = B$.

8. Using only the elements 0 and 1, find the number of 3×3 matrices that are :

(a) diagonal (b) upper triangular (c) non-singular and upper triangular. Generalize to $n \times n$ matrices.

9. Let $D_k = kI$, the scalar matrix belonging to the scalar k , show that :

(a) $D_k A = kA$

(b) $BD_k = kB$

(c) $D_k + D_{k'} = D_{k+k'}$

(d) $D_k D_{k'} = D_{kk'}$

10. Suppose $AB=C$, where A and C are upper triangular.

(a) Find 2×2 nonzero matrices A, B, C where B is not upper triangular.

(b) Suppose A is also invertible. Show that B must also be upper triangular.

11. Find x, y, z such that A is symmetric, where (a) $A = \begin{bmatrix} 2 & x & 3 \\ 4 & 5 & y \\ z & 1 & 7 \end{bmatrix}$ (b) $A = \begin{bmatrix} 7 & -6 & 2x \\ y & z & -2 \\ x & -2 & 5 \end{bmatrix}$.

12. Suppose A and B are symmetric. Show that the following are also symmetric :

(a) $A+B$

(b) kA , for any scalar k

(c) A^2

(d) A^n , for $n > 0$ (e) $f(A)$, for any polynomial $f(x)$.13. Find a 2×2 orthogonal matrix P whose first row is a multiple of (a) $(3, -4)$ (b) $(1, 2)$ 14. Find a 3×3 orthogonal matrix P whose first two rows are multiple of :(a) $(1, 2, 3)$ and $(0, -2, 3)$ (b) $(1, 3, 1)$ and $(1, 0, -1)$ 15. Suppose A and B are orthogonal matrices. Show that A^T, A^{-1}, AB are also orthogonal.16. Which of the following matrices are normal ? $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.17. Find real numbers x, y, z such that A is hermitian, where $A = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$.18. Suppose A is a complex matrix. Show that AA^* and A^*A are hermitian.19. Let A be a square matrix. Show that (a) $A + A^*$ is hermitian, (b) $A - A^*$ is skew-hermitian, (c) $A = B + C$, where B is hermitian and C is skew hermitian.

20. Determine which of the following matrices are unitary :

$$A = \begin{bmatrix} \frac{i}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-i}{2} \end{bmatrix},$$

$$B = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix},$$

$$C = \frac{1}{2} \begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$$

21. Suppose A and B are unitary. Show that A^*, A^{-1}, AB are unitary.22. Determine which of the following matrices are normal $A = \begin{bmatrix} 3+4i & 1 \\ i & 2+3i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1-i & i \end{bmatrix}$.

Answers

1. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 3 \\ 3 & 0 \end{bmatrix}$

3. (a) $\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & n & \frac{1}{2}n(n-1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

4. (a) $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} -2 & -3 \\ 0 & -5 \end{bmatrix}$, $\begin{bmatrix} 2 & -7 \\ 0 & -5 \end{bmatrix}$, $\begin{bmatrix} -2 & 7 \\ 0 & 5 \end{bmatrix}$ (b) none

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5. (a) $k = 2$, (b) $k = -5$ (c) none

6. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ 7. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

8. All entries below the diagonal must be 0 to be upper triangular, and all diagonal entries must be 1 to be non singular. (a) $8; 2^n$ (b) $2^6; 2^{n(n+1)/2}$ (c) $2^3; 2^{n(n-1)/2}$.

10. (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$

11. (a) $x=4, y=1, z=3$, (b) $x=0, y=-6, z$ any real number.

13. (a) $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, (b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$

14. (a) $\begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ 0 & \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{12}{\sqrt{157}} & \frac{-3}{\sqrt{157}} & \frac{-2}{\sqrt{157}} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{22}} & \frac{-2}{\sqrt{22}} & \frac{3}{\sqrt{22}} \end{bmatrix}$

16. A, C

17. $x=3, y=0, z=3$

19. (c) Hint : Let $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2}(A - A^*)$

20. A, B, C

22. A

Sr. No.	Matrix	Definition	A+B	AB	kA	A	tr(A)
1	Diagonal	$a_{ij} = 0, i \neq j$					
2	Scalar	$a_{ij} = \begin{cases} 0 & , i \neq j \\ k & , i = j \end{cases}$					
3	Lower triangular	$a_{ij} = 0, i < j$					
4	Upper triangular	$a_{ij} = 0, i > j$					
5	Super lower triangular	$a_{ij} = 0, i \leq j$					
6	Super upper triangular	$a_{ij} = 0, i \geq j$					
7	Backward diagonal	$a_{ij} = \begin{cases} 0 & , i + j \neq n + 1 \\ a_i & , i + j = n + 1 \end{cases}$					
8	Backward scalar	$a_{ij} = \begin{cases} 0 & , i + j \neq n + 1 \\ k & , i + j = n + 1 \end{cases}$					
9	Symmetric	$A' = A$					
10	Skew-symmetric	$A' = -A$					
11	Hermitian	$A^* = A$					
12	Skew-hermitian	$A^* = -A$					
13	Orthogonal	$AA' = I$					
14	Unitary	$AA^* = I$					
15	Normal	$AA^* = A^*A$					
16	Idempotent	$A^2 = A$					
17	Involutory	$A^2 = I$					
18	Nilpotent	$A^k = 0$					

2.1 Determinantal rank

Def. Minor of a matrix : Let A be an $m \times n$ matrix, then the determinant of every square sub-matrix of A is called a Minor of the matrix A and the order of the sub-matrix is called order of minor.

Def. Determinantal Rank : Let $A = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix, then a non-negative integer r is called the determinantal rank of A iff

- (i) there exists at least one minor of A of order r which is not zero.
- (ii) every minor of A of order greater than r is zero.

In other words, determinantal rank of a matrix is the highest order of a non-zero minor of the matrix. The rank of a matrix is generally denoted by $\rho(A)$.

Results :

1. If A is a matrix of order $m \times n$ then its determinantal rank is less than or equal to minimum of m and n i.e. $\rho(A) \leq \min\{m, n\}$. For example, if a matrix A is of order 3×5 then clearly the largest possible order of any minor of matrix A is 3 and hence $\rho(A) \leq 3$.
2. Determinantal rank of a zero matrix is always 0 because it has no non-zero minor of positive order.
3. Determinantal rank of non-zero matrix is always greater than 0.
4. If A is non-singular square matrix of order n then its determinantal rank is n because the determinant of matrix A i.e. $|A|$ is the non-zero minor of highest order n .
5. If A is a singular square matrix of order n then its rank is less than n because there is only one minor of order n i.e. $|A|$, which is zero. So highest non-zero minor of A will be of order less than n .
6. If every minor of order r of a matrix A vanishes then every minor of order $r + 1$ of A also vanishes.

It is so because every minor of order $r + 1$ is a sum of scalar multiples of minors of order r .

Example 1 : If A is an n -rowed square matrix of rank $(n - 1)$, show that $\text{adj } A \neq 0$.

Solution : As $\rho(A) = n - 1$, so there is atleast one minor of A of order $(n - 1)$ which does not vanish and hence there is atleast one element in A whose cofactor is non-zero. Also we know that elements of $\text{adj } A$ are the co-factors of elements of A and hence $\text{adj } A$ is a non-zero matrix.

Example 2 : Show that no non-zero skew-symmetric matrix is of rank one.

Solution : Let us prove this result for a 4×4 skew symmetric matrix. The method of proof can be used for a skew-symmetric matrix of any order.

Let
$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

If a, b, c, d, e, f are all zero, then $\rho(A) = 0$ which is not the case here. If at least one of these elements say c is non-zero, then the second order minor $\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix}$ of matrix A is $c^2 \neq 0$.

Hence $\rho(A) \geq 2$ and so $\rho(A) \neq 1$.

Example 3 : Find the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{bmatrix}$, x, y, z being all real.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{bmatrix}$, so $|A| = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

Now the following cases arise :

Case (i) : If x, y, z are all different and $x+y+z=0$, then $|A| = 0$ and so $\rho(A) \leq 2$

But $\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = y-x \neq 0$ [$\because x, y$ are different]

So there is a minor of order 2 of A , which does not vanish and hence $\rho(A) = 2$.

Case (ii) : If x, y, z are all different and $x+y+z \neq 0$, then $|A| \neq 0$ and so $\rho(A) = 3$

Case (iii) : If any two of x, y, z are equal (say $x=y$) then $|A| = 0$.

Now $x=y \neq z$, then the minor $\begin{vmatrix} 1 & 1 \\ x & z \end{vmatrix} = z-x \neq 0$ and so $\rho(A) = 2$.

Case (iv) : If $x=y=z$ then
$$A = \begin{bmatrix} 1 & 1 & 1 \\ x & x & x \\ x^2 & x^2 & x^2 \end{bmatrix}$$

Here every minor of order 2 vanishes but there is a minor of order 1, which does not vanish ($\because 1$ is an element of A which is not zero), so $\rho(A) = 1$.

Example 4 : Prove that rank of transpose of a matrix is equal to the original matrix.

Solution : Let A be any matrix of rank r . Then according to the definition, A has at least one non-zero minor of order r , and all minors of order $r+1$ are zero. Now minors of A' are same as that of A because value of a determinant does not change when rows and columns are interchanged. Hence A'

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has also a non-zero minor of order r and all the minors of order $r + 1$ of A' are zero. So,

$$\rho(A') = r = \rho(A)$$

Exercise 2.1

1. Find the rank of the following matrices

(i) $\begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 & 7 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & -9 \\ 3 & 6 & 4 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 5 & 4 & 6 \\ 2 & 7 & 5 & 9 \\ 3 & 9 & 6 & 12 \end{bmatrix}$

(viii) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

(ix) $\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a \end{bmatrix}_{n \times n}$

2. Find the rank of the following matrices

(i) $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 7 & 6 & 2 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 5 & 4 & 6 \\ 2 & 7 & 5 & 9 \\ 3 & 9 & 6 & 12 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 6 & 6 & 1 \\ -8 & 7 & 2 & 3 \\ -2 & 3 & 0 & 1 \\ -3 & 2 & 1 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

3. Find the rank of $A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$

4. Let A and B be two square matrices of order n . If $\rho(A) = \rho(B) = n$, prove that $\rho(AB) = n$ and conversely.

5. If A is a square matrix of order n and $\rho(A) \leq n - 2$ then $\text{adj } A = O$.

6. Find the value of a so that $\rho(A) \leq 2$, where A is the matrix $\begin{bmatrix} 3a-8 & 3 & 3 \\ 3 & 3a-8 & 3 \\ 3 & 3 & 3a-8 \end{bmatrix}$

7. Find x so that $A = \begin{bmatrix} x+p & q & r \\ p & x+q & r \\ p & q & x+r \end{bmatrix}$ has rank 3.

8. Show that rank of a matrix $\geq$ rank of every sub-matrix of it.

9. If $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ find $\rho(A)$ and $\rho(A^2)$.

10. Show that rank of a matrix with each element unity is one.

11. Prove that if A is a skew-symmetric matrix of odd order say n then $\rho(A) < n$.

12. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$. Find the rank of A , B , $A+B$, AB and BA with the

help of minors of the corresponding matrices.

13. Find a so that the rank of the matrix $A = \begin{bmatrix} a & 0 & 1 \\ 1 & 2 & a \\ 1 & 2 & 3 \end{bmatrix}$ is less than 3. Also find the rank for these values

of a .

Answers

1. (i) 1 (ii) 1 (iii) 1 (iv) 3 (v) 3 (vi) 3 (vii) 2 (viii) 2 (ix) 0 if $a = 0$ and n if $a \neq 0$

2. (i) 3 (ii) 2 (iii) 2 (iv) 2 (v) 3 (vi) 1

3. $\rho(A) = \begin{cases} 3 & \text{if } x \neq y, y \neq z, z \neq x \\ 2 & \text{if either } x=y \text{ or } x=z \text{ and } y \neq z \\ 1 & \text{if } x=y=z \end{cases}$

6. $a = \frac{2}{3}, \frac{11}{3}$

7. $x \neq 0, x = -(p+q+r)$

9. $\rho(A) = 3, \rho(A^2) = 2$

12. $\rho(A) = 2, \rho(B) = 1, \rho(A+B) = 2, \rho(AB) = 0, \rho(BA) = 1$

13. $a = 0, 3; \rho(A) = 2$

2.2 Linear Independence and dependence

Def. Linearly dependent vectors : The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent if there exist scalars $a_1, a_2, \dots, a_n$ (not all zero) such that $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$.

OR

Some finite vectors are said to be linearly dependent if their linear combination can be made zero without taking all scalars zero.

Def. Linearly independent vectors : Vectors which are not L.D. are called L.I.

OR

Vectors $v_1, v_2, \dots, v_n$ are called linearly independent if, whenever

$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ for some scalars a_i 's, then $a_1 = a_2 = \dots = a_n = 0$.

OR

Some finite vectors are said to L.I. if the only way to make their linear combination zero is to take all scalars zero.

Def. An infinite subset of V is said to be linearly independent if every finite subset of S is linearly independent, otherwise it is linearly dependent.

Results:

1. The empty set is defined to be linearly independent.
2. A set containing only the zero vector i.e. $\{0\}$ is linearly dependent.
3. A set containing only a non-zero vector is linearly independent.
4. If two vectors are L.D., then one of them is a scalar multiple of other.
5. A set which contain the null vector '0' is linearly dependent.
6. Every super set of a linearly dependent set is linearly dependent.
7. Every subset of a linearly independent set is linearly independent.
8. If a vector v is a linear combination of vectors $v_1, v_2, \dots, v_n$ then the set $\{v, v_1, v_2, \dots, v_n\}$ is linearly dependent.
9. If the set $\{v_1, v_2, \dots, v_n\}$ is L.I. and the set $\{v_1, v_2, \dots, v_n, v\}$ is L.D., then v is a linear combination of the vectors $v_1, v_2, \dots, v_n$.
10. The set of non-zero vectors $v_1, v_2, \dots, v_n$ is L.D. iff one of the vectors v_r ($r > 1$) is a linear combination of the preceding vectors.
11. A square matrix A is singular if and only if its columns (rows) are linearly dependent.

Exercise 2.2

1. Determine which of the following sets of vectors are linearly dependent and which are linearly independent.

(i) $(2, 3, 1), (-1, 4, -2), (1, 18, -4)$

(ii) $(0, 2, -4), (1, -2, -1), (1, -4, 3)$

(iii) $(1, 1, 1), (1, 2, 3), (0, 1, 2)$

(iv) $(2, 3, -1, -1), (1, -1, -2, -4), (3, 1, 3, -2), (6, 3, 0, -7)$

(v) $(1, -5, -2, 3), (1, 0, 0, -1), (1, 0, 2, 4)$

(vi) $(1, -2, 1), (2, 1, -1), (7, -4, 1)$

(vii) $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$

2. (i) Find a if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

(ii) Find p if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ p \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

(iii) Find a if the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ a \end{bmatrix}$ are linearly dependent.

(iv) Find k if the vectors $\begin{bmatrix} 2 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$ are linearly dependent.

3. Show that the three row vectors as well as the three column vectors of the matrix $\begin{bmatrix} 2 & 3 & 6 \\ 0 & 4 & 2 \\ 5 & 2 & 2 \end{bmatrix}$ are linearly independent.

4. Prove that a square matrix A is singular if and only if its column (rows) are linearly dependent.

Answers

1. (i) Linearly dependent (ii) Linearly dependent (iii) Linearly dependent
 (iv) Linearly dependent (v) Linearly independent (vi) Linearly dependent
 (vii) Linearly independent

2. (i) $a = 1$ (ii) $p = -1$ (iii) $a = 6$ (iv) $k = -4$

2.3 Row rank and Column rank

Def. Row rank : The maximum number of linearly independent rows of a matrix A is called the row rank of A and is denoted by $\rho_R(A)$.

Def. Column rank : The maximum number of linearly independent columns of a matrix A is called column rank of A and is denoted by $\rho_C(A)$.

Elementary operations : There are six elementary operations, three elementary row operations and three elementary column operations. These are given below:

Def. Elementary Row operations : The following three types of operations on the rows of a matrix are called elementary row operations.

- The interchange of any two distinct rows of a matrix. If the i th row is interchanged with j th row, then it is denoted by $R_i \leftrightarrow R_j$ or $R_{i,j}$.
- The multiplication of any row (each element of the row) by a fixed non-zero scalar. If i th row is multiplied by $k \neq 0$ then it is denoted by $R_i \rightarrow k R_i$ or $R_i(k)$.
- The addition to the elements of a row, a fixed scalar multiple of the corresponding elements of another row. If k times the elements of j th row are added to the corresponding elements of i th row then it is denoted by $R_i \rightarrow R_i + k R_j$ or $R_{i,j}(k)$.

Def. Elementary column operations : The following three types of operations on the columns of a matrix are called elementary column operations.

- The interchange of any two distinct columns of a matrix. If the i th column is interchanged with j th column, then it is denoted by $C_i \leftrightarrow C_j$ or $C_{i,j}$.
- The multiplication of any column (each element of the column) by a fixed non-zero scalar. If i th column is multiplied by $k \neq 0$ then it is denoted by $C_i \rightarrow k C_i$ or $C_i(k)$.
- The addition to the elements of a column, a fixed scalar multiple of the corresponding elements of another column. If k times the elements of j th column are added to the corresponding elements of i th column then it is denoted by $C_i \rightarrow C_i + k C_j$ or $C_{i,j}(k)$.

Def. Row equivalent matrices : A matrix A is said to be row equivalent to a matrix B if B can be obtained from A by applying, in succession, a finite number of elementary row operations on A and we write $A \sim^R B$.

Def. Column Equivalent matrices : A matrix A is said to be column equivalent to a matrix B if B can be obtained from A by applying in succession a finite number of elementary column operations on A and we write $A \overset{C}{\sim} B$.

Def. Equivalent Matrices : Two matrices A and B are said to be equivalent if one can be obtained from the other by applying a finite number of elementary operations in succession. Then we write $A \sim B$.

Def. Row-echelon form of a matrix : A matrix is said to be in row-echelon form if

- (i) All zero rows, if any, are at the bottom of the matrix.
- (ii) First non-zero element of every row is on the right hand side of the first non-zero element in the preceding row.

Remark : The first non-zero element of any row is called key-element or pivotal element or pivot of that row.

Def. Row reduced echelon form of a matrix : A matrix is said to be in row reduced echelon form if

- (i) It is in row echelon form.
- (ii) Every key element is unity.
- (iii) The elements above the key element in every column are all zero.

Def. Column-echelon form of a matrix : A matrix is said to be in column-echelon form if

- (i) All zero columns, if any, are at the extreme right of the matrix.
- (ii) First non-zero element of every column is below the first non-zero element in the preceding column.

Remark : The first non-zero element of any column is called key-element or pivotal element or pivot of that column.

Def. Column reduced echelon form of a matrix : A matrix is said to be in column reduced echelon form if

- (i) It is in column echelon form.
- (ii) Every key element is unity.
- (iii) The elements to the left of the key element in every row are all zero.

Note : If A is in row echelon form, then its transpose A' is in the column echelon form.

Results :

1. The elementary operations do not alter the rank of a matrix.
2. For any matrix A , $\rho_R(A) = \rho(A) = \rho_C(A)$.

Proof : Let $\rho_R(A) = r$ and let B be the matrix obtained when A is reduced to row-echelon form. Then by definition of row-rank, B contains r key elements (or r non-zero rows) and all the zero rows are at the bottom of B . Removing all the zero rows (if any) and those columns which do not contain a key element, we will obtain a minor of order r of the triangular form

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$$|C| = \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ 0 & c_{22} & \dots & c_{2r} \\ \vdots & & & \\ 0 & 0 & \dots & c_{rr} \end{vmatrix} \quad \text{where } c_{11} \neq 0, c_{22} \neq 0, \dots, c_{rr} \neq 0$$

Then, $|C| = c_{11} c_{22} \dots c_{rr} \neq 0$. Also, any minor of order $r+1$ (if any) will contain at least one zero row at its bottom and hence $\rho(B) = r$. But elementary row operations do not alter the rank and B is obtained from A by elementary operations, so

Therefore, $\rho(A) = r = \rho_R(A)$

Hence rank of A and row rank of A are equal. Similarly, we can prove that rank of A and column rank of A are equal. Therefore $\rho_R(A) = \rho(A) = \rho_C(A)$

3. Let A be a $m \times n$ matrix, then rank of A is at most $\min\{m, n\}$ is $\rho(A) \leq \min\{m, n\}$.

4. $A = 0$ iff $\rho(A) = 0$ or zero matrix is the only matrix whose rank is zero.

5. $A \neq 0$ iff $\rho(A) \geq 1$ or rank of a non zero matrix is at least one.

6. Let A be $n \times n$ matrix, then $\rho(A)$ is n iff $\det A \neq 0$ or $\rho(A) < n$ iff $\det A = 0$

OR

$\rho(A) = n$ iff A is non-singular (invertible) or $\rho(A) < n$ iff A is singular (non-invertible).

7. Let A and B are two matrices of same size, then $|\rho(A) - \rho(B)| \leq \rho(A+B) \leq \rho(A) + \rho(B)$

8. Let A and B are two matrices such that AB is defined, then $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

i.e. $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$

9. Let A be a $m \times n$ matrix and B is a $n \times p$ matrix, then $\rho(AB) = \begin{cases} \rho(A) & \text{if } \rho(B) = n \\ \rho(B) & \text{if } \rho(A) = n \end{cases}$

OR

Let A be a $m \times n$ matrix and B be a $n \times p$ matrix if rank of any one matrix is n , then $\rho(AB)$ is equal to the rank of other matrix.

10. Frobenius inequality of rank : Let A and B are two $n \times n$ matrices, then $\rho(A) + \rho(B) \leq \rho(AB) + n$

Corollary : If A is a nilpotent matrix of size n and index of nilpotency is 2 i.e., $A^2 = 0$ then

$$\rho(A) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof : Put $B = A$ in Frobenius inequality $\rho(A) + \rho(A) \leq \rho(A^2) + n$

$$\Rightarrow 2\rho(A) \leq 0 + n \Rightarrow \rho(A) \leq \frac{n}{2} \Rightarrow \rho(A) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

[$\because \rho$ is always non negative integer]

Def. Nullity of a matrix : The dimension of the null space of matrix is called its nullity.

11. Rank-nullity theorem : Sum of rank and nullity of a matrix is always equal to its number of columns.

Def. Gram Matrix : Let A be any real matrix then the matrix AA' and $A'A$ are called Gram Matrix w.r.t. A .

Remark : The gram matrix is always symmetric.

$$12. \rho(A) = \rho(A') = \rho(\bar{A}) = \rho(A^*) = \rho(AA') = \rho(A'A)$$

$$13. \text{ If } \rho(A) = n-1 \text{ then } \rho(\text{adj } A) = 1.$$

Proof : By Frobenius inequality, $\rho(A) + \rho(B) \leq \rho(AB) + n$

Taking $B = \text{adj } A$, we have

$$\rho(A) + \rho(\text{adj } A) \leq \rho(A \text{adj } A) + n$$

$$\Rightarrow n-1 + \rho(\text{adj } A) \leq \rho(|A|I_n) + n \quad [\text{Since } A \text{adj } A = |A|I_n]$$

$$\Rightarrow n-1 + \rho(\text{adj } A) \leq 0 + n \quad [\text{Since } \rho(A) = n-1, \text{ therefore } |A| = 0.]$$

$$\Rightarrow \rho(\text{adj } A) \leq 1 \quad \dots\dots(1)$$

$$\text{Since, } \rho(A) = n-1 \Rightarrow \text{adj } A \neq 0$$

$$\Rightarrow \rho(\text{adj } A) \geq 1 \quad \dots\dots(2)$$

From (1) and (2), $\rho(\text{adj } A) = 1$

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Exercise 2.3

1. Reduce the following matrices to the row reduced echelon form, Also find their row rank.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -3 & -6 \\ 3 & -1 & 1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 3 & 1 \\ 2 & -2 & 1 & 0 & 2 \\ 1 & 1 & -1 & -3 & 3 \end{bmatrix}$

2. Reduce the following matrices to the column reduced echelon form, Also find their column rank.

(i) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$

3. Find the rank of the following matrices without using minors :

(i) $\begin{bmatrix} 1 & -2 & -1 & 4 \\ 2 & -4 & 3 & 5 \\ -1 & 2 & 6 & -7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} 4 & 4 & 0 & 3 \\ -2 & 3 & -1 & 5 \\ 1 & 4 & 8 & 7 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -4 & 1 & 2 \\ 5 & 2 & 1 & 3 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$

(x) $\begin{bmatrix} 3 & -2 & 0 & -1 & -1 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -3 & 1 \\ 0 & 1 & 1 & 2 & 6 \end{bmatrix}$

(xi) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

4. Using only 0's and 1's, list all possible 2×2 matrices in row reduced echelon form.

5. Using only 0's and 1's, find the number n of possible 3×3 matrices in row reduced echelon form.

Everything you want is on the other side of fear. - Jack Canfield

Answers

1. (i) $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 2 \end{bmatrix}$ and $\rho_R(A)=2$

(ii) $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{9} \\ 0 & 0 & 1 & \frac{11}{9} \end{bmatrix}$ and $\rho_R(A)=3$

(iii) $\begin{bmatrix} 1 & 0 & \frac{7}{4} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, $\rho_R(A)=2$

(iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ and $\rho_R(A)=3$

(v) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\rho_R(A)=4$

(vi) $\begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\rho_R(A)=3$

2. (i) 2 (ii) 2 (iii) 3

3. (i) 2 (ii) 3 (iii) 2 (iv) 2 (v) 3 (vi) 3 (vii) 2 (viii) 3 (ix) 2 (x) 4 (xi) 2

4. 5: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 5. 16

2.4 Normal form

Def. Normal form of a matrix : (Canonical form) : Any of the following form of a matrix are called normal (canonical) form

$$I_r, \begin{bmatrix} I_r & : & O \\ \dots & & \dots \\ O & : & O \end{bmatrix}, [I_r : O], \begin{bmatrix} I_r \\ \dots \\ O \end{bmatrix} \text{ where } I_r \text{ is the identity matrix of order } r.$$

If $A = I_r$ or $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ or $[I_r \ O]$, $\begin{bmatrix} I_r \\ O \end{bmatrix}$ then $\rho(A) = r$

Theorem : If A is an $m \times n$ matrix of rank r , then A is equivalent to the matrix $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ which is normal form.

Proof : If A is null matrix then clearly it is in the normal form so let $A \neq O$

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Let $a_{ij} = \lambda$ be a non-zero element of A . We can bring this to $(1, 1)$ position i.e., top left position by interchanging the first row with the i th row and then the first column with j th column. Thus A is equivalent to matrix B with $\lambda \neq 0$ as its $(1, 1)$ entry.

Now operate $R_1 \left(\frac{1}{\lambda} \right)$ so that first entry of B becomes 1. By subtracting suitable multiples of first row from the remaining rows, we can make all elements 0 in the first column below the first element.

Similarly, by subtracting suitable multiples of the first column from the remaining columns of B , we make all elements 0 except the first element in the first row. Let the new matrix be C .

Thus

$$A \sim C = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} I_1 & O \\ O & D \end{bmatrix}$$

If $D = 0$, we stop, otherwise we repeat the argument with D . The row and column operations will not change the entries in the first row and first column.

Thus, we can obtain

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} I_2 & O \\ O & E \end{bmatrix}$$

Continuing the above process, after a finite number of row and column operations, we have

$$A \sim \begin{bmatrix} I_k & O \\ O & O \end{bmatrix}, \quad \text{which is the normal form.}$$

Since $\rho(A) = r$ and rank of $\begin{bmatrix} I_k & O \\ O & O \end{bmatrix} = k$. Therefore $r = k$

Exercise 2.4

Reduce the following matrices to the normal form and hence find their ranks :

1. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 9 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 4 & 5 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

$$9. \begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix} \quad 10. \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Answers

1. $I_3, \rho=3$ 2. $I_3, \rho=3$ 3. $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \rho=2$
4. $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \rho=2$ 5. $[I_3 \ 0], \rho=3$ 6. $[I_3 \ 0], \rho=3$
7. $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \rho=3$ 8. $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \rho=3$ 9. $[I_3 \ 0], \rho=3$ 10. $I_4, \rho=4$

2.5 Elementary Matrices

Def. A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix (or *E*-matrix).

For example, if we operate $R_1 \leftrightarrow R_2$, $R_3 \rightarrow R_3 - 3R_1$, $R_2 \rightarrow R_2 + 2R_1$ on the unit matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ we obtain the following elementary matrices } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notation for elementary matrices :

- (i) E_{ij} will denote the elementary matrix obtained by interchanging the i^{th} and j^{th} row of a unit matrix.
- (ii) F_{ij} will denote the elementary matrix obtained by interchanging the i^{th} and j^{th} column of a unit matrix.
- (iii) $E_i(k)$ will denote the elementary matrix obtained by multiplying the i^{th} row of a unit matrix by a non-zero number k .
- (iv) $F_i(k)$ will denote the elementary matrix obtained by multiplying the i^{th} column of a unit matrix by a non-zero number k .
- (v) $E_{ij}(k)$ will denote the elementary matrix, obtained by adding to the elements of the i^{th} row of a unit matrix the products by any number k of the corresponding elements of the j^{th} row.

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(vi) $F_{ij}(k)$ will denote the elementary matrix, obtained by adding to the elements of the i^{th} column of a unit matrix the products by any number k of the corresponding elements of the j^{th} column.

Remark : Here it should be noted that matrices obtained by interchanging the i^{th} and j^{th} row of a unit matrix or by interchanging the i^{th} and j^{th} column of a unit matrix are same. Therefore, we have

$$E_{ij} = F_{ij}$$

Similarly, we have $E_i(k) = F_i(k)$

But it is very interesting to note that $E_{ij}(k) = F_{ji}(k)$

Result 1 : The elementary matrices are non-singular.

Proof. We know that there are three type of elementary matrices, namely,

$$E_{ij}, E_i(k) \text{ and } E_{ij}(k)$$

Now, E_{ij} is obtained from identity matrix by interchange of two rows (or columns), so

$$|E_{ij}| = -|I| = -1 \neq 0$$

Again, $E_i(k)$ is obtained from identity matrix by multiplying the i^{th} row (or column) by $k (\neq 0)$, so

$$|E_i(k)| = k|I| = k.1 = k \neq 0$$

Finally, $E_{ij}(k)$ is obtained from identity matrix by multiplying the j^{th} row (or column), by k and then adding it to i^{th} row (or column) so by properties of determinants, we have $|E_{ij}(k)| = |I| = 1 \neq 0$.

Hence, elementary matrices are non-singular.

Result 2 : Let A and B are two matrices of order $m \times n$ and $n \times p$. Then any elementary row operation applied on the pre-factor of AB or on the product AB gives the same result.

Another form of above result : Let $B = PA$, where P and A are the matrices conformable for multiplication. If any elementary row operation is applied on B then on the right hand side it will be applied on pre-factor P only.

Result 3 : If A is an $m \times n$ matrix and E an elementary row matrix of order m , then EA is the same as the matrix obtained from A by applying the elementary row operation to which E corresponds.

Result 4 : Let A and B are two matrices of order $m \times n$ and $n \times p$. Then any elementary column operation applied on the post-factor of AB or on the product AB gives the same result.

Result 5 : If A is an $m \times n$ matrix and F an elementary column matrix of order n , then AF is the same as the matrix obtained from A by applying the elementary column operations to which F corresponds.

Result 6 : The inverse of any elementary matrix is also an elementary matrix of same type.

Result 7 : Inverse of elementary matrices :

$$\begin{aligned} \text{(i)} \quad E_{ij}^{-1} &= E_{ij} & \text{(ii)} \quad E_i^{-1}(k) &= E_i\left(\frac{1}{k}\right) & \text{(iii)} \quad E_{ij}^{-1}(k) &= E_{ij}(-k) \\ \text{(iv)} \quad F_{ij}^{-1} &= F_{ij} & \text{(v)} \quad F_i^{-1}(k) &= F_i\left(\frac{1}{k}\right) & \text{(vi)} \quad F_{ij}^{-1}(k) &= F_{ij}(-k) \end{aligned}$$

Result 8 : Two matrices A and B are equivalent if and only if there exist non-singular matrices P and Q such that $PAQ = B$.

Proof : Let A and B be equivalent. Then, B can be obtained from A by applying in succession a finite number of row and column operations. Let $E_1, E_2, \dots, E_r$; $F_1, F_2, \dots, F_s$ be the elementary matrices corresponding to the row and column operations which reduce A to B . Since each elementary row (column) operation of a matrix can be brought about by pre-multiplication (post-multiplication) with the corresponding elementary matrix, therefore, we have

$$\begin{aligned} E_r \dots E_2 E_1 A F_1 F_2 \dots F_s &= B \\ \Rightarrow PAQ &= B \text{ where } P = E_r \dots E_2 E_1 \text{ and } Q = F_1 F_2 \dots F_s \end{aligned}$$

P and Q are non-singular matrices since the elementary matrices are non-singular and product of non-singular matrices is non-singular.

Cor 1 : If A is an $m \times n$ matrix of rank r , then there exist non-singular matrices P and Q of order m and n respectively, such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

Proof : Since A is of rank r , A is equivalent to the matrix $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ in the normal form.

$\therefore$ there exists non-singular matrices P and Q of order m and n respectively, such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

Cor. 2 : If A is a square non-singular matrix of order n then, there exist non-singular matrices P and Q such that $PAQ = I_n$.

Proof : Since every matrix is equivalent to its normal form so A is equivalent to I_n .

$\therefore$ There exist non-singular matrices P and Q (as in the above theorem) such that $PAQ = I_n$.

Cor. 3 : Every non-singular matrix can be expressed as a product of elementary matrices.

Proof : Let A be a non-singular matrix of order n .

$\therefore$ there exists non-singular matrices P and Q (by above theorem) such that $PAQ = I_n$

Let $P = E_1 E_2 \dots E_r$ and $Q = F_1 F_2 \dots F_s$ where $E_1, E_2, \dots, E_r$ and $F_1, F_2, \dots, F_s$ are elementary matrices.

Thus, $E_1 E_2 \dots E_r A F_1 F_2 \dots F_s = I_n$

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$$\Rightarrow A = E_r^{-1} \dots E_2^{-1} E_1^{-1} F_s^{-1} \dots F_2^{-1} F_1^{-1}$$

$\Rightarrow A$ is the product of elementary matrices since the inverse of an elementary matrix is an elementary matrix.

Cor. 4 : (i) If A is non-singular, then $\rho(AB) = \rho(B)$ and $\rho(BA) = \rho(B)$.

(ii) If P and Q are non-singular, then $\rho(PAQ) = \rho(A)$.

Proof : (i) Every non-singular matrix can be expressed as the product of elementary matrices. Also, elementary row (column) operations are equivalent to pre-multiplication (post multiplication) with the corresponding elementary matrices and elementary matrices do not alter the rank of a matrix. Hence, the result.

(ii) Directly follows from part (i).

Result 9 : The rank of the product of two matrices cannot exceed the rank of either matrix.

i.e.,
$$\rho(AB) \leq \rho(A) \quad \text{and} \quad \rho(AB) \leq \rho(B).$$

Working rule to express a non-singular matrix as a product of elementary matrices :

- (i) First reduce the given matrix A into the unit matrix by using elementary row operations and elementary column operations.
- (ii) Suppose $E_1, E_2, \dots, E_r$ are the elementary matrices corresponding to the elementary row operations which are applied in succession in above step (i). Similarly let $F_1, F_2, \dots, F_s$ be the elementary matrices corresponding to the elementary column operations which are applied in succession in above step (i).

(iii) Then we can write
$$I_n = (E_r \dots E_2 E_1) A (F_1 F_2 \dots F_s) \quad \dots (1)$$

Here it should be noted that the matrices E_i 's are written in reverse order on left of A and the matrices F_i 's are written in direct order on right of A .

(iv) By (1), we have

$$(E_r \dots E_2 E_1)^{-1} I_n (F_1 F_2 \dots F_s)^{-1} = (E_r \dots E_2 E_1)^{-1} (E_r \dots E_2 E_1) A (F_1 F_2 \dots F_s) (F_1 F_2 \dots F_s)^{-1}$$

$$\Rightarrow A = (E_r \dots E_2 E_1)^{-1} (F_1 F_2 \dots F_s)^{-1}$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} \dots E_r^{-1} F_s^{-1} \dots F_2^{-1} F_1^{-1}$$

Example 1 : Express $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ as the product of elementary matrices.

Solution : The given matrix is $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

Operating $R_3 \rightarrow R_3 + 2R_2$, $A \sim \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 7 \end{bmatrix}$

Operating $R_2 \rightarrow R_2 + R_1$, $A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

Operating $C_2 \rightarrow C_2 - 2C_1$, $C_3 \rightarrow C_3 - C_1$, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

Operating $R_2 \leftrightarrow R_3$, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 2 & 3 \end{bmatrix}$

Operating $R_3 \rightarrow R_3 - 2R_2$, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 0 & -11 \end{bmatrix}$

Operating $C_3 \rightarrow C_3 - 7C_2$, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{bmatrix}$

Operating $R_3 \rightarrow -\frac{1}{11} R_3$, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We note that applying the elementary operations

$$R_{3,2}(2), R_{2,1}(1), C_{2,1}(-2), C_{3,1}(-1), R_{2,3}, R_{3,2}(-2), C_{3,2}(-7), R_3\left(-\frac{1}{11}\right)$$

we can reduce A to I_3 .

If $E_{32}(2), E_{21}(1), F_{21}(-2), F_{31}(-1), E_{23}, E_{32}(-2), F_{32}(-7), E_3\left(-\frac{1}{11}\right)$ are the corresponding elementary matrices then

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$$I_3 = \left[E_3 \left(-\frac{1}{11} \right) E_{32}(-2) E_{23} E_{21}(1) E_{32}(2) \right] A \left[F_{21}(-2) F_{31}(-1) F_{32}(-7) \right]$$

$$\Rightarrow A = \left[E_3 \left(-\frac{1}{11} \right) E_{32}(-2) E_{23} E_{21}(1) E_{32}(2) \right]^{-1} \left[F_{21}(-2) F_{31}(-1) F_{32}(-7) \right]^{-1}$$

$$\Rightarrow A = (E_{32}(2))^{-1} (E_{21}(1))^{-1} (E_{23})^{-1} (E_{32}(-2))^{-1} \left(E_3 \left(-\frac{1}{11} \right) \right)^{-1} (F_{32}(-7))^{-1} (F_{31}(-1))^{-1} (F_{21}(-2))^{-1}$$

$$\Rightarrow A = E_{32}(-2) E_{21}(-1) E_{23} E_{32}(2) E_3(-11) F_{32}(7) F_{31}(1) F_{21}(2)$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Working Rule to find P and Q so that PAQ is in normal form :

(i) If A is a $m \times n$ matrix then we write $A = I_m A I_n$ (1)

where I_m and I_n denotes the identity matrices of order m and n respectively.

