



Chapter 5

Oscillations

Course Instructor: Mesfin Tadesse (PhD)

Department of Physics, AAU

Content

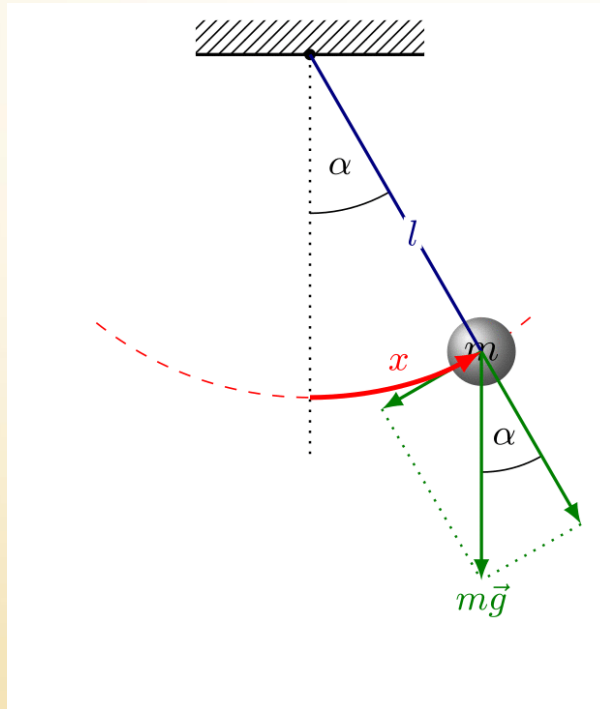
- Simple Harmonic Motion
 - Mass-Spring system
 - The Simple Pendulum

Periodic Motion

- Any motion that repeats itself is called **periodic** motion.
- Examples

A Simple Pendulum
And
A Double Pendulum

Simple Pendulum



Double Pendulum (Chaotic)



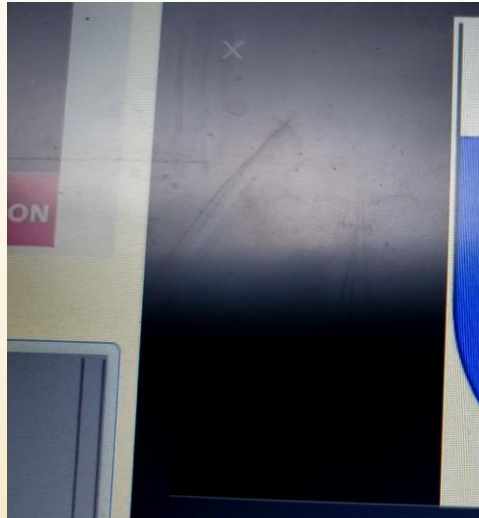
Periodic Motion

- Any motion that repeats itself is called **periodic** motion.
- Examples

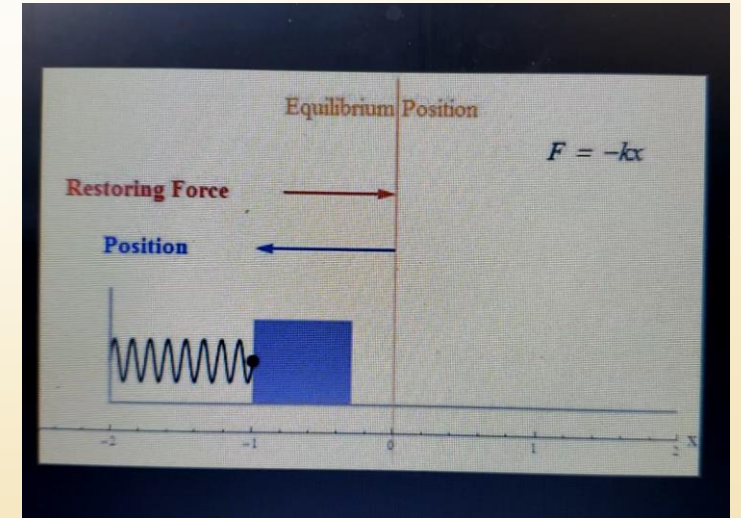
Water in U tube

Spring-Mass System

Water in U-Tube



Spring-Mass System



Periodic Motion

- Any motion that repeats itself is called **periodic** motion.
- Examples

Electrocardiogram



animated-heart-monitor-ekg.webm

Periodic Motion

- Key features of periodic motion
 - Period (T) is measured in seconds
 - Frequency (f) is measures in s⁻¹ (Hz)
 - Amplitude (A)

$$T = \frac{1}{f}$$

Periods and frequencies for a wide range of periodic motions

Table 13.1 Common Periods and Frequencies

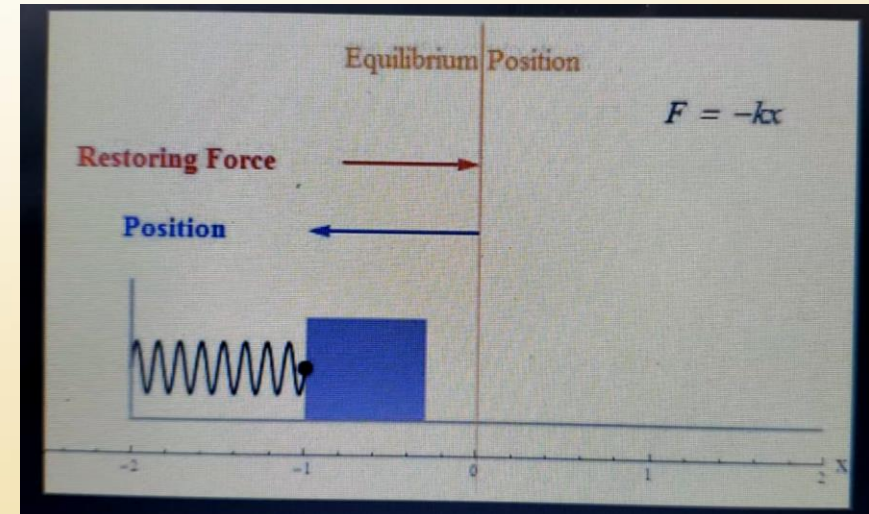
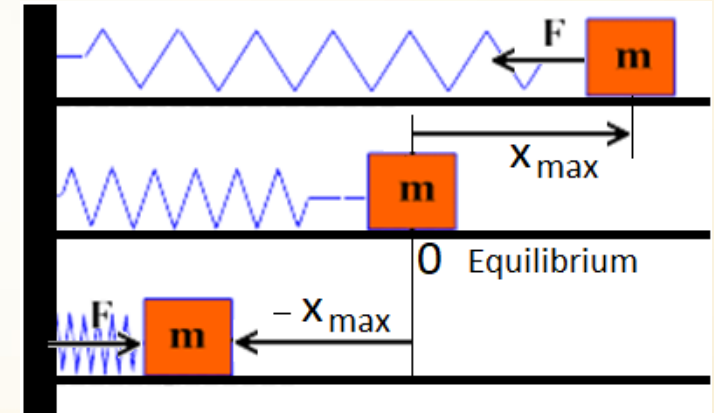
System	Period (s)	Frequency (Hz)
Hour hand of a clock	43,200 (1 cycle per 12 hours)	2.3×10^{-5}
Minute hand of a clock	3600 (1 cycle per hour)	2.8×10^{-4}
Second hand of a clock	60 (1 cycle per minute)	0.017
Pendulum in a grandfather clock	2.0	0.50
Human heartbeat	1.0	1.0
Sound at lower range of human hearing	5.0×10^{-2}	20
Wing beat of a housefly	5.0×10^{-3}	200
Sound at upper range of human hearing	5.0×10^{-5}	20,000
Computer processor	3.1×10^{-10}	3.2×10^9

The Simple Harmonic Motion

- In one type of periodic motion the force is proportional to the displacement from equilibrium.

$$\vec{F} = -k\vec{x}$$

- This is simple harmonic motion (SHM).
- A mass attached to a spring is a classic example of SHM.



The Simple Harmonic Motion

- Applying Newton's second law

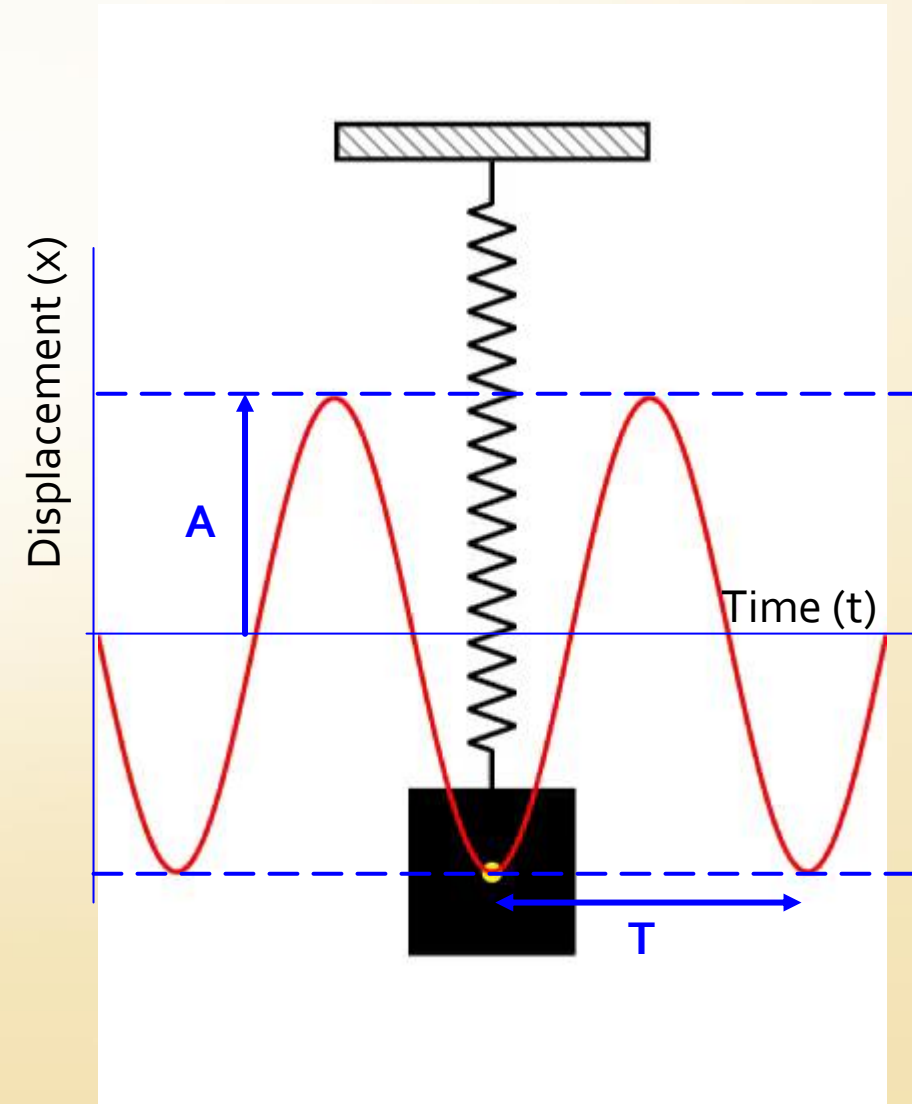
$$ma = -kx \quad a = -\frac{k}{m}x$$

Where k is a measure of the stiffness of the spring

$$\omega = \sqrt{k/m} \quad (\text{unit is } 1/\text{s})$$

$$\omega = \frac{2\pi}{T} = 2\pi f$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$



