

MATRICES

Introduction

Linear algebra begins with sets of linear equations in several variables. The number of variables may be large, or it may be unspecified. In either case we need an alphabet larger than the ones used in ordinary language. Thus we invent synthetic ones. This unit provides an introduction to the basic concepts of matrices and system of linear equations. You will develop operations on matrices and will work with matrices according to the rules they obey; this will enable you to solve system of linear equations and to do other computational problems in a fast and efficient manner.

The concept of matrices has a wide range of application in our life. One of the main applications is to write systems of linear equation in a compact form, which enables you to solve them very easily and efficiently.

Objectives

After successful completion of this unit, you will be able to:

- Define a matrix
- Apply operations on matrices
- Describe properties of matrices
- Reduce matrices into reduced row echelon form
- Find the inverse of a matrix by using elementary row operations
- Solve system of linear equations by using Gaussian elimination
- Find the inverse of a matrix.

2.1. Definition of matrices.

Definition 2.1.1: An $m \times n$ (m by n) matrix A is a rectangular array of mn real (or complex) numbers arranged in m horizontal rows and n vertical columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Remark : A matrix is a rectangular array of numbers, or symbols , like the above. The rows of a matrix are its horizontal lines: for example the second row of the matrix A has entries $a_{21}, a_{22}, \dots, a_{2n}$. Note that the first subscript names the row from which the entry is taken. The columns of the matrix are its vertical lines; the first column of A has entries $a_{11}, a_{21}, \dots, a_{m1}$. The second subscript names the column from which the entry comes. Notation: When the entries of a matrix are named by means of subscripts, the matrix can be abbreviated by telling the range of the subscripts.

The above matrix A will be written as $A = [a_{ij}]_{m \times n}$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$).

Example 2.1.1.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix with } a_{21} = -1, a_{23} = 4$$

$$A = [1 \ 0 \ -2 \ 3] \text{ is a } 1 \times 4 \text{ matrix. with } a_{13} = -2$$

$$A = [1] \text{ is a } 1 \times 1 \text{ matrix and } a_{11} = 1$$

Activity 2.1.1: Let $A = [a_{ij}]$ be a matrix $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$, and $B = [b_{ij}]$ ($i = 1, 2$; $j = 1, 2$) with $b_{ij} = a_{ji}$. Write down B explicitly.

Definition 2.1.2 : Let A be an $m \times n$ matrix. If $m = n$, we say that A is a **Square matrix** of order n .

Example 2.1.2 : $A = \begin{pmatrix} 2 & -1 \\ 4 & 7 \end{pmatrix}$ is a 2×2 or square matrix of order 2

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 13 \end{bmatrix} \text{ is a } 3 \times 3 \text{ or square matrix of order 3.}$$

Definition 2.1.3 : The $m \times n$ zero matrix $0_{m \times n}$ is the matrix with all entries equal to 0.

$$0_{m \times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Definition 2.1.4: Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if $a_{ij} = b_{ij}$ $1 \leq i \leq m$, $1 \leq j \leq n$, that is, if they have the same size and the corresponding components are equal.

Example 2.1.3: The matrices $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 13 \end{bmatrix}$ are not equal since they do not

have the same size.

There are three binary matrix operations These are matrix addition , multiplication of a matrix by a scalar and multiplication of two matrices.

2.2. Operations on matrices.

Matrix Addition

Definition 2.2.1: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices that is A and B two matrices of the same size, then their sum written $A + B$, is the matrix obtained by adding corresponding entries of A and B.

That is : If $A + B = C$, then C is the $m \times n$ matrix where $C = [c_{ij}]$, such that $c_{ij} = a_{ij} + b_{ij}$ $(1 \leq i \leq m, 1 \leq j \leq n)$

Remark : If A and B do not have the same number of rows and columns , their sum is not defined.

Example 2.2.1 : Let

$$A = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix}$$

Then

$$A+B = \begin{bmatrix} 0+6 & -1+4 & 4+(-1) \\ -3+7 & 2+0 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 4 & 2 & 3 \end{bmatrix}$$

The Second operation is multiplication of a matrix by a scalar(That is a number)

Scalar Multiplication:

Definition 2.2.2: If $A = [a_{ij}]$ is an $m \times n$ matrix and α is a scalar, then the scalar multiple of A by α , αA is the $m \times n$ matrix $B = [b_{ij}]$, where $b_{ij} = \alpha a_{ij}$ ($1 \leq i \leq m, 1 \leq j \leq n$).

Remark : B is obtained from A by multiplying each element of A by α .

Example 2.2.2: Let $A = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$ and

$$(a) \text{ If } \alpha = 3, \text{ then } \alpha A = 3A = 3 \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 12 \\ -9 & 6 & 15 \end{bmatrix}.$$

$$(b) \text{ If } \alpha = -1, \text{ then } \alpha A = -1A = (-1) \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 3 & -2 & -5 \end{bmatrix}.$$

Definition 2.2.3:

If A and B are $m \times n$ matrices, then we define

$A - B = A + (-1)B$ and we call this the **difference** of A and B .

