

The State of Eritrea
Ministry of Education

Mathematics

for

Grade 9

Textbook

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Introduction

Dear learners, welcome to the Grade 9 mathematics text book. Its focus is on the basic concepts and techniques of mathematics.

This Grade 9 mathematics textbook is written based on the new national curriculum; that is, learner centred and interactive pedagogical approach. A presentation of mathematics which puts its emphasis on the role of learner rather than on the role of a teacher is likely to lead to far greater satisfaction on the part of the learner and result in greater mathematical competence.

This new approach helps you to widen your thinking ability, master the subject matter of mathematics and solve mathematical problems; and relate this to your daily activities.

The book consists of 8 units. These comprise: set theory, foundation of functions, linear functions, quadratic functions, exponential and logarithmic expressions, right angled triangles and trigonometric ratios, business mathematics, and polynomials.

By the end of Grade 9, you can use and apply the basic concepts of set theory and business transactions in your daily activities; describe independent and dependent quantities in functional relationships; identify the properties and attributes of functions and describe them in a variety of ways and in various representations.

You can interpret and describe the effects of changes in the parameters of quadratic functions and demonstrate an understanding of the elementary attributes of polynomials and distinguish them from other functions; use graphs of polynomials to describe their properties and apply division polynomials in arithmetical simplifications; and develop the trigonometric ratios of a right angled triangle.

You have to begin with solving simple basic problems and then proceed to more difficult ones and acquire a deep knowledge and skill of what is discussed in the class. This is done step by step so that you can understand it easily and apply it in your daily activities.

In order to think critically and apply the knowledge, skills and attitudes practically in every unit different activities accompanied by some illustrations are included. These activities lead the learner to use the relevant learning materials that will be applied. The main concepts and definitions that need to be focused on are highlighted in boxes.

The basic concepts are introduced with sufficient activities and examples before formal definitions are stated. It is hoped that such an introduction will help you to grasp the precise definition of the concepts. For this reason you are encouraged to read them on your own. Moreover, there is a need for continuous assessment to help the overall evaluation of each unit. Review questions are therefore included at the end of each unit and you should work through them regularly.

This book is accompanied by a teacher's guide. The guide includes the level of expected learning outcomes, teaching methodology, and information on mechanism and instruments of assessment.

It is also recommended that parents and interested individuals also play their role in facilitating the process of teaching-learning so as to meet the intended target.

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CHAPTER 1

Set Theory

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- differentiate between finite and infinite sets;
- differentiate between equal and equivalent sets;
- set up union and intersection of two or more sets;
- find subsets from a given set;
- find the complement of a given set;
- find difference of sets; and
- find the Cartesian product of two sets.

1.1 Methods of specifying sets

Activity 1.1

The notion of a set is basic in Mathematics. Look at the following sentences.

- At home you may have a set of forks, a set of spoons, a set of dishes, and so on and so forth.
- The set of all Nations in the world.
- The set of positive integers.
- The set of the days of the week.

What do you think a set is in Mathematics?

George Cantor, (1845-1918) a German mathematician created set theory, a new area of Mathematics. Cantor defined a set as:

A set is a collection of well-defined objects called *elements*.

We usually use capital letters, e.g. A, B, C, ... to denote sets. .

We use braces to enclose the elements of a set. That is,

$$A = \{a, b, c\}$$

is used to indicate the set A whose elements are a, b, and c. We

indicate that *a* is an element of A by writing

$$a \in A.$$

Since d is not an element of A, we write $d \notin A$.

Example 1.1

A = the set of natural numbers less than or equal to 7, i. e.,

$A = \{1, 2, 3, 4, 5, 6, 7\}$, This means that 1, 2, 3, 4, 5, 6, and 7 are elements of set A. So we can write:

- a) $2 \in A$ which means that 2 is an element of A.
- b) $8 \notin A$ which means that 8 is not an element of A.

Activity 1.2

Can you list all the elements of the following sets?

- a) The set of integers between 0 and 10, inclusive.
- b) The set of positive factors of 45.
- c) The set of all letters in the English alphabet.

If the elements of a set A can all be listed, then the set is given by putting the letter A equal to a brace which includes all the elements of A in any order whatsoever. This method of listing is called the **complete listing method**.

Example 1.2

The set A of integers between 0 and 10, inclusive, is given by

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Activity 1.3

Can you list all the elements of the following sets?

- a) The set of integers.
- b) The set of even integers.
- c) The set of all integers that are multiples of 3.

Have you observed that we can not list all the elements of the sets given in a), b), and c)? But they have a certain pattern. If the elements of a set can be indicated by listing some of the elements, usually the first few ones, then the set is described by putting the letter denoting the set equal to a brace This includes these few elements with three dots (...) either after or before or both before or after these few elements. This method of describing a set is called the **partial listing method**. Both, the complete and partial listings are called the **roster method**.

Example 1.3

The set A of integers:

$$A = \{\dots, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Activity 1.4

How can sets with a large number of elements and with no pattern such as the set of real numbers then be described?

For such sets and others, we use the **set builder notation**. This notation requires that we use a statement, given in words or in mathematical form to describe clearly the conditions the elements must satisfy in order to belong to a particular set.

Example 1.4

1. The set A of real numbers.

$$A = \{x: x \text{ is a real number}\}$$

which is read as x such that x is a real number

2. The set B of all rational numbers.

$$B = \{x: x \text{ is a rational number}\}$$

We can also use this notation for finite sets as well.

Example 1.5

The set $S = \{1, 3, 5, 7\}$ can be written as

$$S = \{x: x \text{ is an odd integer and } 1 \leq x \leq 7\}$$

Activity 1.5

1. Can you find the number of elements in the following three sets?
- The set of negative integers greater than -5.
 - The set of integers.
 - The set of integers less than 1000.

A set is **finite** if its elements can be counted and the counting process comes to end. If the elements can be counted, but the counting process carried on indefinitely, we say that the set is **infinite**.

2. List the elements of the following sets.
- $A =$ the set of all real numbers whose square is negative.
 - $B = \{x: x \text{ is a man with blue beard}\}$

Sets A and B in number 2 of Activity 1.5 are special sets with no elements. Such sets are called empty sets.

A set which contains no elements is called the **empty set**. It is denoted by ϕ .

Exercise 1.1

1. Describe each of the following sets using one of the three methods discussed above.
 - a) A is the set whose members are only -1, 0, 1.
 - b) C is the set of positive multiples of 4.
 - c) D is the set of even integers.
 - d) F is the set of the four seasons in a year.
 - e) G is the set of letters in the word “mathematics”.
2. Determine which of the following sets is finite and which is infinite.
 - a) A = the set of even numbers.
 - b) B = the set of letters in the English Alphabet
 - c) $D = \{3, 5, 7, 9, \dots\}$
 - d) E = the set of all positive multiples of 3 less than 30}
 - e) $F = \{1, 2, 3, 4, \dots, 1000\}$
 - f) $A = \{x: x \text{ is a student}\}$.
3. Determine whether each statement is true or false.
 - a) $3 \in \{1, 3, 5, 7\}$
 - b) $10 \in \{1, 2, 3, \dots, 16\}$
 - c) $0 \notin \{x: x \text{ is a multiple of } 5\}$
 - d) $4 \notin \{x: x \text{ is an even integer}\}$
4. Write the following sets using set-builder notation.
 - a) $\{5, 6, 7, 8, 9, 10, \dots, 198, 199\}$
 - b) $\{2, 4, 6, 8, 10, \dots\}$
 - c) $\{1, 3, 5, 7, \dots\}$

1.2 Equal and Equivalent Sets

Activity 1.6

1. Suppose a class contains 40 chairs and one student is sitting in each chair with no one is standing. Can we say that the number of chairs matches with the number of students?

2. Let $A = \{x, y, z, u, v\}$ and $B = \{1, 2, 3, 4, 5\}$
 - a) Compare the number of elements in set A and in set B.
 - b) Do the two sets match?
1. Compare the following pair of sets and tell your observations.
 - a) $A = \{a, b, c\}$, $B = \{b, a, c\}$
 - b) $C = \{w, x, y, z\}$, $D = \{u, x, y, z\}$

Let us consider the following two sets A and B.

$$A = \{1, 2, 3\}$$

$$B = \{a, b, c\}$$

Each member of set A is matched exactly with one and only one member of set B and each member of set B is matched exactly with one and only one member of set A. This is what we call a **one-to-one correspondence** between sets.

We can use pictures like the following to see if two sets are in one-to-one correspondence or not.

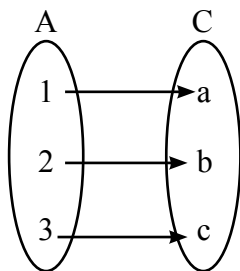


Fig. 1.1

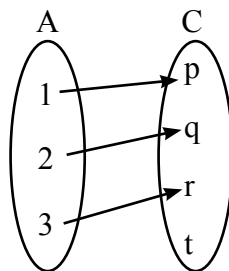


Fig. 1.2

Observe there is a one to one correspondence between sets A and B. But there is no one to one correspondence between sets A and C.

Two sets A and B are said to be **equivalent sets** if there is a one-to-one correspondence between them (i.e. if they have the same number of elements).

Example 1.6

1. $A = \{1, 3, 5, 7\}$ and $B = \{x: x \text{ is an odd natural number less than } 8\}$ are equivalent sets.
2. $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ are equivalent sets.
3. $C = \{1, 3, 5, 7\}$ and $D = \{2, 4, 6, 8, 10\}$ are not equivalent sets.

Two sets A and B are said to be **equal sets** if they contain exactly the same elements. We denote the equality of sets A and B by $A = B$.

Example 1.7

$$A = \{1, 3, 5, 7\}$$

$$B = \{x: x \text{ is an odd natural number less than } 8\}$$

Every element in set A is found in set B and every element in B is in set A. Thus $A = B$.