The Simple Harmonic Motion

- From conservation of energy

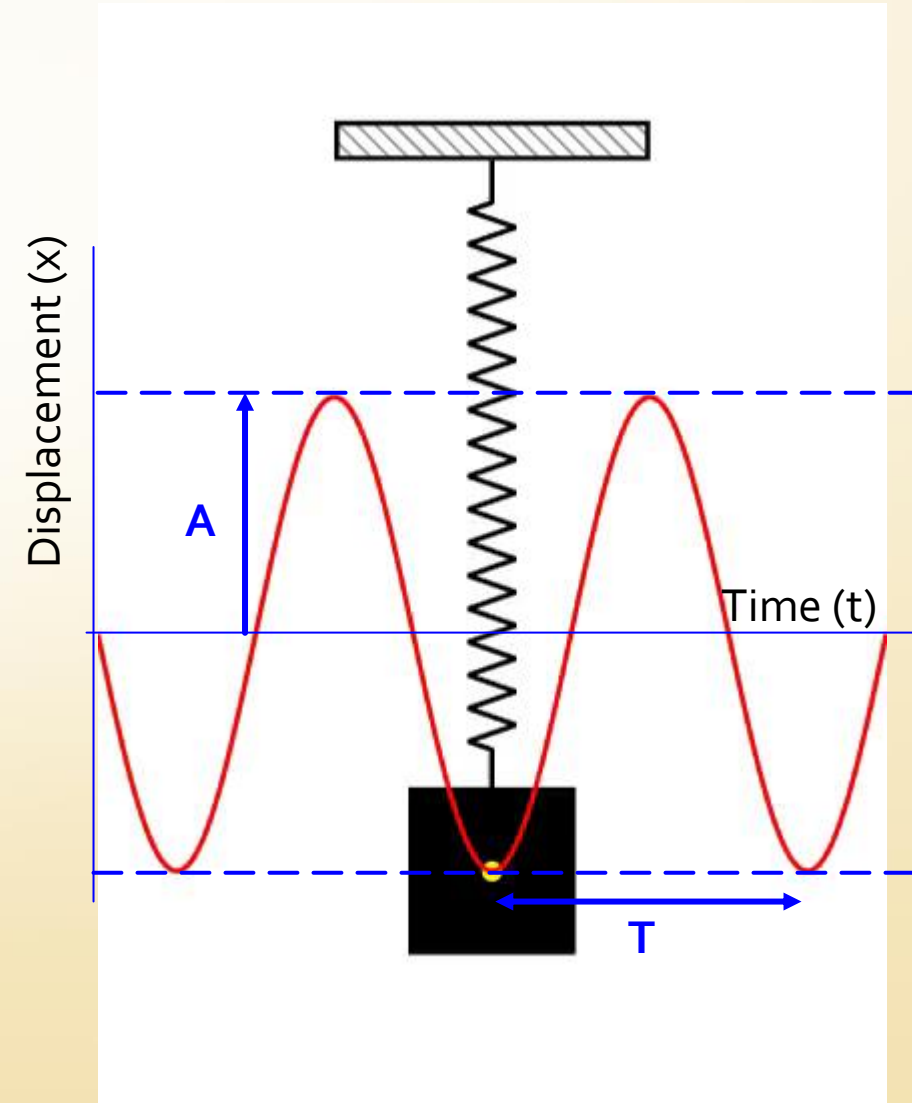
$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$m\omega^2x^2 + mv^2 = m\omega^2A^2$$

$$v^2 = \omega^2(A^2 - x^2)$$

$$v = \pm\omega\sqrt{A^2 - x^2}$$

$$v_{max} = \omega A$$



Examples

A 0.22-kg air-track cart is attached to a spring with a spring constant of 12 N/m and set into motion. What is the period of its oscillation?

Solution

Substituting for m and k in the period equation for a mass on a spring yields

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} \\ &= 2\pi\sqrt{\frac{0.22 \text{ kg}}{12 \text{ N/m}}} \\ &= \boxed{0.85 \text{ s}} \end{aligned}$$

Application of SHM in Space

- Astronauts in orbit around the Earth can't use a bathroom scale to determine their mass because of **weightlessness**.
- The mass of an astronaut can be measured using a **Body Mass Measurement Device (BMMD)**.
- The device is essentially a chair attached to a spring having a known spring constant k .
- The mass of the astronaut in the figure is determined by measuring the period of oscillations as she rocks back and forth.



Quiz

- Suppose you have two identical mass–spring systems. If one mass is given twice the amplitude of the other, how will its period change?
 - A. It will be twice the period of the second system
 - B. It will increase the period of the second system
 - C. It will not be affected
- Increasing the amplitude increases the force on the mass. The mass then moves more rapidly and covers the greater distance in precisely the same time.
- This result is predicted by the formula for the period, which does not contain the amplitude, A .

Quiz

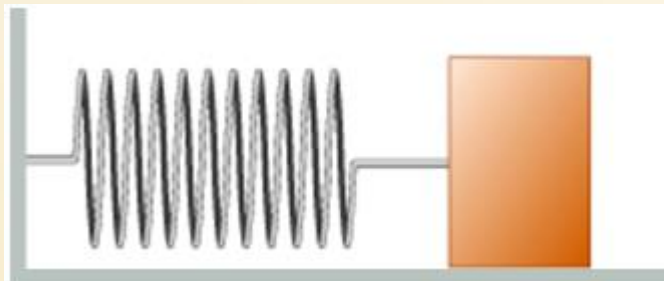
- An object moves with simple harmonic motion. If the amplitude and the period are both increased by a factor of 2, the object's maximum speed is
 - A. decreased by factor of $\frac{1}{4}$.
 - B. decreased by factor of $\frac{1}{2}$.
 - C. increased by factor of 4.
 - D. increased by factor of 2.
 - E. unchanged.

$$T = \frac{2\pi}{\omega}$$
$$v_{max} = \omega A$$

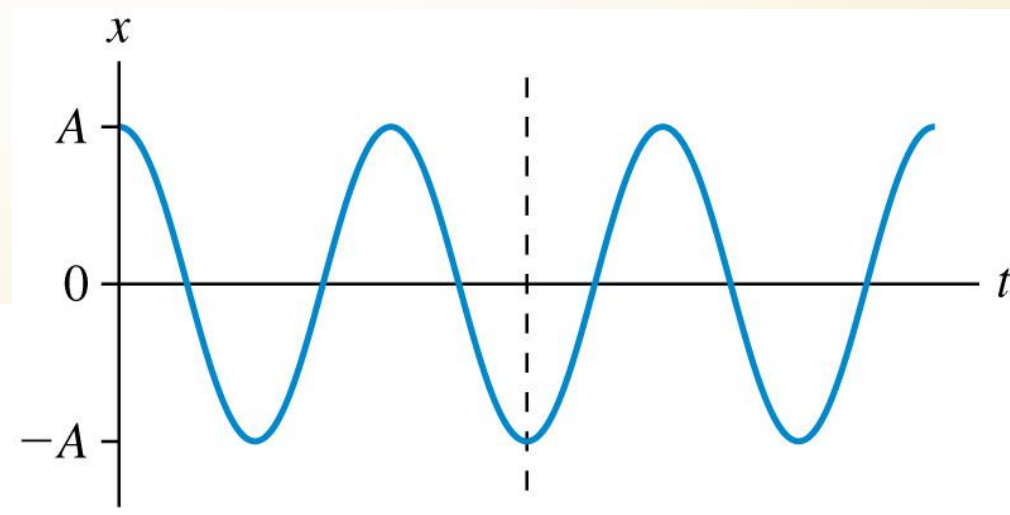
Combining,

$$v_{max} = 2\pi \frac{A}{T}$$

Quiz

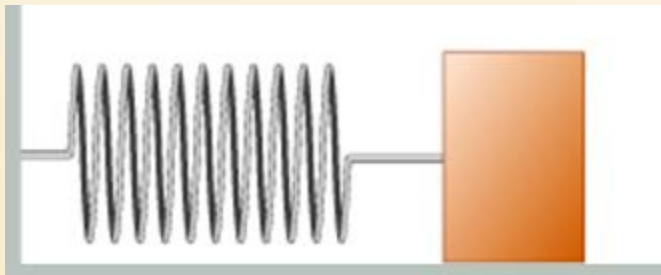


This is the position graph of a mass on a spring. What can you say about the velocity and the force at the instant indicated by the dotted line?



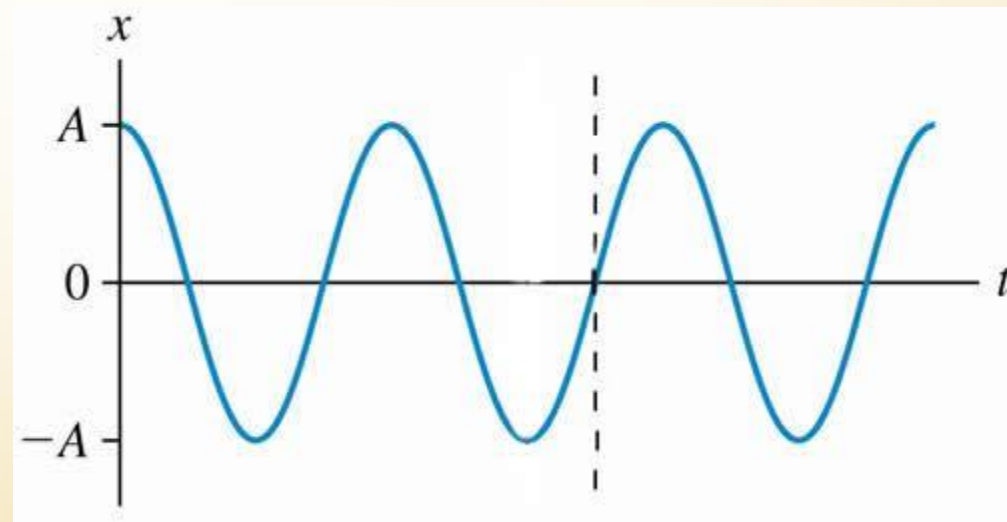
- A. Velocity is positive; force is zero.
- B. Velocity is negative; force is zero.
- C. Velocity is negative; force is to the right.
- D. Velocity is zero; force is to the right.**
- E. Velocity is zero; force is to the left.