Example 2.2.3 : Let $A = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix}$

Then find $A - B$

Solution

$$A - B = A + (-1)B$$

$$(-1)B = \begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 1 \\ -7 & 0 & 2 \end{bmatrix}$$

$$A - B = A + (-1)B = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} -6 & -4 & 1 \\ -7 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 5 \\ -10 & 2 & 7 \end{bmatrix}$$

Activity 2.2.1

Let $A = \begin{bmatrix} 2 & -1 & -8 \\ -3 & 12 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -3 \\ 5 & -1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 3 & -2 \\ 0 & 1 & 6 \end{bmatrix}$. Then Find

- $A+B$
- $A - B$
- $A + C$
- $A-2B$

Next we state several elementary properties of matrix addition and scalar multiplication

Theorem 2.2.1: Properties of Matrix Addition and scalar Multiplication

Let A, B, C can be an $m \times n$ matrices and $\alpha, \beta \in \mathbb{R}$. Then

M1. $A+B = B+A$ (Commutativity of Addition)

M2. $A+(B+C) = (A+B)+C$ (Associativity of Addition)

M3. $A+0 = 0+A = A$

M4. $A + (-A) = (-A)+A = 0$

M5. $\alpha (A+B) = \alpha A + \alpha B$

M6. $(\alpha + \beta)A = \alpha A + \beta A$

M7. $\alpha (\beta A) = (\alpha \beta)A$

M8. $1A = A$

M9. $0A = \mathbf{0}$

Proof:

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ where $1 \leq i \leq m, 1 \leq j \leq n$

$$M1. A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$$

$$\begin{aligned} M2. A+(B+C) &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] = [a_{ij} + b_{ij}] + [c_{ij}] = (A + B) + C \end{aligned}$$

$$M3. A + 0 = [a_{ij} + 0] = [a_{ij}] = A$$

$$M4. A + (-A) = [a_{ij} + (-a_{ij})] = [0] = 0$$

$$M5. \alpha(A+B) = \alpha[a_{ij} + b_{ij}] = [\alpha a_{ij} + \alpha b_{ij}] = [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha A + \alpha B$$

$$\begin{aligned} M6. (\alpha + \beta)A &= (\alpha + \beta)[a_{ij}] \\ &= [(\alpha + \beta)a_{ij}] \\ &= [\alpha a_{ij} + \beta a_{ij}] \\ &= [\alpha a_{ij}] + [\beta a_{ij}] \\ &= \alpha[a_{ij}] + \beta[a_{ij}] \\ &= \alpha A + \beta A \end{aligned}$$

M7, M8 and M9 are left as an exercise.

Matrix Multiplication

Definition 2.2.4 : Let $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{jk}]$ is an $n \times p$ matrix, then the **product** of A and B, AB, is the $m \times p$ matrix $C = [c_{ik}]$, defined by

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk} = \sum_{j=1}^n a_{ij}b_{jk} \quad (1 \leq i \leq m, 1 \leq k \leq p).$$

Note that the product of A and B is defined only when the number of columns of A is equal to the number of rows of B .

Example 2.2.4 Let $A = \begin{pmatrix} 2 & -1 \\ 4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find $A.B$

Solution

A is a 2×2 matrix and B is a 2×1 matrix hence multiplication of matrices is defined .

AB is a 2×1 matrix .Let $C = (C_{ik})$ where $(1 \leq i \leq 2, 1 \leq k \leq 1)$ be AB .

$$C = (C_{ik}) \quad (1 \leq i \leq 2, 1 \leq k \leq 1)$$

$$i=1, k=1 \quad C_{11} = a_{11}.b_{11} + a_{12}.b_{21} = 2.(-1) + (-1)1 = -3$$

$$i=2, k=1 \quad C_{21} = a_{21}.b_{11} + a_{22}.b_{21} = 4.(-1) + 7.1 = 3$$

$$\text{Therefore } AB = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

Example 2.2.5 Let $A = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix}$

Find a. AB

b. BA

Solution

A is a 3×3 matrix and B is a 3×3 matrix hence multiplication of matrices is defined . AB

is a 3×3 matrix .Let $C = (C_{ik})$ where $(1 \leq i \leq 3, 1 \leq k \leq 3)$ be AB .

$$C = (C_{ik}) \quad (1 \leq i \leq 3, 1 \leq k \leq 3)$$

$$C_{ik} = a_{i1}.b_{1k} + a_{i2}.b_{2k} + a_{i3}.b_{3k}$$

$$i=1, k=1, \quad C_{11} = a_{11}.b_{11} + a_{12}.b_{21} + a_{13}.b_{31} = 0.1 + (-2)(-2) + (-1)2 = 2$$

$$i=1, k=2, \quad C_{12} = a_{11}.b_{12} + a_{12}.b_{22} + a_{13}.b_{32} = 0.0 + (-2)1 + (-1)1 = -3$$

$$i=1, k=3, \quad C_{13} = a_{11}.b_{13} + a_{12}.b_{23} + a_{13}.b_{33} = 0.3 + (-2)2 + (-1)4 = -8$$

$$i=2, k=1, \quad C_{21} = a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} = 1.1 + 0(-2) + 4.2 = 9$$

$$i=2, k=2, \quad C_{22} = a_{21}.b_{12} + a_{22}.b_{22} + a_{23}.b_{32} = 1.0 + 0.1 + 4.1 = 4$$

$$i=2, k=3, \quad C_{23} = a_{21}.b_{13} + a_{22}.b_{23} + a_{23}.b_{33} = 1.3 + 0.2 + 4.4 = 19$$

$$i=3, k=1, \quad c_{31} = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} = 5 \cdot 1 + 1(-2) + 1 \cdot 2 = 5$$

$$i=3, k=2, \quad c_{32} = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} = 5 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$i=3, k=3, \quad c_{33} = a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} = 5 \cdot 3 + 1 \cdot 2 + 1 \cdot 4 = 21$$

Hence

$$\mathbf{A} \mathbf{B} = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & -8 \\ 9 & 4 & 19 \\ 5 & 2 & 21 \end{bmatrix}$$

Similarly

$$\mathbf{B} \mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 2 \\ 11 & 6 & 8 \\ 21 & 0 & 6 \end{bmatrix}$$

From the above example we can conclude that $\mathbf{AB} \neq \mathbf{BA}$.

Remark : Matrix multiplication is not commutative.

