New Lomax-G family of distributions: Statistical properties and applications

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ABSTRACT

This research article introduces a new family of distributions developed using the innovative beta-generated transformation technique. Among these distributions, the focus is on the inverse exponential power distribution, which exhibits unique reverse-J, inverted bathtub, or monotonically increasing hazard functions. This paper thoroughly investigates the distribution's key characteristics and utilizes the maximum likelihood estimation method to determine its associated parameters. To assess the accuracy of the estimation procedure, the researchers conducted a simulation experiment, revealing diminishing biases and mean square errors with increasing sample sizes, even when working with small samples. Moreover, the practical applicability of the proposed distribution is demonstrated by analyzing real-world COVID-19 and medical datasets. The article establishes that the proposed model outperforms existing models by using model selection criteria and conducting goodness-of-fit test statistics. The potential applications of this research extend to various fields where modeling and analyzing hazard functions or survival data are crucial. Additionally, the study contributes to advancing probability theory and statistical inferences.

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I. INTRODUCTION

Statistical distributions find extensive application in describing real-world occurrences. Given their utility, the theory behind statistical distributions is extensively researched, leading to the development of novel distributions. The statistical profession continues to show a keen interest in creating even more adaptable distribution models. Statistical models play a crucial role in representing and analyzing datasets in practical applications. While traditional distributions, such as Weibull, Lomax, Uniform, gamma, log-normal, exponential, and beta, have been widely used, they may not always provide a satisfactory fit for complex datasets. To address this limitation, researchers have been actively working on developing new asymmetrical models that offer greater adaptability and flexibility. These advancements often involve techniques, such as exponentiation, transformation, modification, T–X, beta, and gamma generator

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approaches, to generate more flexible asymmetrical distributions. In this research, we concentrate on an alternative approach introduced by Eugene *et al.*[1] This approach provides a distinct blend of high skewness and flexibility to the base distribution.

There has been a growing fascination with defining novel generators for univariate continuous distributions in the past few years. This involves introducing one or more extra shape parameters to the original distribution. Including these parameters has proven beneficial for examining tail properties and enhancing the goodness-of-fit of the generated distribution. Several renowned generators fall into this category: The Weibull-G family (Bourguignon *et al.*),[2] generalized transmuted-G family (Nofal *et al.*),[3] Gompertz-G family (Alizadeh *et al.*),[4] Zubair-G family (Ahmad),[5] odd kappa-G family (Al-Shomrani and Al-Arfaj),[6] shifted exponential-G family (Eghwerido *et al.*),[7] Odd Chen-G family (Anzagra *et al.*),[8] new sine-G family (Benchiha *et al.*),[9] and Topp-Leone-type II exponentiated half logistic-G family (Gabanakgosi and Oluyede).[10] For more papers, see Haq *et al.*,[11] Alotaibi *et al.*,[12] Jamal *et al.*,[13] Bantan *et al.*,[14] Al-Babtain *et al.*,[15] Bakr *et al.*,[16] Afify *et al.*,[17] Ahmad *et al.*,[18] and ZeinEldin *et al.*[19]

This paper's primary aim is to develop a new flexible G-family using the Lomax distribution, which is a skewed distribution with a long right tail and is characterized by two main parameters: the shape parameter (α) and the scale parameter (β) and is also a special case of the Pareto distribution. Recently, some generalizations of the Lomax distribution can be found in the literature. The Lomax distribution has undergone significant developments and extensions to cater to diverse scenarios and applications. This literature review delves into various studies that have introduced novel extensions and uses of the Lomax distribution. Oguntunde *et al.*[20] presented a generalized model of the Lomax distribution, demonstrating its applicability in engineering data and real-world scenarios beyond traditional statistical applications. Cordeiro *et al.*[21] introduced the odd Lomax-G family of distributions, a unique generalization that combines elements from both Lomax and Maxwell distributions. This expansion provides researchers with a broader selection of distributional choices for data analysis. In a separate work, Oguntunde *et al.*[22] proposed a new model by generalizing the Lomax distribution using the Topp–Leone-G family, further enriching the collection of generalized Lomax distributions suitable for specific applications and data types. Falgore and Doguwa[23] explored the inverse Lomax-G family, offering insights into the Lomax distribution's potential in practical engineering applications. Almongy *et al.*[24] introduced the four-parameter Marshall–Olkin alpha power Lomax distribution, presenting a novel perspective and enabling the study of heavy-tailed phenomena. Abu El Azm *et al.*[25] extended the Lomax distribution by introducing a five-parameter generalized form, which was successfully applied to COVID-19 data, enhancing the distribution's ability to fit complex datasets. Abiodun and Ishaq[26] introduced a new generalization of the Lomax distribution and Maxwell distributions using the T–X technique, allowing for more flexibility in modeling diverse datasets. Sapkota and Kumar[27] contributed to the Lomax family by introducing the odd Lomax generalized exponential distribution, which enables the modeling of complex data with heavy-tailed characteristics. Fayomi *et al.*[28] made significant contributions by introducing the unit-exponentiated Lomax distribution and its application using COVID-19 data. In addition, Fayomi *et al.*[29] defined the bivariate Lomax distribution, showcasing its versatility in handling multi-dimensional datasets in medical, environmental, and computer science data. Additionally, Al-Essa *et al.*[30] extended the Lomax distribution by defining a four-parameter form, encompassing the bivariate Lomax distribution as a special case, making it suitable for a broader range of data types and research domains. These advancements further enrich the applications and understanding of the Lomax distribution.

We have organized the rest of this work as follows: The Lomax-G family is introduced in Sec. II, while its particular member, the Lomax-inverse exponential distribution, is presented in Sec. III. Some statistical properties are discussed in Secs. IV and V, where we discuss statistical inferences from the suggested distribution. The simulation experiment, application, and conclusion of the suggested model are presented in Secs. VI–VIII, respectively.

II. LOMAX-G FAMILY OF DISTRIBUTIONS

Let $X > 0$ be a random variable following the Lomax distribution with shape parameter α and scale parameter β, then its cumulative distribution function (CDF) is

$$F(x;\alpha,\beta) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}; \quad \alpha > 0, \quad \beta > 0. \tag{1}$$

The probability distribution function (PDF) corresponding to Eq. (1) is

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}; \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

According to Eugene *et al.*,[1] the generating class of univariate continuous distribution can be obtained by using

$$\begin{aligned} G(x) &= 1 - \int_0^{-\log[B(x)]} b(t)dt \\ &= 1 - \int_0^{-\log[B(x)]} \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1} dx. \end{aligned} \tag{3}$$

Here, $B(x)$ and $b(t)$ represent CDF and PDF of any continuous distribution. In this work, we have chosen $b(t)$ as the PDF of Lomax distribution. After some algebra, we get the CDF $G(x)$ of the Lomax-G family of distributions as

$$G(x) = \beta^{\alpha}[\beta - \log\{B(x)\}]^{-\alpha}; \quad x \in \Re, \quad \alpha > 0, \quad \beta > 0. \tag{4}$$

The PDF of the Lomax-G family of distributions corresponding to Eq. (4) can be expressed as

$$g(x) = \alpha\beta^{\alpha}\frac{b(x)}{B(x)}[\beta - \log\{B(x)\}]^{-(\alpha+1)}; \quad x \in \Re, \tag{5}$$

where $b(x)$ and $B(x)$ are PDF and CDF of any continuous distribution.

