

Partial Differential Equations :

1) Order : order of the highest order derivative

Degree : degree of the HOD.

2) Linear PDE : (i) dep't variable z & its PD's occur in 1st degree only

(ii) No product of dependent variable or PD's.

Semi-Linear : $P(x, y) \frac{\partial z}{\partial x} + Q(x, y) \frac{\partial z}{\partial y} = R(x, y, z) + z^2$

Quasi-Linear : $P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z)$

3) $\frac{\partial z}{\partial x} = p$ $\frac{\partial z}{\partial y} = q$ $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial x \partial y} = s$ $\frac{\partial^2 z}{\partial y^2} = t$

4) Formulation of PDE :

(i) Elimination of arbitrary constants

$f(x, y, z, a, b) = 0$ where a & b are constants

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \quad \text{--- (1)}$$

Eliminate using (1) & (11).

NOTE : If (no of arbitrary constants = no of indept variables)
then PDE is of 1st order only. (11)

eg. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ a, b, c are constants.

We get $\frac{x}{a^2} + \frac{z}{c^2} = 0$ $\& \quad \frac{y}{b^2} + \frac{z}{c^2} = 0 \Rightarrow c^2 = -\frac{a^2 z}{x}$ or $-\frac{b^2 z}{y}$
(Put in (1)) or (Put in (11))

$$\frac{1}{a^2} + \frac{1}{c^2} p^2 + \frac{z}{c^2} r = 0 \quad \text{--- (1)}$$

$$\frac{1}{b^2} + \frac{1}{c^2} q^2 + \frac{z}{c^2} t = 0 \quad \text{--- (11)}$$

$$-\frac{p^2}{x} + p^2 + z r = 0 \quad \text{Similarly another PDE}$$

$$x z \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x} \right)^2 - \frac{z \partial z}{\partial x} = 0$$

(ii) Elimination of arbitrary function :

u, v, w are functions of x, y, z and they have a relation $f(u, v) = 0$

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0$$

compare the $\frac{\partial f}{\partial u} \mid -\frac{\partial f}{\partial v}$ ratios from both

we get $P \cdot p + Q \cdot q = R$ where $P = \frac{\partial(u, v)}{\partial(y, z)}$

$$Q = \frac{\partial(u, v)}{\partial(z, x)}$$

$$R = \frac{\partial(u, v)}{\partial(x, y)}$$

This is a linear PDE.

For more than 1 function, higher order may be there and it might not be linear PDE

Example: ST PDE for all cones with vertex at origin is

$$px + qy = z$$

cone: $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$

$$\Rightarrow (ax + hy + gz) + (gz + fy + cz)p = 0 \quad \text{--- (I)}$$

$$(hx + by + fz) + (gx + fy + cz)q = 0 \quad \text{--- (II)}$$

$$\text{Do } \textcircled{I} \times x + \textcircled{II} \times y \Rightarrow (gx + fy + cz)(px + qy) + (ax^2 + by^2 + 2hxy + gxz + fzy) = 0$$

Remember such manipulations

$$\Rightarrow (gx + fy + cz)(px + qy) + (-cz^2 - gxz - fzy) = 0$$

$$\Rightarrow (gx + fy + cz)(px + qy - z) = 0$$

$$\Rightarrow px + qy = z$$

Example : $\phi(x+y+z, x^2+y^2-z^2)=0$

$$\begin{aligned} \frac{\partial \phi}{\partial u}(1+p) + \frac{\partial \phi}{\partial v}(2x-2zp) &= 0 \\ \times \frac{\partial \phi}{\partial u}(1+q) + \frac{\partial \phi}{\partial v}(2y-2zq) &= 0 \end{aligned} \Rightarrow \frac{1+p}{x-zp} = \frac{1+q}{y-zq}$$

$$\Rightarrow y - pzq + py - zq = x - zp - \cancel{zq} + xz$$

$$\Rightarrow p(y+z) - q(x+z) = x-y$$

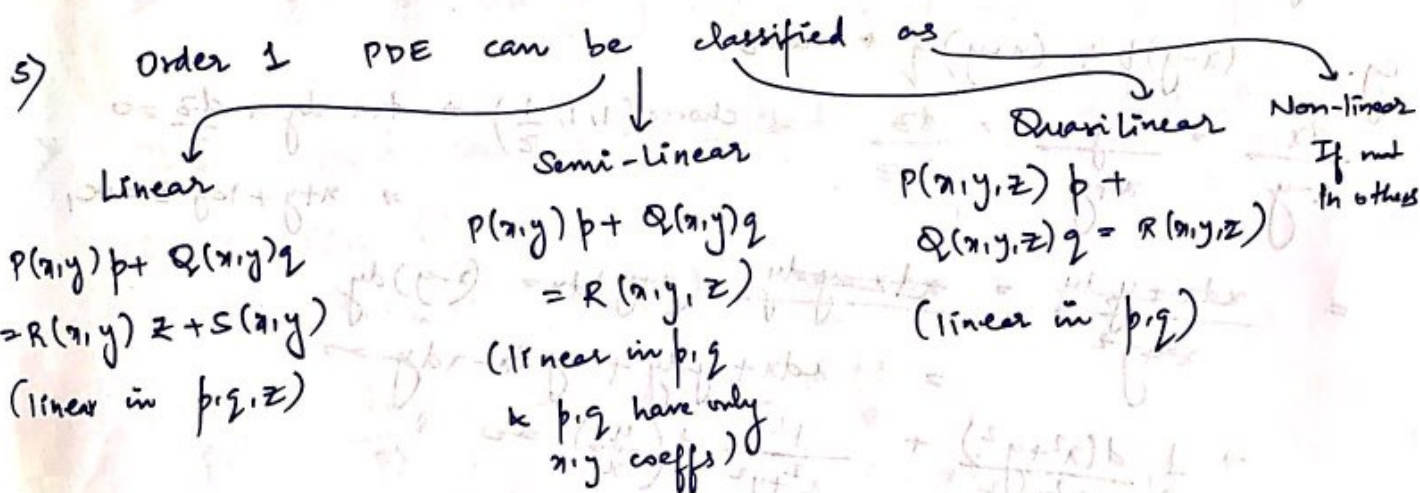
(iii) Equations solvable by direct integration:

eg. $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

$\Rightarrow$ Integrate twice w.r.t x keeping y const & then w.r.t y

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^2y^2 - \frac{1}{4}\sin(2x-y) = x f(y) + g(y)$$

$$\Rightarrow z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x \underbrace{\int f(y)dy}_{u(y)} + \underbrace{\int g(y)dy}_{v(y)} + h(x)$$



6) Lagrange's Linear Equation : $Pp + Qq = R$ P, Q, R are fns of (x, y, z)

The general solution is $f(u, v) = 0$ where u & v are fns of (x, y, z)

Eliminating this arbitrary 'f' gives the above PDE.