(ii) Check the entry a_{11} . If $a_{11} \neq 0$, its okay but if $a_{11} = 0$, then make it non-zero by suitable row operations or column operations. Any row operation which is applied on L.H.S. of (1) is also to be applied on pre-factor I_m of R.H.S. of (1). Similarly any column operation which is applied on L.H.S. of (1) is also to be applied on post factor I_n of R.H.S.

(iii) Make a_{11} equal to 1 by multiplying first row by $\frac{1}{a_{11}}$ and apply the same row operation on the pre-factor of R.H.S.

(iv) Make 0's below a_{11} by applying row operation and apply the same row operation on pre-factor of R.H.S.

(v) Make 0's on the right of a_{11} by applying column operations and apply the same column operations on the post factor of R.H.S.

(vi) Repeat the above process on $a_{22}, a_{33}, \dots$ until all non-diagonal elements on the L.H.S.

become zero and suppose we have obtained $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix} = P A Q$ and then r is the rank of the matrix A .

Example 2 : Find the non-singular matrices P and Q such that PAQ is in the normal form, where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ the matrix A is of order 3×4 , so we consider

$$A = I_3 A I_4 \quad \text{i.e.} \quad \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \rightarrow R_2 + (-2)R_1$, $R_3 \rightarrow R_3 + (-3)R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \rightarrow R_3 + (-1)R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \rightarrow C_2 + (-2)C_1$, $C_3 \rightarrow C_3 + (-3)C_1$, $C_4 \rightarrow C_4 + 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Operating } C_2 \rightarrow \left(-\frac{1}{6}\right)C_2, \text{ we get } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -3 & 2 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Operating $C_3 \rightarrow C_3 + 5C_2$, $C_4 \rightarrow C_4 + (-7)C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} I_2 & : & O \\ \dots & & \dots \\ O & : & O \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.5

1. Find the elementary matrices corresponding to the following elementary operations. Also find the inverse of the elementary matrices and verify your answer.

- (i) $R_{1,3}$ (ii) $R_2(3)$ (iii) $R_{1,2}(4)$ (iv) $C_{2,3}$ (v) $C_3(5)$ (vi) $C_{3,1}(7)$

2. Express the following matrices as a product the elementary matrices

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -2 \\ 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 & -1 \\ 4 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

3. For the following find non-singular matrices P and Q such that PAQ is in normal form and hence determine the rank of A .

(i) $\begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

Answers

$$2. (i) \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv)

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Although the matrices P and Q are not unique here we are giving one of the answers

$$(i) P = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1/3 & 8/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}; \rho(A) = 2$$

$$(ii) P = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}; \rho(A) = 2$$

$$(iii) P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \rho(A) = 2$$

$$(iv) P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \rho(A) = 2$$

$$(v) P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1 & -2/3 & 1/3 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 4/7 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}; \rho(A) = 2$$

$$(vi) P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2/3 & 1/3 & -4/3 \\ 0 & 1/3 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \rho(A) = 3$$

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$$(vii) P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 0 & -1/3 & 0 \\ 5/16 & -3/16 & -1/8 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & -1/3 & -3/4 \\ 0 & 1 & -5/3 & -3/4 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \rho(A) = 3$$

3.1 System of linear equations

System of non-homogeneous linear equations : Consider a system of m linear non-homogeneous equations in n unknowns $x_1, x_2, \dots, x_n$

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \right\} \dots (1)$$

This system can be written as a single matrix equation $AX=B$

where $A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

The matrix A is called the co-efficient matrix of the system of equations (1). Any set of values of $x_1, x_2, \dots, x_n$ which simultaneously satisfy all these equations is called a solution of system (1). When the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

The matrix $[A : B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \dots & \dots & \dots & \dots & : & \dots \\ \dots & \dots & \dots & \dots & : & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$ is called the **Augmented matrix** of the given

system.

Results:

1. The system of linear equations $AX = B$ is consistent i.e., possesses a solution if and only if the co-efficient matrix A and the augmented matrix $[A : B]$ are of the same rank.
2. If A be an $n \times n$ square matrix and X and B be $n \times 1$ matrices then the system of equations $AX = B$ possesses a unique solution if the matrix A is non-singular (i.e., if A^{-1} exists).

3. Working rule for finding a solution of a system of non-homogeneous linear equations $AX = B$:
Write the augmented matrix $[A : B]$ and reduce it to row echelon form to find the rank of $[A : B]$ and A . Now
- (i) If $\rho[A : B] \neq \rho(A)$, the system of equations is inconsistent.
 - (ii) If $\rho[A : B] = \rho(A) = \text{number of unknowns}$, the system is consistent and has a unique solution.
 - (iii) If $\rho[A : B] = \rho(A) < \text{number of unknowns}$, the system is consistent and has an infinite number of solutions. In this case if there are n unknowns and $\rho(A) = r$, then by giving arbitrary values to $n - r$ unknowns, we get the values of r unknowns.

Exercise 3.1

1. (i) Show that the system of equations
 $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$ is not consistent.
- (ii) Solve the equations :
 $x + y + 2z + w = 5$, $2x + 3y - z - 2w = 2$, $4x + 5y + 3z = 7$
- (iii) Check whether the following system equations is consistent or not :
 $2x - y + z = 4$, $3x - y + z = 6$, $4x - y + 2z = 7$, $-x + y - z = 9$
2. Check the consistency of the following system of equations and find the solution if it is consistent :
 - (i) $2x - 3y + z = 9$, $x + y + z = 6$, $x - y + z = 2$
 - (ii) $4x + 3y + 2z = -7$, $2x + y - 4z = -1$, $x + 2y + z = 1$
 - (iii) $x + 2y + 3z + 4t = 0$, $2x + 3y + 4z = 1$, $3x + 4y + t = 2$, $4x + z + 2t = 3$
3. Check the consistency of the following system of equations and solve if it is consistent :
 - (i) $x + 3y - z = 4$, $2x + y + z = 7$, $2x - 4y + 4z = 6$, $3x + 4y = 11$
 - (ii) $x - 4y - 3z = -16$, $2x + 7y + 12z = 48$, $4x - y + 6z = 16$, $5x - 5y + 3z = 0$
 - (iii) $2x - y + 3z = 3$, $x + 2y - z - 5w = 4$, $x + 3y - 2z - 7w = 5$
4. For what values of a and b do the system of equations
 $x + y + 5z = 0$, $x + 2y + 3az = b$, $x + 3y + az = 1$
 have (i) no solution (ii) unique solution (iii) infinitely many solutions.
5. For what value of a and b do the system of equations
 $x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have
 (i) no solution (ii) a unique solution (iii) more than one solution?
6. Investigate the values of λ and μ so that the equations
 $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have
 (i) no solution (ii) a unique solution (iii) an infinite number solution.

7. For what value of λ , the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case.
8. Show that the equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ have no solution unless $a + b + c = 0$ in which case they have infinitely many solutions. Find the solution when $a = 1$, $b = 1$, $c = -2$.

9. For what value of λ , does the system $\begin{bmatrix} 1 & -2 & -1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ has

(i) no solution (ii) unique solution (iii) More than one solution

10. Solve the system of equations $\lambda x + 2y - 2z = 1$, $4x + 2\lambda y - z = 2$, $6x + 6y + \lambda z = 3$ considering specially the case when $\lambda = 2$.

11. Determine whether each of the following systems is linear :

(a) $3x - 4y + 2yz = 8$ (b) $ex + 3y = \pi$ (c) $2x - 3y + kz = 4$

12. Solve : (a) $\pi x = 2$ (b) $3x + 2 = 5x + 7 - 2x$ (c) $6x + 2 - 4x = 5 + 2x - 3$

13. Solve each of the following systems :

(a) $2x + 3y = 1$ (b) $4x - 2y = 5$ (c) $2x - 4 = 3y$ (d) $2x - 4y = 10$
 $5x + 7y = 3$ $-6x + 3y = 1$ $5y - x = 5$ $3x - 6y = 15$

14. Consider each of the following systems in unknowns x and y :

(a) $x - ay = 1$ (b) $ax + 3y = 2$ (c) $x + ay = 3$
 $ax - 4y = b$ $12x + ay = b$ $2x + 5y = b$

For which values of a does each system have a unique solution, and for which pairs of values (a, b) does each system have more than one solution ?

15. Solve :

(a) $x + y + 2z = 4$ (b) $x - 2y + 3z = 2$ (c) $x + 2y + 3z = 3$
 $2x + 3y + 6z = 10$ $2x - 3y + 8z = 7$ $2x + 3y + 8z = 4$
 $3x + 6y + 10z = 17$ $3x - 4y + 13z = 8$ $5x + 8y + 19z = 11$

16. Solve :

(a) $x - 2y = 5$ (b) $x + 2y - 3z + 2t = 2$ (c) $x + 2y + 4z - 5t = 3$
 $2x + 3y = 3$ $2x + 5y - 8z - 6t = 5$ $3x - y + 5z + 2t = 4$
 $3x + 2y = 7$ $3x + 4y - 5z + 2t = 4$ $5x - 4y + 6z + 9t = 2$

17. Solve :

$$(a) \quad 2x - y - 4z = 2$$

$$(b) \quad x + 2y - z + 3t = 3$$

$$4x - 2y - 6z = 5$$

$$2x + 4y + 4z + 3t = 9$$

$$6x - 3y - 8z = 8$$

$$3x + 6y - z + 8t = 10$$

18. Consider each of the following systems in unknowns x, y, z :

$$(a) \quad x - 2y = 1$$

$$(b) \quad x + 2y + 2z = 1$$

$$(c) \quad x + y + az = 1$$

$$x - y + az = 2$$

$$x + ay + 3z = 3$$

$$x + ay + z = 4$$

$$ay + 9z = b$$

$$x + 11y + az = b$$

$$ax + y + z = b$$

For which values of a does the system have a unique solution, and for which pairs of values (a, b) does the system have more than one solution? The value of b does not have any effect on whether the system has a unique solution. Why?

19. Write v as a linear combination of u_1, u_2, u_3 , where :

$$(a) \quad v = (4, -9, 2), \quad u_1 = (1, 2, -1), \quad u_2 = (1, 4, 2), \quad u_3 = (1, -3, 2);$$

$$(b) \quad v = (1, 3, 2), \quad u_1 = (1, 2, 1), \quad u_2 = (2, 6, 5), \quad u_3 = (1, 7, 8);$$

$$(c) \quad v = (1, 4, 6), \quad u_1 = (1, 1, 2), \quad u_2 = (2, 3, 5), \quad u_3 = (3, 5, 8).$$

Answers

1. (i) Inconsistent system

(iii) Inconsistent

$$2. (i) \quad x = 11, \quad y = 2, \quad z = -7$$

$$(ii) \quad x = -\frac{13}{4}, \quad y = \frac{5}{2}, \quad z = -\frac{3}{4}$$

$$(iii) \quad x = \frac{9}{11}, \quad y = -\frac{1}{11}, \quad z = -\frac{1}{11}, \quad t = -\frac{1}{11}$$

$$3. (i) \quad x = \frac{17}{5} - \frac{4}{5}z, \quad y = \frac{1}{5} + \frac{3}{5}z \quad (ii) \quad x = -\frac{9}{5}k + \frac{16}{3}, \quad y = -\frac{6}{5}k + \frac{16}{3}, \quad z = k$$

$$(iii) \quad x = 2 - k_1 + k_2, \quad y = 1 + k_1 + 2k_2, \quad z = k_1, \quad w = k_2$$

$$4. (i) \quad a = 1, \quad b \neq \frac{1}{2} \quad (ii) \quad a \neq 1, \quad b \text{ is arbitrary} \quad (iii) \quad a = 1, \quad b = \frac{1}{2}$$

$$5. (i) \quad a = 8, \quad b \neq 15 \quad (ii) \quad a \neq 8, \quad b \text{ may have any value} \quad (iii) \quad a = 8, \quad b = 15$$

$$6. (i) \quad \lambda = 5, \quad \mu \neq 9 \quad (ii) \quad \lambda \neq 5, \quad \mu \text{ arbitrary} \quad (iii) \quad \lambda = 5, \quad \mu = 9$$

$$7. \text{ For } \lambda = 1; \quad x = 1 + 2k, \quad y = -3k, \quad z = k \quad , \text{ for } \lambda = 2; \quad x = 2k, \quad y = 1 - 3k, \quad z = k$$

$$8. \quad x = z - 1, \quad y = z - 1 \quad 9. (i) \quad \lambda = 1, (ii) \quad \lambda \neq 1 (iii) \text{ More than one solution is not possible.}$$

$$10. \quad x = \frac{1}{2} - k, \quad y = k, \quad z = 0$$

11. (a) no,

(b) yes,

(c) linear x, y, z , not linear in x, y, z, k

- Man is not the creature of circumstances; circumstances are the creatures of man. - Benjamin Disraeli*

Results :

1. Since zero solution is always a solution of a homogeneous system so the system of homogeneous linear equations is always consistent.
2. Linear combination of solutions of a homogeneous system is also a solution.

Suppose X_1 and X_2 are two solutions of (2). Then their linear combination $k_1X_1 + k_2X_2$ is also a solution of (2), where k_1 and k_2 are any arbitrary numbers. Since X_1 and X_2 are solutions of (2)

$$\therefore AX_1 = O \text{ and } AX_2 = O \Rightarrow A(k_1X_1 + k_2X_2) = k_1AX_1 + k_2AX_2 = k_1(O) + k_2(O) = O$$

Hence $k_1X_1 + k_2X_2$ is also a solution of (2).

Remark : Linear combination of solutions of a non-homogeneous system need not be a solution.

3. A homogeneous system of n linear equations in n unknowns, whose determinant of co-efficients does not vanish, has only trivial solution.
4. A system of m homogeneous equations in n unknowns has a solution other than trivial solution if and only if rank r of the coefficient matrix A is less than n , the number of unknowns.
5. The number of linearly independent solutions of m homogeneous linear equations in n variables $AX = O$ is $(n - r)$, where r is the rank of the matrix A .
6. Working rule for finding the solution of the system of homogeneous equation $AX = O$:
Reduce the coefficient matrix A to echelon form by applying elementary row transformations only and find the rank of the matrix A . Let it be r then
 - (i) If $\rho(A) = n$, the number of unknowns, then zero solution (trivial solution) is the only solution.
 - (ii) If $\rho(A) < n$, the number of unknowns, then the system will have infinitely many solutions of which $(n - r)$ solutions are linearly independent. These linearly independent solutions are obtained by giving arbitrary values to $(n - r)$ unknowns and solving for remaining r unknowns.
7. Dimension of solution space of the homogeneous system $AX = O$ is $n - r$ where n is number of unknowns and r is the rank of the matrix A .
8.
$$\left(\begin{array}{c} \text{General solution of non} \\ \text{homogeneous system} \end{array} \right) = \left(\begin{array}{c} \text{Particular solution of non} \\ \text{homogeneous system} \end{array} \right) + \left(\begin{array}{c} \text{General solution of corresponding} \\ \text{homogeneous system} \end{array} \right)$$

Remark : The set of all solutions of a non-homogeneous system does not form a vector space and therefore, there is no question of its basis and dimension.

9. In a homogeneous system,

- (i) If no. of variables $>$ no. of equations, then the system has infinitely many solutions.
- (ii) If no. of variables $\leq$ no. of equations, then the system has either unique or infinitely many solutions.

Proof: (i) Let $AX = O$ be a system such that A is an $m \times n$ matrix, where $n > m$. Then,

$$r = \text{rank of } A \leq \min\{m, n\} = m < n$$

$$\Rightarrow n-r > 0 \quad \Rightarrow n-r \geq 1 \quad \Rightarrow \text{system has atleast 1 free variable}$$

$\Rightarrow$ system has infinitely many solutions.

(ii) Let $AX = 0$ be a system such that A is an $m \times n$ matrix, where $n \leq m$,

$$\text{then } r = \text{rank of } A \leq \min\{m, n\} = n \Rightarrow n-r \geq 0.$$

If $n-r = 0$, then there is no free variable and hence system has a unique solution i.e.,

$$x = 0, y = 0, z = 0 \text{ (only trivial solution)}$$

If $n-r > 0$, then there is atleast 1 free variable and hence the system has infinitely many solutions.

10. In a non-homogeneous system,

(i) If no. of variables $>$ no. of equations, then the system has either no solution or infinitely many solutions.

(ii) If no. of variables $\leq$ no. of equations, then the system may have no solution, unique solution or infinitely many solutions.

11. (i) If $AX = 0$ has infinitely many solutions, then $AX = B$ has either no solution (if relation is not maintained) or infinitely many solutions (if relation is maintained).

(ii) If $AX = 0$ has a unique solution, then $AX = B$ has either unique solution or no solution.

Exercise 3.2

1. Solve the following system of equations and show they can have only trivial solution:

(i) $x - y + z = 0$

$$x + 2y - z = 0$$

$$2x + y + 3z = 0$$

(ii) $x - y + z = 0$

$$-3x + y - 4z = 0$$

$$7x - 3y - 9z = 0$$

$$4x - 2y + 5z = 0$$

2. Solve the following system of equations :

(i) $x - y + 2z - 3w = 0$

$$3x + 2y - 4z + w = 0$$

$$4x - 2y + 9w = 0$$

(ii) $x_1 + 2x_2 + x_3 + x_4 = 0$

$$3x_1 + 2x_2 + 3x_3 - 4x_4 = 0$$

$$4x_1 + x_2 - x_4 = 0$$

3. Solve the following system of equations completely :

(i) $3x - y - z + 2w = 0$

$$-6x - 2y + 2z + 4w = 0$$

$$12x + 3y - 4z - 6w = 0$$

(ii) $x - 2y + z - w = 0$

$$x + y - 2z + 3w = 0$$

$$4x + y - 5z + 8w = 0$$

$$5x - 7y + 2z - w = 0$$

(iii) $4x + 2y + z + 3u = 0$

$$6x + 3y + 4z + 7u = 0$$

$$2x + y + u = 0$$

4. (i) Find the value of k such that the system of equations $x + ky + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ has a non-trivial solution.

(ii) Find the value of k such that the system of equations

$$(3k - 8)x + 3y + 3z = 0$$

$$3x + (3k - 8)y + 3z = 0$$

$$3x + 3y + (3k - 8)z = 0$$

has a non-trivial solution.

5. Find the dimension and a basis of the general solution W of each of the following homogeneous systems :

(a) $x - y + 2z = 0$

(b) $x + 2y - 3z = 0$

(c) $x + 2y + 3z + t = 0$

$2x + y + z = 0$

$2x - 5y + 2z = 0$

$2x + 4y + 7z + 4t = 0$

$5x + y + 4z = 0$

$3x - y - 4z = 0$

$3x + 6y + 10z + 5t = 0$

6. Find the dimension and a basis of the general solution W of each of the following systems :

(a) $x_1 + 3x_2 + 2x_3 - x_4 - x_5 = 0$

(b) $2x_1 - 4x_2 + 3x_3 - x_4 + 2x_5 = 0$

$2x_1 + 6x_2 + 5x_3 + x_4 - x_5 = 0$

$3x_1 - 6x_2 + 5x_3 - 2x_4 + 4x_5 = 0$

$5x_1 + 15x_2 + 12x_3 + x_4 - 3x_5 = 0$

$5x_1 - 10x_2 + 7x_3 - 3x_4 + 18x_5 = 0$

Answers

1. (i) $x = y = z = 0$

(ii) $x = y = z = 0$

2. (i) $x = k$, $y = \frac{13}{2}k$, $z = \frac{17}{4}k$, $w = k$ (ii) $x_1 = \frac{11}{16}k$, $x_2 = -\frac{7}{4}k$, $x_3 = \frac{29}{16}k$, $x_4 = k$

3. (i) $x = \frac{1}{3}k_1$, $y = k_2$, $z = k_1$, $w = \frac{1}{2}k_2$

(ii) $x = k_1 - \frac{5}{3}k_2$, $y = k_1 - \frac{4}{3}k_2$, $z = k_1$, $w = k_2$

(iii) $x = k_1$, $u = k_2$, $y = -2k_1 - k_2$, $z = -k_2$

4. (i) $k = 0, \frac{9}{2}$ (ii) $k = \frac{11}{3}$ or $\frac{2}{3}$

5. (a) $\dim W = 1$, $u_1 = (-1, 1, 1)$ (b) $\dim W = 0$, no basis, (c) $\dim W = 2$, $u_1 = (-2, 1, 0, 0)$, $u_2 = (5, 0, -2, 1)$

6. (a) $\dim W = 3$, $u_1 = (-3, 1, 0, 0, 0)$, $u_2 = (7, 0, -3, 1, 0)$, $u_3 = (3, 0, -1, 0, 1)$

(b) $\dim W = 2$, $u_1 = (2, 1, 0, 0, 0)$, $u_2 = (-12, 0, 12, 14, 1)$

4.1 Vector Space (Definition and Examples)

Def. Vector Space : Let F be a field. A non empty set V together with two binary operations called vector addition '+' (internal composition in V) and scalar multiplication '.' (external composition in V over F), is called a vector space over the field F , if following axioms are satisfied :

I. V is an additive abelian group :

- (i) V is closed under addition : $u + v \in V$ for all $u, v \in V$.
- (ii) Associativity : $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$
- (iii) Existence of additive identity : There exists an element $0 \in V$ such that $u + 0 = u = 0 + u$ for all $u \in V$. This element '0' is called zero vector.
- (iv) Existence of additive inverse : For each $u \in V$, there exist an element $v \in V$ such that $u + v = 0 = v + u$. v is called additive inverse of u and is denoted by ' $-u$ '.
- (v) Commutativity : $u + v = v + u$ for all $u, v \in V$.

II. V is closed under scalar multiplication : $au \in V$ for all $a \in F, u \in V$.

III. Scalar multiplication satisfies the following properties :

- (i) $a(u + v) = au + av$ for all $u, v \in V, a \in F$
- (ii) $(a + b)u = au + bu$ for all $a, b \in F, u \in V$
- (iii) $(ab)u = a(bu)$ for all $a, b \in F, u \in V$
- (iv) $1.u = u$ for all $u \in V$ where '1' is the multiplicative identity of F .

We, then, write that V is a vector space over F or $V(F)$ is a vector space.

Remarks :

1. The elements of V are called vectors and the elements of F are called scalars.

2. Here, by the word 'vector', we do not mean a quantity having magnitude and direction both.

Similarly, by the word 'scalar', we do not mean a quantity having magnitude only and no direction.

These two definitions of vector and scalar are commonly used in physics and applied mathematics. In a vector space $V(F)$, by the word 'vector', we just mean the elements of V and by the word 'scalar', we just mean the elements of F .

3. In all, a vector space has ten properties. We have divided these ten properties into three parts. In part I, we have five properties of additive abelian group, in part II we have one property that scalar multiplication is closed and in part III, we have four properties of scalar multiplication.

Properties of vector space : If 0 denotes the zero element of F and $\bar{0}$ denotes the zero element of vector space V over F , then

- (i) $a\bar{0} = \bar{0}$ for all $a \in F$
- (ii) $0u = \bar{0}$ for all $u \in V$
- (iii) $(-1)u = -u$ for all $u \in V$
- (iv) $a(-u) = -(au) = (-a)u$ for all $a \in F$ and $u \in V$
- (v) $a(u-v) = au - av$ for all $a \in F$ and $u, v \in V$
- (vi) $au = \bar{0} \Rightarrow a = 0$ or $u = \bar{0}$
- (vii) $au = bu$ and $u \neq \bar{0} \Rightarrow a = b$
- (viii) $au = av$ and $a \neq 0 \Rightarrow u = v$.

Exercise 4.1

1. (Number system) : Which of the following are vector spaces ?

- | | | | |
|---------------------------------|--------------------------------|--------------------------------|---------------------------------|
| (i) $\mathbb{Z}(\mathbb{Z})$ | (ii) $\mathbb{Q}(\mathbb{Z})$ | (iii) $\mathbb{R}(\mathbb{Z})$ | (iv) $\mathbb{C}(\mathbb{Z})$ |
| (v) $\mathbb{Z}(\mathbb{Q})$ | (vi) $\mathbb{Q}(\mathbb{Q})$ | (vii) $\mathbb{R}(\mathbb{Q})$ | (viii) $\mathbb{C}(\mathbb{Q})$ |
| (ix) $\mathbb{Z}(\mathbb{R})$ | (x) $\mathbb{Q}(\mathbb{R})$ | (xi) $\mathbb{R}(\mathbb{R})$ | (xii) $\mathbb{C}(\mathbb{R})$ |
| (xiii) $\mathbb{Z}(\mathbb{C})$ | (xiv) $\mathbb{Q}(\mathbb{C})$ | (xv) $\mathbb{R}(\mathbb{C})$ | (xvi) $\mathbb{C}(\mathbb{C})$ |

2. (n -tuples) : Show that the following are the vector spaces :

- | | | |
|---------------------------------|---------------------------------|----------------------------------|
| (i) $\mathbb{C}^n(\mathbb{C})$ | (ii) $\mathbb{C}^n(\mathbb{R})$ | (iii) $\mathbb{C}^n(\mathbb{Q})$ |
| (iv) $\mathbb{R}^n(\mathbb{R})$ | (v) $\mathbb{R}^n(\mathbb{Q})$ | (vi) $\mathbb{Q}^n(\mathbb{Q})$ |

Notation : $M_{m,n}(F) = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} : a_{ij} \in F \right\}$ = Set of all $m \times n$ matrices whose entries are from F .

3. (Matrices) : Show that the following are the vector spaces :

- | | | |
|--|--|---|
| (i) $M_{m,n}(\mathbb{C})$ over $\mathbb{C}$ | (ii) $M_{m,n}(\mathbb{C})$ over $\mathbb{R}$ | (iii) $M_{m,n}(\mathbb{C})$ over $\mathbb{Q}$ |
| (iv) $M_{m,n}(\mathbb{R})$ over $\mathbb{R}$ | (v) $M_{m,n}(\mathbb{R})$ over $\mathbb{Q}$ | (vi) $M_{m,n}(\mathbb{Q})$ over $\mathbb{Q}$ |

Notation : $P_n(F) = \{a_0 + a_1x + \dots + a_nx^n : a_i \in F \text{ and } n \text{ is a fixed non negative integer}\}$

= Set of all polynomials of degree atmost n with coefficients from F and the zero polynomial.

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4. (Polynomials of degree atmost n) : Show that the following are the vector spaces :

- (i) $P_n(\mathbb{C})$ over $\mathbb{C}$. (ii) $P_n(\mathbb{C})$ over $\mathbb{R}$ (iii) $P_n(\mathbb{C})$ over $\mathbb{Q}$
(iv) $P_n(\mathbb{R})$ over $\mathbb{R}$ (v) $P_n(\mathbb{R})$ over $\mathbb{Q}$ (vi) $P_n(\mathbb{Q})$ over $\mathbb{Q}$

Notation : $P(F) = \{a_0 + a_1x + \dots + a_nx^n : a_i \in F \text{ and } n \text{ is any non negative integer}\}$
= Set of all polynomials with coefficients from F .

5. (All polynomials) : Show that the following are the vector spaces :

- (i) $P(\mathbb{C})$ over $\mathbb{C}$. (ii) $P(\mathbb{C})$ over $\mathbb{R}$ (iii) $P(\mathbb{C})$ over $\mathbb{Q}$
(iv) $P(\mathbb{R})$ over $\mathbb{R}$ (v) $P(\mathbb{R})$ over $\mathbb{Q}$ (vi) $P(\mathbb{Q})$ over $\mathbb{Q}$

6. (Functions) : Show that the set V of all real valued continuous functions of x defined on interval $[0, 1]$ is a vector space over the field $\mathbb{R}$ of real numbers with respect to vector addition and scalar multiplication defined by $(f_1 + f_2)x = f_1(x) + f_2(x)$ for all $f_1, f_2 \in V$

$$(af_1)x = af_1(x) \text{ for all } a \in \mathbb{R}, f_1 \in V.$$

7. (Finite vector spaces) : Show that the following are the vector spaces :

- (i) $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$ (ii) $M_{m,n}(\mathbb{Z}_p)$ over $\mathbb{Z}_p$ (iii) $P_n(\mathbb{Z}_p)$ over $\mathbb{Z}_p$

8. Find the number of elements in the vector spaces given in above question.

9. Write all the elements of following vector spaces :

- (i) $\mathbb{Z}_2^2$ over $\mathbb{Z}_2$ (ii) $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$ (iii) $\mathbb{Z}_3^2$ over $\mathbb{Z}_3$
(iv) $\mathbb{Z}_3^3$ over $\mathbb{Z}_3$ (v) $M_{2,2}(\mathbb{Z}_2)$ over $\mathbb{Z}_2$ (vi) $M_{2,3}(\mathbb{Z}_2)$ over $\mathbb{Z}_2$
(vii) $P_2(\mathbb{Z}_2)$ over $\mathbb{Z}_2$ (viii) $P_2(\mathbb{Z}_3)$ over $\mathbb{Z}_3$ (ix) $P_3(\mathbb{Z}_2)$ over $\mathbb{Z}_2$

10. Show that every field is a vector space over itself.

11. Show that every field is a vector space over its subfield .

12. Let $\mathbb{R}^+$ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as : $u + v = u.v$ for all $u, v \in \mathbb{R}^+$ and $au = u^a$ for all $u \in \mathbb{R}^+, a \in \mathbb{R}$. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$.

13. Let $V = \{(a, b) : a, b \in \mathbb{R}\}$ then show that V is not a vector space over reals under addition and scalar multiplication defined as in each one of the following cases :

- (i) $(a, b) + (c, d) = (0, b + d)$ and $k(a, b) = (ka, kb)$

(ii) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (0, kb)$

(iii) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, b)$

(iv) $(a, b) + (c, d) = (a + d, b + c)$ and $k(a, b) = (ka, kb)$

(v) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (a, b)$

(vi) $(a, b) + (c, d) = (0, 0)$ and $k(a, b) = (ka, kb)$

(vii) $(a, b) + (c, d) = (ac, bd)$ and $k(a, b) = (ka, kb)$

14. Prove that the set of all vectors in a plane over the field of real numbers is a vector space over R with respect to vector addition and scalar multiplication.

15. Prove that the set of all matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, where $x, y \in \mathbb{C}$, the set of complex numbers, is a vector space over $\mathbb{C}$ with respect to matrix addition and scalar multiplication.

16. Show that the set $Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$ with respect to the vector addition and scalar multiplication defined as $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$ and $\alpha(a + b\sqrt{2}) = a\alpha + b\alpha\sqrt{2}$ where a, b, c, d and α are all rational numbers.

17. Show that the set V of all polynomials with constant term 2 over $\mathbb{R}$ is not a vector space over $\mathbb{R}$. Further tell which fixed value can be assigned to the constant term so that V can become a vector space.

18. Let V be the set of ordered pairs (a, b) of real numbers with addition in V and scalar multiplication on V defined by $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, 0)$. Show that V satisfies all the axioms of a vector space except $1 \cdot v = v$. Hence this axiom is not a consequence of the other axioms.

19. Show that the axiom $u + v = v + u$ of a vector space can be derived from the other axioms.

5.1 Subspaces

Def. Subspace : Let $V(F)$ be a vector space. A non empty subset W of V is said to be a subspace of V if W itself is a vector space over F with the same vector addition and scalar multiplication as for V .

Remarks :

1. For every vector space V , the entire space V and zero space $\{0\}$ are always subspaces of V and are called improper subspaces. All subspaces other than these two are called proper subspaces.
2. It is clear that in any subset W of V , associativity of addition, commutativity of addition and four properties of scalar multiplication are always satisfied. i.e., out of ten properties of a vector space, six are always satisfied in every subset. So to prove W a subspace our main emphasis is on the four remaining properties, namely, closure of addition, existence of additive identity, existence of additive inverse and closure of scalar multiplication.

Def. Linear sum of subspaces : Let W_1 and W_2 be two subspaces of a vector space V , then linear sum of W_1 and W_2 is denoted by $W_1 + W_2$ and is defined as :

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Further a vector space V is said to be linear sum of its subspaces W_1 and W_2 if every $v \in V$ can be expressed as $v = w_1 + w_2$; $w_1 \in W_1, w_2 \in W_2$.

Def. Direct sum of subspaces : A vector space $V(F)$ is said to be direct sum of its two subspaces W_1 and W_2 if every $v \in V$ can be uniquely expressed as

$$v = w_1 + w_2 ; w_1 \in W_1, w_2 \in W_2$$

We denote the direct sum of W_1 and W_2 by $W_1 \oplus W_2$.

Def. Disjoint subspaces : Let $V(F)$ be a vector space. Two subspaces W_1 and W_2 of V are said to be disjoint if their intersection is the zero subspace, i.e. $W_1 \cap W_2 = \{0\}$.

Remark : By two disjoint sets A and B we mean that their intersection is empty. That intuition should not be mixed up with the definition of disjoint subspaces.

Results :

1. A necessary and sufficient condition for a non empty subset W of a vector space V to be a subspace of V is that W is closed under addition and scalar multiplication.
2. A necessary and sufficient condition for a non – empty subset W of a vector space $V(F)$ to be a subspace of V is that $au + v \in W$ for all $u, v \in W$ and $a \in F$.

3. A necessary and sufficient conditions for a non empty subset W of a vector space $V(F)$ to be a subspace are : (i) $u - v \in W$ for all $u, v \in W$ (ii) $au \in W$ for all $a \in F, u \in W$.
4. A necessary and sufficient condition for a non empty subset W of a vector space $V(F)$ be a subspace of V is that $au + bv \in W$ for all $u, v \in W ; a, b \in F$.
5. The intersection of two subspaces of a vector space $V(F)$ is again a subspace of $V(F)$.
6. The union of two subspaces of a vector space $V(F)$ may not be a subspace of $V(F)$.
e.g., Consider the vector space $V_3(R)$ where $V_3 = \{(x, y, z) : x, y, z \in R\}$ then it can be easily checked that $W_1 = \{(x, 0, 0) : x \in R\}$ and $W_2 = \{(0, y, 0) : y \in R\}$ are both subspace of V_3 .
Now consider, $W_1 \cup W_2 = \{(x, 0, 0) \text{ or } (0, y, 0) \text{ where } x, y \in R\}$
Now $(1, 0, 0)$ and $(0, 1, 0)$ belongs to $W_1 \cup W_2$ but $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W_1 \cup W_2$
Hence $W_1 \cup W_2$ is not closed under addition and so it is not a subspace of $V_3(R)$.
7. The union of two subspaces is a subspace iff one is contained in the other.
8. Linear sum of two subspaces is a subspace.
9. The necessary and sufficient conditions for a vector space $V(F)$ to be a direct sum of its subspaces W_1 and W_2 are that (i) $V = W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$.

OR

The necessary and sufficient condition for a vector space V to be direct sum of its two subspaces are that V is linear sum of these two subspaces and they are disjoint.

Exercise 5.1

1. Let V be a vector space given by $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ then which of the following are subspaces of V over R ?
 - (i) $W = \{(x, y, z) : 3x + y - z = 0, x, y, z \in R\}$
 - (ii) $W = \{(x, y, z) : x + y - z = 0, 2x + 3y - z = 0, x, y, z \in R\}$
 - (iii) $W = \{(x, y, z) : x + y \geq 0, x, y, z \in R\}$
 - (iv) $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, x, y, z \in R\}$
2. Let V be the vector space of all n - square matrices over reals .Which of the following sets of matrices are subspaces of V .
 - (i) all scalar matrices
 - (ii) all singular matrices
 - (iii) all diagonal matrices
 - (iv) all upper triangular matrices
 - (v) all skew symmetric matrices.

3. Prove that the intersection of any family of subspaces of a vector space is a subspace .
4. Let $V = \mathbb{R}^3$ be the vector space over $\mathbb{R}$. Prove that W is a vector sub space over $\mathbb{R}$ where
- $$W = \{(x, y, z) : ax + by + cz = 0 ; x, y, z \in \mathbb{R}\}.$$

OR

Show that any plane passing through the origin is a subspace of $\mathbb{R}^3$.

5. Let $V(\mathbb{R})$ be the vector space of all function from $\mathbb{R}$ to $\mathbb{R}$ (reals). Prove that the following subsets are the subspaces of $V(\mathbb{R})$
- (a) set of all odd functions (b) set of all even functions (c) set of all continuous functions.
6. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ (reals). Let W_1 and W_2 denotes the subspaces of all even functions and all odd functions respectively . Prove that V is the direct sum of W_1 and W_2 .
7. Let $M_{2,2}(\mathbb{R})$ denotes the vector space of all 2×2 matrices over reals. Determine which of the following are subspaces of $M_{2,2}(\mathbb{R})$.
- (i) $W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$
- (ii) $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$
- (iii) $W = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$
- (iv) $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$
- (v) $W = \{A \in M_{2,2}(\mathbb{R}) : A^2 = A\}$
- (vi) $W = \{A \in M_{2,2}(\mathbb{R}) : A^2 = I\}$
- (vii) $W = \{A \in M_{2,2}(\mathbb{R}) : A \text{ is singular}\}$
8. Let $P_4(x)$ denotes the vector space of all polynomials of degree at the most 4 over the field of real numbers. Determine which of the following are subspaces of $P_4(x)$.

- (i) $W = \{f(x) : \text{constant term of } f(x) = 1\}$
- (ii) $W = \{f(x) : \deg f(x) = 2\}$
- (iii) $W = \{f(x) : \deg f(x) \leq 2\}$
- (iv) $W = \{f(x) : \text{coeff. of } x \text{ in } f(x) \text{ is zero}\}$
- (v) $W = \{f(x) : f(2) = f(4) = 0\}$
- (vi) $W = \{f(x) : \text{constant term of } f(x) = 0\}$

9. In the vector space $V = \mathbb{R}^2(\mathbb{R})$. Consider $W_1 = \{(x, 0) : x \in \mathbb{R}\}$, $W_2 = \{(0, x) : x \in \mathbb{R}\}$,

$W_3 = \{(x, x) : x \in \mathbb{R}\}$. Prove that W_1, W_2, W_3 are subspaces of $\mathbb{R}^2(\mathbb{R})$ and prove that

- (i) $V = W_1 \oplus W_2$
- (ii) $V = W_1 \oplus W_3$

10. Let V be a vector space given by : $V = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$. Which of the following are subspaces of V over $\mathbb{R}$?