Activity 2.2.2

$$\text{Let } \mathbf{A} = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 5 \\ 2 & -1 & 1 \end{bmatrix}$$

Compute $\mathbf{AB}$

Theorem 2.2.2: Properties of Matrix multiplication

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are compatible matrices under multiplication and addition, then

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Example 2.2.6

$$\text{Let } \mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix}$$

Compute

- a. $\mathbf{B} + \mathbf{C}$
- b. $\mathbf{A}(\mathbf{B} + \mathbf{C})$
- c. $\mathbf{AB}$
- d. $\mathbf{AC}$

Solution

$$a. B+C = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 1+4 & -1+3 & 5+2 \\ 3+0 & 1+(-3) & -3+11 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 3 & -2 & 8 \end{bmatrix}$$

$$b. A(B+C) = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 5 & 2 & 7 \\ 3 & -2 & 8 \end{bmatrix} = \begin{bmatrix} 3.5 + (-1)3 & 3.2 + (-1)(-2) & 3.7 + (-1)8 \\ 4.5 + 5.3 & 4.2 + 5.(-2) & 4.7 + 5.8 \end{bmatrix} \\ = \begin{bmatrix} 12 & 8 & 13 \\ 35 & -2 & 68 \end{bmatrix}$$

$$c. AB = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 3.1 + (-1)3 & 3(-1) + (-1)1 & 3.5 + (-1)(-3) \\ 4.1 + 5.3 & 4.(-1) + 5.1 & 4.5 + 5.(-3) \end{bmatrix} \\ = \begin{bmatrix} 0 & -4 & 18 \\ 19 & 1 & 5 \end{bmatrix}$$

$$e. AC = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 12 & 12 & -17 \\ 16 & -3 & 47 \end{bmatrix}$$

From the above $AB + AC = A(B+C)$

Activity 2.2.3

$$\text{Let } A = \begin{bmatrix} 1 & 9 & -2 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \text{ and } C = \begin{pmatrix} 0 & -1 \\ 2 & 3 \\ 5 & 1 \end{pmatrix}$$

Compute

- $A + B$
- $(A + B)C$
- AC
- BC

Activity 2.2.4

Suppose A , B and C be matrices such that

$$AC = \begin{bmatrix} 6 & -1 & 0 \\ -4 & 0 & 2 \end{bmatrix} \text{ and } BC = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 2 & -3 \end{bmatrix}. \text{ Find } (A+B)C$$

Definition 2.2.5: Suppose that A is a square matrix and let n be a positive integer, then we

$$\text{define } A^n = AA^{n-1}$$

That is $A^2 = AA$, $A^3 = AA^2$, $A^4 = AA^3$ and so on $A^n = AA^{n-1}$

Remark: $A^0 = I$

2.3. Types of matrices

The Transpose of a Matrix

There is another operation on matrices , this one is a unary operation.

Definition 2.3.1 :If $A = [a_{ij}]$ is an $m \times n$ matrix, then the $n \times m$ matrix

$A^t = [b_{ij}]$, where $b_{ij} = a_{ji}$ ($1 \leq i \leq m, 1 \leq j \leq n$) is called the **transpose** of A.

Remark : If A is an $m \times n$ matrix , then A^t is an $n \times m$ matrix.

Example 2.3.1: Let $A = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$. Find A^t

Solution

Since A is an 3×2 matrix , A^t is a 2×3 matrix.

$A^t = [b_{ij}]$ where $b_{ij} = a_{ji}$ ($1 \leq i \leq 3, 1 \leq j \leq 2$)

$b_{11} = a_{11} = 1$, $b_{12} = a_{21} = 3$, $b_{13} = a_{31} = -2$,

$b_{21} = a_{12} = 4$, $b_{22} = a_{22} = -1$, $b_{23} = a_{32} = 2$

Therefore

$$A^t = [b_{ij}] = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 4 & -1 & 2 \end{pmatrix}$$

Remark : The transpose of A is obtained by interchanging the rows by the columns of A

Activity 2.3.1:

Let $A = \begin{pmatrix} 2 & -1 & 0 & 6 \\ 4 & 3 & 5 & -2 \end{pmatrix}$. Find A^t

Theorem 2.3.1 Properties of Transpose

If α is a scalar and A and B are matrices, then

- T1. $(A^t)^t = A$
- T2. $(A+B)^t = A^t + B^t$
- T3. $(AB)^t = B^t A^t$
- T4. $(\alpha A)^t = \alpha A^t$

Remark : By using property T3 , one can show that $(ABC)^t = C^t B^t A^t$ and in general $(A_1 A_2 \dots A_n)^t = A_n^t \dots A_2^t A_1^t$

Example 2.3.2: Let

$$\text{Let } A = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix}$$

Compute

1. A^t
2. B^t
3. $B^t A^t$
4. $(AB)^t$

Solution

$$\text{a. } A^t = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$$

$$\text{b. } B^t = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 5 & -3 \end{bmatrix}$$

$$\text{c. } B^t A^t = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 5 & -3 \end{bmatrix} \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} = \begin{bmatrix} 0 & 19 \\ -4 & 1 \\ 18 & 5 \end{bmatrix}$$

$$\text{d. } AB = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 18 \\ 19 & 1 & 5 \end{bmatrix}$$

$$(AB)^t = B^t A^t = \begin{bmatrix} 0 & 19 \\ -4 & 1 \\ 18 & 5 \end{bmatrix}$$

Definition 2.3.2: A square matrix $A = (a_{ij})$ is called an **identity matrix** if $a_{ij} = 1$

whenever

$i = j$ and $a_{ij} = 0$ whenever $i \neq j$. It is then denoted by I_n or I when the order is clear from the context.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 2.3.3 :A square matrix in which all the diagonal elements are 0 is called a **diagonal matrix**..That is $A = (a_{ij})$ is a diagonal matrix iff $a_{ij} = 0$ whenever $i \neq j$.