Here u & v are solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — ①

This is called the Auxiliary / Subsidiary Equation

7) Solving the simultaneous equations : $\frac{dx}{P} = \frac{dy}{Q} ; \frac{dx}{P} = \frac{dz}{R} ; \frac{dy}{Q} = \frac{dz}{R}$

- ① Directly integrating if it is variable separable
- ② If one of them can be found directly, then use it to find the other one
- ③ using Componendo - Dividendo :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = \frac{l_2 dx + m_2 dy + n_2 dz}{l_2 P + m_2 Q + n_2 R}$$

now if $l_1 dx + m_1 dy + n_1 dz = 0$ great (try to get Den^r = 0)
 else try $\frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = d\phi$ & then proceed.

Try to remember some smart manipulations.

eg. $(x-y)p + (x+y)q = 2xz$

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$$

choose $(1, 1, \frac{1}{z}) \Rightarrow dx + dy + \frac{dz}{z} = 0$
 $\Rightarrow x + y + \log z = c_1$

$$= \frac{x dx + y dy}{x^2 + y^2} = \frac{x dx + y dy}{x^2 + y^2} \quad (x+y) dx = (x-y) dy$$

$$= 2x dx + y dy + y dx - x dy = 0$$

$$\Rightarrow \frac{1}{2} \frac{d(x^2 + y^2)}{x^2 + y^2} + \frac{1}{x^2 + y^2} d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{\log(x^2 + y^2)}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c$$

$$(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)r$$

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} = \frac{x dx + y dy + z dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{(x-y)(x+y+z)(y-z)(x+y+z)}$$

$$\Rightarrow \frac{(x-y)}{(y-z)} = c_1$$

$$\frac{1}{2} \frac{d(x^2 + y^2 + z^2)}{(x+y+z)} = d(\log(x+y+z))$$

$$\Rightarrow x^2 + y^2 + z^2 = (x+y+z)^2 + c_2$$

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$[x^3 + y^3 + z^3 - 3xyz = x^2 + y^2 + z^2 - xy - yz - zx]$$

$$\cos(x+y)p + \sin(x+y)q = z$$

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} = \frac{d(x+y)}{\cos(x+y) + \sin(x+y)} = \frac{d(x-y)}{\cos(x+y) - \sin(x+y)}$$

$$\text{Last -2} \Rightarrow d(x-y) = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} d(x+y)$$

$$\Rightarrow (x-y) = \log |\cos(x+y) + \sin(x+y)| + C_1$$

$$\Rightarrow \cos(x+y) + \sin(x+y) = C_1 e^{x-y}$$

$$\text{Also } \frac{dz}{z} = \frac{d(x+y)}{\cos(x+y) + \sin(x+y)} = \frac{dt}{\cos t + \sin t} \quad t = x+y$$

$$\frac{dz}{z} = \frac{1}{\sqrt{2}} \sec\left(t - \frac{\pi}{4}\right) dt \Rightarrow z = \frac{1}{\sqrt{2}} \log \left| \sec\left(t - \frac{\pi}{4}\right) + \tan\left(t - \frac{\pi}{4}\right) \right|$$

$$\Rightarrow \log z = \frac{1}{\sqrt{2}} \log \left| \sec\left(x+y - \frac{\pi}{4}\right) + \tan\left(x+y - \frac{\pi}{4}\right) \right|$$

$$\Rightarrow z^{\sqrt{2}} = \sec\left(x+y - \frac{\pi}{4}\right) + \tan\left(x+y - \frac{\pi}{4}\right) C_2$$

⇒ This method can be generalised for > 2 independent variables PDE also.

$$P_1 p_1 + P_2 p_2 + \dots + P_n p_n = R$$

$$\Rightarrow \frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

$$\triangleright \text{GS is } f(u_1, u_2, \dots, u_n) = 0.$$

eg. $x_1^3 + x_2^3 + x_3^3 = 1$ When $z=0$ then soln of $(s-x_1)p_1 + (s-x_2)p_2 + (s-x_3)p_3 = s-z$ can be given as $s^3[(x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3]^4 = (x_1+x_2+x_3-3z)^4$ where $s = x_1+x_2+x_3+z$

Write Lagrange & obtain: $\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} = \frac{ds}{3s} \Rightarrow (x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = a$

$$\text{Also } \frac{d(x_1-z)^3}{-3(x_1-z)^3} = \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} = \frac{d[(x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3]}{-3[(x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3]} = \frac{d(\sum x_i - 3z)}{-(\sum x_i - 3z)}$$

$$\Rightarrow \sum (x_i - z)^3 = b(\sum x_i - 3z)^3 \Rightarrow b = 1/(\sum x_i)^3 \quad [\sum z=0]$$

No do manipulations from here! $[(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = (x_1+x_2+x_3)^4] \wedge [(x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3] = \frac{(x_1+x_2+x_3-3z)^3}{(x_1+x_2+x_3)^3}$

9) Integral Surfaces : through a given curve :

(a) Say curve is in parametric form then $u_1(x(t), y(t), z(t)) = c_1$
 $\& u_2(x(t), y(t), z(t)) = c_2$
 Eliminate 't' to get relation in c_1, c_2 & then put $u_1 = c_1$
 $\& u_2 = c_2$

(b) Curve is given as $\phi(x, y, z) = 0$ & $\psi(x, y, z) = 0$

Eliminate x, y, z from u_1, u_2, ϕ & ψ & then obtain relation
 by c_1 & c_2

eg. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ passing through $x=1, y=0$
 $\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$ choose $(x, y, \frac{1}{2}) \Rightarrow x^2 + y^2 + z = c_1$
 $\& (z, 1, x) \Rightarrow y + xz = c_2$
 $\Rightarrow c_1 - c_2 = 1$
 $\therefore (x^2 + y^2 + z) - (y + xz) = 1$

10) Surfaces orthogonal to a given system of surfaces :

Given system is $f(x, y, z) = c$
 Normal DR's at (x, y, z) to f are $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

Let the $\perp$ surface be $z = \phi(x, y)$
 Normal DR's are $(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1)$ i.e. $(p, q, -1)$

$\therefore \frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q - \frac{\partial f}{\partial z} = 0 \Rightarrow$ Lagrange form

$$\Rightarrow \frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{dz}{\frac{\partial f}{\partial z}}$$

eg. $f = z(x+y) = c(3z+1) \therefore f = \frac{z(x+y)}{(3z+1)} = c$

$$\therefore \frac{dx}{\frac{z}{3z+1}} = \frac{dy}{\frac{z}{3z+1}} = \frac{dz}{\frac{(x+y)}{(3z+1)^2}} \Rightarrow \begin{aligned} x-y &= c_1 \\ \frac{(x+y)^2}{2} - 2z^3 + z^2 &= c_2 \end{aligned}$$

⇒ Non-Linear Equations :

1) $f(x, y, z, a, b) = 0 \rightarrow F(x, y, z, p, q)$ upon elimination of a & b .

Complete solution : contains as many constants as the no of independent variables. eg. given F we get f as complete solⁿ.

