

3.2 Motion in a plane

3.2.1 Projectile motion

A **projectile** is an object that has been launched into the air by the action of a force. The action of the force ceases after the projectile is launched.

- 1] A ball that is thrown or a football which is kicked into the air are both projectiles.
- 2] Javelins thrown in athletics and bullets fired from guns are also projectiles.
- 3] If the force continues, for example, in a rocket, the object is not a projectile. However, if the force stops, the rocket then becomes a projectile.

Trajectory: the path a moving object follows through space.

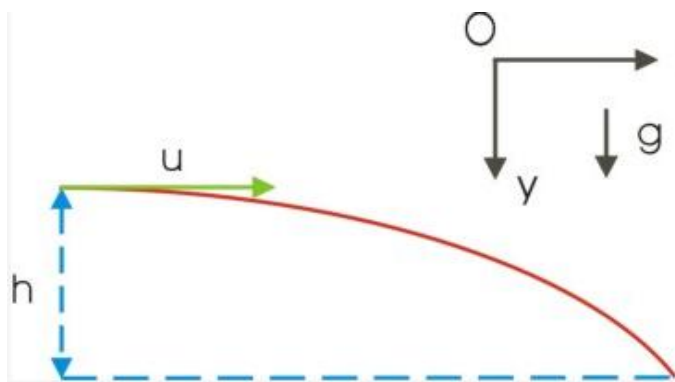
A body can be projected in two ways:

A] Horizontal projection: when the body is given an initial velocity in the horizontal direction only

B] Angular projection: when the body is thrown with an initial velocity at an angle to the horizontal direction.

A] Horizontal projection

A body is thrown with an initial velocity u along the horizontal direction. We will study the motion along x and y axis separately. We will take the starting point to be at the origin.



Along x-axis	Along y-axis
1. Component of initial velocity along x-axis. $u_x = u$	1. Component of initial velocity along y-axis. $u_y = 0$
2. Acceleration along x-axis $a_x = 0$ (Because no force is acting along the horizontal direction)	2. Acceleration along y-axis $a_y = g = 9.8 \text{ m/s}^2$ It is directed downwards.
3. Component of velocity along the x-axis at any instant t . $v_x = u_x + a_x t$ $= u + 0$ $v_x = u$ This means that the horizontal component of velocity does not change throughout the projectile motion.	3. Component of velocity along the y-axis at any instant t . $v_y = u_y + a_y t$ $= 0 + gt$ $v_y = gt$
4. The displacement along x-axis at any instant t $x = u_x t + (1/2) a_x t^2$ $x = u_x t + 0$ $x = u t$	4. The displacement along y-axis at any instant t $y = u_y t + (1/2) a_y t^2$ $y = 0 + (1/2) a_y t^2$ $y = 1/2 gt^2$

Equation of a trajectory(path of a projectile)

We know at any instant $x = ut$

$$t = x/u$$

Also, $y = (1/2)gt^2$

Substituting for t we get

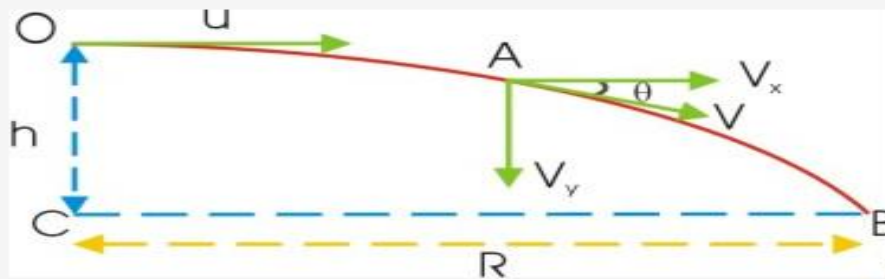
$$y = (1/2)g(x/u)^2$$

$$y = (1/2)(g/u^2)x^2$$

$$y = kx^2 \text{ where } k = g/(2u^2)$$

This is the **equation of a parabola** which is symmetric about the y-axis. Thus, the path of projectile, projected horizontally from a height above the ground is a parabola.

Net velocity at any instant of time t



We know, at any instant t

$$v_x = u$$

$$v_y = gt$$

$$v = (v_x^2 + v_y^2)^{1/2} = [u^2 + (gt)^2]^{1/2}$$

Direction of v with the horizontal at any instant :

$$(\text{angle}) = \tan^{-1} (v_y/v_x) = \tan^{-1} (gt/u)$$

Time of flight (T):

It is the total time for which the projectile is in flight (from O to B in the diagram above)

To find T we will find the time for vertical fall

From $y = u_y t + (1/2) gt^2$

When $y = h$, $t = T$

$$h = 0 + (1/2) gt^2$$

$$T = (2h/g)^{1/2}$$

Range (R) :

It is the horizontal distance covered during the time of flight T.