Example 2.3.3: $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix

Definition 2.3.4 :A **diagonal** matrix in which all diagonal elements are equal is called a scalar matrix.

Example 2.3.4: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a scalar matrix

Remark : Scalar matrices can be written as αI for some scalar α and behaves like a scalar. That is $\alpha I + \beta I = (\alpha + \beta)I$ and $\alpha I.A = \alpha A$

Definition 2.3.5: A square matrix $A = (a_{ij})$ is called an **upper triangular** matrix if all the elements below diagonal are 0 , that is $a_{ij} = 0$ whenever $i > j$

Definition 2.3.6: A square matrix $A = (a_{ij})$ is called a **lower triangular** matrix if all the elements above diagonal are 0 , that is $a_{ij} = 0$ whenever $i < j$

Definition 2.3.7: A square matrix $A = (a_{ij})$ is called **triangular** matrix if it is either upper triangular or lower triangular .

Example 2.3.5 : In the following two matrices

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ -2 & 9 & 4 \end{bmatrix}$$

A is an upper triangular matrix while B is a lower triangular matrix

Definition 2.3.8:A matrix $A = [a_{ij}]$ is called **symmetric** if $A^t = A$.

That is, A is symmetric if it is a square matrix for which $a_{ij} = a_{ji}$ for all i, j .

Example : $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 3 & 7 \\ 4 & 7 & 8 \end{bmatrix}$ is a symmetric matrix.

Note Symmetry of a matrix refers to symmetry about the principal diagonal

Example 2.3.6: If A and B are symmetric matrices of the same order and α is a scalar then αA and $A + B$ are symmetric matrices.

Solution:

a. $(\alpha A)^t = \alpha A^t = \alpha A$. Hence αA is a symmetric matrix.

b. $(A + B)^t = A^t + B^t = A + B$. Hence $A + B$ is a symmetric matrix.

Remark : AB may not be symmetric. For example

Consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$ which are symmetric matrices. But $AB = \begin{pmatrix} 10 & 9 \\ 19 & 8 \end{pmatrix}$ is not symmetric.

Definition 2.3.9: A matrix $A = [a_{ij}]$ is called **skew symmetric** if $A^t = -A$.

That is, A is skew symmetric if it is a square matrix for which $a_{ij} = -a_{ji}$ for all i, j .

Example 2.3.7 : $A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 5 \\ -4 & -5 & 0 \end{bmatrix}$ is a skew symmetric matrix since $A^t = -A$.

Note: The main diagonal elements of a skew symmetric matrix must all be zero.

Example 2.3.8: Let A be any square matrix. Then

- $A + A^t$ is symmetric
- $A - A^t$ is skew symmetric
- A can be written as a sum of a symmetric and skew symmetric matrices.

Solution

a) $(A + A^t)^t = A^t + (A^t)^t = A^t + A$

b) $(A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$.

Hence $A - A^t$ is a skew symmetric matrix.

c) Let $B = \frac{1}{2}(A + A^t)$, $C = \frac{1}{2}(A - A^t)$. Then $A = B + C$. Moreover since B is symmetric and C is skew symmetric, Any matrix can be written as a sum of symmetric and skew symmetric.

Activity 2.3.2

Give an example of a matrix which is

- An upper triangular matrix
- Whose transpose is an upper triangular matrix
- A lower triangular matrix
- Whose transpose is a lower triangular matrix

- e) Symmetric
- f) Skew symmetric

2.4 Elementary Row Operations and Inverse of a matrix.

In this section We shall study some operations called elementary operations which can be used to reduce any given matrix to one with a simple form thereby facilitating the solution of some problems to be solved for the original matrix.

Row echelon form

Definition 2.4.1: An $m \times n$ matrix is said to be in **row echelon form** when it satisfies the following properties:

- R1) All rows consisting entirely of zeros, if any, are at the bottom of the matrix.
- R2) The leading non zero number in a row is 1.
- R3) If two successive rows that do not consist entirely of zeros, then the leading entry of the lower row is farthest to the right than the leading entry of the upper row.

Example 2.4.1: The matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in row echelon form.

Example 2.4.2 The matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ are not in row}$$

echelon form. Because in case of Matrix A Condition R1 failed , Matrix B Condition R2 failed and Matrix C Condition R2 failed

Activity 2.4.1

Give different examples of matrices which are in row echelon form.

Definition 2.4.2 An $m \times n$ matrix is said to be in reduced row echelon if its in the **row echelon form** and the elements above the first non zero entry any row are 0(the elements below are already 0).

Example 2.4.3 : Consider

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

A is in reduced row echelon form while B is not in reduced row echelon form.

Elementary operation

Definition 2.4.3: An elementary row operation on a matrix is any one of the following operations:

- Interchange two rows .
- Multiply a row by a non zero constant
- Replacing a row by the sum of that row and a scalar multiple of another row.

Notation :

$A_{i,j}$ Interchanging the i^{th} with j^{th} row of A.

$A_i \rightarrow \alpha A_i$, Multiply the i^{th} row of A by α .

$A_i \rightarrow A_i + \alpha A_j$ Add α times the j^{th} row of A to the i^{th} row of A.

Remark: An elementary column operation on a matrix is defined similarly. By an elementary operation we mean an elementary row operation or an elementary column operation.

Definition 2.4.4: An elementary column operation on a matrix is any one of the following operations:

- Interchange two columns .
- Multiply a column by a non zero constant
- Replacing a column by the sum of that column and a scalar multiple of another column.

Definition 2.4.5: We say two matrices are **equivalent** if one is obtained from the other by a finite sequence of elementary operations.