Quiz



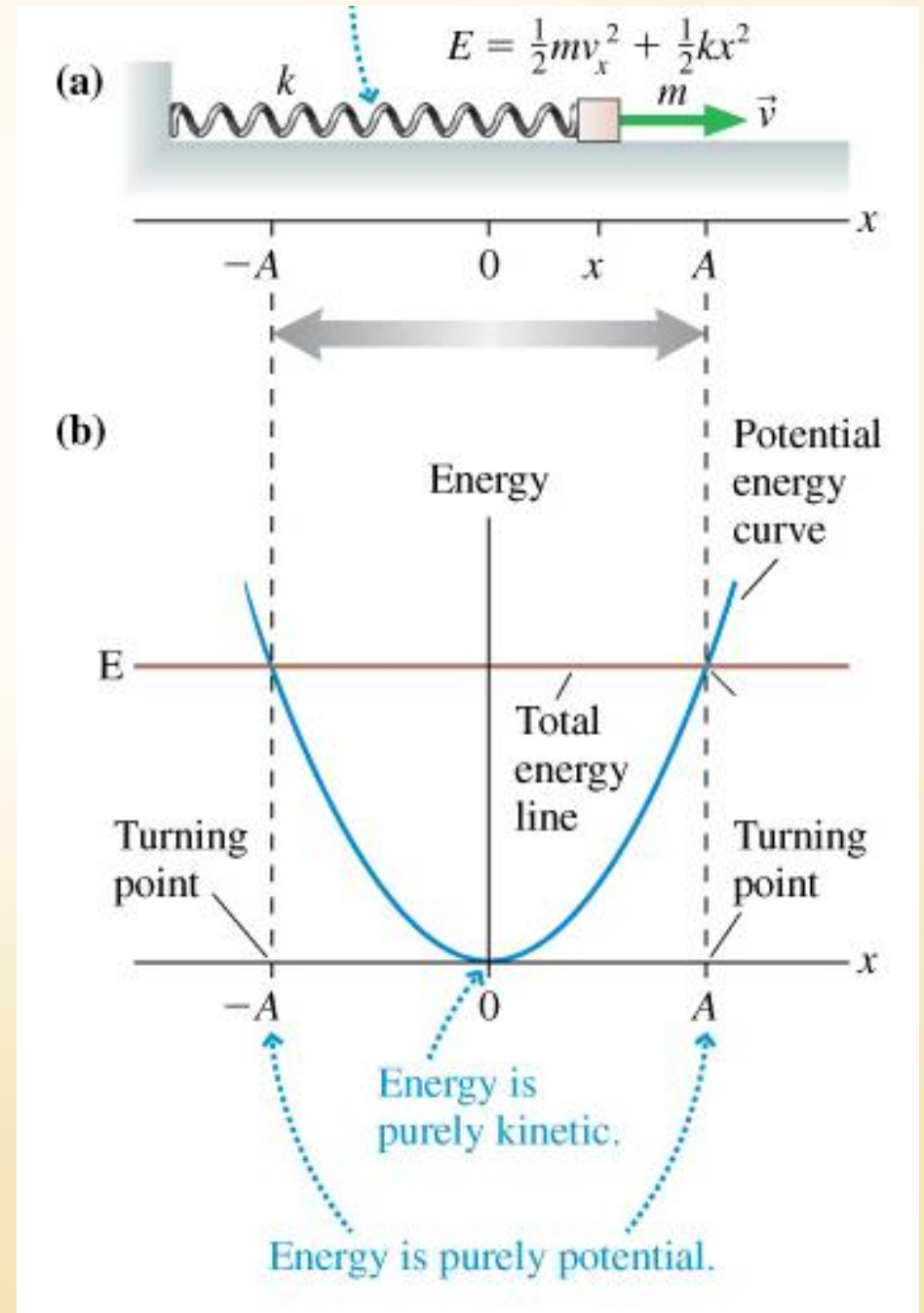
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- A. Velocity is positive; force is zero.
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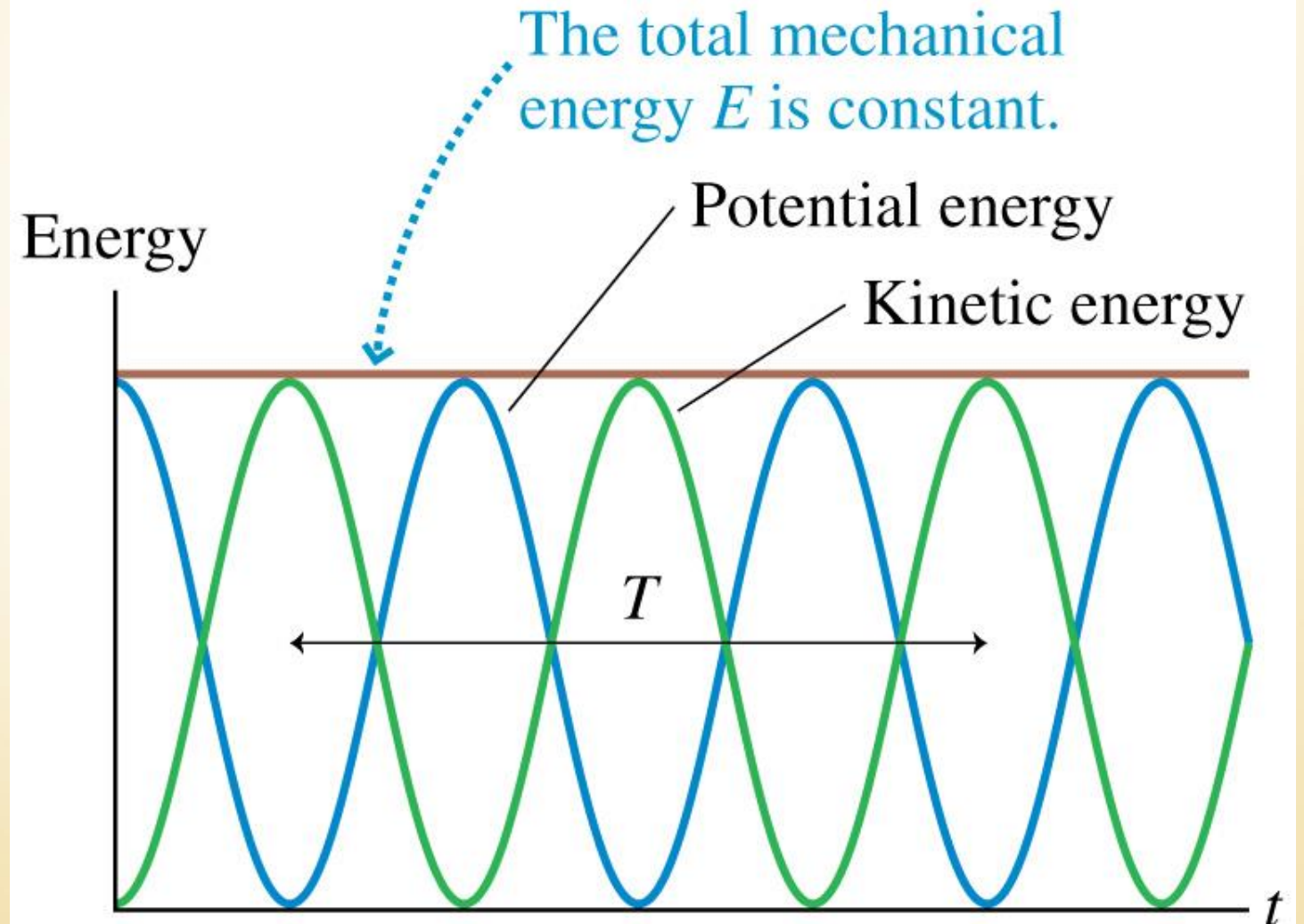
Energy consideration in SHM

- Energy is transferred between kinetic and potential.
- The total mechanical energy E does not change
- $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$



Energy consideration in SHM

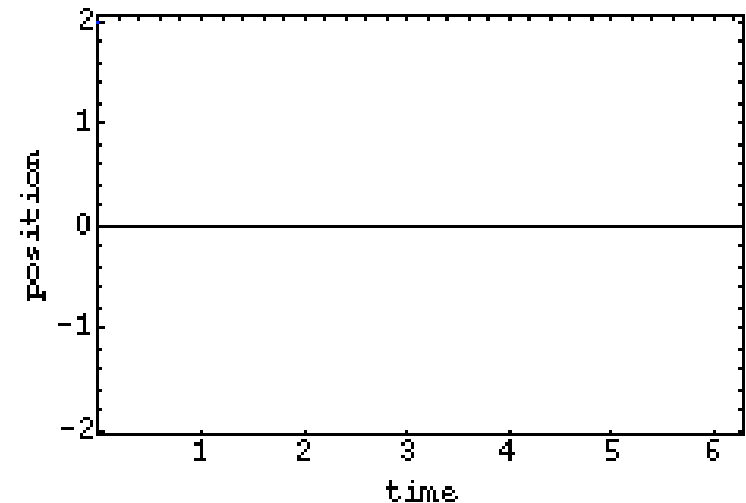
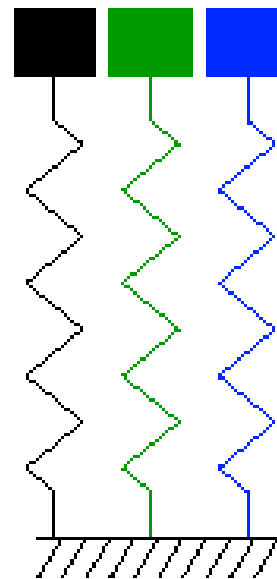
- $K(t) = \frac{1}{2}mv_t^2$
- $U(t) = \frac{1}{2}kx_t^2$



Quiz:

Compare the following spring-mass systems

- Period
- Frequency
- Spring constant
- KE
- PE



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EXAMPLE

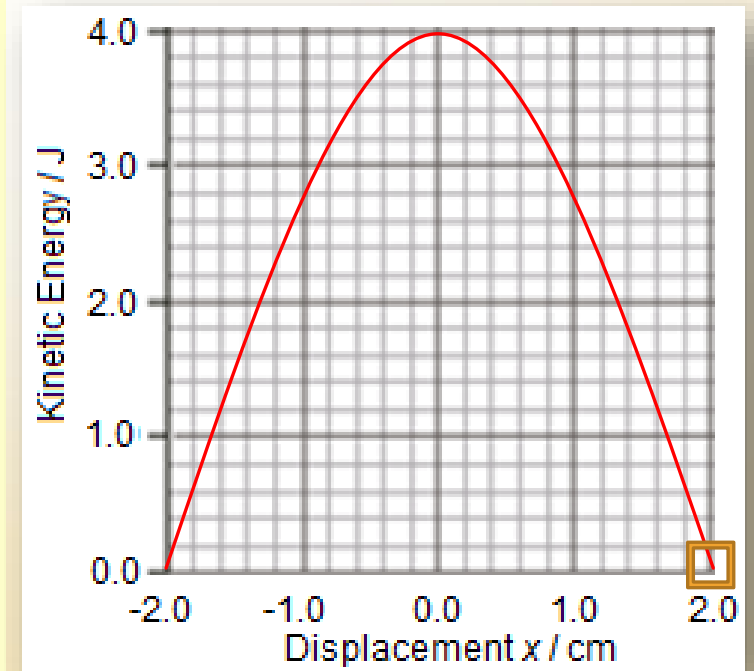
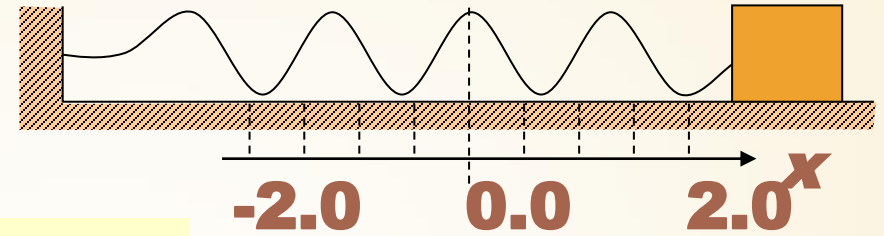
The kinetic energy vs. displacement for a system undergoing SHM is shown in the graph. The system consists of a 0.125-kg mass on a spring.

(a) Determine the maximum velocity of the mass.

SOLUTION:

When the kinetic energy is maximum, the velocity is also maximum. Thus $4.0 = \frac{1}{2}mv_{max}^2$ so that

$$4.0 = \frac{1}{2}(0.125)v_{max}^2 \quad v_{max} = 8.0 \text{ m s}^{-1}.$$



EXAMPLE

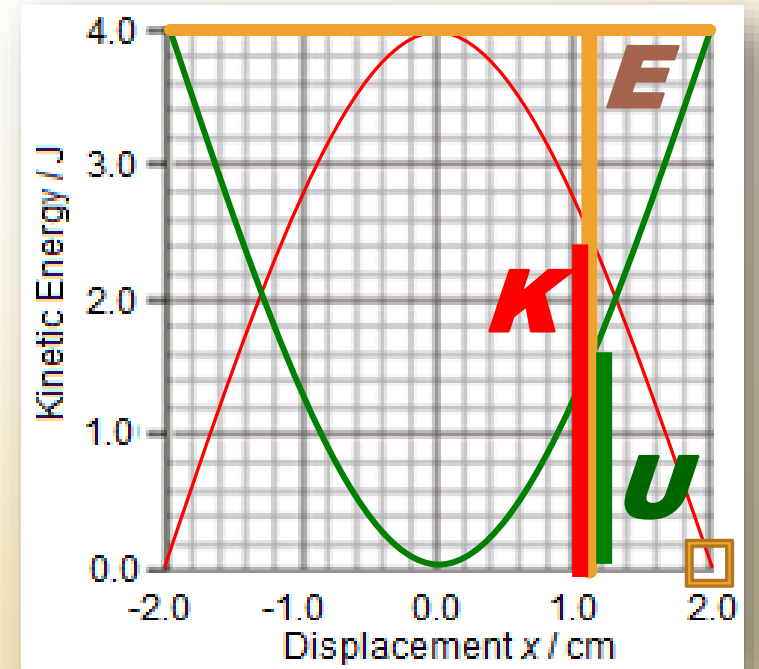
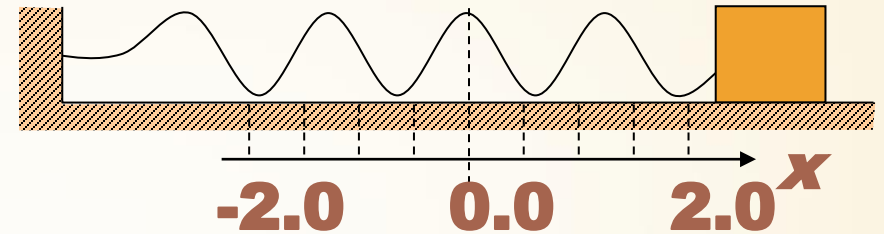
The kinetic energy vs. displacement for a system undergoing SHM is shown in the graph. The system consists of a 0.125-kg mass on a spring.

(b) Sketch E and determine the total energy of the system.

SOLUTION:

$K + U = E$ and since $U = 0$ when K is max,
 $E = K_{max} = 4.0 \text{ J}$.

The U graph will be the “inverted” K graph.



EXAMPLE

The kinetic energy vs. displacement for a system undergoing SHM is shown in the graph. The system consists of a 0.125-kg mass on a spring.