Exercise 1.2

1. Determine if the following pair of sets are equal or equivalent?
 - a) A is the set of whole numbers less than 8.
B is the set of the distinct numbers in the telephone number 07251634.
 - b) A is the set of Zobas in Eritrea.
B is the set of Zoba administrators.
 - c) $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 1, 2, 3, 4\}$
 - d) $A = \{1, 2, 3, 4\}$, $B = \{4, 3, 2, 1\}$
 - e) $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5\}$
2. Are equal sets equivalent? Explain your answer.

1.3 Subsets and Proper Subsets

Activity 1.7

Let us consider the learners in your class as set A. We can form another group or collections from the considered set A. Determine if the following sets are contained within set A, or not. (Assuming the class contains both boys and girls.)

- a) B = the set of female learners in your class.
- b) C = the set of male learners in your class.
- c) E = the set of learners in your class who are 15 years of age.

Set A is said to be a **subset** of set B, if every member of A is also a member of B and is denoted by $A \subseteq B$.

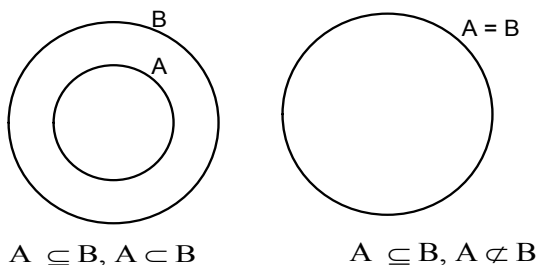


Fig. 1.3

If A is a subset of B, but B contains some elements which are not in A, we say A is a **proper subset** of B and is denoted by $A \subset B$.

Example 1.8

1. Let $A = \{1, 2, 5, 9\}$ and $B = \{1, 2, 5, 7, 9\}$. Is $A \subset B$?
2. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4\}$, $C = \{1, 2\}$, and $D = \{2, 1, 4, 3\}$. Determine whether each statement is true or false.
 - a) $C \subseteq A$
 - b) $A \subseteq B$
 - c) $C \subset B$
 - d) $D \subseteq A$
 - e) $A \subseteq D$

Solution:

1. Yes, since each element of A is an element of B and B contains an element that is not an element of A.
2.
 - a) True, every element of C is an element of A.
 - b) False, 1 is in A but not in B.
 - c) False, 1 is in C but not in B.
 - d) True, every element of D is an element of A.
 - e) True, every element of A is an element of D.

Every set is a **subset** of itself, and the empty set is a subset of every set.

Activity 1.8

- List all possible subsets of $\{1, 2\}$. How many are there?
- List all possible subsets of $\{1, 2, 3\}$. How many are there?
- Guess how many subsets there are of $\{1, 2, 3, 4\}$. Verify your answer by listing all the possible subsets.
- Make a table of your findings in a), b), and c)
- Complete the following table

Number of elements in a set	1	2	3	4	5	6	7
Number of subsets	2	4	8	16			

- Can you generalise the case of a set A with exactly n elements, for any positive integer n ?

Exercise 1.3

- Determine which of the following statements are true and which are false.
 - $\{\text{Abraham, Azieb}\} \subset \{\text{Ali, Abraham, Azieb}\}$
 - $\{x: x \text{ is a prime number}\} \subset \{x: x \text{ is an odd integer}\}$
 - $\{x: x \text{ is a prime number}\} \subset \{x: x \text{ is a rational number}\}$
- Let $N =$ The set of natural numbers
 $Z =$ The set of integers
 $Q =$ The set of rational numbers
 $R =$ The set of real numbers

Determine which of the following statements are true, and which are false.

- $N \subseteq Z$
- $Q \subseteq Z$
- $Z \subseteq Q$
- $Q \subset R$

- Are the following statements true?
 - If $A = B$, then $A \subseteq B$ and $B \subseteq A$
 - If $A \subseteq B$ and $B \subseteq A$, then $A = B$

1.4 Union and Intersection of Sets

1.4.1 Union of Sets

Activity 1.9

1.

- a) Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{0, 2, 4, 6, 8\}$. What do you get if the elements of A and B are combined?
- b) Let $M = \{1, 2, 3\}$ and $N = \{1, 3, 4\}$. Combine these two sets M and N.
- c) Let $V = \{x: x \text{ is a vowel in the English alphabet}\}$
 $W = \{x: x \text{ is a consonant in the English alphabet}\}$
 Do you agree that the union of V and W gives the set of all English alphabets?

2. If $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, a\}$, then find the union of the two sets.

Note: When we find the union of sets we avoid repeating the elements which occur in both sets.

The **union** of two sets A and B is the set of all elements that belong to set A or set B or to both sets A and B. The union of the sets A and B is denoted by $A \cup B$.

Example 1.9

Let $A = \{0, 2, 3\}$, $B = \{2, 3, 7\}$, and $C = \{0, 7, 8\}$ List the elements in each of the following sets.

- a) $A \cup B$
- b) $A \cup C$

Solution:

- a) $A \cup B = \{0, 2, 3, 7\}$
- b) $A \cup C = \{0, 2, 3, 7, 8\}$

1.4.2 Intersection of Sets

Activity 1.10

- Let $A = \{a, b, c, d, e\}$ and, $B = \{a, e, i, o, u\}$
Which of the elements are common to both sets?
- Let $D = \{x: x \text{ is a counting number less than } 9\}$
 $E = \{x: x \text{ is a prime number less than } 15\}$
List the common elements in sets D and E.
- Let $A = \{1, 2, 3\}$ and $B = \{5, 7\}$.
Is there any element which belongs to both A and B?

The **intersection** of two sets A and B is the set consisting of all elements that belong to both sets A and B. A intersection B is denoted as $A \cap B$.

Two sets A and B are said to be **disjointed**, if they do not have any element in common.

Example 1.10

Let $A = \{0, 2, 3\}$, $B = \{2, 3, 7\}$, and $C = \{7, 8\}$. List the elements in each of the following sets.

- $A \cap B$
- $B \cap C$
- $A \cap C$

Solution:

- $A \cap B = \{2, 3\}$
- $B \cap C = \{7\}$
- $A \cap C = \emptyset$

1.4.3 Venn Diagrams

Sets and set problems are mostly represented by diagrams called **Venn diagrams**. A Venn diagram is a simple closed curve to denote the set of elements within the curve. It is a useful way to visualise relationships between sets, and set operations.

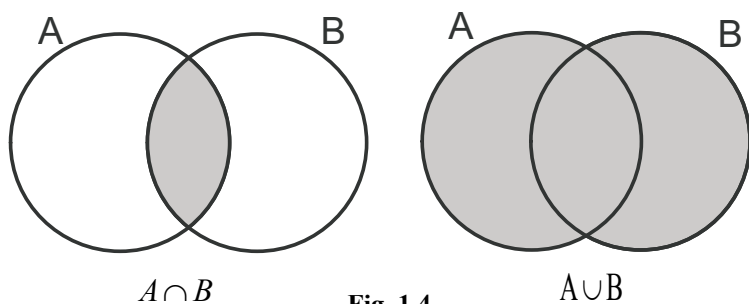


Fig. 1.4

Activity 1.11

In a survey, 150 high school learners were asked if they owned a bicycle, a calculator, or both. The result showed that 63 owned a bicycle, 84 owned a calculator, and 27 owned both. Construct a Venn diagram with two circles, one for learners owning a bicycle, and one for learners owning a calculator. Fill in the number for each region in the diagram.

- Construct a Venn diagram with two circles, one for learners owning a bicycle, and one for learners owning a calculator. Fill in the number for each region in the diagram.
- How many learners owned none of the items mentioned in the survey?
- How many learners owned a calculator only?
- How many learners owned a bicycle only?
- How many learners owned a bicycle or a calculator?

The **cardinality** of a finite set A is the number of elements of the set, and is denoted by $n(A)$.

Example 1.11

- If $A = \{1, 2, 3\}$, then its cardinality is given by $n(A) = 3$.
- If $B = \{x: x \text{ is a vowel in the English alphabet}\}$, then its cardinality is $n(B) = 5$.

The **cardinal number** of the union of two finite sets A and B is given by $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Exercise 1.4

- Suppose $A = \{0, 1, 2\}$, $B = \{3, 4, 5, 6\}$, $D = \{2, 4, 6\}$ and $E = \{1, 3\}$. Write each of the following sets.
 - $A \cap B$
 - $A \cup B$
 - $A \cap D$
 - $B \cap E$
 - $D \cup E$
- If $A = \{2, 3, 5, 7\}$, $B = \{2, 5, 7\}$, $C = \{3, 5, 7, 11\}$, then find $A \cap (B \cup C)$.
- If $B \subseteq A$, find a simpler expression for each of the following:
 - $A \cap B$
 - $A \cup B$

1.5 Complement and Difference of Sets

1.5.1 Complement of a Set

Activity 1.12

- Consider the following Venn diagram

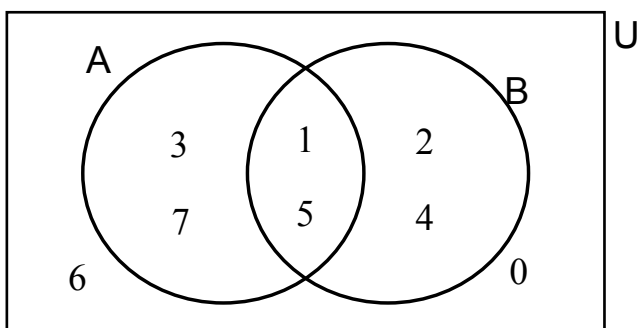


Fig. 1.5

- List the elements of A .
- List the elements of B .
- List all the elements we are discussing in this problem.

- d) List the elements which are not in A.
- e) List the elements which are not in B.

A **universal** set is a set that contains all the elements being considered in a given discussion or problem.