- (i) $W = \{(x, y, z) : x - 3y + 4z = 0, x, y, z \in \mathbb{R}\}$ (ii) $W = \{(x, 2y, 3z) : x, y, z \in \mathbb{R}\}$
 (iii) $W = \{(x, x, x) : x \in \mathbb{R}\}$ (iv) $W = \{(x, y, z) : x, y, z \in \mathbb{Q}\}$
 (v) $W = \{(x, y, z) : x \geq 0, x, y, z \in \mathbb{R}\}$ (vi) $W = \{(x, y, z) : xy = 0, x, y, z \in \mathbb{R}\}$

11. Let V be a vector space of polynomials of degree ≤ 6 . Which of the following are subspaces of V ?
 Here $f(x) \in V$

- (i) $W = \{f(x) : f(0) = 1\}$ (ii) $W = \{f(x) : \deg f(x) \leq 4\}$
 (iii) $W = \{f(x) : f(1) = 0, f(3) = 0\}$ (iv) $W = \{f(x) : \text{coefficient of } x^2 \text{ is } 1 \text{ or } -1\}$
 (v) $W = \{f(x) : \text{having positive coefficient}\}$

12. If a vector space V is the set of all real valued continuous functions over $\mathbb{R}$, then show that the set

W of solutions of differential equations $3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$ is a subspace of V .

13. Give an example to show that the linear sum of two subspaces need not be always the direct sum .

14. Let V be the vector space of all $n \times n$ real matrices. Let W_1 and W_2 be the set of all $n \times n$ real symmetric and skew-symmetric matrices respectively. Then prove that

- (i) W_1 is a subspace of V .
 (ii) W_2 is a subspace of V .
 (iii) $V = W_1 \oplus W_2$.

----- S C Q -----

1. Under which one of the following condition does the system of equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 1 & 2 & a-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ a \end{bmatrix}$$

have a unique solution ?

- (a) for all $a \in \mathbb{R}$ (b) $a = 8$
(c) for all $a \in \mathbb{Z}$ (d) $a \neq 8$

2. The system of equations

$$x + y + z = 1$$

$$2x + 3y - z = 5$$

$$x + 2y - kz = 4$$

where $k \in \mathbb{R}$, has an infinite number of solutions for

- (a) $k=0$ (b) $k=1$ (c) $k=2$ (d) $k=3$
(CSIR NET Dec 2012)

3. What is the dimension of the vector space formed by the solution of the system of the following equations ?

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + 2x_2 = 0$$

$$x_2 - x_3 = 0$$

- (a) 1 (b) 2 (c) 3 (d) 0

4. If the system of equations $x - 2y - 3z = 1$,
 $(p+2)z = 3$, $(2p+1)y + z = 2$ is inconsistent, then what is the value of p ?

- (a) -2 (b) $-\frac{1}{2}$ (c) 0 (d) 2

5. The system of equation and $kx + y + z = 1$,
 $x + ky + z = k$, $x + y + kz = k^3$ does not have a solution, if k is equal to

- (a) 0 (b) 1 (c) -1 (d) -2

6. The system of equations

$$4x_1 + x_2 - 3x_3 - x_4 = 0$$

$$2x_1 + 3x_2 + x_3 - 5x_4 = 0$$

$$x_1 - 2x_2 - 2x_3 + 3x_4 = 0 \text{ has}$$

- (a) no solution
(b) only one solution (0,0,0,0)
(c) infinite number of solutions
(d) only two solutions

7. The system of equations

$$x - y + 3z = 4$$

$$x + z = 2$$

$$x + y - z = 0$$

- (a) a unique solution
(b) finitely many solutions
(c) infinitely many solutions
(d) no solution

8. Consider the equation $AX = B$, where

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \text{ then}$$

- (a) the equation has no solution.
(b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution of the equation
(c) there exists a non-zero unique solution
(d) the equation has infinitely many solution

9. The system of the equations

$$2x + y = 5$$

$$x - 3y = -1$$

$3x + 4y = k$ is consistent, when k is

- (a) 1 (b) 2 (c) 5 (d) 10

10. The value of α for which the system of equation

$$x + y + z = 0$$

$$y + 2z = 0$$

$\alpha x + z = 0$ has more than one solution is

- (a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

11. The system of equations

$$x + 2y + z = 9$$

$$2x + y + 3z = 7$$

can be expressed as

$$(a) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

(d) None of these

12. If $3x + 2y + z = 0$

$$x + 4y + z = 0$$

$$2x + y + 4z = 0$$

be a system of equation, then

(a) it is inconsistent

(b) it has only the trivial solution $x = 0$,
 $y = 0, z = 0$

(c) it can be reduced to a single equation and so a solution does not exist

(d) the determinant of the matrix of coefficient is zero

13. Let M be a $m \times n$ ($m < n$) matrix with rank m .

Then,

(a) for every b in $\mathbb{R}^m$, $Mx = b$ has unique solution

(b) for every b in $\mathbb{R}^m$, $Mx = b$ has a solution but it is not unique

(c) there exists $b \in \mathbb{R}^m$ for which $Mx = b$ has no solution

(d) None of the above

14. Let A be a $m \times n$ matrix with row rank $= r =$ column rank. The dimension of the space of solutions of the system of linear equation $AX = 0$ is

(a) r

(b) $n - r$

(c) $m - r$

(d) $\min(m, n) - r$

15. Let A be an $m \times n$ matrix where $m < n$.

Consider the system of linear equation

$AX = \underline{b}$, where $\underline{b}$ is an $m \times 1$ column vector

and $\underline{b} \neq 0$. Which of the following is always true?

(a) The system of equations has no solution.

(b) The system of equations has a solution if

and only if it has infinitely many solutions.

(c) The system of equations has a unique solution.

(d) The system of equations has atleast one solution.

16. The set of all solutions to the system of equations $(1 - i)x_1 - ix_2 = 0$,

$2x_1 + (1 - i)x_2 = 0$ is given by

(a) $(x_1, x_2) = (0, 0)$

(b) $(x_1, x_2) = (1, 1)$

(c) $(x_1, x_2) = c \left(1, \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) \sqrt{2} \right)$

where c is any complex number

(d) $(x_1, x_2) = c \left(\cos \frac{3\pi}{4}, i \sin \frac{3\pi}{4} \right)$, where c is any complex number.

17. Let S be the solution space of a set of m homogeneous linear equations with real coefficients in n unknowns. If A is the matrix of this system of equations, then

(a) dimension $S = n - \text{rank } A$

(b) dimension S is always n

(c) dimension S is infinite

(d) dimension $S = n + \text{rank } A$

18. Consider the system of linear equations

$$x + y + z = 3, x - y - z = 4, x - 5y + kz = 6$$

Then, the value of k for which this system has an infinite number of solutions is

(a) $k = -5$

(b) $k = 0$

(c) $k = 1$

(d) $k = 3$

19. Find the value of k for which the following simultaneous equations $x + y + z = 3$;

$x + 2y + 3z = 4$; $x + 4y + kz = 6$ will not have a unique solution.

(a) 0

(b) 5

(c) 6

(d) 7

20. In the matrix equation $AX = B$, which of the following is a necessary condition for the existence of atleast one solution for the unknown vector X ?

(a) Augmented matrix $[A : B]$ must have the same rank as matrix A .

(b) Vector B must have only non-zero elements.

- (c) Matrix A must be singular.
(d) Matrix A must be square.
21. Solution for the system defined by the set of equations $4y + 3z = 8$; $2x - z = 2$ and $3x + 2y = 5$ is
- (a) $x = 0; y = 1; z = \frac{4}{3}$
(b) $x = 0; y = \frac{1}{2}; z = 2$
(c) $x = 1; y = \frac{1}{2}; z = 2$
(d) not exist
22. Consider the system of simultaneous equations
 $x + 2y + z = 6$; $2x + y + 2z = 6$; $x + y + z = 5$
The system has
- (a) unique solution
(b) infinite number of solution
(c) no solution
(d) exactly two solutions
23. Consider the system
 $x + y + z = 0$; $x - y - z = 0$
then the system of equation have
- (a) no solution
(b) infinite number of solution
(c) unique solution
(d) None of the above
24. For the set of equations
 $x_1 + 2x_2 + x_3 + 4x_4 = 2$
 $3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$
The following statement is true
- (a) only the trivial solution
 $x_1 = x_2 = x_3 = x_4 = 0$ exists
(b) there are no solution
(c) a unique non-trivial solution exists
(d) multiple non-trivial solutions exists
25. Let A be a 4×3 matrix whose columns form a linearly independent set which conclusion is justified i.e., which are true?
- (a) The set of rows in A is linearly dependent.
(b) The equation $Ax = b$ is consistent for every b in $\mathbb{R}^4$.
(c) The equation $Ax = 0$ has a non-trivial solution.
(d) There is a matrix B s.t. $AB = I_4$.
26. Let A be an $m \times n$ matrix, $\bar{b} \in \mathbb{R}^m$, then the system of equations $A\bar{x} = \bar{b}$ has a solution if and only if
- (a) $\bar{b} \in$ column space of A
(b) $\bar{b} \in$ row space of A
(c) $\bar{b} = 0$
(d) $\text{rank}[A, \bar{b}] > \text{rank } A$
27. Let A be a 3×3 matrix and consider the system of equation $AX = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then,
- (a) if the system is consistent, then it has a unique solution.
(b) if A is singular, then the system has infinitely many solution
(c) if the system is consistent, then the $|A| \neq 0$
(d) if the system has a unique solution, then A is non-singular
28. A system of linear equations
 $x + 2y - z = 11$, $3x + y - 2z = 10$, $x - 3y = 5$
- (a) no solution
(b) exactly one solution
(c) exactly 3 solution
(d) infinitely many solution
29. Let A be an $m \times n$ matrix and $b = (b_1, b_2, \dots, b_m)'$ be a fixed vector. Consider a system of m linear equations $Ax = b$, where $x = (x_1, x_2, \dots, x_n)'$.
Consider the following statements.
I. If $\text{rank } n$ the system has a unique solution.
II. If $\text{rank} < n$, the system has infinitely many solutions.

III. If $b = 0$, the system has atleast one solution.

Which of the following is correct?

- (a) I and II are true
- (b) I and III are true
- (c) Only I is true
- (d) Only III is true

30. A homogeneous system of 5 linear equation in 6 variables admits

- (a) no solution in $\mathbb{R}^6$
- (b) a unique solution in $\mathbb{R}^6$
- (c) infinitely many solutions in $\mathbb{R}^6$
- (d) finite, but more than 2 solutions in $\mathbb{R}^6$

31. Consider the system of equations,

$Ax = 0$ where A is a 2×3 matrix. Then, it must be true that

- (a) there is a solution $x \neq 0$
- (b) there are atleast two linearly independent solutions
- (c) any solution x satisfies $x_1 + x_2 + x_3 = 0$
- (d) any two solutions are linearly dependent

32. Consider the non-homogeneous system of linear equations $Ax = b$, where A is $m \times n$, $X \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, then

- I. the corresponding homogeneous system $Ax = 0$ has unique solution $\Rightarrow Ax = b$ has infinitely many solutions for each vector b in $\mathbb{R}^m$.
- II. the corresponding homogeneous system $Ax = 0$ has a non zero solution $\Rightarrow$ there is a vector b in $\mathbb{R}^m$ for which $Ax = b$ has no solution

Choose the correct code

- (a) I and II are true
- (b) I is true but II is false
- (c) I is false but II is true
- (d) both I and II are false

33. The system of equations

$$x + y + z = 0$$

$$3x + 6y + z = 0$$

$$\alpha x + 2y + z = 0$$

has infinitely many solutions, then α is equal to

- (a) 7
- (b) $\frac{7}{5}$
- (c) $\frac{5}{7}$
- (d) 4

34. Let A be a 5×4 matrix with real entries such that $A\underline{x} = \underline{0}$ if and only if $\underline{x} = \underline{0}$ where $\underline{x}$ is a 4×1 vector and $\underline{0}$ is null vector.

Then, the rank of A is

- (1) 4
- (2) 5
- (3) 2
- (4) 1

(CSIR NET Dec 2013)

35. Consider the equation $2x + 2y = 1$ and

$2x - 2y = 1$ over Z_3 . What is the solution of (x, y) ?

- (a) (1,1) but not (2,0)
- (b) (2,0) but not (1,1)
- (c) Both (1,1) and (2,0)
- (d) $(\frac{1}{2}, 0)$

36. Let A be an $m \times n$ matrix of rank n with real entries. Choose the correct statement.

- 1. $Ax = b$ has a solution for any b .
- 2. $Ax = 0$ does not have a solution.
- 3. If $Ax = b$ has a solution, then it is unique.
- 4. $y'A = 0$ for some nonzero y , where y' denotes the transpose of the vector y .

(CSIR NET June 2015)

37. Let A be a $n \times m$ matrix and b be a $n \times 1$ vector (with real entries). Suppose the equation $Ax = b$, $x \in \mathbb{R}^m$ admits a unique solution. Then we can conclude that

- 1. $m \geq n$
- 2. $n \geq m$
- 3. $n = m$
- 4. $n > m$

(CSIR NET June 2016)

----- MCQ -----

- 1. Consider a homogenous system of linear equation $Ax=0$, where A is an $m \times n$ real matrix and $n > m$. Then which of the following statements are always true ?
 - (1) $Ax=0$ has a solution
 - (2) $Ax=0$ has no nonzero solution
 - (3) $Ax=0$ has a nonzero solution
 - (4) Dimension of the space of all solutions is at least $n-m$.

(CSIR NET June 2014)

- 2. Let A be a 3×4 and b be a 3×1 matrix with integer entries. Suppose that the system $Ax=b$ has a complex solution. Then

- (1) $Ax=b$ has an integer solution.
- (2) $Ax=b$ has a rational solution.
- (3) The set of real solutions to $Ax=0$ has a basis consisting of rational solutions.

- (4) If $b \neq 0$ then A has positive rank.
(CSIR NET Dec 2014)

----- S C Q -----

1. Consider the real vector space $V = \mathbb{R}^3$ and following of its subsets

I. $S = \{(x, y, z) \in V : x = y = 0\}$

II. $T = \{(x, y, z) \in V : x = 0\}$

III. $W = \{(x, y, z) \in V : z \neq 0\}$

Which one of the following statement is correct?

- (a) S, T and W are subspaces
- (b) Only S and W are subspaces
- (c) Only T and W are subspaces
- (d) Only S and T are subspaces

2. If V is the real vector space of all mapping from $\mathbb{R}$ to

$\mathbb{R}$, $V_1 = \{f \in V : f(-x) = f(x)\}$ and

$V_2 = \{f \in V : f(-x) = -f(x)\}$, then which one of the following is correct?

- (a) Neither V_1 nor V_2 is a subspace of V .
- (b) V_1 is a subspace of V , but V_2 is not a subspace of V .
- (c) V_1 is not a subspace of V , but V_2 is a subspace of V .
- (d) Both V_1 and V_2 are subspace of V .

3. Let R be the set of all real number and $\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. Then, which one of the following is a subspace of $\mathbb{R}^2$ over $\mathbb{R}$?

- (a) $\{(x_1, x_2) : x_1 > 0, x_2 > 0\}$
- (b) $\{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 > 0\}$
- (c) $\{(x_1, x_2) : x_1 < 0, x_2 < 0\}$
- (d) $\{(x_1, 0) : x_1 \in \mathbb{R}\}$

4. Suppose $X = \{(a, b, c, d) : b - c = 4\}$;

$Y = \{(a, b, c, d) : a = b + c\}$

$Z = \{(a, b, c, d) : b = 0, c = d\}$

Which of these subset of the vector space

$\mathbb{R}^4$ is /are subspace(s) ?

- (a) X only
- (b) Y and Z
- (c) X, Y and Z
- (d) X and Z

5. Let W_1, W_2, W_3 be subspace of a vector space V such that $W_3 \subset W_1$. Then, which one of the following is correct?

- (a) $W_1 \cap (W_2 + W_3) = W_2 + W_1 \cap W_3$
- (b) $W_1 \cap (W_2 + W_3) = W_3 + W_1 \cap W_2$
- (c) $W_1 \cap (W_2 + W_3) = W_2 + W_1 \cap W_3$
- (d) None of the above

6. Which one of the following is correct?

- (a) R is a vector space over N .
- (b) R is a vector space over C .
- (c) R is a vector space over Z .
- (d) None of the above.

7. The set $V = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ is

- (a) a vector subspace of $\mathbb{R}^2$
- (b) not a vector subspace of $\mathbb{R}^2$, since every element does not have an inverse in V
- (c) not a vector subspace of $\mathbb{R}^2$, since it is not closed under scalar multiplication
- (d) not a vector subspace of $\mathbb{R}^2$, since it is not closed under vector addition

(GATE 2004)

8. If V and W are subspaces of $\mathbb{R}^n$, then

- (a) $V \cup W$ is necessarily a subspace of $\mathbb{R}^n$
- (b) $V \cup W$ is never a subspace of $\mathbb{R}^n$
- (c) $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if one of V, W is contained in the other
- (d) $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if one of V, W in $\{0\}$

9. Consider the set of all functions f defined on $[0, 1]$ such that

I. $f(x) = 0$ at a finite number of points in $[0, 1]$

II. f has a local maxima at $x = 1/2$

III. f has a local extrema at $x = 1/2$

Then, the correct code is

- (a) only (I) is vector space over R
 (b) only (II) is vector space over R
 (c) only (III) is vector space over R
 (d) None of the above
10. Which of the following sets are subspaces of P (the vector space of all polynomials)
 (a) $\{p \in P : \text{degree of } p = 4\}$
 (b) $\{p \in P : \text{degree of } p \geq 3\}$
 (c) $\{p \in P : \text{degree of } p \leq 4 \text{ and } p'(0) = 1\}$
 (d) None of the above
11. Which of the following sets are subspaces of the space $C(a, b)$ of all real-valued continuous functions defined on the open interval (a, b)
 (a) $\{f \in C(a, b) : f(x_0) = 0, x_0 \in (a, b)\}$
 (b) $\{f \in C(a, b) : f'(x) \neq 0, \text{ for all } x \in (a, b)\}$
 (c) $\left\{f \in C(a, b) : f\left(\frac{a+b}{2}\right) = 1\right\}$
 (d) None of the above
12. Which of the following sets are subspaces of the space $C(a, b)$ of all real-valued continuous functions defined on the open interval (a, b)
 (a) $\{f \in C(a, b) : f(x) = x^2 f(x)\}$
 (b) $\{f \in C(a, b) : 2f'''(x) + 3xf^2(x) - f'(x) + x^2 f(x) = 0\}$
 (c) $\{f \in C(a, b) : \int_a^b f(x) dx = 0\}$
 (d) All of the above
13. Which of the following sets is not a subspace of $C(a, b)$
 (a) $\{f \in C(a, b) : \lim_{x \rightarrow \frac{a+b}{2}} f(x) = 0\}$
 (b) $\{f \in C(a, b) : \lim_{x \rightarrow \frac{a+b}{2}} f(x) = 1\}$
 (c) $\{f \in C(-\infty, \infty) : f(x) = f(-x)\}$
 (d) All of the above
14. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Then, the set W , consisting of all matrices A for which $A^2 = A$,
 (a) is a subspace of V
 (b) is not a subspace of V as it is not closed with respect to vector addition

- (c) is not a subspace of V as it is not closed w.r.t. scalar multiplication
 (d) Both (b) and (c)

15. Let V be the vector space of all $n \times n$ matrices over R . Then, the set W , consisting of all matrices which commute with a given matrix M i.e., $W = \{M \in V : AM = MA\}$
 (a) is a subspace of V
 (b) is not a subspace of V as it is not closed w.r.t. vector addition
 (c) is not a subspace of V as it is not closed w.r.t. scalar multiplication
 (d) None of the above
16. Which of the following is not a vector subspace of the vector space of polynomials with real coefficient
 (a) W consists of all polynomials divisible by x
 (b) $W = \{p(x) \in V : p(3) = 0\}$
 (c) $W = \{p(x) \in V : p(a) = p(1-a), a \in R\}$
 (d) W consists of all polynomials with integral coefficient

17. Consider the set of all functions f defined on $[0, 1]$ such that

$$\text{I. } f\left(\frac{1}{2}\right) = 0 \quad \text{II. } f\left(\frac{3}{4}\right) = 1$$

$$\text{III. } f(x) = xf(x) \quad \text{IV. } f(0) = f(1)$$

Then, the correct code is

- (a) only I is vector space.
 (b) only I and IV are vector spaces.
 (c) only I, III and IV are vector spaces.
 (d) all are vector spaces.

----- M C Q -----

1. V be the vector space of all $n \times n$ real matrices over R and W_1, W_2 be the collection of all symmetric and skew symmetric matrices respectively in V , then
 (a) W_1 is a subspace of V
 (b) W_1 is not a subspace of V
 (c) W_2 is a subspace of V
 (d) W_2 is not a subspace of V
2. V be the vector space of all $n \times n$ complex matrices over C and W_1, W_2 be the collection of all hermitian and skew hermitian matrices

respectively in V , then

- (a) W_1 is a subspace of V
- (b) W_1 is not a subspace of V
- (c) W_2 is a subspace of V
- (d) W_2 is not a subspace of V

3. V be the vector space of all $n \times n$ complex Matrices over R and W_1, W_2 be the collection of all hermitian and skew hermitian matrices respectively in V , then

- (a) W_1 is a subspace of V
- (b) W_1 is not a subspace of V
- (c) W_2 is a subspace of V
- (d) W_2 is not a subspace of V

4. In the vector space $\mathbb{R}^2(\mathbb{R})$, consider the subsets

$$A = \{(x, y) : x + y = 0\}, \quad B = \{(x, y) : x - y = 0\}$$

$$C = \{(x, y) : xy = 0\}, \quad D = \left\{(x, y) : \frac{x}{y} = 0, y \neq 0\right\}$$

Then

- (a) A is a subspace
- (b) B is a subspace
- (c) C is a subspace
- (d) D is a subspace

5. In the vector space $\mathbb{R}^2(\mathbb{R})$, consider the subsets

$$A = \{(x, y) : x^2 + y^2 = 0\}$$

$$B = \{(x, y) : x^2 - y^2 = 0\}$$

Then

- (a) A is a subspace
- (b) B is a subspace
- (c) A is not a subspace
- (d) B is not a subspace

6. Let V be the vector space of all real valued functions defined on $[0, 1]$. Let W_1, W_2, W_3 be the collection of all continuous functions, differentiable functions, monotonic functions respectively in V , then

- (a) W_1 is a subspace
- (b) W_2 is a subspace
- (c) W_3 is a subspace
- (d) All are subspaces

7. Let M be the vector space of all 3×3 real

matrices and let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Which of the following are subspaces of M ?

- (a) $\{X \in M : XA = AX\}$
- (b) $\{X \in M : X + A = A + X\}$
- (c) $\{X \in M : \text{trace}(AX) = 0\}$
- (d) $\{X \in M : \det(AX) = 0\}$

(CSIR NET June 2011)

8. Let $V = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n} : a_{ij} \in \mathbb{R} \right\}$ be the real vector space and $W_1 = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n} : a_{ij} = a_{ji} \right\}$ and

$W_2 = \left\{ \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n} : a_{ij} = -a_{ji} \right\}$ then

- (a) $V = W_1 + W_2$
- (b) $V = W_1 \oplus W_2$
- (c) $W_1 \cap W_2 = \{0\}$
- (d) $V = W_1 \cup W_2$

9. If V is the real vector space of all mapping from $\mathbb{R}$ to $\mathbb{R}$, $W_1 = \{f \in V : f(-x) = f(x)\}$ and $W_2 = \{f \in V : f(-x) = -f(x)\}$, then

- (a) $V = W_1 + W_2$
- (b) $V = W_1 \oplus W_2$
- (c) $W_1 \cap W_2 = \{0\}$
- (d) $V = W_1 \cup W_2$

10. Let V_1, V_2 be subspaces of a vector space V .

Which of the following is necessarily a subspace of V ?

- (a) $V_1 \cap V_2$
- (b) $V_1 \cup V_2$
- (c) $V_1 + V_2 = \{x + y : x \in V_1, y \in V_2\}$
- (d) $V_1 \setminus V_2 = \{x \in V_1 \text{ and } y \notin V_2\}$

(CSIR NET June 2012)

11. For arbitrary subspaces U , V and W of a finite dimensional vector space, which of the following hold

- (a) $U \cap (V+W) \subset U \cap V + U \cap W$
- (b) $U \cap (V+W) \supset U \cap V + U \cap W$
- (c) $(U \cap V) + W \subset (U+W) \cap (V+W)$
- (d) $(U \cap V) + W \supset (U+W) \cap (V+W)$

(CSIR NET Dec 2014)

12. Which of the following are subspaces of the vector space $\mathbb{R}^3$?

- (a) $\{(x, y, z): x + y = 0\}$
- (b) $\{(x, y, z): x - y = 0\}$
- (c) $\{(x, y, z): x + y = 1\}$
- (d) $\{(x, y, z): x - y = 1\}$

(CSIR NET June 2015)

13. Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$ with $b \neq 0$

1. The set of all real solutions of $Ax = b$ is a vector space.
2. If u and v are two solutions of $Ax = b$, then $\lambda u + (1-\lambda)v$ is also a solution of $Ax = b$ for any $\lambda \in \mathbb{R}$.
3. For any two solutions u and v of $Ax = b$, the linear combination $\lambda u + (1-\lambda)v$ is also a solution of $Ax = b$ only when $0 \leq \lambda \leq 1$.
4. If rank of A is n , then $Ax = b$ has at most one solution.

(CSIR NET Dec 2015)

6.1 Linear combinations

Def. Linear Combination : Let $V(F)$ be a vector space. A vector $v \in V$ is said to be Linear Combination (L.C) of the vectors $v_1, v_2, \dots, v_n \in V$ if v can be expressed as

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \text{ where } a_1, a_2, \dots, a_n \in F \text{ are scalars.}$$

Def. Linear Span : Let S be a non empty subset of vector space $V(F)$ then the set of all linear combinations of finite number of elements of S is called linear span (or span) of S . It is denoted by

$$\langle S \rangle \text{ or } L(S) \text{ i.e. } L(S) = \langle S \rangle = \left\{ \sum_{i=1}^n a_i v_i : a_i \in F, v_i \in S, n \text{ finite} \right\}.$$

Remarks :

- (i) Here it should be noted very carefully that the set S may be finite or infinite but $L(S)$ contains linear combinations of only finite number of elements of S . In fact, linear combination of infinite elements is not defined, in general.
- (ii) Every non empty set is always contained in its linear span. Let S be a non empty subset of vector space V , for any $v \in S$, we can write $v = 1.v \in L(S) \Rightarrow S \subseteq L(S)$
- (iii) The linear span of empty set ϕ is defined to be $\{0\}$.

Results :

1. The set of all linear combinations of a non-empty set S of a vector space $V(F)$ is the smallest subspace of $V(F)$ containing S .

OR

In a vector space V , $L(S)$ is the smallest sub-space containing S .

2. $L(W) = W$ iff W is a subspace of V .

Def. Generating (Spanning) set : A non empty set S of vector space V is called a generating set of V if every element $v \in V$ is a linear combination of finite number of elements of S i.e. $V = L(S)$.

Def. Finitely generated vector space : As clear by its name, a vector space V over F is said to be finitely generated if it is generated by finite number of vectors.

e.g. $\mathbb{R}^n$ and $P_n(\mathbb{R})$ are finitely generated vector space.

Def. Infinitely generated vector space : A vector space V over a field F is called infinitely generated if it can not be generated by finite number of vectors e.g. the vector space $P(\mathbb{R})$ of all polynomials

over a field is infinitely generated vector space. The vectors $\{1, x, x^2, \dots, x^n, \dots\}$ generates $P(\mathbb{R})$ but no finite set can generate $P(\mathbb{R})$.

Exercise 6.1

- Express the vector $v = (4, -5, 9, -7)$ as a linear combination of vectors $v_1 = (1, 1, -2, 1)$, $v_2 = (3, 0, 4, -1)$, $v_3 = (-1, 2, 5, 2)$
- Is it possible to express the vector $v = (2, -5, 4)$ as a linear combination of $v_1 = (1, -3, 2)$ and $v_2 = (2, -1, 1)$.
- Consider the vector $v = (1, -2, k)$ in $\mathbb{R}^3(\mathbb{R})$. For what value of k (if any) the vector v can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$?
- Express the vector $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.
- Find the condition on a, b, c such that the matrix $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ is a linear combination of $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.
- If S and T are two non-empty subsets of a vector space $V(F)$, then show that
 - $S \subseteq T \Rightarrow L(S) \subseteq L(T)$
 - $L(S \cup T) = L(S) + L(T)$
 - $L(L(S)) = L(S)$
- Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = (2, -1, 3, 2)$, $v_2 = (-1, 1, 1, -3)$ and $v_3 = (1, 1, 9, -5)$?
- Let $V_3 = \mathbb{R}^3(\mathbb{R})$ and $S = \{v_1 = (1, 1, 0), v_2 = (0, -1, 1), v_3 = (1, 0, 1)\}$. Prove that $(a, b, c) \in L(S)$ iff $a = b + c$.
- In the complex vector space $\mathbb{C}^2(\mathbb{C})$ does $(1 + i, 1 - i)$ belongs to $\langle (1 + i, 1), (1, 1 - i) \rangle$?
- Which of the following polynomials belong to the vector space generated by $\{x^3, x^2 + 2x, x^2 + 2, 1 - x\}$.
 - $3x^2 + x + 5$
 - $2x^3 + 3x^2 + 3x + 7$
 - $x^4 + 7x + 2$
- Consider the vectors $u = (1, 2, 3)$ and $v = (2, 3, 1)$ in $\mathbb{R}^3$.
 - Write $w = (1, 3, 8)$ as a linear combination of u and v .

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- (b) Write $w = (2, 4, 5)$ as a linear combination of u and v .
- (c) Find k so that $w = (1, k, 4)$ is a linear combination of u and v .
- (d) Find conditions on a, b, c so that $w = (a, b, c)$ is a linear combination of u and v .
12. Write the polynomial $f(t) = at^2 + bt + c$ as a linear combination of the polynomials $p_1 = (t-1)^2$, $p_2 = t-1$, $p_3 = 1$.
13. Show that $\text{span}(S) = \text{span}(S \cup \{0\})$. That is, by joining or deleting the zero vector from a set, we do not change the space spanned by the set.
- Answers**
1. $v = -3v_1 + 2v_2 - v_3$ 2. No 3. $k = -8$ 4. $v = 2v_1 - v_2 + 2v_3$
5. $a + b + 2c = 0$ 7. No 9. Yes 10. only (i)
11. (a) $w = 3u_1 - u_2$ (b) Impossible (c) $k = \frac{11}{5}$ (d) $7a - 5b + c = 0$
12. Using $f = xp_1 + yp_2 + zp_3$, we get $x = a$, $y = 2a + b$, $z = a + b + c$.

6.2 Linear independence and dependence

Def. Linearly dependent (L.D.) vectors : Let V be a vector space over a field F . The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent if there exist scalars $a_1, a_2, \dots, a_n$ (not all zero) such that $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$.

OR

Some finite vectors are said to be linearly dependent if their linear combination can be made zero without taking all scalars zero.

Def. Linearly independent (L.I.) vectors : Vectors which are not L.D. are called L.I.

OR

Vectors $v_1, v_2, \dots, v_n$ are called linearly independent if, whenever

$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ for some scalars a_i 's, then $a_1 = a_2 = \dots = a_n = 0$.

OR

Success comes to those who dare to begin.

Some finite vectors are said to be L.I. if the only way to make their linear combination zero is to take all scalars zero.

Def. An infinite subset of V is said to be linearly independent if every finite subset of S is linearly independent, otherwise it is linearly dependent.

Results :

1. The empty set is defined to be linearly independent.
2. (i) A set containing only the zero vector i.e. $\{0\}$ is linearly dependent.
(ii) A set containing only a non-zero vector is linearly independent.
3. A set which contains the null vector '0' is linearly dependent.
4. If two vectors are L.D., then one of them is scalar multiple of other.
5. Every super set of a linearly dependent set is linearly dependent.
6. Every subset of a linearly independent set is linearly independent.
7. If a vector v is a linear combination of vectors $v_1, v_2, \dots, v_r$, then the set $\{v, v_1, v_2, \dots, v_r\}$ is linearly dependent.
8. If the set $\{v_1, v_2, \dots, v_r\}$ is L.I. and the set $\{v_1, v_2, \dots, v_r, v\}$ is L.D., then v is a linear combination of the vectors $v_1, v_2, \dots, v_r$.
9. The set of non-zero vectors $v_1, v_2, \dots, v_r$ of a vector space $V(F)$ is L.D. iff one of the vectors $v_r (r > 1)$ is a linear combination of the preceding vectors.
10. To determine that given vectors are L.D. or L.I.

Input : Vectors $v_1, v_2, \dots, v_n$ are given.

Output : $v_1, v_2, \dots, v_n$ L.D. or L.I.

Working Steps :

(i) Take n scalars $a_1, a_2, \dots, a_n$ and form the system $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ (1)

(ii) Solve the system (1) to find the values of $a_1, a_2, \dots, a_n$.

(iii) If $a_1 = a_2 = \dots = a_n = 0$ is the only solution of (1), then $v_1, v_2, \dots, v_n$ are linearly independent.

(iv) If system (1) has a non-trivial solution i.e. atleast one a_i is non-zero, then $v_1, v_2, \dots, v_n$ are linearly dependent.

Remark : In above method, we are not required to find the actual values of $a_1, a_2, \dots, a_n$ rather our only need is to decide whether the system has a non-trivial solution or not.

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11. To determine that given vectors are L.D. or L.I.

Input : Vectors $v_1, v_2, \dots, v_n$ are given.

Output : $v_1, v_2, \dots, v_n$ L.D. or L.I.

Working Steps :

(i) Construct a matrix A by writing the given vectors in the column.

(ii) Reduce the matrix A into row echelon form by using elementary row operation.

(iii) Number of non-zero rows gives the rank of A i.e. $\rho(A)$. If $\rho(A) =$ number of columns of A, then given vectors are L.I. and if $\rho(A) <$ number of columns of A, then given vectors are L.D.

12. Vectors constructed from given L.I. vectors : Let $v_1, v_2, \dots, v_n$ are L.I. vectors and suppose

$w_1, w_2, \dots, w_m$ are constructed as

$$w_1 = a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n$$

$$w_2 = a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n$$

$$\dots\dots\dots$$

$$w_m = a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n$$

Let $A = [a_{ij}]_{m \times n}$ denotes the coefficient matrix of above system, then

(i) if $m > n$ then $w_1, w_2, \dots, w_m$ are L.D.

(ii) if $m < n$ then $w_1, w_2, \dots, w_m$ are L.I. if $\rho(A) = m$ and L.D. if $\rho(A) < m$.

Exercise 6.2

1. Determine whether the following set of vectors are linearly dependent or independent

(i) $\{(1, 1, 1), (1, 2, 3), (0, 1, 2)\}$ in $\mathbb{R}^3(\mathbb{R})$

(ii) $\{(1, -2, 1), (2, 1, -1), (7, -4, 1)\}$ in $\mathbb{R}^3(\mathbb{R})$

(iii) $\{(1, -2, 3), (2, 3, 4), (0, 1, 2)\}$ in $\mathbb{R}^3(\mathbb{R})$

(iv) $\{(2, 3, -1, -1), (1, -1, -2, -4), (3, 1, 3, -2), (6, 3, 0, -7)\}$ in $\mathbb{R}^4(\mathbb{R})$

(v) $\{(1, 2, 3, 4), (0, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 0)\}$ in $\mathbb{R}^4(\mathbb{R})$

(vi) $\{(1, 1, 0, 0), (0, 1, -1, 0), (0, 0, 0, 3)\}$ in $\mathbb{R}^4(\mathbb{R})$

(vii) $\{(0, 1, 0, 1, 1), (1, 0, 1, 0, 1), (0, 1, 0, 1, 1), (1, 1, 1, 1, 1)\}$ in $\mathbb{R}^5(\mathbb{Q})$

Dream your wildest dreams and you will live a wild life.

(viii) $\{(1+i, 2i), (1, 1+i)\}$ in $\mathbb{C}^2(\mathbb{R})$

(ix) $\{(1+i, 2i), (1, 1+i)\}$ in $\mathbb{C}^2(\mathbb{C})$

2. In the vector space of polynomials of degree ≤ 4 , which of the following vectors are L.I.

(i) $x^2 + x + 1, x, 1$

(ii) $x^3 + 2x + 1, x^3 - x + 1, x + 1$

(iii) $x^4 + x^3, x^4 - 1, x^3 + x^2, x^2 + x, x + 1$

(iv) $x^4 - x, x^3 + 1, x^3 - 1$

3. Show that the three row vectors as well as the three column vectors of the matrix $\begin{bmatrix} 2 & 3 & 6 \\ 0 & 4 & 2 \\ 5 & 2 & 2 \end{bmatrix}$ are

linearly independent.

4. Prove that the four vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and $(1, 1, 1)$ are linearly dependent but any three of them are linearly independent.

5. Prove that the non-zero rows in an echelon matrix form a linearly independent set.

6. Find α if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \alpha \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

7. If u, v, w are linearly independent, then show that the vectors $u + v, v - w, u - 2v + w$ are linearly independent.

8. Prove that the vectors (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2(\mathbb{R})$ are linearly dependent if $a_1 b_2 - a_2 b_1 = 0$.

9. If v_1 and v_2 are vectors in V and $a, b \in F$, show that the set $\{v_1, v_2, av_1 + bv_2\}$ is linearly dependent.

10. Show that if $\{v_1, v_2, \dots, v_n\}$ is a linearly independent set of vectors and $a_1, a_2, \dots, a_{n-1}$ are any scalars, then the set $\{v_1 - a_1 v_n, v_2 - a_2 v_n, \dots, v_{n-1} - a_{n-1} v_n\}$ is also linearly independent.

11. If $v_1, v_2, \dots, v_n$ are linearly independent vectors of a vector space V , then show that none of them can be a zero vector.

12. Let S be a set of four vectors such that any three of them are linearly independent. Does it follow that the four vectors are linearly independent?

13. In the vector space $\mathbb{R}(\mathbb{Q})$, prove that the set $\{1, u\}$ is L.I. iff u is irrational.

14. Under what conditions on scalar $a \in \mathbb{C}$, the vectors $(1+a, 1-a)$ and $(1-a, 1+a)$ in $\mathbb{C}^2(\mathbb{C})$ are linearly dependent.