Notation : If A and B are equivalent then we write $A \sim B$

Example 2.4.4: Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \xrightarrow{A_{1,2}} \begin{bmatrix} -1 & 4 & 0 & 5 & 7 \\ 1 & 3 & 5 & 1 & 4 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \xrightarrow{A_{2,3}} \begin{bmatrix} -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \\ 1 & 3 & 5 & 1 & 4 \end{bmatrix}$$

$$\text{Hence, } \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \\ 1 & 3 & 5 & 1 & 4 \end{bmatrix} \text{ Similarly}$$

$$A = \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \xrightarrow{A_2 \rightarrow 2A_2} \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -2 & 8 & 0 & 10 & 14 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \xrightarrow{A_3 \rightarrow A_3 + -1A_1}$$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -2 & 8 & 0 & 10 & 14 \\ 2 & -5 & 0 & 5 & -1 \end{bmatrix} \xrightarrow{A_{2,3}} \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 2 & -5 & 0 & 5 & -1 \\ -2 & 8 & 0 & 10 & 14 \end{bmatrix}$$

Hence ,

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 2 & -5 & 0 & 5 & -1 \\ -2 & 8 & 0 & 10 & 14 \end{bmatrix}$$

Note: A is column equivalent to B if and only if A^t is row equivalent to B^t .
Next we state a theorem without proof

Theorem 2.4.1: Any matrix can be reduced to a matrix in reduced echelon form by elementary operations.

Example 2.4.5: Reduce $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$

- In to row echelon form.
- In to reduced row echelon form.

Solution

$$\text{a. } A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \xrightarrow{A_{2,1}} \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix} \xrightarrow{A_1 \rightarrow (-1)A_1} \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$$

$$\xrightarrow{A_2 \rightarrow A_2 + (-3)A_1} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix} \xrightarrow{A_2 \rightarrow (1/11)A_2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is the row echelon form of A.

$$\text{b. } A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{A_1 \rightarrow A_1 + (2)A_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the reduced row echelon form of A.

Example 2.4.6: Reduce $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

- In to row echelon form.
- In to reduced row echelon form.

Solution

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{A_2 \rightarrow A_2 + (2)A_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{A_2 \rightarrow (-1)A_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A_3 \rightarrow A_3 + (-1)A_2 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{A_3 \rightarrow (1/5)A_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ is the row echelon form of A.

b.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{A_1 \rightarrow A_1 + (1)A_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 \rightarrow A_2 + (3)A_3 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{A_1 \rightarrow A_1 + (1)A_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Activity 2.4.1

Find the reduced row echelon form of $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$

The Inverse of a Matrix

Definition 2.4.6: An $n \times n$ matrix A is called **nonsingular** (or **invertible**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

The matrix B is called an **inverse** of A denoted by A^{-1} . If there exists no such matrix B , then A is called **singular** (or **noninvertible**).

Example 2.4.7: Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

Since $AB = BA = I_2$, we conclude that B is the *inverse* of A and hence that A is invertible..

Theorem 2.4.2 :If a matrix is invertible , then its inverse is unique.

Proof:Let B and C be inverses of A. Then

$$CA = AB = I_n.$$

Therefore,

$$C = CI_n = C(AB) = (CA)B = I_n B = B$$

Remark : Since the inverse of a matrix if it exists is unique, We write the inverse of A by A^{-1} . Therefore, $A^{-1} = A^{-1}A = I_n$.

Example 2.4.8: Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. To find A^{-1} , let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then we must have

$$A A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a+2c & b+2d \\ a+3c & b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the above two matrices to be equal the corresponding components should be equal. Hence

$$\begin{aligned} a+2c &= 1 & a+3c &= 0 \\ b+2d &= 0 & b+3d &= 1 \end{aligned}$$

Collecting terms

$$\begin{aligned} a+2c &= 1 & b+2d &= 0 \\ a+3c &= 0 & b+3d &= 1 \end{aligned}$$

Solving simultaneously

$$a = 3, c = -1, b = -2, d = 1$$

Therefore

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Activity 2.4.2 :

By matrix multiplication show that $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

Theorem 2.4.3

1. If A is an invertible (nonsingular) matrix, then A^{-1} is invertible (nonsingular) and $(A^{-1})^{-1} = A$.
2. If A is an invertible (nonsingular) matrix, then $(A^t)^{-1} = (A^{-1})^t$.
3. If A and B are invertible (nonsingular) matrices, then AB is invertible (nonsingular) and $(AB)^{-1} = B^{-1} A^{-1}$.

Proof

1. Let A be an invertible matrix . there exists a matrix B such that $AB=BA=I_n$. $B^{-1}=A$. Since $B = A^{-1}$, $B^{-1} = A$ implies that $(A^{-1})^{-1} = A$
2. $(A^t)(A^{-1})^t = (A^{-1}A)^t = I^t = I$ and $(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I$
Therefore , $(A^t)^{-1} = (A^{-1})^t$
3. $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI A^{-1} = A A^{-1} = I$. Similarly
 $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$
Therefore , $(AB)^{-1} = B^{-1}A^{-1}$

Remark: By repeated application the preceding theorem ,It can be shown that if

$A_1, \dots, A_n$ are invertible matrices of the same order then
 $(A_1A_2 \dots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \dots A_2^{-1} A_1^{-1}$

Procedure for finding the inverse of a matrix.

The procedure for computing the inverse of matrix A is as follows.

Step1. Form the augmented matrix $B = [A \quad I_n]$ obtained by adjoining the identity matrix I_n to the given matrix A .

Step2. By using elementary row operations on B ,Transform the matrix A in to reduced row echelon form.

Step3.

a.If the reduced row echelon form. of A is the identity matrix , A is invertible and its inverse is the matrix to the right of the vertical bar..

b.If the reduced row echelon form. of A is not the identity matrix , A is not invertible.