Particular integral : giving specific values to a & b .

Singular integral : eliminating a & b b/w f , $\frac{\partial f}{\partial a}$ & $\frac{\partial f}{\partial b}$.

General integral : if $b = \phi(a)$, then we have a subfamily of solution. The envelope of this subfamily obtained

by eliminating 'a' from $f(x, y, z, a, \phi(a)) = 0$ & $\frac{\partial f}{\partial a} = 0$ is the general integral.

2) CHARPIT'S METHOD : (1st order PDE) :

Charpit's Auxillary Equations :

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-\frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{d\phi}{0}$$

Working :

a) put in the standard format $f(x, y, z, p, q) = 0$ & form the aux. eqn.

b) Select 2 simplest solution relation to get p & q in terms of (x, y, z) [atleast one of p, q HAS TO BE there].

Sometimes if you get 'p', directly put in f to get 'q' instead of solving another relation.

c) Put in $dz = p dx + q dy$ to get complete integral.

Example: $p^2x(x-1) + q^2y(y-1) + 2pqxy - 2pxz - 2qyz + z^2 = 0$

Simplified An eqⁿ give $\frac{dp}{-p^2} = \frac{dq}{-q^2} = \frac{1}{p} dp = \frac{1}{q} dq$

$$= \frac{\frac{1}{p} dp - \frac{1}{q} dq}{-p+q}$$

$$\text{Also } \frac{\frac{1}{2} dx - \frac{1}{2} dy}{2(p-q)}$$

Remember this.

$$\Rightarrow \frac{1}{2} \left(\frac{1}{x} dx - \frac{1}{y} dy \right) \Rightarrow \frac{1}{p} dp - \frac{1}{q} dq = 0$$

$$\Rightarrow \frac{1}{2} \log \left(\frac{x}{y} \right) + \log \left(\frac{p}{q} \right) = \log a.$$

$$\therefore \left(\frac{x}{y} \right)^{1/2} = \frac{qa}{p} \Rightarrow p = qa \left(\frac{y}{x} \right)^{1/2}$$

Also PDE $\equiv (px + qy - z)^2 = p^2x + q^2y$

$$\Rightarrow px + qy - z = \sqrt{p^2x + q^2y}$$

$$\Rightarrow q \left[y + a(xy)^{1/2} - (a^2x)^{1/2} y^{1/2} \right] = z$$

$$\Rightarrow q = \frac{z}{y + a(xy)^{1/2} - (1+a^2)^{1/2} y^{1/2}}$$

$$\therefore dz = p dx + q dy \Rightarrow dz = \frac{a z dx}{x^{1/2} [y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}]} + \frac{z dy}{y^{1/2} [y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}]}$$

$$\Rightarrow \frac{dz}{z} = \frac{ay^{1/2} dx + x^{1/2} dy}{(xy)^{1/2} (y^{1/2} + a x^{1/2} - (1+a^2)^{1/2})}$$

$$\Rightarrow \log z = 2 \log [y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}] + \log b$$

$$\Rightarrow z = b [y^{1/2} + a x^{1/2} - (1+a^2)^{1/2}]^2$$

Find C.I, S.I, & I for $(p^2 + q^2)y = qz$ $f_x = 0$ $f_y = p^2 + q^2$ $f_z = -1$
 $f_p = 2py$ $f_q = 2qy - z$

$$\therefore \frac{dx}{-2py} = \frac{dy}{z} = \frac{dp}{-p^2} = \frac{dq}{p^2} = \frac{dz}{-2py + qz - 2qy}$$

$$\Rightarrow p^2 + q^2 = a^2 \Rightarrow q = \frac{a^2 y}{z} \quad p = a \sqrt{1 - \frac{a^2 y^2}{z^2}} \Rightarrow dz = p dx + q dy$$

$$\therefore dz = a^2 \left[1 - \frac{a^2 y^2}{z^2} \right] dx + \frac{a^2 y dy}{z} \Rightarrow \frac{1}{2} d(z^2 - a^2 y^2) = a^2 dx \sqrt{z^2 - a^2 y^2} \Rightarrow (z^2 - a^2 y^2)^{3/2} = \dots$$

$$C1: (z^2 - a^2 y^2) = (ax + b)^2 \quad \therefore \phi = a^2 x^2 + b^2 + 2abx + a^2 y^2 - z^2$$

$$\frac{\partial \phi}{\partial a} = 2ax^2 + 2bx + 2ay = 0$$

$$= ax^2 + bx + ay = 0$$

$\therefore z=0$ is the SI.

$$\frac{\partial \phi}{\partial b} = 2b + 2ax = 0$$

$$b + ax = 0$$

$$\Rightarrow a = b = 0.$$

$$G1: \phi: z^2 - a^2 y^2 = (ax + \phi(a))^2$$

$$\therefore \frac{\partial \phi}{\partial a} = -2ay^2 = 2(ax + \phi(a))(x + \phi'(a))$$

Eliminate a from ϕ & $\frac{\partial \phi}{\partial a}$ to get G1.

3) Some special forms of f :

(a) $f(p, q) = 0$

$$\frac{dp}{p} = \frac{dq}{q} = \dots$$

$$\Rightarrow p = a, f(q, q) = 0$$

$$\hookrightarrow q = \phi(a)$$

$$\therefore z = ax + \phi(a)y + b$$

Try if $pz, px, etc.$ are multiplied.

Sometimes subⁿ is needed to reduce to $f(p, q) = 0$ form. *Very Very useful.*

eg. $z^2 p^2 y + 6zpxy + 2zq^2 x^2 + 4x^2 y = 0$

Put $x dx = dx$ & $y dy = dy$
 $z dz = dz$

(b) $f(z, p, q) = 0$

$$\frac{dp}{p} = \frac{dq}{q} = \dots \Rightarrow p = aq$$

$$= dz = p d(ax + y)$$

$$dz = p dx \quad x = ax + y$$

$$p = \frac{dz}{dx} \quad q = \frac{1}{a} \frac{dz}{dx}$$

$$\therefore f\left(z, \frac{dz}{dx}, \frac{1}{a} \frac{dz}{dx}\right) = 0$$

(c) Separable $f_1(x, p) = f_2(y, q)$

$$\frac{dx}{\frac{\partial f_1}{\partial p}} = \frac{dp}{-\frac{\partial f_1}{\partial x}} \Rightarrow \frac{\partial f_1}{\partial p} dp + \frac{\partial f_1}{\partial x} dx = 0$$

$$\Rightarrow df_1 = 0 \Rightarrow f_1 = \text{const} = a$$

$$\therefore f_1 = f_2 = a, \text{ solve to get}$$

$$p = F_1(x, a) \quad q = F_2(y, a)$$

$$\text{then } z = \int F_1(x, a) dx + \int F_2(y, a) dy + b$$

(d) Clairaut Eqⁿ: $z = px + qy + f(p, q)$

$$\frac{dp}{p} = \frac{dq}{q} = \dots \quad p = a \quad q = b$$

$$\therefore z = ax + by + f(a, b)$$

eg. $4xyz = pq + 2p^2 xy + 2q^2 xy^2$

$$z = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + \frac{\partial z \cdot x^2}{2x \partial x} + \frac{\partial z \cdot y^2}{2y \partial y}$$

$$z = pX + qY + pQ \quad X = x^2 \quad Y = y^2$$

$$\therefore z = ax^2 + by^2 + ab \Rightarrow \begin{cases} x^2 + b = 0 \\ y^2 + a = 0 \end{cases}$$

$$\therefore z = -xy^2 \text{ is SI}$$

Ψ Solution through given curve: **CRAM!!!**

We get the CI: $f(x, y, z, a, b) = 0$ & the curve C (parametrised by t).