15. Let u, v, w be the vectors of $V(F)$ and $a, b \in F$. Show that the set $\{u, v, w\}$ is L.D. if the set $\{u + av + bw, v, w\}$ is L.D.

16. If u, v, w are L.I. in $V(F)$, then show that $u + v, v + w, w + u$ are L.I.

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17. Show that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}(\mathbb{C})$ but $\{1, i\}$ is linearly independent in the vector space $\mathbb{C}(\mathbb{R})$.
18. Determine whether the following vectors in $\mathbb{R}^4$ are linearly dependent or independent :
(a) $(1, 2, -3, 1), (3, 7, 1, -2), (1, 3, 7, -4), (1, 3, 1, -2), (2, 5, -1, 3), (1, 3, 7, -2)$.
19. Determine whether the following polynomials u, v, w in $\mathbb{P}(t)$ are linearly dependent or independent
(a) $u=t^3-4t^2+3t+3, v=t^3+2t^2+4t-1, w=2t^3-t^2-3t+5$
(b) $u=t^3-5t^2-2t+3, v=t^3-4t^2-3t+4, w=2t^3-7t^2-7t+9$.
20. Show that the following functions f, g, h are linearly independent :
(a) $f(t)=e^t, g(t)=\sin t, h(t)=t^2$ (b) $f(t)=e^t, g(t)=e^{2t}, h(t)=t$.
21. Show that $u=(a, b)$ and $v=(c, d)$ in $\mathbb{R}^2$ are linearly dependent if and only if $ad-bc=0$.
22. Suppose u, v, w are linearly independent vectors. Prove that S is linearly independent where :
(a) $S=\{u+v-2w, u-v-w, u+w\}$ (b) $S=\{u+v-3w, u+3v-w, u+w\}$.
23. Suppose $\{u_1, \dots, u_r, w_1, \dots, w_s\}$ is a linearly independent subset of V . Show that $\text{span}(u_i) \cap \text{span}(w_j) = \{0\}$.
24. Suppose $v_1, v_2, \dots, v_n$ are linearly independent. Prove that S is linearly independent where
(a) $S=\{a_1v_1, a_2v_2, \dots, a_nv_n\}$ and each $a_i \neq 0$.
(b) $S=\{v_1, \dots, v_{k-1}, w, v_{k+1}, \dots, v_n\}$ and $w=\sum_i b_i v_i$ and $b_k \neq 0$.

Answers

1. (i) Linearly dependent (ii) Linearly dependent (iii) Linearly Independent
(iv) Linearly dependent (v) Linearly dependent (vi) Linearly Independent
(vii) Linearly dependent (viii) Linearly Independent (ix) Linearly dependent
2. (i) Linearly Independent (ii) Linearly Independent (iii) Linearly dependent
(iv) Linearly Independent 6. $\alpha = 1$
18. (a) Dependent (b) Independent 19. (a) Independent (b) Dependent

6.3 Basis and dimension

Def. Basis of a vector space : Let V be a vector space over the field F . A set of vectors $v_1, v_2, \dots, v_n \in V$ is called a basis of V if

- (i) $v_1, v_2, \dots, v_n$ are L.I. and (ii) $v_1, v_2, \dots, v_n$ span V .

Remark : The basis for a vector space is not unique.

Def. Minimal generating set : A subset S of a vector space V is said to be minimal generating set if

- (i) S is a generating set of V . (ii) Any proper subset of S can not generate V .

Def. Maximal Linearly independent set : A subset S of a vector space V is said to be maximal linearly independent set if

- (i) S is linearly independent.
(ii) No proper superset of S is linearly independent i.e. if $S \subsetneq S'$ then S' must not be linearly independent.

Results :

1. A set of vectors $v_1, v_2, \dots, v_n \in V$ is a basis of V if each element of V can be uniquely expressed as a linear combination of the vectors $v_1, v_2, \dots, v_n$.
2. If $S = \{v_1, v_2, \dots, v_n\}$ spans a vector space $V(F)$, then there exists a subset of S which is a basis of V .
3. A minimal generating set is always a basis in a finitely generated vector space.
4. A maximal linearly independent set is always a basis in a finitely generated vector space.
5. If $S = \{u_1, u_2, \dots, u_n\}$ spans a vector space $V(F)$, then any $n+1$ vectors in V are linearly dependent.
6. If V is a finitely generated vector space, then any two basis of V have same number of elements.

Def. Dimension of a vector space : The numbers of elements in any basis of a vector space $V(F)$ is called dimension of V and is denoted by $\dim V$.

If $\dim V = n$, then V is called n dimensional vector space. A vector space of finite dimension is called finite dimensional vector space.

7. If $\dim V = n$ and $S = \{v_1, v_2, \dots, v_n\}$ is L.I. subset of V then S is a basis of V .

OR

A L.I. set of n vectors in a n -dimensional vector space is always a basis.

8. If $\dim V = n$ and $S = \{v_1, v_2, \dots, v_n\}$ generates V then S is a basis of V .

OR

A generating set of n vectors in a n -dimensional vector space is a basis.

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Tables of basis :

Table 1

Sr. No.	Vector space	Standard basis	General basis	Dimension
1	$\mathbb{C}(\mathbb{C})$	$\{1\}$	$\{a+ib\}$ s.t. $a+ib \neq 0$	1
2	$\mathbb{C}(\mathbb{R})$	$\{1, i\}$	$\{a+ib, c+id\}$ s.t. $ad-bc \neq 0$	2
3	$\mathbb{C}(\mathbb{Q})$	---	---	∞
4	$\mathbb{R}(\mathbb{R})$	$\{1\}$	$\{a\}; a \neq 0$	1
5	$\mathbb{R}(\mathbb{Q})$	---	---	∞
6	$\mathbb{Q}(\mathbb{Q})$	$\{1\}$	$\left\{\frac{p}{q}\right\}; p, q \neq 0$	1

Table 2

Sr. No.	Vector space	Standard basis	General basis	Dimension
1	$\mathbb{C}^n(\mathbb{C})$	$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$	Any n L.I. vectors	n
2	$\mathbb{C}^n(\mathbb{R})$	$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$ $\{(i, 0, \dots, 0), (0, i, \dots, 0), \dots, (0, 0, \dots, i)\}$	Any $2n$ L.I. vectors	$2n$
3	$\mathbb{C}^n(\mathbb{Q})$	---	---	∞
4	$\mathbb{R}^n(\mathbb{R})$	$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$	Any n L.I. vectors	n
5	$\mathbb{R}^n(\mathbb{Q})$	---	---	∞
6	$\mathbb{Q}^n(\mathbb{Q})$	$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$	Any n L.I. vectors	n

Never let your fears be the boundaries of your dreams.

Table 3

Sr. No.	Vector space	Standard basis	General basis	Dimension
1	$M_{m,n}(\mathbb{C})$ over $\mathbb{C}$	$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ $\dots$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$	Any mn L.I. vectors	mn
2	$M_{m,n}(\mathbb{C})$ over $\mathbb{R}$	$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ $\dots$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ $\begin{bmatrix} i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & i \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ $\dots$ $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ i & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & i & \dots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & i \end{bmatrix}$	Any $2mn$ L.I. vectors	$2mn$
3	$M_{m,n}(\mathbb{C})$ over $\mathbb{Q}$	----	---	∞
4	$M_{m,n}(\mathbb{R})$ over $\mathbb{R}$	Same as 1	Any mn L.I. vectors	mn
5	$M_{m,n}(\mathbb{R})$ over $\mathbb{Q}$	---	---	∞
6	$M_{m,n}(\mathbb{Q})$ over $\mathbb{Q}$	Same as 1	Any mn L.I. vectors	mn

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Table 4

Sr. No.	Vector space	Standard basis	General basis	Dimension
1	$P_n(\mathbb{C})$ over $\mathbb{C}$	$\{1, x, x^2, \dots, x^n\}$	Any $(n+1)$ L.I. vectors	$n+1$
2	$P_n(\mathbb{C})$ over $\mathbb{R}$	$\left\{ \begin{matrix} 1, x, x^2, \dots, x^n \\ i, ix, ix^2, \dots, ix^n \end{matrix} \right\}$	Any $2(n+1)$ L.I. vectors	$2(n+1)$
3	$P_n(\mathbb{C})$ over $\mathbb{Q}$	---	---	∞
4	$P_n(\mathbb{R})$ over $\mathbb{R}$	$\{1, x, x^2, \dots, x^n\}$	Any $(n+1)$ L.I. vectors	$n+1$
5	$P_n(\mathbb{R})$ over $\mathbb{Q}$	---	---	∞
6	$P_n(\mathbb{Q})$ over $\mathbb{Q}$	$\{1, x, x^2, \dots, x^n\}$	Any $(n+1)$ L.I. vectors	$n+1$

Table 5

Sr. No.	Vector space	Standard basis	General basis	Dimension
1	$P(\mathbb{C})$ over $\mathbb{C}$	$\{1, x, x^2, \dots\}$	---	∞
2	$P(\mathbb{C})$ over $\mathbb{R}$	$\left\{ \begin{matrix} 1, x, x^2, \dots \\ i, ix, ix^2, \dots \end{matrix} \right\}$	---	∞
3	$P(\mathbb{C})$ over $\mathbb{Q}$	---	---	∞
4	$P(\mathbb{R})$ over $\mathbb{R}$	$\{1, x, x^2, \dots\}$	---	∞
5	$P(\mathbb{R})$ over $\mathbb{Q}$	---	---	∞
6	$P(\mathbb{Q})$ over $\mathbb{Q}$	$\{1, x, x^2, \dots\}$	---	∞

Never give up! Failure and rejection are only the first step to succeeding. -Jimmy Valvano

Exercise 6.3

1. Which of the following set are L.I. generating set, basis of the vector space $\mathbb{C}(\mathbb{R})$.

- | | | | |
|---------------------|----------------------|------------------------|---|
| (i) $\{1, i\}$ | (ii) $\{1, 2, i\}$ | (iii) $\{2, 3i\}$ | (iv) $\left\{2, 3i, \frac{7}{2}i\right\}$ |
| (v) $\{2i, 3i\}$ | (vi) $\{2, 2+3i\}$ | (vii) $\{2, 3, 4-5i\}$ | (viii) $\{1+i, 1-i\}$ |
| (ix) $\{i, 2+3i\}$ | (x) $\{2+3i, 4+6i\}$ | (xi) $\{2+3i, 3+4i\}$ | (xii) $\{1, i, 1+i\}$ |
| (xiii) $\{3, 1+i\}$ | (xiv) $\{4, 3i\}$ | (xv) $\{i, -i\}$ | (xvi) $\{1+i, 1-i, 2\}$ |

2. Determine whether or not the following sets form a basis of $\mathbb{R}^3(\mathbb{R})$.

- | | |
|--|--|
| (i) $\{(4, 3, 2), (2, 1, 0), (-1, 1, -1)\}$ | (ii) $\{(1, -3, 2), (2, 4, 1), (1, 1, 1)\}$ |
| (iii) $\{(1, 0, 0), (1, 1, 0), (4, 5, 0)\}$ | (iv) $\{(2, 4, -3), (0, 1, 1), (0, 1, -1)\}$ |
| (v) $\{(1, 2, 1), (5, 0, -1), (2, -1, 0), (5, 2, 1)\}$ | |

3. Determine whether or not the following sets form a basis of $\mathbb{R}^4(\mathbb{R})$.

- | |
|---|
| (i) $\{(1, 0, 0, 0), (1, 2, 0, 0), (1, 2, 3, 0), (1, 2, 0, 4)\}$ |
| (ii) $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ |
| (iii) $\{(1, -2, 5, -3), (0, 7, -9, 2), (0, 0, 1, 0), (0, 0, 0, 1)\}$ |

4. Let $\{u, v, w\}$ be a basis for the vector space $\mathbb{R}^3$. Prove that the sets $\{u+v, v+w, w+u\}$ and $\{u, u+v, u+v+w\}$ are also bases of $\mathbb{R}^3$.

5. Show that the set $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{C}^3(\mathbb{C})$ but not a basis of $\mathbb{C}^3(\mathbb{R})$ or $\mathbb{C}^3(\mathbb{Q})$.

6. Let V be the vector space of solutions of the differential equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$. Show that

$V(\mathbb{R})$ is a 3-dimensional real vector space. Find a basis of this vector space.

7. Let $\mathcal{V} = M_{n,n}(\mathbb{C})$ be the vector space over the field $\mathbb{C}$. Which of the following subsets W of V are subspaces. In case of a subspace find the dimension and a basis of W .

- W = set of all diagonal matrices in V .
- W = set of all scalar matrices in V .
- W = set of all lower triangular matrices in V .
- W = set of all upper triangular matrices in V .
- W = set of all super lower triangular matrices in V .
- W = set of all super upper triangular matrices in V .
- W = set of all backward diagonal matrices in V .

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- (viii) W = set of all backward scalar matrices in V .
- (ix) W = set of all symmetric matrices in V .
- (x) W = set of all skew symmetric matrices in V .
- (xi) W = set of all hermitian matrices in V .
- (xii) W = set of all skew hermitian matrices in V .
- (xiii) W = set of all orthogonal matrices in V .
- (xiv) W = set of all unitary matrices in V .
- (xv) W = set of all normal matrices in V .
- (xvi) W = set of all idempotent matrices in V .
- (xvii) W = set of all involutory matrices in V .
- (xviii) W = set of all nilpotent matrices in V .

8. Let $V = M_{n,n}(\mathbb{C})$ be the vector space over the field $\mathbb{R}$. Which of the following subsets W of V are subspaces. In case of a subspace find the dimension and a basis of W .

- (i) W = set of all diagonal matrices in V .
- (ii) W = set of all scalar matrices in V .
- (iii) W = set of all lower triangular matrices in V .
- (iv) W = set of all upper triangular matrices in V .
- (v) W = set of all super lower triangular matrices in V .
- (vi) W = set of all super upper triangular matrices in V .
- (vii) W = set of all backward diagonal matrices in V .
- (viii) W = set of all backward scalar matrices in V .
- (ix) W = set of all symmetric matrices in V .
- (x) W = set of all skew symmetric matrices in V .
- (xi) W = set of all hermitian matrices in V .
- (xii) W = set of all skew hermitian matrices in V .
- (xiii) W = set of all orthogonal matrices in V .
- (xiv) W = set of all unitary matrices in V .
- (xv) W = set of all normal matrices in V .
- (xvi) W = set of all idempotent matrices in V .
- (xvii) W = set of all involutory matrices in V .
- (xviii) W = set of all nilpotent matrices in V .

9. Let $V = M_{n,n}(\mathbb{R})$ be the vector space over the field $\mathbb{R}$. Which of the following subsets W of V are subspaces. In case of a subspace find the dimension and a basis of W .

- (i) W = set of all diagonal matrices in V .
- (ii) W = set of all scalar matrices in V .
- (iii) W = set of all lower triangular matrices in V .
- (iv) W = set of all upper triangular matrices in V .
- (v) W = set of all super lower triangular matrices in V .
- (vi) W = set of all super upper triangular matrices in V .
- (vii) W = set of all backward diagonal matrices in V .
- (viii) W = set of all backward scalar matrices in V .
- (ix) W = set of all symmetric matrices in V .
- (x) W = set of all skew symmetric matrices in V .
- (xi) W = set of all hermitian matrices in V .
- (xii) W = set of all skew hermitian matrices in V .
- (xiii) W = set of all orthogonal matrices in V .
- (xiv) W = set of all unitary matrices in V .
- (xv) W = set of all normal matrices in V .
- (xvi) W = set of all idempotent matrices in V .
- (xvii) W = set of all involutory matrices in V .
- (xviii) W = set of all nilpotent matrices in V .

10. Find the dimension of following vector space over $\mathbb{R}$?

- (i) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i + j = n + 1 \right\}$
- (ii) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i = 1 \text{ or } n \right\}$
- (iii) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i = 1 \text{ or } n \text{ or } j = 1 \text{ or } n \right\}$
- (iv) $V = \left\{ [a_{ij}]_{n \times n} : a_{1j} = a_{2j} \text{ for } 1 \leq j \leq n \right\}$
- (v) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ and } a_{ij} = a_{ji} \forall ij \right\}$
- (vi) $V = \left\{ (x_1, x_2, \dots, x_{100}) : x_1 + x_3 + \dots + x_{99} = 0 : x_i \in \mathbb{R} \right\}$
- (vii) $V = \left\{ (x_1, x_2, \dots, x_{100}) : x_1 = x_2 = \dots = x_{50} \text{ and } x_{51} + x_{52} + x_{100} = 0, x_i \in \mathbb{R} \right\}$
- (viii) $V = \left\{ (x_1, x_2, \dots, x_n) : x_1 + x_3 + x_5 + \dots + x_k = 0 \text{ where } k \text{ is the largest odd integer less than } n \right\}$
- (ix) $V = \left\{ (x_1, x_2, \dots, x_n) : x_3 = x_{3k} \text{ for } k \geq 1 \right\}$

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- (x) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = a_{kl} \text{ if } i+j = k+l \right\}$
- (xi) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i \text{ divides } j \right\}$
- (xii) $V = \{ p(x) \in P_4(\mathbb{R}) : p(1) = p(2) = p(3) \}$
- (xiii) $V = \{ p(x) \in P_4(\mathbb{R}) : p(1) = p(2) = p(3) = 0 \}$
- (xiv) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = a_{ji}, a_{1j} = a_{2j} \quad \forall i, j \right\}$
- (xv) $V = \{ (x_1, x_2, \dots, x_n) : x_1 = x_4 = x_9 = \dots = 0 \}$
- (xvi) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = -a_{ji} \text{ and } a_{ij} = 0 \text{ if } j=1 \right\}$
- (xvii) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } |i-j| = 1 \right\}$
- (xviii) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i+j \text{ is even} \right\}$
- (xix) $V = \{ a_0 + a_1x + \dots + a_{100}x^{100} : a_i = 0 \text{ if } i=3n \text{ for some integer } n \}$
- (xx) $V = \left\{ [a_{ij}]_{n \times n} : a_{ij} = 0 \text{ if } i \text{ or } j = 1 \text{ and } a_{ij} = a_{ji} \right\}$
- (xxi) $V = \{ (x, y, z, t) : x+y+2t=0, 2x-y+z=0, 3x+2t+z=0 \}$
- (xxii) $V = \{ (x, y, z, t) : x-y=0, z-t=0 \}$
- (xxiii) $V = \{ (x, y, z, t) : 2x+y=0, x+y+z-7t=0 \}$
- (xxiv) $V = \{ (x_1, x_2, x_{100}) : x_1+x_3=0, x_3+x_5=0, \dots, x_{97}+x_{99}=0 \}$
- (xxv) $V = \{ (x_1, x_2, \dots, x_{2015}) : x_i = 0 \text{ if } i \text{ is a perfect square} \}$

11. Find a subset of u_1, u_2, u_3, u_4 that gives a basis for $W = \text{span}(u_i)$ of $\mathbb{R}^5$ where :

- (a) $u_1 = (1, 1, 1, 2, 3), u_2 = (1, 2, -1, -2, 1), u_3 = (3, 5, -1, -2, 5), u_4 = (1, 2, 1, -1, 4)$
- (b) $u_1 = (1, -2, 1, 3, -1), u_2 = (-2, 4, -2, -6, 2), u_3 = (1, -3, 1, 2, 1), u_4 = (3, -7, 3, 8, -1)$
- (c) $u_1 = (1, 0, 1, 0, 1), u_2 = (1, 1, 2, 1, 0), u_3 = (2, 1, 3, 1, 1), u_4 = (1, 2, 1, 1, 1)$
- (d) $u_1 = (1, 0, 1, 1, 1), u_2 = (2, 1, 2, 0, 1), u_3 = (1, 1, 2, 3, 4), u_4 = (4, 2, 5, 4, 6)$

12. Consider the subspaces $U = \{ (a, b, c, d) : b - 2c + d = 0 \}$ and $W = \{ (a, b, c, d) : a = d, b = 2c \}$ of $\mathbb{R}^4$.

Find a basis and the dimension of : (a) U , (b) W , (c) $U \cap W$

13. Find a basis and the dimension of the solution space W of each of the following homogeneous systems :

(a) $x + 2y - 2z + 2s - t = 0$

(b) $x + 2y - z + 3s - 4t = 0$

$x + 2y - z + 3s - 2t = 0$

$2x + 4y - 2z - s + 5t = 0$

$2x + 4y - 7z + s + t = 0$

$2x + 4y - 2z + 4s - 2t = 0$

14. Find a homogeneous system whose solution space is spanned by the following sets of three vectors :

(a) $(1, -2, 0, 3, -1), (2, -3, 2, 5, -3), (1, -2, 1, 2, -2)$

(b) $(1, 1, 2, 1, 1), (1, 2, 1, 4, 3), (3, 5, 4, 9, 7)$.

15. Determine whether each of the following is a basis of the vector space $\mathbb{P}_n(t)$:

(a) $\{1, 1+t, 1+t+t^2, 1+t+t^2+t^3, \dots, 1+t+t^2+\dots+t^{n-1}+t^n\}$,

(b) $\{1+t, t+t^2, t^2+t^3, \dots, t^{n-2}+t^{n-1}, t^{n-1}+t^n\}$.

16. Find a basis and the dimension of the subspace W of $\mathbb{P}(t)$ spanned by :

(a) $u=t^3+2t^2-2t+1, v=t^3+3t^2-3t+4, w=2t^3+t^2-7t-7$,

(b) $u=t^3+t^2-3t+2, v=2t^3+t^2+t-4, w=4t^3+3t^2-5t+2$

17. Find a basis and the dimension of the subspace W of $V = M_{2,2}$ spanned by

$$A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$$

Answers

1. (i) L.I., generating set, basis (ii) L.D., generating set, not a basis
 (iii) L.I., generating set, basis (iv) L.D., generating set, not a basis
 (v) L.D., not a generating set, not a basis (vi) L.I., generating set, basis
 (vii) L.D., generating set, not a basis (viii) L.I., generating set, basis
 (ix) L.I., generating set, basis (x) L.D., not a generating set, not a basis
 (xi) L.I., generating set, basis (xii) L.D., generating set, not a basis
 (xiii) L.I., generating set, basis (xiv) L.I., generating set, basis
 (xv) L.D., not a generating set, not a basis (xvi) L.D., generating set, not a basis
2. (i) Yes (ii) Yes (iii) No (iv) Yes (v) No
3. (i) Yes (ii) Yes (iii) Yes
7. (i) Yes ; n (ii) Yes ; 1 (iii) Yes ; $\frac{n(n+1)}{2}$ (iv) Yes ; $\frac{n(n+1)}{2}$
 (v) Yes ; $\frac{n(n-1)}{2}$ (vi) Yes ; $\frac{n(n-1)}{2}$ (vii) Yes ; n (viii) Yes ; 1

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- (ix) Yes ; $\frac{n(n+1)}{2}$ (x) Yes ; $\frac{n(n-1)}{2}$ (xi) No (xii) No
- (xiii) No (xiv) No (xv) No (xvi) No
- (xvii) No (xviii) No
8. (i) Yes ; $2n$ (ii) Yes ; 2 (iii) Yes ; $n(n+1)$ (iv) Yes ; $n(n+1)$
- (v) Yes ; $n(n-1)$ (vi) Yes ; $n(n-1)$ (vii) Yes ; $2n$ (viii) Yes ; 2
- (ix) Yes ; $n(n+1)$ (x) Yes ; $n(n-1)$ (xi) Yes ; n^2 (xii) Yes ; n^2
- (xiii) No (xiv) No (xv) No (xvi) No
- (xvii) No (xviii) No
9. (i) Yes ; n (ii) Yes ; 1 (iii) Yes ; $\frac{n(n+1)}{2}$ (iv) Yes ; $\frac{n(n+1)}{2}$
- (v) Yes ; $\frac{n(n-1)}{2}$ (vi) Yes ; $\frac{n(n-1)}{2}$ (vii) Yes ; n (viii) Yes ; 1
- (ix) Yes ; $\frac{n(n+1)}{2}$ (x) Yes ; $\frac{n(n-1)}{2}$ (xi) Yes ; $\frac{n(n+1)}{2}$ (xii) Yes ; $\frac{n(n-1)}{2}$
- (xiii) No (xiv) No (xv) No (xvi) No
- (xvii) No (xviii) No
10. (i) $n^2 - n$ (ii) $n^2 - 2n$ (iii) $(n-2)^2$ (iv) $n^2 - n$
- (v) $\frac{n(n-1)}{2}$ (vi) 99 (vii) 50 (viii) $n-1$
- (ix) $n - \left\lfloor \frac{n}{3} \right\rfloor + 1$ (x) $2n-1$ (xi) $n^2 - \left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \dots + \left\lfloor \frac{n}{n} \right\rfloor \right)$
- (xii) 3 (xiii) 2 (xiv) $\frac{n(n-1)}{2}$ (xv) $n - \left\lfloor \sqrt{n} \right\rfloor$
- (xvi) $\frac{(n-2)(n-1)}{2}$ (xvii) $n^2 - 2n + 2$ (xviii) $\left\lfloor \frac{n^2}{2} \right\rfloor$ (xix) 67
- (xx) $\frac{n(n-1)}{2}$ (xxi) 2 (xxii) 2 (xxiii) 2
- (xxiv) 51 (xxv) 1971
11. (a) u_1, u_2, u_4 ; (b) u_1, u_2, u_3 ; (c) u_1, u_2, u_4 ; (d) u_1, u_2, u_3

Your dreams minus your doubts equal your true worth.

12. (a) $\dim U = 3$, (b) $\dim W = 2$, (c) $\dim(U \cap W) = 1$
 13. (a) Basis : $\{(2, -1, 0, 0, 0), (4, 0, 1, -1, 0), (3, 0, 1, 0, 1)\}$; $\dim W = 3$.
 14. (a) $5x + y - z - s = 0, x + y - z - t = 0$; (b) $3x - y - z = 0, 2x - 3y + s = 0, x - 2y + t = 0$
 15. (a) Yes. (b) No, since $\dim \mathbb{P}_n(t) = n + 1$, but the set contains only n elements
 16. (a) $\dim W = 3$ (b) $\dim W = 3$ 17. $\dim W = 2$

6.4 Basis and subspaces of finite vector spaces

Results :

1. Let $V_n(\mathbb{Z}_p)$ be a finite vector space of dimension n over the field $\mathbb{Z}_p$ then the total number of

distinct basis is
$$\frac{(p^n - 1)(p^n - p) \dots (p^n - p^{n-1})}{n!}$$

2. Let $V_n(\mathbb{Z}_p)$ be a finite vector space of dimension n over the field $\mathbb{Z}_p$ then the total number of

subspaces of dimension r is
$$\frac{(p^n - 1)(p^n - p) \dots (p^n - p^{n-1})}{(p^r - 1)(p^r - p) \dots (p^r - p^{r-1})}$$

Exercise 6.4

1. Find the number of distinct basis of the following vector spaces :
 (i) $\mathbb{Z}_2^3(\mathbb{Z}_2)$ (ii) $\mathbb{Z}_3^2(\mathbb{Z}_3)$ (iii) $\mathbb{Z}_5^2(\mathbb{Z}_5)$ (iv) $\mathbb{Z}_3^3(\mathbb{Z}_3)$
 (v) $M_{2,3}(\mathbb{Z}_2)$ (vi) $M_{2,2}(\mathbb{Z}_3)$ (vii) $P_3(\mathbb{Z}_2)$ (viii) $P_2(\mathbb{Z}_3)$
2. Find the number of 1-dimensional and 2-dimensional subspaces of the following vector spaces.
 Hence find the total number of proper subspaces and subspaces. Also write all the distinct basis and all subspaces.
 (i) $\mathbb{Z}_2^3(\mathbb{Z}_2)$ (ii) $P_2(\mathbb{Z}_3)$ (iii) $\mathbb{Z}_3^3(\mathbb{Z}_3)$

Answers

1. (i) 28 (ii) 24 (iii) 240 (iv) 1872
 (v) 27998208 (vi) 1010880 (vii) 840 (viii) 1872
2. (i) 1-dimensional 7, 2-dimensional 7, total subspaces 16, no. of distinct basis 28.
 (ii) 1-dimensional 13, 2-dimensional 13, total subspaces 28, no. of distinct basis 1872.
 (iii) 1-dimensional 13, 2-dimensional 13, total subspaces 28, no. of distinct basis 1872..

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7.1 Row space, column space and null space of a matrix

Def. Row space of a matrix : The space generated by rows of a matrix is called its row space. If A is a $m \times n$ real matrix then its row space is a subspace of $\mathbb{R}^n$.

Def. Column space of a matrix : The space generated by columns of a matrix is called its column space. If A is a $m \times n$ real matrix then its column space is a subspace of $\mathbb{R}^m$.

If $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ then

$$\text{row space of } A = \langle (a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{m1}, a_{m2}, \dots, a_{mn}) \rangle$$

$$\text{column space of } A = \langle (a_{11}, a_{21}, \dots, a_{m1}), (a_{12}, a_{22}, \dots, a_{m2}), \dots, (a_{1n}, a_{2n}, \dots, a_{mn}) \rangle$$

Result : The dimension of row space and of the column space of a matrix are both equal to the rank of the matrix.

Def. Null space of a matrix : Let A be a $m \times n$ matrix then null space of A is defined as

$$\text{Null space of } A = \{ \bar{x} : A\bar{x} = \bar{0} \}, \text{ where } \bar{x} \text{ is a } n \times 1 \text{ vector and } \bar{0} \text{ is a } m \times 1 \text{ vector.}$$

Clearly null space of A is a subspace of $\mathbb{R}^n$.

Result : If A is a $m \times n$ matrix then dimension of null space of A is $n - r$, where r is the rank of the matrix A .

Identical subspaces : Let W_1 and W_2 be two subspaces of a finite dimensional vector space V and let S_1 and S_2 be their spanning sets respectively. Then W_1 and W_2 are identical i.e. $W_1 = W_2$ if every vector of S_1 can be expressed as linear combination of vectors of S_2 and every vector of S_2 can be expressed as linear combination of vectors of S_1 .

Results :

1. Suppose A and B are two matrices such that AB is defined. Then row space of AB is contained in the row space of B and the column space of AB is contained in the column space of A .
2. Suppose A and B are row reduced echelon matrices. Then A and B have the same row space if and only if they have the same non zero rows.

Exercise 7.1

1. Find the dimension of row space and column space of each of the following matrices :

$$(a) \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}, \quad (b) \begin{bmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{bmatrix}, \quad (c) \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

2. For $k = 1, 2, \dots, 5$, find the number n_k of linearly independent subsets consisting of k columns for each of the following matrices :

$$(a) A = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 2 & 5 \\ 1 & 3 & 0 & 2 & 7 \end{bmatrix},$$

$$(b) B = \begin{bmatrix} 1 & 2 & 1 & 0 & 2 \\ 1 & 2 & 3 & 0 & 4 \\ 1 & 1 & 5 & 0 & 6 \end{bmatrix}.$$

$$3. \text{ Let (a) } A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 6 \\ 2 & 4 & 3 & 8 & 3 & 15 \\ 1 & 2 & 2 & 5 & 3 & 11 \\ 4 & 8 & 6 & 16 & 7 & 32 \end{bmatrix},$$

$$(b) B = \begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 2 & 4 & 5 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 4 & 6 \\ 3 & 6 & 7 & 7 & 9 & 10 \end{bmatrix}.$$

For each matrix (where $C_1, \dots, C_6$ denote its column) :

(i) Find its row canonical form M .

(ii) Find the columns that are linear combinations of preceding columns.

(iii) Find columns (excluding C_6) that form a basis for the column space.

(iv) Express C_6 as a linear combination of the basis vectors obtained in (iii).

4. Determine which of the following matrices have the same row space :

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 10 \\ 3 & -5 & 1 \end{bmatrix}$$

5. Determine which of the following subspaces of $\mathbb{R}^3$ are identical :

$$W_1 = \text{span}[(1, 1, -1), (2, 3, -1), (3, 1, -5)], \quad W_2 = \text{span}[(1, -1, -3), (3, -2, -8), (2, 1, -3)]$$

$$W_3 = \text{span}[(1, 1, 1), (1, -1, 3), (3, -1, 7)]$$

6. Determine which of the following subspaces of $\mathbb{R}^4$ are identical :

$$W_1 = \text{span}[(1, 2, 1, 4), (2, 4, 1, 5), (3, 6, 2, 9)], \quad W_2 = \text{span}[(1, 2, 1, 2), (2, 4, 1, 3)],$$

$$W_3 = \text{span}[(1, 2, 3, 10), (2, 4, 3, 11)]$$

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7. Find two different basis for (i) the row space and (ii) the column space of each matrix M :

$$(a) M = \begin{bmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix}, \quad (b) M = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 3 \\ 3 & 6 & 5 & 2 & 7 \\ 2 & 4 & 1 & -1 & 0 \end{bmatrix}$$

8. Show that if any row is deleted from a matrix in echelon (respectively, row reduced echelon) form, then the resulting matrix is still in echelon (respectively, row reduced echelon) form.

9. Let $r = \text{rank}(A+B)$. Find 2×2 matrices A and B such that :

$$(a) r < \text{rank}(A), \text{rank}(B) \quad (b) r = \text{rank}(A) = \text{rank}(B) \quad (c) r > \text{rank}(A), \text{rank}(B).$$

Answers

$$1. (a) 3 \quad (b) 2 \quad (c) 3$$

$$2. (a) n_1=4, n_2=5, n_3=n_4=n_5=0 \quad (b) n_1=4, n_2=6, n_3=3, n_4=n_5=0$$

$$3. (a) (i) M = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (ii) C_2, C_4, C_6$$

$$(iii) C_1, C_3, C_5$$

$$(iv) C_6 = 3C_1 + C_3 + 2C_5.$$

$$(b) (i) M = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (ii) C_2, C_5, C_6$$

$$(iii) C_1, C_3, C_4$$

$$(iv) C_6 = C_1 - C_3 + 2C_4$$

$$4. A \text{ and } C \text{ are row equivalent to } \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}, \text{ but not } B.$$

$$5. W_1 \text{ and } W_2 \text{ are row equivalent to } \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \text{ but not } W_3.$$

$$6. W_1 \text{ and } W_3 \text{ are row equivalent to } \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \text{ but not } W_2.$$

$$7. (a) (i) (1, 3, 1, 2, 1), (0, 0, 1, -1, -1), (0, 0, 0, 4, 7); C_1, C_3, C_4.$$

(b) (i) $(1, 2, 1, 0, 1), (0, 0, 1, 1, 2)$ (ii) C_1, C_3 .

9. (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

7.2 Sums, direct sums and intersections

Results :

- If W_1 and W_2 are two subspaces of a vector space $V(F)$, where V is finite dimensional.
Then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.
- If a finite dimensional vector space V is a direct sum of its two subspaces W_1 and W_2 , then
 $\dim V = \dim W_1 + \dim W_2$.
- If S_1 is a generating set of W_1 and S_2 is a generating set of W_2 then $S_1 \cup S_2$ is a generating set of $W_1 + W_2$.
- If B_1 is a basis of W_1 and B_2 is a basis of W_2 then $B_1 \cup B_2$ is a generating set of $W_1 + W_2$, but not necessarily a basis.
- Suppose $W_1, W_2, \dots, W_r$ are subspaces of a vector space V , then
 - $\text{span}(W_1, W_2, \dots, W_r) = W_1 + W_2 + \dots + W_r$.
 - If S_i spans W_i for $i=1, \dots, r$, then $S_1 \cup S_2 \cup \dots \cup S_r$ spans $W_1 + W_2 + \dots + W_r$.

Exercise 7.2

- If W_1 and W_2 are two-dimensional subspaces of $\mathbb{R}^3$, then show that $W_1 \cap W_2 \neq \{0\}$.
- Suppose W_1 and W_2 are subspaces of V such that $\dim W_1 = 4$, $\dim W_2 = 5$, and $\dim V = 7$. Find the possible dimensions of $W_1 \cap W_2$.
- Let W_1 and W_2 be subspaces of $\mathbb{R}^3$ for which $\dim W_1 = 1$, $\dim W_2 = 2$, and $W_1 \not\subseteq W_2$. Show that $\mathbb{R}^3 = W_1 \oplus W_2$.
- Consider the following subspaces of $\mathbb{R}^5$:

$$W_1 = \text{span}[(1, -1, -1, -2, 0), (1, -2, -2, 0, -3), (1, -1, -2, -2, 1)]$$

$$W_2 = \text{span}[(1, -2, -3, 0, -2), (1, -1, -3, 2, -4), (1, -1, -2, 2, -5)]$$
 - Find two homogeneous system whose solution spaces are W_1 and W_2 , respectively.
 - Find a basis and the dimension of $W_1 \cap W_2$.

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5. Let W_1, W_2, W_3 be the following subspaces of $\mathbb{R}^3$:

$$W_1 = \{(a, b, c) : a = c\}, \quad W_2 = \{(a, b, c) : a + b + c = 0\}, \quad W_3 = \{(0, 0, c)\}$$

Show that : (a) $\mathbb{R}^3 = W_1 + W_2$, (b) $\mathbb{R}^3 = W_2 + W_3$, (c) $\mathbb{R}^3 = W_1 + W_3$. When is the sum direct ?

6. Suppose W_1, W_2, W_3 are subspaces of a vector space V . Show that

$$(W_1 \cap W_2) + (W_1 \cap W_3) \subseteq W_1 \cap (W_2 + W_3). \text{ Find subspaces of } \mathbb{R}^2 \text{ for which equality does not hold.}$$

7. Let V be the vector space of n -square matrices. Let W_1 be the subspace of upper triangular matrices, and let W_2 be the subspace of lower triangular matrices. Find : (a) $W_1 \cap W_2$ (b) $W_1 + W_2$.

Answers

2. $\dim(W_1 \cap W_2) = 2, 3 \text{ or } 4$

4. (a) (i) $3x + 4y - z - t = 0$

$$4x + 2y + s = 0$$

(ii) $4x + 2y - s = 0$

$$9x + 2y + z + t = 0$$

(b) Basis : $\{(1, -2, -5, 0, 0), (0, 0, 1, 0, -1)\}$; $\dim(U \cap W) = 2$

5. The sum is direct in (b) and (c)

6. In $\mathbb{R}^2$, let W_1, W_2, W_3 be, respectively, the line $y = x$, the x -axis, the y -axis.

7. (a) Diagonal matrices, (b) V

7.3 Coordinates

Def. Co ordinate Vector : If the set $B = \{v_1, v_2, \dots, v_n\}$ is a basis of a vector space $V(F)$, then a vector $v \in V$ can be expressed uniquely as $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ for some scalars

$$a_1, a_2, \dots, a_n.$$

The coefficients $a_1, a_2, \dots, a_n$ in the linear combination of v are called co - ordinates of v relative to the basis B . The vector $(a_1, a_2, \dots, a_n)$ is called the coordinate vector of v relative to the basis of B .

