Rotational motion

*A **rigid object** is one that is nondeformable—that is, the relative locations of all particles of which the object is composed remain constant. All real objects are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.*

6.1. Rotation about affixed axis

A rotating object spins about its **axis of rotation**. For example, the Earth's axis of rotation goes between the geographic north and south poles.

In everyday applications this rate of rotation may be described in terms such as the number of rotations per minute. In all theoretical work in physics, however, we specify it in radians per second. Remember that there are 2π radians in one complete turn.

If the flywheel turns through an angle θ in a time t, its angular velocity ω is defined by

$$\omega = \theta/t$$

In Unit 4, you learnt that the linear speed v and angular speed ω of a point on a rotating body are connected by the equation:

$$v = r\omega$$

where r is the distance of the point from the axis of rotation.

Remember that radians are dimensionless units and so are ignored when working out the dimensions of a quantity.

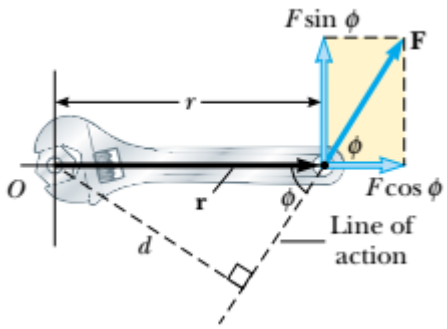


6.2. **Torque and angular acceleration**

Imagine trying to rotate a door by applying a force of magnitude *F* perpendicular to the door surface but at various distances from the hinges. You will achieve a more rapid rate of rotation for the door by applying the force near the doorknob than by applying it near the hinges.

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** $\boldsymbol{\tau}$ (Greek tau). Torque is a vector.

Consider the wrench pivoted on the axis through *O* in Figure 10.13. The applied force **F** acts at an angle ϕ to the horizontal. We define the magnitude of the torque associated with the force **F** by the expression



$$\tau \equiv rF\sin\phi = Fd$$

where *r* is the distance between the pivot point and the point of application of **F** and *d* is the perpendicular distance from the pivot point to the line of action of **F**. (The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force. The dashed line extending from the tail of **F** in Figure above is part of the line of action of **F**.) From the right triangle in Figure above that has the wrench as its hypotenuse, we see that $d = r\,sin\phi$. The quantity *d* is called the **moment arm** (or *lever arm*) of **F**.

Since torque is a vector quantity it can be expressed in vector form

The torque vector $\boldsymbol{\tau}$ is related to the two vectors **r** and **F**. We can establish a mathematical relationship between $\boldsymbol{\tau}$, **r**, and **F** using a mathematical operation called the **vector product,** or **cross product:**

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}$$

$$\boldsymbol{\tau} = (r_y F_z - r_z F_y)\hat{\mathbf{i}} + (r_z F_x - r_x F_z)\hat{\mathbf{j}} + (r_x F_y - r_y F_x)\hat{\mathbf{k}}$$

In linear terms the work done by a force is **F.s**, the force **F** multiplied by the displacement **s** it moves in that direction. The rotational equivalent of linear displacement is angular displacement. Angular displacement is θ. The angular equivalent of force is torque, τ. So, when a torque turns a body, the work it does is the torque multiplied by the angle it turns through. Substitute τ and θ into the equation for work done ($W = \mathbf{F} \bullet \mathbf{s}$):

$$W = \tau \bullet \theta$$

Torque has the same units as work done, newton metres (N m), but the units are not called joules, unlike for work done. The energy in work done is always a scalar (coming from the scalar product of force and displacement) but torque is a vector because it is the vector product of force and displacement. The displacement is the angle through which the torque acts.

N.B

The cross product of any two vectors **A** and **B** can be expressed in the following determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{\mathbf{k}}$$

Expanding these determinants gives the result

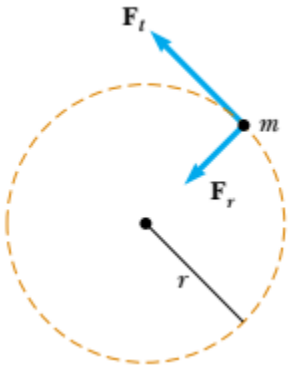
$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} - (A_x B_z - A_z B_x)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}} \qquad (11.8)$$

Torque should not be confused with force. Forces can cause a change in linear motion, as described by Newton's second law. Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces, in the combination that we call *torque*. Torque has units of force times length—newton · meters in SI units—and should be reported in these units. Do not confuse torque and work, which have the same units but are very different concepts.