The pts of intersection are $f(x(t), y(t), z(t), a, b) = 0$. This should have equal roots, i.e., $b^2 - 4ac = 0$ or f & $\frac{\partial f}{\partial t}$ having common root.

Using which we get relation $\Psi(a, b) = 0$. Use it to get one-parameter subsystem of the CI. Its envelope is the reqd solⁿ [Do $b^2 - 4ac$ again on this subsystem]

example: CI of PDE: $(p^2 + q^2)x = pz$ & deduce the soln through $x=0, z=4y$

We get CI as $z^2 = a^2x^2 + (ay+b)^2$ — (1)
 $\therefore C$ is $(0, t^2, 2t)$

$$\therefore 4t^2 = 0 + (at^2 + b)^2 \Rightarrow a^2t^4 + (2ab - 4)t^2 + b^2 = 0$$

$$\text{equal roots} \Rightarrow (2ab - 4)^2 - 4(a^2b^2) = 0 \Rightarrow 4 - 4ab = 0 \Rightarrow ab = 1$$

$\therefore z^2 = a^2x^2 + (ay + \frac{1}{a})^2$ is the one parameter sub-system of (1)

$$\Rightarrow (x^2 + y^2)a^4 + (2y - z^2)a^2 + 1 = 0$$

whose envelope we get by $(2y - z^2)^2 - 4(x^2 + y^2) = 0$
 $\Rightarrow z^2 = 2y - 2\sqrt{x^2 + y^2}$

5> Deriving one CI from another:

Suppose wkt $f(x, y, z, a, b) = 0$ is CI for $F(x, y, z, p, q) = 0$ — (1)

& we want to show $g(x, y, z, h, k) = 0$ is also CI. — (2)

choose on (2), a curve C in whose eqn the constants h, k appear as independent parameters, then find the envelope of one parameter subfamily of (1) touching C . Since this soln contains 2 arbitrary constants, it's a CI

eg ST $x^2p^2 + y^2q^2 = 1$ has CI (a) $(z+b)^2 = 4(ax+y)$
 (b) $kx(z+h) = ky^2 + x^2$

& deduce (b) from (a)

Write aux. eqn & solve twice to get (a) & (b) $\Rightarrow$ straightforward

Now choose $C = y=0, x=k(z+h)$ on (b). Get the relation b/w a, b for a PI touching C , ie, $(z+b)^2 = 4ak(z+h)$
 & do $b^2 - 4ac = 0$ for equal roots

We get $b = ak + h$.

Put this in (a) to get one-parameter sub-family. Now eliminate (a) from quad. in 'a' $[z+(h+ak)]^2 = 4(ax+y)$
 on doing $D=0$ we get the cond (b). HP

g) Complete integral satisfying other condition such as circumscribing a given surface:

* 2 surfaces circumscribe each other if they touch along a curve. PDE is $F(x, y, z, p, q) = 0$ — (1)

We want CI $f(x, y, z, a, b)$ to circumscribe $\psi(x, y, z) = 0$
 $f(x, y, z, a, b) = 0$ — (2) & $\psi(x, y, z) = 0$ — (3) is the surface.

* (2) touches (3) iff (2), (3) & $\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z}$ are consistent

Use these 3 to eliminate x, y, z & get relation in $\{a, b\}$

* Using $\{a, b\}$ get a one-parameter subsystem of (2) whose envelope is the required surface.

eg ST only integral surface of the equation $2q(z - px - qy) = 1 + q^2$ which is circumscribed about the paraboloid $2x = y^2 + z^2$ is the

enveloping cylinder which touches it along its section by plane $y + 1 = 0$

$$F(x, y, z, p, q) = z - px + qy + \frac{1+q^2}{2q}$$

$$\therefore f(x, y, z, a, b) = z - ax + by + \frac{1+b^2}{2b} \quad [\because \text{charant form}]$$

$$\psi(x, y, z) = 2x - y^2 - z^2 \quad \therefore \frac{a}{2} = \frac{b}{-2y} = \frac{-1}{-2z} \Rightarrow z = \frac{1}{a}, y = \frac{1}{a}$$

$$\therefore 2x - \frac{b^2}{a^2} - \frac{1}{a^2} = 0 \Rightarrow x = \frac{b^2+1}{2a^2}$$

$$\therefore \frac{1}{a} = \frac{b^2+1}{2a} - \frac{b^2}{a} + \frac{1+b^2}{2b} \quad \frac{1+b^2}{2a} = \frac{1+b^2}{2b} \Rightarrow \boxed{b=a}$$

$$\therefore \text{one parameter form: } z = ax + ay + \frac{1+a^2}{2a} \Rightarrow 2a^2(x+y) + 1 + a^2 - 2az = 0$$

$$\Rightarrow 4z^2 - 4[2(x+y) + 1] = 0 \Rightarrow \boxed{z^2 = 2(x+y) + 1}$$

This is the reqd circumscribing surface.

Since it touches $\psi \Rightarrow 2(x+y) + 1 = 2x - y^2$

$$\Rightarrow y^2 + 2y + 1 = 0$$

$$\Rightarrow (y+1)^2 = 0$$

$$\Rightarrow y+1=0$$

⇒ Jacobi's Method :

Solving PDE with 3 (or) more than 3 independent variables =

eg $F(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ — (1) w/o variable 'z'

Jacobi's Aux eqn: $\frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}} = \frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}}$

Solving gives 2 additional eqn. $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ & $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$

Verify that $(F_1, F_2) = \sum_{r=1}^3 \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0$ — (7)

If (6) is satisfied, solve (1), (2), (3) get p_1, p_2, p_3 & put in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$

For 4 independent variables: verify $(F_1, F_2), (F_2, F_3), (F_3, F_1)$ separately.

eg $p_1^2 + p_2^2 + p_3^2 = 1$

JAE: $\frac{dx_1}{-2p_1} = \frac{dx_2}{-2p_2} = \frac{dx_3}{-2p_3} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$

$\therefore p_1 = a_1, p_2 = a_2 \quad \therefore F_1 = p_1 - a_1, F_2 = p_2 - a_2$
 $(F_1, F_2) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$\therefore p_1 = a_1, p_2 = a_2$ & put in PDE $\Rightarrow p_3 = \sqrt{1 - a_1^2 - a_2^2}$