The **complement** of set A is the set of all elements in the universal set that are not in A. The complement of set A is denoted by A' .

$$A' = \{x \in U \text{ and } x \notin A\}$$

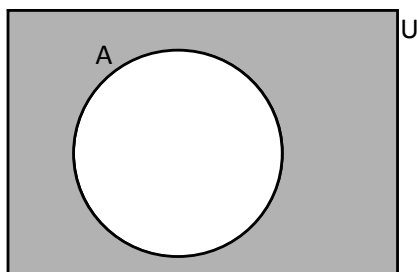


Fig. 1.6: The shaded region is A'

1.5.2 Difference of Sets

Activity 1.13

If $A = \{2, -2, 5\}$ and $B = \{2, -2, 0, -3\}$.

- a) What are the elements which are in A but not in B?
- b) What are the elements of B which are not in A?

The **difference** of a set A with respect to set B is the set of elements in B that are not in A. It is denoted as $B-A$.

$$B-A = \{x: x \in B, x \notin A\}$$

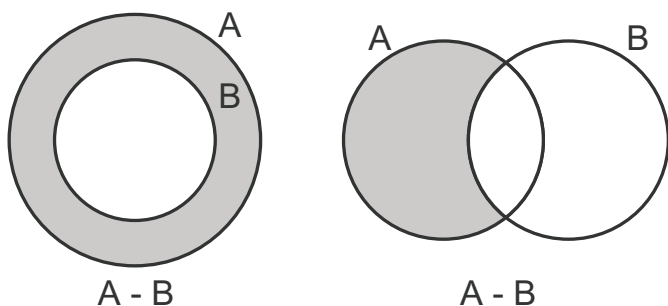


Fig. 1.7

Exercise 1.5

- Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$ and $C = \{3, 6\}$. Then find
 - A'
 - B'
 - C'
- Given that $U = \{4, 5, 6, 7, 8, a, e, i, o, u\}$, $A = \{5, 6, 7, a\}$, $B = \{4, 7, e, a\}$, $C = \{6, 7, 8, i, o\}$, identify the following:
 - A', B', C'
 - $(A \cup B)'$
 - $A' \cap B'$
 - $(A \cap B)'$
 - $A' \cup B'$
- The equation $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ are referred to as **Demorgan's laws**.
 - Use Venn diagrams to show that $(A \cup B)' = A' \cap B'$
 - Use Venn diagrams to show that $(A \cap B)' = A' \cup B'$
- If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$ and $C = \{3, 6\}$ then answer each of the following questions.
 - $A - B$
 - $A - C$
 - $B - C$
 - $B - A$
 - $C - A$

- f) Are $A-B$ and $B-A$ equal?
 - g) Can you find an example such that $A-B = B - A$?
 - h) Have you observed that $A' = U - A$?
5. Let U = The set of natural numbers
- $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $B = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- a) Find $B - A$.
 - b) Find $A - B$.
 - c) Is $B-A = A-B$?

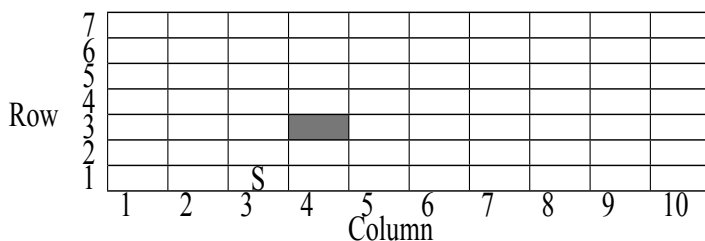
1.6 Cartesian Product

Activity 1.14

Draw a plan of the seating in your classroom, putting all the desks together. Represent each desk by a square. In each square, write the name of the student occupying the desk, which the square represents. The position of each student in the classroom is described precisely by using two independent informations:

- a) The column in which she or he sits.
- b) The row in which she or he sits.

If you are sitting on the desk lying in the 4th column and 3rd row (represented by the shaded square in the table below), your position could be written as (4, 3) first writing the column number and then the row number. Is this the same as (3, 4)? Write down the names and positions of the other students in your class. For example if Saba is sitting in the 3rd column and 1st row, write S(3,1). Note that the teacher's desk is not part of your sitting plan.



A pair of numbers in which x is the first and y is the second is called an **ordered pair** and is written as (x, y) .

Since the order in which the numbers occur is considered as well as the numbers, it follows that (y, x) and (x, y) are different ordered pairs unless $y = x$.

Activity 1.15

Let $A = \{1, 2, 3\}$ and $B = \{0, 2\}$

- Can you obtain a third set by using one of the set operations?
- Can you obtain a third set whose elements are ordered pairs?
- Do you agree the following is one which is obtained from the given sets A and B

$$C = \{(1, 0), (2, 0), (3, 0), (1, 2), (2, 2), (3, 2)\}$$

- Find another set whose elements are ordered pairs obtained from the given sets A and B with the first entry from set B and the second entry from set A .

Set C in Activity 1.15 is a set of ordered pairs and contains six elements. Moreover, all the first entries are elements of A while the second entries are elements of B .

Given two sets A and B , the **Cartesian product** of A and B , is the set consisting of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. It is denoted by $A \times B$.

Example 1.12

If $A = \{2, 4, 5\}$ and $B = \{1, 3, 6\}$, then

$$A \times B = \{(2, 1), (2, 3), (2, 6), (4, 1), (4, 3), (4, 6), (5, 1), (5, 3), (5, 6)\}$$

Exercise 1.6

1. Given $C = \{-1, -2\}$ and $D = \{0, -1\}$
 - a) Find $C \times D$ and $D \times C$
 - b) Is $C \times D = D \times C$?
2. If set A has 3 elements and set B has 2 elements, how many elements does set $A \times B$ have?
3. How about if A contains n elements and B contains m elements?

Review Exercise 1

1. List the elements in each set.
 - a) $\{x: x \text{ is an even natural number less than } 10\}$
 - b) $\{x: x \text{ is a natural number greater than } 6\}$
 - c) $\{x: x \text{ is an odd natural number between } 12 \text{ and } 57\}$
 - d) $\{x: x \text{ is an integer greater than } -5\}$
2. Write each set using set-builder notation.
 - a) $\{3, 4, 5, 6\}$
 - b) $\{5, 7, 9, 11, \dots\}$
 - c) $\{9, 11, 13, 15, \dots, 51\}$
3. Determine which of the following statements are true, and which are false.
 - a) $4 \in \{3, 4, 5, 6\}$
 - b) $0 \notin \{1, 2, 3, \dots\}$
 - c) $\{\text{Abraham, Ibrahim}\} \subseteq \{\text{Abraham, Ibrahim, Halima}\}$
 - d) If $A = \{x: x \text{ is an even integer between } 1 \text{ and } 11\}$ and $B = \{2, 4, 5, 6, 8, 10\}$, then $A \subseteq B$.
 - e) $3 \subseteq \{1, 2, 3\}$
 - f) $\{3\} \in \{1, 2, 3\}$
 - g) $\phi = \{\phi\}$

4. If $U = \{0, 1, 2, \dots, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{2, 3, 4, 5, 6\}$, and $C = \{0, 2, 4, 6, 8\}$, then find

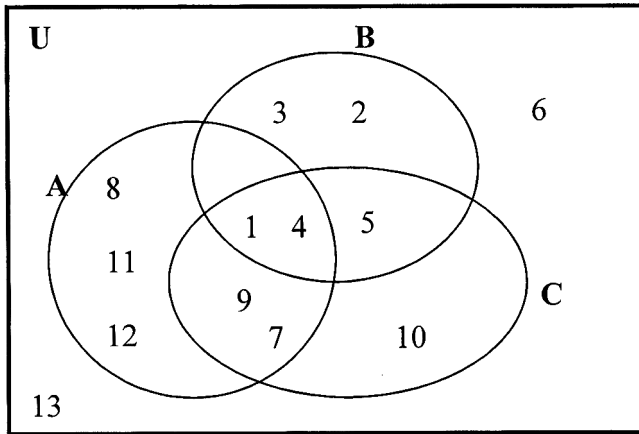
- | | |
|-------------------------------|------------------------|
| a) $A \cap B$ | f) $A \cap (B \cup C)$ |
| b) $B \cap C$ | g) $A \cap \phi$ |
| c) $A \cap B \cap (A \cap C)$ | h) A' |
| d) $A \cup \phi$ | i) B' |
| e) $A - B$ | j) $A - C$ |

5. If $A = \{-2, 0, 5\}$, $B = \{4, 5, 6\}$, $C = \{-2, 4, 6\}$, find

- | | |
|-----------------|-----------------|
| a) $A \times B$ | d) $C \times A$ |
| b) $A \times C$ | |
| c) $B \times C$ | |

6. In a group of 200 learners each one speaks Tigre or Tigrigna. If 50 speak both and 110 speak Tigre, how many speak Tigrigna only?
7. Use set builder notation to denote the set whose elements satisfy the following conditions.
- x is a real number greater than -1 and less than or equal to 10.
 - x is a real number less than 0 or greater than 5.
 - x is a real number less than -3 or greater than 2.
8. How many subsets does a set with 10 elements have?
9. How many proper subsets does a set with 10 elements have?

10. Consider the following Venn Diagram.



Find

- $A \cap B$
- $A \cap C$
- $A \cap (B \cap C)$
- $(A \cap B) \cap C$
- $(A \cap B)'$
- $A \cup B$
- $A \cup C$
- $A \cup (B \cup C)$
- $U - A$
- $A - B$
- $A - (B \cup C)$

CHAPTER 2

Foundation of Functions

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- describe a function as a dependence of one quantity on another;
- express functional relationships using visual and algebraic methods;
- solve equations and inequalities in linear and absolute value forms;
- identify the difference between direct and inverse variations.

2.1 Describing Functions

2.1.1 Dependent and Independent Quantities

Activity 2.1

Abrehet has a small number of poultry in Keren. She sells eggs every Saturday. This coming Saturday, she wishes to get income by selling 45 eggs. The selling price of an egg is 3.00 Nakfa.