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III. LOMAX-INVERSE EXPONENTIAL POWER (LIEP) DISTRIBUTION

To introduce a new member of the Lomax-G family, we have chosen inverse exponential power (IEP) distribution as a base distribution defined by Chaudhary *et al.*[31] having CDF,

$$B(x) = \exp\left[1 - e^{(\lambda/x)^{\delta}}\right]; \quad x > 0, \quad \delta > 0, \quad \lambda > 0. \tag{6}$$

Substituting Eq. (6) in Eq. (4), we get the CDF of Lomax-inverse exponential power (LIEP) distribution having four parameters $\xi = (\alpha, \beta, \delta, \lambda)$ as

$$G(x;\xi) = \beta^{\alpha}\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]^{-\alpha}; \quad x > 0, \quad \xi > 0. \tag{7}$$

The PDF of LIEP can be obtained by using Eq. (7) as

$$g(x;\xi) = \alpha\beta^{\alpha}\delta\lambda^{\delta}x^{-(\delta+1)}e^{(\lambda/x)^{\delta}}\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]^{-(\alpha+1)}. \tag{8}$$

A. Hazard rate function (HRF)

$$h(x;\xi) = \frac{\alpha\beta^{\alpha}\delta\lambda^{\delta}x^{-(\delta+1)}e^{(\lambda/x)^{\delta}}\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]^{-(\alpha+1)}}{1 - \beta^{\alpha}\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]^{-\alpha}}.$$

The LIEP distribution has a density plot that can take on a diversity of shapes, including symmetrical, left-skewed, right-skewed, or decreasing, and Fig. 1 (left) shows some examples of these shapes. The hazard rate function (HRF), on the other hand, can take the shape of an increasing, a j, or a reverse-j, and Fig. 1 (right) shows some examples of these shapes.

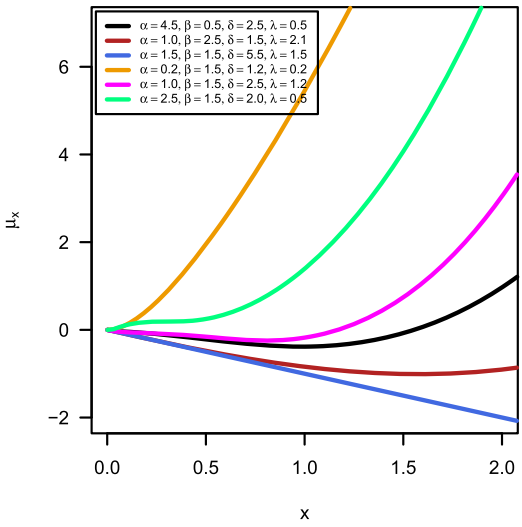


FIG. 2. MRLF plot for the different combinations of the parameters.

B. Quantile function

The quantile function of LIEP for q can be obtained by using Eq. (6) as

$$Q(q) = \lambda\left[\log\left(\beta q^{-1/\alpha} - \beta + 1\right)\right]^{-1/\delta}. \tag{9}$$

Reliability function,

$$R(x;\xi) = 1 - \beta^{\alpha}\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]^{-\alpha}; \quad x > 0, \quad \xi > 0.$$

Reverse hazard function,

$$Rh(x;\xi) = \frac{\alpha\delta\lambda^{\delta}x^{-(\delta+1)}e^{(\lambda/x)^{\delta}}}{\left[\beta - 1 + e^{(\lambda/x)^{\delta}}\right]}; \quad x > 0, \quad \xi > 0.$$

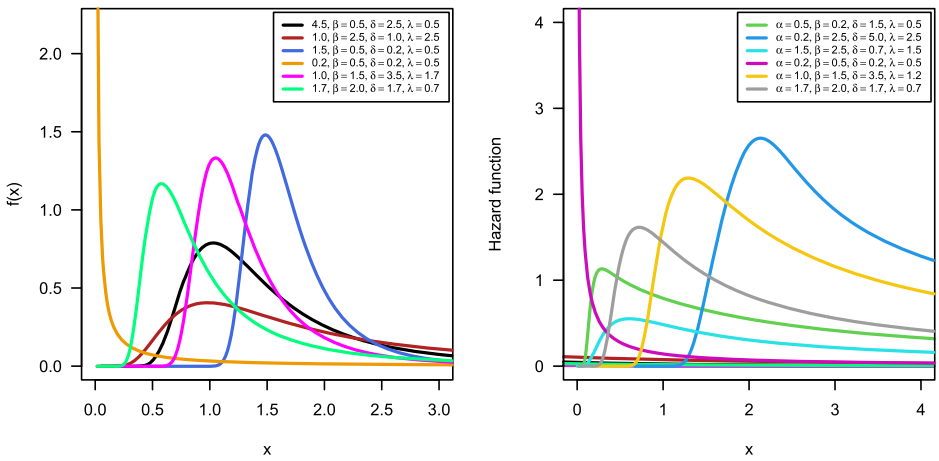


FIG. 1. Shapes of PDF and HRF of LIEP distribution.

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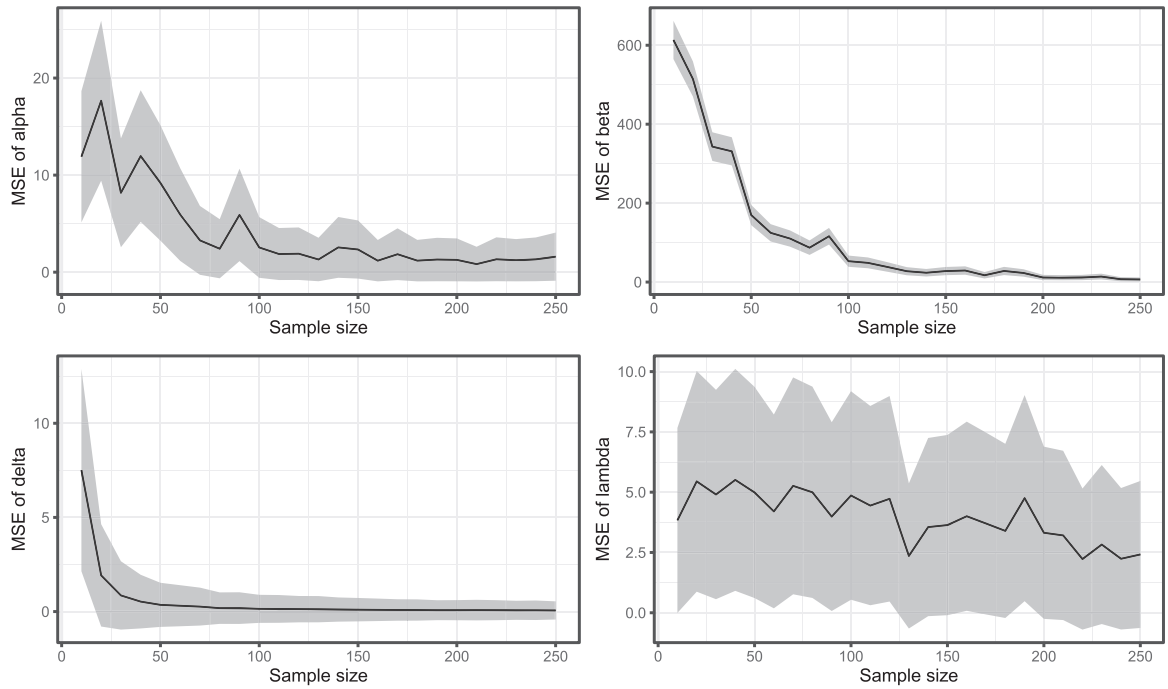


FIG. 3. MSEs for the combination of the parameters (set 1).