From $x = ut$

When $t = T$, $x = R$

$$R = uT$$

$$R = u(2h/g)^{1/2}$$

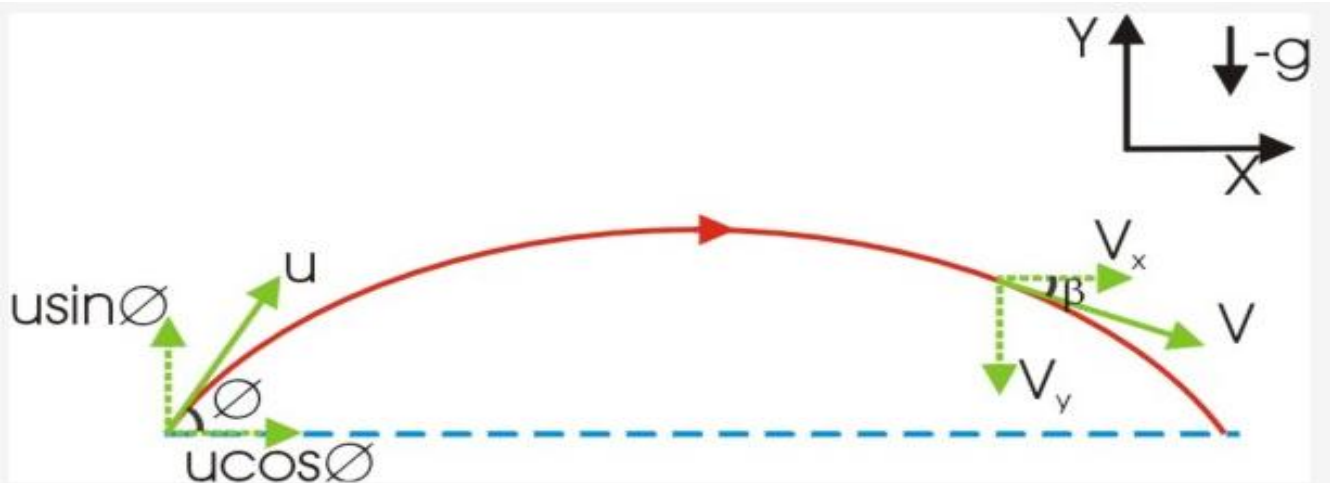
B] Angular projection

We will now consider the case when the object is projected with an initial velocity u at an angle to the horizontal direction.

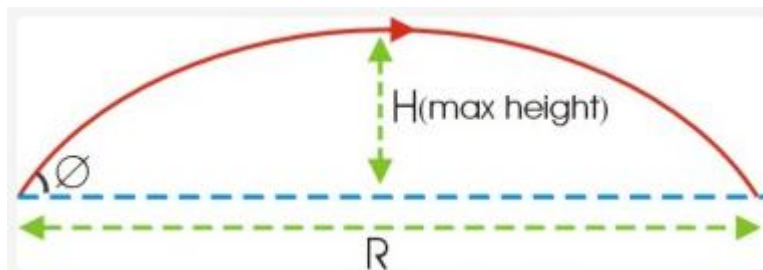
We assume that there is no air resistance. Also since the body first goes up and then comes down after reaching the highest point, we will use the Cartesian convention for signs of different physical quantities.

The acceleration due to gravity 'g' will be negative as it acts downwards.

We will separate the motion into horizontal motion (motion along x-axis) and vertical motion (motion along y-axis). We will study x-motion and y-motion separately.



X axis	Y axis
1. Component of initial velocity along x-axis. $u_x = u \cos \phi$	1. Component of initial velocity along y-axis. $u_y = u \sin \phi$
2. Acceleration along x-axis $a_x = 0$ (Because no force is acting along the horizontal direction)	2. Acceleration along y-axis $a_y = -g = -9.8 \text{ m/s}^2$ (g is negative as it is acting in the downward direction)
3. Component of velocity along the x-axis at any instant t . $v_x = u_x + a_x t$ $= u \cos \phi + 0 = u \cos \phi$ $v_x = u \cos \phi$ This means that the horizontal component of velocity does not change throughout the projectile motion.	3. Component of velocity along the y-axis at any instant t . $v_y = u_y + a_y t$ $v_y = u \sin \phi - gt$
4. The displacement along x-axis at any instant t $x = u_x t + (1/2) a_x t^2$ $x = u \cos \phi \cdot t$	4. The displacement along y-axis at any instant t $y = u_y t + (1/2) a_y t^2$ $y = u \sin \phi \cdot t - (1/2) g t^2$



