

# Unit - IV

## Curve Tracing

Introduction :- The purpose of curve tracing is to find the approximate shape of the curve without plotting of large number of points. We can trace the curve using different theories viz. symmetry, tangents, asymptotes, region of absence etc.

To understand methods of curve tracing we divide curves into following types.

a) Cartesian equation :-  $f(x, y)$ .

i) Explicit Relation

ii) Implicit Relation.

b) Polar equation :-  $f(r, \theta)$

c) Parametric equation :-  $x = f(t), y = g(t)$

### 4.1 Rules for Tracing cartesian curve :-

Rule 1 : Symmetry :-

(i) About  $x$ -axis :- If the given equation has even powers of  $y$  everywhere, then curve is symmetric about  $x$ -axis.

(ii) About  $Y$ -axis :- If the given equation has even powers of  $x$  everywhere, then curve is symmetric about  $Y$ -axis.

(iii) About both the axis :- If all powers of  $x$  and  $y$  are even, everywhere, then curve is symmetric about both the axis.

(iv) About the line  $y=x$  :- If we replace  $y$  by  $x$  and  $x$  by  $y$ , equation remains the same, curve symmetric about the line  $y=x$  line.

(v) About the line  $y=-x$  :- If we replace  $x$  by  $-y$  and  $y$  by  $-x$ , equation remains the same, curve is symmetric about the line  $y=-x$ .

(vi) In opposite quadrant :- If we replace  $x$  by  $-x$  and  $y$  by  $-y$ , equation remains the same, curve symmetric in opposite quadrant.

[Note that to trace curve in cartesian form we have to ~~remember~~ state at least any one of the above symmetry so that we get an idea to trace the curve]

Rule 2 :- Origin :- put  $x=0$  or  $y=0$

If we get ~~to~~

$y=0$  or  $x=0$  i.e. we get  $(0,0)$  point on the curve, then ~~we~~ we say that curve passing through the origin

If the curve passing through the origin then we find tangent at origin

(i) Tangent at origin :-

To calculate tangent at origin we equate lowest degree term to zero.

Rule 3 :- Intersection with co-ordinate axis

(i)  $x$ -intercept :- put  $y=0$  we get  $x$  intercept.

(ii)  $y$ -intercept :- put  $x=0$  we get  $y$  intercept.

Rule 4 :- Asymptotes :-

Asymptotes are tangents to the curve at infinity.

(i) Asymptote parallel to  $x$  axis :-

To calculate asymptote parallel to  $x$  axis we equate coefficient of highest power of  $x$  to zero.

(ii) Asymptote parallel to  $y$  axis :-

To calculate asymptote parallel to  $y$ -axis we equate coefficient of highest power of  $y$  to zero.

(iii) oblique Asymptote &

Let  $y = mx + c$  be oblique Asymptote.  
put  $y = mx + c$  in given equation and  
by comparing L.H.S. = R.H.S. to find the  
values of  $m$  and  $c$ . we get oblique  
Asymptote.

Note That Generally we find Asymptote  
parallel to either  $x$  or  $y$  axis only.

Rule 5 Region of absence of curves

If possible write the equation of  
the curve in either  $y^2 = f(x)$  or  
 $x^2 = f(y)$ . Then for those values  
of  $x$ ,  $y^2$  becomes negative or for  
those values of  $y$ ,  $x^2$  becomes negative  
then in that region curve does not  
exist.

Rule 6 & Tangents at special points.

To find tangents at special points,  
we find  $\frac{dy}{dx}$  from given eqn of curve.  
& then.

(i) If  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$  then tangent  
is parallel to  $x$  axis  
at  $(x_1, y_1)$

(i) If  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$ , then tangent is parallel to Y-axis at  $(x_1, y_1)$ .

(ii) If  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = +ve$  then tangent makes acute angle with x-axis at  $(x_1, y_1)$ .

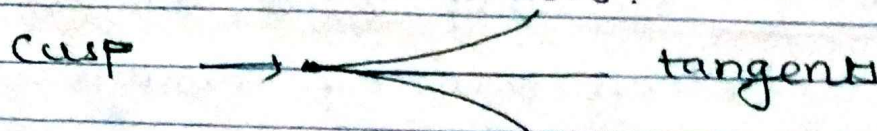
(iii) If  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -ve$ , then tangent makes obtuse angle with Y-axis at  $(x_1, y_1)$ .

\* Note that the following Important points.

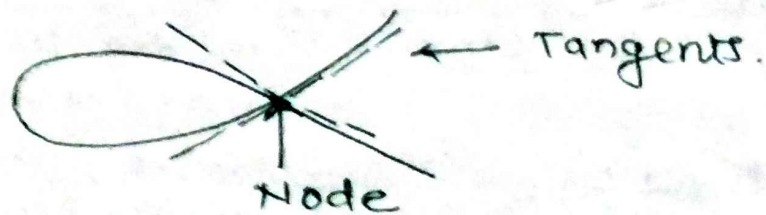
(i) Double Point :- A point from which two or more branches of curve passing through it is called double point.



(ii) Cusp :- A double point is called cusp if tangents at are real and coincides.



(ii) Node :- A double point is called a node if the tangents are at real and distinct



4.1.1 Trace the curve  
 $y^2(2a-x) = x^3$

Solving

We have

$$y^2(2a-x) = x^3$$

$$\text{i.e. } y^2 = \frac{x^3}{(2a-x)}$$

(i) Symmetry :- since all powers of  $y$  are even, curve symmetric about  $x$ -axis.

(ii) Origin :- put  $y=0$ , we get  $x=0$   
 $\therefore$  curve passing through origin.

(a) Tangent at origin :-

We equate lowest degree term to zero.

i.e.  $2ay^2 = 0$

i.e.  $y^2 = 0 \quad \therefore y = 0, y = 0$

Thus  $x$ -axis is a tangent at origin.  
which coincides.