- a) What will be her income if she sells 10 eggs?
- b) What will be her income if she sells 18 eggs?
- c) Verbally, describe the relationship between the income she earns and the number of eggs she sells.
- d) What do you think Abrehet should do to increase her income?



Fig. 2.1

The income Abrehet earns depends on the number of eggs she sells. In this relationship, the income earned is called the **dependent** quantity while the number of eggs sold is called the **independent** quantity.

Activity 2.2

Kelifa is sitting for a Mathematics examination. The question paper contains 30 multiple-choice questions. Each question carries 2 marks.

- What will be his score in the examination, if the number of questions he answered correctly is 5? 14?
- Verbally, describe the relationship between the score he obtains and the number of questions he answers correctly.
- What do you think Kelifa should do to increase his score in the examination?
- In this relationship, identify the dependent and independent quantities.

In a functional relationship between two quantities:

- The set of all possible values of the independent quantity is called the **domain** of the function.
- The set of all possible values of the dependent quantity is called the **range** of the function.

For the functional relationship in Activity 2.2,

Domain = $\{0, 1, 2, 3, \dots, 30\}$, and

Range = $\{0, 2, 4, 6, \dots, 60\}$.

Exercise 2.1

- In each of the following, two quantities are given. Determine whether there is a relationship between them or not. If a relationship exists, identify the dependent and independent quantities.
 - Using a public telephone, the total cost you are charged and the number of minutes you use the telephone.
 - The distance a car travels at a constant speed and the time it travels.
 - The wage a person earns and the number of hours the person works.
 - The area of a square region and the length of its side.

2. In each of the following situations, describe verbally the functional relationship between the given quantities. Identify the domain and the range of the function.
 - a) A car travels at a constant speed of 40 k.m per hour. The distance the car travels and the time it travels.
 - b) A company wishes to buy tyres for its cars. Each car needs 5 new tires (including one spare tyre). The number of cars the company owns and the total number of tyres it needs to buy.
 - c) A person earns a fixed amount of 50.00 Nakfa plus 5.00 Nakfa per each hour the person works. The total earnings of the person and the number of hours the person works.
 - d) Ali is 3 years older than his sister Keria. Ali's age given the age of his sister Keria.
 - e) The area of a square region and the length of its side.

2.1.2 Describing Relationships using Tables and Graphs

Activity 2.3

1. Abrehet noted that if she sold 1 egg for 3 Nakfa, then she would get 3 Nakfa. If she sold 2 eggs, then she would get 6 Nakfa, and so on. Record the relationship between the number of eggs sold and the income earned.
2. Abdu used the following **table of values** to describe the functional relationship between the number of questions Kelifa answers correctly and the score he gets in the examination.

No. of correct answers	0	1	2	5	14	19		27	
Total score	0	2	4	10	28		42		58

- a) Complete the table of values.
- b) Write the ordered pairs in the table of values.
- c) In the ordered pairs, what does the first number represent? How about the second number?
- d) Are the ordered pairs (1, 2) and (2, 1) the same? Why?

Functional relationships can be described as a set of ordered pairs. Hence, a graph on a coordinate plane can be used to describe a functional relationship visually. The horizontal axis is used to represent the values of the independent quantity. The vertical axis is used to represent the values of the dependent quantity.

Example 2.1

Draw the graph of the functional relationship that relates the score Kelifa obtained and the number of questions he answered correctly.

Solution:

Akberet and Ali constructed the following table of values which displays some of the ordered pairs in this functional relationship.

No. of correct answers	0	1	2	5	6	8	10
Total score	0	2	4	10	12	16	20

Now plotting the ordered pairs on a coordinate plane, they obtained the graph shown in Fig. 2.2.

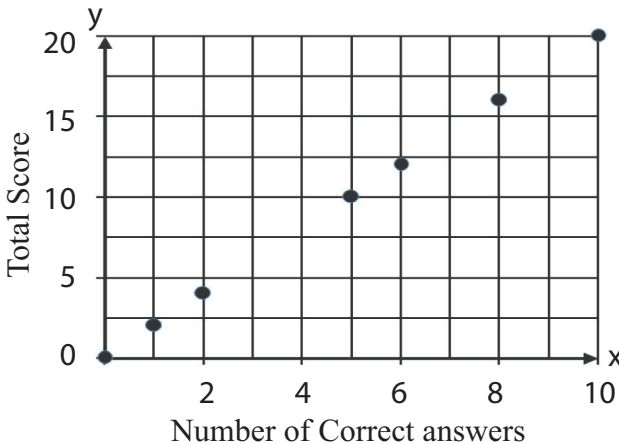


Fig. 2.2

The graph in Fig. 2.2 is called a **discrete graph**. A discrete graph is a graph in which the quantities involved in the functional relationship take on only limited real numbers within their limits.

Example 2.2

A person lives 25 km away from his workplace. One day, he travelled to his work on foot at a constant speed of 5 km per hour. Draw the graph of the functional relationship that relates the remaining distance to reach his workplace and the time he travelled.

Solution:

Kalid constructed the following table of values considering only the integral minutes.

Time travelled (hours)	0	1	2	3	4	5
Remaining distance to reach work	25	20	15	10	5	0

Now plotting the ordered pairs on a coordinate plane, he obtained the graph shown in Fig. 2.3. Next, he considered smaller and smaller fractions of time between the integral hours like 0.1 hour, 0.5 hour, 3.5 hours, etc. Finding the corresponding remaining distances, he plots the ordered pairs on the number plane. He obtained the graph in Fig. 2.4 which is a straight line.

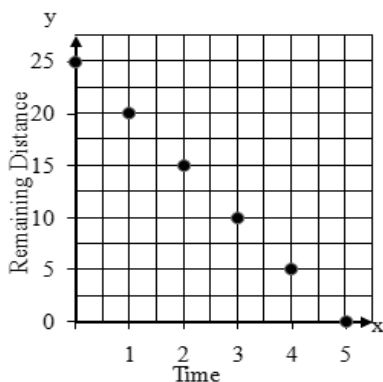


Fig. 2.3

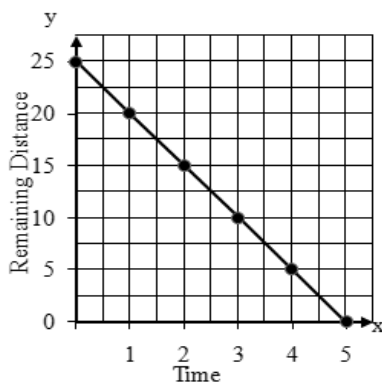


Fig. 2.4

The graph in Fig. 2.4 is called a continuous graph. A **continuous graph** is a graph in which the quantities involved in the functional relationship take on all real numbers within their limits.

Exercise 2.2

1. The drama club of Barka Secondary School is putting on a play next Saturday. The school charges each person an entrance fee of 0.50 Nakfa. The maximum capacity of the hall is 125 seats.
 - a) Find the school's total income, if the number of people who attend the play is 23? 50? 60? 90? 121?
 - b) Use a table of values to describe the data in a).
 - c) Graph the functional relationship that relates the total income and the number of people who attend the show.
 - d) Is the graph discrete or continuous? Why?
2. Ali is 3 years older than his sister Keria.
 - a) What is Ali's age just at the birth of Keria? When she is 1 year old? 3 years old? 4 years old? 6 years old? 8 years old?
 - b) Use a table of values to describe the data in a).
 - c) Graph the functional relationship that relates Ali's age and his sister Keria, taking Keria's age from 0 to 10 years.
 - d) Is the graph discrete or continuous? Why?
 - e) Using your graph, how old is Ali when Keria is 2 years old? 5 years old? 7 years old?
3. The following tables describe functional relationships between two quantities. For each table, complete the missing values.

a)

Independent	-3	-2	-1	0	1	3	4	5
Dependent	-6	-4		0	2			10

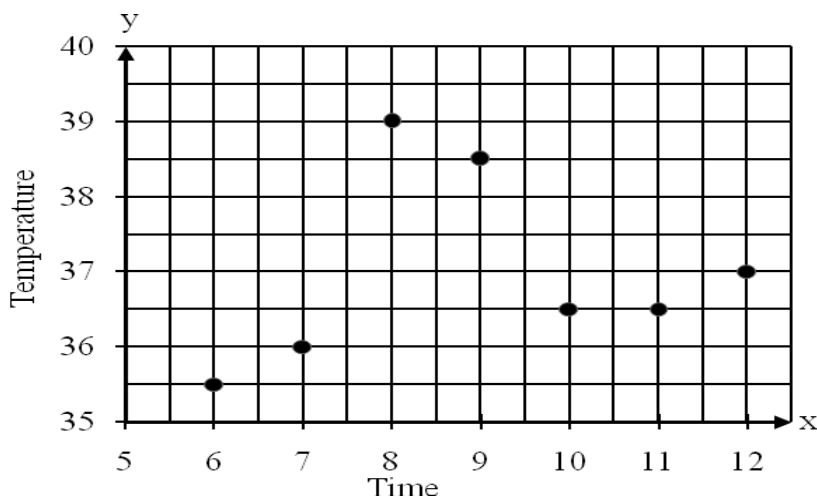
b)

Independent	-2	-1	0	1	2	3	4	5
Dependent	-4		-2	-1			2	

c)

Independent	-3	-2	-1	0	1	2	3	4
Dependent		4	1	0	1		9	

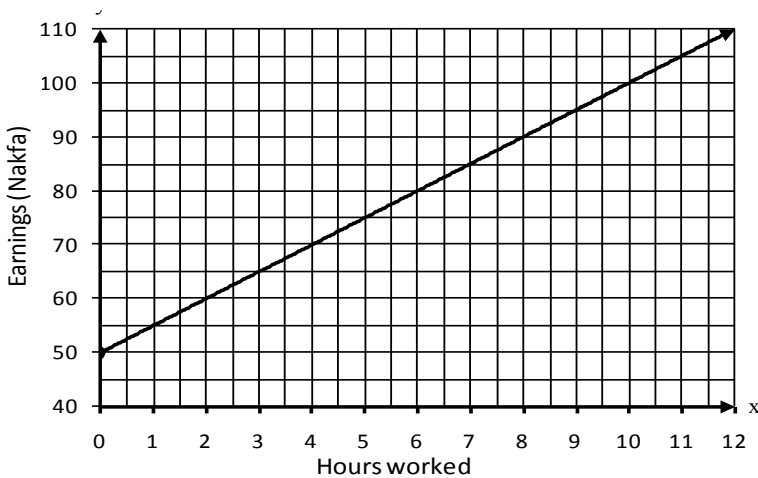
4. Consider the functional relationship between the area of a square region and the length of its side.
 - a) Find the area of the square region if the length of its side is 0.2cm? 0.5cm? 0.6cm? 1cm? 1.5cm? 3cm?
 - b) Use a table of values to describe the data in a).
 - c) Graph the functional relationship that relates the area of a square region and the length of its side taking the length of the side from 0.1cm to 3cm.
 - d) Is the graph discrete or continuous? Why?
 - e) Using your graph, what is the area of the square regions if the length of its side is 0.4cm? 2cm? 2.5cm?
5. The following graph shows the temperature of a patient in a hospital recorded every hour in the morning of one day.