$\therefore dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^2 - a_2^2) dx_3$

$z = a_1 x_1 + a_2 x_2 + (1 - a_1^2 - a_2^2) x_3 + a_3$

eg $p_1 p_2 p_3 = z^3 x_1 x_2 x_3$

We have to remove z , so put $z = \ln z \quad X_i = \frac{x_i^2}{2}$

so PDE becomes $P_1 P_2 P_3 = 1$ whose soln is $z = a_1 x_1 + a_2 x_2 + \frac{1}{a_1 a_2} x_3^2$

$\therefore \ln z = \frac{a_1 x_1^2}{2} + \frac{a_2 x_2^2}{2} + \frac{x_3^2}{2 a_1 a_2} + a_3$

⇒ Cauchy's Method of Characteristics for non-linear PDE:

1) Given $f(x, y, z, p, q) = 0$ we wish to find integral surface touching a given curve $x = f_1(\lambda)$ $y = f_2(\lambda)$ $z = f_3(\lambda)$ λ is parameter

2) We first want to find initial values x_0, y_0, z_0, p_0, q_0 , to They are needed to determine the arbitrary constants from the solution of characteristics strip PDEs.

Take $x_0 = f_1(\lambda)$ $y_0 = f_2(\lambda)$ $z_0 = f_3(\lambda)$, then p_0, q_0 are determined using $f'_3(\lambda) = p_0 f'_1(\lambda) + q_0 f'_2(\lambda)$ & $f(x_0, y_0, z_0, p_0, q_0) = 0$

3) Once we have initial values, then solve following 5 equations:

$$\frac{dp}{dt} = -f_x - pf_z \quad \frac{dq}{dt} = -f_y - qf_z \quad [\because \text{res of charpit denrs}]$$

$$\frac{dx}{dt} = +fp \quad \frac{dy}{dt} = +fq \quad \frac{dz}{dt} = pf + qf$$

4) We find 5-6 solutions by combining different equations. With each solution we get an arbitrary constant. It is determined using x_0, y_0, z_0, p_0, q_0 found in (2).

Sometimes we get t as part of the above function: initial value to is entirely our own choice for convenience.

⇒ So we get $x = \phi_1(\lambda, t)$ $y = \phi_2(\lambda, t)$ $z = \phi_3(\lambda, t)$

Eliminate λ, t to get relation b/w x, y, z .

Example: $z = \frac{1}{2}(p^2 + q^2) + (p-x)(q-y)$ passing through n -axis

Parametric eqⁿ of C : $(x=\lambda, y=0, z=0)$

Let initial values be $x_0 = \lambda, y_0 = 0, z_0 = 0$

then $f_3'(x) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \Rightarrow 0 = p_0 + 0 \Rightarrow \boxed{p_0 = 0}$

$\& 0 = \frac{1}{2}(p_0^2 + q_0^2) + (p_0 - \lambda)(q_0 - 0) \Rightarrow \boxed{q_0 = 2\lambda}$

now $f_x = y - q, f_y = x - p, f_z = -1, f_p = p + q - y, f_q = p + q - x$

$\therefore \frac{dp}{dt} = p + q - y, \frac{dq}{dt} = p + q - x, \frac{dz}{dt} = (p + q)^2 - py - qx$

$\frac{dx}{dt} = p + q - y, \frac{dy}{dt} = p + q - x$

$\therefore \begin{cases} p - x = c_1 \\ q - y = c_2 \end{cases} \text{ use } p_0 = 0 = y_0 \Rightarrow \begin{cases} c_1 = -\lambda \\ c_2 = 2\lambda \end{cases} \Rightarrow \begin{cases} x = p + \lambda \\ y = q - 2\lambda \end{cases} \quad \text{--- (1)}$

Also $\frac{d(p + q - x)}{dt} = p + q - x \Rightarrow p + q - x = c_3 e^t$
 $\frac{d(p + q - y)}{dt} = p + q - y \Rightarrow p + q - y = c_4 e^t$
 $\lambda = c_3 e^0 \Rightarrow c_3 = \lambda$
 $2\lambda = c_4 e^0 \Rightarrow c_4 = 2\lambda$
 take to

$\therefore p + q - x = \lambda e^t \quad \& \quad p + q - y = 2\lambda e^t$

using (1), $q = \lambda(1 + e^t), p = 2\lambda(e^t - 1)$

$\therefore \boxed{x = \lambda(2e^t - 1)} \quad \boxed{y = \lambda(e^t - 1)}$

Also $\frac{dz}{dt} = p[p + q - y] + q[p + q - x] = 4\lambda^2(e^{2t} - e^t) + \lambda^2(e^{2t} + e^t) = 5\lambda^2 e^{2t} - 3\lambda^2 e^t$

$\therefore z = \frac{5}{2}\lambda^2 e^{2t} - 3\lambda^2 e^t + C_5$

$0 = \frac{5\lambda^2}{2} - 3\lambda^2 + C_5 \Rightarrow C_5 = \lambda^2/2$

$\therefore \boxed{z = \frac{\lambda^2}{2} [5e^{2t} - 6e^t + 1]}$

Now $e^t = \frac{y + \lambda}{\lambda} \Rightarrow x = \lambda \left[\frac{2(y + \lambda)}{\lambda} - 1 \right] \Rightarrow \lambda = x - 2y \quad \& \quad e^t = \frac{x - y}{x - 2y}$

$\therefore z = \frac{(x - 2y)^2}{2} \left[5 \left(\frac{x - y}{x - 2y} \right)^2 - 6 \left(\frac{x - y}{x - 2y} \right) + 1 \right]$

⇒ Linear PDE with constant coefficients :

1) Linear Homogeneous : $[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n] z = f(x, y)$

Complementary Function : $f(D, D') z = f(x, y)$

then CF is soln to $f(D, D') z = 0$

(1) Replace $D \rightarrow m$ & $D' \rightarrow 1$ & get the

Aux eqn $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$

& solve for 'm'

(i) Distinct roots $m_1, m_2, \dots, m_n$

then $CF = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$

(ii) Repeated roots $m = m' (n \text{ times})$

then $CF = \phi_1(y + m x) + x \phi_2(y + m x) + \dots + x^{n-1} \phi_n(y + m x)$

Note all ϕ_i are diff t.

(iii) If D^m is a factor, then

$CF = \phi_1(y) + x \phi_2(y) + \dots + x^{m-1} \phi_m(y)$

(iv) If D'^m is a factor, then

$CF = \phi_1(x) + y \phi_2(x) + \dots + y^{m-1} \phi_m(x)$

[OR]

(b) Factorise $F(D, D')$ as $(bD - aD')$ factors ,
 then for $(bD - aD') \equiv \phi(by + ax)$ is the part of CF

Esay: remember $(D - \frac{a}{b} D')$ has $m = \frac{a}{b}$
 $\therefore \phi(y + mx) \equiv \phi(by + ax)$

Q) Particular Integral : $PI = \frac{1}{F(D, D')} f(x, y)$

$\frac{1}{D} \equiv$ partial integration wrt x $\frac{1}{D'} \equiv$ wrt y .