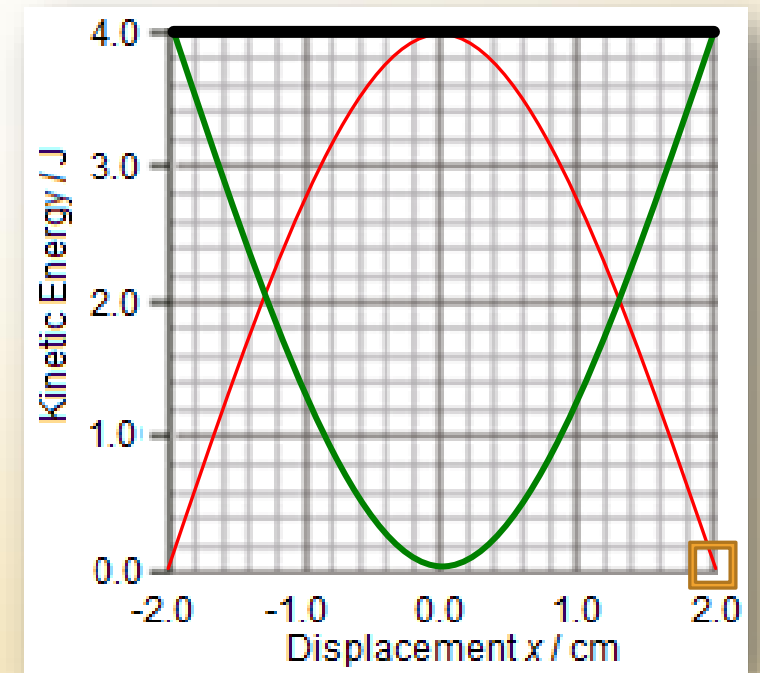
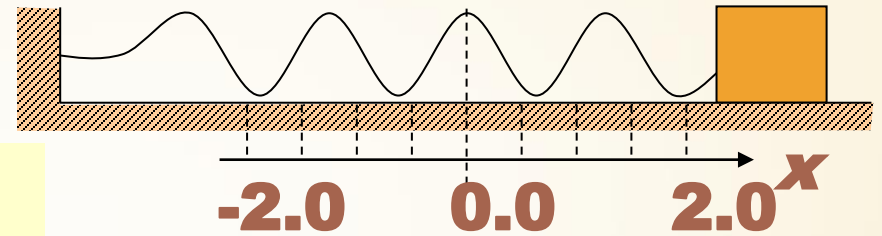
(c) Determine the spring constant k of the spring.

SOLUTION: Recall $U = \frac{1}{2}kx^2$.

Note that $K = 0$ at $x = A = 2.0 \text{ cm}$. Thus

$$K + U = E \Rightarrow E = 0 + \frac{1}{2}kA^2$$

$$4.0 = \frac{1}{2}k(0.020^2) \Rightarrow k = 20000 \text{ Nm}^{-1}.$$



EXAMPLE

The kinetic energy vs. displacement for a system undergoing SHM is shown in the graph. The system consists of a 0.125-kg mass on a spring.

(d) Determine the acceleration of the mass at $x = 1.0$ cm.

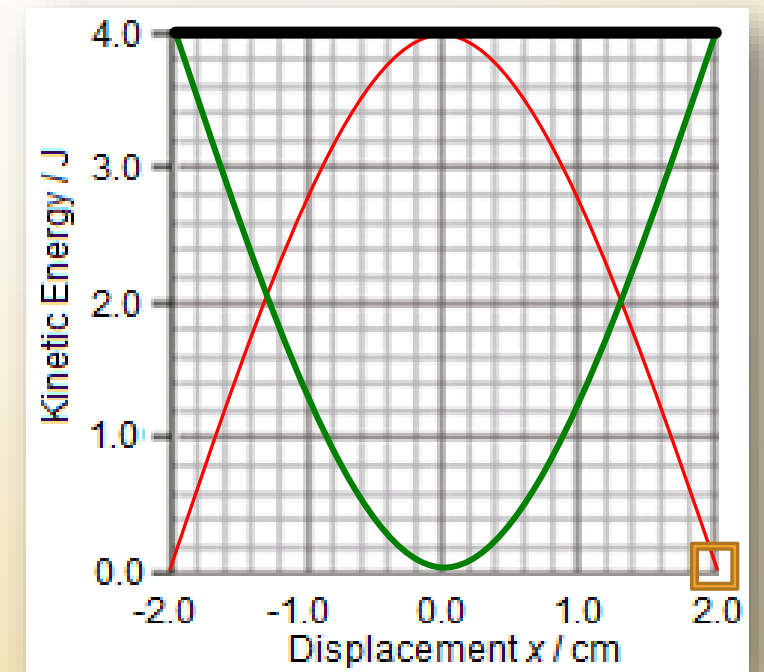
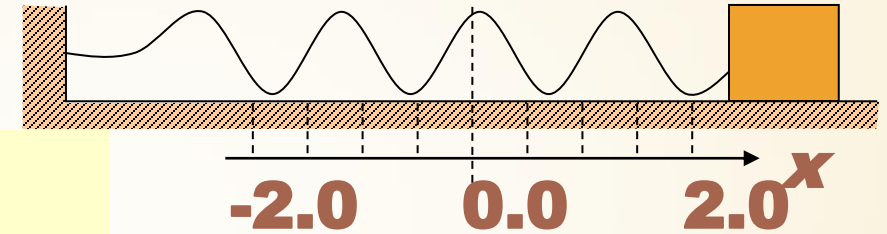
SOLUTION:

From Hooke's law, $F = -kx$

We get: $F = -20000(0.01) = -200$ N.

From $F = ma$

we get $a = \frac{F}{m} = -\frac{200}{0.125} = -1600$ m/s².



EXAMPLE

A 4.0-kg mass is placed on a spring's end and displaced 2.0 m to the right. The spring force F vs. its displacement x from equilibrium is shown in the graph.

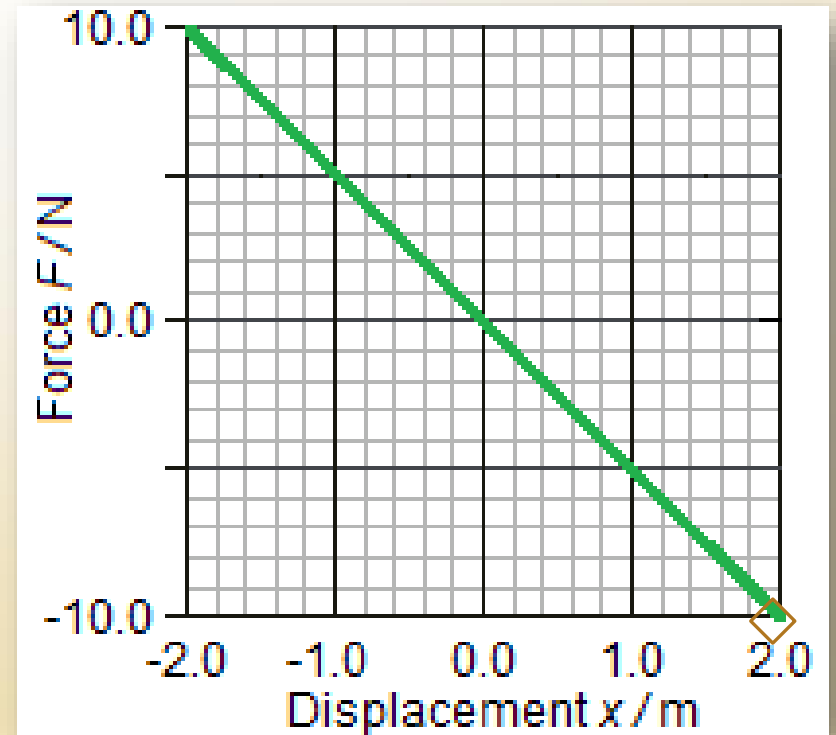
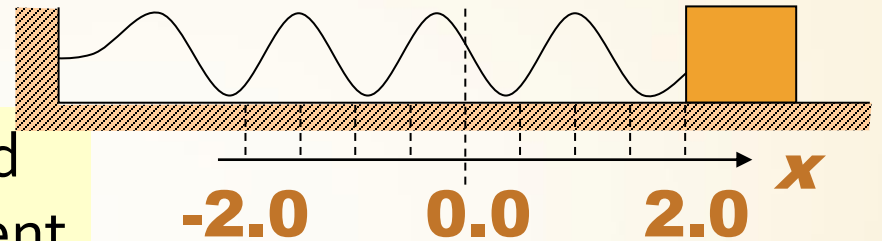
(a) How do you know that the mass is undergoing SHM?

SOLUTION:

In SHM, $F \propto -x$.

The graph also shows that $F \propto -x$.

Thus we have SHM.



EXAMPLE

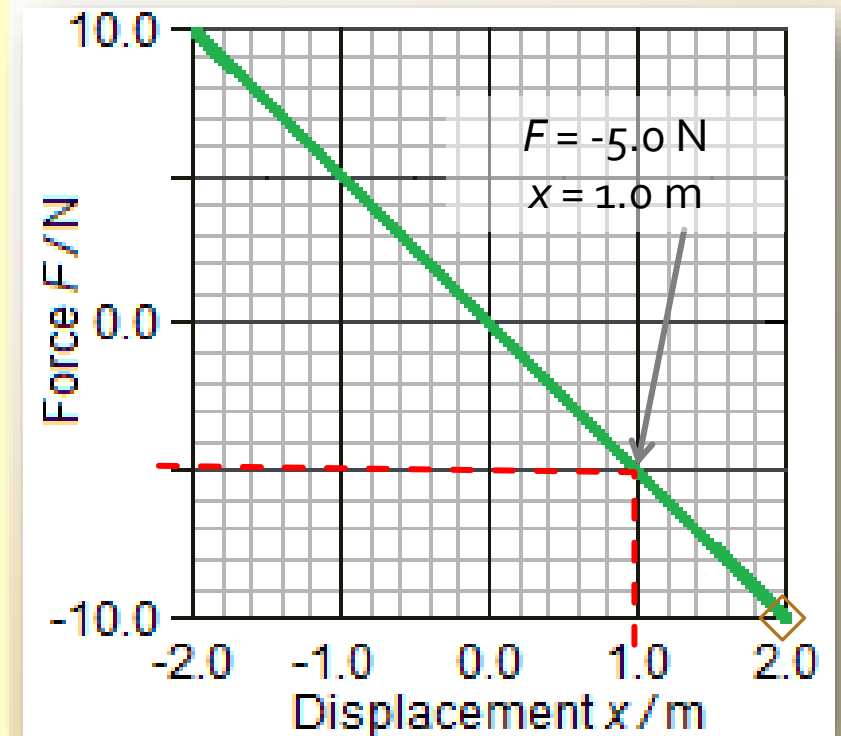
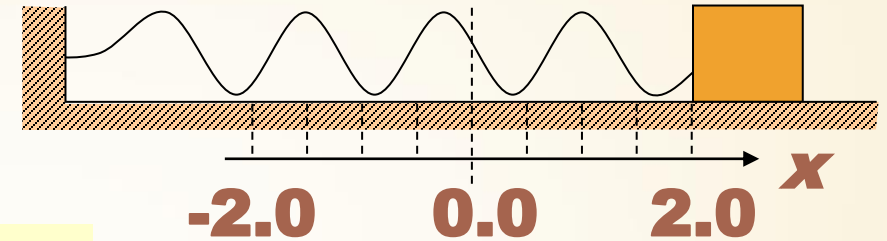
A 4.0-kg mass is placed on a spring's end and displaced 2.0 m to the right. The spring force F vs. its displacement x from equilibrium is shown in the graph.

(b) Find the spring constant of the spring.

SOLUTION: Use Hooke's law: $F = -kx$

Pick any F and any x . e.g., $F = -5.0 \text{ N}$, $x = 1.0 \text{ m}$

$$\text{Then } k = -\frac{F}{x} = -\frac{-5.0 \text{ N}}{1.0 \text{ m}} = 5.0 \text{ Nm}^{-1}$$



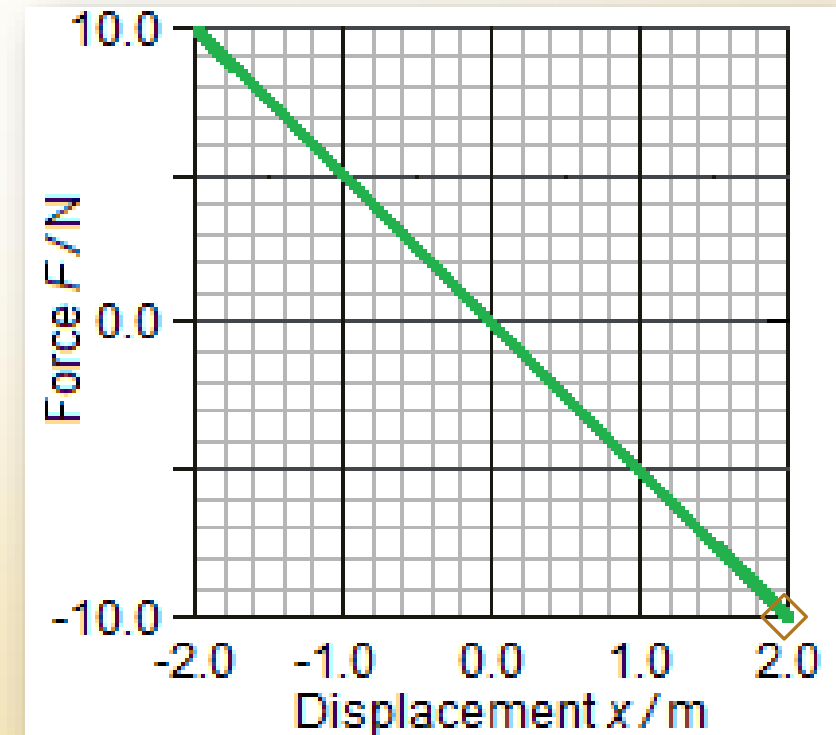
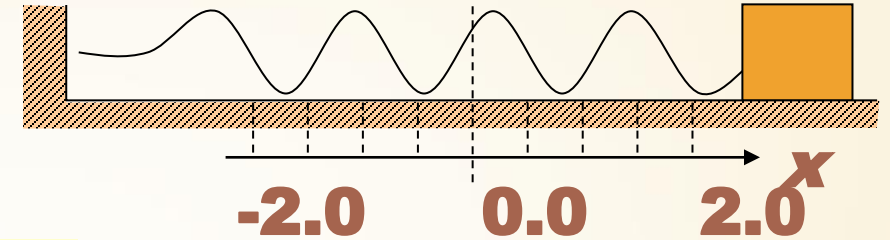
EXAMPLE

A 4.0-kg mass is placed on a spring's end and displaced 2.0 m to the right. The spring force F vs. its displacement x from equilibrium is shown in the graph.