Equation of Trajectory (Path of projectile)

At any instant **t**

$$x = u \cos \phi \cdot t$$

$$t = x / (u \cos \phi)$$

Also, $y = u \sin \phi \cdot t - (1/2)gt^2$

Substituting for **t**

$$y = u \sin \phi \cdot x / (u \cos \phi) - (1/2)g[x / (u \cos \phi)]^2$$

$$y = x \cdot \tan \phi - [(1/2)g \cdot \sec^2 \phi \cdot x^2] / u^2$$

This equation is of the form $y = ax + bx^2$ where 'a' and 'b' are constants. This is the equation of a parabola. Thus, the path of a projectile is a parabola.

Net velocity of the body at any instant of time t

$$v_x = u \cos \phi$$

$$v_y = u \sin \phi - gt$$

$$v = (v_x^2 + v_y^2)^{1/2}$$

$$\phi = \tan^{-1}(v_y / v_x) \text{ Where } \phi \text{ is the angle that the resultant velocity (v) makes with the horizontal at any instant.}$$



Time of flight T

Angular Projectile motion is symmetrical about the highest point. The object will reach the highest point in time **T/2**. At the highest point, the vertical component of velocity **v_y** becomes equal to zero.

$$v_y = u \sin \phi - gt$$

At $t = T/2$, $v_y = 0$

$$0 = u \sin \phi - gT/2$$

$$T = (2u \sin \phi) / g$$

Maximum height H

Equation for vertical distance (y component)

$$y = u_y t - (1/2)gt^2$$

At $t = T/2$, $y = H$

$$H = u \sin \phi \cdot T/2 - (1/2)g(T/2)^2$$

substituting **T**

$$H = u \sin \phi \cdot u \sin \phi / g - (1/2)g(u \sin \phi / g)^2$$

$$= (u^2 \sin^2 \phi) / g - (u^2 \sin^2 \phi) / 2g$$

$$H = (u^2 \sin^2 \phi) / 2g$$

Range R

Range is the total horizontal distance covered during the time of flight.

From equation for horizontal motion, $x = u_x t$

When $t = T$, $x = R$

$$R = u_x T = u \cos \phi \cdot 2u \sin \phi / g$$

$$= u^2 2 \sin \phi \cos \phi / g = u^2 \sin 2\phi / g \text{ using } 2 \sin \phi \cos \phi = \sin 2\phi$$

$$R = (u^2 \sin 2\phi) / g$$

Activity

- 1] A golf ball is hit off a tee at the edge of a cliff. Its x and y coordinates as functions of time are given by the following expressions:

$$x = (18 \text{ m/s})t$$

and

$$y = (4 \text{ m/s})t - (5 \text{ m/s}^2)t^2$$

Determine the position, the velocity, and the acceleration of the golf ball, all at $t = 3\text{s}$.

- 2] The vector position of a particle varies in time according to the expression $\mathbf{r} = (3 \hat{i} - 6t^2 \hat{j}) \text{ m}$.
- Find expressions for the velocity and acceleration as functions of time.
 - Determine the particle's position and velocity at $t = 1 \text{ s}$.
- 3] To start an avalanche on a mountain slope, an artillery shell is fired with an initial velocity of 300 m/s at 53° above the horizontal. It explodes on the mountainside 40 s after firing. What are the x and y coordinates of the shell where it explodes, relative to its firing point?
- 4] An aircraft flying at a steady velocity of 70 m/s eastwards at a height of 800 m drops a package of supplies.
- How long will it take for the package to reach the ground?
 - How fast will it be going as it lands? Express your answer as a vector.
- 5] A stone is thrown upwards with an initial velocity of 25 m/s at an angle of 37° to the ground.
- Find the vertical and horizontal components of the velocity at the start.
 - Calculate the maximum height of the stone.
 - How long will it take for the stone to reach its maximum height?
 - How long will it take between when the stone was thrown and when it comes back to the ground?
 - Work out the stone's horizontal range.
- 6] A ball is tossed from an upper-story window of a building. The ball is given an initial velocity of 8 m/s at an angle of 37° below the horizontal. It strikes the ground 3 s later.
- How far horizontally from the base of the building does the ball strike the ground?
 - Find the height from which the ball was thrown.
 - How long does it take the ball to reach a point 10 m below the level of launching?
- 7] A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?