(i) $F(D, D') z = \phi(ax+by)$ where $F(D, D')$ is homogeneous with degree n

(a) $F(a, b) \neq 0$

then $PI = \boxed{\frac{1}{F(a, b)} \int \int \int \dots \int \phi(v) dv \cdot dv \cdot dv \dots}$
n times

(b) $F(a, b) = 0 \Rightarrow (bD - aD')$ is a factor

then $\frac{1}{(bD - aD')^n} \phi(ax+by) = \boxed{\frac{x^n}{b^n n!} \phi(ax+by)}$

(ii) $F(D, D') z = x^m y^n$ or some algebraic function of x^m, y^n

Expand $\frac{1}{F(D, D')}$ as $\left[\frac{D'}{D} \right]$ if $n < m$ or $\left[\frac{D}{D'} \right]$ if $m < n$

Basically try to differentiate smaller power

Example: $x + 5y + 6z = (y - 2x)^{-1}$
 $= (D + 2D')(D + 3D') z = (y - 2x)^{-1} \Rightarrow z_c = \phi_1(y - 2x) + \phi_2(y - 3x)$

$z_p = \frac{1}{D + 2D'} \left[\frac{1}{D + 3D'} (y - 2x)^{-1} \right] = \frac{1}{D + 2D'} \log(y - 2x)$

$= \frac{x^1}{1! \cdot 1!} \cdot \log(y - 2x) = x \log(y - 2x)$

$(D^2 - 6DD' + 9D'^2) z = 12x^2 + 36xy \Rightarrow z_c = \phi_1(y + 3x) + x \phi_2(y + 3x)$

$z_p = \frac{1}{(D^2 - 3D')^2} (12x^2 + 36xy) = \frac{1}{D^2} \left[1 - \frac{3D'}{D} \right]^{-2} [12x^2 + 36xy]$

$= \frac{1}{D^2} \left[1 + \frac{6D'}{D} + \frac{27D'^2}{D^2} \right] [12x^2 + 36xy] = \frac{1}{D^2} [12x^2 + 36xy + 108x^2] = \frac{1}{D^2} [4x^3 + 108x^2 + 36xy]$
 $= x^3 + 6x^2y + 9x^2 + 6xy^2$

3) general method: $\frac{1}{(D-mD')} \phi(x,y) = \int \phi(x, a-mx) dx$

after integration put $a = y + mx$.

use for sin, cos functions repeatedly.

Example: $(D^2 + DD' - 6D'^2) z = y \sin x \Rightarrow (D+3D')(D-2D') z = y \sin x$

$$z_c = \phi_1(y-3x) + \phi_2(y+2x)$$

$$z_p = \frac{1}{(D+3D')(D-2D')} y \sin x = \frac{1}{D+3D'} \int (a-2x) \sin x dx$$

$$= \frac{1}{D+3D'} [-2 \cos x (a-2x) + 2 \cos x]$$

might have error!

$$= \frac{1}{D+3D'} [2 \cos y - y \cos x] = \int 2 \cos(a+3x) - (a+3x) \cos x dx$$

$$= \frac{2}{3} \sin(a+3x) - (a+3x) \sin x - 3 \cos x$$

just proceed like this

4) Non-homogeneous Linear PDE with constant coefficients:

CF:

(a) $F(D, D')$ can be resolved as linear factors.

if $F(D, D') = [(D-m_1D'-k_1)(D-m_2D'-k_2) \dots (D-m_nD'-k_n)] z$

$$CF = e^{k_1x} \phi_1(y+m_1x) + e^{k_2x} \phi_2(y+m_2x) + \dots + e^{k_nx} \phi_n(y+m_nx)$$

Repeated factor $(D-mD'-k)$

$$CF = e^{kx} [\phi_1(y+mx) + x \phi_2(y+mx) + \dots + x^{r-1} \phi_{r-1}(y+mx)]$$

(b) If factor is $(\alpha D + \beta D' + \gamma) z = 0 \Rightarrow$ CF part is $e^{\frac{-\gamma}{\alpha}x} \left[\phi\left(\frac{\alpha y - \beta x}{\sqrt{\alpha^2 + \beta^2}}\right) \right]$

(b) If can't factorise into linear factors:

Use hit & trial (mostly e^{ax+by})

$$\text{eg } (2D^4 - 3D^2D' + D'^2) z = 0 \Rightarrow (2D^2 - D')(D^2 - D') z = 0$$

$$\text{let } z = A e^{hx+ky} \Rightarrow A h^2 e^{hx+ky} - A k e^{hx+ky} \Rightarrow h^2 = k \quad [\text{using } 2D^2 = D']$$

$$\text{using } D^2 = D' \Rightarrow z = \sum B e^{h^2x + 2h^2y} \Rightarrow \text{GS: } z = \sum A e^{hx+h^2y} + \sum B e^{h^2x+2h^2y}$$

Particular Integral : (similar to ODE) :

$$(a) \quad \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by} \quad [F(a, b) \neq 0]$$

$$(b) \quad \frac{1}{F(D, D')} \sin(ax+by) \quad \text{Put } D^2 = -a^2 \quad D'^2 = -b^2 \\ DD' = -ab$$

$$(c) \quad f(x, y) = x^m y^n \quad \text{use binomial expansion}$$

$$(d) \quad f(x, y) = e^{ax+by} V = \frac{1}{F(D, D')} f(x, y) = \frac{e^{ax+by}}{F(D+a, D'+b)} V$$

5) Reducible to Linear PDE with constant coefficients :

$$\text{if } A_0 x^n \frac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

$$\text{Put } x = e^u \quad y = e^v \quad (\text{like ODE})$$

$$\text{This will reduce } x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n} = \frac{D(D-1)\dots(D-(m-1)) \cdot D'(D'-1)\dots(D'-(n-1))}{D'(D'-1)\dots(D'-(n-1))} z$$

$$\text{eg. } x^2 z - y^2 z = xy \quad \Rightarrow [D(D-1) - D'(D'-1)] z = e^{x+y} \quad \text{Let } e^u = x \quad e^v = y$$

$$\Rightarrow (D-D')(D+D'-1) z = e^{x+y}$$

$$z = \phi_1(x+y) + \phi_2(x+y)e^x + x e^{x+y}$$

$$z = \psi_1(xy) + x \psi_2\left(\frac{y}{x}\right) + (\log x) xy$$

6) Finding surface satisfying some conditions & PDE.