(c) Find the total energy of the system.

SOLUTION: Use $E = \frac{1}{2}kA^2$. Then

$$E = \frac{1}{2}(5.0)2.0^2 = 10 \text{ J}$$



EXAMPLE

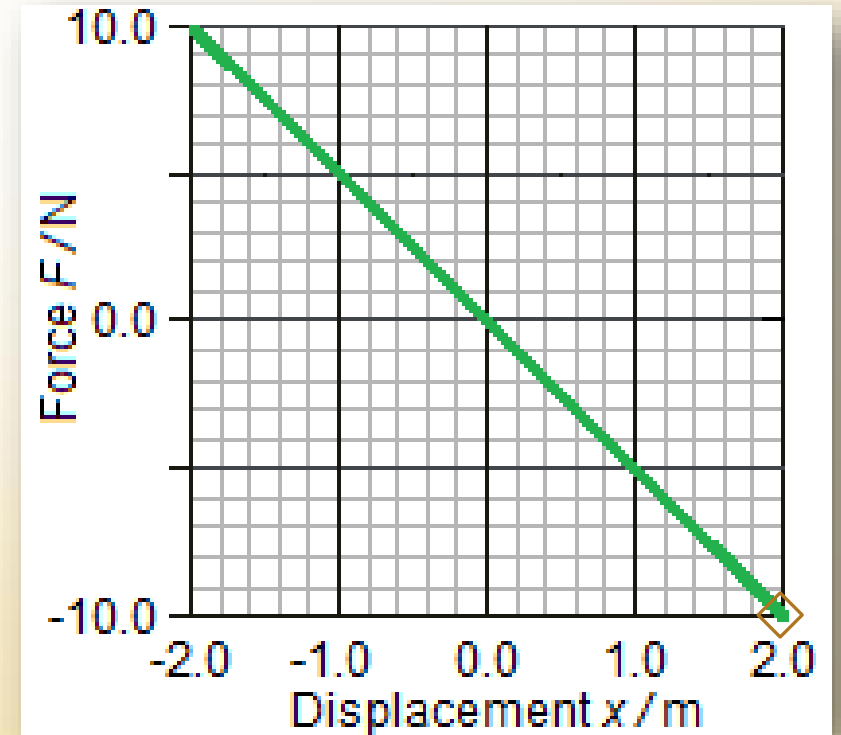
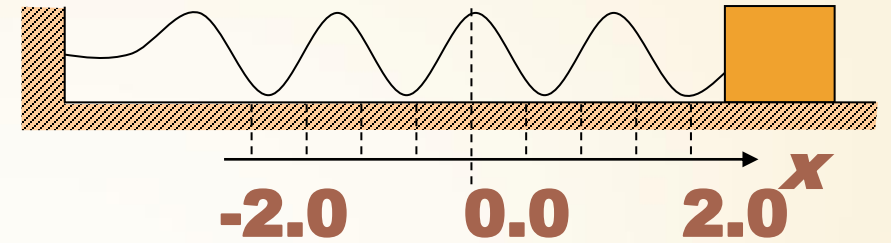
A 4.0-kg mass is placed on a spring's end and displaced 2.0 m to the right. The spring force F vs. its displacement x from equilibrium is shown in the graph.

(d) Find the maximum speed of the mass.

SOLUTION: Use $E = \frac{1}{2}mv_{max}^2$.

$$10 = \frac{1}{2}(4.0)v_{max}^2$$

$$v_{max} = 2.2 \text{ m s}^{-1}.$$



EXAMPLE

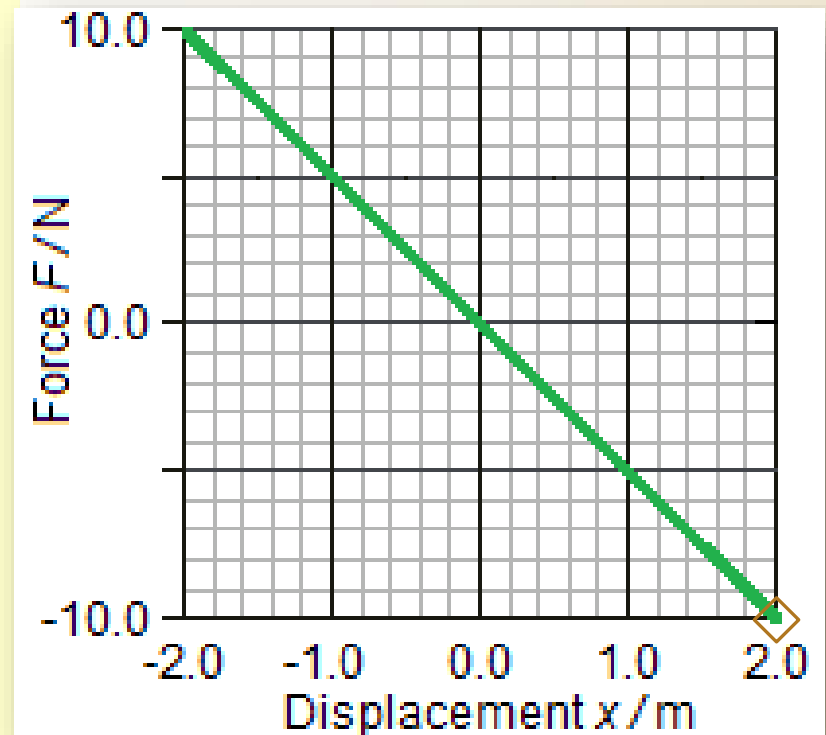
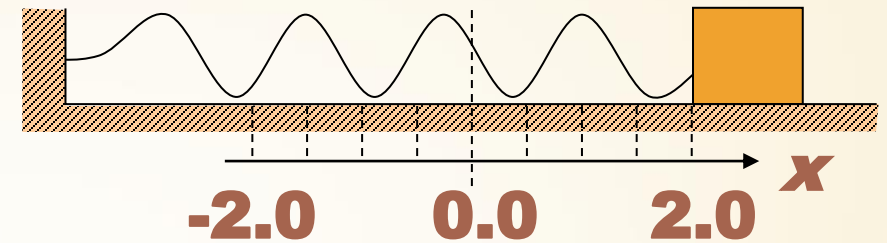
A 4.0-kg mass is placed on a spring's end and displaced 2.0 m to the right. The spring force F vs. its displacement x from equilibrium is shown in the graph.

(e) Find the speed of the mass when its displacement is 1.0 m.

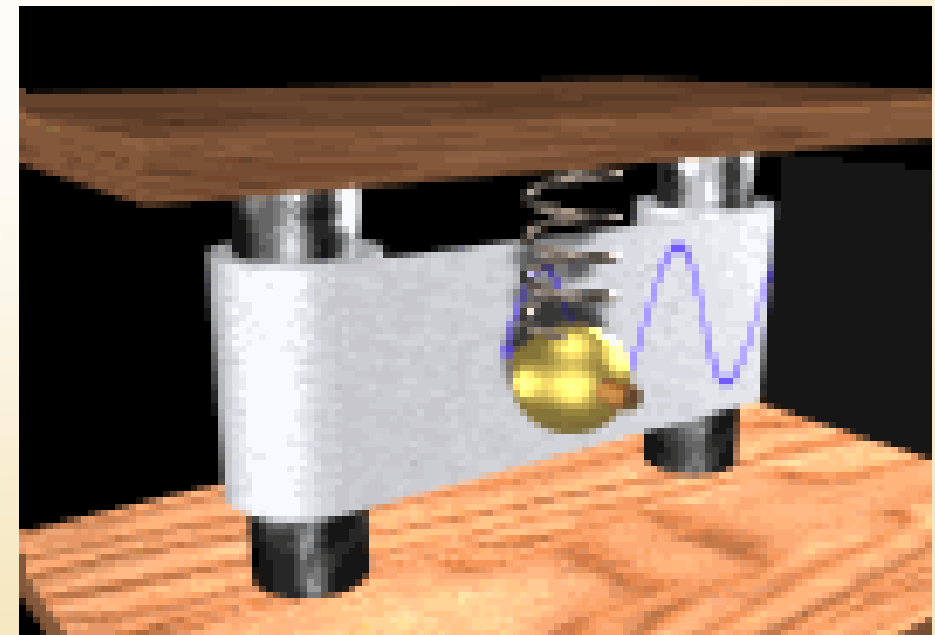
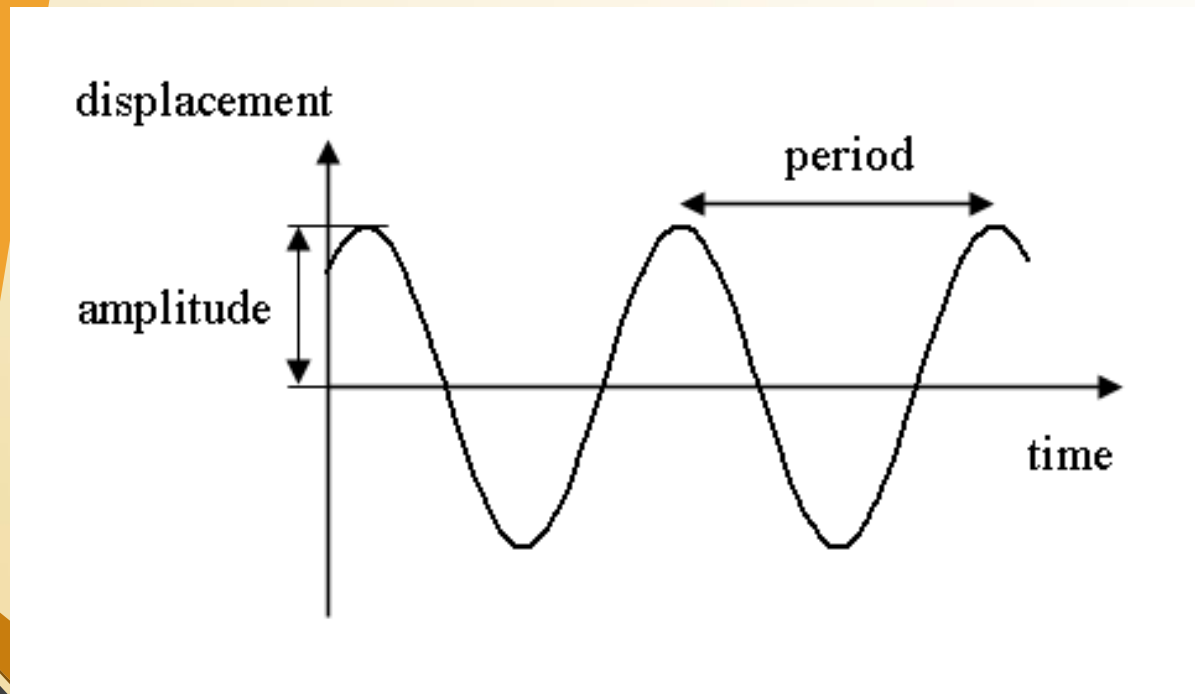
SOLUTION: Use $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. Then