(iii) Intercepts :-

(i)  $x$ -intercept: Put  $y = 0$  we get  
 $x = 0$ . Thus we  
have  $(0, 0)$  is  $x$ -intercept

(ii)  $y$ -intercept: Put  $x = 0$  we get  
 $y = 0$ . Thus we  
have  $(0, 0)$  is  $y$ -intercept.

(iv) Asymptote :-

(a) Parallel to  $x$  axis :-  
equate coeff. of highest power of  
 $x$  to zero.

i.e.  $1 = 0$ . which is absurd.

Thus asymptote does not exist

(b) Parallel to  $y$ -axis :-

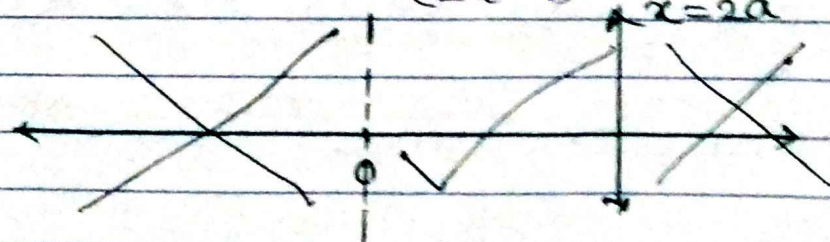
Equate coeff. of highest power of  
 $y$  to zero,

i.e.  $(2a - x) = 0 \quad \therefore x = 2a$ .

is the line parallel  $y$  axis is an asymptote

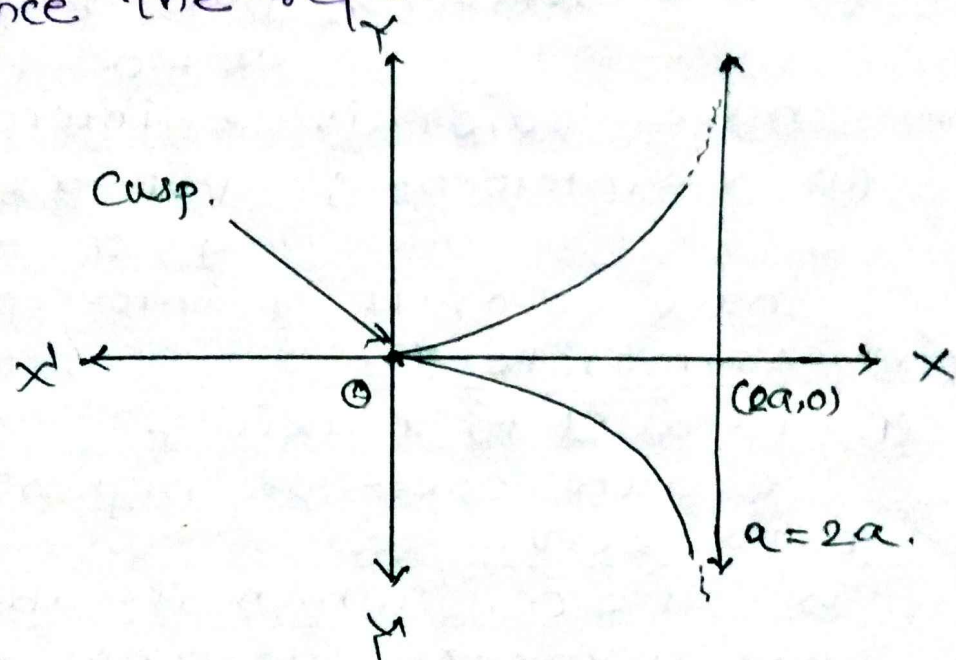
(v) Region of absence of the curve :-

Here  $y^2 = \frac{x^3}{(2a-x)}$



Here we observe that for  $x < 0$  and  $x > 2a$  curve does not exist and  $0 < x < 2a$  curve is exist.

Hence the required curve is.



Home-work

- i) Trace the curve  $y^2(a^2 - x^2) = a^3x$
- ii) Trace the curve  $y^2x = 4a^2(2a - x)$
- iii) Trace the curve  $y^2 = x^5(2a - x)$
- iv) Trace the curve  $ay^2 = x(a^2 - x)$
- v) Trace the curve  $y = x^3$

## 4.2 Parametric curves :-

Consider the parametric equations are given by  $x = f(t)$  and  $y = g(t)$

### Rules for tracing parametric equations :-

#### Rule 1: Symmetry

- (i) If  $x = f(t)$  is even function i.e.  $f(-t) = f(t)$  and  $y = g(t)$  is odd function i.e.  $g(-t) = -g(t)$ , then curve is symmetric about  $x$ -axis.
- (ii) If  $x = f(t)$  is odd function i.e.  $f(-t) = -f(t)$  and  $y = g(t)$  is even function i.e.  $g(-t) = g(t)$  then curve is symmetric about  $y$ -axis.
- (iii) If both  $x = f(t)$  and  $y = g(t)$  both are odd, then curve is symmetric in opposite quadrants.

#### Rule 2: Origin

For some value of  $t$ , if both  $x$  and  $y$  becomes zero, then curve passing through origin.

To check this we follow the following steps.

1. Put  $x = 0$  and find  $t$ .
2. Put this values of  $t$  in  $y$

(iii) If we get  $y=0$  for this value of  $t$ . then we get curve passing through origin.

Rule 3 : Tangents at special points

ie.  $\frac{dy}{dx}$

find  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Rule 4 : Prepare a table for values of  $t, x, y, \frac{dy}{dx}$

4.2.1. Trace the cycloid

$$x = a(t + \sin t)$$

$$y = a(1 + \cos t)$$

Solving

(i) Symmetry :- since  $x$  is odd and  $y$  is an even function. Hence curve symmetric about  $y$ -axis.

(ii) Origin :- put  $x=0$

$$\text{ie. } a(t + \sin t) = 0$$

$$\Rightarrow t = 0$$

$$\therefore y = a(1 + \cos 0) = a(1 + 1) = 2a.$$

Thus curve passing through  $(0, 2a)$   
ie. curve not passing through origin.