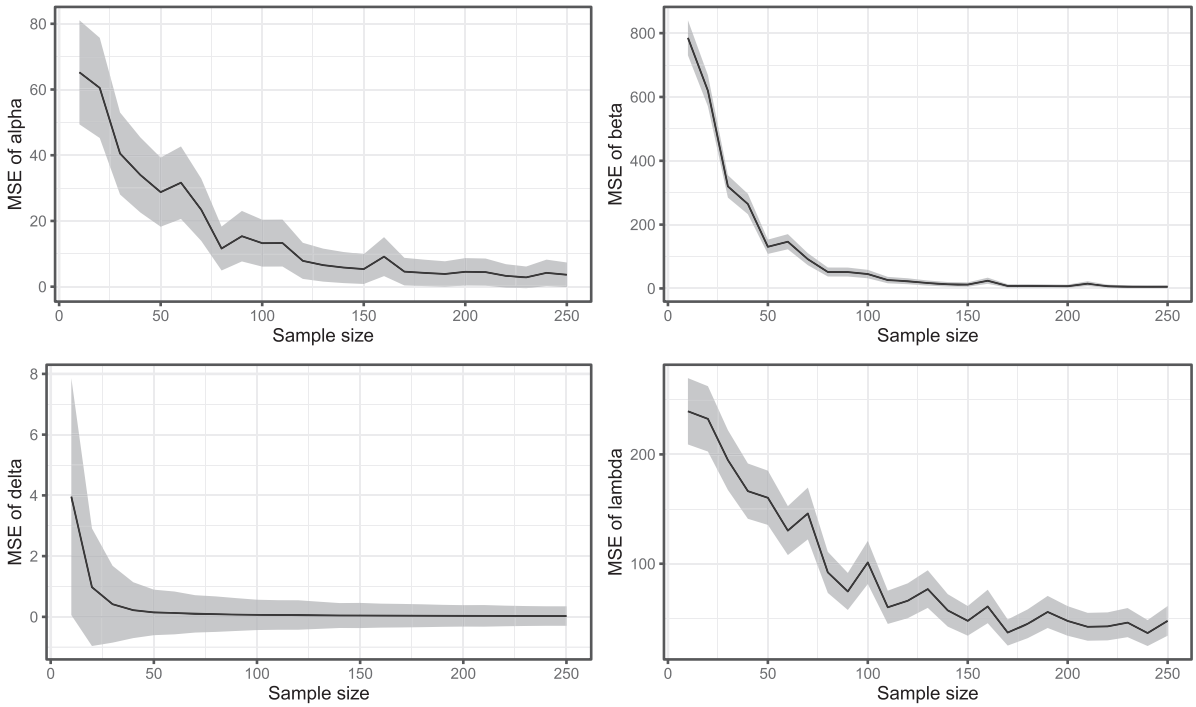


FIG. 4. MSEs for the combination of the parameters (set 2).

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IV. STATISTICAL PROPERTIES OF LIEP DISTRIBUTION

A. The linear form of LIEP distribution

The CDF defined in Eq. (7) can be expressed in the linear form as

$$G(x;\xi) = \sum_{k=0}^{\infty} \varphi_k e^{-(\alpha+k)\lambda^{\delta}x^{-\delta}}, \tag{10}$$

where $\varphi_k = (-1)^k \beta^{\alpha}(\beta-1)^k \binom{\alpha+k-1}{k}$. Now, Eq. (10) can be expressed as the linear mixture of inverse Weibull distribution as

$$G(x;\xi) = \sum_{k=0}^{\infty} \varphi_k D(x;\theta,\delta),$$

where $D(x;\ \theta,\delta)$ is the CDF of inverse Weibull distribution with scale parameter $\theta = (\alpha+k)\lambda^{\delta}$ and shape parameter δ. In addition, for $\delta = 2$, Eq. (10) can be expressed as the linear mixture of inverse Rayleigh distribution. Now differentiating Eq. (10) with respect to x, we get the linear form of PDF as

$$g(x;\xi) = \sum_{k=0}^{\infty} \varphi_k^* x^{-(\delta+1)} e^{-(\alpha+k)\lambda^{\delta}x^{-\delta}}, \tag{11}$$

where $\varphi_k^* = \delta(\alpha+k)\lambda^{\delta}\varphi_k$.

B. Moments

The rth moment of LIEP distribution can be calculated as

$$\mu_r' = \sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{r}{\delta}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{r}{\delta}}}; \quad \forall \delta > r. \tag{12}$$

Now mean and variance of LIEP distribution can be expressed as

$$E[X] = \sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{1}{\delta}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{1}{\delta}}}; \quad \delta > 1$$

and

$$V[X] = \sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{2}{\delta}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{2}{\delta}}} - \left[\sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{1}{\delta}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{1}{\delta}}}\right]^2; \quad \delta > 2.$$

C. Moment generating function (MGF)

For any real number t, the moment generating function (MGF) of LIEP distribution can be defined as

TABLE I. Simulation results for the combination of the parameters (set 1).