$$10 = \frac{1}{2}(4)v^2 + \frac{1}{2}(5)1^2$$

$$v = 1.9 \text{ ms}^{-1}.$$



Graphical Treatment Equations of SHM

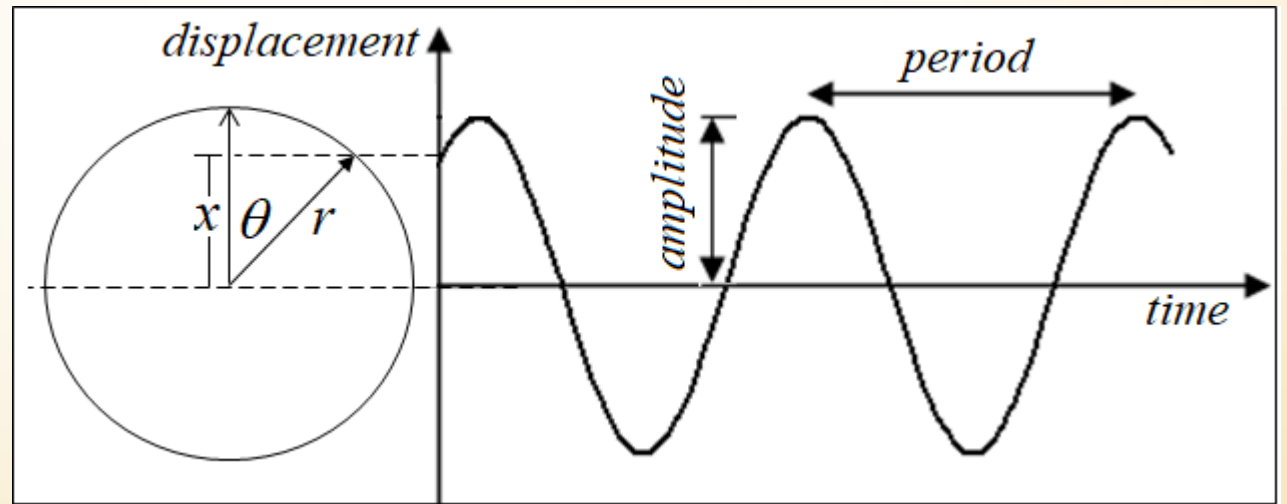


Displacement as a function of time

$$x(t) = A \cos \omega t$$

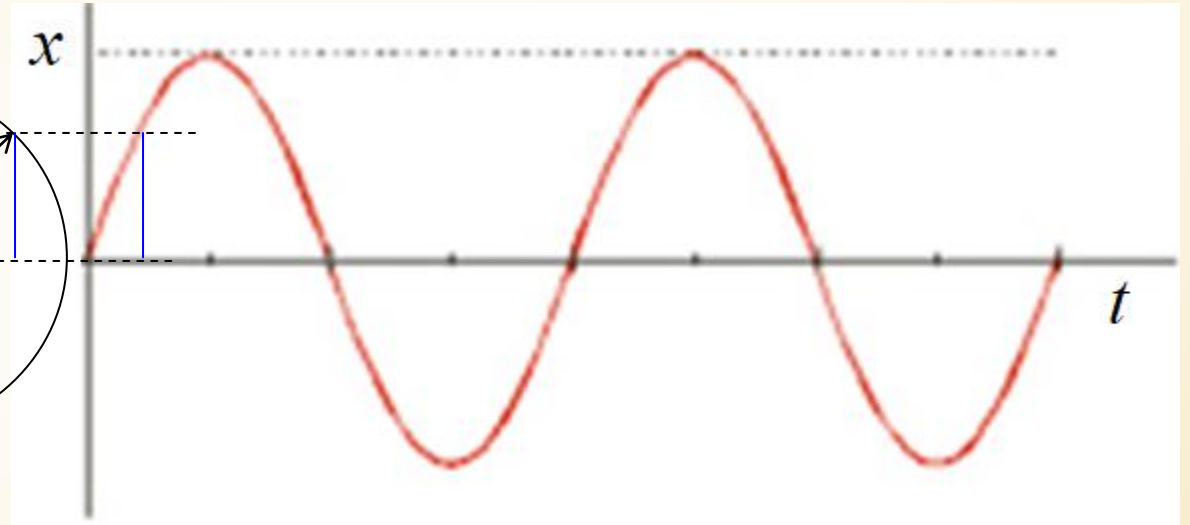
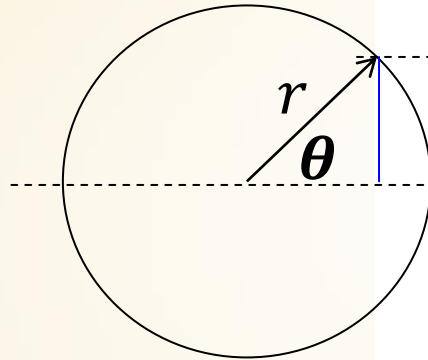
When $t = 0$, $x(0) = A$

Mass starts from maximum displacement



Displacement vs time: $x = A \sin \omega t$

- Mass starts from equilibrium position at $t = 0$



- $v_0 = 2\pi r/T$ but $2\pi/T = \omega$ so
- $v_0 = \omega r$ but $r = A$ and $v_0 = v_{max}$.
- $v_{max} = \omega A$.

- *using* $v = \pm \omega \sqrt{A^2 - x^2}$
- $v = \pm \omega \sqrt{A^2 - A^2 \sin^2 \omega t}$
- $v = \pm \omega A \cos \omega t = \pm v_{max} \cos \omega t$

Velocity against time: $v(t) = v_0 \cos \omega t$

Starting where?

Equilibrium position where velocity is maximum.



Equations of SHM

- $x = A \cos \omega t$
 - $v = -\omega A \sin \omega t$
 - $a = -\omega^2 A \cos \omega t = -\omega^2 x$
 - Released from max displacement
- $x = A \sin \omega t$
 - $v = \omega A \cos \omega t$
 - $a = -\omega^2 A \sin \omega t = -\omega^2 x$
 - Shot from equilibrium

Equations for SHM

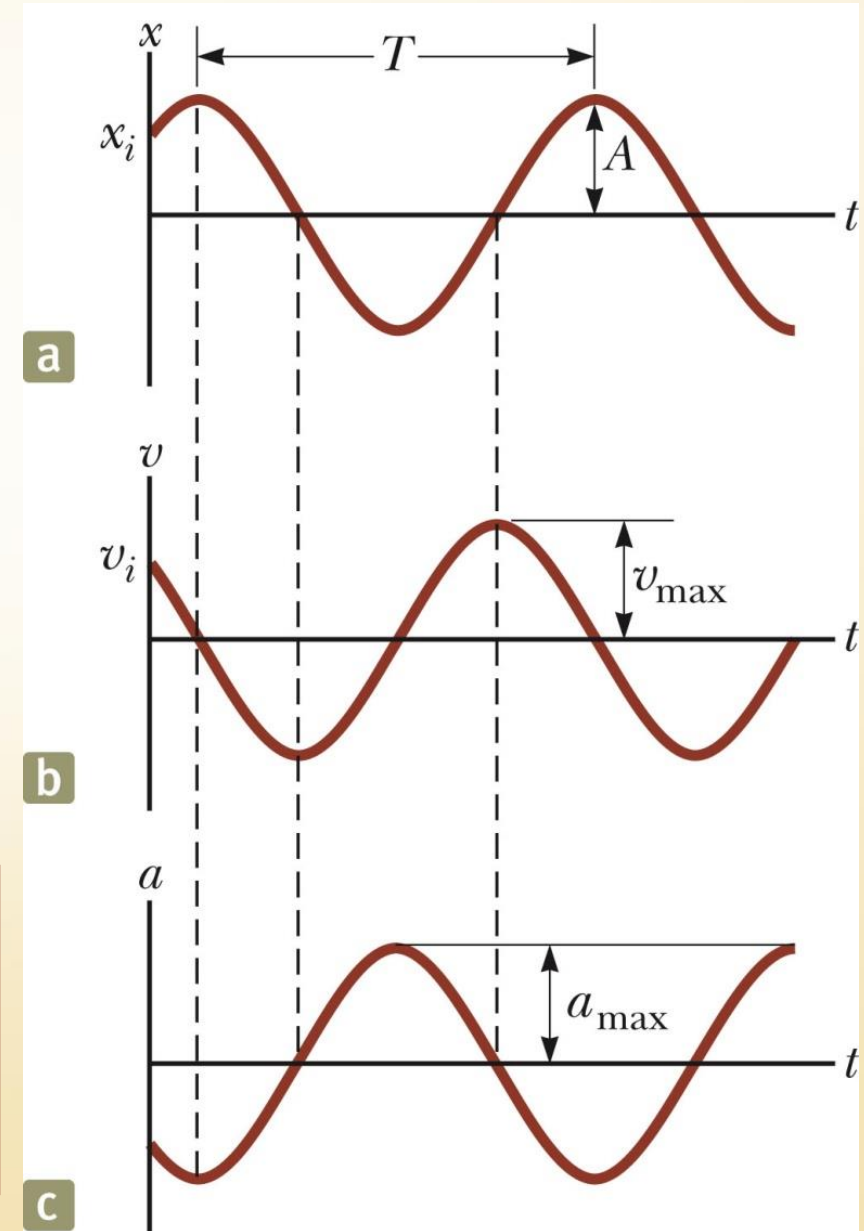
$$x(t) = A \cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x$$

The velocity is $\pi/2$ rad (90°) out of phase with the displacement and the acceleration is (π rad) 180° out of phase with the displacement.

SEE SIMULATIONS



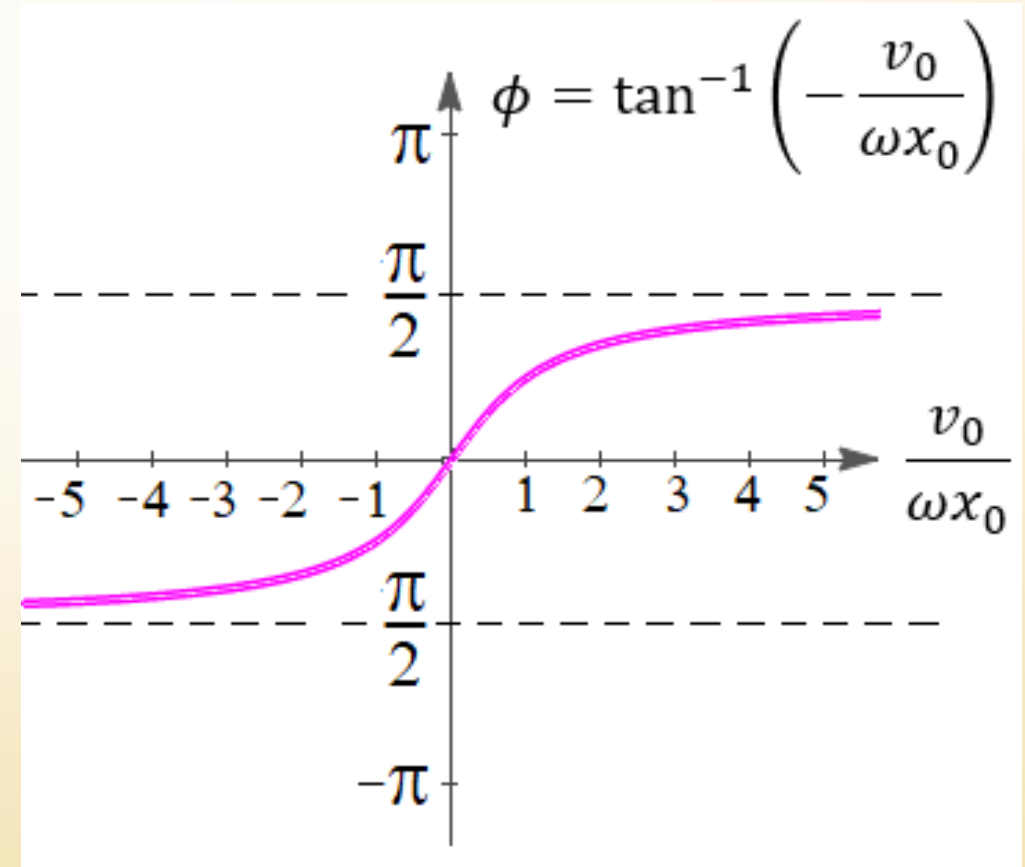
Determining the amplitude and phase from the initial conditions

- At $t = 0$ are
 - $x(0) = x_0 = A \cos \phi$
 - $v(0) = v_0 = -\omega A \sin \phi$

- Solving for A and ϕ :