Exercise 7.3

- In $\mathbb{R}^3(\mathbb{R})$, find the coordinates of vector u w.r.t. a basis (u_1, u_2, u_3) where
 - $u = (4, 5, 6)$; $u_1 = (-1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 0, -1)$
 - $u = (1, 0, -1)$; $u_1 = (0, 1, -1)$, $u_2 = (1, 1, 0)$, $u_3 = (1, 0, 2)$.
- In $\mathbb{C}^3(\mathbb{C})$, find the coordinates of $(2i, 3+4i, 5)$ w.r.t. the basis $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.
- The vectors $u_1 = (1, -2)$ and $u_2 = (4, -7)$ form a basis S of $\mathbb{R}^2$. Find the coordinates vector $[v]$ of v relative to S where : (a) $v = (5, 3)$, (b) $v = (a, b)$.
- The vectors $u_1 = (1, 2, 0)$, $u_2 = (1, 3, 2)$, $u_3 = (0, 1, 3)$ form a basis S of $\mathbb{R}^3$. Find the coordinates vector $[v]$ of v relative to S where (a) $v = (2, 7, -4)$, (b) $v = (a, b, c)$.
- $S = \{t^3 + t^2, t^2 + t, t + 1, 1\}$ is a basis of $P_3(\mathbb{R})$. Find the coordinates vector $[v]$ of v relative to S where (a) $v = 2t^3 + t^3 - 4t + 2$ (b) $v = at^3 + bt^2 + ct + d$.
- Let $V = M_{2,2}$. Find the coordinates vector $[A]$ of A relative to S where :
 $S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and (a) $A = \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$, (b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Answers

- (i) $(15, -10, 9)$ (ii) $(-3, 3, -2)$
- $-3-2i, -2+4i, 5$ 3. (a) $[-47, 13]$, (b) $[-7a-4b, 2a+b]$
- (a) $[-11, 13, -10]$, (b) $[c-3b+7a, -c+3b-6a, c-2b+4a]$
- (a) $[2, -1, -3, 5]$ (b) $[a, b-c, c-b+a, d-c+b-a]$
- (a) $[7, -1, -13, 10]$ (b) $[d, c-d, b+c-2d, a-b-2c+2d]$

7.4 Extension to basis

Extension theorem : If V is a finitely generated vector space, then any set of linearly independent vectors can be extended to a basis of V .

Method : To extend a given linearly independent set of V to form a basis.

Input : A linearly independent set $S = \{v_1, v_2, \dots, v_m\}$ of a vector space V is given where $\dim V = n$.

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Output : A basis B of V such that $S \subseteq B$.

Working Steps :

- (i) Let $S' = \{e_1, e_2, \dots, e_n\}$ denotes the standard basis of V .
- (ii) As $\dim V = n$ and S already contains m elements, so we have to choose $n - m$ vectors out of $S' = \{e_1, e_2, \dots, e_n\}$.
- (iii) Select the vector e_1 out of S' and check the linear independence of the vectors $v_1, v_2, \dots, v_m, e_1$. If these are linearly dependent then remove e_1 and if linearly independent then take e_1 .
- (iv) Now select the vector e_2 and check the linear independence of e_2 together with the set obtained in step (iii). If these are linearly dependent then remove e_2 and if linearly independent then take e_2 .
- (v) Repeat the above process for $e_3, e_4, \dots$ until $n - m$ vectors out of S' are selected. These $n - m$ vectors together with S form a basis B of V .

Exercise 7.4

1. Extend the following sets of vectors to form a basis of $\mathbb{R}^3$.
(i) $\{(1, 2, 3), (2, -2, 0)\}$ (ii) $\{(0, 1, 2), (2, -1, 4)\}$ (iii) $\{(1, 1, 1), (1, 0, 0)\}$
2. Extend the set $\{(0, 0, 0, 3), (1, 1, 0, 0), (0, 1, -1, 0)\}$ to form a basis of $\mathbb{R}^4$.
3. Extend the set $S = \{(1, 1, 0)\}$ to form two different bases of $\mathbb{R}^3(\mathbb{R})$.
4. Extend the set $\{(1, 2, 3, 4, 5), (2, 3, 6, 1, 2)\}$ to form a basis of $\mathbb{R}^5$.

Answers

1. The answer is not unique, however one answer of each part is given
(i) $\{(1, 2, 3), (2, -2, 0), (1, 0, 0)\}$
(ii) $\{(0, 1, 2), (2, -1, 4), (1, 0, 0)\}$
(iii) $\{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$
2. The answer is not unique, however one answer is given
 $\{(0, 0, 0, 3), (1, 1, 0, 0), (0, 1, -1, 0), (0, 1, 0, 0)\}$.
3. $\{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$, $\{(1, 1, 0), (0, 1, 0), (0, 0, 1)\}$

7.5 Complementary subspace

Def. Complementary Subspace : Let W be a subspace of a finite dimensional vector space $V(F)$. If there exist a subspace W' of V such that V is direct sum of W and W' i.e. $V = W \oplus W'$, then W' is called complementary subspace (or simply complement) of W .

Results :

1. Every subspace W of a finite dimensional vector space $V(F)$ has a complementary subspace W' and $\dim W' = \dim V - \dim W$.
2. Complement of a subspace need not be unique.
3. Method : To determine a complement of a subspace.
Input : A subspace W is given.
Output : Complementary subspace W' .
Working Steps :
(i) First find a basis of W .
(ii) Extend this basis to form a basis of V .
(iii) Generate a subspace W' by the additional vectors used to extend the basis.
(iv) This W' is a complement of W .

Exercise 7.5

1. Find a complement of a subspace W generated by $(1, -2, 5, -3)$, $(2, 3, 1, -4)$ and $(3, 8, -3, -5)$ in $\mathbb{R}^4(\mathbb{R})$.

Answers

1. $W' = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle$

7.6 Quotient spaces

Def. Coset : Let $V(F)$ be a vector space and W be a subspace of V , then for any $v \in V$, then the set $v + W = \{v + w : w \in W\}$ is called left coset of W formed with v or generated by v and the set $W + v = \{w + v : w \in W\}$ is called right coset of W formed with v or generated by v . Since the vector space $V(F)$ is commutative w.r.t addition, so $W + v = v + W$ and so it is known as simply coset of W formed with v .

Results :

1. Let W be a subspace of a vector space $V(F)$ and let V/W be the set of all cosets of W in V i.e.

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$V/W = \{v + W : v \in V\}$ then V/W is a vector space under the operations defined by

$$(u + W) + (v + W) = (u + v) + W \text{ and } a(u + W) = au + W$$

and V/W is called quotient space of V w.r.t. W .

2. Let W be a sub space of a finite dimensional vector space $V(F)$, then

$$\dim(V/W) = \dim V - \dim W.$$

Remark : To find the basis of V/W we first extend the basis of W to form a basis of V and then take the cosets of W formed with additional vectors which are used for extension of basis. These cosets form a basis of V/W .

Exercise 7.6

1. Determine $\dim V/W$, where $V = \mathbb{C}(\mathbb{R})$, $W = \mathbb{R}(\mathbb{R})$.
2. Determine $\dim V/W$ where $V = \mathbb{C}^2(\mathbb{R})$, $W = \mathbb{R}^2(\mathbb{R})$.
3. Let $A = \{(x, y, 0), x, y \in \mathbb{R}\}$ and $B = \{(0, y, z) : y, z \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^3(\mathbb{R})$.

Verify that $\dim((A + B)/A) = \dim(B/A \cap B)$.

4. If W is a subspace of $\mathbb{R}^3(\mathbb{R})$ generated by $\{(1, 0, 0), (1, 1, 0)\}$, find basis and dimension of V/W .
5. If V is a vector space of all 2×2 matrices over $\mathbb{R}$ and W is the set of all 2×2 diagonal matrices over $\mathbb{R}$, then show that W is a subspace of V . Also find $\dim(V/W)$.
6. Let V be the vector space of all 2×2 matrices over $\mathbb{C}$. Let $W_1 = \left\{ \begin{bmatrix} a & -a \\ b & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$,

$W_2 = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ be two subsets of V . Verify that W_1 and W_2 are subspaces of V

$$\text{and } \dim\left(\frac{W_1 + W_2}{W_2}\right) = \dim\left(\frac{W_1}{W_1 \cap W_2}\right).$$

Answers

1. 1 2. 2 4. Basis = $\{W + (0, 0, 1)\}$, $\dim = 1$ 5. 2

True-False questions :

Answer true or false. If false, prove it with a counterexample.

1. If u_1, u_2, u_3 span V , then $\dim V = 3$.
2. If A is a 4×8 matrix, then any six columns are linearly dependent.
3. If u_1, u_2, u_3 are linearly independent, then u_1, u_2, u_3, w are linearly dependent.
4. If u_1, u_2, u_3, u_4 are linearly independent then $\dim V \geq 4$.
5. If u_1, u_2, u_3 span V , then w, u_1, u_2, u_3 span V .
6. If u_1, u_2, u_3, u_4 are linearly independent, then u_1, u_2, u_3 are linearly independent.
7. If any column is deleted from a matrix in echelon form, then the resulting matrix is still in echelon form.
8. If any column is deleted from a matrix in row canonical form, then the resulting matrix is still in row reduced echelon form.
9. If any column without a pivot is deleted from a matrix in row canonical form, then the resulting matrix is in row reduced echelon form.

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8.1 Linear transformations

Def. Linear Transformation : Let U and V be two vector spaces over a same field F , then a function $T : U \rightarrow V$ is called a linear transformation or vector space homomorphism if

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in U$

(ii) $T(au) = a T(u)$ for all $a \in F, u \in U$

Remark : The above two properties in the definition of L.T. can be combined into a single property, namely, $T(au+bv) = aT(u)+bT(v)$ for all $a, b \in F$ and $u, v \in U$

Def. Linear Operator : A linear transformation is called a linear operator if the vector space on both sides is same.

Def. Linear Functional : A linear transformation is called a linear functional if the vector space on the right hand side is the field of scalars.

Def. Injective (one – one) Linear Transformation : A linear transformation $T : U \rightarrow V$ is called injective or one-one if for any $u, v \in U$, $T(u) = T(v) \Rightarrow u = v$

OR $u \neq v \Rightarrow T(u) \neq T(v)$ i.e. different elements has different images.

Def. Surjective (onto) Linear Transformation : A linear transformation $T : U \rightarrow V$ is called surjective or onto if for each $v \in V$ there is an element $u \in U$ s.t. $T(u) = v$ i.e. each element of V has a pre-image in U .

Def. Bijective Linear Transformation : A linear transformation $T : U \rightarrow V$ is called bijective if it is one – one and onto both. A bijective linear transformation is also known as vector space isomorphism.

Exercise 8.1

1. Determine which of the following mappings are linear transformation. .

(i) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (z, 2x+y)$

(ii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x^2, y)$

(iii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by $T(x, y, z) = (2x+y, z, x, y+1)$

(iv) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (|x|, y, z)$

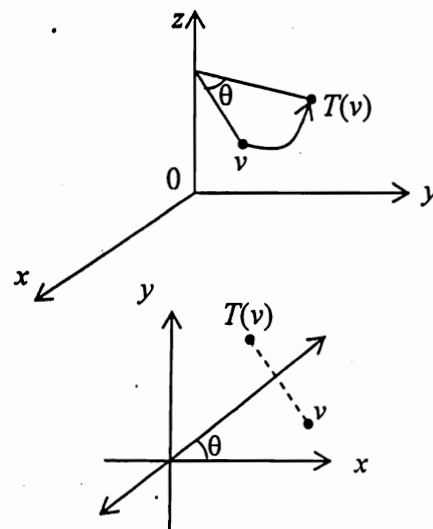
(v) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = x^2 + y^2$

2. Determine which of the following mappings are linear transformation. Further if a mapping is linear transformation then find whether it is one-one or onto or neither or both.

Take a risk-jump out of your comfort zone!

- (i) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$
- (ii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x - y, y - z, z - x)$
- (iii) $T: \mathbb{C}(\mathbb{R}) \rightarrow \mathbb{C}(\mathbb{R})$ defined by $T(x + iy) = x - iy$
- (iv) $T: \mathbb{C}(\mathbb{R}) \rightarrow \mathbb{C}(\mathbb{R})$ defined by $T(x + iy) = x$
- (v) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = |2x + 3y|$
- (vi) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(x, y) = xy$
- (vii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, 3z, 0)$
- (viii) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, ay, z)$, a is a non-zero real number.

3. Zero mapping : Let U and V be any vector spaces then the mapping $O: U \rightarrow V$ defined by $O(u) = 0$ for all $u \in U$ is called zero mapping. Prove that zero mapping is a linear transformation.
4. Identity mapping : Let V be any vector space then the mapping $I: V \rightarrow V$ defined by $I(v) = v$ for all $v \in V$ is called identity mapping. Prove that identity mapping is a linear transformation.
5. Projection mapping : The mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$ is called projection of space onto the xy -plane. Prove that T is a linear transformation.
6. Define the following projection mappings and prove that they are linear :
- Projection mapping of space onto the yz -plane.
 - Projection mapping of space onto the xz -plane.
 - Projection mapping of space onto x -axis.
 - Projection mapping of space onto y -axis.
 - Projection mapping of space onto z -axis.
7. Translation mapping : The mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + a, y + b, z + c)$ is called translation mapping, where a, b, c are real constants. Prove that T is not a linear transformation.
8. Rotation mapping : The mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$ rotates a vector v about the z -axis through an angle θ in anti clockwise sense. (as shown in the adjoining figure.)
Prove that T is a linear transformation.
9. Reflection mapping : The mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x \cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$ reflects a vector v about the line passing through origin and making an angle θ with positive x -axis.



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(as shown in the adjoining figure.)

Prove that T is a linear transformation.

10. Matrix mapping : Let A be a $m \times n$ matrix over a field F . The mapping $T: F^n \rightarrow F^m$ defined by

$T(v) = A(v)$ for all $v \in F^n$ is called matrix mapping. Prove that T is a linear transformation.

11. Derivative mapping : Let $P(\mathbb{R})$ denotes the vector space of all real polynomials. A mapping

$D: P(\mathbb{R}) \rightarrow P(\mathbb{R})$ defined by $D(f(x)) = f'(x)$ for all $f(x) \in P(\mathbb{R})$ is called derivative mapping.

Show that D is a linear operator on $P(\mathbb{R})$. Further tell whether D is one - one or onto or neither or both.

12. Integral mapping : Let V be the vector space of all integrable functions on $\mathbb{R}$. The mapping $T:$

$V \rightarrow \mathbb{R}$ defined by $T(f) = \int_a^b f(x) dx$ for all $f \in V$ and $a, b \in \mathbb{R}$ is called integral mapping.

Show that T is a linear functional.

13. Let V be the vector space of all polynomials over $\mathbb{R}$ and $g(t)$ is a fixed polynomial in V .

A mapping $T: V \rightarrow \mathbb{R}$ is defined by $T(f) = \int_a^b f(t) \cdot g(t) dt$ where $a, b \in \mathbb{R}$ and $f \in V$

Show that T is linear functional.

14. Let $M_n(F)$ be the vector space of all $n \times n$ matrices over a field F . Let M be a fixed $n \times n$ matrix.

Which of the following mappings $T: M_n(F) \rightarrow M_n(F)$ are linear :

(i) $T(A) = AM$ for all $A \in M_n(F)$

(ii) $T(A) = MA$ for all $A \in M_n(F)$

(iii) $T(A) = AM + MA$ for all $A \in M_n(F)$

(iv) $T(A) = MA + AM$ for all $A \in M_n(F)$

(v) $T(A) = AM - MA$ for all $A \in M_n(F)$

(vi) $T(A) = A + M$ for all $A \in M_n(F)$

(vii) $T(A) = M + A$ for all $A \in M_n(F)$

15. Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices and $P_2(\mathbb{R})$ be a vector space of

all polynomial of degree at most 2 with real coefficients. A mapping $T: M_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ is

defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b+c)x + dx^2$. Show that T is linear transformation. Further

We will either find a way or make one. -Hannibal

tell whether T is injective or surjective or both or none.

16. Consider the linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (3x + 5y, 2x + 3y)$ and the unit circle $S = \{(x, y): x^2 + y^2 = 1\}$ in $\mathbb{R}^2$, then find (i) $T(S)$ (ii) $T^{-1}(S)$.
17. Consider the linear map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y + z, y - 2z, y - 3z)$ and the unit sphere $S = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, then find (i) $T(S)$ (ii) $T^{-1}(S)$.
18. Let H be the plane $x + 2y - 3z = 4$ in $\mathbb{R}^3$ and let T be the linear map in problem 17, then find :
- (i) $T(H)$ (ii) $T^{-1}(H)$.

Answers

1. (i) Yes (ii) No (iii) No (iv) No (v) No.
2. (i) L.T. and both one – one and onto. (ii) L.T. but neither one – one nor onto.
- (iii) L.T. and both one – one and onto. (iv) L.T. but neither one – one nor onto.
- (v) not a L.T. (vi) not a L.T.
- (vii) L.T. but neither one – one not onto. (viii) L.T. and both one – one and onto.
14. (i), (ii), (iii), (iv), (v)
16. (i) $13x^2 - 42xy + 34y^2 = 1$, (ii) $13x^2 + 42xy + 34y^2 = 1$
17. (i) $x^2 - 8xy + 26y^2 + 6xz - 38yz + 14z^2 = 1$, (ii) $x^2 + 2xy + 3y^2 + 2xz - 8yz + 14z^2 = 1$
18. (i) $x - y + 2z = 4$, (ii) $x + 6z = 4$

8.2 Properties and construction of linear transformations

Results :

1. Let $T: U \rightarrow V$ be a linear transformation. Then prove the following :
- (i) $T(0) = 0'$, where 0 on L.H.S. is of U and $0'$ on R.H.S. is of V .
- (ii) $T(-u) = -T(u)$ for all $u \in U$
- (iii) $T(u_1 - u_2) = T(u_1) - T(u_2)$ for all $u_1, u_2 \in U$
- (iv) $T(a_1u_1 + a_2u_2 + \dots + a_nu_n) = a_1T(u_1) + a_2T(u_2) + \dots + a_nT(u_n)$ where $u_i \in U$ and a_i 's are scalars.
2. Let $T: U \rightarrow V$ be a linear transformation and vectors $u_1, u_2, \dots, u_n \in U$ are linearly dependent, then prove that $T(u_1), T(u_2), \dots, T(u_n)$ are also linearly dependent.

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OR

A linear transformation $T : U \rightarrow V$ carries L.D. vectors of U into L.D. vectors of V .

3. Images of L.I. vectors under a linear transformation need not be L.I.
4. Let $T : U \rightarrow V$ be a linear transformation. Then prove that T is one – one iff it carries linearly independent vectors to linearly independent vectors.
5. Let $T : U \rightarrow V$ be a linear transformation and $T(u_1), T(u_2), \dots, T(u_n)$ are L.I., then show that $u_1, u_2, \dots, u_n$ are also L.I.
6. Pre – images of L.D. vectors under a linear transformation need not to be L.D.
7. Let $T : U \rightarrow V$ be an onto linear transformation. Show that T is one-one iff pre-images of L.D. vectors under T are L.D. vectors.

Def. Equality of two linear transformations : Let $T_1, T_2 : U \rightarrow V$ be two linear transformation. Then T_1 and T_2 are said to be equal if $T_1(u) = T_2(u)$ for all $u \in U$ and then we write $T_1 = T_2$

8. Let $T_1, T_2 : U \rightarrow V$ be two linear transformations and $u_1, u_2, \dots, u_n$ be a basis of U . If $T_1(u_i) = T_2(u_i)$ for all i , then $T_1 = T_2$

OR

If two linear transformations agree on a basis of their domain, then they are equal.

9. Let U and V be two vector spaces, where $\dim U$ is finite. Let $\{u_1, u_2, \dots, u_n\}$ be a basis of U and $v_1, v_2, \dots, v_n$ be any n vectors in V . Then there is a unique linear transformation $T : U \rightarrow V$ such that $T(u_i) = v_i, i = 1, 2, \dots, n$.
10. Two finite dimensional vector spaces over the same field are isomorphic if and only if they have the same dimension.
11. Every n – dimensional vector space $V(F)$ is isomorphic to F^n .
12. For any field $F, F^n \cong F^m$ iff $n = m$.
13. There is no one – one onto linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ where $m \neq n$.

METHOD 8.2.1 : To find T when image of a basis set is given.

Input : Let $T : U \rightarrow V$ be a linear transformation. A basis $\{u_1, u_2, \dots, u_n\}$ of U is given and

$T(u_1) = v_1, T(u_2) = v_2, \dots$ and $T(u_n) = v_n$ are given.

Output : Defining formula for T .

Working Steps :

- (i) Take any arbitrary vector u of U and let $u = a_1u_1 + a_2u_2 + \dots + a_nu_n$ (1)
- (ii) Solving (1), find the values of $a_1, a_2, \dots, a_n$.
- (iii) Substitute the values of $a_1, a_2, \dots, a_n$, in (1) itself and then apply T on both sides to obtain the relation $T(u) = a_1T(u_1) + a_2T(u_2) + \dots + a_nT(u_n)$
- (iv) Now use the given values of $T(u_1), T(u_2), \dots, T(u_n)$ to get $T(u) = a_1v_1 + a_2v_2 + \dots + a_nv_n$ and thus T is found.

METHOD 8.2.2 : To find T when image of a subset of a basis is given

Input : Let $T : U \rightarrow V$ be a linear transformation and $\dim U = n$. A L.I. set $\{u_1, u_2, \dots, u_m\}$, where $m < n$ is given and $T(u_1) = v_1, T(u_2) = v_2, \dots, T(u_m) = v_m$ are given.

Output : Defining formula for T .

Working Steps :

- (i) First extend the set $\{u_1, u_2, \dots, u_m\}$ to a basis of U and let the basis obtained be $\{u_1, u_2, \dots, u_m, u_{m+1}, \dots, u_n\}$
- (ii) There is no condition on $T(u_{m+1}), \dots, T(u_n)$ so for our simplicity we assume that $T(u_{m+1}) = \dots = T(u_n) = 0$
- (iii) Now find T by using Method 8.2.1.

Remark : $T(u_{m+1}), \dots, T(u_n)$ in above method can be taken any vectors of V . So the answer of this problem is not unique but there are infinitely many linear transformations satisfying the conditions given in the example. However, in case images of vectors of a whole basis are given, then T is unique.

Exercise 8.2

1. Give an example of a linear transformation to show that a linear transformation not necessarily carries a L.I. set to a L.I. set.
2. What is the necessary and sufficient condition for a linear transformation to carry a L.I. set to a L.I. set.
3. Which of the following vector spaces are isomorphic to each other :

(i) $P_5(\mathbb{R})$	(ii) $M_{2,3}(\mathbb{R})$	(iii) $\mathbb{R}^6(\mathbb{R})$	(iv) $\mathbb{C}^3(\mathbb{R})$	(v) $M_{2,4}(\mathbb{R})$
(vi) $P_7(\mathbb{R})$	(vii) $P_8(\mathbb{R})$	(viii) $\mathbb{C}^4(\mathbb{R})$	(ix) $\mathbb{R}^8(\mathbb{R})$	(x) $M_{4,2}(\mathbb{R})$

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4. Find a linear transformation in each of the following cases and verify your answer.
- (i) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(2, 3) = (4, 5)$ and $T(1, 0) = (0, 0)$
 - (ii) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(2, -5) = (-1, 2, 3)$ and $T(3, 4) = (0, 1, 5)$
 - (iii) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, 1, 1) = (2, 1)$, $T(1, 1, 0) = (2, 1)$, $T(1, 0, 0) = (2, 1)$
 - (iv) $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ such that $T(1+x) = 1+x$, $T(2+x) = x+3x^2$ and $T(x^2) = 0$
5. Find a linear transformation in the following cases, which maps
- (i) $(1, 1, 1)$ and $(1, 1, 2)$ in $\mathbb{R}^3$ to $(1, 0)$ and $(1, -1)$ in $\mathbb{R}^2$ respectively.
 - (ii) $(1, 2, 3)$ and $(0, 1, 2)$ in $\mathbb{R}^3$ to $(1, 1)$ and $(1, 2)$ in $\mathbb{R}^2$ respectively.
- Also verify your answer in both (i) and (ii).
6. Show that there exists no linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1, 1) = (1, 2)$, $T(1, 2) = (1, 3)$, $T(2, 3) = (2, 4)$.
7. Does there exist a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1, 1) = (1, 2)$, $T(1, 2) = (1, 3)$, $T(2, 3) = (2, 5)$. If yes, find it.

4. (i) $T(x, y) = \left(\frac{4y}{3}, \frac{5y}{3} \right)$ (ii) $T(x, y) = \left(\frac{-4x+3y}{23}, \frac{13x-4y}{23}, \frac{37x+y}{23} \right)$
- (iii) $T(x, y, z) = (2x, x)$ (iv) $T(a+bx+cx^2) = (-a+2b)+bx+3(a-b)x^2$.
5. The answer is not unique, but by taking image of third vector as $(0, 0)$, the answers are
- (i) $T(x, y, z) = (x, x-z)$ (ii) $T(x, y, z) = (-x+y, -3x+2y)$

Def. Ordered basis : A basis of a vector space is called an ordered basis if its vectors are written in a certain specific order. e.g. Let $B = \{v_1, v_2, v_3\}$ is a basis of a vectors space V . Let us change the order of its vectors and let $B' = \{v_2, v_1, v_3\}$, then B and B' are same basis but different ordered basis since order of three vectors in B and B' is different.

Note : It may be difficult for a curious reader to digest above definition, because he or she may think that if we change the order of vectors in a basis, then how this can affect our subject. The answer to this curiosity is : As far as our previous work of linear transformation is concerned, the order of

Hope doesn't guarantee anything-hard work does.

vectors is a useless thing, but in the associated matrix of a linear transformation (which we are going to study now), the order of vectors certainly has an effect. We will see it soon.

Def. Matrix of a Linear Transformation relative to ordered basis :

Let $B = \{u_1, u_2, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively and let $T : U \rightarrow V$ be a linear transformation.

Since $T(u_1), T(u_2), \dots, T(u_n) \in V$ and $\{v_1, v_2, \dots, v_m\}$ is a basis of V , each $T(u_i)$ can be expressed uniquely as a linear combination of the vectors $v_1, v_2, \dots, v_m$.

$$\text{Let } T(u_1) = a_{11}v_1 + a_{12}v_2 + \dots + a_{1m}v_m$$

$$T(u_2) = a_{21}v_1 + a_{22}v_2 + \dots + a_{2m}v_m$$

$$\dots$$

$$T(u_n) = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nm}v_m \text{ where } a_{ij} \in F.$$

The coefficient matrix of the above system is a $n \times m$ matrix given by

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

The transpose of this matrix is a $m \times n$ matrix called the matrix of T w.r.t. ordered basis B and B' ,

denoted by $[T : B, B']$ and is given by $[T : B, B'] = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$

Remarks : (1) It should be noted that the order of matrix of a linear transformation

$$T : U \rightarrow V \text{ is } m \times n \text{ i.e. } \dim V \times \dim U.$$

- (2) Whenever basis B and B' are not mentioned in any problem, they are understood to be taken as standard basis.
- (3) If $U = V$ and B is the basis used on both sides, then the matrix of T is denoted by $[T : B]$ instead of $[T : B, B]$.
- (4) Sometimes matrix of T is denoted by $[T]$ or $m(T)$, when the bases are understood.
- (5) The matrix $[T : B, B']$ depends very sensitively on the ordered basis B and B' of U and V . The order (arrangement) of the vectors of B determines the order (arrangement) of the columns of matrix $[T : B, B']$. Similarly, order of the vectors of B' determines the order of the rows of matrix $[T : B, B']$.

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Short cut method : To find the matrix of a linear transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ in standard basis on both sides, there is a simple technique

Pick up the first coordinate in the defining formula of T . The coefficients of x, y, z etc. in this coordinate are written in the first row of the matrix.

Similarly, pick up the second coordinate in the defining formula of T . The coefficients of x, y, z etc. in this coordinate are written in the second row of the matrix, and so on.

Warning : This method is only applicable when the basis used on both sides are standard.

METHOD : To find T when matrix is given

Input : Let $T: U \rightarrow V$ be a linear transformation where $\dim U = n$ and $\dim V = m$.

Two ordered basis $B = \{u_1, u_2, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_m\}$ of U and V respectively are given and the matrix $[T: B, B']$ is given.

Output : The defining formula of T .

Working Steps :

- (i) $T(u_1)$ is equal to linear combination of $v_1, v_2, \dots, v_m$ by using scalars of first column of $[T: B, B']$.

Similarly, $T(u_2)$ is equal to linear combination of $v_1, v_2, \dots, v_m$ by using scalars of second column of $[T: B, B']$ and so on

- (ii) We are ready with T – images of a basis of U . Now find T by using Method 8.2.1.

Shortcut method : If we are given the matrix $[T: B, B']$ and we are required to find defining formula for T , then follow these steps :

Multiply the scalars in the first row by x, y, z etc. and add. This gives the first coordinate of defining formula.

Similarly, the second, third and other co – ordinates can be found.

Exercise 8.3

1. Find the matrix $[T: B, B']$ for the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y) = (x, -y) \text{ where}$$

(i) $B = (e_1, e_2)$, $B' = \{(1, 1), (1, -1)\}$

(ii) $B = \{(1, 1), (1, 0)\}$, $B' = \{(2, 3), (4, 5)\}$

You cannot have everything, but you can try.

2. Find the matrix representing the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by $T(x, y, z) = (x + y + z, 2x + z, 2y - z, 6y)$ relative to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.
3. Let $\mathbb{C}(\mathbb{R})$ denotes the vector space of complex numbers over real numbers and T be a linear operator on $\mathbb{C}(\mathbb{R})$ defined by $T(x + iy) = x - iy$ for all $x, y \in \mathbb{R}$. Determine the matrix of T relative to the basis $B = \{1 + i, 1 + 2i\}$
4. Write the matrix of linear transformation $T : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3 + (a_2 + a_3)x + (a_0 + a_1)x^2$ relative to the bases $B = \{1, x - 1, (x - 1)^2, (x - 1)^3\}$ and $B' = \{1, x, x^2\}$ respectively.
5. Find the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ whose matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}$ relative to ordered basis : $B = \{(1, 1), (-1, 1)\}$ and $B' = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$.
6. Describe the linear operator T on $\mathbb{R}^2$ determined by the matrix $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{2}{3} & 4 \end{bmatrix}$, relative to the ordered basis $B = \{(1, 0), (1, 1)\}$.
7. Describe the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ determined by the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$, relative to the ordered basis $B = \{(1, 1), (0, 2)\}$ and $B' = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.
8. If the matrix of a linear transformation T on $\mathbb{R}^3$ relative to the ordered basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$; Find the matrix relative to the basis $B' = \{(0, 1, -1), (-1, 1, 0), (1, -1, 1)\}$.
9. Let V be a vector space of some particular real functions and $B = \{1, t, e^t, te^t\}$ is a basis for V . Let $D : V \rightarrow V$ be the differential operator on V i.e. $D(f) = \frac{df}{dt}$. Find the matrix $[D : B]$.
10. Let $M_2(\mathbb{R})$ be the vector space of all 2×2 real matrices and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a fixed matrix.

Find the matrix $[T]$ of each of the following linear operators T relative to the standard basis B of

$M_2(\mathbb{R})$ given by $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Also find trace and determinant of $[T]$.

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- (i) $T(A) = MA$ (ii) $T(A) = AM$ (iii) $T(A) = MA + AM$
(iv) $T(A) = MA - AM$ (v) $T(A) = AM - MA$

11. Find the matrix in standard basis of the linear transformations :

- (i) Zero linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.
(ii) Identity operator on $\mathbb{R}^3$.
(iii) Projection mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$
(iv) Rotation mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$
(v) Reflection mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x \cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$
(vi) Matrix mapping $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by $T(u) = A(u)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 8 & 7 \end{bmatrix}$
(vii) $T: P_3(\mathbb{R}) \rightarrow \mathbb{R}^4$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_1, a_1 + 2a_2, 2a_2 + 3a_3, 3a_3)$
(viii) $T: M_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b + c + d)x + (a - d)x^2 + cx^3$

12. For each linear transformation L on $\mathbb{R}^2$, find the matrix A representing L (relative to the usual basis of $\mathbb{R}^2$) :

- (i) L is the rotation in $\mathbb{R}^2$ counterclockwise by 45° .
(ii) L is reflection in $\mathbb{R}^2$ about the line $y = x$.
(iii) L is defined by $L(1, 0) = (3, 5)$ and $L(0, 1) = (7, -2)$.
(iv) L is defined by $L(1, 1) = (3, 7)$ and $L(1, 2) = (5, -4)$.

13. Let $\mathbb{D}$ denote the differential operator ; that is, $\mathbb{D}(f(t)) = df/dt$. Each of the following sets is a basis of a vector space V of functions. Find the matrix representing $\mathbb{D}$ in each basis :

- (i) $\{e^t, e^{2t}, te^{2t}\}$. (ii) $\{1, t, \sin 3t, \cos 3t\}$ (iii) $\{e^{5t}, te^{5t}, t^2e^{5t}\}$.

Answers

1. (i) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} -\frac{9}{2} & -\frac{5}{2} \\ \frac{5}{2} & \frac{3}{2} \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$ 3. $[T : B] = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

$$4. \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

$$5. T(x, y) = (2y - x, y, 3y - 3x)$$

$$6. T(x, y) = \left(\frac{7x + 23y}{6}, \frac{2x + 10y}{3} \right) \quad 7. T(x, y) = (2y - 4x, x, y - 2x)$$

$$8. \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$9. [D : B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$10. (i) [T] = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}, \text{ trace of } [T] = 2(a + d) = 2 \text{ trace}(M), \det[T] = (ad - bc)^2 = (\det M)^2$$

$$(ii) [T] = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}, \text{ trace of } [T] = 2(a + d) = 2 \text{ trace}(M), \det[T] = (ad - bc)^2 = (\det M)^2$$

$$(iii) [T] = \begin{bmatrix} 2a & c & b & 0 \\ b & a + d & 0 & b \\ c & 0 & a + d & c \\ 0 & c & b & 2d \end{bmatrix}, \text{ trace of } [T] = 4(a + d) = 4 \text{ trace}(M), \det[T] = 4(ad - bc)(a + d)^2 = 4(\det M)(\text{trace } M)^2$$

$$(iv) [T] = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a - d & 0 & b \\ c & 0 & d - a & -c \\ 0 & c & -b & 0 \end{bmatrix}, \text{ trace of } [T] = 0, \det[T] = 0$$

$$(v) [T] = \begin{bmatrix} 0 & c & -b & 0 \\ b & d - a & 0 & -b \\ -c & 0 & a - d & c \\ 0 & -c & b & 0 \end{bmatrix}, \text{ trace of } [T] = 0, \det[T] = 0$$

$$12. (i) \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 3 & 7 \\ 5 & -2 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 1 & 2 \\ 18 & -11 \end{bmatrix}$$

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13. (i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

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9.1 Range space and Null space

Def. Null Space or Kernel of a Linear Transformation : Let $T : U \rightarrow V$ be a linear transformation. The null space (kernel) of T is denoted by $N(T)$ or $\text{Ker}(T)$ and is defined as $N(T) = \ker T = \{u \in U : T(u) = 0\}$. In words, the null space is the subset of U consisting of all vectors whose image under T is 0.

Def. Range or image of a Linear Transformation Let $T : U \rightarrow V$ be a linear transformation, then range of T or image of T is denoted by $R(T)$ or $T(U)$ and is defined as $R(T) = \{T(u) : u \in U\}$. In words range of T is the set of all those vectors of V which are images of vectors of U under T .

Results :

1. If $T : U \rightarrow V$ is a linear transformation, then $N(T)$, the null space of T , is a subspace of U .
2. Let $T : U \rightarrow V$ be a linear transformation, then T is one-one if and only if $N(T) = \{0\}$.
3. Let $T : U \rightarrow V$ be a linear transformation, then range of T is a subspace of V .

Def. Rank and Nullity of a Linear Transformation : Let $T : U \rightarrow V$ be a linear transformation. The rank of T is denoted by $\rho(T)$ and is defined as the dimension of $R(T)$. The nullity of T is denoted by $\mu(T)$ and is defined as the dimension of $N(T)$.

4. **Sylvester's law :** If $T : U \rightarrow V$ is a linear transformation, then

$$\text{Rank } T + \text{Nullity } T = \dim U \quad \text{i.e.} \quad \rho(T) + \mu(T) = \dim U$$

5. Let $T : U \rightarrow V$ be a linear transformation where U and V are finite dimensional vector spaces of same dimension, then T is one-one iff T is onto.
6. If $T : V \rightarrow V$ is a linear operator on V , then $R(T) \cap N(T) = \{0\}$ iff $T(T(v)) = 0 \Rightarrow T(v) = 0$.
7. **Fundamental theorem of vector space homomorphism :** Let $T : U \rightarrow V$ be a linear transformation, then $U/\text{Ker } T \cong T(U)$.
8. If $T : U \rightarrow V$ is a onto linear transformation, then $U/\text{Ker } T \cong V$.
9. If W_1 and W_2 are two subspaces of a vector space then $\frac{W_1 + W_2}{W_1} \cong \frac{W_2}{W_1 \cap W_2}$.

Method 9.1 : To find range space of a Linear Transformation

Input : A linear transformation $T : U \rightarrow V$ is given.

Output : Range space $R(T)$ and rank of T i.e. $\rho(T)$

Working Steps :

(i) To find range space $R(T)$, we need a basis of U and for the sake of simplicity we consider the standard basis $e_1, e_2, \dots, e_n$.

(ii) Range space is generated by T -images of vectors of basis i.e.

$$R(T) = \langle T(e_1), T(e_2), \dots, T(e_n) \rangle$$

(iii) Now we have a generating set of $R(T)$ so find a basis of $R(T)$ by techniques of vector spaces i.e. either by selecting a maximal L.I. set out of $\{T(e_1), T(e_2), \dots, T(e_n)\}$ or by row echelon method.