Miscellaneous Example: Surface is drawn satisfying $r+t=0$ & touching

$x^2+z^2=1$ along its section by $y=0$. Find its equation in

the form $z^2(x^2+z^2-1) = y^2(x^2+z^2)$

$$(D^2+D'^2)z=0 \quad \Rightarrow \quad z = \phi_1(y+ix) + \phi_2(y-ix)$$

$$p = \frac{\partial z}{\partial x} = i [\phi_1'(y+ix) - \phi_2'(y-ix)]$$

$$q = \frac{\partial z}{\partial y} = \phi_1'(y+ix) + \phi_2'(y-ix)$$

$$\text{Also } x^2 + z^2 = 1 \Rightarrow z = \sqrt{1-x^2} \quad \therefore p = \frac{-x}{\sqrt{1-x^2}} \quad q = 0$$

Since they touch, equate p 's & q 's $\Rightarrow \phi_1'(y+ix) + \phi_2'(y-ix) = 0$

$$\Rightarrow \phi = 2i \phi_1'(y+ix) \quad \text{where } y=0$$

$$\therefore \phi_1'(ix) = \frac{x i}{2\sqrt{1+(ix)^2}} \quad \phi_2'(-ix) = \frac{-ix}{2\sqrt{1+(-ix)^2}}$$

$$\therefore \phi_1(u) = \frac{1}{2} \sqrt{1+u^2} + c_1 \quad \phi_2(v) = \frac{1}{2} \sqrt{1+v^2} + c_2$$

$$\therefore z = \frac{1}{2} \left\{ \sqrt{1+(y+ix)^2} + \sqrt{1+(y-ix)^2} \right\} + c \quad \text{--- (1)}$$

Also 2 values of z should be same from $x^2 + z^2 = 1$ & (1) when $y=0$

$$\Rightarrow \sqrt{1-x^2} = \sqrt{1-x^2} + c \Rightarrow c=0$$

$$\therefore 2z = \sqrt{1+(y+ix)^2} + \sqrt{1+(y-ix)^2} \quad [\text{which can be written in the desired form}]$$

7) CANONICAL FORM :

Let PDE be $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

This can be converted into canonical form by substitutions

$$x \rightarrow u = u(x, y) \quad y \rightarrow v = v(x, y) \quad [u, v \text{ are independent}]$$

To find u & v :-

(a) Solve $R\lambda^2 + S\lambda + T = 0$ $\xrightarrow{\lambda_1, \lambda_2}$ & solve $\frac{dy}{dx} + \lambda_1 = 0$ & $\frac{dy}{dx} + \lambda_2 = 0$

(b) Let solutions be $f_1(x, y) = c_1$ & $f_2(x, y) = c_2$. Then
but $u = f_1(x, y)$ & $v = f_2(x, y)$ $u = xy$, then take $v = x$
 $J(u, v) \neq 0$ hence independent

(c) Find p, q, r, s, t in terms of u & v & rewrite the PDE. We get the canonical form

(d) In case of equal roots, choose any 'v' independent of 'u'. Also
In case of complex roots $u = \alpha + i\beta$ & $v = \alpha - i\beta$ choose α, β as new coordinates

Example: $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow r + x^2 t = 0$

$\therefore R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda^2 + x^2 = 0 \Rightarrow \lambda = \pm ix$

$\therefore \frac{dy}{dx} \pm ix = 0 \Rightarrow y \pm i \frac{x^2}{2} = u$

$u = y + i \frac{x^2}{2} \quad v = y - i \frac{x^2}{2}$

Use $\alpha = y$ & $\beta = \frac{x^2}{2}$ as the transformation

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} = x q'$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = p'$

DO THE HARD WORK, ELSE ERROr CHANCES

$r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(x q') = q' + x \frac{\partial q'}{\partial x} = q' + x \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \beta} \right) = q' + x \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \beta} \right) \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial z}{\partial \beta} \right) \frac{\partial \beta}{\partial x}$
 $= q' + x^2 t'$

$t = \frac{\partial q}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta} \left(\frac{\partial z}{\partial \alpha} \right) \cdot \frac{\partial \beta}{\partial y} = r'$

$\therefore r + x^2 t = q' + x^2 t' + x^2 r' = 0$
 $q' + (r' + t') (2\beta) = 0$

$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta} \frac{\partial z}{\partial \beta} \Rightarrow \text{canonical form}$

b) Product Method or variable separation method:

Assume $z = X(x) Y(y)$ as the solution

eg. $\frac{\partial^2 z}{\partial x^2} - 2x \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \Rightarrow X''Y - 2X'Y + XY' = 0 \Rightarrow \frac{X'' - 2X'}{X} = -\frac{Y'}{Y}$

$\Rightarrow X'' - 2X' - \alpha X = 0 \Rightarrow X = c_1 e^{(1+\sqrt{1+\alpha})x} + c_2 e^{(1-\sqrt{1+\alpha})x}$

$Y' + \alpha Y = 0 \Rightarrow Y = c_3 e^{-\alpha y}$

$\Rightarrow z = XY = k_1 (e^{(1+\sqrt{1+\alpha})x} + k_2 e^{(1-\sqrt{1+\alpha})x}) e^{-\alpha y} \Rightarrow \text{soln}$

This is used in wave, heat & laplace equations!

⇒ Applications of PDE:

1) Standard Equations:

1-D wave equation: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

1-D heat flow equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

2-D heat eqⁿ: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2-D wave eqⁿ: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

Laplace's eqⁿ in 3D: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

2) Fourier sine series: F-expansion of $f(x)$ in $0 < x < l$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$n = 1, 2, 3, \dots$

Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$n = 1, 2, 3, \dots$

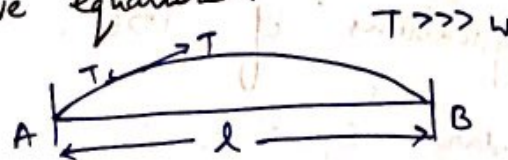
Double Fourier sine series: $f(x, y)$ expansion in $0 \leq x \leq a$ $0 \leq y \leq b$

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$

$m = 1, 2, 3, \dots; n = 1, 2, 3, \dots$

3) Stretched elastic string - wave equation:



$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$y(x, t) = X(x) T(t) \Rightarrow X'' T'' = c^2 X'' T \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T}$$

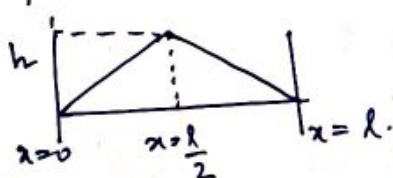
Initial condition: $y(0, t) = y(l, t) = 0 \forall t$ & $\frac{dy}{dt}(x, 0) = 0 \forall x$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = a \quad \begin{cases} a=0 \Rightarrow X = a_1 x + a_2 & T = a_3 t + a_4 \\ a=\lambda^2 \Rightarrow X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} & T = b_3 e^{\lambda c t} + b_4 e^{-\lambda c t} \\ a=-\lambda^2 \Rightarrow X = c_1 \cos \lambda x + c_2 \sin \lambda x & T = c_3 \cos \lambda c t + c_4 \sin \lambda c t \end{cases}$$

Among these 3, choose the one satisfying the initial conditions:

Std soln (for above initial values) $\Rightarrow y(x, t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi c t}{l} + F_n \sin \frac{n\pi c t}{l} \right) \sin \left(\frac{n\pi x}{l} \right)$

Example: A taut string of length l has its end $x=0$ & $x=l$ fixed. Mid point is taken to a small height 'h' & released from rest at $t=0$. Find $y(x, t)$.