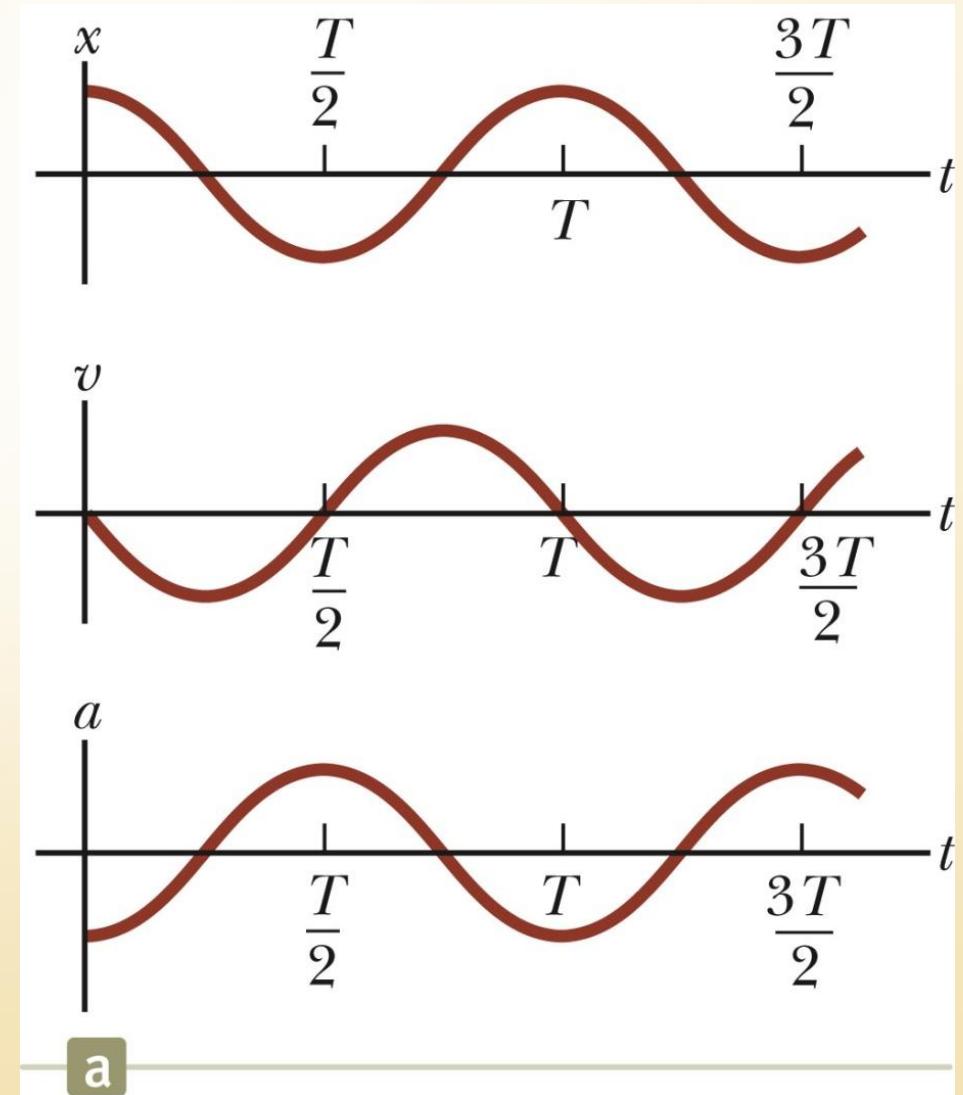
$$A^2 = x_0^2 + \frac{v_0^2}{\omega^2}$$

$$\phi = \tan^{-1} \left(-\frac{v_0}{\omega x_0} \right)$$



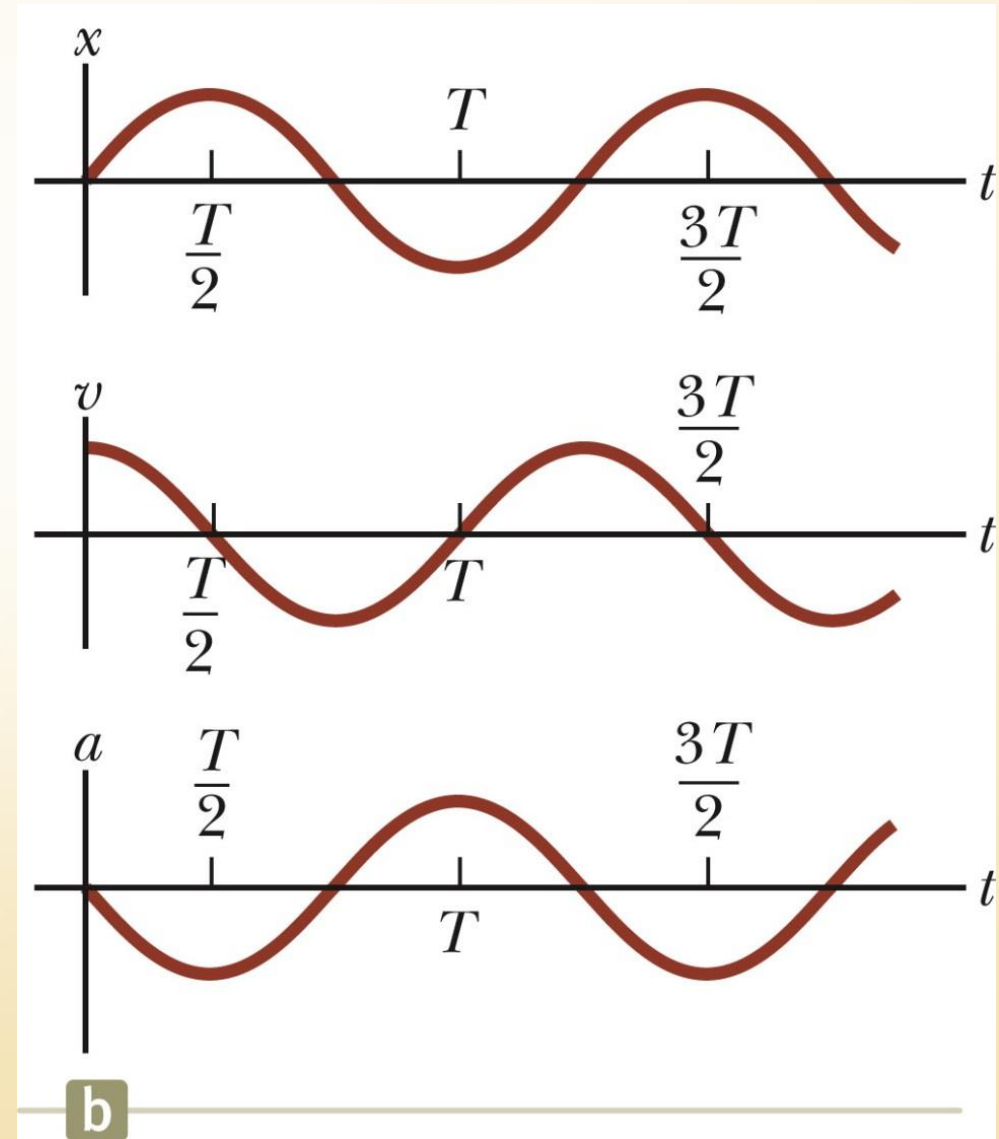
SHM Example 1

- Initial conditions at $t = 0$ are
 - $x(0) = A$
 - $v(0) = 0$
- This means $\phi = 0$
- The acceleration reaches extremes of $\pm\omega^2 A$ at $\pm A$.
- The velocity reaches extremes of $\pm\omega A$ at $x = 0$.



SHM Example 2

- Initial conditions at $t = 0$ are
 - $x(0) = 0$
 - $v(0) = v_{max}$
- This means $\phi = -\frac{\pi}{2}$
- The graph is shifted one-quarter cycle to the right compared to the graph of $x(0) = A$.



EXAMPLE

A mass on a spring is oscillating with $f = 0.2$ Hz and $A = 3$ cm. What is the displacement of the mass 10.66 s after its release from the maximum displacement?

- $x(t) = A \cos \omega t$ $A = 3$ cm
 - $\omega = 2\pi f = 2\pi(0.2) \text{ Hz} = 1.26 \text{ rad/s.}$
 - $t = 10.66 \text{ s}$
- $x = 0.03 \cos (1.26 \times 10.66) = 0.019 \text{ m}$

EXAMPLE

A mass attached to a spring oscillates with a period of 0.20 s. What is its acceleration at a point where its displacement is 1.5 cm?

- $a = -\omega^2 x$
- $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.20} = 3.14 \text{ rad/s}$
- $a = -\left(3.14 \frac{\text{rad}}{\text{s}}\right)^2 (1.5 \text{ cm}) = 14.80 \text{ cm/s}^2.$

Example

- A 2.00-kg block is placed on a frictionless surface. A spring with a force constant of $k=32.00\text{N/m}$ is attached to the block, and the opposite end of the spring is attached to the wall. The equilibrium position is marked as $x=0.00\text{m}$. The block is pulled out to $x=+0.02\text{m}$ and then released from rest. The period of the motion is 1.57 s . Determine the equations of motion.
- **SOLUTION**
 - $\omega = 2\pi/T = 2\pi/1.57 = 4.00/\text{s}, \quad A = 0.02\text{ m}, \quad \phi = 0 \text{ (from rest at } x_{\text{max}})$
 - $v_{\text{max}} = \omega A = 4.00 (0.02) = 0.08\text{ m/s},$
 - $a_{\text{max}} = \omega^2 A = 4.00^2(0.02) = 0.32\text{ m/s}^2$

Example

- A 2.00-kg block is placed on a frictionless surface. A spring with a force constant of $k=32.00\text{N/m}$ is attached to the block, and the opposite end of the spring is attached to the wall. The equilibrium position is marked as $x=0.00\text{m}$. The block is pulled out to $x=+0.02\text{m}$ and then released from rest. The period of the motion is 1.57 s. Determine the equations of motion.
- **SOLUTION (equations of motion)**
 - $x(t) = A \cos (\omega t + \phi) = 0.02 \text{ m} \cos (4.00t)$
 - $v(t) = -\omega A \sin (\omega t + \phi) = -0.08 \text{ m/s} \sin (4.00t)$
 - $a(t) = -\omega^2 A \cos (\omega t + \phi) = -0.32 \text{ m/s}^2 \cos (4.00t)$

Exercises

1. The period of oscillation of an object in an ideal mass-spring system is 0.50 sec and the amplitude is 5.0 cm. What is the speed at the equilibrium point?
2. The diaphragm of a speaker has a mass of 50.0 g and responds to a signal of 2.0 kHz by moving back and forth with an amplitude of 0.18 mm at that frequency.

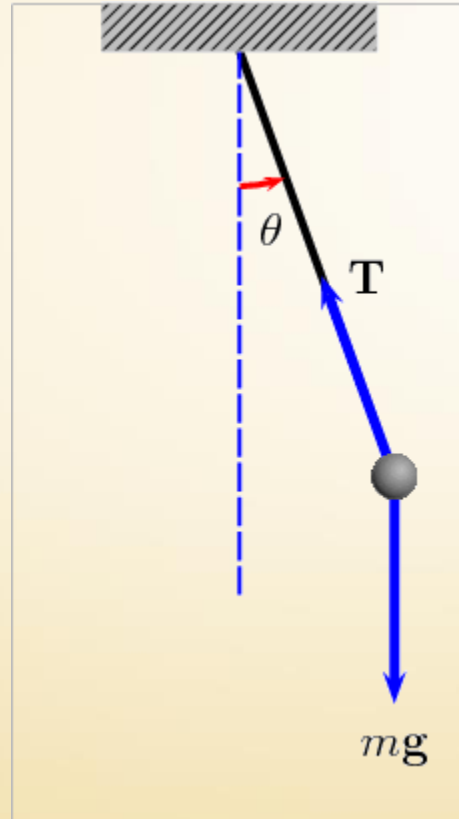
- a) What is the maximum force acting on the diaphragm?
- b) What is the mechanical energy of the diaphragm?

3. The displacement of an object in SHM is given by:

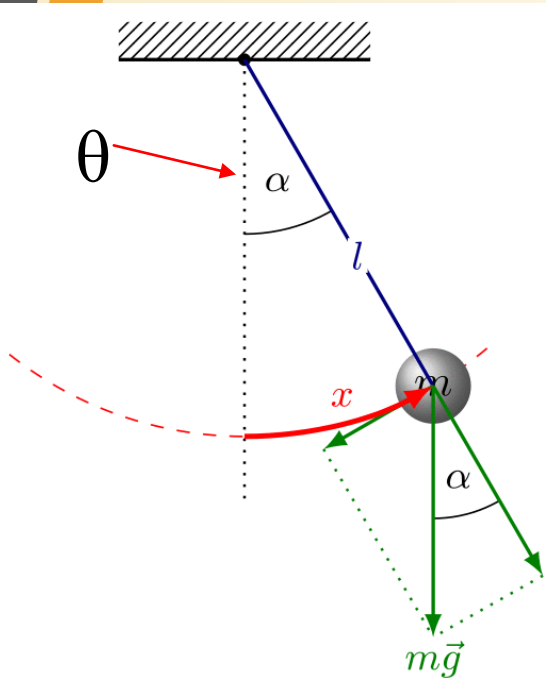
$$y(t) = (8.00 \text{ cm}) \sin[(1.57 \text{ rads/sec}) t]$$

- a) What is the frequency of the oscillations?
 - b) What are the maximum speed and maximum acceleration?
4. A particle executes simple harmonic motion with an amplitude of 3.00 cm. At what displacement from the midpoint of its motion does its speed equal one half of its maximum speed?