(iv) The number of elements in the basis gives the rank of T i.e. $\rho(T)$.

Alternative Method 9.1 : Range space of a linear transformation is the column space of its matrix.

Method 9.2 : To find null space of a linear transformation

Input : A linear transformation $T : U \rightarrow V$ is given.

Output : Null space $N(T)$ and nullity of T i.e. $\mu(T)$.

Working Steps :

(i) Take any arbitrary vector $u \in U$ and substitute $T(u) = 0$. Here a system of equations is obtained.

(ii) Solve the system of equations obtained in step (i). There are two possible cases.

Case (a) : If $u = 0$ is only solution of above system then $N(T) = \{0\}$ and so $\mu(T) = 0$.

Case (b) : If system has non-trivial solutions, then number of free variables, say r , gives the dimension of $N(T)$ and so $\mu(T) = \text{nullity of } T = r$. Further, let $u_1, u_2, \dots, u_r$ be the solutions by setting one of the free variable equal to 1 (or any non-zero constant) and the remaining free variables equal to 0, then $\{u_1, u_2, \dots, u_r\}$ is a basis of $N(T)$.

Alternative Method 9.2 : Null space of a linear transformation is the null space of its matrix.

Method 9.3 : To find a linear transformation when a generating set of range space is given.

Input : Let $T : U \rightarrow V$ be a linear transformation and $\dim U = n$. A generating set $\{v_1, v_2, \dots, v_k\}$ where $k \leq n$ of $R(T)$ is given.

Output : Defining formula of T .

Working Steps :

(i) Take any basis of U , for simplicity, we can take the standard basis $\{e_1, e_2, \dots, e_n\}$.

(ii) Two possibilities are there

Case (a) : If $k = n$ then take $T(e_1) = v_1, T(e_2) = v_2, \dots, T(e_k) = v_k$ and find T by Method 8.2.1

Case (b) : If $k < n$ then take $T(e_1) = v_1, T(e_2) = v_2, \dots, T(e_k) = v_k$ and $T(e_{k+1}) = \dots = T(e_n) = 0$ and find T by Method 8.1.

Remember, if you want a different result, do something different.

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Remark : In above method, T does not exist if $k > n$.

Method 9.4 : To find a L.T. when a generating set of null space is given.

Input : Let $T : U \rightarrow V$ be a linear transformation and $\dim(U) = n$.

A generating set $\{u_1, u_2, \dots, u_k\}$ of $N(T)$ is given.

Output : Defining formula of T .

Working Steps :

(i) First check that the set $\{u_1, u_2, \dots, u_k\}$ is L.I. or not. If yes, its okay. If no, then reduce it to a maximal L.I. set.

(ii) Let the L.I. set obtained in step (i) be $S = \{u_1, u_2, \dots, u_r\}$ where $r \leq n$.

Clearly, S is a basis of $N(T)$.

(iii) Now extend the set S to a basis B of U and let $B = \{u_1, u_2, \dots, u_r, u_{r+1}, \dots, u_n\}$. Out of the vectors of B only $u_1, u_2, \dots, u_r$ belong to null space and others do not

belong to $N(T)$, so we assume

$$T(u_1) = 0$$

$$T(u_2) = 0$$

.....

$$T(u_r) = 0$$

and $T(u_{r+1}) = \text{any non-zero vector, say } v_{r+1}$

$T(u_{r+2}) = \text{any non-zero vector, say } v_{r+2}$ such that $\{v_{r+1}, v_{r+2}\}$ is a L.I. set.

.....

$T(u_n) = \text{any non-zero vector say } v_n$ such that $\{v_{r+1}, v_{r+2}, \dots, v_n\}$ is a L.I. set.

(iv) Now find T by Method 8.2.1

Exercise 9.1

1. Find $R(T)$, $\text{rank}(T)$, $N(T)$ and nullity (T) for the following linear transformations and

Verify the Sylvester law.

(i) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (x+y, y+z)$

(ii) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(x, y) = (x+y, x-y, y)$

You can turn negative consequences into positive rewards simply by changing your habits now.

2. A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is as $T(1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$.
Find the defining formula for T and then find range space, rank, null space and nullity.
3. Find range space, rank, null space and nullity for zero operator $O : V \rightarrow V$ defined by $O(x) = 0$ for all x and identity operator $I : V \rightarrow V$ defined by $I(x) = x$ for all x , where V is a finite dimensional vector space.
4. (i) Find a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose range space is generated by $(1, 0, -1)$ and $(1, 2, 2)$.
(ii) Find a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ whose range space is generated by $(1, 2, 3)$ and $(4, 5, 6)$.
5. (i) Find a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ whose null space is spanned by $(2, 3, 4, 1)$ and $(1, 0, 1, 1)$.
(ii) Find a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ whose null space is generated by $(0, 1, -3)$ and $(0, -3, 4)$.
6. Let V be the vector space of all 2×2 real matrices and let $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ be a fixed matrix. Let $T : V \rightarrow V$ be a function defined by $T(A) = MA$ for all $A \in V$. Then
(i) Prove that T is a linear transformation.
(ii) Find range space and rank of T .
(iii) Find null space and nullity of T .
(iv) Verify the Sylvester law.
7. Let $T : V \rightarrow V$ be a linear map such that $R(T) = N(T)$, where V is finite dimensional. Prove that $\dim V$ is even.
8. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is a linear transformation such that $\mu(T) = 2$, then find $\dim R(T)$.
9. Is there a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that $\rho(T) = 3$ and $\mu(T) = 2$.
10. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by $T(x, y) = (y, 0)$. Prove that $R(T) = N(T)$.
11. Let V be the vector space of all 2×2 real matrices and let $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ be a fixed matrix. Let $T : V \rightarrow V$ be the linear map defined by $T(A) = AM - MA$. Find $R(T)$, rank, $N(T)$ and nullity. Then verify the Sylvester Law.
12. Find range space, null space, rank and nullity of the following linear transformations :
(i) Zero linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.
(ii) Identity operator on $\mathbb{R}^3$.

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(iii) Projection mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$

(iv) Rotation mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$

(v) Reflection mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x \cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$

(vi) Matrix mapping $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by $T(u) = A(u)$ where
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 8 & 7 \end{bmatrix}$$

(vii) $T: P_3(\mathbb{R}) \rightarrow \mathbb{R}^4$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_1, a_1 + 2a_2, 2a_2 + 3a_3, 3a_3)$

(viii) $T: M_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b + c + d)x + (a - d)x^2 + cx^3$

13. Each of the following matrices determines a linear map from $\mathbb{R}^4$ into $\mathbb{R}^3$:

(i) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$,

(ii) $B = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 0 & -5 & 3 \end{bmatrix}$.

Find a basis as well as the dimension of the kernel and the image of each linear map.

14. Let $V = \mathbb{P}_{10}(t)$, the vector space of polynomials of degree ≤ 10 . Consider the linear map

$\mathbb{D}^4: V \rightarrow V$, where $\mathbb{D}^4$ denotes the fourth derivative $d^4(f)/dt^4$. Find a basis and the

dimension of (i) the image of $\mathbb{D}^4$ (ii) the kernel of $\mathbb{D}^4$

Answers

1. (i) $R(T) = \mathbb{R}^2$, $\rho(T) = 2$ and $N(T) = \langle (1, -1, 1) \rangle$ and $\mu(T) = 1$

Basis of $N(T)$ is not unique, it can be multiple of $(1, -1, 1)$ as well.

(ii) $R(T) = \langle (1, 1, 0), (1, -1, 1) \rangle$, $\rho(T) = 2$ and $N(T) = \{0\}$, $\mu(T) = 0$.

Basis of $R(T)$ is not unique.

2. $T(x, y) = (2y - x, y - 3x, 7x - y)$, $R(T) = \langle (1, 3, -7), (2, 1, -1) \rangle$,

$\rho(T) = 2$ and $N(T) = \{0\}$, $\mu(T) = 0$.

3. $R(O) = \{0\}$, $\rho(O) = 0$ and $N(O) = V$, $\mu(O) = \dim V$; $R(I) = V$, $\rho(I) = \dim V$ and

$N(I) = \{0\}$, $\mu(I) = 0$

4. (i) $T(x, y, z) = (x + y, 2y, -x + 2y)$ (ii) $T(x, y, z) = (x + 4y, 2x + 5y, 3x + 6y)$

Ask yourself, "Am I now ready to make some changes." –Jack Canfield

5. (i) Answer is not unique. It depends upon the vectors used to extend the basis and their images. However, if we take $T(0, 0, 1, 0) = (1, 0, 0)$ and $T(0, 0, 0, 1) = (0, 1, 0)$

then answer is $T(x, y, z) = (z - x - \frac{2y}{3}, t - x + \frac{y}{3}, 0)$

- (ii) Answer is not unique. However, with $T(1, 0, 0) = (1, 0, 0, 0)$ the answer is

$$T(x, y, z) = (x, 0, 0, 0)$$

6. (ii) $R(T) = \left\langle \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\rangle$ and $\rho(T) = 2$ (iii) $N(T) = \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$ and $\mu(T) = 2$.

8. 3 9. No.

13. (i) $\dim(\text{Ker } A) = 2$, $\{(4, -2, -5, 0), (1, -3, 0, 5)\}$; $\dim(\text{Im } A) = 2$, $\{(1, 2, 1), (0, 1, 1)\}$

(ii) $\dim(\text{Ker } B) = 1$, $\{(-1, \frac{2}{3}, 1, 1)\}$; $\text{Im } B = \mathbb{R}^3$.

14. (i) $\{1, t, t^2, \dots, t^6\}$, (ii) $\{1, t, t^2, t^3\}$

9.2 Singular and non-singular transformation

Def. Singular Linear Transformation : A linear transformation $T : U \rightarrow V$ is said to be singular if there exists a vector $u \neq 0$ in U such that $T(u) = 0$.

In words, image of at least one non-zero vector of U under T is the zero vector.

Def. Non Singular Linear Transformation : A linear transformation $T : U \rightarrow V$ is said to be non-singular if it is not singular.

OR

A linear transformation $T : U \rightarrow V$ is said to be non-singular if $T(u) = 0 \Rightarrow u = 0$.

OR

A linear transformation $T : U \rightarrow V$ is said to be non-singular if $0 \in U$ is the only vector whose image under T is 0.

Results :

1. A linear transformation $T : U \rightarrow V$ is non-singular iff T is one-one.
2. A linear transformation $T : U \rightarrow V$ is non-singular iff $N(T) = \{0\}$.
3. A linear transformation $T : U \rightarrow V$ is non-singular iff T carries linearly independent vectors of U into linearly independent vectors of V .
4. Let $U(F)$ is a finite dimensional vector space and let $T : U \rightarrow V$ be a linear transformation. Then U and $R(T)$ have the same dimensions iff T is non-singular.

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5. Let $T: U \rightarrow V$ be a linear transformation, then the following conditions are equivalent :

- (i) T is non-singular.
- (ii) T is one-one.
- (iii) $N(T) = \{0\}$
- (iv) T carries linearly independent vectors to linearly independent vectors.
- (v) $\dim U = \dim R(T)$.

Remark : If $\dim U = \dim V = \text{finite}$ then, we can add condition (vi) to above theorem, namely,

- (vi) T is onto

Def. Full rank matrix : A matrix is said to be of full rank if its rank is as large as possible. In other words a $m \times n$ matrix A is said to be of full rank if $\text{rank}(A) = \min\{m, n\}$.

Results :

1. Let $T: U \rightarrow V$ be a linear transformation where $\dim U < \dim V$ and A be the matrix of T .

Then T cannot be onto and T is one-one if and only if the matrix A is of full rank.

2. Let $T: U \rightarrow V$ be a linear transformation where $\dim U > \dim V$ and A be the matrix of T .

Then T cannot be one-one and T is onto if and only if the matrix A is of full rank.

3. Let $T: U \rightarrow V$ be a linear transformation where $\dim U = \dim V$ and A be the matrix of T .

Then T is either both one-one and onto or neither one-one nor onto. Further T is one-one and onto if and only if the matrix A is of full rank.

Remark : In above results U and V are finite dimensional.

Exercise 9.2

1. Which of the following linear transformations are singular or non-singular. In case of a singular linear transformation, find a non zero vector u such that $T(u) = 0$.

- (i) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x+z, x-z, y)$
- (ii) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(x, y) = (x+y, x-y, y)$
- (iii) $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by $T(x, y, z, t) = (2x, y, z+t, 0)$
- (iv) Zero linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.
- (v) Identity operator on $\mathbb{R}^3$.
- (vi) Projection mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$
- (vii) Rotation mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$

Experience tells you what to do. Confidence allows you to do it. -Stan Smith

(viii) Reflection mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x \cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$

(ix) Matrix mapping $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by $T(u) = A(u)$ where
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 5 & 8 & 7 \end{bmatrix}$$

(x) $T: P_3(\mathbb{R}) \rightarrow \mathbb{R}^4$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_1, a_1 + 2a_2, 2a_2 + 3a_3, 3a_3)$

(xi) $T: M_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b + c + d)x + (a - d)x^2 + cx^3$

2. Given the linear transformation $Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} X$, show that

(i) it is singular

(ii) the images of linearly independent vectors $X_1 = (1, 1, 1)$, $X_2 = (2, 1, 2)$ and $X_3 = (1, 2, 3)$ are linearly dependent.

3. Set up linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ which carries e_1 into $(1, 2, 3)$, e_2 into $(3, 1, 2)$, e_3 into $(2, -1, -1)$. Show that the linear transformation is singular.

4. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by

$T(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$. Prove that T is non-singular iff

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Answers

- | | | |
|------------------------------------|---------------------|--------------------------------------|
| 1. (i) Non-singular | (ii) Non-singular | (iii) Singular ; $u = (0, 0, 1, -1)$ |
| (iv) Singular ; $u = (x, y, z, t)$ | (v) Non-singular | (vi) Singular ; $u = (0, 0, z)$ |
| (vii) Non-singular | (viii) Non-singular | (ix) Singular ; $u = (1, -2, 1, 0)$ |
| (x) Non-singular | (xi) Non-singular | |

9.3 Algebra of linear transformations

Def. Sum of Linear Transformations : Let U and V be two vector spaces over the field F . Let $T_1: U \rightarrow V$ and $T_2: U \rightarrow V$ be two linear transformations. We define the sum of T_1 and T_2 by $T_1 + T_2: U \rightarrow V$ by setting $(T_1 + T_2)u = T_1(u) + T_2(u)$ for all $u \in U$.

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Def. Scalar multiplication of linear transformation : Let $T : U \rightarrow V$ be a linear transformation and $a \in F$ be any scalar then scalar multiplication of T with a is denoted by $aT : U \rightarrow V$ and is defined as $(aT)u = a(T(u))$ for all $u \in U$.

Results :

1. Sum of two linear transformations from a vector space $U(F)$ to a vector $V(F)$ is also a linear transformation.
2. Scalar multiplication of a linear transformation from a vector space $U(F)$ to a vector $V(F)$ is also a linear transformation.

Def. Composition (Product) of two Linear Transformations : Let U, V, W be three vector spaces over a field F . Let $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ be two linear transformations. We define $T_2 T_1 : U \rightarrow W$, called product (composition) of T_2 and T_1 by setting

$$(T_2 T_1)(u) = T_2(T_1(u)) \text{ for all } u \in U.$$

3. Composition of two linear transformations is again a linear transformation.

Remark : (1) T_1 and T_2 be two linear transformations, then $T_2 T_1$ is defined under the condition that range of $T_1 \subseteq$ domain of T_2 .

Similarly, $T_1 T_2$ is defined under the condition that range of $T_2 \subseteq$ domain of T_1 .

These two conditions can be remembered by the technique that composition of two linear transformations is defined when range of post factor transformation $\subseteq$ domain of pre-factor transformation.

Def. Invertible Linear Transformation : A linear transformation $T : U \rightarrow V$ is called invertible if T is one - one and onto and inverse of T is written as $T^{-1} : V \rightarrow U$ such that

$$T(u) = v \text{ iff } u = T^{-1}(v).$$

4. Suppose a linear transformation $T : U \rightarrow V$ is one - one and onto. Then the inverse mapping $T^{-1} : V \rightarrow U$ is also a linear transformation.
5. Let $T : U \rightarrow V$ be an invertible linear transformation. Then
 - (i) T^{-1} is also invertible and $(T^{-1})^{-1} = T$
 - (ii) aT is also invertible for any scalar $a \neq 0$ and $(aT)^{-1} = a^{-1} T^{-1}$.
6. If T_1 and T_2 are two invertible linear operators on a vector space $V(F)$, then $T_1 T_2$ is also invertible and $(T_1 T_2)^{-1} = T_2^{-1} T_1^{-1}$.

We cannot always control what goes on outside, but we can control what goes on inside.

7. Let U and V be two vector spaces over the field F . Let $T_1 : U \rightarrow V$ and $T_2 : U \rightarrow V$ be two linear transformations then $\rho(T_1 + T_2) \leq \rho(T_1) + \rho(T_2)$.
8. Let $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ be two linear transformations. Then
- $T_2 T_1$ is one - one , then T_1 is one - one
 - if $T_2 T_1$ is onto , then T_2 is onto.
9. Let $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ be two linear transformations. Then
- $\rho(T_2 T_1) \leq \rho(T_2)$
 - $\rho(T_2 T_1) = \rho(T_2)$, if T_1 is invertible (non- singular).
10. Let T_1 and T_2 be two linear operators on a vector space $V(F)$ and $[T_1]$ and $[T_2]$ denote the matrices of T_1 and T_2 relative to a ordered basis $B = \{v_1, v_2, \dots, v_n\}$. Then the matrices of the operator $T_1 + T_2$, kT_1 and $T_1 T_2$ with respect to the basis B are given by
- $[T_1 + T_2] = [T_1] + [T_2]$
 - $[kT_1] = k[T_1]$
 - $[T_1 T_2] = [T_1][T_2]$.

Remark : Some times, the matrix of a linear operator T is denoted by $m(T)$. In view of this notation the above theorem can be given as :

- $m(T_1 + T_2) = m(T_1) + m(T_2)$
 - $m(kT_1) = km(T_1)$
 - $m(T_1 T_2) = m(T_1) m(T_2)$
11. Let $T : V \rightarrow V$ be a linear operator and A be any matrix of T w.r.t. some ordered basis and $f(x)$ be any polynomial. Then $f(A)$ is the matrix of $f(T)$ w.r.t. the same ordered basis.

Exercise 9.3

- $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $T_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformations defined by $T_1(x, y, z) = (y, x + z)$, $T_2(x, y, z) = (2z, x - y)$ and $T_3(x, y) = (y, 2x)$. Find the defining formulae, if exist , for the following linear transformations.
 (i) $T_1 T_3$ (ii) $T_2 T_3$ (iii) $T_3 T_1$ (iv) $T_3 T_2$ (v) $T_3 T_1 + T_3 T_2$.
- Let $T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined as $T_1(x, y, z) = (3x, 4y - 2z)$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as $T_2(x, y) = (-x, y)$. Compute $T_1 T_2$ and $T_2 T_1$.
- Let T_1 and T_2 be linear operators on $\mathbb{R}^2$ defined by $T_1(x, y) = (0, x)$ and $T_2(x, y) = (y, x)$. Compute $T_1 + T_2$, $2T_2 - 3T_1$, $T_2 T_1$, $T_1 T_2$, T_2^2 , T_1^2 .
- Let T and S be linear operators on $\mathbb{R}^3$ defined by $T(x, y, z) = (x - 3y - 2z, y - 4z)$, $S(x, y, z) = (2x, 4x - y, 2x + 3y)$. Show that $ST \neq TS$.
- Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (0, x, y)$. Show that $T \neq O$, $T^2 \neq O$ but $T^3 = O$.

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6. Show that each of the following operators T on $\mathbb{R}^3$ is invertible and find T^{-1}
- $T(x, y, z) = (3x, x - y, 2x + y + z)$
 - $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$
 - $T(x, y, z) = (x + z, x - z, y)$
7. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is linear operator defined by $T(x, y, z) = (0, 0, x)$ Show that $T \neq O$ but $T^2 = O$.
8. Give an example of two linear transformations T_1 and T_2 such that $T_1 T_2 = O$ but $T_2 T_1 \neq O$.
9. Let $P(\mathbb{R})$ be the vector space of all real polynomials. Consider the derivative mapping D and integral mapping T defined as $D(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(x) dx$. Show that $TD \neq I$ but $DT = I$
10. Let $T : V \rightarrow V$ be a linear operator such that $\text{rank}(T^2) = \text{rank}(T)$, where V is finite dimensional. Prove that $R(T) \cap N(T) = \{0\}$.

Answers

- not defined
 - not defined
 - $T_3 T_1(x, y, z) = (x + z, 2y)$
 - $T_3 T_2(x, y, z) = (x - y, 4z)$
 - $(T_3 T_1 + T_3 T_2)(x, y, z) = (2x - y + z, 2y + 4z)$
- $(T_2 T_1)(x, y, z) = (-3x, 4y - 2z)$ and $T_1 T_2$ is not defined.
- $(T_1 + T_2)(x, y) = (y, 2x)$ $(2T_2 - 3T_1)(x, y) = (2y, -x)$; $T_2 T_1(x, y) = (x, 0)$
 $T_1 T_2(x, y) = (0, y)$; $T_2^2(x, y) = (x, y)$ $T_1^2(x, y) = (0, 0)$.
- $T^{-1}(x, y, z) = \left(\frac{x}{3}, \frac{x}{3} - y, z - x + y \right)$
 - $T^{-1}(x, y, z) = (x + 3y + 14z, y + 4z, z)$
 - $T^{-1}(x, y, z) = \left(\frac{x}{2} + \frac{y}{2}, \frac{x}{2} - \frac{y}{2}, z \right)$

9.4 Coordinate vectors

Def . Coordinate Vector : If $B = \{u_1, u_2, \dots, u_n\}$ is a basis for a vector space $V(F)$, then a vector $u \in U$ can be expressed as $u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$. The coordinate vector $(a_1, a_2, \dots, a_n)$ is

expressed as a column vector $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is denoted by $[u, B]$ and is called coordinate vector of u relative

to the basis B .

Results :

1. Let U and V be finite dimensional vector spaces and let $B = \{u_1, u_2, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_m\}$ be ordered basis for U and V respectively. If $T : U \rightarrow V$ be a linear transformation, then for any $u \in U$, $[T(u), B'] = [T : B, B'] [u, B]$

Remark : If $U = V$ and $B = B'$, then above theorem takes the form $[T(u), B] = [T : B][u, B]$.

Exercise 9.4

1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator defined as $T(x_1, x_2, x_3) = (2x_2 + x_3, x_1 - 4x_2, 3x_1)$. Find the matrix of T relative to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and then verify that $[T, B] [u, B] = [T(u), B]$ for all $u \in \mathbb{R}^3$.
2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation defined by $T(x, y, z) = (2x + y - z, 3x - 2y + 4z)$. Find the matrix of T relative to ordered basis $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3$ and $B_2 = \{(1, 3), (1, 4)\}$ of $\mathbb{R}^2$. Also verify that : $[T : B_1, B_2] [u, B_1] = [T(u), B_2]$
3. Let T be a linear operator on $\mathbb{R}^2$ whose matrix relative to the basis $B = \{(1, 1), (-1, 0)\}$ is $\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$. Without finding the defining formula for T , find the image of the vector $(0, 2)$ under T .

Answers

1. $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$

3. $(-4, 2)$

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9.5 Change - of - basis

Def . Change - of - basis or Transition matrix : Let V be a vector space of dimension n and $B_1 = \{u_1, u_2, \dots, u_n\}$ and $B_2 = \{v_1, v_2, \dots, v_n\}$ be its two bases. Then each vectors in B_2 can be written uniquely as a linear combination of the vectors in B_1 , so let

$$\begin{aligned}v_1 &= a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n \\v_2 &= a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n \\&\vdots \\v_n &= a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n\end{aligned}$$

Let P denotes the transpose of the matrix of coefficients in above system

i.e.
$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

This matrix P is called the change -of - basis matrix or transition matrix from the basis B_1 to B_2 . Since the vectors $v_1, v_2, \dots, v_n$ are linearly independent, the matrix P is invertible and its inverse P^{-1} is called change - of - basis matrix from B_2 to B_1 .

Results :

- Let V be a vector space of dimension n and $B_1 = \{u_1, u_2, \dots, u_n\}$ and $B_2 = \{v_1, v_2, \dots, v_n\}$ be its two bases. Let P be the change - of - basis matrix from B_1 to B_2 . Then for any vector $v \in V$, we have $P[v, B_2] = [v, B_1]$ and $[v, B_2] = P^{-1}[v, B_1]$.

Remark : The above theorem can be easily remembered, if written in words, as follows.

$$\begin{pmatrix} \text{Co - ordinate vector} \\ \text{of } u \text{ in } B_1 \end{pmatrix} = \begin{pmatrix} \text{Transition matrix} \\ \text{from } B_1 \text{ to } B_2 \end{pmatrix} \begin{pmatrix} \text{Co - ordinate vector} \\ \text{of } u \text{ in } B_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \text{Co - ordinate vector} \\ \text{of } v \text{ in } B_2 \end{pmatrix} = \begin{pmatrix} \text{Inverse of transition} \\ \text{matrix from } B_1 \text{ to } B_2 \end{pmatrix} \begin{pmatrix} \text{Co - ordinate vector} \\ \text{of } u \text{ in } B_1 \end{pmatrix}$$

Def. Similar matrices : Two matrices A and B are said to be similar if there exists a non-singular matrix C such that $A = C^{-1}BC$.

2. Similar matrices have same determinant.

3. Let T be the transition matrix from the basis $B_1 = \{x_i\}$ to the basis $B_2 = \{y_i\}$ in a vector space V . Then for any linear operator T on V , $[T, B_2] = P^{-1}[T, B_1]P$.

OR

Let $T : V \rightarrow V$ be a linear operator. Then matrices of T relative to any two ordered basis of V are always similar.

Def. Determinant of a linear operator : Let V be a finite dimensional vector space and $T : V \rightarrow V$ be a linear operator. Then $\det [T]$ is defined to be determinant of matrix of T relative to any ordered basis of V .

Def. Similar linear operators : Two linear operators T_1 and T_2 on a vector space $V(F)$ are said to be similar if there exists an invertible linear operator P s.t. $T_2 = P^{-1}T_1P$

4. If T_1, T_2 are similar linear operators on a finite dimensional vector space $V(F)$, then $\det [T_1] = \det [T_2]$.

Exercise 9.5

1. Let B_1 denotes the standard basis of $\mathbb{R}^3$ and $B_2 = \{(1, 0, 1), (2, 1, 2), (1, 2, 2)\}$ be another basis of $\mathbb{R}^3$. Then

(i) Find the change-of-basis matrix from B_1 to B_2

(ii) Find the change-of-basis matrix from B_2 to B_1

(iii) Find the coordinate vector of $(1, 0, 3)$ relative to the basis B_2 , using change of basis matrix.

2. Let $B_1 = \{(1, -2), (3, -4)\}$ and $B_2 = \{(1, 3), (3, 8)\}$ be two bases of $\mathbb{R}^2$. Then

(i) Find the change-of-basis matrix P from B_1 to B_2 .

(ii) Find the change-of-basis matrix Q from B_2 to B_1

(iii) Find the coordinate vector $[v, B_1]$ of $v = (x, y)$ relative to B_1

(iv) Find the coordinate vector $[v, B_2]$ of $v = (x, y)$ by using change-of-basis matrix.

(v) Directly find the coordinate vector $[v, B_2]$ of $v = (x, y)$.

(vi) Compare the answers of (iv) and (v). Are they equal?

3. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear operator defined by $T(x, y) = (5x + y, 3x - 2y)$ and let $B_1 = \{(1, 2), (2, 3)\}$ and $B_2 = \{(1, 3), (1, 4)\}$ be two ordered basis of T

(i) Find the matrix of T relative to basis B_1 .

The only thing you have to fear is not giving 100%.

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- (ii) Find the change – of – basis matrix from B_1 to B_2 .
- (iii) Find the matrix of T relative to basis B_2 , using part (i) and (ii).
- (iv) Directly find the matrix of T relative to the basis B_2 .
- (v) Compare the answers of part (iii) and part (iv). Are they same ?
4. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$.
- (i) Find $B = P^{-1}AP$. (ii) Verify that $\text{tr}(B) = \text{tr}(A)$ (iii) Verify that $\det(B) = \det(A)$.
5. Find the trace and determinant of each of the following linear maps on $\mathbb{R}^3$:
- (i) $F(x, y, z) = (x + 3y, 3x - 2z, x - 4y - 3z)$
- (ii) $G(x, y, z) = (y + 3z, 2x - 4z, 5x + 7y)$
6. Suppose $S = \{u_1, u_2\}$ is a basis of V and $T: V \rightarrow V$ is defined by $T(u_1) = 3u_1 - 2u_2$ and $T(u_2) = u_1 + 4u_2$. Suppose $S' = \{w_1, w_2\}$ is a basis of V for which $w_1 = u_1 + u_2$ and $w_2 = 2u_1 + 3u_2$.
- (i) Find the matrices A and B representing T relative to the bases S and S' respectively.
- (ii) Find the matrix P such that $B = P^{-1}AP$.

Answers

1. (i) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$
2. (i) $P = \begin{bmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{bmatrix}$ (ii) $Q = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$ (iii) $\begin{bmatrix} -2x - \frac{3}{2}y \\ x + \frac{y}{2} \end{bmatrix}$
- (iv) $\begin{bmatrix} -8x + 3y \\ 3x - y \end{bmatrix}$ (v) $\begin{bmatrix} -8x + 3y \\ 3x - y \end{bmatrix}$ (vi) yes.
3. (i) $\begin{bmatrix} -23 & -39 \\ 15 & 26 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} 35 & 41 \\ -27 & -32 \end{bmatrix}$

To be successful you must decide exactly what you want to accomplish, and then resolve to pay the price to get it.

(iv) $\begin{bmatrix} 35 & 41 \\ -27 & -32 \end{bmatrix}$

(v) yes.

4. (i) $[-34, 57; -19, 32]$

(ii) $\text{tr}(B)=\text{tr}(A)=-2$

(iii) $\det(B)=\det(A)=-5$

5. (i) $\text{tr}(F)=-2, \det(F)=13$

(ii) $\text{tr}(G)=0, \det(G)=22$

6. (i) $A=[3, 1; -2, 4], B=[8, 11; -2, -1]$ (ii) $P=[1, 2; 1, 3]$

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10.1 Eigen values and eigen vectors(Definition)

Def. Eigen values and eigen vectors of a linear operator : Let T be a linear operator on a vector space $V(F)$. If there exists a non zero vector $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in F$, then v is called an eigen vector of T corresponding to λ and λ is called an eigen value of T corresponding to v .

Remark : Eigen vector is also called characteristic vector and eigen value is also called characteristic value (root).

Def. Eigen values and eigen vectors of a matrix : Let A be a square matrix of order n over a field F . If there exists a non - zero column vector $X \in F^n$ such that $AX = \lambda X$ for some $\lambda \in F$, then X is called an eigen vector of A corresponding to λ and λ is called an eigen value of A corresponding to X .

Exercise 10.1

1. Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$, which of the following vectors are eigen vectors of A .

(i) $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (ii) $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ (iii) $v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (iv) $v_1 + v_2$ (v) $v_2 + v_3$ (vi) $v_1 + v_3$

2. True or false : Sum of two eigen vectors of a matrix is again an eigen vector.

3. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator defined by

$T(x, y, z) = (3x - y + z, 7x - 5y + z, 6x - 6y + 2z)$ then which of the following vectors are eigen vectors of T

(i) $v_1 = (1, 1, 0)$ (ii) $v_2 = (0, 4, 4)$ (iii) $v_3 = (1, 0, 2)$
(iv) $v_1 + v_2$ (v) $v_2 + v_3$ (vi) $v_1 + v_3$

4. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, then which of the following vectors are eigen vectors of A

(i) $v_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ (ii) $v_2 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$ (iii) $v_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (iv) $v_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (v) $v_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If you want your dreams to come true then WAKE UP!

5. Show that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix}$ are eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$ but their sum is

not an eigen vector.

6. If A and B are similar matrices such that $B = P^{-1}AP$ and if X is an eigen vector of A corresponding to eigen value λ , then prove that $P^{-1}X$ is an eigen vector of B corresponding to eigen value λ .

7. If A is a $n \times n$ matrix given by $A = \begin{bmatrix} a & 1 & 0 & \dots & \dots & 0 \\ 1 & a & 1 & \dots & \dots & 0 \\ 0 & 1 & a & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & a & 1 \\ 0 & 0 & 0 & \dots & 1 & a \end{bmatrix}$ i.e. A is a matrix in

which diagonal elements are a and entries above and below the diagonal are 1 and zero ;

then show that the column matrices $X_i (i = 1, 2, \dots, n)$ with j^{th} entry $\sin\left(\frac{ij\pi}{n+1}\right)$

are eigen vectors of A. Also find the corresponding eigen values of A.

[Hint : Use the formulae $\sin 2A = 2 \sin A \cos A$ and $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$]

Answers

7. $a + 2 \cos\left(\frac{i\pi}{n+1}\right)$ are the eigen values.

10.2 Evaluation of eigen values and eigen vectors

Part I

Result 1 : Let $T : V(F) \rightarrow V(F)$ be a linear transformation, then $\lambda \in F$ is an eigen value of T iff $T - \lambda I$ is singular.

Result 2 : Let A be a $n \times n$ matrix over a field F. Then $\lambda \in F$ is eigen value of A iff $A - \lambda I$ is singular i.e. $|A - \lambda I| = 0$.

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Def. Characteristic polynomial : Let A be a $n \times n$ matrix over a field F . The function

$f(\lambda) = |A - \lambda I|$ is a polynomial in λ of degree n . This polynomial is called characteristic polynomial of A .

Def. Characteristic equation : Characteristic equation of a matrix A is given by $|A - \lambda I| = 0$ or $|\lambda I - A| = 0$

Result 3 : Roots of characteristic equation of a matrix are the eigen values of that matrix.

Result 4 : Let T be a linear operator on a finite dimensional vector space $V(F)$. If A is the matrix of T with respect to any ordered basis B , then a scalar $\lambda \in F$ is an eigen value of T iff λ is an eigen value of A .

Result 5 : Let T be a linear operator on a finite dimensional vector space $V(F)$. If A is the matrix of T with respect to some ordered basis B of V , then a vector $v \in V$ is an eigen vector of T corresponding to the eigen value λ iff its coordinate vector X relative to the basis B is an eigen vector of A corresponding to its eigen value λ .

Result 6 : We know that the coordinate vector of a vector in the standard basis is equal to itself. So by above result we can say that if we are working in the standard basis then a vector is eigen vector of T if and only if it is eigen vector of A , where A is the matrix of T relative to the standard basis.

Result 7 : It is clear that, to find the eigen values and eigen vectors of a linear operator T we just have to find out the eigen values and eigen vectors of matrix of T relative to standard basis.

Part II

Result 1 : Corresponding to a characteristic vector of a linear operator, there exists one and only one characteristic value.

Result 2 : Corresponding to a characteristic root of a linear operator there exists more than one characteristic vectors. In fact every non zero scalar multiple of a characteristic vector is also a characteristic vector.

Remark : All eigen vectors together with zero vector form a subspace of V called eigen space. Let us define it formally.

Def. Eigen Space : Let λ be an eigen value of a linear operator T then the eigen space with respect to λ is denoted by E_λ and is given by $E_\lambda = \{v : T(v) = \lambda v\}$.

Result 3 : Let T be a linear operator on $V(F)$. If λ is an eigen value of T , then the eigen space of λ i.e. E_λ is a subspace of $V(F)$.

Def. Algebraic multiplicity of an eigen value is the number of times it appears as the root of the characteristic equation.

Def. Geometric multiplicity of an eigen value is defined to be the dimension of its eigen space.

Result 4 : $1 \leq \text{G.M.} \leq \text{A.M.}$ for each eigen value.

Remark : (i) If A.M. of an eigen value is 1, then clearly its G.M. is also 1.

(ii) If A.M. is 2, then G.M. is 1 or 2

(iii) If A.M. is 3, then G.M. = 1 or 2 or 3

Result 5 :

(i) For a 2×2 matrix A , the characteristic equation is $\lambda^2 - (\text{tr} A)\lambda + \det A = 0$.

(ii) For a 3×3 matrix A , characteristic equation is $\lambda^3 - (\text{tr} A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det A = 0$,

where A_{ii} denotes the co-factor of a_{ii} .

(iii) For a $n \times n$ matrix A , characteristic equation is $\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \dots + (-1)^n S_n = 0$,

where S_k is the sum of the principal minors of order k .

Result 6 : Let T be a linear operator on a finite dimensional vector space $V(F)$. Let

$\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct eigen values of T and $v_1, v_2, \dots, v_n$ be their corresponding eigen vectors, then $v_1, v_2, \dots, v_n$ are linearly independent.

OR

On a finite dimensional vector space, the characteristic vectors corresponding to distinct characteristic values are always linearly independent.

Exercise 10.2

1. For each of the following matrices find all eigen values (real or complex), construct their A.M.- G.M. table and find a basis for each eigen space.

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -13 \\ 1 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

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$$(vii) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \quad (viii) \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. For each of the following linear operators, find all eigen values (real or complex), construct their A.M.-G.M. table and a basis for each eigen space.
- (i) $T(x, y) = (3x + 3y, x + 5y)$
- (ii) $T(x, y) = (y, -x)$
- (iii) $T(x, y, z) = (3x - y + z, 7x - 5y + z, 6x - 6y + 2z)$
- (iv) $T(x, y, z) = (4x + y - z, 2x + 5y - 2z, x + y + 2z)$
3. Let T be a linear operator on a finite dimensional vector space $V(F)$. Then 0 is the characteristic value of T iff T is singular.
4. Let A be a real matrix of type $n \times n$ and λ be its real eigen value. Show that there exists an eigen vector X of A corresponding to eigen value λ such that X is also real.