(iii) Find  $\frac{dy}{dx}$

Since  $y = a(1 + \cos t)$

$$\therefore \frac{dy}{dt} = a(-\sin t) = -a \sin t$$

f  $x = a(t + \sin t)$

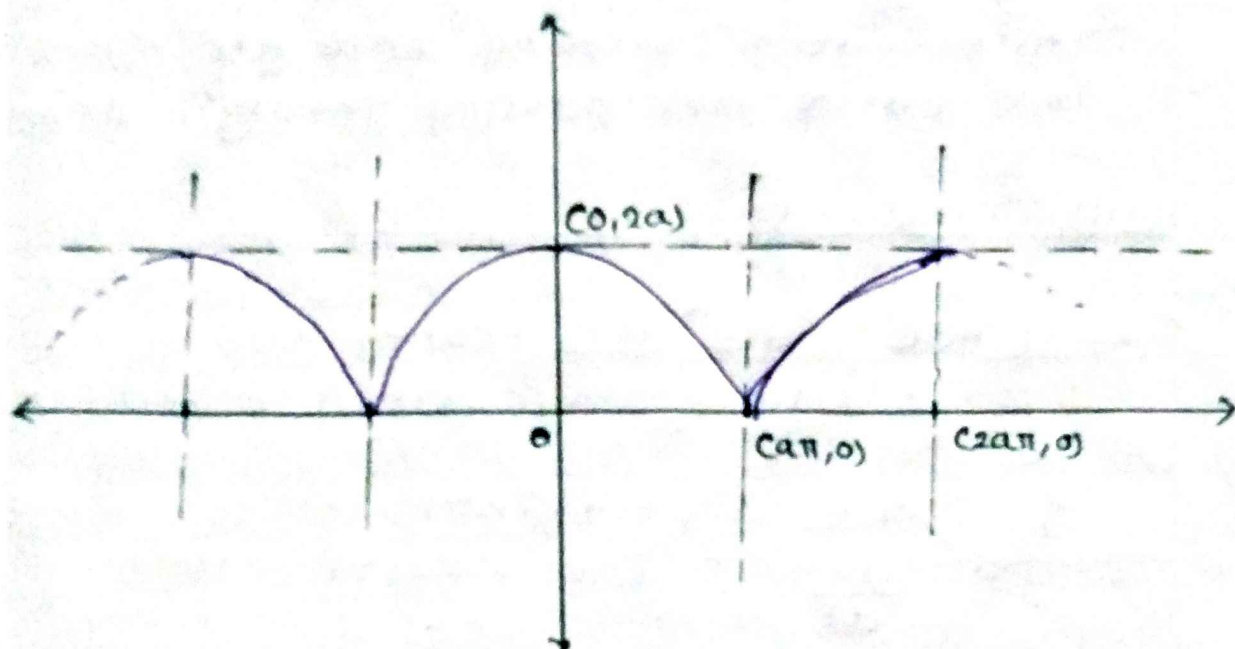
$$\therefore \frac{dx}{dt} = a(1 + \cos t)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 + \cos t)} \\ &= \frac{-2 \sin t/2 \cdot \cos t/2}{2 \cos^2 t/2} \\ &= -\tan \frac{t}{2}. \end{aligned}$$

(iv) Prepare a table for  $t$ ,  $x$ ,  $y$ , and  $\frac{dy}{dx}$

| $t$             | 0    | $\frac{\pi}{2}$        | $\pi$     | $\frac{3\pi}{2}$        | $2\pi$   |
|-----------------|------|------------------------|-----------|-------------------------|----------|
| $x$             | 0    | $a(\frac{\pi}{2} + 1)$ | $a\pi$    | $a(\frac{3\pi}{2} + 1)$ | $2\pi a$ |
| $y$             | $2a$ | $a$                    | 0         | $a$                     | $2a$     |
| $\frac{dy}{dx}$ | 0    | -                      | $-\infty$ | -                       | 0        |

Thus the required curve is,



Homework :-

i) Trace the curve

$$a) \quad x = a(t + \sin t) \\ y = a(t - \cos t)$$

$$b) \quad x = a(t - \sin t) \\ y = a(1 - \cos t)$$

$$c) \quad x = a(t - \sin t) \\ y = a(1 + \cos t)$$

ii) Trace the curve  $x^{2/3} + y^{2/3} = a^{2/3}$

[Hint: It's parametric eqn is

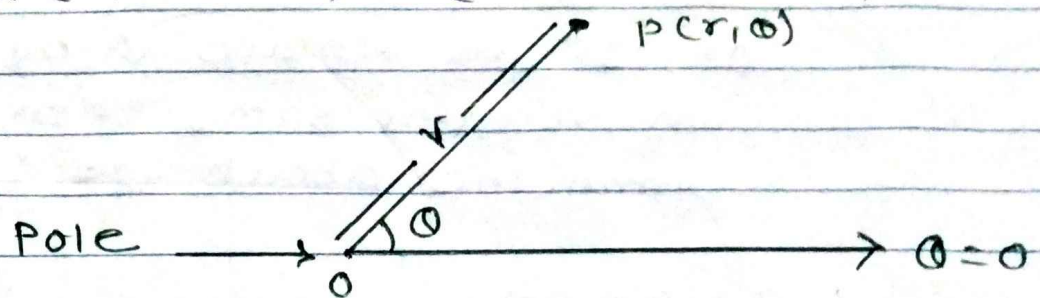
$$x = a \cos^3 t, \quad y = a \sin^3 t]$$

is called four cusped / Astroide.

### 4.3 Polar Curves:

There are two types of polar curves viz.

- i)  $r = a \cos n\theta$  /  $a \sin n\theta$ . (Rose curves)
- (ii)  $r = a(1 \pm \cos \theta)$   
 $r = a(1 \pm \sin \theta)$  (Cardioids).



Here the point  $P(x, y)$  is in polar form  $P(r, \theta)$  related by the equations  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

Here  $r$  is called radius vector of point  $P$  and  $\theta$  is the angle made by radius vector with initial line ( $\theta = 0$  line).