n	$bias_\alpha$	$bias_\beta$	$bias_\delta$	$bias_\lambda$	mse_α	mse_β	mse_δ	mse_λ	CIL_α	CIU_α	CIL_β	CIU_β	CIL_δ	CIU_δ	CIL_λ	CIU_λ
10	0.6159	11.9722	1.6324	0.5085	11.8924	613.4513	7.5131	3.831	0.0186	11.1900	0.0000	65.6017	0.6337	8.2593	0.1441	5.9404
20	0.6787	10.6695	0.6492	0.9038	17.6635	514.6765	1.9213	5.4487	0.0149	11.4493	0.0000	66.9041	0.6238	5.3543	0.1665	7.5358
30	0.4351	7.5788	0.4190	0.8142	8.1788	343.2281	0.8572	4.9044	0.0171	7.9878	0.0015	59.7210	0.6260	3.5534	0.1716	6.7417
40	0.5589	7.5363	0.2716	0.9034	11.9626	331.2176	0.5287	5.5102	0.0156	5.9711	0.0050	61.495	0.6021	3.1192	0.1896	8.0561
50	0.4599	4.9664	0.2135	0.7436	9.1917	169.8033	0.3568	4.9891	0.0160	4.2097	0.0106	43.3183	0.6113	2.7496	0.1813	7.4324
60	0.4599	4.0187	0.1835	0.6185	5.9366	124.7502	0.3094	4.203	0.0170	6.2882	0.0221	38.1904	0.6266	2.6268	0.1546	6.9496
70	0.2267	3.7786	0.1583	0.7956	3.2746	110.1580	0.2638	5.2624	0.0167	1.6668	0.0229	36.5936	0.6374	2.4671	0.1967	8.0655
80	0.2319	3.2846	0.1041	0.8514	2.4137	87.1337	0.1820	4.9956	0.0144	1.8742	0.0474	32.1014	0.6302	2.2464	0.1981	8.457
90	0.2905	3.1433	0.1200	0.6911	5.9001	116.0877	0.184	3.9881	0.0163	1.7413	0.0496	32.9314	0.6979	2.2227	0.2066	7.0032
100	0.1699	2.4048	0.1002	0.7725	2.5449	52.9193	0.1444	4.8627	0.0158	1.3070	0.0610	26.7874	0.7003	2.0798	0.2167	7.4621
110	0.2213	2.2130	0.0939	0.6694	1.8688	48.6787	0.1413	4.4445	0.0156	1.6642	0.0706	21.8758	0.7103	2.1203	0.1872	7.3497
120	0.2249	1.8750	0.0877	0.6891	1.9031	38.2733	0.1273	4.7266	0.0163	1.8342	0.0710	17.9855	0.7089	2.0101	0.2070	7.3671
130	0.1868	1.5476	0.0880	0.4953	1.3063	27.4983	0.1264	2.3567	0.0246	1.5631	0.0724	15.3982	0.7329	2.0326	0.2016	5.6462
140	0.167	1.5941	0.0711	0.6432	2.5525	23.6388	0.1075	3.5532	0.0211	1.2076	0.0892	16.2715	0.7798	1.9429	0.2212	7.0503
150	0.2211	1.5373	0.0602	0.6437	2.335	27.8362	0.1004	3.637	0.0181	1.7462	0.0886	15.4694	0.761	1.9266	0.1684	7.1952
160	0.1715	1.4012	0.0616	0.5667	1.1835	29.2635	0.0927	4.0035	0.0234	1.5192	0.1066	13.058	0.7894	1.8835	0.2158	5.4868
170	0.2036	1.0978	0.0692	0.5866	1.8504	17.2413	0.0844	3.7010	0.0164	1.2857	0.1219	10.5726	0.8122	1.9056	0.1917	6.9637
180	0.1729	1.5115	0.0365	0.5915	1.1889	28.1242	0.0829	3.3949	0.0229	1.5509	0.1232	13.9589	0.77	1.8724	0.2185	5.9555
190	0.1665	1.2383	0.0442	0.6914	1.3027	23.0566	0.0728	4.7536	0.0127	1.3452	0.1328	8.8382	0.8251	1.8241	0.1886	7.9621
200	0.1648	1.0434	0.0355	0.5827	1.2682	11.5416	0.0741	3.3177	0.018	1.2186	0.1512	9.4991	0.7981	1.7786	0.2310	5.8810
210	0.1472	0.9531	0.0498	0.5178	0.831	11.0711	0.0778	3.2078	0.0192	1.3431	0.1367	9.1961	0.8093	1.8497	0.2333	5.7696
220	0.1634	0.7307	0.0588	0.4291	1.3284	11.5116	0.0736	2.2272	0.0233	1.1491	0.1384	6.957	0.8439	1.8400	0.2247	5.1529
230	0.1663	0.8388	0.0434	0.5008	1.2318	13.9202	0.0667	2.8292	0.0197	1.1863	0.1611	7.2517	0.8591	1.8072	0.2341	5.5172
240	0.1771	0.8572	0.0348	0.4423	1.323	7.2117	0.0698	2.2394	0.0214	1.2547	0.1622	7.3253	0.8328	1.8116	0.2353	5.1476
250	0.1905	0.6928	0.0393	0.4219	1.5978	6.6089	0.0604	2.4180	0.0259	1.3409	0.1746	7.0306	0.8329	1.7770	0.2145	4.6022

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$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!}\mu_r' = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^p \varphi_k^*}{p!} \frac{\Gamma\left(1-\frac{r}{\delta}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{r}{\delta}}}; \quad \forall \delta > r. \tag{13}$$

D. Conditional moment

The conditional moment for LIEP distribution is given by

$$E\left[X^n/X > x\right] = \frac{1}{R(x;\xi)} \sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{n}{\delta}, (\alpha+k)\lambda^{\delta}x^{-\frac{1}{\delta}}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{n}{\delta}}},$$

where $R(x;\xi)$ is reliability function, and $\Gamma(a,b) = \int_b^{\infty} x^{a-1}e^{-x}dx.$ is the upper incomplete gamma function.

E. Mean residual life function (MRLF)

The Mean Residual Life (MRL) function is a concept used in survival analysis and reliability engineering to understand the remaining life expectancy of a system or entity, given that it has survived up to a certain point in time. In simpler terms, it provides an average estimation of the additional lifespan for an individual or system beyond a specific time point, given that it has already survived up to that point. We have displayed the possible shapes of mean residual life function (MRLF) in Fig. 2.

Mathematically, the mean residual life for LIEP distribution is given by

$$\mu_x = \frac{1}{R(x;\xi)} \sum_{k=0}^{\infty} \varphi_k^* \frac{\Gamma\left(1-\frac{1}{\delta}, (\alpha+k)\lambda^{\delta}x^{-\frac{1}{\delta}}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{1}{\delta}}} - x\,; \quad \delta > 1,$$

where $R(x;\xi)$ is the reliability function, X represents the lifetime random variable (time to failure, death, or event of interest), and x is a specific point in time at which we want to calculate the mean residual life.

F. Average past lifetime function

Let $a(x)$ denote the average past lifetime function of an item or system of items, and it can be obtained as

$$a(x) = x - \frac{1}{G(x;\xi)} \sum_{k=0}^{\infty} \varphi_k^* \frac{\gamma\left(1-\frac{1}{\delta}, (\alpha+k)\lambda^{\delta}x^{-\frac{1}{\delta}}\right)}{\left\{(\alpha+k)\lambda^{\delta}\right\}^{1-\frac{1}{\delta}}}; \quad \delta > 1,$$

where $G(x;\xi)$ is the CDF of LIEP distribution.