The Simple Pendulum



Equation of Motion for the Simple Pendulum



- Along the radial direction: $F - mg \cos \theta = m \frac{v^2}{l}$
- Along the tangential direction: $-mg \sin \theta = ma_t$
- If the angle α is small, $\sin \theta \approx \theta$ (in radian measures)
- and with $a_t = l\alpha$
- We get $l\alpha = -g\theta$ or $\alpha = -\frac{g}{l}\theta = -\omega^2\theta$ or $a_t = -\omega^2x$
- Where $\omega = \sqrt{\frac{g}{l}}$ So, $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$

Example

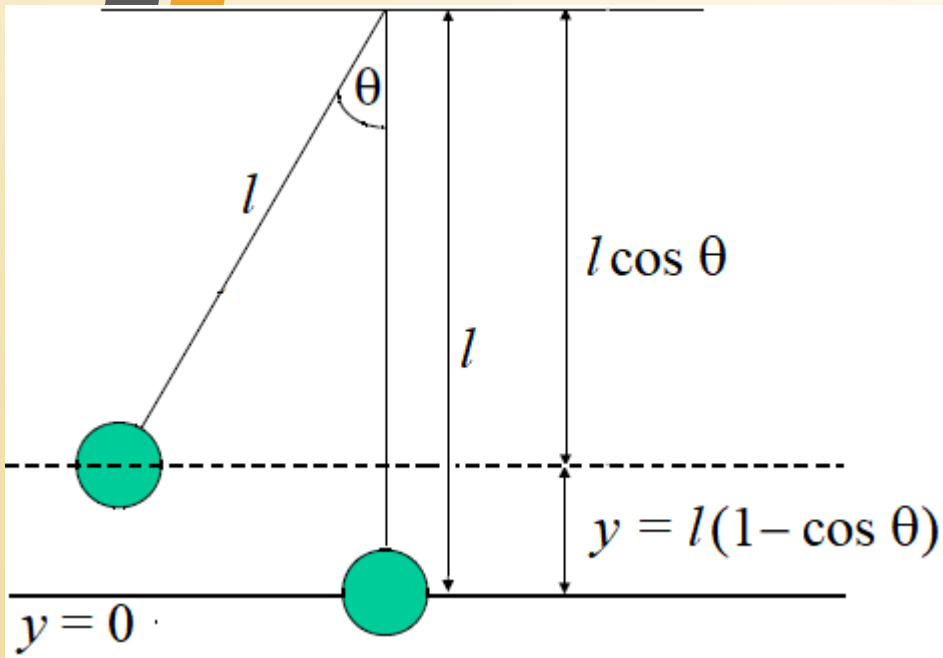
- A clock has a pendulum that performs one full swing every 1.0 sec. The object at the end of the string weighs 10.0 N. What is the length of the pendulum?
- SOLUTION

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$$

- Solving for l

$$l = g \frac{T^2}{4\pi^2} = 9.8 \times \frac{(1.0)^2}{4\pi^2} = 0.25 \text{ m}$$

Energy Considerations



- Conservation of mechanical energy gives

$$\frac{1}{2}mv^2 + mgl(1 - \cos \theta) = mgl(1 - \cos \theta_{max})$$

$$v^2 = l^2 \dot{\theta}^2 = 2gl(\cos \theta - \cos \theta_{max})$$

$$\dot{\theta} = \pm \sqrt{2 \frac{g}{l} \left[\left(1 - \frac{\theta^2}{2}\right) - \left(1 - \frac{\theta_{max}^2}{2}\right) \right]} \quad \text{for } \theta \ll 1 \text{ rad}$$

$$\dot{\theta} = \pm \sqrt{\frac{g}{l} (\theta_{max}^2 - \theta^2)} = \pm \omega \sqrt{A^2 - \theta^2}$$

- At equilibrium, $\dot{\theta}_{max} = \omega \theta_{max}$
- At maximum displacement, $\dot{\theta} = 0$

Example

- A pendulum of length 0.5 m swings with $f = 0.5$ Hz. What is the size and direction of the tangential/angular acceleration when the bob has a displacement of 2 cm to the right?
- **SOLUTION**

$$a_t = -\omega^2 x \quad \text{and} \quad \omega = 2\pi f = 2\pi \left(\frac{1}{2}\right) = \pi$$

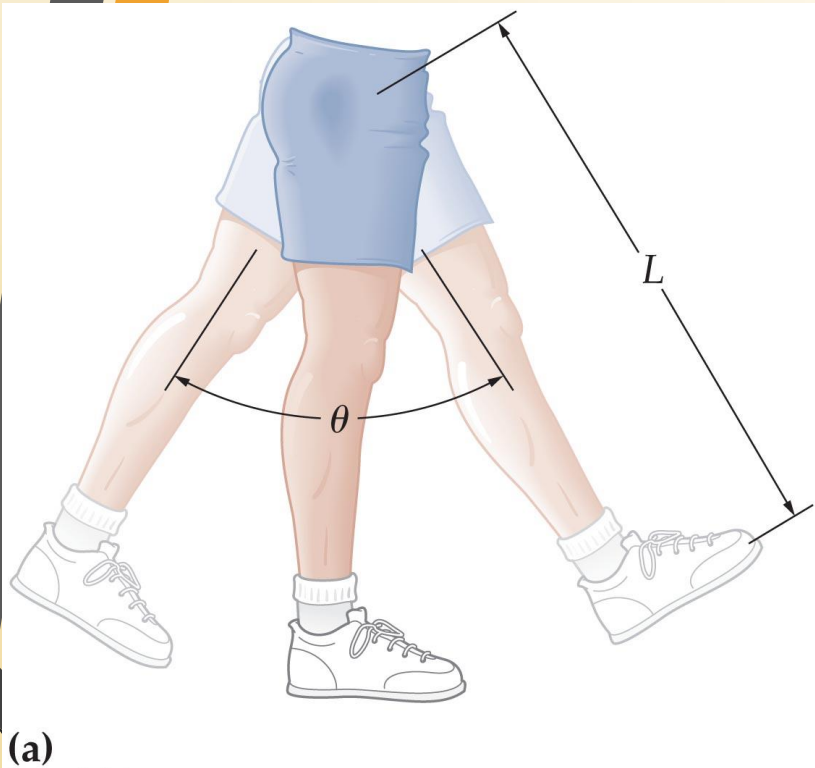
$$a_t = -\pi^2 (0.02 \text{ m}) = -0.20 \text{ m/s}^2 \text{ left.}$$

$$\alpha = \frac{a_t}{l} = \frac{-0.20}{0.5} = 0.4 \text{ s}^{-2}$$

Example

- A pendulum swings with $f = 1$ Hz and amplitude 3 cm. At what position will its speed be maximum and what is the speed?
- SOLUTION
- Use $v = \pm\omega\sqrt{A^2 - x^2}$
- *Speed is maximum at $x = 0$.*
- $\omega = 2\pi f = 2\pi(1) = 2\pi \text{ rad/s}$
- $v_{\max} = (2\pi \text{ s}^{-1})(0.03 \text{ m}) = 0.2 \text{ m/s}$

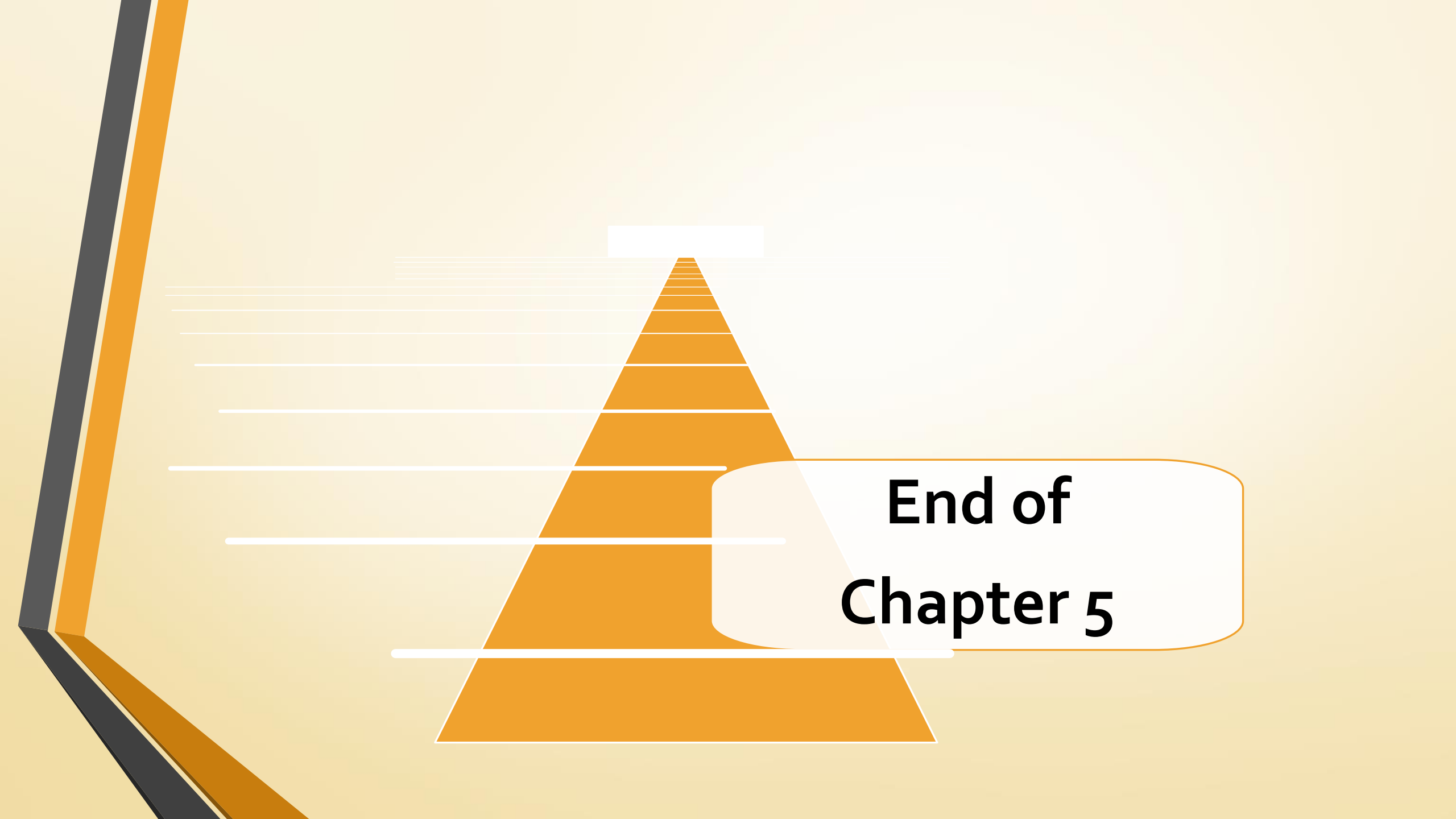
Brainteaser: The leg as a pendulum



- The longer/shorter the legs, the longer/shorter the period.
- The longer/shorter the legs, the longer/shorter the steps.
- Do animals with longer or shorter legs have greater walking speeds?
- Assuming the leg is oscillating about the hip like a simple pendulum with half length ($l = \frac{L}{2}$), estimate your own walking speed.

Exercises

1. A simple pendulum has a period of 2.50 s. (a) What is its length? (b) What would its period be on the Moon, where $g_{\text{Moon}} = 1.67 \text{ m/s}^2$?
2. If a simple pendulum with length 1.50 m makes 72.0 oscillations in 180 s, what is the acceleration of gravity at its location?
3. A simple pendulum having a length of 2.23 m and a mass of 6.74 kg is given an initial speed of 2.06 m/s at its equilibrium position. Assume it undergoes simple harmonic motion and determine its (a) period, (b) total energy, and (c) maximum angular displacement.
4. Martha has two pendula, one has a length of 1 meter, and the other one is longer. She sets them both swinging at the same time. when the 1 meter pendulum has completed 12 oscillations, the longer one has only completed 11. How long is the longer pendulum?



**End of
Chapter 5**