- a) What was the patient's temperature at 10 a.m?
- b) When was the patient's temperature 38.5°C ?
- c) What was the patient's highest recorded temperature?
When did it happen?
- d) What was the patient's lowest recorded temperature?
When did it happen?

- e) A patient is required to take a tablet if the body temperature of the patient is deviated from the normal body temperature by more than 1°C . When did this patient take the tablet?

The accompanying graph illustrates the functional relationship between a person's earnings and the number of hours the person works per day.



- a) Is the graph discrete or continuous?
b) What are the person's earnings if the person works for 1 hour? for 2 hours? 4 hours? 9 hours?

2.1.3 Describing Relationships using Equations

Activity 2.4

In Activity 2.2, let the variable x represent the number of questions Kelifa answers correctly in the examination and let the variable y represent the score he obtains.

- Use an equation to describe the relationship between x and y .
- Use your equation to find y when x is 2, 5, 17 and 25.

A functional relationship between two quantities can also be described using an equation of two variables. The variable that represents the independent quantity is called the **independent variable**. The variable that represents the dependent quantity is called the **dependent variable**.

Activity 2.5

- A water tank contains 10ℓ of water. The tank is leaking at a constant rate of 0.2ℓ per hour.
 - Use an equation to describe the amount of water in the tank at any given time in hours.
 - Use your equation to determine the amount of water after 5 hours? 10 hours? 12 hours? 15 hours? 2 days?
 - Draw the graph of the functional relationship.
 - Is the graph discrete or continuous?
 - Identify the domain and the range.
- In the functional relationship described by the following table of values, let x represent the independent variable and y represent the dependent variable. The domain is the set of all real numbers.

x	1	2	3	4	5	6
y	0.6	0.9	1.2	1.5	1.8	2.1

- Use an algebraic equation to describe the relationship between x and y .
- Use your equation to determine the value of y when x is 0, 1.5, 10 and 15.

Exercise 2.3

1. Consider a functional relationship given by the following ordered pairs: $(-2, 5)$, $(0, 1)$, $(4, -7)$, $(7, -13)$.

- Describe the functional relationship in words.
- Find an equation that describes the function in which x represents a domain element and y represents the corresponding range element.
- Complete the following table.

x	-3	-1	1	2	3	5	6	8
y								

- Graph the function for real numbers x from -3 to 8 .

2. Himbol Stationery has a photocopy machine and charges 1.50 Nakfa per page. The amount charged is a function of the number of pages copied.

- What are the amounts charged for the following numbers of pages?

Number of pages	0	1	2	3	4	5
Total cost	0.00					

- Graph this function for the numbers of pages copied from 0 to 10.
 - Is the graph discrete or continuous?
 - Let x be the number of pages copied and y be the total amount charged. Find an equation that describes the function.
 - Use the equation to determine the amount charged for 13 pages, 21 pages, 32 pages.
3. The area of a square region is the square of the length of its side. The area of the square region is a function of the length of its side.
- What are the areas of the square regions whose sides in centimetre are the following numbers?

Side length of the square	1	2	3	4	5	6
Area of the square region	1					

- b) Graph this function for square regions of sides from 1 to 6 cm.
 - c) Let x be the length of the side of a square region and y be the area of the square region. Find an equation that describes the function.
 - d) Use the equation to determine the areas of the square regions whose sides are 7cm; 6.5cm; 3.5cm.
 - e) Find the domain and range of the function.
4. The table below shows a functional relationship between two quantities x and y , where x is the independent variable and y is the dependent variable. The domain is the set of real numbers.

x	0	1	2	3	4	5
y	2	3	4	5	6	7

- a) Use an algebraic equation to describe the function.
 - b) Graph this function for values of x from -4 to 5.
5. Draw the graph of each of the following functions described by the given equations where x is a real number from -3 to 3.
- a) $y = x$
 - b) $y = 2x - 1$
 - c) $y = -x + 2$
 - d) $y = x^2$

2.2 Solving Equations and Inequalities

2.2.1 Solving Equations

Activity 2.6

Recall Abrehet in Activity 2.1. One Saturday, Abrehet returned home after selling a certain number of eggs. She found 50.00 Nakfa in her handbag. However, she could not remember the number of eggs she had sold. She could remember that she had 15.00 Nakfa when she left home for the market. Besides, she did not spend any money that day other than 10.00 Nakfa for transportation. Could you help her to find out the number of eggs she sold that day?

Example 2.3

Solve the equations $2x+3 = 13$ and $2x=10$.

Solution:

Beyan said that the solution set of the equation $2x+3=13$ is $\{5\}$. Kibret said that the solution set of the equation $2x=10$ is $\{5\}$. Did you notice that both equations have the same solution set?

Two equations that have exactly the same solution set are called **equivalent equations**.

To solve an equation, we perform algebraic operations that isolate the variable on the left hand side (**LHS**) and the numbers on the right hand side (**RHS**) of the equation. However, we need to maintain the equality. That is, the resulting equation must be equivalent to the original equation.

We can think of an equation as a balance scale, with the equal sign as the balance point.

Activity 2.7

Consider the balance scale in Fig. 2.5. The weights on both plates are balanced. What actions can you do on the balance scale that maintain the balance? Assume all the boxes as well as all the balls have the same weight.

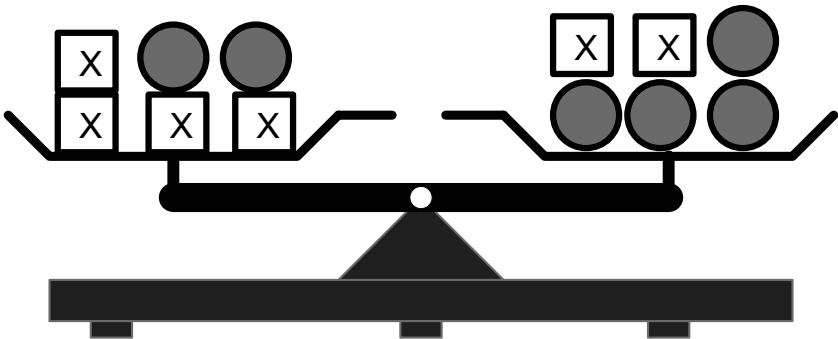


Fig. 2.5

In a balance scale, the same amount must be put on or removed from each side of the balance scale to maintain the balance.

Similarly, the same operation must be performed on each side of the equation to maintain equality.

The following operations are some of the several methods that are used to isolate the variable while maintaining the equality in an equation.

Transformations that produce equivalent equations:

1. Adding the same quantity to both sides
2. Subtracting the same quantity from both sides.
3. Multiplying both sides by the same nonzero quantity.
4. Dividing both sides by the same nonzero quantity.
5. Replacing an expression by an equivalent expression.
6. Switching the left and righthand sides of the equation.

Example 2.4

Solve $x + 5 = -6$.

Solution:

To solve the equation $x + 5 = -6$, we need to get x alone on the LHS of the equation. We do it as follows:

$x + 5 = -6$	Given equation
$x + 5 - 5 = -6 - 5$	Subtracting 5 from both sides
$x = -11$	Simplifying both sides

Therefore, the solution set is $\{-11\}$.

Activity 2.8

To solve the equation $15 = 3y - 9$, you need to get y alone on the LHS of the equation.

- a) What will you do to the RHS of the equation to get $3y$ alone?
- b) What will you do to the LHS of the equation to keep the equation balanced?
- c) What will be the resulting equivalent equation?
- d) What will you do next to get y alone on the LHS?
- e) What is the solution set of the equation?

Exercise 2.4

1. Solve each of the following equations.

a) $x - 5 = 4$	f) $15 + x = -3$
b) $7 = 14 + x$	g) $14 = 12y + 8$
c) $-z = 6$	h) $11 = 1 - 4z$
d) $3y = -15$	i) $2x + 18 = 8$
e) $48x = 12$	j) $-4x - 8 = -8$

2. Each equal side of an isosceles triangle is three times as long as the third side. If the perimeter of the triangle is 56cm, find the length of the three sides of the triangle.

2.2.2 Solving Linear Equations with Variables on both Sides**Activity 2.9**

To solve the equation $2x + 6 = 4x - 2$, you need to get x alone on the LHS of the equation.

- a) What will you do to clear of $4x$ from the RHS of the equation?
- b) What will you do to the LHS to keep the equality?
- c) What will be the resulting equivalent equation?
- d) How will you proceed next to get x alone on the LHS of the resulting equation?
- e) What is the solution set of the equation?
- f) How do you verify your solution is correct?

Example 2.5

Solve $x + 23 = 2x + 45$.