TABLE II. Simulation results for the combination of the parameters (set 2).

n	$bias_\alpha$	$bias_\beta$	$bias_\delta$	$bias_\lambda$	mse_α	mse_β	mse_δ	mse_λ	CIL_α	CIU_α	CIL_β	CIU_β	CIL_δ	CIU_δ	CIL_λ	CIU_λ
10	3.2497	13.7331	1.1917	3.2252	65.2266	785.2765	3.9599	239.4232	0.0418	24.3196	0.0000	69.3113	0.3823	5.7847	0.2551	54.5491
20	2.8096	10.2410	0.5084	3.7787	60.4839	619.3419	0.9768	232.4825	0.0595	26.4776	0.0000	73.7202	0.3899	3.7510	0.2982	50.2138
30	2.025	7.2348	0.2828	4.2208	40.5300	319.7019	0.4162	194.8114	0.0751	23.9277	0.0020	59.9218	0.4001	2.2840	0.2896	47.8040
40	1.7238	5.9450	0.1767	3.9876	34.0027	264.4628	0.2203	166.3895	0.0799	20.5098	0.0072	60.9739	0.4040	1.9149	0.2830	38.7120
50	1.6971	4.3925	0.1264	3.864	28.7622	130.5248	0.1463	160.3201	0.0897	19.6055	0.0173	39.0024	0.4284	1.7004	0.2778	41.0453
60	1.7843	4.3752	0.1194	3.4395	31.6448	146.3839	0.1294	130.3052	0.0947	19.6715	0.0239	38.7895	0.4236	1.6140	0.2424	34.5850
70	1.5105	3.5471	0.0900	3.6620	23.3973	91.6674	0.0984	145.9433	0.1013	17.8330	0.0370	29.8204	0.4342	1.5107	0.2688	35.4930
80	0.9363	2.4836	0.0840	3.2545	11.6101	51.4609	0.0889	92.0926	0.0977	12.2756	0.0511	19.3528	0.4384	1.4368	0.2927	32.2271
90	1.1924	2.4966	0.0691	2.6211	15.3612	51.3151	0.0759	74.6438	0.1131	13.3142	0.0555	19.9952	0.4560	1.4032	0.2423	31.0449
100	1.0899	2.4423	0.0602	2.9753	13.2409	45.0917	0.0649	101.1476	0.1207	12.7537	0.0697	18.4998	0.4452	1.3446	0.3007	32.7062
110	1.0575	1.9014	0.0622	2.5318	13.2803	26.1450	0.0617	60.1396	0.1078	12.8646	0.0830	16.0069	0.4662	1.3531	0.2662	27.0719
120	0.8453	1.7831	0.0535	2.7358	7.8617	22.5123	0.0612	66.1544	0.117	10.0152	0.0916	14.6046	0.4587	1.3151	0.2638	28.5436
130	0.7201	1.6019	0.0471	3.2083	6.5607	16.7968	0.052	76.8065	0.0960	8.9941	0.0966	13.1309	0.4790	1.2693	0.2922	32.9036
140	0.6038	1.3216	0.0367	2.6577	5.8233	12.9636	0.0438	57.3258	0.1142	8.3120	0.1280	11.1151	0.4652	1.2022	0.2467	25.0007
150	0.6734	1.223	0.0448	2.2071	5.3545	12.0491	0.0447	47.7593	0.1116	8.9325	0.1134	11.0336	0.4748	1.2092	0.2596	24.411
160	0.8096	1.5987	0.0249	2.9072	9.1443	24.0892	0.0406	60.9081	0.1084	10.1352	0.145	12.1752	0.4643	1.2004	0.2428	25.9819
170	0.6142	0.9636	0.0455	1.8037	4.5796	7.7142	0.0389	37.0525	0.1141	7.6173	0.1413	9.5415	0.4993	1.1952	0.2752	20.7868
180	0.5491	1.0310	0.0326	2.2303	4.1921	7.8154	0.0369	45.0861	0.1287	8.4949	0.1411	9.4799	0.4910	1.1770	0.2581	22.6821
190	0.514	0.9913	0.0304	2.6404	3.8689	7.3374	0.0341	55.8966	0.1065	7.2329	0.1649	9.1146	0.5005	1.1521	0.2941	22.9412
200	0.5369	0.9341	0.0305	2.3904	4.5248	6.8511	0.0323	47.6563	0.1063	8.3562	0.1639	8.7169	0.5056	1.1363	0.2804	21.4735
210	0.5119	0.9554	0.0372	2.1834	4.4355	14.9601	0.0330	42.3169	0.1140	8.0295	0.1775	8.3203	0.4940	1.1488	0.2979	21.9983
220	0.4816	0.8948	0.0247	2.1744	3.2862	6.8952	0.0300	42.6978	0.1143	6.6034	0.1983	7.9415	0.5000	1.1237	0.2495	21.0309
230	0.4233	0.7979	0.0239	2.1198	2.8358	5.1637	0.0280	46.1394	0.1162	6.0536	0.2024	8.2942	0.4981	1.0970	0.2728	21.2731
240	0.5168	0.7197	0.0281	2.0346	4.2106	5.1542	0.0269	36.5820	0.1063	7.0171	0.1826	7.5305	0.5131	1.1219	0.2249	19.1257
250	0.5194	0.7933	0.0230	2.1214	3.6298	5.2081	0.0267	47.7135	0.1182	6.5214	0.2035	7.8465	0.5198	1.1154	0.2634	21.7215

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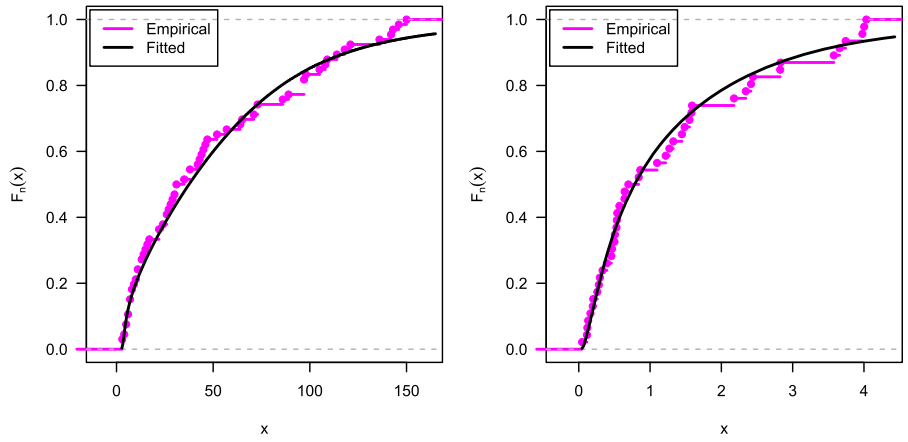


FIG. 5. Graphs of the KS plot of the LIEP distribution (left for dataset-I and right for dataset-II).

G. Skewness and kurtosis

To determine the coefficient of skewness and kurtosis based on the quartiles using the quantile function (9), we can use the following expressions:

$$S_k(B) = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

and

$$K_u(M) = \frac{Q(0.875) + Q(0.375) - Q(0.125) - Q(0.625)}{Q(3/4) - Q(1/4)},$$

respectively.