### \* Rules to tracing Polar curves.

#### Rule 1: Symmetry &

##### a) About initial line ( $\theta = 0$ or $x$ -axis.)

If we replace  $\theta$  by  $-\theta$ , equation remains the same curve is symmetric about initial line or  $\theta = 0$  i.e.  $x$ -axis.

b) About the line  $\theta = \frac{\pi}{2}$  (i.e. y axis).

If we replace  $\theta$  by  $-\theta$  and  $r$  by  $-r$ , equation remains the same curve is symmetric about  $\theta = \frac{\pi}{2}$  line

or If we replace  $\theta$  by  $\pi - \theta$ , equation remains same then also curve symmetric about  $\theta = \frac{\pi}{2}$  line.

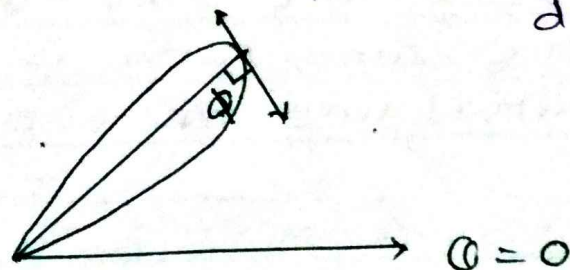
### Rule 2: Pole (origin)

The curve will pass through the pole (origin), if for some values of  $\theta$ ,  $r$  becomes zero.

To check that put  $r=0$ , find  $\theta$ . Thus these value of  $\theta$  lines are tangents at origin.

### Rule 3: Tangents

If  $\phi$  be the angle between the radius vector and tangents to the curve then  $\tan \phi = r \cdot \frac{d\theta}{dr}$



Rule 4: Make the table for different values of  $r$ ,  $\theta$  and  $\tan \phi$ .

| $r$         | 0 | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|-------------|---|---------|-------|----------|--------|
| $\theta$    |   |         |       |          |        |
| $\tan \phi$ |   |         |       |          |        |

A) For rows curves. i.e.  
 $r = a \cos n\theta / a \sin n\theta$ .

Note that if the values of  $n$  is even then in that curves ' $2n$ ' no. of loops & if  $n$  is odd then it has ' $n$ ' no. of loops.

4.3.1. Trace the curve  $r = a \cos 2\theta$ .

Solving

(i) symmetry :- If we replace  $\theta$  by  $-\theta$  equation remains the same curve symmetric about  $\theta = 0$  line, i.e. initial line.

(ii) Pole :- Put  $r = 0$

$$\text{i.e. } a \cos 2\theta = 0$$

$$\therefore \cos 2\theta = 0$$

$$\therefore 2\theta = \cos^{-1}(0)$$

$$\text{i.e. } 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$$

are tangents at origin (Pole)

(iii) Tangents :- If  $\phi$  is the angle between radius vector and tangents then

$$\tan \phi = r \cdot \frac{d\theta}{dr}$$

$$\therefore r = a \cos 2\theta$$

$$\therefore \frac{dr}{d\theta} = -2a \sin 2\theta$$

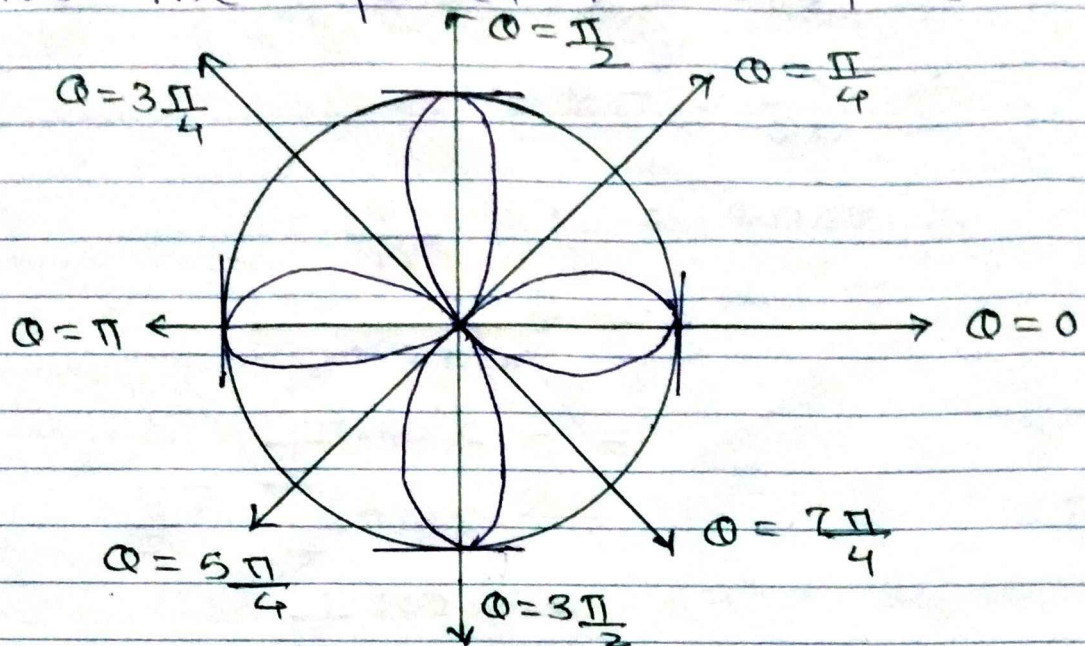
$$\begin{aligned} \therefore \tan \phi &= \frac{a \cos 2\theta}{-2a \sin 2\theta} \\ &= -\frac{1}{2} \cot 2\theta \end{aligned}$$

(iv) Make the table for different values of  $\theta$ ,  $r$ ,  $\tan \phi$ .