Solution:

The equation is solved as follows:

$x + 23 = 2x + 45$	Given equation
$x - x + 23 = 2x - x + 45$	Subtracting x from both sides
$23 = x + 45$	Simplifying both sides
$23 - 45 = x + 45 - 45$	Subtracting 45 from both sides
$-22 = x$	Simplifying both sides
$x = -22$	Switching both sides

Therefore, $\{-22\}$ is the solution set of the equation.

Exercise 2.5

1. Solve each of the following equations
 - a) $12x - 13 = 3x + 14$
 - b) $10z - 17 = 3z - 11$
 - c) $2(x - 10) = 3(2x + 7)$
 - d) $-3(y + 2) = y - 2(2y + 3)$
 - e) $20 - (x + 5) = 9(2x + 8)$
 - f) $4(x + 3) = 5 - 3(x - 1)$
 - g) $3(2x + 1) + 7(x - 1) = 5x + 4$
 - h) $5(2x - 3) = 2(3x - 2) - (5x + 2)$
 - i) $6(z + 4) - 8(z - 6) = 5(11 - z) + 17$
 - j) $2y - 7 = -2(1 - y)$
2. If 4 is subtracted from twice a number, the result is 10 less than the number. Find the number.

2.2.3 Solving Linear Equations with Fractional and Decimal Coefficients

Activity 2.10

To solve the equation, $\frac{x-2}{4} - \frac{x+5}{6} = -2$ you need to have x alone on the LHS of the equation.

- a) What are the denominators of the fractions in the equation?
- b) What is the LCM of the denominators of the fractions?
- c) What will be the resulting equivalent equation if you multiply both sides of the equation by the LCM?
- d) Solve the resulting equivalent equation.
- e) How do you verify that your solution is correct?

Example 2.6

Solve $0.2x + 2 = 2.6$.

Solution:

The equation is solved as follows:

$$0.2x + 2 = 2.6$$

Given equation

$$10(0.2x + 2) = 10(2.6)$$

Multiplying both sides by 10
to remove the decimals

$$2x + 20 = 26$$

Simplifying both sides

$$2x + 20 - 20 = 26 - 20 \quad \text{Subtracting 20 from both sides}$$

$$2x = 6 \quad \text{Simplifying both sides}$$

$$\frac{2x}{2} = \frac{6}{2} \quad \text{Dividing both sides by 2}$$

$$x = 3 \quad \text{Simplifying both sides}$$

Therefore, the solution set of the equation is $\{3\}$.

Activity 2.11

To solve the equation $-5.4 = 2x + 2.6$, you need to get x alone on the LHS of the equation.

- How do you clear of the decimal points from the equation?
- What will be the resulting equivalent equation?
- Solve the resulting equivalent equation.
- How do you verify that your solution is correct?

Exercise 2.6

- Solve each of the following equations.

a) $\frac{2}{3}x + 5 = 7$

f) $\frac{x-5}{2} - \frac{x+1}{4} = 3$

b) $\frac{x}{5} + \frac{x}{3} = \frac{8}{15}$

g) $\frac{y}{3} + 7 = \frac{y}{5} - 3$

c) $0.2z - 5 = 7$

h) $0.13z - 1.4 = 0.08z + 7.6$

d) $1.5y + 1.7 = y$

i) $0.4x + 0.3(2000 - x) = 750$

e) $\frac{y+1}{5} + \frac{3y-1}{2} = 4$

j) $0.06(x - 5) = 0.04(x + 8)$

- On a field trip of a Grade 9 class, two-third of the learners in the class arrived on time for departure. Three more learners arrived late and brought the number to 25. What was the total number of learners in the class?

2.2.4 Solving Linear Inequalities

Activity 2.12

In a national examination, there are 50 questions. Each question carries 2 marks. The passing mark is at least 50. How many questions should a learners have to answer to pass the examination?

Solving a linear inequality in one variable is much like solving a linear equation in one variable. The goal is to isolate the variable on one side of the inequality using transformations that produce equivalent inequalities. The box below presents some of these transformations.

Transformations that produce equivalent inequalities:

1. Add the same quantity to both sides.
2. Subtract the same quantity from both sides.
3. Multiply both sides by the same positive quantity.
4. Divide both sides by the same positive quantity.
5. Replace an expression by an equivalent expression.
6. Multiply both sides by the same negative quantity and reverse the inequality.
7. Divide both sides by the same negative quantity and reverse the inequality.
8. Switch the LHS and RHS and reverse the inequality.

Example 2.7

Solve $3y - 8 < 10$.

Solution:

To solve the inequality $3y - 8 < 10$, we need to get y alone on the LHS of the inequality. We proceed as follows:

$3y - 8 < 10$	Given inequality
$3y - 8 + 8 < 10 + 8$	Adding 8 to both sides
$3y < 18$	Simplifying both sides
$\frac{3y}{3} < \frac{18}{3}$	Dividing both sides by 3
$y < 6$	Simplifying both sides

Therefore, the solution set is $\{y: y < 6\}$.

Example 2.8Solve $3x - 2 \leq 5x - 4$.**Solution:**We solved the inequality $3x - 2 \leq 5x - 4$ as follows:

$3x - 2 \leq 5x - 4$	Given inequality
$-5x + 3x - 2 \leq -5x + 5x - 4$	Adding $-5x$ to both sides
$-2x - 2 \leq -4$	Simplifying both sides
$-2x - 2 + 2 \leq -4 + 2$	Adding 2 to both sides
$-2x \leq -2$	Simplifying both sides
$\frac{-2x}{-2} \geq \frac{-2}{-2}$	Dividing both sides by -2 and reversing the inequality
$x \geq 1$	Simplifying both sides

Therefore, the solution set is $\{x: x \geq 1\}$.**Activity 2.13**To solve the equation $9 \geq -5x - 1$, you need to get x alone on the LHS of the inequality.

- What shall you do to the inequality to get $-5x$ alone in the RHS?
- What will be the resulting equivalent inequality?
- What shall you do to the inequality in b) to get x alone in the RHS?
- What will be the resulting equivalent inequality? Do you need to reverse the inequality? Why?
- What shall you do next to get x alone on the LHS? Do you need to reverse the inequality?
- What is the solution of the inequality? How many solutions do you get? Can you mention 5 of them?

Exercise 2.7

1. Solve each of the following inequalities.

- | | |
|--------------------|-------------------------------|
| a) $x + 4 \leq 19$ | e) $\frac{1}{3}x - 1 \leq 3$ |
| b) $x - 3 > -3$ | f) $2x + \frac{1}{2} \geq -3$ |
| c) $3 - 2x > 15$ | g) $-0.1x + 5 < 0.3x + 1$ |
| d) $-1 < -2x + 1$ | |

h) $5x + 4 \geq 11 - 2x$

j) $\frac{z+2}{8} \leq \frac{z}{2} + 2$

i) $\frac{x-2}{4} - \frac{x-4}{6} \geq \frac{2}{3}$

2. Three times a number is less than 8 plus twice of the number.
Find the number.

2.3 Variations

2.3.1 Direct Variation

Activity 2.14

A man travels at a constant speed of 5km per hour.

- How far does he travel in 1 hour? 2 hours? 5.5 hours? 8 hours?
- What happens to the distance travelled, if the time of travel increases?
- What happens to the distance travelled, if the time of travel decreases?

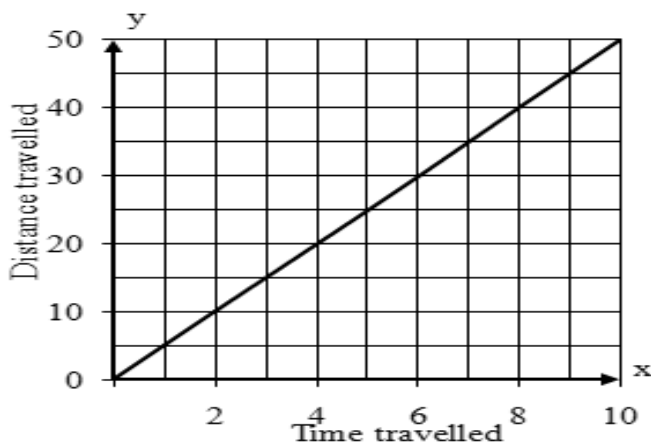


Fig. 2.6

Fig. 2.6 is the graph of the functional relationship between the distance y the man travels and the duration of time x he travels. As shown in the graph, if the value of x increases, y also increases. If x decreases, so does the value of y . Thus, the two variables change in the same manner. In this case, we say that y **varies directly** with x , and express it algebraically as:

$$y = kx$$

or

$$\frac{y}{x} = k$$

where the constant k is called the **constant of proportionality**. That is, two quantities vary directly if their ratio is a constant.

Example 2.9

Suppose y varies directly with x , and $y = 2$ when $x = 5$. Find the value of y when $x = 15$.

Solution:

First, we find the value of k , using $y = 2$ when $x = 5$ and $y = kx$ as follows.

$$y = kx$$

y varies directly with x

$$2 = 5k$$

Substituting $y = 2$ and $x = 5$

$$k = \frac{2}{5}$$

Solving for k

Next, we use this constant of proportionality in the equation $y = kx$ for the situation when $x = 15$ as follows.

$$y = kx$$

y varies directly with x

$$y = \frac{2}{5}x$$

Substituting $k = \frac{2}{5}$

$$y = \frac{2}{5} \times 15$$

Substituting $x = 15$

$$y = 6$$

Simplifying

Therefore, $y = 6$ when $x = 15$.

Activity 2.15

The number of calories a person burns and the time the person spends performing an activity vary directly. A certain person can burn off 75 calories by sitting in class for 50 minutes.

- What are the quantities involved in this problem?
- Assign variable to each of them.
- Write an equation that relates the quantities.
- How long must the person sit in class to burn off 545 calories?