V. ESTIMATION OF LIEP PARAMETERS

In this section, we estimate the parameters of the LIEP model using the maximum likelihood method. Let $X = (x_1, \ldots, x_n)^T$ be a vector of size n of independent random variables from the LIEP distribution. Then, the log-likelihood is given by

$$l(x;\xi) = n \log(\alpha\beta^{\alpha}\delta\lambda^{\delta}) - (\delta+1)\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} (\lambda/x_i)^{\delta} - (\alpha+1)\sum_{i=1}^{n} \log\left(\beta - 1 + e^{(\lambda/x_i)^{\delta}}\right). \quad (14)$$

Partially differentiating Eq. (14) with respect to $\xi = (\alpha, \beta, \delta, \lambda)$, we get

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \beta^{\alpha} \log \beta - \sum_{i=1}^{n} \log\left(\beta - 1 + e^{(\lambda/x_i)^{\delta}}\right),$$

$$\frac{\partial l}{\partial \beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n} \left(\beta - 1 + e^{(\lambda/x_i)^{\delta}}\right)^{-1},$$

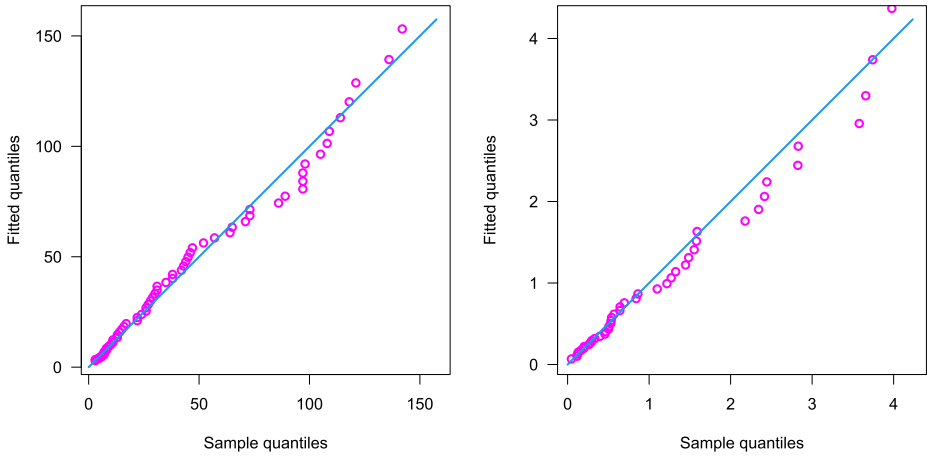


FIG. 6. Graphs of the QQ plot of the LIEP distribution (left for dataset-I and right for dataset-II).

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TABLE III. Estimated parameters using the MLE method for the data set-I.

Model	Parameter	SE	Parameter	SE	Parameter	SE	Parameter	SE
LIEP$(\alpha,\beta,\delta,\lambda)$	0.2133	0.1278	2.00×10^{-4}	0.0011	3.1575	1.5185	6.9404	1.3488
LogisticL(α,β,λ)	1.0641	0.1161	4.00×10^{-4}	0.0000	53.2133	3.4958	⋯	⋯
LIL(α,δ,λ)	2.1295	0.1999	112.6553	15.7231	0.4701	0.0064	⋯	⋯
GE(α,λ)	1.1470	0.1885	0.0219	0.0034	⋯	⋯	⋯	⋯
PIL(α,β,θ)	1.3162	0.1856	209.2384	6.3717	9.1540	2.8594	⋯	⋯
PEP(α,β,λ)	0.2127	0.0534	0.6356	0.1844	11.5982	5.6232	⋯	⋯
ATGE(α,β,δ)	0.0194	0.7206	1.1454	0.1885	0.0219	0.0034	⋯	⋯

$$\frac{\partial l}{\partial \delta} = \frac{n}{\delta} + \lambda^{\delta}\log\lambda - \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}(\lambda/x_i)^{\delta}\log(\lambda/x_i) - (\alpha+1) \times \sum_{i=1}^{n}\left(\beta - 1 + e^{(\lambda/x_i)^{\delta}}\right)^{-1}(\lambda/x_i)^{\delta}e^{(\lambda/x_i)^{\delta}}\log(\lambda/x_i),$$

$$\frac{\partial l}{\partial \lambda} = \frac{n\delta}{\lambda} + \delta\sum_{i=1}^{n}\lambda^{\delta-1}(1/x_i)^{\delta} - (\alpha+1)\delta\sum_{i=1}^{n}\lambda^{\delta-1}(1/x_i)^{\delta}e^{(\lambda/x_i)^{\delta}}\left(\beta - 1 + e^{(\lambda/x_i)^{\delta}}\right)^{-1}.$$

The MLEs of $\xi = (\alpha,\beta,\delta,\lambda)$ can be obtained by maximizing the above nonlinear equations, with the aid of some suitable software.

VI. SIMULATION STUDY

In our research study, we employed the maxLik R package, developed by Henningsen and Toomet,[32] to generate samples from the quantile function described in Eq. (9) for various combinations of parameters of the PiPHLIW distribution. The MLEs were then computed for each sample using the `maxLik()` function and the BFGS algorithm. This analysis allowed us to investigate issues related to parameter estimation and determine the direction and magnitude of bias in the MLEs, whether it was overestimation or underestimation.

In our simulation, we utilized sample sizes ranging from 10 to 250 with increments of 10 for the two sets of parameter values set 1 = $(\alpha = 0.25, \beta = 1.0, \delta = 1.25, \lambda = 0.75)$ and set 2 = $\alpha = 0.5$, $\beta = 1.25, \delta = 0.75, \lambda = 2.75$, respectively. The entire process was repeated 1000 times in order to obtain estimates of the mean square error (MSE). The MSEs for the two different parameter combinations are presented in Figs. 3 and 4 with a 95% confidence bound and these numerical values are presented in Tables I and II. The results demonstrate that the corresponding MSEs decrease toward zero as the sample size increases, but there is a problem with the

TABLE IV. Estimated parameters using the MLE method for the data set-II.

Model	Parameter	SE	Parameter	SE	Parameter	SE	Parameter	SE
LIEP$(\alpha,\beta,\delta,\lambda)$	0.0601	0.029	143.1119	11.9212	0.6501	0.0667	58.4111	5.4464
LogisticL(α,β,λ)	1.0561	0.1372	0.0185	0.0067	40.5734	13.0684	⋯	⋯
LIL(α,δ,λ)	3.9951	0.5015	28.3396	4.5638	0.1973	0.0128	⋯	⋯
GE(α,λ)	1.1172	0.2203	0.8101	0.1522	⋯	⋯	⋯	⋯
PIL(α,β,θ)	1.1205	0.0305	19.2337	1.5772	21.6935	10.3783	⋯	⋯
PEP(α,β,λ)	0.1916	0.0778	1.4341	0.234	14.2485	12.641	⋯	⋯
ATGE(α,β,δ)	4.00×10^{-4}	0.7548	1.1172	0.2203	0.8102	0.1522	⋯	⋯

TABLE V. Model selection and goodness of fit statistics for the data set-I.