Relationship Between Torque and Angular Acceleration

Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$



The magnitude of the torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t) r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$, the torque can be expressed as

$$\tau = (mr\alpha) r = (mr^2)\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the particle about the *z* axis passing through the origin, so that

$$\tau = I\alpha$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. Note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

6.3. Rotational Kinetic Energy and Rotational inertia

Rotational inertia

In linear motion, when a force F acts on a body of mass m, the body has an acceleration a, according to the equation $F = ma$.

When a body is rotating at a constant angular velocity and is acted on by a torque τ, this will produce an acceleration, α, so

$$\tau = I\alpha$$

where I is known as the **rotational inertia**, or moment of inertia. It is the rotational equivalent of mass.

The moment of inertia of a body is defined as:

$$I = \Sigma m_i r_i^2$$

where m_i is the mass of a point of the body and r_i is the distance of the point from the axis of rotation (see Figure 6.8).

The moment of inertia of a body is a measure of the manner in which the mass of that body is distributed in relation to the axis about which that body is rotating. It depends on the:

- mass of the body
- size of the body
- which axis is being considered.

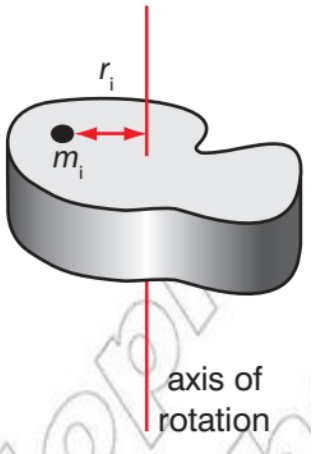


Figure 6.8

If we put the dimensions in the equation $\tau = I\alpha$, we can work out its dimensions.

$$I = \tau/\alpha = \text{N m/(rad/s}^2\text{)}$$

When considering dimensions we ignore radians, and as the dimension of newtons are also kg m/s^2, the dimensions of I are:

$$I = \text{kg m/s}^2 \times \text{m} \times \text{s}^2 = \text{kg m}^2.$$

The moments of inertia of some bodies are:

A disc of mass M and radius R (as with the first flywheel):

$$I = \tfrac{1}{2}MR^2$$

A sphere of mass M and radius R: $I = \frac{2}{5}MR^2$

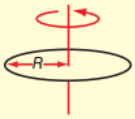
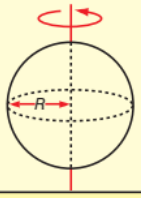
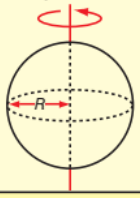
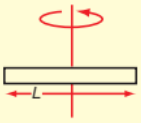
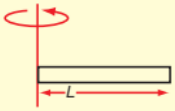
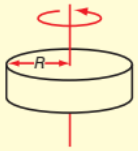
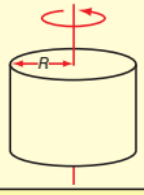
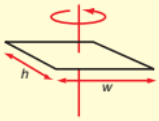
A thin rod or bar: $I = \frac{1}{12}ML^2$ where L is the length of the bar.

Note: these moments of inertia only apply when the axis of rotation goes through the centre of mass of the body.

Thin rod or bar being rotated about its end: $I = \frac{1}{3}ML^2$, where L is the length of the bar.

The moment of inertia of a body is not constant for that body because it depends on the axis chosen for it to rotate around. As indicated above, it may be defined for a given body about a given axis as the sum of mr^2 for every particle in that body (where m is the mass of the particle and r is its distance from the axis).