Example 2.4.9 : Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\text{Step 1. } B = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right)$$

$$\text{Step 2: } B = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right) \xrightarrow{B_2 \rightarrow B_2 - B_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

$$\xrightarrow{B_1 \rightarrow B_1 - 2B_2} \left(\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

Therefore $\left(\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array} \right)$ is the reduced row echelon form of B .

Step 3 : Since the reduced row echelon form of A is the identity matrix , A is invertible

and its inverse is $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Example 2.4.10 : Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$$

$$\text{Step 1. } B = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 3 & 3 & -1 & 0 & 0 & 1 \end{array} \right)$$

$$\text{Step 2: } B = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 3 & 3 & -1 & 0 & 0 & 1 \end{array} \right) B_3 \rightarrow B_3 + (-3)B_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 6 & -7 & -3 & 0 & 1 \end{array} \right) B_3 \rightarrow B_3 + (-6)B_2 \quad \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 11 & -3 & -6 & 1 \end{array} \right)$$

$$B_3 \rightarrow \frac{1}{11}B_3 \quad \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3/11 & -6/11 & 1/11 \end{array} \right)$$

$$B_2 \rightarrow B_2 + (3)B_3 \quad \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -9/11 & -7/11 & 3/11 \\ 0 & 0 & 1 & -3/11 & -6/11 & 1/11 \end{array} \right)$$

$$B_1 \rightarrow B_1 + (-2)B_3 \quad \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 17/11 & 12/11 & -2/11 \\ 0 & 1 & 0 & -9/11 & -7/11 & 3/11 \\ 0 & 0 & 1 & -3/11 & -6/11 & 1/11 \end{array} \right)$$

$$B_1 \rightarrow B_1 + (1)B_2 \begin{pmatrix} 1 & 0 & 0 & : & \frac{8}{11} & \frac{5}{11} & \frac{1}{11} \\ 0 & 1 & 0 & : & -\frac{9}{11} & -\frac{7}{11} & \frac{3}{11} \\ 0 & 0 & 1 & : & -\frac{3}{11} & -\frac{6}{11} & \frac{1}{11} \end{pmatrix}$$

Step 3 Since the reduced row echelon form. of A is the identity matrix , A is invertible

and its inverse is $\begin{bmatrix} \frac{8}{11} & -\frac{5}{11} & \frac{1}{11} \\ -\frac{9}{11} & -\frac{7}{11} & \frac{3}{11} \\ -\frac{3}{11} & -\frac{6}{11} & \frac{1}{11} \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} \frac{8}{11} & -\frac{5}{11} & \frac{1}{11} \\ -\frac{9}{11} & -\frac{7}{11} & \frac{3}{11} \\ -\frac{3}{11} & -\frac{6}{11} & \frac{1}{11} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix}$

Activity 2.4.3

Find the inverse of

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$

2.5 Rank of a matrix

Definition 2.5.1 :The **rank** of a matrix is the number of non zero rows the row echelon form

of the matrix.

Example 2.5.1: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the rank of A.

Solution:

A is in row echelon form. A has two non zero rows. Hence the rank of A is 2.

Activity 2.5.1:

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find the rank of A

Activity 2.5.2:

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Find the rank of A

Remark : If A is an $m \times n$ matrix , the the rank of A is less than or equal to the $\min(m,n)$

2.6 System of Linear Equations and methods of solving.

System of linear equations

Consider the following m system of linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad (*)$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Hence linear system (*) above can be written in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

Note : If b is the zero matrix then the system is said to be homogeneous otherwise non homogeneous.

The matrix A is called the **coefficient matrix** of the equations in (*), and the matrix

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

Obtained by adjoining **b** to A, is called the **augmented matrix** of (*). Like inverses of a matrix , the augmented matrix of (*) will be written as $[A \quad \mathbf{b}]$.

Example 2.6.1 :Consider the following system of three linear equations in two unknowns.

$$x - 2y + z = 12$$

$$-2x + \quad + z = 5$$

a. Write in the form of $AX = b$

b. Write the augmented matrix B

Solution

a. The system can be written as.

$$x - 2y + z = 12$$

$$-2x + 0y + z = 5.$$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$b. \quad B = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 12 \\ -2 & 0 & 1 & 5 \end{array} \right)$$

Activity 2.6.1 :Consider the following system of three linear equations in two unknowns.

$$3x_1 + 2x_2 + 7x_3 + 4x_4 = 0$$

$$x_1 - 3x_2 - 6x_3 + x_4 = 3$$

$$8x_1 + x_3 - 3x_4 = 2$$

a) Write in the form of $AX = b$

b) Write the augmented matrix B

Solving Systems of linear equations

Consider the following m system of linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

(*)

Our aim is now to solve the above system of linear equations

If $b_1 = b_2 = \dots = b_n = 0$, the system is called homogeneous otherwise it is non **homogenous system**.

A solution of the above equation (*) is an n – tuple that satisfies the system. The solution set S is the family of all such n -tuples .Solving the system means describing the set S in some way that enables us to tell easily whether a given n – tuple belongs to S or not.

Remark : A homogenous system always has a solution where $x_1 = x_2 = \dots = x_n = 0$, which is called a trivial solution.

Gaussian elimination, named for the great Mathematician C.F.Guass , is the process of reducing a matrix to row echelon form.

Operations that leads to equivalent systems of equations

Each of the following operations performed on a system of linear equations produces an equivalent system .

- i) Interchange two equations
- ii) Multiply an equation by a non zero constant
- iii) Add a multiple any equation to any other equation.

Given a system of m linear equations in n unknowns. Three questions arise

- (i) Does the system have any solution
- (ii) If so, how many?
- (iii) What are the solutions?