Exercise 2.8

1. In each of the following, the values of x and y vary directly. One pair of values for x and y are given. Write an equation that relates x and y .
 - a) $x = 2, y = 5$
 - b) $x = 12, y = 3$
 - c) $x = 0.1, y = 0.9$
 - d) $x = 5, y = 10$
2. Suppose v varies directly with g , and $v = 30$ when $g = 70$. Find v when $g = 14$.
3. In a factory, the profit P varies directly with the inventory I . If $P = 100$ when $I = 20$, find P when $I = 50$.
4. The distance d a car travels and the amount of petrol p it needs vary directly. A certain car travels 205kms on 20ℓ of petrol. How much petrol will be needed for a journey of 369km?
5. The velocity v of a falling object varies directly with the time t of the fall. If after 2 seconds, the velocity of the object is 19.6 meters per second, what will its velocity be after 3 seconds?

2.3.2 Inverse Variation

Activity 2.16

Nejat is a Grade 9 student at Semhar Secondary School, Massawa. The distance from her home to the school is 10km. She used to go to school on bicycle. Please help her to answer the following questions.

- a) How long will it take her to reach school if she rides the bicycle at 10km per hour? 15km per hour? 20km per hour?
- b) If she increases the speed at which she rides the bicycle, what will happen to the time she needs to reach school?
- c) If she decreases the speed at which she rides the bicycle, what will happen to the time she needs to reach school?
- d) Do the speed and the time change in the same manner?

- e) Let x represent the time she needs to reach school and y represent the speed at which she rides the bicycle. Write an equation that relates x and y .
- f) How long will it take her to reach school, if she rides the bicycle at 25 km per hour? 17.5 km per hour?

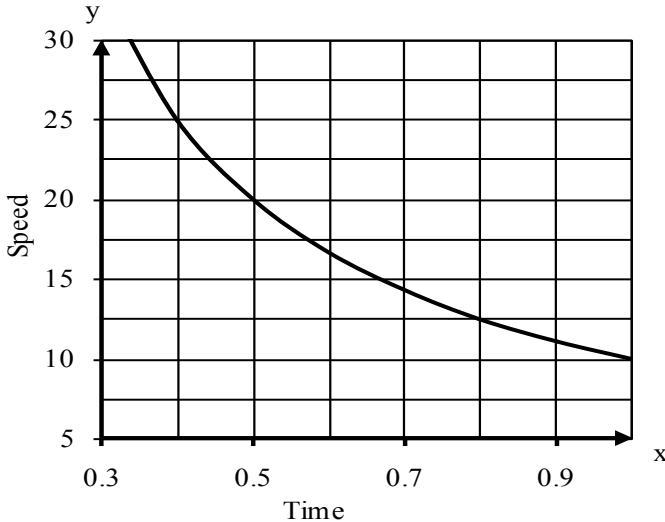


Fig. 2.7

The graph in Fig. 2.7 describes the functional relationship between the time x it takes her to reach school and the velocity y at which Nejat rides her bicycle. As shown in the graph, if the value of x gets larger, the value y gets smaller. If the value of x gets smaller, the value of y gets larger. Therefore, the two quantities change in an opposite manner. In this case, we say that y **varies inversely** as x , and express it algebraically as:

$$xy = k$$

or

$$y = \frac{k}{x}$$

where the constant k is called the **constant of proportionality**. That is, two quantities vary inversely if their product is a constant.

Example 2.10

If z varies inversely as p and z is 200 when $p = 4$, find z when $p = 10$.

Solution:

Dawd first found the value of k , using $z = 200$ when $p = 4$ and $zp = k$ as follows.

$$\begin{array}{ll} zp = k & z \text{ varies inversely with } p \\ 200 \times 4 = k & \text{Substituting } z = 200 \text{ and } p = 4 \\ k = 800 & \text{Solving for } k \end{array}$$

Next, he used this constant of proportionality in the equation

$z = k/p$ for the situation when $p = 10$ as follows.

$$\begin{array}{ll} z = \frac{k}{p} & z \text{ varies inversely with } p \\ z = \frac{800}{p} & \text{Substituting } k = 800 \\ z = \frac{800}{10} & \text{Substituting } p = 10 \\ z = 80 & \text{Simplifying} \end{array}$$

Then, he concluded that $z = 80$ when $p = 10$.

Exercise 2.9

- In each of the following, determine whether the two given quantities vary directly or inversely.
 - The weight of a basket and the number of boxes put in it.
 - The time required to do a specific job and the number of people assigned to do the job.
 - The number of eggs Abrehet sells and the income she generates from it.
- In each of the following, the values of x and y vary inversely. One pair of values for x and y are given. Write an equation that relates x and y .
 - $x = 3, y = 2$
 - $x = 4, y = 6$
 - $x = 5, y = 1$
 - $x = 0.2, y = 0.25$

3. R varies inversely with T. If R is 168 when $T = 24$, find R when $T = 30$.
4. The number of hours h it takes for a block of ice to melt varies inversely with temperature T . If it takes 2 hours for the block of ice to melt at 65°F , find the constant of proportionality.

2.4 Absolute Value

2.4.1 Absolute Values of Numbers

Activity 2.17

Use a ruler to draw a number line that corresponds to the real numbers from -4 to 4. Label the points on the line so that the distance between consecutive integers is 1 cm. See Fig. 2.8.

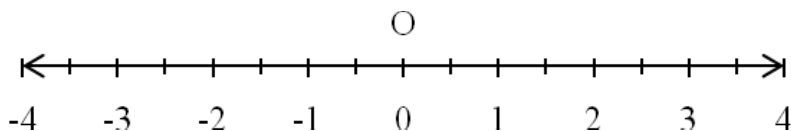


Fig. 2.8

- a) If M is a point on the number line whose coordinate is 3, then what is the distance of M from the origin O?
- b) If N is a point on the number line whose coordinate is -3, then what is the distance of N from the origin O?
- c) Are the points M and N equidistant from the origin O? Why?
- d) Are the points M and N located on the same side of the origin O or on opposite side of the origin O? Why?
- e) If P is a point on the number line to the left of the origin and its distance from the origin O is 4 cm, then what is the coordinate of P?
- f) If Q is a point on the number line to the right of the origin and its distance from the origin O is 1.5 cm, then what is the coordinate of Q?
- g) If R is a point on the number line whose distance from the origin O is 4 cm, then what is the coordinate of R?
- h) If S is a point on the number line whose distance from the origin O is 1.5 cm, then what is the coordinate of S?

The **absolute value** of a real number x is the distance from the origin to the number x on a number line. Since distance is never negative, the absolute value of a real number is never negative. Using algebraic notations, the absolute value of a real number is defined as follows.

If x is a real number, the absolute value of x , denoted by the symbol $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 2.11

Find $|-5|$.

Solution:

The number within the absolute value bars is -5 . Since -5 is negative, to get its absolute value, we multiply it by -1 . Therefore, $|-5| = -1(-5) = 5$.

Example 2.12

Find $|6|$.

Solution:

Meriem did it as follows. The number within the absolute value bars is 6 . Since 6 is positive, its absolute value is the number itself. Therefore, $|6| = 6$.

Exercise 2.10

1. Evaluate each of the following absolute values.

a) $|0|$

f) $|3 - \sqrt{5}|$

b) $|-9|$

c) $|9|$

g) $|4| - 7|$

d) $|5 - 14|$

h) $5x + 4 \geq 11 - 2x$

e) $|2 - \sqrt{5}|$

i) $\frac{z+2}{8} \leq \frac{z}{2} + 2$

2. If the absolute value of a number is 3, then what is the number?
3. For each of the following values of x and y , evaluate the expressions $|x - y|$ and $|y - x|$.
 - a) $x = 0, y = 2$
 - b) $x = -1, y = 3$
 - c) $x = 3, y = 4$
 - d) $x = -2, y = -1$
 - e) $x = -2, y = 0$
 - f) $x = 5, y = -3$
 - g) What can you conclude about the expressions?
4. Repeat problem 3 for the expressions $|xy|$ and $|x||y|$.
5. Repeat problem 3 for the expressions $|x + y|$ and $|x| + |y|$.

2.4.2 Absolute Value Equations

A teacher gave the following task to students: P is a point on a number line which is 4 units away from the origin. What is the coordinate of P? Halima and Ibrahim are discussing the task.

Halima: Is P to the right or to the left of the origin?

Ibrahim: No information is given about that.

Halima: I see. So, let us represent the coordinate of P by the letter x . Now, how do we get the distance of P from the origin?

Ibrahim: We take the absolute value of the coordinate of P.

Halima: Exactly. The distance is given by $|x|$. How far is P from the origin?

Ibrahim: According to the problem, it is 4 units.

Halima: You are right. So, we have the equation $|x| = 4$. Now, what is the number x whose absolute value is 4?

Ibrahim: I think we have two such numbers. If x is negative, then it is -4. If x is positive, then it is 4.

Halima: You are right. If P is to the right of the origin on the number line, then its coordinate is 4. If P is to the left of the origin, then its coordinate is -4.

Equations such as

$$|x| = 4$$

$$|x + 2| = 1$$

$$|2x - 13| = 7$$

are called **Absolute value equations**.

If k is a positive real number and u is any algebraic expression, then

$$|u| = k \text{ is equivalent to } u = -k \text{ or } u = k$$

That is, to solve the equation $|u| = k$, we solve the equations $u = -k$ and $u = k$, and then take the union of their solution sets as the solution set of the original equation.

Example 2.13

Solve the equation $|2x - 5| = 9$.

Solution:

Fating solved this equation as follows.

$$|2x - 5| = 9$$

Given equation

$$2x - 5 = -9 \text{ or } 2x - 5 = 9$$

Equivalent form

$$x = -2 \text{ or } x = 7$$

Solving for x

Then, he concludes that the solution set of the equation is $\{-2, 7\}$.

Can you verify the solutions?

Exercise 2.11

Solve each of the following equations.