$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$  are tangents at origin. Hence we increase the angle by  $\frac{\pi}{4}$ ,

| $\theta$ | $r$ | $\tan \phi$ | $\theta$ | $r$ | $\tan \phi$ |
|----------|-----|-------------|----------|-----|-------------|
| 0        | a   | $\infty$    | $5\pi/4$ | 0   | -           |
| $\pi/4$  | 0   | -           | $3\pi/2$ | -a  | 0           |
| $\pi/2$  | -a  | $\infty$    | $7\pi/4$ | 0   | -           |
| $3\pi/4$ | 0   | -           | $2\pi$   | a   | $\infty$    |
| $\pi$    | a   | $\infty$    |          |     |             |
|          |     |             |          |     |             |
|          |     |             |          |     |             |

Thus the required trace of curve is



4.3.2. Trace the curve  $r = a(1 + \cos \theta)$

Soln

- (i) Symmetry : Curve symmetric about  $\theta = 0$  line (i.e. initial line).
- (ii) Pole : Put  $r = 0$  we get,  
 $0 = a(1 + \cos \theta)$   
 $\therefore \cos \theta = -1$   
 $\therefore \theta = \pi, 3\pi, \dots$   
 are tangents at origin.

(iii) Tangents :

If  $\phi$  be the angle between radius vector and tangents at curve then  $\tan \phi = r \frac{d\theta}{dr}$ .

$$\therefore r = a(1 + \cos \theta)$$

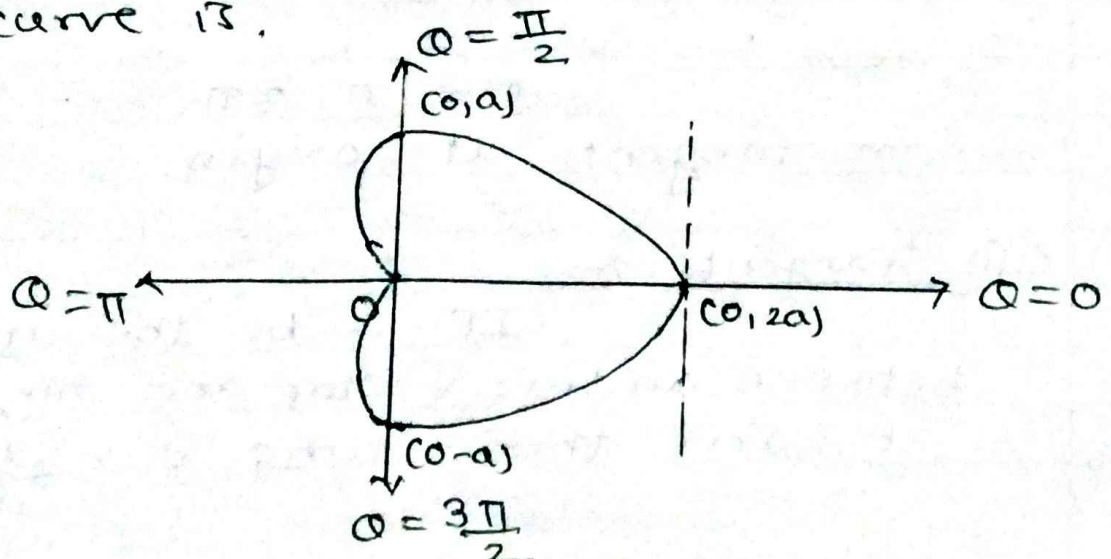
$$\therefore \frac{dr}{d\theta} = -a \sin \theta$$

$$\begin{aligned} \therefore \tan \phi &= r \cdot \frac{d\theta}{dr} \\ &= \frac{a(1 + \cos \theta)}{-a \sin \theta} \\ &= -\frac{2 \cos^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}} \\ &= -\cot \frac{\theta}{2}. \end{aligned}$$

(iv) Make the table for different values of  $r, \theta, \tan \phi$ .

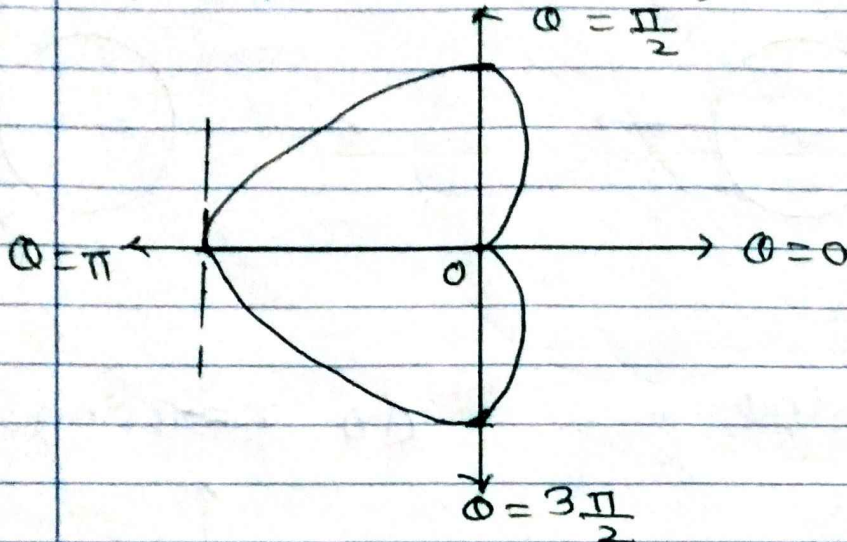
|             |          |                 |       |                  |          |  |  |
|-------------|----------|-----------------|-------|------------------|----------|--|--|
| $\theta$    | 0        | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$   |  |  |
| $r$         | $2a$     | $a$             | 0     | $a$              | $2a$     |  |  |
| $\tan \phi$ | $\infty$ | -               | 0     | -                | $\infty$ |  |  |

Thus the required trace of curve is,

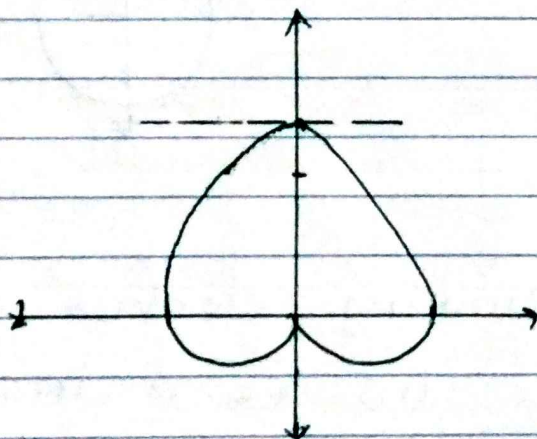


\* Some Important polar curves & Traces.