Answers

1. (i)

λ	G.M.	A.M.
-1	1	1
4	1	1

 ; basis of $E_{-1} = \{(-1, 1)\}$
basis of $E_4 = \{(2, 3)\}$

(ii)

λ	G.M.	A.M.
4	1	2

 ; basis of $E_4 = \{(2, 1)\}$

(iii)

λ	G.M.	A.M.
$2i$	1	1
$-2i$	1	1

 ; basis of $E_{2i} = \{(3 + 2i, 1)\}$
basis of $E_{-2i} = \{(3 - 2i, 1)\}$

(iv)

λ	G.M.	A.M.
$1+i$	1	1
$1-i$	1	1

 ; basis of $E_{1+i} = \{(1, i)\}$
basis of $E_{1-i} = \{(1, 1)\}$

λ	G.M.	A.M.
0	1	1
3	1	1
15	1	1

(v) basis of $E_0 = \{(1, 2, 2)\}$
 ; basis of $E_3 = \{(2, 1, -2)\}$
 basis of $E_{15} = \{(2, -2, 1)\}$

λ	G.M.	A.M.
-2	2	2
4	1	1

(vi) basis of $E_{-2} = \{(1, 1, 0), (-1, 0, 1)\}$
 ; basis of $E_4 = \{(1, 1, 2)\}$

λ	G.M.	A.M.
1	1	1
i	1	1
$-i$	1	1

(vii) basis of $E_1 = \{(1, 0, 0)\}$
 ; basis of $E_i = \{(i, -1-i, 2)\}$
 basis of $E_{-i} = \{(i, 1-i, -2)\}$

λ	G.M.	A.M.
i	1	2
$-i$	1	2

(viii) basis of $E_i = \{(-1, i, -1, i)\}$
 ; basis of $E_{-i} = \{(-i, 1, -i, 1)\}$

λ	G.M.	A.M.
2	1	1
6	1	1

2. (i) basis of $E_2 = \{(3, -1)\}$
 ; basis of $E_6 = \{(1, 1)\}$

λ	G.M.	A.M.
i	1	1
$-i$	1	1

(ii) basis of $E_i = \{(-i, 1)\}$
 ; basis of $E_{-i} = \{(i, 1)\}$

λ	G.M.	A.M.
2	1	2
-4	1	1

(iii) basis of $E_2 = \{(1, 1, 0)\}$
 ; basis of $E_{-4} = \{(0, 1, 1)\}$

λ	G.M.	A.M.
3	2	2
5	1	1

(iv) basis of $E_3 = \{(1, 0, 1), (1, -1, 0)\}$
 ; basis of $E_5 = \{(1, 2, 1)\}$

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10.3 Diagonalization

Def. Diagonalizable operator : A linear operator T on a finite dimensional vector space is said to be diagonalizable if there exist an ordered basis B of V such that matrix of T with respect to B is a diagonal matrix.

Def. Diagonalizable matrix : A $n \times n$ matrix ' A ' is said to be diagonalizable if there exist a non-singular matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

OR

A square matrix is said to be diagonalizable if it is similar to a diagonal matrix.

Result 1 : A matrix (or linear operator) is diagonalizable if and only if A.M. = G.M. for each eigen value.

Result 2 : A matrix whose all eigen values are distinct is always diagonalizable.

Result 3 : A $n \times n$ matrix (or linear operator) is diagonalizable if and only if it has n linearly independent eigen vectors.

Result 4 : A linear operator T on a finite dimensional vector space $V(F)$ is diagonalizable iff there exist a basis of V in which all vectors are eigen vectors of T .

Result 5 : Let A be a $n \times n$ diagonalizable matrix and let $X_1, X_2, \dots, X_n$ be n linearly independent eigen vectors of A . If $P = [X_1 \ X_2 \ \dots \ X_n]$ is a $n \times n$ matrix formed by taking $X_1, X_2, \dots, X_n$ as columns, then $P^{-1}AP$ is a diagonal matrix in which elements on the diagonal are the eigen values of A .

Exercise 10.3

1. Which of the following matrices A are diagonalizable (a) over $\mathbb{R}$ (b) $\mathbb{C}$. In case A is diagonalizable find a matrix P and a matrix D such that $P^{-1}AP = D$.

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -13 \\ 1 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

To be number one, you have to train like you are number two.

$$(vii) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$$(viii) \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Which of the following operators T are diagonalizable (a) over $\mathbb{R}$ (b) $\mathbb{C}$. In case T is diagonalizable find a matrix P and a matrix D such that $P^{-1}AP = D$.

(i) $T(x, y) = (3x + 3y, x + 5y)$

(ii) $T(x, y) = (y, -x)$.

(iii) $T(x, y, z) = (3x - y + z, 7x - 5y + z, 6x - 6y + 2z)$

(iv) $T(x, y, z) = (4x + y - z, 2x + 5y - 2z, x + y + 2z)$

3. Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

(i) Find eigenvalues and corresponding eigenvectors.

(ii) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal.

(iii) Find A^8 and $f(A)$ where $f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5$.

(iv) Find a matrix B such that $B^2 = A$.

Answers

1. (i) diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ both.

(ii) neither diagonalizable over $\mathbb{R}$ nor over $\mathbb{C}$.

(iii) not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$.

(iv) not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$.

(v) diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ both.

(vi) diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ both.

(vii) not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$.

(viii) neither diagonalizable over $\mathbb{R}$ nor over $\mathbb{C}$.

2. (i) diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ both.

(ii) not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$.

(iii) neither diagonalizable over $\mathbb{R}$ nor over $\mathbb{C}$.

(iv) diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ both.

3. (i) $\lambda = 1, u = (1, 1); \lambda = 4, v = (1, -2)$

(ii) $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

(iii) $f(A) = \begin{bmatrix} 19 & -13 \\ -26 & 32 \end{bmatrix}, A^8 = \begin{bmatrix} 21846 & -21845 \\ -43690 & 43691 \end{bmatrix}$

(iv) $B = \begin{bmatrix} 4/3 & -1/3 \\ -2/3 & 5/3 \end{bmatrix}$

10.4 Properties of eigen values

Results :

1. Eigen values of a lower triangular, upper triangular and diagonal matrices are just the diagonal elements.
2. Sum of all eigen values of a matrix is equal to its trace.
3. Product of all eigen values of a matrix is equal to its determinant.
4. All eigen values of a non singular matrix are non zero.
5. Atleast one eigen value of singular matrix is 0.
6. If sum of elements of each row of a matrix is k , then k is an eigen value of that matrix.
7. If sum of elements of each column of a matrix is k , then k is an eigen value of that matrix.
8. The number of eigen values (counting multiplicities) of a $n \times n$ matrix is always n .
9. The eigen values of A and A' are same.
10. If λ is an eigen value of A , then λ^k is an eigen value of A^k , where k is a positive integer.
11. If λ is an eigen value of matrix A corresponding to the eigen vector v then λ^n is an eigen value of A^n corresponding to the eigen vector v , where n is a positive integer.
12. If λ is an eigen values of a non singular matrix A , then $\frac{1}{\lambda}$ is an eigen value of A^{-1} .
13. If λ is an eigen values of a non singular matrix A , then $\frac{\det A}{\lambda}$ is an eigen value of adjoint of A i.e. $\text{adj } A$.

Exercise 10.4

1. If $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$, then find the eigen values A , A^{-1} , A^2 , A^3 , A' and $\text{adj } A$.

2. Find the eigen values of the following matrices :

(i) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 6 & -1 \\ -8 & 2 & 11 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & -2 \end{bmatrix}$

3. If eigen values of a 3×3 matrix are $-7, 8, 9$ then find the value of x in the matrix

$$\begin{bmatrix} 4 & * & * \\ * & 5 & * \\ * & * & x \end{bmatrix}$$

4. Find the eigen values of the following matrices :

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(iii) $A = [a_{ij}]_{m \times n}$ such that $a_{ij} = 1$ for all i, j .

5. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find the eigen values and eigen vectors of A and A' .

What do you notice.

Answers

5. Eigen values of A and A' are always same but eigen vectors may be different.

10.5 Characteristic and minimal polynomial

Def. Annihilating polynomial of a matrix : A polynomial $f(x)$ is said to be an annihilating polynomial of a matrix A if $f(A) = 0$.

Remark : Let $f(x)$ be an annihilating polynomial of A and $g(x)$ be any other polynomial, then $f(x) \cdot g(x)$ is also an annihilating polynomial of A .

Proof : $f(A) = 0$ (given), Consider $f(A) \cdot g(A) = 0 \cdot g(A) = 0 \Rightarrow f(x)g(x)$ is also an annihilating polynomial of A .

Remark : By above remark, it is clear that we can construct infinitely many annihilating polynomials by using one annihilating polynomial.

Def. Minimal polynomial of a linear operator(matrix) : The minimal polynomial $m(x)$ of a linear operator T is defined to be monic polynomial of lowest degree of which T is a root.

We know that if A is a matrix of T and $f(x)$ be any polynomial, then $f(A)$ is the matrix of $f(T)$. In particular, $f(T) = 0$ if and only if $f(A) = 0$. Therefore, we can say that T and A has same minimal polynomials.

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Result 1 : Cayley Hamilton Theorem : Every square matrix satisfies its characteristic polynomial.

OR

If $c(x)$ is the characteristic polynomial of a matrix A , then $c(A) = 0$.

OR

Characteristic polynomial is an annihilating polynomial.

Result 2 : If A is a $n \times n$ matrix and $k \geq n$ is a positive integer then A^k can be expressed as the linear combination of $I, A, A^2, \dots, A^{n-1}$.

Result 3 : Let A be a $n \times n$ matrix and $W = \langle A^n : n \geq 0 \rangle$ then $\dim W \leq n$.

Result 4 : If A is a $n \times n$ non-singular matrix then A^{-m} can be expressed as the linear combination of $I, A, A^2, \dots, A^{n-1}$ for all positive integers m .

Result 5 : Let A be a $n \times n$ non-singular matrix and $W = \langle A^n : n \in \mathbb{Z} \rangle$ then $\dim W \leq n$.

Result 6 : The minimal polynomial of a matrix is unique.

Result 7 : If $m(x)$ is the minimal polynomial of a matrix A and $f(x)$ be an annihilating polynomial then $m(x)$ divides $f(x)$.

OR

If $f(x)$ is a polynomial such that $f(A) = 0$, and if $m(x)$ is the minimal polynomial of A , then there exists a polynomial $q(x)$ such that $f(x) = m(x)q(x)$.

Result 8 : The minimal polynomial of a matrix divides its characteristic polynomial.

Result 9 : Let $m(x)$ and $c(x)$ denotes the minimal and characteristic polynomial of a matrix respectively. Then, $m(x)$ and $c(x)$ have the same distinct linear factors.

OR

If $c(x) = (x - a_1)^{\alpha_1} (x - a_2)^{\alpha_2} \dots (x - a_k)^{\alpha_k}$, then $m(x)$ is of the form $(x - a_1)^{\beta_1} (x - a_2)^{\beta_2} \dots (x - a_k)^{\beta_k}$ where $1 \leq \beta_i \leq \alpha_i$.

Result 10 : Let $m(x)$ and $c(x)$ denotes the minimal and characteristic polynomial of a $n \times n$ matrix respectively then $c(x)$ divides $(m(x))^n$.

Result 11 : A scalar λ is an eigen value of a matrix A if and only if λ is a root of the minimal polynomial of A .

Result 12 : A matrix is non singular iff constant term in the minimal polynomial is non zero.

Result 13 : Minimal polynomial of a matrix is of degree 1 iff the matrix is a scalar matrix.

Result 14 : A matrix is diagonalizable if and only if its minimal polynomial has no repeated linear factors (i.e. all factors must be with single power in $m(x)$).

Exercise 10.5

1. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$, then find

- (i) $c(x)$ (ii) $m(x)$ (iii) Is A diagonalizable
(iv) Find five different annihilating polynomials of A.

2. Find $c(x)$ and $m(x)$ of each of the following matrices and then find which of them are diagonalizable :

(i) $\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(v) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. Find $c(x)$ and $m(x)$ of each of the following linear transformations and then find which of them are diagonalizable :

(i) $T(x, y) = (2x, 4x - 5y)$

(ii) $T(x, y, z) = (x + y, 2x + 3y + z, -2x + 3y + 5z)$

(iii) $T(x, y, z) = (x + y + z, 2y + z, 2y + 3z)$

4. Find the possible candidates for $m(x)$ if

(i) $c(x) = (x-1)^2(x+2)$ (ii) $c(x) = (x+2)^3(x-7)^2(x+4)$

5. Find the eigen values of $n \times n$ idempotent, involutory and nilpotent matrices. Also find $c(x)$ and $m(x)$.

6. Let A be a 100×100 idempotent matrix such that 0 is an eigen value of A with algebraic multiplicity (A.M.) is 40, then find the other eigen values of A.

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Answers

2. (i) $c(x) = x^2 - 3x + 2$, $m(x) = x^2 - 3x + 2$
(ii) $c(x) = x^3 - 4x^2 + 5x - 2$, $m(x) = x^2 - 3x + 2$
(iii) $c(x) = x^3 - 4x^2 + 5x - 2$, $m(x) = x^3 - 4x^2 + 5x - 2$
(iv) $c(x) = x^3 - 6x^2 - 3x + 18$, $m(x) = x^3 - 6x^2 - 3x + 18$
(v) $c(x) = x^3 - 12x^2 + 36x - 32$, $m(x) = x^2 - 10x + 16$
3. (i) $c(x) = x^2 + 3x - 10$, $m(x) = x^2 + 3x - 10$
(ii) $c(x) = x^3 - 9x^2 + 18x$, $m(x) = x^3 - 9x^2 + 18x$
(iii) $c(x) = x^3 - 6x^2 + 9x - 4$, $m(x) = x^2 - 5x + 4$
6. 1 with A.M. 60

10.6 Diagonalizability of some special matrices

Result 1 : Let A be a real symmetric matrix. If u and v are eigen vectors corresponding to distinct eigen values then u and v are orthogonal. i.e. $(u, v) = 0$.

Result 2 : Let A be a real symmetric matrix, then there exists an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

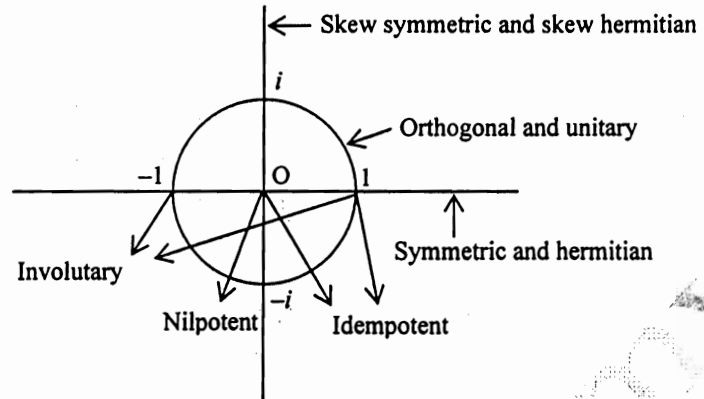
✓ **Result 3 :** If two matrices are similar, then their D C T E R M are same,

where D—determinant ; C—characteristic polynomial ; T—trace ; E—eigen values ; R—rank ;

M—minimal polynomial

✓ **Result 4 :** If A and B are two $n \times n$ matrices such that anyone (or both) of them is non singular then AB and BA are similar and therefore their D C T E R M are same.

✓ **Result 5 :** Let A and B are any two $n \times n$ matrices then D C T E of AB and BA are necessarily same but R M may or may not be same.

Result 6 : Spectrum of eigen values :**Result 7 : Table of eigen values and diagonalizability :**

S.No.	Matrix	Eigen values	Diagonalizable
1.	Real symmetric	Real	over $\mathbb{R}$
2.	Hermitian	Real	over $\mathbb{R}$
3.	Real skew-symmetric	0 or purely imaginary	over $\mathbb{C}$
4.	Skew-Hermitian	0 or purely imaginary	over $\mathbb{C}$
5.	Orthogonal	of modulus unity	over $\mathbb{C}$
6.	Unitary	of modulus unity	over $\mathbb{C}$
7.	Idempotent	0 or 1	over $\mathbb{R}$
8.	Involutory	-1 or 1	over $\mathbb{R}$
9.	Nilpotent	0	not diagonalizable

Result 8: A non zero nilpotent matrix is never diagonalizable. In other words, if a nilpotent matrix A is diagonalizable then $A=0$.

Exercise 10.6

1. For each of the following symmetric matrices A , find its eigenvalues, a maximal orthogonal set S of eigenvectors, and an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal :

(i) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & 17 \end{bmatrix}$

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2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Show that A and B have different characteristic

polynomials (and so are not similar), but have the same minimal polynomial. Thus nonsimilar matrices may have the same minimal polynomial.

Answers

1. (i) $\lambda = -1, 2$; $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(ii) $\lambda = 1, 2, 2$; $P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{21}} & \frac{2}{\sqrt{21}} & \frac{4}{\sqrt{21}} \end{bmatrix}$

10.7 Block Matrices and companion matrices

Using a system of horizontal and vertical (dashed) lines, we can partition a matrix A into submatrices called blocks (or cells) of A . Clearly, a given matrix may be divided into blocks in different ways.

Def. Block square matrix : A square matrix is said to be block square matrix if there are same number of horizontal and vertical lines and they are placed symmetrically..

Def. Block diagonal matrix : A block square matrix is said to be block diagonal matrix if all the non-diagonal blocks are zero matrices.

Def. Block upper triangular matrix : A block square matrix is said to be block upper triangular matrix if all the blocks below the diagonal are zero matrices.

Def. Block lower triangular matrix : A block square matrix is said to be block lower triangular matrix if all the blocks above the diagonal are zero matrices.

Def. Block triangular matrix : A block square matrix is said to be block triangular matrix if either it is block upper triangular or block lower triangular matrix.

The matrices A , B and C (given below) are block lower triangular, block upper triangular and block diagonal matrices respectively.

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 8 & 6 & 5 & 4 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & 3 & 4 & 8 & 6 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Remark : Clearly, every block diagonal matrix (B.D.M.) is block triangular matrix (B.T.M.)

CPT Result : Let A is a block triangular matrix with diagonal blocks $A_{11}, A_{22}, \dots, A_{rr}$ and let $c_i(x)$ be the characteristic polynomial of A_{ii} and $c(x)$ be the characteristic polynomial of A then

$$c(x) = c_1(x)c_2(x) \dots c_r(x).$$

In words characteristic polynomial of a block triangular matrix is the product of the characteristic polynomials of the diagonal blocks.

MLD Result : Let A is a block diagonal matrix with diagonal blocks $A_{11}, A_{22}, \dots, A_{rr}$ and let $m_i(x)$ be the minimal polynomial of A_{ii} and $m(x)$ be the minimal polynomial of A then

$$m(x) = \text{l.c.m.}[m_1(x), m_2(x), \dots, m_r(x)].$$

In words minimal polynomial of a block diagonal matrix is the l.c.m. of the minimal polynomials of the diagonal blocks.

DPT Result : Let A is a block triangular matrix with diagonal blocks $A_{11}, A_{22}, \dots, A_{rr}$ then

$$\det(A) = \det(A_{11})\det(A_{22}) \dots \det(A_{rr}).$$

Def. Companion matrix : Let $f(x)$ be a monic polynomial of degree n i.e.

$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ then the following four matrices are known as companion matrix of $f(x)$.

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_2 & 0 & 0 & \dots & 1 & 0 \\ -a_1 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

CMCM Result : Let A be companion matrix of a monic polynomial $f(x)$ then characteristic and minimal polynomials of A are $f(x)$ itself.

Exercise 10.7

1. Find the characteristic polynomial, minimal polynomial and determinant of the following

matrices (i) $A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. If $M = \left[\begin{array}{cc|cc} 2 & 1 & 3 & 4 \\ 2 & 2 & 1 & 0 \\ \hline 6 & 6 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right]$ then show that $|M| \neq |A||D| - |B||C|$.

3. Find a matrix A whose minimal polynomial is :

(i) $x^3 - 5x^2 + 6x + 8$ (ii) $x^4 - 5x^3 - 2x^2 + 7x + 4$

Answers

1. (i) $c(x) = (x-2)^3(x-7)^2$, $m(x) = (x-2)^2(x-7)$
- (ii) $c(x) = (x-3)^5$, $m(x) = (x-3)^3$
- (iii) $c(x) = (x-2)^2(x-4)^2(x-5)$, $m(x) = (x-2)(x-4)(x-5)$

10.8 Jordan Canonical Forms

Def. Basic Jordan Block : $B_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ 0 & & & \lambda \end{bmatrix}$

i.e λ (eigen value) on diagonal entries, 1's on super-diagonal and 0's elsewhere.

Def. Jordan Block : $J_n(\lambda) = \begin{bmatrix} B_{n_1}(\lambda) & & 0 \\ & B_{n_2}(\lambda) & \\ & & \ddots \\ 0 & & & B_{n_k}(\lambda) \end{bmatrix}$, where $n = n_1 + n_2 + \dots + n_k$.

i.e. basic Jordan blocks on diagonal entries and 0's elsewhere.

Def. Jordan Canonical form : $J_n = \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & J_{n_2}(\lambda_2) & \\ & & \ddots \\ 0 & & & J_{n_k}(\lambda_k) \end{bmatrix}$, where $n = n_1 + n_2 + \dots + n_k$.

i.e Jordan blocks on diagonal entries and 0 elsewhere.

Remark : $\boxed{\text{Matrix}} \xrightarrow{\text{I}} \boxed{c(x), m(x)} \xrightarrow{\text{II}} \boxed{\text{Blocks and their sizes}} \xrightarrow{\text{III}} \boxed{\text{J.C.F.}}$

Some results :

1. Diagonal matrices are the most beautiful matrices of this world.
2. Jordan canonical form of a matrix is always an upper triangular matrix.
3. Jordan canonical forms are not the diagonal matrices but they are very close to the diagonal matrices. In a Jordan Canonical Form, all non diagonal elements are 0 except some or all 1's on the super diagonal.
4. If a matrix is diagonalizable, then its Jordan canonical form is a diagonal matrix.

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5. Every matrix is not diagonalizable over $\mathbb{C}$ but every matrix has a Jordan canonical form over $\mathbb{C}$. In other words, we can say that every matrix is triangularizable over $\mathbb{C}$.

Jordan Canonical forms of nilpotent Matrices :

Result 1 : 0 is the only eigen value of a nilpotent matrix.

Result 2 : Let A be a $n \times n$ nilpotent matrix with index of nilpotency k , then

$$c(x) = x^n \quad \text{and} \quad m(x) = x^k.$$

Result 3 : Let A be a nilpotent matrix, then the number of basic Jordan blocks is equal to the nullity of A .

OR

For a $n \times n$ nilpotent matrix with index of nilpotency ' k ' and rank ' r ' we have,

(i) $c(x) = x^n$

(ii) $m(x) = x^k$

(iii) Decomposition for the basic Jordan blocks is

$$n = k + k_1 + k_2 + \dots + k_m \text{ where } k \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 1 \text{ and } m+1 = n-r$$

Result 4 : Let A and B are any two $n \times n$ matrices. If AB is nilpotent then BA is also nilpotent.

Result 5 : Let A and B are any two $n \times n$ nilpotent matrices such that $AB=BA$ then $A+B$ and AB are also nilpotent.

Exercise 10.8

1. Construct the following basic Jordan blocks :

(i) $B_4(-1)$

(ii) $B_4(1)$

(iii) $B_4(0)$

(iv) $B_3(-1)$

(v) $B_3(1)$

(vi) $B_3(0)$

(vii) $B_2(-1)$

(viii) $B_2(1)$

(ix) $B_2(0)$

(x) $B_1(-1)$

(xi) $B_1(1)$

(xii) $B_1(0)$

2. Determine all the possible Jordan canonical forms with the following data :

(i) $c(x) = (x-7)^3(x+1)^2$; $m(x) = (x-7)^2(x+1)$

(ii) $c(x) = (x-1)^4(x+1)^2$; $m(x) = (x-1)^3(x+1)^2$

(iii) $c(x) = (x+2)^4(x-3)^3$; $m(x) = (x+2)^3(x-3)^2$

(iv) $c(x) = (x-1)^5(x+2)^4$; $m(x) = (x-1)^3(x+2)^2$

3. Find the Jordan canonical forms for the following matrices :

$$(i) A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$(ii) B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$(iii) C = \begin{bmatrix} 1 & 2+3i \\ 2-3i & 2 \end{bmatrix}$$

$$(iv) D = \begin{bmatrix} 2i & 1+2i \\ -1+2i & -3i \end{bmatrix}$$

$$(v) E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(vi) F = \frac{1}{5} \begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$$

$$(vii) G = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

$$(viii) H = \begin{bmatrix} 5 & 4 \\ -\frac{21}{4} & -5 \end{bmatrix}$$

$$(ix) J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4. Find the Jordan canonical forms for the following nilpotent matrices :

$$(i) A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(ii) B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iii) C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iv) D = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(v) E = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vi) F = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vii) G = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(viii) H = \begin{bmatrix} 1 & 2 & 3 \\ -5 & -10 & -15 \\ 3 & 6 & 9 \end{bmatrix}$$

Answers

$$1. (i) \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iv) \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(v) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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(vii) $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

(viii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(ix) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(x) $[-1]$

(xi) $[1]$

(xii) $[0]$

2. (i) $\begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ OR $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ OR

$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ OR $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

3. (i) $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1+\sqrt{7} & 0 \\ 0 & 0 & 1-\sqrt{7} \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & i\sqrt{14} & 0 \\ 0 & 0 & -i\sqrt{14} \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{3+\sqrt{53}}{2} & 0 \\ 0 & \frac{3-\sqrt{53}}{2} \end{bmatrix}$

$$(iv) \begin{bmatrix} \frac{(3\sqrt{5}-1)i}{2} & 0 \\ 0 & \frac{-(3\sqrt{5}+1)i}{2} \end{bmatrix}$$

$$(v) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$(vi) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \text{ where } \lambda_1 = \frac{(i-3) + \sqrt{-398+294i}}{10} \text{ and } \lambda_2 = \frac{(i-3) - \sqrt{-398+294i}}{10}$$

$$(vii) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(viii) \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$(ix) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$4. (i) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(v) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vi) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(vii) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(viii) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

11.1 Quadratic Forms (Definition and examples)

Def. Quadratic Form : An expression q in n variables is called a quadratic form if it is an homogeneous expression of degree two.

Def. Matrix of a quadratic form : Let q be a quadratic form in variables $x_1, x_2, \dots, x_n$ over a field F . A $n \times n$ matrix A is said to be matrix of q if

(i) A is symmetric.

(ii) $q(X) = X'AX$, where $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X$

Def. Real Quadratic Form : A quadratic form is said to be real if its matrix A is over the real field $\mathbb{R}$ and the column vector X is in the vector space $\mathbb{R}^n$.

Note : We shall confine our study to real quadratic forms only. Therefore, unless otherwise stated, a quadratic form will be understood to be a real quadratic form.

Method : To write the matrix of a quadratic form.

Input : A quadratic form $q(X)$ is given.

Output : Symmetric matrix of $q(X)$.

Working Steps : The matrix of the quadratic form is given by

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2} \text{coeff of } x_1 x_2 & \cdots & \frac{1}{2} \text{coeff of } x_1 x_n \\ \frac{1}{2} \text{coeff of } x_2 x_1 & \text{coeff of } x_2^2 & \cdots & \frac{1}{2} \text{coeff of } x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} \text{coeff of } x_n x_1 & \frac{1}{2} \text{coeff of } x_n x_2 & \cdots & \text{coeff of } x_n^2 \end{bmatrix}$$

Method : To find a quadratic form when its matrix is given .

Input : Matrix of a quadratic form $A = [a_{ij}]_{n \times n}$ is given.

Working Steps : (i) Take $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(ii) Compute $X'AX$ and we get the required quadratic form.

Exercise 11.1

1. Which of the following are quadratic forms :

(i) $x^2 + y^2 - xy + yz + 4y$

(ii) $x_1^2 - 2x_1x_2 + x_3^2 - 4x_1x_2x_3$

(iii) $x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2$

(iv) $x^2 + y^2 + z^2 - 4xy + 2xz + 3yz + 5$

(v) $x^2 + y^2 + z^2 - 2xy$

(vi) $x_1^2 + x_1x_2 - x_1x_3$

2. Write the symmetric matrices of the following quadratic forms :

(i) $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + x_2x_3 + 4x_1x_3$

(ii) $(x+y)^2 + (z+t)^2$

(iii) $x_1x_2 + 3x_2x_3 + 2x_1x_3$

(iv) $ax^2 + 2hxy + by^2$

(v) $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

(vi) $x_1^2 - 2x_2^2 + 3x_3^2 - 4x_2x_3 + 6x_1x_3$

(vii) $2x^2 + 4y^2 + 3z^2 - 2xy$

(viii) $x_1x_2 + x_2x_3 + x_3x_1$

3. Write the quadratic forms corresponding to following symmetric matrices :

(i) $\begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 5 & 3 \\ 5 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & \frac{7}{2} \\ 4 & \frac{7}{2} & 1 \end{bmatrix}$

(v) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 2 & 0 & 1 \\ 5 & 0 & 3 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$

Answers

1. (i) No (ii) No (iii) yes (iv) No (v) Yes (vi) Yes

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2. (i) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{3}{2} & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$

(v) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(viii) $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

3. (i) $x_1^2 + 8x_1x_2 + 2x_2^2$

(ii) $x_1^2 + 3x_2^2 + 7x_3^2 + 4x_1x_2 + 8x_1x_3 + 10x_2x_3$

(iii) $2x_1^2 + x_3^2 + 10x_1x_2 + 6x_1x_3 + 8x_2x_3$

(iv) $x_1^2 + x_2^2 + x_3^2 + 6x_1x_2 + 8x_1x_3 + 7x_2x_3$

(v) $ax^2 + by^2 + cz^2$

(vi) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 10x_1x_3 + 4x_1x_4 + 2x_2x_4 + 4x_3x_4$

11.2 Diagonalization

Def. Discriminant of a quadratic form : The discriminant of a quadratic form is defined to be the determinant of its matrix i.e. for $q(X) = X'AX$, the discriminant of $q(X)$ is $\det(A)$.

Def. Rank of a quadratic form : Rank of a quadratic form is defined to be the rank of its matrix.

Def. Singular and non – singular quadratic forms : A quadratic form is said to be singular if its discriminant is zero ; otherwise it is called non – singular.

Def. Let $q(X) = X'AX$ be any quadratic form. A substitution $X = BY$ in it is called a linear transformation of the quadratic form. Further , if B is singular then the linear transformation $X = BY$ is called singular and if B is non – singular then it is called non – singular.

Result 1 : A quadratic form remains a quadratic form when subjected to a linear transformation.

Result 2 : The rank of a quadratic form is invariant under a non – singular linear transformation.

Def. Diagonal quadratic form : A quadratic form $q(X) = X'AX$ is called diagonal if matrix A is of the type $\begin{bmatrix} D_r & O \\ O & O \end{bmatrix}$ where D_r is a diagonal matrix of order r .

OR

A quadratic form q in the variables $x_1, x_2, \dots, x_n$ is called canonical or diagonal if it is represented as $q = a_1x_1^2 + a_2x_2^2 + \dots + a_rx_r^2$, where $r \leq n$.

Def. Diagonalization : The process of transforming a quadratic form into a diagonal form is called diagonalization.

There are many methods to diagonalize a quadratic form. Among them we shall discuss two methods. Before those we see a theorem first.

Result 3 : Every real quadratic form of rank r can be reduced to the form

$d_1y_1^2 + d_2y_2^2 + \dots + d_ry_r^2$, $d_i \neq 0$ by the application of a linear transformation.

OR

Every real quadratic form on a field can be reduced to a diagonal form.

Method : To diagonalize a quadratic form.

Input : A quadratic form.

Output : Equivalent diagonal quadratic form and equations of transformation.

Working Steps :

- (i) Find the symmetric matrix A of the given quadratic form.
- (ii) Form the augmented matrix $[A : I]$.
- (iii) Check the entry a_{11} . If $a_{11} \neq 0$ its okay but if $a_{11} = 0$ then make it non - zero by suitable row operations and corresponding column operations.
- (iv) Makes 0's below a_{11} by applying row operations.
- (v) Apply the corresponding column operations of step (iv) to make 0's on the right of a_{11} .
- (vi) Repeat the above process on $a_{22}, a_{33}, \dots$ so on untill A is diagonalized. Then augmented matrix $[A : I]$ is transformed to a matrix, say, $[D : P']$.
- (vii) The equations of transformation are $X = PY$ and diagonal form of given quadratic form is $d_{11}y_1^2 + d_{22}y_2^2 + \dots + d_{nn}y_n^2$.

Def. Rank , Index and Signature of a quadratic form : The number of non - zero terms in the diagonal form of a quadratic form is called its rank and is denoted by r .

The number of positive terms in the diagonal form of a quadratic form is called its index and

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is denoted by p .

The number $2p - r$ is called signature of a quadratic form and is denoted by s .

Example 1 : Reduce to diagonal form and find the rank, index and signature of the following quadratic forms : $x^2 + y^2 + 4z^2 - 9t^2 - 2xy - 4yz + 6yt - 6tx - 12tz$.

Solution : The matrix of the given quadratic form is $A = \begin{bmatrix} 1 & -1 & 0 & -3 \\ -1 & 1 & -2 & 3 \\ 0 & -2 & 4 & -6 \\ -3 & 3 & -6 & -9 \end{bmatrix}$

Consider the augmented matrix $[A : I] = \begin{bmatrix} 1 & -1 & 0 & -3 & : & 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 3 & : & 0 & 1 & 0 & 0 \\ 0 & -2 & 4 & -6 & : & 0 & 0 & 1 & 0 \\ -3 & 3 & -6 & -9 & : & 0 & 0 & 0 & 1 \end{bmatrix}$

Operating $R_2 \rightarrow R_2 + R_1$, $R_4 \rightarrow R_4 + 3R_1$, we get $\sim \begin{bmatrix} 1 & -1 & 0 & -3 & : & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & -6 & : & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & -18 & : & 3 & 0 & 0 & 1 \end{bmatrix}$

Operating $C_2 \rightarrow C_2 + C_1$, $C_4 \rightarrow C_4 + 3C_1$, we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & -6 & : & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & -18 & : & 3 & 0 & 0 & 1 \end{bmatrix}$

Operating $R_2 \leftrightarrow R_3$, $C_2 \leftrightarrow C_3$, we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & -6 & : & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & -6 & 0 & -18 & : & 3 & 0 & 0 & 1 \end{bmatrix}$

Operating $R_2 \rightarrow \frac{1}{2}R_2$, $C_2 \rightarrow \frac{1}{2}C_2$, we get $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & : & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -18 & : & 3 & 0 & 0 & 1 \end{bmatrix}$

Operating $R_3 \rightarrow R_3 + 1R_2$, $R_4 \rightarrow R_4 + 3R_2$, we get ~

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -3 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -27 & : & 3 & 0 & \frac{3}{2} & 1 \end{bmatrix}$$

Operating $C_3 \rightarrow C_3 + 1.C_2$, $C_4 \rightarrow C_4 + 3C_2$, we get ~

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -3 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -3 & -27 & : & 3 & 0 & \frac{3}{2} & 1 \end{bmatrix}$$

$R_4 \rightarrow R_4 + (-3)R_3$, $C_4 \rightarrow C_4 + (-3)C_3$, we get ~

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -18 & : & 0 & -3 & 0 & 1 \end{bmatrix} = [D : P']$$

Thus the diagonal form is $y_1^2 + y_2^2 - y_3^2 - 18y_4^2$.

and rank = 4, index = 2, signature = 0.

Equations of transformations are given by $X = PY$ i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

i.e. $x_1 = y_1 + y_3$, $x_2 = y_3 - 3y_4$, $x_3 = \frac{1}{2}y_2 + \frac{1}{2}y_3$ and $x_4 = y_4$.

Def. Canonical quadratic form : A quadratic form $q(X) = X' A X$ is said to be canonical if

(i) it is diagonal.

(ii) all non-zero elements on the diagonal of A are either 1 or -1.

OR

A quadratic form $q(X)$ in the variables $\{x_1, x_2, \dots, x_n\}$ is called in canonical form if

$$q(X) = \pm x_1^2 \pm x_2^2 \pm \dots \pm x_r^2 \text{ where } r \leq n$$

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Example 2 : Reduce the quadratic form $x^2 + y^2 + 4z^2 - 9t^2 - 2xy - 4yz + 6yt - 6tx - 12tz$ into canonical form and also find the equations of transformation and rank, index, signature.

Solution : For the given quadratic form we have already obtained in example (3) that

$$[A : I] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -18 & : & 0 & -3 & 0 & 1 \end{bmatrix}$$

Operating $R_4 \rightarrow \frac{1}{\sqrt{18}}R_4$, $C_4 \rightarrow \frac{1}{\sqrt{18}}C_4$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & : & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{18}} \end{bmatrix} = [D : P']$$

Thus the canonical form of quadratic form is $y_1^2 + y_2^2 - y_3^2 - y_4^2$.

Equations of transformations are given by

$$X = PY \quad \text{i.e.} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

i.e. $x_1 = y_1 + y_3$, $x_2 = y_3 - \frac{1}{\sqrt{2}}y_4$, $x_3 = \frac{1}{2}y_2 + \frac{1}{2}y_3$ and $x_4 = \frac{1}{\sqrt{18}}y_4$.

Also, rank, index and signature are given by same manner as in the diagonal form.

So Rank = 4, Index = 2 and Signature = 0.

Def. Canonical quadratic form over complex field : A quadratic form $q(X) = X' A X$ is said to be in canonical form if matrix A is of the type $A = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, where I_r denotes the identity matrix of order r .

OR

A quadratic form $q(X)$ in the variables $\{x_1, x_2, \dots, x_n\}$ is called in canonical form over C $q(X) = x_1^2 + x_2^2 + \dots + x_r^2$, $r \leq n$

Example 3 : Reduce the quadratic form $x^2 + y^2 + 4z^2 - 9t^2 - 2xy - 4yz + 6yt - 6tx - 12tz$ into canonical form over complex field and also find the equations of transformation.

Solution : For the given quadratic form we have already obtained in Example 5 that

$$[A : I] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & : & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & : & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{18}} \end{bmatrix}$$

Operating $R_3 \rightarrow iR_3$, $C_3 \rightarrow iC_3$, $R_4 \rightarrow iR_4$, $C_4 \rightarrow iC_4$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & : & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & : & i & i & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & 1 & : & 0 & -\frac{1}{\sqrt{2}}i & 0 & \frac{1}{\sqrt{18}}i \end{bmatrix} = [D : P']$$

Thus the canonical form of quadratic form is $y_1^2 + y_2^2 + y_3^2 + y_4^2$.