The solution Set S of (*) is not changed to the standard form by these three operations

- a. Two equations are interchanged
- b. One equation is multiplied by a number that is not 0.
- c. One equation is changed by adding another one to it

The operations (a,b,c) have counterparts for matrices. Interchanging two linear equations is equivalent to interchanging two rows of the matrix of the system. Multiplying an equation by a number corresponds to multiplying the associated row of the matrix by the same number. Finally adding one equation to another is equivalent to adding one row to another row.

Gaussian – Jordan method for solving Systems of linear equations

Consider the following m system of linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

To solve the System

Step 1. Write in the form of $Ax = b$.

Step2. Form the augmented matrix $B = [A \quad b]$.

Step3. Transform the augmented matrix to reduced row echelon form by using elementary row operations.

Step4. Compare the rank of A and the rank of B

- a. If rank of A < rank of B , the system has no solution
- b. If rank of A = rank of B , the system has a solution
 - (i) rank of A = rank of B = n , the system has exactly one solution.
 - (ii) rank of A = rank of B < n , the system has infinitely many solutions.

Step 5. Use back substitution

The above method is called Gauss-Jordan elimination.

Example 2.6.2: Solve

$$\begin{aligned}x - 2y + z &= 0 \\ -2x + \quad + z &= -1 \\ 2x - y + z &= 2\end{aligned}$$

Solution

$$\text{Step 1: } A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\text{Step 2 : } B = \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ -2 & 0 & 1 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix}$$

$$\text{Step 3: } B = \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ -2 & 0 & 1 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix} \quad B_2 \rightarrow B_2 + 2B_1 \quad \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & -4 & 3 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix}$$

$$B_3 \rightarrow B_3 + (-2)B_1 \quad \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & -4 & 3 & : & -1 \\ 0 & 3 & -1 & : & 2 \end{pmatrix} \quad B_2 \rightarrow -\frac{1}{4}B_2 \quad \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 3 & -1 & : & 2 \end{pmatrix}$$

$$B_3 \rightarrow B_3 + (-3)B_2 \quad \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & \frac{5}{4} & : & \frac{5}{4} \end{pmatrix} \quad B_3 \rightarrow \frac{4}{5}B_3 \quad \begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Now B is changed in to row echelon form.

Step 4. The rank of A = 3 (The left hand of the vertical bar) and the rank of B = 3 , the system has a solution. Since

rank of A = rank of B = 3 = n , the system has exactly one solution.

Step 5. By Using back substitution

$$\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$$z = 1, \quad y + -\frac{3}{4}z = \frac{1}{4} \Rightarrow y = 1$$

$$z = 1, \quad y = 1, \quad x - 2y + z = 0 \Rightarrow x = 1$$

Hence x = 1, y = 1, z = 1

Remark: The linear systems with at least one solution are called consistent, otherwise the system is said to be inconsistent. A consistent system has either one solution or infinitely many solutions.

Theorem 2.6.1 : A system of linear equations with fewer equations than variables must have either an infinite number of solutions or no solution. (Such a system cannot have a unique solution)

Proof: $m < n$. Rank $A = r \leq \min(m,n) = m < n$

$$\Rightarrow \text{rank } A < n$$

$\Rightarrow$ It has no unique solution

Note: In a consistent system rank $A = r < n$, $n - r$ of the unknowns are assigned any values whatever, the other r unknowns are uniquely determined.

Example 2.6.3 : Solve

$$2x + y + 2z = 3$$

$$3x - y + 4z = 7$$

$$x + 3y = -1$$

Solution

$$\text{Step 1: } A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & -1 & 4 \\ 1 & 3 & 0 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$$

$$\text{Step 2: } B = \begin{pmatrix} 2 & 1 & 2 & : & 3 \\ 3 & -1 & 4 & : & 7 \\ 1 & 3 & 0 & : & -1 \end{pmatrix}$$

$$\text{Step 3: } B = \begin{pmatrix} 2 & 1 & 2 & : & 3 \\ 3 & -1 & 4 & : & 7 \\ 1 & 3 & 0 & : & -1 \end{pmatrix} \quad B_{1,3} \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 3 & -1 & 4 & : & 7 \\ 2 & 1 & 2 & : & 3 \end{pmatrix}$$

$$B_3 \rightarrow B_3 + (-2)B_1 \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 3 & -1 & 4 & : & 7 \\ 0 & -5 & 2 & : & 5 \end{pmatrix} \quad B_2 \rightarrow B_2 + (-3)B_1 \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & -10 & 4 & : & 10 \\ 0 & -5 & 2 & : & 5 \end{pmatrix}$$

$$B_2 \rightarrow B_2 + (-2)B_3 \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 0 & 0 & : & 0 \\ 0 & -5 & 2 & : & 5 \end{pmatrix} \quad B_{3,2} \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & -5 & 2 & : & 5 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

$$B_2 \rightarrow -1/5 B_2 \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 1 & -2/5 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

Now B is changed in to row echelon form.

Step 4. The rank of $A = 2$ (The left hand of the vertical bar) and the rank of $B = 2$, the system has a solution. Since

rank of A = rank of B < 3 = n , the system has infinitely many solutions.

Step 5. By Using back substitution

$$r = \text{rank of A} = 2, \quad n = 3$$

$n - r = 3 - 2 = 1$ unknown will be assigned any value let us say t.