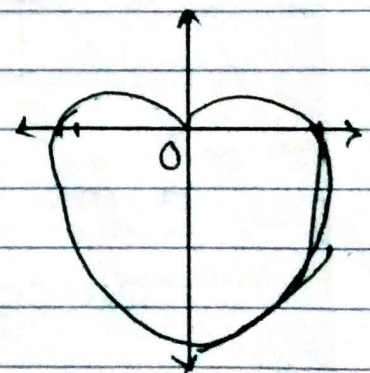
i)  $r = a(1 - \cos\theta)$



(ii)  $r = a(1 + \sin\theta)$



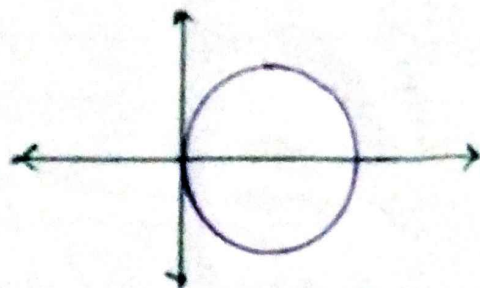
(iii)  $r = a(1 - \sin\theta)$



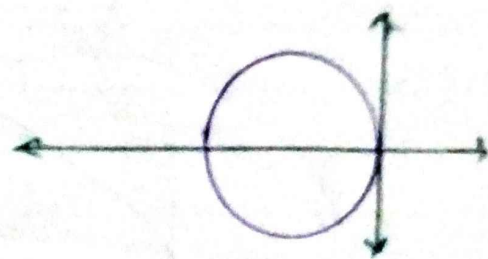
This types of curves are known as cardioids or Heart shape.

## Equations of circles & shapes.

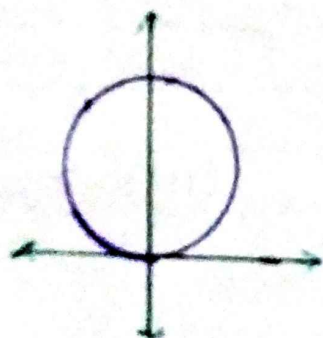
(i)  $r = a \cos \theta$



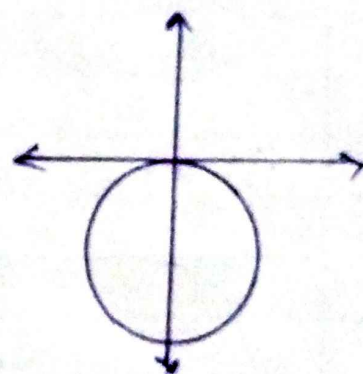
(ii)  $r = -a \cos \theta$



(iii)  $r = a \sin \theta$



(iv)  $r = -a \sin \theta$



### Home work :-

Trace the following curves.

(i)  $r = a \cos 3\theta$

(ii)  $r = a \sin 2\theta$

(iii)  $r = a \cos 5\theta$

(iv)  $r = a \sin 3\theta$

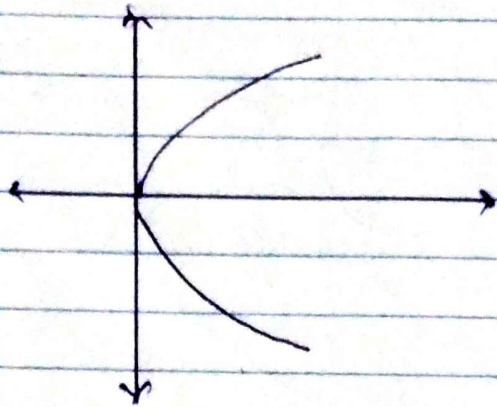
(v)  $r = a(1 - \cos \theta)$

(vi)  $r = a(1 + \sin \theta)$

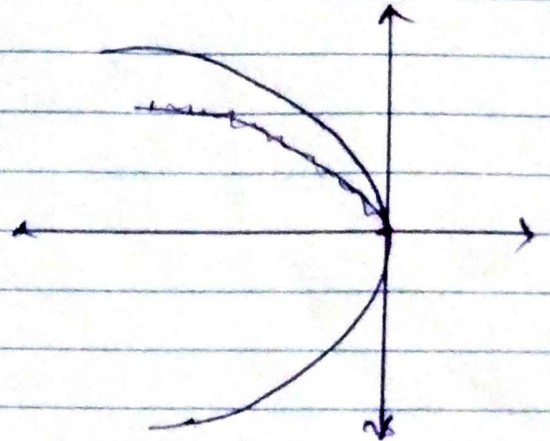
(vii)  $r = a(1 - \sin \theta)$

Some Important cartesian curves & shapes.

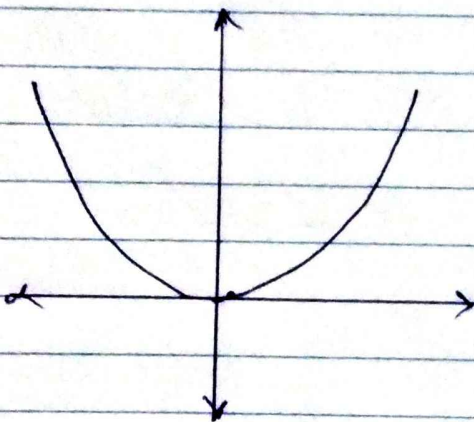
(i)  $y^2 = 4ax$  (Parabola)