Equations of transformations are given by

$$X = PY \quad \text{i.e.} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 0 & i & -\frac{1}{\sqrt{2}}i \\ 0 & \frac{1}{2} & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{18}}i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

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i.e. $x_1 = y_1 + i y_3$, $x_2 = i y_3 - \frac{1}{\sqrt{2}} i y_4$, $x_3 = \frac{1}{2} y_2 + \frac{1}{2} i y_3$ and $x_4 = \frac{1}{\sqrt{18}} i y_4$.

Lagrange's method of diagonalization :

Input : A quadratic form $q(X) = X' A X$.

Output : Diagonal form of $q(X)$.

Working steps :

- (i) If quadratic form is given in the matrix notation then first change it into polynomial expression form. Let the variables used are $x_1, x_2, \dots, x_n$.
- (ii) Collect the terms containing x_1 in one bracket and take the coefficient of x_1^2 common outside the bracket to make it unity.
- (iii) Complete the square by adding and subtracting $\left(\frac{1}{2} \text{ coefficient of } x_1\right)^2$.
- (iv) Repeat the above process for $x_2, x_3, \dots$ and so on.
- (v) Give the name $a_1 y_1^2 + a_2 y_2^2 + \dots + a_r y_r^2$ to the last obtained expression, this is the required diagonal form and the equations of transformation are given by values of $y_1, y_2, \dots, y_r$ in terms of $x_1, x_2, \dots, x_n$.

Example 4 : Apply Lagrange's method to reduce the following to the diagonal form

$$2x^2 + 2y^2 + 3z^2 - 4yz + 2xy - 4zx.$$

Solution : The given quadratic form can be expressed as $(2x^2 + 2xy - 4xz) + 2y^2 - 4yz + 3z^2$

$$\begin{aligned} &= 2[x^2 + x(y - 2z)] + 2y^2 - 4yz + 3z^2 \\ &= 2\left[x^2 + x(y - 2z) + \frac{1}{4}(y - 2z)^2\right] + 2y^2 - 4yz + 3z^2 - \frac{1}{2}(y - 2z)^2 \\ &= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + 2y^2 - 4yz + 3z^2 - \frac{1}{2}(y^2 + 4z^2 - 4yz) \\ &= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}y^2 - 2yz + z^2 \\ &= 2\left[x + \frac{1}{2}(y - 2z)\right]^2 + \frac{3}{2}\left[y^2 - \frac{4}{3}yz\right] + z^2 \end{aligned}$$

Success could be described as 50/50-50% vision and 50% action.

$$\begin{aligned}
&= 2\left[x + \frac{1}{2}(y-2z)\right]^2 + \frac{3}{2}\left[y^2 - \frac{4}{3}yz + \frac{4}{9}z^2\right] + z^2 - \frac{2}{3}z^2 \\
&= 2\left[x + \frac{1}{2}(y-2z)\right]^2 + \frac{3}{2}\left[y - \frac{2}{3}z\right]^2 + \frac{1}{3}z^2 \\
&= 2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2
\end{aligned}$$

where $y_1 = x + \frac{1}{2}y - z$, $y_2 = y - \frac{2}{3}z$, $y_3 = z$.

Exercise 11.2

1. Diagonalize the following quadratic forms and also find the rank, index and signature.

(i) $x^2 + 2y^2 - 7z^2 - 4xy + 8xz$

(ii) $xy + yz - 2xz$

(iii) $X' \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix} X$

(iv) $2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_1x_3 + 2x_1x_2$

(v) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$

2. Reduce the following quadratic forms into canonical form and also find the equations of transformations. Also find rank, index and signature.

(i) $x_1^2 + 3x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_1x_3 + 10x_2x_3$

(ii) $X' \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -4 & 6 & 2 \\ 3 & 6 & 9 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix} X$

(iii) $2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_1x_3 + 2x_1x_2$

3. Reduce the following quadratic forms into form $\sum d_i y_i^2$ and also find the equations of transformations.

(i) $2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_1x_3 + 2x_1x_2$

(ii) $X' \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix} X$

(iii) $x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_3x_4$

4. By using the Lagrange's method diagonalize the following quadratic forms.

(i) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

(ii) $2x_1^2 + 5x_2^2 + 19x_3^2 - 24x_4^2 + 8x_1x_2 + 12x_1x_3 + 8x_1x_4 + 18x_2x_3 - 8x_2x_4 - 16x_3x_4$

5. Reduce the real quadratic form

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$$x_1^2 + 3x_2^2 + 8x_3^2 - 8x_4^2 + 4x_1x_2 + 6x_3x_1 + 4x_1x_4 + 10x_2x_3 + 16x_2x_4 + 20x_3x_4$$

to canonical form over (i) R (ii) C . Also find the rank and index and write the matrix of transformation.

Answers

1. (i) $y_1^2 - 2y_2^2 + 9y_3^2$, rank = 3 index = 2 signature = 1.

(ii) $y_1^2 - \frac{1}{4}y_2^2 + 2y_3^2$, rank = 3 index = 2 signature = 1.

(iii) $y_1^2 - 6y_2^2 + \frac{14}{3}y_3^2$, rank = 3 index = 2 signature = 1.

(iv) $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$, rank = 3 index = 3 signature = 3.

(v) $y_1^2 - 2y_2^2 - 4y_3^2$, rank = 3 index = 1 signature = -1.

2. (i) $y_1^2 - y_2^2 + y_3^2$, rank = 3, index = 2, signature = 1

Equations are $x_1 = y_1 - 2y_2 - 2\sqrt{2}y_3$, $x_2 = y_2 + \frac{1}{\sqrt{2}}y_3$, $x_3 = \frac{1}{\sqrt{2}}y_3$

(ii) $y_1^2 - y_2^2$, rank = 2 , Index = 1 , signature = 0.

Equations are $x_1 = y_1 - \frac{y_2}{\sqrt{2}} - 3y_3 - y_4$, $x_2 = \frac{y_2}{\sqrt{2}}$, $x_3 = y_3$, $x_4 = y_4$

(iii) $y_1^2 + y_2^2 + y_3^2$, rank = 3 , index = 3 , signature = 3.

3. (i) $2y_1^2 + \frac{3}{2}y_2^2 + \frac{1}{3}y_3^2$, equations are $x_1 = y_1 - \frac{1}{2}y_2 + \frac{2}{3}y_3$, $x_2 = y_2 + \frac{2}{3}y_3$, $x_3 = y_3$

(ii) $y_1^2 - 2y_2^2 + 9y_3^2$, equations are $x_1 = y_1 + 2y_2 + 4y_3$, $x_2 = y_2 + 4y_3$, $x_3 = y_3$

(iii) $y_1^2 - \frac{1}{4}y_2^2 - y_3^2 - \frac{3}{4}y_4^2$, equations are

$x_1 = y_1 - \frac{1}{2}y_2 - y_3 - \frac{1}{2}y_4$, $x_2 = y_1 + \frac{1}{2}y_2 - y_3 - \frac{1}{2}y_4$, $x_3 = y_3 - \frac{1}{2}y_4$, $x_4 = y_4$

4. (i) $y_1^2 - 2y_2^2 + 9y_3^2$

(ii) $2y_1^2 - 3y_2^2 + 4y_3^2$

5. (i) $y_1^2 + y_2^2 - y_3^2$; rank = 3 ; index = 2 ;

$$\begin{bmatrix} 1 & -5 & -2 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

(ii) $y_1^2 + y_2^2 + y_3^2$;

$$\begin{bmatrix} 1 & -5 & -2i & -1 \\ 0 & 2 & i & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

11.3' Definiteness of the quadratic forms

Def. Positive definite quadratic form : A real quadratic form $q = X'AX$ is called positive definite if q is always positive for all non-zero real values of X and is zero only if $X = 0$.

A quadratic form $q(x_1, x_2, \dots, x_n)$ is positive definite if in its diagonal form (over reals)

all the variables $x_1, x_2, \dots, x_n$ are present with positive coefficients i.e.

$$q = a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 \text{ where } a_1, a_2, \dots, a_n \text{ are all positive real numbers.}$$

Further in this form we see that rank = index = n i.e. $p = r = n$.

Def. Negative definite quadratic form : A real quadratic form $q = X'AX$ is called negative definite if q is always negative for all non-zero real values of X and is zero only when $X = 0$.

A quadratic form $q(x_1, x_2, \dots, x_n)$ is negative definite if in its diagonal form (over reals) all the variables $x_1, x_2, \dots, x_n$ are present with negative coefficients i.e.

$$q = -a_1x_1^2 - a_2x_2^2 - \dots - a_nx_n^2 \text{ where } a_1, a_2, \dots, a_n \text{ are all positive real numbers.}$$

Further in this form we see that rank = n and index = 0 i.e. $r = n$ and $p = 0$.

Def. Positive semi - definite quadratic form : A real quadratic form $q = X'AX$ is called positive semi - definite if q is always ≥ 0 for all real values of X .

A quadratic form $q(x_1, x_2, \dots, x_n)$ is positive semi - definite if in its diagonal form

(over reals) at least one variable out of $x_1, x_2, \dots, x_n$ is absent and all remaining

variables are present with positive coefficients i.e.

$$q = a_1x_1^2 + a_2x_2^2 + \dots + a_rx_r^2, \quad r \leq n, \text{ where } a_1, a_2, \dots, a_n \text{ are all positive real numbers.}$$

Further in this form we see that rank = index $< n$ i.e. $p = r < n$.

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Def. Negative semi - definite quadratic form : A real quadratic form $q = X'AX$ is called negative semi - definite if q is always ≤ 0 for all real values of X .

A quadratic form $q(x_1, x_2, \dots, x_n)$ is negative semi - definite if in its diagonal form (over reals) at least one variable out of $x_1, x_2, \dots, x_n$ is absent and all remaining variables are present with negative coefficients i.e.

$$q = -a_1x_1^2 - a_2x_2^2 - \dots - a_rx_r^2, \quad r \leq n, \text{ where } a_1, a_2, \dots, a_n \text{ are all positive real numbers.}$$

Further in this form we see that $\text{rank} < n$ and $\text{index} = 0$ i.e. $r < n$ and $p = 0$.

Def. Indefinite quadratic form : A real quadratic form $q = X'AX$ is called indefinite if q can assume both positive and negative values for different real values of X .

A quadratic form $q(x_1, x_2, \dots, x_n)$ is indefinite if in its diagonal form (over reals) at least one negative and one positive term is present.

Def. Let $q = X'AX$ be a real quadratic form. The definiteness of the matrix A is defined to be the definiteness of the quadratic form q .

Example : Determine the definiteness of the quadratic form.

$$6x^2 + 25y^2 + 61z^2 - 24xy - 36xz + 76yz.$$

Solution : The matrix of the given quadratic form is $A = \begin{bmatrix} 6 & -12 & -18 \\ -12 & 25 & 38 \\ -18 & 38 & 61 \end{bmatrix}$

We have already obtained the diagonal form of this quadratic form in Example 4, namely, $6y_1^2 + y_2^2 + 3y_3^2$. Clearly this canonical form contains all the three variables and also with positive coefficients, therefore the given quadratic form positive definite.

Exercise 11.3

1. Determine the definiteness of the following quadratic forms using the diagonalization process

(i) $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$

(ii) $9x^2 + y^2 + 4z^2 + 6xy - 4yz - 12zx$

(iii) $-21x^2 - 11y^2 - 2z^2 + 30xy + 8yz - 12zx$

(iv) $-y^2 - 2xy + 2yz$

(v) $2x_1^2 + 4x_2^2 + 8x_3^2 - 2x_2x_3 - 2x_1x_3 + 2x_1x_2$

(vi) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$

(vii) $-3x^2 - 4y^2 + 2z^2 - 2yz + xz + 7xy$

2. Prove that $-y^2 + 2yz - 2xy$ is indefinite.

Answers

1. (i) Positive Definite (ii) Positive Semi definite (iii) Negative Semi definite
 (iv) Indefinite (v) Positive definite (vi) Indefinite
 (vii) Indefinite
-

11.4 Sylvester's Criterion

Example : Write down all the minors, principal minors and the leading principal minor of order 1,2,3 for a 3×3 matrix.

Solution : Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(i) Minors of order 1 : $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.

Principal minors of order 1 : a_{11}, a_{22}, a_{33} .

Leading principal minor of order 1 : a_{11} .

(ii) Minors order of 2 : $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{23} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Principal minors of order 2 : $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{32} & a_{23} \end{vmatrix}$

Leading Principal Minor of order 2 : $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(iii) Minor of order 3 = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

There is only one minor of order 3 which acts as principal minor and leading principal minor as well.

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Result : Let $A = [a_{ij}]_{n \times n}$ be a matrix, then

- (i) Number of all minors of order $r \times r$ is $({}^nC_r)^2$
- (ii) Number of all principal minors of order $r \times r$ is ${}^nC_r$.
- (iii) Number of leading principal minor of $r \times r$ is 1.

Theorem 1 : A quadratic form $q(X) = X^T A X$ is positive definite iff all the leading principal minors of A are positive.

Particular Cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive definite iff $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive definite iff

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$$

Theorem 2 : A quadratic form $q(X) = X^T A X$ is negative definite iff all the leading principal minors are alternatively negative and positive, starting with first.

Particular cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is negative definite iff $a_{11} < 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is negative definite iff

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0$$

Theorem 3 : A quadratic form $q(X) = X^T AX$ is positive semi definite iff A is singular i.e

$|A| = 0$ and all other principal minors are non-negative.

Particular cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive semi-definite iff $a_{11} \geq 0$, $a_{22} \geq 0$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive semi-definite iff

$$a_{11} \geq 0, a_{22} \geq 0, a_{33} \geq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \geq 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \geq 0 \text{ and } |A| = 0$$

Theorem 4 : A quadratic form $q(X) = X^T AX$ is negative semi definite iff $|A| = 0$ and all principal minors of even order of A are non-negative while those of odd order are non-positive.

Particular Cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is negative semi-definite iff $a_{11} \leq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is negative semi-definite iff

$$a_{11} \leq 0, a_{22} \leq 0, a_{33} \leq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \geq 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \geq 0 \text{ and } |A| = 0.$$

Theorem 5 : A quadratic form $q(X) = X^T AX$ is indefinite iff at least one of the following condition is satisfied.

(i) A has negative principal minor of even order.

(ii) A has positive principal minor of odd order and negative principal minor of even order.

Particular Cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is indefinite iff $|A| < 0$ and a_{11} and a_{22} are of opposite signs.

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(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is indefinite iff

$$(i) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} < 0 \text{ or } \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} < 0 \text{ or } \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} < 0$$

(ii) Out of $a_{11}, a_{22}, a_{33}, |A|$ at least one is positive and at least one is negative.

Theorem 6 : A quadratic form $q(X) = X^T A X$ is positive semi-definite if $|A| = 0$ and the other leading principal minors are positive.

Particular Cases :

(i) The matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is positive semi-definite if $a_{11} > 0$, $\det(A) = 0$

(ii) The matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is positive semi-definite if $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ and $\det(A) = 0$

Theorem 7 : A quadratic form $q(X) = X^T A X$ is Negative Semi-definite if $\det(A) = 0$ and the other Principal Minor of odd order are negative and of even order are positive.

Eigen Value Theorem For definiteness : A quadratic form $q(X) = X^T A X$ is

(i) Positive definite iff all eigen value of A are positive.

(ii) Negative definite iff all eigen value of A are negative.

(iii) Positive semi-definite iff at-least one eigen value is zero and all other eigen values are non-negative.

(iv) Negative semi-definite iff at-least one eigen value is zero and all other eigen values are non-positive.

(v) Indefinite iff at-least one eigen value is positive and at-least one eigen value is negative.

Exercise 11.4

1. Determine the definiteness of the following quadratic forms using the Sylvester's Criterion .

(i) $6x^2 + 3y^2 + 14z^2 + 4xy + 4yz + 18zx$ (ii) $x^2 + y^2 + z^2 - xy + yz - zx$

(iii) $-x^2 - 2y^2 - 2z^2 + 2xy + 2yz$ (iv) $-x^2 - 3y^2 - 2z^2$

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$$(v) -21x_1^2 + 30x_1x_2 - 12x_1x_3 - 11x_2^2 + 8x_2x_3 + 2x_3^2$$

2. Prove that the quadratic form $ax^2 + 2hxy + by^2$ is positive definite iff $a > 0$ and $h^2 < ab$.
3. For a positive definite quadratic form $X'AX$, prove that $|A| > 0$.
4. If two matrices A and B are positive definite then what can you say about the positive definiteness of the following :

- | | | | |
|------------|--------------|-------------|------------|
| (i) $A+B$ | (ii) AB | (iii) A^2 | (iv) ABA |
| (v) $A'BA$ | (vi) $A^T A$ | (vii) AA' | |

Answers

- | | | |
|--------------------------|----------------------------|-------------------------|
| 1. (i) Positive Definite | (ii) Positive definite | (iii) Negative definite |
| (iv) Negative definite | (v) Negative semi-definite | |

Range of A Quadratic form $\Rightarrow$ The set of all values which can be attained by a Quadratic form is called its range.

Result $\Rightarrow$ (i) Range of P.D and PSD quadratic form is $[0, \infty)$

(ii) Range of ND and NSD quadratic form is $(-\infty, 0]$

(iii) Range of I.D Quadratic form is $(-\infty, \infty)$

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12.1 Inner product space (Definition and examples)

Def. Inner Product : Let V be a vector space V over F where F is the field of complex numbers or real numbers. A mapping $(,) : V \times V \rightarrow F$ is called an inner product if it satisfies the following axioms :

- (i) $(u, v) = \overline{(v, u)}$, the complex conjugate of (v, u)
- (ii) $(u, u) \geq 0$ and $(u, u) = 0$ iff $u = 0$
- (iii) $(au + bv, w) = a(u, w) + b(v, w)$ for all $u, v, w \in V$ and $a, b \in F$.

The space V equipped with an inner product is called an inner product space.

OR

A vector space V over F (F being the field of complex numbers or real numbers) is called an inner product space if for any two elements $u, v \in V$, there is defined an element of F , denoted by (u, v) , satisfying the following axioms :

- (i) $(u, v) = \overline{(v, u)}$, the complex conjugate of (v, u)
- (ii) $(u, u) \geq 0$ and $(u, u) = 0$ iff $u = 0$
- (iii) $(au + bv, w) = a(u, w) + b(v, w)$ for all $u, v, w \in V$ and $a, b \in F$.

Remarks :

- (1) Axioms (i) , (ii) and (iii) in the definition of Inner Product Space are called Symmetry , Non-negativity and Linearity respectively.
- (2) If F is the field of real numbers , then V is called real inner product space or Euclidean space. In this case the axiom (i) is stated as : $(u, v) = (v, u)$
- (3) If F is the field of complex numbers , then V is called complex inner product space or Unitary space.
- (4) It should be noted that by axiom (i) , we have , $(u, u) = \overline{(u, u)}$ so that (u, u) is a real number and hence $(u, u) \geq 0$ in axiom (ii) is justified.

Some standard inner products :

- 1. Let $u = (x_1, x_2, \dots, x_n)$ and $v = (y_1, y_2, \dots, y_n)$ be any two vectors in $\mathbb{R}^n$ then the standard inner product is defined as $(u, v) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

If you worry about yesterday's failures, then today's successes will be few.

Exercise 12.1

- For $u = (x_1, x_2)$ and $v = (y_1, y_2) \in \mathbb{C}^2$ show that $(u, v) = 2x_1\overline{y_1} + x_1\overline{y_2} + x_2\overline{y_1} + x_2\overline{y_2}$ is an inner product on $\mathbb{C}^2$.
- Consider the vector space $\mathbb{R}^2(\mathbb{R})$. For $u = (x_1, x_2)$ and $v = (y_1, y_2)$, show that (u, v) defined by $(u, v) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$ is an inner product on $\mathbb{R}^2$.
- For $u = (x_1, x_2)$ and $v = (y_1, y_2)$, show that $(u, v) = x_1\overline{y_2} + x_2$ does not define an inner product on $\mathbb{C}^2(\mathbb{C})$.
- Let V be the vector space over $\mathbb{C}$ of all complex valued continuous functions on the interval $[a, b]$. Define the inner product on V as $(f(t), g(t)) = \int_a^b f(t) \overline{g(t)} dt$. Show that V is an inner product space.
- In $\mathbb{C}^3(\mathbb{C})$, let $u = (1+i, i, -1)$, $v = (1+2i, 1-i, 2i)$. Find (u, v) and (v, u) and verify that :
 (i) $(u, v) = \overline{(v, u)}$ (ii) $(u, v) + (v, u) = 2\operatorname{Re}(u, v)$ (iii) $(u, v) - (v, u) = 2\operatorname{Im}(u, v)$
- (a) Let $u = (1, 2)$, $v = (-1, 1) \in \mathbb{R}^2$. If w is a vector s.t. $(u, w) = -1$ and $(v, w) = 3$, find w .
 (b) If $u_1 = (1, -3, 1)$, $u_2 = (2, 1, -4)$, $u_3 = (6, -7, 8) \in \mathbb{R}^3$, find $u \in \mathbb{R}^3$ such that $(u, u_1) = -1$, $(u, u_2) = -1$, $(u, u_3) = 7$.
- (a) Let V be the vector space of all real polynomials of degree ≤ 2 , show that $(f(x), g(x)) = \int_{-1}^1 f(x)g(x) dx$ for all $f(x), g(x) \in V$ is an inner product on V .
 (b) If $f(x) = x^2 + x - 4$ and $g(x) = x - 1$, then find $(f(x), g(x))$ and $(g(x), g(x))$.

Answers

6. (a) $\left(-\frac{7}{3}, \frac{2}{3}\right)$ (b) $(1, 1, 1)$ 7. (b) $8, \frac{8}{3}$.

12.2 Properties of an inner product

Let $V(F)$ be an inner product space then for all $a, b, c \in F$ and $u, v, w \in V$, we have

- | | |
|--|-------------------------------------|
| (i) $(au, v) = a(u, v)$ | (ii) $(u + v, w) = (u, w) + (v, w)$ |
| (iii) $(u - v, w) = (u, w) - (v, w)$ | (iv) $(u, av) = \overline{a}(u, v)$ |
| (v) $(u, bv + cw) = \overline{b}(u, v) + \overline{c}(u, w)$ | (vi) $(u, v + w) = (u, v) + (u, w)$ |
| (vii) $(u, v - w) = (u, v) - (u, w)$ | (viii) $(0, v) = (u, 0) = 0$ |

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$$(ix) (u, v) = 0 \text{ for all } v \in V \Rightarrow u = 0$$

$$(x) (u, w) = (v, w) \text{ for all } w \in V \Leftrightarrow u = v$$

Def. Norm of a vector : Let V be an inner product space and $v \in V$. Then norm (length) of v , denoted by $\|v\|$, is defined as $\|v\| = \sqrt{(v, v)} = (v, v)^{1/2}$

We take only non-negative value of square root and so norm of a vector is always a non – negative real number.

Def. Unit Vector : A vector v of an inner product space V is called a unit vector if $\|v\| = 1$.

Result 1 : In an inner product space V over the field F , then

$$(i) \|av\| = |a| \|v\| \text{ for all } a \in F, v \in V$$

$$(ii) \frac{v}{\|v\|} \text{ is always a unit vector.}$$

Result 2 : Cauchy schwarz inequality : Let V be an inner product space. Then for arbitrary vectors $u, v \in V$, we have $|(u, v)| \leq \|u\| \|v\|$

Further, equality holds if and only if u and v are linearly dependent.

Result 3 : Triangle's inequality : Let V be an inner product space, then for all $u, v \in V$, we have

$$\|u + v\| \leq \|u\| + \|v\|.$$

Further if $\|u + v\| = \|u\| + \|v\|$, then u and v are linearly dependent.

Remark : In the above result, we have that if $\|u + v\| = \|u\| + \|v\|$, then u and v are linearly dependent.

Now, we give an example to show that converse of this is not true, in general, i.e., if u and v are linearly dependent, then it is not necessary that $\|u + v\| = \|u\| + \|v\|$.

For this, consider standard inner product on $\mathbb{R}^3$.

Let $u = (1, -2, 2)$, $v = (-2, 4, -4) \in \mathbb{R}^3$ then u and v are linearly dependent as $v = -2u$

$$\text{We have : } \|u\| = \sqrt{(u, u)} = \sqrt{1.1 + (-2)(-2) + 2.2} = \sqrt{1+4+4} = 3$$

$$\text{and } \|v\| = \sqrt{(v, v)} = \sqrt{(-2)(-2) + 4.4 + (-4)(-4)} = \sqrt{4+16+16} = 6$$

$$\text{Thus } \|u\| + \|v\| = 3 + 6 = 9$$

Also, $u + v = (-1, 2, -2)$ and

$$\|u + v\| = \sqrt{(u + v, u + v)} = \sqrt{(-1)(-1) + 2.2 + (-2)(-2)} = \sqrt{1+4+4} = 3$$

Clearly $\|u + v\| \neq \|u\| + \|v\|$.

Result 4 : Parallelogram law : If u and v are vectors in an inner product space, then

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Result 5 : Every inner product space is a metric space.

Result 6 : Every inner product space is a normed linear space.

Exercise 12.2

1. Let $V(\mathbb{R})$ be a vector space of all real polynomial with inner product defined by

$$(f, g) = \int_0^1 f(x)g(x) dx;$$

If $f(x) = x^2 + 1$ and $g(x) = x - 1$ then find (f, g) and $\|g\|$.

2. For $x_i, y_i \in \mathbb{R}$, prove that $\left| \sum_{i=1}^n x_i y_i \right| \leq \left\{ \sum_{i=1}^n |x_i|^2 \right\}^{1/2} \left\{ \sum_{i=1}^n |y_i|^2 \right\}^{1/2}$.

3. Prove that : $\left| \int_0^1 f(x) \overline{g(x)} dx \right| \leq \left\{ \int_0^1 |f(x)|^2 dx \right\}^{1/2} \left\{ \int_0^1 |g(x)|^2 dx \right\}^{1/2}$ where $f(x)$ and $g(x)$ are continuous

complex valued functions on $[0, 1]$.

4. If $u = (-2, -3, 5)$ and $v = (4, 6, -10) \in \mathbb{R}^2$, verify that $|(u, v)| = \|u\| \cdot \|v\|$. Why does the equality occur here?
5. If u, v are vectors in a real inner product space s.t. $\|u\| = \|v\|$, prove that $(u + v, u - v) = 0$.

Answers

1. $-\frac{7}{12}, \frac{1}{\sqrt{3}}$.

12.3 Orthogonality

Def. Orthogonal Vectors : Let V be an inner product space and u, v be vectors in V . Then u is said to be orthogonal to v (written $u \perp v$) if $(u, v) = 0$.

Here note that

- (i) If $u \perp v$, then $(u, v) = 0$ and so $(v, u) = \overline{(u, v)} = 0$. Hence $u \perp v$ iff $v \perp u$.
- (ii) Since $(0, u) = 0$ for all $u \in V$ so the zero vector is orthogonal to every vector.
- (iii) Let $u \in V$ such that $u \perp u$ then $(u, u) = 0 \Rightarrow u = 0$. Hence the zero vector is the only vector which is orthogonal to itself.

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(iv) If $(u, v) = 0$, then $(au, v) = a(u, v) = a \cdot 0 = 0$. Hence if $u \perp v$, then any scalar multiple of u is also orthogonal to v .

Def. Orthogonal Set : A subset S of inner product space is said to be orthogonal if $(u, v) = 0$ for all $u, v \in S$ such that $u \neq v$ i.e. every two distinct vectors of S are orthogonal.

Def. Orthonormal Set : A subset S of an inner product space $V(F)$ is called an orthonormal set if (i) $\|u\| = 1$ for all $u \in S$

(ii) $(u, v) = 0$ for all $u, v \in S$ such that $u \neq v$

i.e. if every element of S is a unit vector and every two different vectors of S are orthogonal.

Obviously, if a set $\{u_1, u_2, \dots, u_n\}$ is orthonormal, then we must have $(u_i, u_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Result 1 : In an inner product space, an orthogonal set of non-zero vectors is linearly independent.

Result 2 : In an inner product space, an orthonormal set is linearly independent.

Result 3 : Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal basis of an inner product space V and $u \in V$ be an

arbitrary vector, then $u = \sum_{i=1}^n (u, u_i) u_i = (u, u_1) u_1 + (u, u_2) u_2 + \dots + (u, u_n) u_n$

OR

Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal basis of an inner product space V and $u \in V$ be an arbitrary vector, then the coordinates of u relative to this basis are (u, u_i) , $1 \leq i \leq n$.

Result 4 : Let $\{u_1, u_2, \dots, u_n\}$ be an orthogonal basis of an inner product space V and $u \in V$ be an

arbitrary vector, then $u = \sum_{i=1}^n \frac{(u, u_i)}{\|u_i\|^2} u_i = \frac{(u, u_1)}{\|u_1\|^2} u_1 + \frac{(u, u_2)}{\|u_2\|^2} u_2 + \dots + \frac{(u, u_n)}{\|u_n\|^2} u_n$

Result 5 : Bessel's inequality : If $\{u_1, u_2, \dots, u_n\}$ is any finite orthonormal set in an inner

product space $V(F)$ and if v is any vector in $V(F)$, then $\sum_{i=1}^n |(v, u_i)|^2 \leq \|v\|^2$.

Further, the equality holds iff v is in the subspace spanned by $\{u_1, u_2, \dots, u_n\}$.

Result 6 : If $\{u_1, u_2, \dots, u_n\}$ is an orthogonal set of non zero vectors in an inner product space V , then for any vector $v \in V$

$$\sum_{i=1}^n \frac{|(v, u_i)|^2}{\|u_i\|^2} \leq \|v\|^2 \quad \text{i.e.} \quad \frac{|(v, u_1)|^2}{\|u_1\|^2} + \frac{|(v, u_2)|^2}{\|u_2\|^2} + \dots + \frac{|(v, u_n)|^2}{\|u_n\|^2} \leq \|v\|^2.$$

Be absolutely determined to enjoy what you do. -Gerry Sikorski

Result 7 : Gram-schmidt orthogonalisation process : Every finite dimensional inner product space has an orthonormal basis.

Result 8 : Construction : If an ordinary basis $\{u_1, u_2, u_3\}$ of an inner product space is given then an orthonormal basis $\{v_1, v_2, v_3\}$ is constructed in the following manner :

$$\begin{array}{|c|} \hline u_1 \\ \hline u_2 \\ \hline u_3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline w_1 \\ \hline w_2 \\ \hline w_3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline v_1 \\ \hline v_2 \\ \hline v_3 \\ \hline \end{array} ; \quad \begin{array}{l} w_1 = u_1 \\ w_2 = u_2 - (u_2, v_1)v_1 \\ w_3 = u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2 \end{array} ; \quad \begin{array}{l} v_1 = \frac{w_1}{\|w_1\|} \\ v_2 = \frac{w_2}{\|w_2\|} \\ v_3 = \frac{w_3}{\|w_3\|} \end{array}$$

Exercise 12.3

- Construct an orthonormal basis of $\mathbb{R}^3(\mathbb{R})$ with standard inner product given the basis $\{u_1, u_2, u_3\}$ where
 (i) $u_1 = (1, 1, 1)$, $u_2 = (1, -2, 1)$, $u_3 = (1, 2, 3)$
 (ii) $u_1 = (3, 0, 4)$, $u_2 = (-1, 0, 7)$, $u_3 = (2, 9, 11)$.
- Construct an orthonormal basis of $\mathbb{C}^3(\mathbb{C})$ with standard inner product given the basis $\{u_1, u_2, u_3\}$ where $u_1 = (1+i, i, 1)$, $u_2 = (2, 1-2i, 2+i)$, $u_3 = (1-i, 0, -i)$.
- Construct an orthonormal basis w.r.t. standard inner product for the subspace of $\mathbb{R}^4$ generated by $(1, 1, -1, 1)$ and $(1, 0, 1, 2)$.
- Let V be the vector space over $\mathbb{R}$ of all real polynomials of degree ≤ 2 . Define an inner product on V as $(f, g) = \int_0^1 f(x)g(x) dx$ for $f, g \in V$. Construct an orthonormal basis of V .
- Let V be the vector space of real functions satisfying $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$. In V , inner product is defined as $(u, v) = \int_{-\infty}^0 u \cdot v dx$ $u, v \in V$. Find an orthonormal basis of V .
- Let V be the vector space of real functions satisfying $\frac{d^3 y}{dx^3} - 6\frac{d^2 y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$. In V , inner product is defined as $(u, v) = \int_{-\infty}^0 u \cdot v dx$ $u, v \in V$. Find an orthonormal basis of V .
- Prove that two vectors u and v in a real inner product space V are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (This is known as Pythagoras theorem)
- Let V be a complex inner product space. If u and v are orthogonal, then

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$\|u + v\|^2 = \|u\|^2 + \|v\|^2$. However, the converse need not be true.

9. Show that if V is a complex inner product space, then $u, v \in V$ are orthogonal if

and only if $\|au + bv\|^2 = \|au\|^2 + \|bv\|^2$ for all $a, b \in \mathbb{C}$.

10. If $\{u, v\}$ is an orthonormal set in an inner product space V , then $\|u - v\| = \sqrt{2}$.

Answers

1. (i) $\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\}$ (ii) $\left\{ \left(\frac{3}{5}, 0, \frac{4}{5} \right), \left(-\frac{4}{5}, 0, \frac{3}{5} \right), (0, 1, 0) \right\}$

2. $\left(\frac{1}{2}(1+i), \frac{i}{2}, \frac{1}{2} \right), \left(\frac{1}{2\sqrt{3}}, \frac{1-5i}{4\sqrt{3}}, \frac{3+3i}{4\sqrt{3}} \right), \left(\frac{7-i}{2\sqrt{30}}, \frac{-5}{2\sqrt{30}}, \frac{-6+3i}{2\sqrt{30}} \right)$

3. $\left\{ \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) \right\}$

4. $\left\{ 1, (2x-1)\sqrt{3}, \left(x^2 - x + \frac{1}{6} \right) 3\sqrt{20} \right\}$

5. $\{ 2e^{2x}, \sqrt{6}(5e^{3x} - 4e^{2x}) \}$

6. $\{ \sqrt{2}e^x, 6e^{2x} - 4e^x, \sqrt{6}(10e^{3x} - 12e^{2x} + 3e^x) \}$

12.4 Orthogonal complement

Def. Orthogonal subspaces : Let $V(F)$ be an inner product space. Two subspaces

W_1 and W_2 of V are said to be orthogonal to each other if $(w_1, w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$

Def. Orthogonal complement : Let $V(F)$ be an inner product space and W be a subspace of V .

The orthogonal complement of W , denoted by $W^\perp$, is given by

$W^\perp = \{v \in V : (v, w) = 0 \text{ for all } w \in W\}$ i.e. orthogonal complement of W is collection of all

those vectors of V which are orthogonal to every element of W . Clearly, we have

$(w_1, w_2) = 0$ for all $w_1 \in W$ and $w_2 \in W^\perp$.

Thus W and $W^\perp$ are orthogonal to each other.

Result 1 : Let W be a subspace of an inner product space $V(F)$, then $W^\perp$ is a subspace of V . i.e. orthogonal complement of a subspace is also a subspace.

There is no achievement without goals. –Robert J. Mckaine

Result 2 : Let $V(F)$ be any inner product space , and W be a subspace of V , then prove that

$$(i) \{0\}^\perp = V \quad (ii) V^\perp = \{0\} \quad (iii) W \cap W^\perp = \{0\} \quad (iv) W_1 \subseteq W_2 \Rightarrow W_2^\perp \subseteq W_1^\perp$$

Result 3 : Let W be a subspace of an inner product space $V(F)$. If $\{u_1, u_2, \dots, u_n\}$ is a basis of W , then $w \in W^\perp$ iff $(w, u_i) = 0 \quad 1 \leq i \leq n$. In other words, w is orthogonal to each element of W iff w is orthogonal to all the elements in a basis of W .

Result 4 : If W is a subspace of a finite dimensional inner product space V , then $V = W \oplus W^\perp$ i.e. V is a direct sum of W and $W^\perp$.

Result 5 : Let $V(F)$ be a finite dimensional inner product space and W be a subspace of V . If $\{u_1, u_2, \dots, u_n\}$ is an orthonormal basis of W and $\{v_1, v_2, \dots, v_m\}$ is an orthonormal basis of $W^\perp$ then $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m\}$ is an orthonormal basis of V .

Result 6 : Let V be an inner product space and W be a subspace of V , then

$$(i) W \subseteq W^{\perp\perp} \quad (ii) W = W^{\perp\perp}, \text{ if } V \text{ is finite dimensional} \quad (iii) W^\perp = W^{\perp\perp\perp}$$

Result 7 : If W_1 and W_2 are subspaces of a finite dimensional inner product space , then

$$(i) (W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp \quad (ii) (W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$$

Def. Orthogonal projection : If W is a subspace of a finite dimensional inner product space V , then we know that $V = W \oplus W^\perp$. By definition of direct sum , every element $v \in V$ can be uniquely expressed as $v = w + w'$, where $w \in W$ and $w' \in W^\perp$.

The vectors w and w' are called orthogonal projections of v on the subspaces W and $W^\perp$.

Exercise 12.4

1. Let W be a subspace of $\mathbb{R}^4(\mathbb{R})$ generated by the vectors $u_1 = (1, 2, 3, -2)$ and $u_2 = (2, 4, 5, -1)$. Obtain a basis for $W^\perp$.
2. Consider $\mathbb{R}^3(\mathbb{R})$ with standard inner product defined in it. Let W be a subspace of $\mathbb{R}^3$ spanned by the vector $w = (2, -1, 6)$. Find the projection of the vector $v = (4, 1, 2)$ on W and $W^\perp$.

Result 1 : Let $V(F)$ be an inner product space and S be a subspace of V . If $L(S)$ denotes the linear span of S , then prove that

- (i) $S^\perp = [L(S)]^\perp$
- (ii) $L(S) \subseteq S^{\perp\perp}$
- (iii) $L(S) = S^{\perp\perp}$ if V is finite dimensional.

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Result 2 : Let $W_1, W_2, \dots, W_n$ be pairwise orthogonal subspaces of an inner product space

$V(F)$. If $w = \sum_{i=1}^n w_i$ where $w_i \in W_i$ for $i = 1, 2, \dots, n$, then show that $\|w\|^2 = \sum_{i=1}^n \|w_i\|^2$.

Result 3 : If $V(F)$ is a finite dimensional inner product space and $\{u_1, u_2, \dots, u_n\}$ is an

orthonormal set in $V(F)$ s.t. $\sum_{i=1}^n |(v, u_i)|^2 = \|v\|^2$ for all $v \in V$ then $\{u_1, u_2, \dots, u_n\}$ is a basis of V .