Let $z = t$

$$y + \frac{2}{5}z = -2 \Rightarrow y + \frac{2}{5}t = -2 \Rightarrow y = -2 + \frac{2}{5}t$$

$$x + 3y = -1 \Rightarrow x = -1 + -3y \Rightarrow x = -1 + -3(-2 + \frac{2}{5}t) \Rightarrow x = -7 + \frac{6}{5}t$$

Hence the solution is $x = -7 + \frac{6}{5}t$, $y = -2 + \frac{2}{5}t$, $z = t$ where $t \in \mathbb{R}$

Example 2.6.4 : Solve

$$x - y + z = 2$$

$$2x + y + z = 1$$

$$3x + 2z = 5$$

Solution

$$\text{Step 1: } A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

$$\text{Step 2: } B = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 2 & 5 \end{array} \right)$$

$$\text{Step 3: } B = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 2 & 5 \end{array} \right) \xrightarrow{B_3 \rightarrow B_3 + (-3)B_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 0 & 3 & -1 & -1 \end{array} \right)$$

$$B_2 \rightarrow B_2 + (-2)B_1 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -1 & -3 \\ 0 & 3 & -1 & -1 \end{array} \right) \xrightarrow{B_3 \rightarrow B_3 + (-1)B_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -1 & -3 \\ 0 & 0 & 0 & 2 \end{array} \right)$$

$$B_2 \rightarrow \frac{1}{3}B_2 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 2 \end{array} \right) \xrightarrow{B_3 \rightarrow \frac{1}{2}B_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

Now B is changed in to row echelon form.

Step 4. The rank of A = 2 (The left hand of the vertical bar) and the rank of B = 3, the system has a solution. Since

rank of A < rank of B , the system has no solution

Activity 2.6.2

1. Solve

$$\begin{aligned} 2x - y + z &= 3 \\ x + y - 3z &= 1 \\ x + 2y - 2z &= -11 \end{aligned}$$

2. . Solve

$$\begin{aligned} x - y + z &= 6 \\ 2x + y + 3z &= 1 \\ x + y - 3z &= 1 \end{aligned}$$

3. . Solve

$$\begin{aligned} x + y - z &= 1 \\ 3x - y &= 5 \\ 2x - 2y + z &= 3 \end{aligned}$$

Inverse of a matrix for solving Systems of linear equations

Suppose A is an $n \times n$ matrix. Our aim is now to solve the equation $A\mathbf{x} = \mathbf{b}$ which is a system of n equations in n unknowns. Suppose that A is invertible . Then A^{-1} exists and we can multiply $A\mathbf{x} = \mathbf{b}$ by A^{-1} on both sides. We obtain

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Moreover, $\mathbf{x} = A^{-1}\mathbf{b}$ is clearly a solution to the given linear system. Thus if A is invertible , the solution is unique .

Using the above property ,

Theorem 2.6.2. If A is an $n \times n$ matrix, the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if A is noninvertible.

Example 2.6.5: Solve

$$\begin{aligned} x + 2z &= 1 \\ x + 3z &= -1 \end{aligned}$$

Solution

First write in the form of $A\mathbf{x} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

By using elementary row operations , we can find

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Hence $x = 5, z = -2$.

Example 2.6.6 : Solve

$$\begin{aligned} x - y + 2z &= 1 \\ y - 3z &= 2 \\ 3x + 3y - z &= 0 \end{aligned}$$

Solution

First write in the form of $\mathbf{AX} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

By using elementary row operations , we can find

$$\mathbf{A}^{-1} = \frac{1}{11} \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{11} \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -2 \\ -23 \\ -15 \end{pmatrix}$$

Activity 2.6.3 Solve the following systems of linear equations using the inverse of a matrix.

1. Solve

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned}$$

2. $x + 2y - z = -2$

$$2x - y + z = 5$$

$$-x + y + z = 2$$

exercises

1. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$.

Find a. AB.

b. A^t

c. B^t

d. $B^t A^t$

e. The rank of A

2. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ -2 & 7 & 4 \end{bmatrix}$$

is A a symmetric matrix?

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -3 & 5 \\ -2 & 6 & 1 \end{bmatrix}$$

Write A as a sum of symmetric and antisymmetric matrices..

4. Determine whether the matrices are in row or reduced row echelon forms.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Find the inverse of the following matrices.

a. $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

b. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \\ 3 & -4 & 2 \end{bmatrix}$

6. Find the rank of the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & -2 & 0 & 1 \\ 5 & -3 & 1 & 2 \end{pmatrix}$$

7. Solve

$$2x + y + 2z = 3$$

$$3x - y + 4z = 7$$

$$4x + 3y + 6z = 5$$

8. Find for what values of α and β , the system

$$2x + 4y + (\alpha + 3)z = 2$$

$$X + 3y + z = 2$$

$$(\alpha - 20)x + 2y + 3z = \beta$$

is consistent .

9. Solve

$$2x + 6y + z = -1$$

$$3x + 9y + 2z = -1$$

$$-y + 3z = 4$$

10. Let A be a 3x5 matrix and B be a 5x5 matrix .What is the size of AB?

11. Let A and B square matrices of the same order which of the following is true?

A. $(A + B)^{-1} = A^{-1} + B^{-1}$

B. $AB = 0 \Rightarrow |A| = 0$ or $|B| = 0$

C. $AB = 0 \Rightarrow |A| = 0$ and $|B| = 0$

D. $(AB)^{-1} = A^{-1} B^{-1}$

12. Solve

$$x + 2y = 2$$

$$3x + 6y - z = 8$$

$$x + 2y + z = 0$$

$$2x + 5y - 2z = 9$$

13.a. List all possible forms of 2x 2 reduced row echelon forms.

b. List all possible forms of 3 x 3 reduced row echelon forms

14. If $\begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 4 & 2 \end{pmatrix}$, find a,b,c,d

15. Verify that $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are two symmetric matrices such that AB is

Skew symmetric.

16. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where $a \neq 0$, $b \neq 0$ and $c \neq 0$. Find the inverse of A.

17. If $A^{-1} = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$, find A

18. Show that $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ is the inverse of $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$.

19. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find

a. $A \cdot A^t$

b. $A^t \cdot A$

20. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$. Find a) A^2 b) A^3

21. Let $A = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}$. Find a non zero vector $u = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $Au = 3u$

22. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Find A^n

23. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$