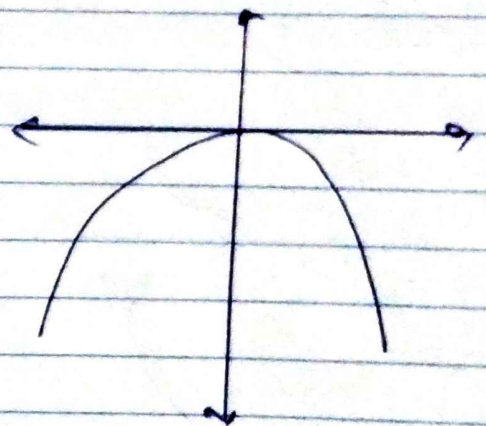
(ii)  $y^2 = -4ax$



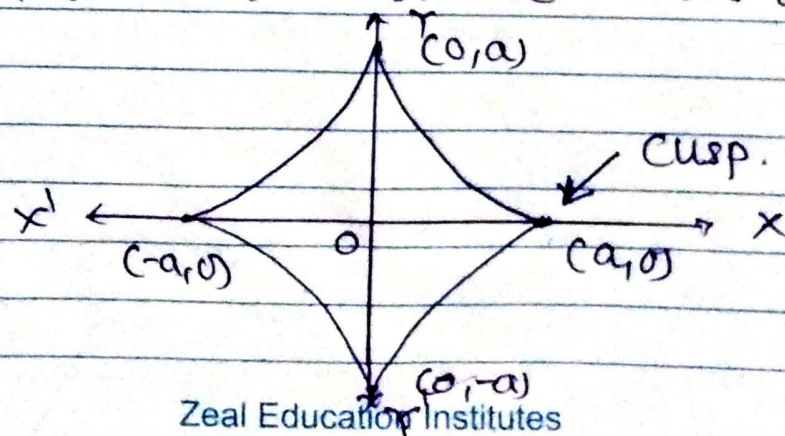
(iii)  $x^2 = 4ay$



(iv)  $x^2 = -4ay$



(v)  $x^{2/3} + y^{2/3} = a^{2/3}$  or  $x = a \cos^3 t, y = a \sin^3 t$



Astroid  
four cusp.

#### 4.4 Rectification :-

The process of finding the length of an arc of the curve is called Rectification.

The curve having finite length is called rectifiable curve.

Following are the formulae of length of the curve (s) in various forms.

- (i) The length of curve  $y = f(x)$  between  $x = x_1$  to  $x_2$  is given by

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

- (ii) The length of the curve  $x = f(y)$  between  $y = y_1$  to  $y_2$  is given by,

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

- (iii) The length of the curve  $r = f(\theta)$  between  $\theta = \theta_1$  to  $\theta_2$  is given by,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

(iv) The length of the curve  $\theta = f(r)$  between  $r = r_1$  to  $r_2$  is given by

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} dr$$

(v) The length of the curve  $x = f(t)$ ,  $y = g(t)$  between  $t = t_1$  to  $t_2$  is given by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Ex. 4.4.1 Find the circumference of circle of radius 'a'.

Soln Equation of circle with centre at origin and radius 'a' is,

$$x^2 + y^2 = a^2$$

$$\text{i.e. } y^2 = a^2 - x^2$$

$$\therefore y = \sqrt{a^2 - x^2} = f(x)$$

diff. w.r.t.  $x$  we get.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2-x^2}} (-2x)$$

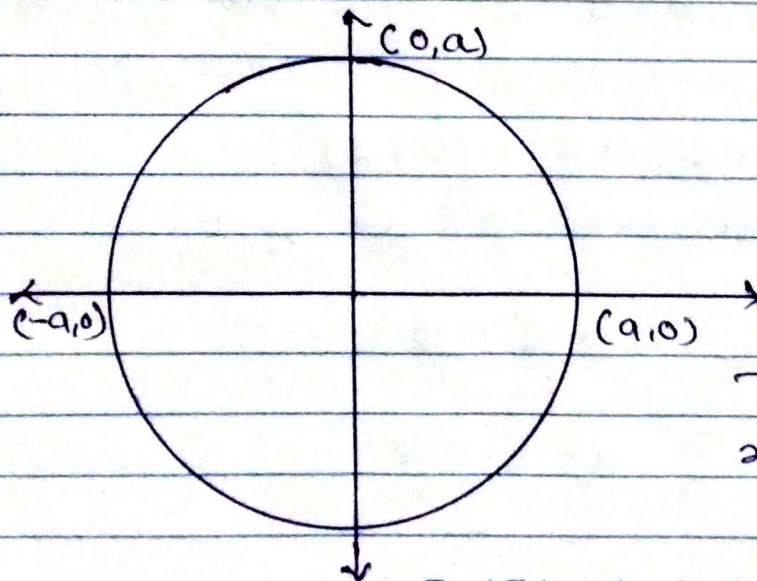
$$= \frac{-x}{\sqrt{a^2-x^2}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{a^2-x^2}$$

$$\begin{aligned} \therefore 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{x^2}{a^2-x^2} \\ &= \frac{a^2-x^2+x^2}{a^2-x^2} \\ &= \frac{a^2}{a^2-x^2} \end{aligned}$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{a}{\sqrt{a^2-x^2}}$$

Let  $s$  be the circumference of this circle as shown below.



Because of symmetry the length in upper half is same as in lower half

Thus  $x$  varies from  $x = -a$  to  $x = a$ .

$$\begin{aligned}
 \therefore S &= 2 \int_{-a}^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= 4 \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= 4 \int_0^a \frac{a}{\sqrt{a^2 - x^2}} dx \\
 &= 4a \left[ \sin^{-1} \frac{x}{a} \right]_0^a \\
 &= 4a \left[ \sin^{-1}(1) - \sin^{-1}(0) \right] \\
 &= 4a \left[ \frac{\pi}{2} - 0 \right] \\
 &= 2\pi a.
 \end{aligned}$$

4.4.2 Find the perimeter of the cardioid  $r = a(1 + \cos \theta)$  and show that the line  $\theta = \frac{\pi}{2}$  divides the upper half of the cardioid.