Table 6.1 *Moments of inertia for different bodies*

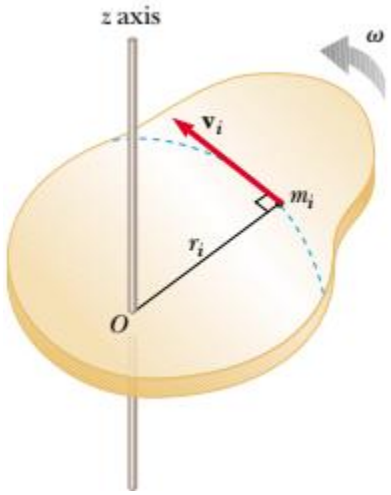
Body							
disc	solid sphere	hollow sphere	thin rod or bar	thin rod or bar	thin cylindrical shell	solid cylinder	rectangular slab
							
Moment of inertia when axis of rotation goes through centre of body							
$\frac{1}{2}MR^2$	$\frac{2}{5}MR^2$	$\frac{2}{3}MR^2$	$\frac{1}{12}ML^2$	$\frac{1}{3}ML^2$ (when rotated about the end of the thin rod or bar)	MR^2	$\frac{1}{2}MR^2$	$\frac{1}{12}M(h^2 + w^2)$

Rotetional kinetic energy

An object rotating about a fixed axis remains stationary in space, so there is no kinetic energy associated with translational motion. The individual particles making up the rotating object, however, are moving through space—they follow circular paths. Consequently, there should be kinetic energy associated with rotational motion.

Let us consider an object as a collection of particles and assume that it rotates about a fixed *z* axis with an angular speed &. Figure 10.7 shows the rotating object and identifies one particle on the object located at a distance *ri* from the rotation axis. Each such particle has kinetic energy determined by its mass and tangential speed. If the mass of the i^{th} particle is m_i and its tangential speed is v_i, its kinetic energy is

$$K_i = \frac{1}{2} m_i v^2{}_i$$



To proceed further, recall that although every particle in the rigid object has the same angular speed &, the individual tangential speeds depend on the distance r_i from the axis of rotation according to the expression $vi = r_i\omega$. The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \tfrac{1}{2} m_i v_i{}^2 = \tfrac{1}{2} \sum_i m_i r_i{}^2 \omega^2$$

We can write this expression in the form

$$K_R = \tfrac{1}{2} \left(\sum_i m_i r_i{}^2 \right) \omega^2$$

where we have factored ω^2 from the sum because it is common to every particle. We simplify this expression by defining the quantity in parentheses as the **moment of inertia *I*:**

$$I \equiv \sum_i m_i r_i{}^2$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 (kg · m^2 in SI units). With this notation, the above Equation becomes

$$K_R = \frac{1}{2} I\omega^2$$

Although we commonly refer to the quantity as **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy because it is derived from a sum over individual kinetic energies of the particles contained in the rigid object.

It is important that you recognize the analogy between kinetic energy associated with linear motion and rotational kinetic energy . The quantities *I* and & in rotational motion are analogous to *m* and *v* in linear motion, respectively. (In fact, *I* takes the place of *m* and & takes the place of *v* every time we compare a linear-motion equation with its rotational counterpart.) The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.

6.4 Rotational dynamics of a rigid body

In our study of linear motion, we found that the simplest form of accelerated motion to analyze is motion under constant linear acceleration. Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration.

the lenear quantities and their rotational equivalent are given in table below

Linear motion		Rotational motion	
Quantity	**Unit**	**Quantity**	**Unit**
displacement, *s*	m	angular displacement, θ	rad
velocity, *v*	m/s	angular velocity, ω	rad/s
acceleration, *a*	m/s^2	angular acceleration, α	rad/s^2
mass, *m*	kg	rotational inertia, *I*	kg m^2
force, *F*	N	torque, τ	N m

Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

N.B. In rotational equations, we must use angles expressed in radians. Don't fall into the trap of using angles measured in degrees in rotational equations.

Example

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

(A) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

Solution We can use Figure 10.2 to represent the wheel. We arrange Equation 10.7 so that it gives us angular displacement:

$$\Delta\theta = \theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2$$

$$= (2.00\ \text{rad/s})(2.00\ \text{s}) + \tfrac{1}{2}(3.50\ \text{rad/s}^2)(2.00\ \text{s})^2$$

$$= 11.0\ \text{rad} = (11.0\ \text{rad})(57.3°/\text{rad}) = 630°$$

(B) Through how many revolutions has the wheel turned during this time interval?

Solution We multiply the angular displacement found in part (A) by a conversion factor to find the number of revolutions:

$$\Delta\theta = 630° \left(\frac{1\ \text{rev}}{360°}\right) = 1.75\ \text{rev}$$

(C) What is the angular speed of the wheel at *t* =2.00 s?