IV $\Rightarrow y(0, t) = y(l, t) = 0 \forall t$

$$\therefore y(x, t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi c t}{l} + F_n \sin \frac{n\pi c t}{l} \right) \sin \left(\frac{n\pi x}{l} \right)$$

Also $\frac{dy}{dt}(x, 0) = 0 \forall x \Rightarrow \frac{dy}{dt} = \sum_{n=1}^{\infty} \left(-\frac{E_n x n\pi c}{l} \sin \frac{n\pi c t}{l} + \frac{F_n c n\pi x}{l} \cos \frac{n\pi c t}{l} \right) \sin \left(\frac{n\pi x}{l} \right) \Rightarrow F_n = 0$

Initially at rest

$$y(x, t) = \sum_{n=1}^{\infty} E_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

Now $y(x, 0) = g(x) = \begin{cases} \frac{2hx}{l} & x < l/2 \\ -\frac{2h}{l}(l-x) & x > l/2 \end{cases} = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{l}$

$$\therefore E_n = \frac{2}{l} \int_0^l g(x) \sin \left(\frac{n\pi x}{l} \right) dx = \frac{2}{l} \left[\int_0^{l/2} \frac{2hx}{l} \sin \left(\frac{n\pi x}{l} \right) dx + \int_{l/2}^l -\frac{2h}{l}(l-x) \sin \left(\frac{n\pi x}{l} \right) dx \right]$$

This gives E_n

4) One-Dimensional heat equation:

$u(x,t)$ is temperature function, then $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

Take $u(x,t) = X(x) T(t) \Rightarrow XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = a$

$a=0 \Rightarrow u = (a_1 x + a_2)(a_3 t)$

$a=-\lambda^2 \Rightarrow u = (C_1 \cos \lambda x + C_2 \sin \lambda x) C_3 e^{-k\lambda^2 t}$ → generally we

$a=\lambda^2 \Rightarrow u = (b_1 e^{\lambda x} + b_2 e^{-\lambda x}) b_3 e^{k\lambda^2 t}$ choose this at $e^{-k\lambda^2 t}$ ensure $u \rightarrow 0$ as $t \rightarrow \infty$.

Now discard as per the initial conditions

* Situations: ① Both ends at zero & initial temperature is given

$$u(0,t) = u(a,t) = 0 \quad \forall t \quad \& \quad u(x,0) = f(x).$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-C_n^2 t} \quad \text{where } C_n^2 = \frac{n^2 \pi^2 k}{a^2}$$

$$\& E_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Example: $u(0,t)=2$ & $u(1,t)=3$
& $u(x,0) = x(1-x)$ Find $u(x,t)$.

Consider $g(x) = c_1 x + c_2 \Rightarrow g(0)=2 \quad g(1)=3 \Rightarrow c_1=1 \quad c_2=2$
 $\therefore g(x) = x+2$

Consider $v(x,t) = u(x,t) - g(x)$ $\therefore v(0,t)=0 \quad v(1,t)=0$
 $v(x,0) = x(1-x) - x - 2 = -(x^2+2)$

Now proceed as normally.

② Both ends are insulated and initial temperature is given

INSULATION $\Rightarrow \frac{\partial u}{\partial x} \Big|_{x=0} = 0 \quad \therefore u_x(0,t) = u_x(a,t) = 0 \quad \forall t$
& $u(x,0) = f(x)$.

$$\Rightarrow u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{a}\right) e^{-\frac{k n^2 \pi^2 t}{a^2}} \quad E_0 = \frac{2}{a} \int_0^a f(x) dx$$

$$E_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

NOTE: while derivation we need to consider $k \neq 0$ case otherwise $\frac{E_0}{2}$ will be missed!!

③ One end is insulated & other at constant temperature:

eg $u(x,0) = 1-x \quad 0 < x < 1 \quad u(0,t)=10 \quad u_x(1,t)=0$.

Consider $y(x,t) = u(x,t) - 10 \Rightarrow y(x,0) = -(x+9) \quad y(0,t)=0$
 $y_x(1,t)=0$
Now solve we get $u(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-C_n^2 t}$

④ Insulated rod, steady state achieved & some conditions

$$\text{STEADY STATE} = \frac{\partial u}{\partial t} = 0 \quad \text{PDE} = \frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \text{Ans}$$

For steady state $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$

Temp at ends is given $\Rightarrow a = c_1(0) + c_2$ & $b = c_1(L) + c_2$
 $\Rightarrow c_2 = a$ & $c_1 = \frac{b-a}{L}$

$$\therefore u(x, 0) = f(x) = \frac{(b-a)x}{L} + a$$

Also given $u(0, t) = u(L, t) = 0 \quad \forall t \geq 0$

Now proceed like in case (1).

5) 2D-Heat eqn + Laplace equation in steady state

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k} \frac{\partial u}{\partial t}$$

take $u(x, y, t) = X(x) Y(y) T(t)$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{k} \frac{T'}{T}$$

Take $\frac{X''}{X} = -m^2 \quad \frac{Y''}{Y} = -n^2 \quad \frac{1}{k} \frac{T'}{T} = -(m^2 + n^2)$

$$\therefore u(x, y, t) = \sum \sum X_m(x) Y_n(y) T_{mn}(t)$$

$$X_m(x) = A_m \cos mx + B_m \sin mx$$

$$Y_n(y) = A_n \cos ny + B_n \sin ny$$

$$T_{mn}(t) = F_{mn} e^{-(m^2 + n^2)kt}$$

so for a plate $0 \leq x \leq a$, $0 \leq y \leq b$ with edges kept at 0°C & $u(x, y, 0) = f(x, y)$

$$\therefore u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$$

$$\therefore u_{mn}(x, y, t) = F_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}$$

$$F_{mn} = \frac{4}{ab} \iint f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

Ques Find CI for $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

Put $x+y = X^2$ & $x-y = Y^2 \Rightarrow p+q = 2X$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial x} = \frac{1}{2X} \frac{\partial z}{\partial X} + \frac{1}{2Y} \frac{\partial z}{\partial Y}$$

$$q = \frac{1}{2X} \frac{\partial z}{\partial X} - \frac{1}{2Y} \frac{\partial z}{\partial Y} \Rightarrow p+q = \frac{1}{X} \frac{\partial z}{\partial X} \quad p-q = \frac{1}{Y} \frac{\partial z}{\partial Y}$$

$\therefore e_1 \Rightarrow P^2 + Q^2 = 1 \Rightarrow Z = aX + \sqrt{1-a^2}Y + b$
 $z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + b.$