Solving

Here  $r = a(1 + \cos \theta)$

diff. w.r.t.  $\theta$  we get,

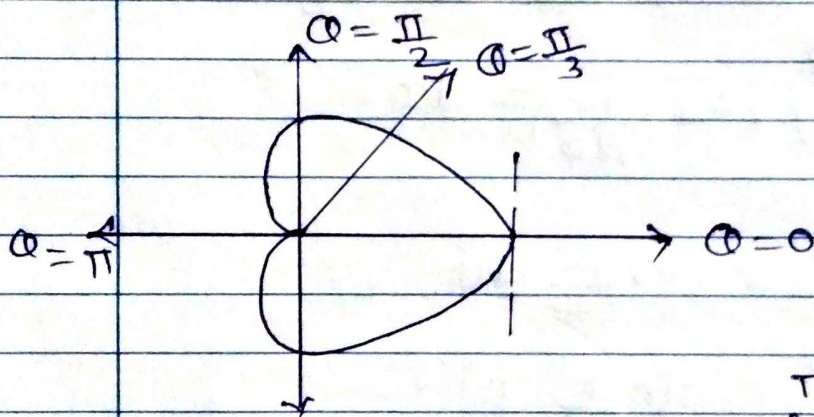
$$\frac{dr}{d\theta} = -a \sin \theta.$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin^2 \theta$$

$$\begin{aligned}
 \therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 &= a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta \\
 &= a^2 [1 + 2\cos\theta + \cos^2\theta + \sin^2\theta] \\
 &= a^2 [2 + 2\cos\theta] \\
 &= 2a^2 (1 + \cos\theta) \\
 &= 2a^2 \left[ \frac{2\cos^2\frac{\theta}{2}}{2} \right] \\
 &= 4a^2 \cos^2\frac{\theta}{2}.
 \end{aligned}$$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a \cos\frac{\theta}{2}.$$

Now let 'S' be the length of cardioid.  
 $r = a(1 + \cos\theta)$



Because of symmetry the length in upper half is same as in lower half.

$$\begin{aligned}
 \therefore S &= 2 \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \\
 &= 2 \int_0^{\pi} 2a \cos\frac{\theta}{2} d\theta.
 \end{aligned}$$

$$\begin{aligned}
 \therefore S &= 4a \int_0^{\pi} \cos \frac{\theta}{2} \cdot d\theta. \\
 &= 4a \left[ \frac{\sin \theta/2}{1/2} \right]_0^{\pi} \\
 &= 8a \left[ \sin \theta/2 \right]_0^{\pi} \\
 &= 8a \left[ \sin \frac{\pi}{2} - 0 \right] \\
 &= 8a.
 \end{aligned}$$

This is the total length of the cardioid. Hence the length in upper half will be 4a.

Now to show that  $\theta = \frac{\pi}{3}$  divides upper half. i.e. to show that length from  $\theta = 0$  to  $\theta = \frac{\pi}{3}$  is  $2a$ .

$$\begin{aligned}
 \therefore S_1 &= \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \\
 &= \int_0^{\pi/3} 2a \cos \frac{\theta}{2} d\theta \\
 &= 2a \left[ \frac{\sin \theta/2}{1/2} \right]_0^{\pi/3} \\
 &= 4a \left[ \sin \theta/2 \right]_0^{\pi/3} \\
 &= 4a \left[ \sin \frac{\pi}{6} - \sin 0 \right] \\
 &= 4a \left[ \frac{1}{2} - 0 \right] = 2a.
 \end{aligned}$$

Hence the proof.

4.4.3 Find the complete arc length of the curve  $x^{2/3} + y^{2/3} = a^{2/3}$

Soln

Here  $x^{2/3} + y^{2/3} = a^{2/3}$  be an equation of astroid.

consider its parametric equations,

$$x = a \cos^3 t \quad \& \quad y = a \sin^3 t.$$

$$\therefore \frac{dx}{dt} = 3a \cos^2 t \cdot (-\sin t) = -3a \cos^2 t \sin t$$

$$\frac{dy}{dt} = 3a \sin^2 t \cdot \cos t$$

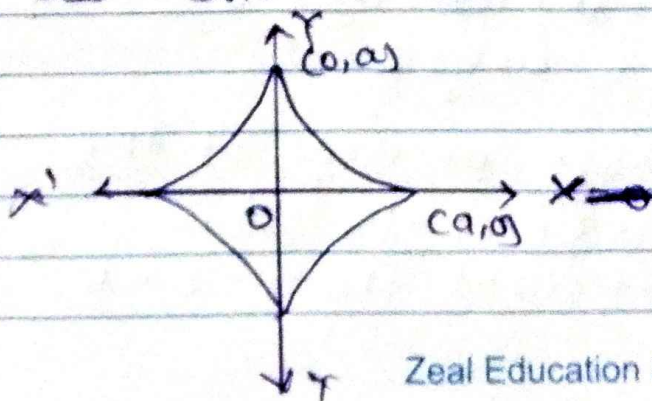
$$\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t.$$

$$= 9a^2 \cos^2 t \sin^2 t$$

$$= 9a^2 \cos^2 t \sin^2 t$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3a \cos t \sin t.$$

Let  $s$  be the length of the astroid as shown below.



Because of symmetry length of astroid in 1st quadrant is same as in each quadrant

$$\therefore S = 4 \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 4 \int_0^{\frac{\pi}{2}} 3a \cos t \cdot \sin t \cdot dt$$

$$= 12a \int_0^{\frac{\pi}{2}} \sin t \cdot \cos t dt$$

$$= 12a \cdot \frac{1}{2} \beta(1, 1)$$

$$= 6a \frac{\Gamma(1) \Gamma(1)}{\Gamma(2)}$$

$$= 6a.$$

Thus the required length of Astroid is  $6a$ .

### Home work

(i) Find the length of the parabola  $y^2 = 4x$  cut off by the line  $3y = 8x$

(ii) Find the length of the cardioid  $r = a(1 - \cos \theta)$  which lies outside the circle  $r = a \cos \theta$ .

(iii) Find the length of arc of the curve  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$  from  $t = 0$  to  $t = 2\pi$ .