Model	$-2\log L$	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
LIEP	641.3373	649.3373	652.7983	0.0699	0.9037	0.0485	0.8867	0.4076	0.8407
LogisticL	647.6865	653.6865	656.2822	0.0888	0.6750	0.0810	0.6877	0.6613	0.5914
LIL	653.4832	659.4832	662.0789	0.9621	0.0000	0.1027	0.5733	0.8319	0.4581
GE	647.0264	651.0264	652.7569	0.0917	0.6358	0.0989	0.5918	0.7484	0.5192
PIL	650.3487	656.3487	658.9444	0.0940	0.6037	0.1193	0.4993	0.9077	0.4091
PEP	648.7473	654.7473	657.3430	0.0914	0.6401	0.1041	0.5667	0.7950	0.4842
ATGE	647.0273	653.0273	655.6230	0.0917	0.6359	0.0984	0.5944	0.7457	0.5213

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TABLE VI. Model selection and goodness of fit statistics for the data set-II.

Model	−2 log L	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
LIEP	116.5425	124.5425	127.2826	0.0753	0.9393	0.0485	0.8876	0.3645	0.8825
LogisticL	117.8684	123.8684	125.9235	0.1024	0.6825	0.0754	0.7212	0.5081	0.7377
LIL	120.0992	126.0992	128.1542	0.9763	0.0000	0.0729	0.7364	0.4930	0.7530
GE	117.5570	121.5570	122.9270	0.1125	0.5671	0.0905	0.6356	0.5728	0.6732
PIL	118.0559	124.0559	126.1110	0.1064	0.6359	0.0818	0.6840	0.5274	0.7181
PEP	117.8152	123.8152	125.8702	0.0826	0.8867	0.0675	0.7697	0.4577	0.7894
ATGE	117.5570	123.557	125.612	0.1124	0.5675	0.0905	0.6357	0.5728	0.6732

parameter lambda for small values (for $\lambda < 10$). This finding suggests that the MLE method exhibits asymptotic efficiency and consistency and follows the invariance property.

VII. MODEL VALIDATION AND APPLICATION

In this section, we demonstrate the application of the LIEP distribution using two real datasets. The datasets utilized for applying the suggested distribution are presented below.

A. First dataset

These data are reported by Chipepa *et al.*,[33] which represent daily new deaths due to COVID-19 in China from January 23, 2020 to March 28, 2020. The data are: "8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6,9, 7, 4, 6, 5, 3, 5."

B. Second dataset

The original data are a subset of data that were reported by Stablein *et al.*[34] and also used by Bekker *et al.*[35] and represent the survival times in years of a group of patients given chemotherapy treatment. The data consisting of 46 survival times (in years) for 46 patients are: "0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033."

C. Model analysis

We computed several well-known goodness-of-fit statistics to analyze both datasets. The fitted models were evaluated using various metrics, including the log-likelihood value (−2 log L), Akaike information criterion (AIC), the Hannan–Quinn information criterion (HQIC), Anderson–Darling (AD), Kolmogorov–Smirnov (KS), and Cramer–von Mises (CVM), with corresponding p-values. All the essential computations and graphical plots were performed using the R software (Wickham and Grolemund)[36] and the R Core Team.[37] To compare the fitting capability of the LIEP model, we have selected several models, such as the logistic Lomax (LL) (Chaudhary and Kumar),[38] logistic inverse Lomax (LIL) (Joshi and Kumar),[39] generalized exponential (GE) (Gupta and Kundu),[40]

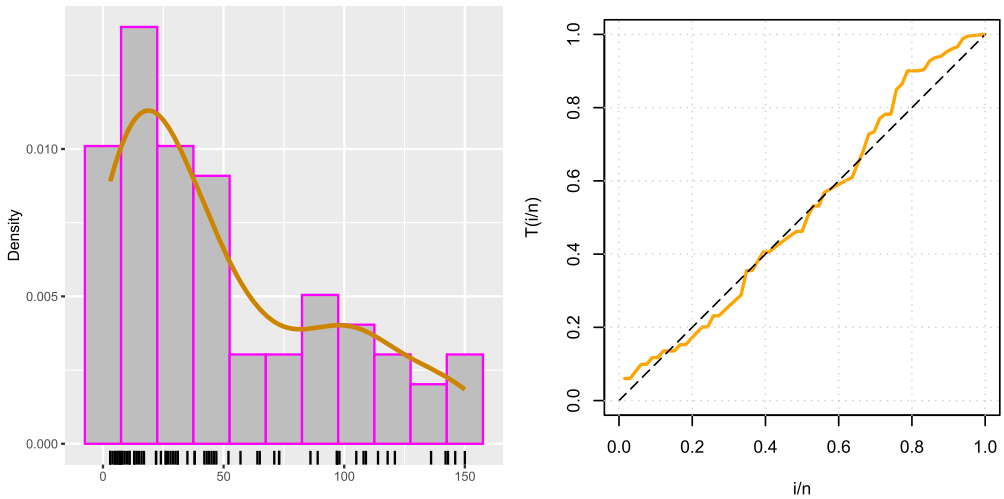


FIG. 7. Histogram and TTT plot of the LIEP distribution (dataset-I).

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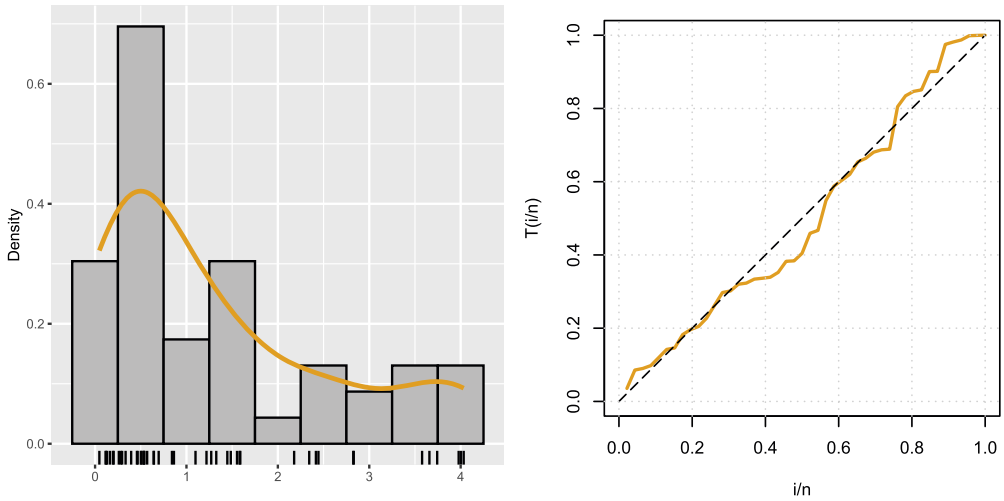


FIG. 8. Histogram and TTT plot of the LIEP distribution (dataset-II).