Solution Because the angular acceleration and the angular speed are both positive, our answer must be greater than 2.00 rad/s. Using Equation 10.6, we find

$$\omega_f = \omega_i + \alpha t = 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00\ \text{s})$$

$$= 9.00\ \text{rad/s}$$

We could also obtain this result using Equation 10.8 and the results of part (A). Try it!

2. Suppose a particle moves along a straight line with a constant acceleration of 3.50 m/s^2. If the velocity of the particle is 2.00 m/s at $t_i = 0$, through what displacement does the particle move in 2.00 s? What is the velocity of the particle at t =2.00 s?

Answer Notice that these questions are translational analogs to parts (A) and (C) of the original problem. The mathematical solution follows exactly the same form. For the displacement,

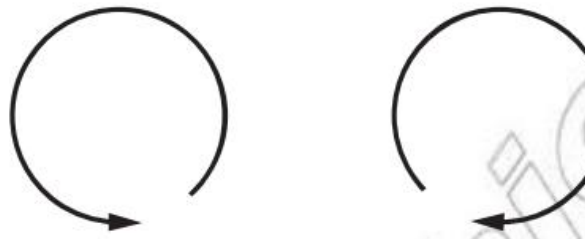
$$\begin{aligned}\Delta x &= x_f - x_i = v_i t + \tfrac{1}{2}at^2 \\ &= (2.00\ \text{m/s})(2.00\ \text{s}) + \tfrac{1}{2}(3.50\ \text{m/s}^2)(2.00\ \text{s})^2 \\ &= 11.0\ \text{m}\end{aligned}$$

and for the velocity,

$$v_f = v_i + at = 2.00\ \text{m/s} + (3.50\ \text{m/s}^2)(2.00\ \text{s}) = 9.00\ \text{m/s}$$

Note that there is no translational analog to part (B) because translational motion is not repetitive like rotational motion.

In rotational motion, positive angular displacement, velocity and acceleration are in the anticlockwise direction, and negative angular displacement, velocity and acceleration are in the clockwise direction (Figure 6.13).



Work done = torque × angle moved through = $\tau\theta$

We can also calculate the power, which is work done divided by time.

Power = $\tau\theta/t$

Now θ/t is also angular velocity ω, so power is

$P = \tau\omega$

summery of part one

Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If α = constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If a = constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Exercise

1. A dentist's drill starts from rest. After 3.20 s of constant angular acceleration, it turns at a rate of 2.51×10^4 rev/min. (a) Find the drill's angular acceleration. (b) Determine the angle (in radians) through which the drill rotates during this period.
2. A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.
3. An airliner arrives at the terminal, and the engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2 000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take the rotor to come to rest?
4. An electric motor rotating a grinding wheel at 100 rev/min is switched off. With constant negative angular acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn while it is slowing down?
5. A centrifuge in a medical laboratory rotates at an angular speed of 3 600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.
6. A wheel 2.00 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4.00 rad/s^2. The wheel starts at rest at $t = 0$, and the radius vector of a certain point P on the rim makes an angle of 57.3° with the horizontal at this time. At $t = 2.00\ s$, find (a) the angular speed of the wheel, (b) the tangential speed and the total acceleration of the point P, and (c) the angular position of the point P

7. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming that no slipping occurs. (b) What is the final angular speed of a tire in revolutions per second?
8. A particle is located at the vector position $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j})\ m$, and the force acting on it is $\boldsymbol{F} = (3\boldsymbol{i} + 2\boldsymbol{j})\ N$. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?
9. A 1.50-kg particle moves in the *xy* plane with a velocity of $\boldsymbol{v} = (4.20\boldsymbol{i} - 3.60\boldsymbol{j})\ m/s$. Determine the angular momentum of the particle when its position vector is $\boldsymbol{r} = (1.50\boldsymbol{i} + 2.20\boldsymbol{j})\ m$.
10. Show that the kinetic energy of an object rotating about a fixed axis with angular momentum $L = I\omega$) can be written as $K = L^2/2I$.
11. A uniform solid sphere of radius 0.500 m and mass 15.0 kg turns counterclockwise about a vertical axis through its center. Find its vector angular momentum when its angular speed is 3.00 rad/s.

PART TWO