Poisson inverted Lomax (PIL) (Joshi and Kumar),[41] Poisson exponential power (PEP) (Joshi and Kumar),[42] and arctan generalized exponential (ATGE) (Chaudhary *et al.*).[43]

We have presented the KS plots, which provide an estimate of the CDF based on both datasets under study, in Fig. 5 (left for dataset-I and left for dataset-II), and also we have displayed the quantile–quantile (QQ) plots in Fig. 6 (left for dataset-I and left for dataset-II). The KS and QQ plots indicate that the suggested model is capable of fitting both datasets under study. The estimated values of the parameters and their associated standard errors (SE) for both datasets are presented in Tables III and IV, which were obtained using the MLE method. Additionally, Tables V and VI showcase model selection criteria, such as log-likelihood, HQIC, and AIC, and goodness of fit statistics, such as KS, along with p-values for both datasets. Our observations show that the LIEP model has the lowest statistics compared to the LL, LIL, GE, PIL, PEP, and ATGE distributions, along with the highest p-values. This indicates that the LIEP model is more flexible and provides a good fit. Histogram with estimated PDF for the two real data and TTT plots set are presented in Figs. 7 and 8. Furthermore, we have provided graphical illustrations of the fitted models in Fig. 9 (left) and Fig. 10 (left), which support our findings that the LIEP model outperforms the other candidate models. We have also displayed the graphs of the Kaplan–Meier estimate of the LIEP model in Fig. 9 (right) and Fig. 10 (right), which provide an estimate of the survival function based on the available data under study.

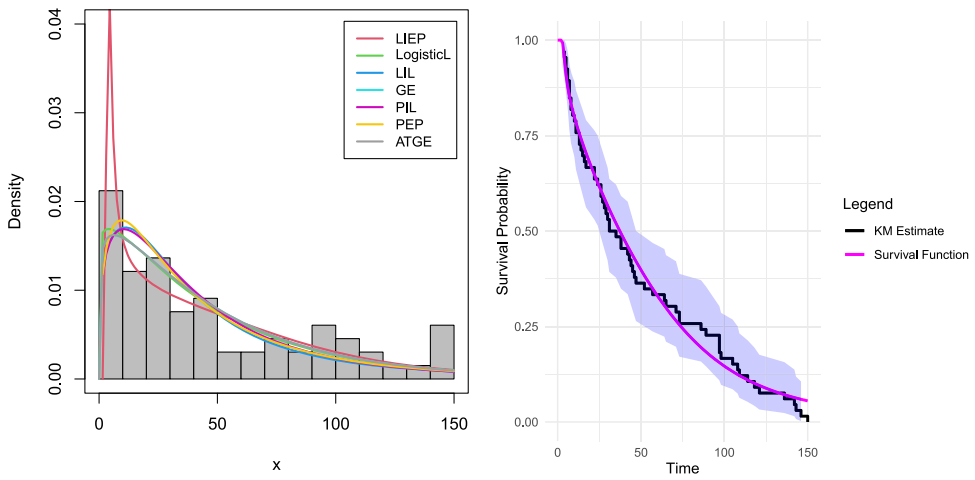


FIG. 9. Graphs of the fitted PDF and KM estimate plot of the LIEP distribution (dataset-I).

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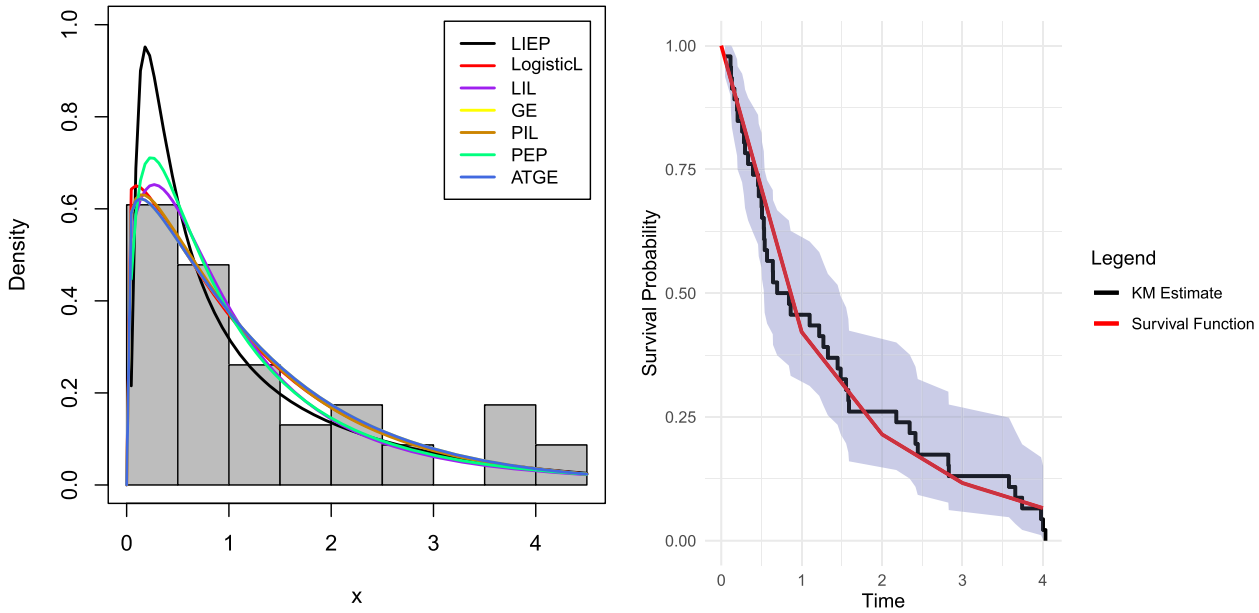


FIG. 10. Graphs of the fitted PDF and KM estimate plot of the LIEP distribution (dataset-II).

VIII. CONCLUSION

In this study, we have introduced an innovative distribution family called the new Lomax-G family. We have selected the inverse exponential power distribution as the base distribution for this new family. The new LIEP distribution offers a wide range of hazard function shapes, including increasing, inverted-bathtub, and reverse-J-shaped. By employing the maximum likelihood estimation technique, we explored the statistical properties of this distribution and estimated its parameters. To assess the accuracy of our estimation method, we conducted a Monte Carlo simulation. The results revealed that the mean square errors decrease as the sample size increases, even when dealing with small samples. To demonstrate the practical utility of the LIEP distribution, we applied it to two real COVID-19 and medical datasets. Through model selection criteria and goodness-of-fit tests, we compared its performance against seven existing models. Our findings strongly support the superiority of the LIEP distribution over the alternative models, suggesting its potential application in various fields, such as medical and life sciences, reliability engineering, actuarial science, and survival analysis. We have proposed not just a singular distribution but rather a whole family of distributions. This approach allows for the potential creation of new models down the line. Moreover, these distributions find utility across a range of disciplines, including reliability engineering, medical and life sciences, and so on. Furthermore, this suggested distribution family holds exciting potential as a basis for crafting innovative models in the times ahead. Expanding upon this work in the future could enable the exploration of Bayesian analysis and inferences, regression analysis, and even extensions into bivariate analyses.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Authors have worked equally to write and review the manuscript.

Laxmi Prasad Sapkota: Conceptualization (equal); Data curation (equal); Resources (equal); Software (equal); Writing – review & editing (equal). **Vijay Kumar**: Investigation (equal); Methodology (equal); Project administration (equal). **Ahmed M. Gemeay**: Software (equal); Supervision (equal); Validation (equal). **M. E. Bakr**: Formal analysis (equal); Resources (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **Oluwafemi Samson Balogun**: Formal analysis (equal); Resources (equal); Validation (equal). **Abdisalam Hassan Muse**: Data curation (equal); Visualization (equal); Writing – original draft (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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