1. $|2x| = 6$

7) $\left| \frac{x}{2} - \frac{1}{3} \right| = 1$

2. $|x + 4| = 13$

3. $|2y - 1| = 7$

8) $|2 - x| = 3$

4. $|5x + 4| = 2$

5. $|-y + 9| = 12$

9) $5 = |3 - 2x|$

6. $|2z - 14| = 0$

10) $\left| \frac{x}{3} + \frac{2}{5} \right| = 2$

2.4.3 Absolute Value Inequalities

Hayat and Akberet are trying to solve the following problem: “ P is a point on a number line whose distance from the origin is less than 3 units. What are the possible coordinates of P ?”

Hayat: Let x represent the coordinate of the point P. Then, the distance of P from the origin is given by $|x|$. Since the distance of P from the origin is less than 3 units, we have $|x| < 3$. Am I right?

Akberet: Yes, you are right. Then, what is the value of x ?

Hayat: We are looking for a number x whose absolute value is less than 3. Since $|2| = 2$ and $|-2| = 2$, the values of x are -2 and 2.

Akberet: How about 1 and -1?

Hayat: Let me see. $|1| = 1$ and $|-1| = 1$. Both are less than 3. So, 1 and -1 are also possible values of x . Does x have four possible values?

Akberet: Not only four. Any number between -3 and 3 is a possible value of x .

Hayat: I see. So, we can conclude that the point P can have any coordinate between -3 and 3.

Akberet: Exactly.

Inequalities such as

$$|x| < 3$$

$$|x - 5| \geq 1$$

$$|2x + 1| < 4$$

are called **absolute value inequalities**.

If k is a positive real number and u is any algebraic expression, then

1. $|u| < k$ is equivalent to $u > -k$ and $u < k$.
2. $|u| \leq k$ is equivalent to $u \geq -k$ and $u \leq k$.
3. $|u| > k$ is equivalent to $u < -k$ or $u > k$.
4. $|u| \geq k$ is equivalent to $u \leq -k$ or $u \geq k$.

If inequalities are joined with the word ‘or,’ then the solution set of the compound inequality is the union of the solutions of each inequality. If inequalities are joined with the word ‘and,’ then the solution set of the compound inequality is the intersection of the solutions of each inequality.

Example 2.14

Solve the inequality $|2x - 3| < 5$.

Solution:

Keria solved this inequality as follows.

$ 2x - 3 < 5$	Given inequality
$2x - 3 > -5$ and $2x - 3 < 5$	Equivalent form
$x > -1$ and $x < 4$	Solving for x

Then, she concludes that the solution set of the inequality is $\{x: -1 < x \text{ and } x < 4\} = \{x: -1 < x < 4\}$.

Example 2.15

Solve the inequality $|x + 2| \geq 4$.

Solution:

Jemal solved this inequality as follows.

$ x + 2 \geq 4$	Given inequality
$x + 2 \leq -4$ or $x + 2 \geq 4$	Equivalent form
$x \leq -6$ or $x \geq 2$	Solving for x

Then, he concludes that the solution set of the inequality is $\{x: x \leq -6 \text{ or } x \geq 2\}$.

Exercise 2.12

Solve each of the following inequalities.

- | | |
|--|---|
| 1. $ 2x < 8$ | 7) $ 1 - 2z > 3$ |
| 2. $ 3x \geq 12$ | 8) $ -y - 2 \leq 1$ |
| 3. $ x - 3 < 2$ | 9) $ 5 + 2z < 7$ |
| 4. $ x - 5 > 9$ | |
| 5. $ 2y - 1 \geq 5$ | 10) $\left \frac{3}{4}x + 1 \right < \frac{5}{2}$ |
| 6) $\left \frac{x}{2} - 3 \right \geq 7$ | |

Review Exercise 2

1. Harnet Secondary School hires a bus for 150.00 Nakfa to take a group of Grade 9 students to Asmara Zoo. The school also pays each student's 2.00 Nakfa admission fee.
 - a) What is the total cost if the number of students in the group is 12 students? 20 students? 35 students?
 - b) Describe verbally the functional relationship between the total cost and the number of students in the group.
 - c) Identify the dependent and independent quantities in this functional relationship.
 - d) Let c be the total cost and n be the number of students in the group. Write an equation to describe the functional relationship between c and t .
 - e) Draw the graph of the functional relationship for $n = 5, 10, 15, 20, 25, 30, 35, 40$ and 45 .
 - f) Is the graph discrete or continuous?
 - g) Find the domain and range of the functional relationship.
 2. Solve each of the following equations.
 - a) $12x + 3 = 7x + 13$
 - b) $1 - 2y = -2.1y + 1.2$
 - c) $\frac{1}{2}x = \frac{1}{6}(x + 2) + 4$
 - d) $|2x - 1| = 4$
 - e) $\left| \frac{3}{4}z - \frac{4}{3} \right| = 2$
 3. Solve each of the following inequalities.
 - a) $3x - 2 \geq -x + 6$
 - b) $5x + 4 < 3x + 2$
 - c) $-\frac{x}{3} + 1 > 4$
 - d) $|2x + 6| \leq 2$
 - e) $5 < |3 - x|$
 4. The number of units produced by a machine working at a constant rate varies directly with the amount of time the machine is in operation. Machine A produces 18 units in a 10 hour period. Machine B works at the same rate but works for 15 hours. Find the number of units produced by Machine B.
 5. T varies inversely as S , and T is 50 when S is 5. Find T when S is 20.
-

CHAPTER 3

Linear Functions

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- represent linear functions in different ways;
- determine whether or not given situations can be represented by linear functions;
- differentiate linear functions from nonlinear functions;
- interpret and describe the effects of change in parameters such as the slope and y-intercept of linear functions in real-world and mathematical situations
- formulate and solve linear equations and inequalities which model real-life problems using graphical and algebraic methods.

3.1 Representing Linear Functions

Activity 3.1

1. Tirhas saved 5 Nakfa at the end of every week in her saving box for 7 consecutive weeks.
 - a) Copy and then complete the table below.

Number of weeks (N)	1	2	3	4	5	6	7
Total saving in Nakfa (s)	5					30	

- b) Is her total saving a function of the number of weeks? What are its domain and range?
- c) If S represents the total saving in N weeks, what is the equation that relates S and N?
- d) Omer's answer to the question in c) is $S = 5N$. Do you agree with him?
- e) Omer's graph is shown in Fig. 3.1.

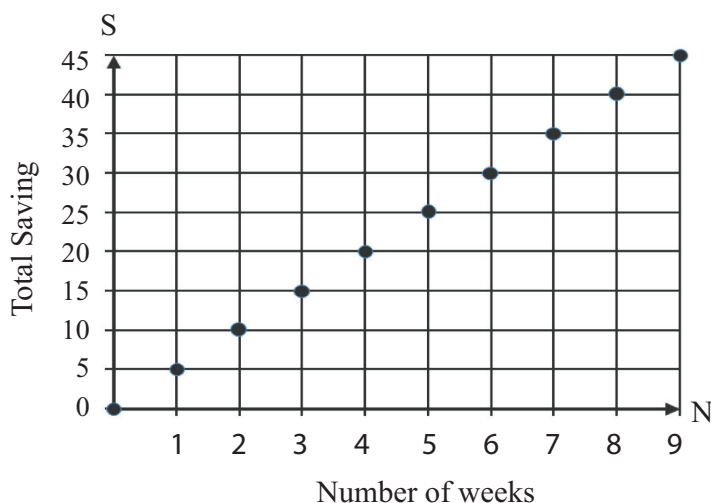


Fig. 3.1

2. a) Why is the graph not a straight line?
b) What is the domain of the function?
3. Consider a rectangle whose length is 5 units and width w units.
 - a) If A represents the area of the rectangle, what is the equation that relates A and w .
 - b) Choose any 4 possible values for w , and then copy and complete the table below.

W				
A				

- c) Is A a function of w ?
 - d) Plot the four ordered pairs (w, A) of the above table. Do the points lie on a straight line?
 - e) Can w be negative? Explain why or why not.
 - f) Do the graph pass through the point $(0, 0)$? Why?
4. Tell at least two differences between the function $S = 5N$ of task 1 and the function $A = 5w$ of this task.

5. Consider the function in which the number y is one less than twice the number x .

- Write the equation that relates x and y .
- What are the domain and range of the function?
- Select any 4 values for x between -2 and 3 and complete the table below.

x				
y				

- Plot the ordered pairs (x, y) you obtained in the table above
- Do the points lie on a straight line? Draw the line.

Points that lie on a straight line are **collinear**.

The three functions you saw in the above activities are examples of linear functions.

A linear function is a function that can be written in the form

$$y = mx + b$$
 where m and b are constant real numbers.

Example 3.1

Determine whether the following equations represent a linear function. Justify your answer.

- $y = -3x + 2$
- $2y - 3x = 6$

Solution:

- $y = -3x + 2$ is already in the form of $y = mx + b$.
- $2y - 3x = 6$ can be written as $2y - 3x = 6$,
 Which is equivalent to $2y = 3x + 6$ (Adding $3x$ to both sides)

$$y = \frac{3}{2}x + 3 \text{ (Dividing both sides by 2), which is in the form}$$

$y = mx + b$. Hence it is linear function.

A linear function can have also the form

$$Ax + By = C$$

where A, B and C are constant real numbers but $B \neq 0$.

Exercise 3.1

- Let x represent the first number and y represent the second number, and then write an equation that relates x and y .
 - The second number is three more than the first.
 - The second number is twice the first.
 - The second number is the product of -3 and the first number.
 - The sum of the numbers is 10 .
- Determine whether each of the following equations represent a linear function. Justify your answer.
 - $-3x + 7y = 9$
 - $A = \pi x^2$
 - $y = 2^x$
 - $y = 3x^2 - 12$
 - $2y = -3x + 6$
 - $5x = 2y + 10$
- The table below shows the relationship between Dahlak's age and her brother Aron's age.

Dahlak's age	1	2	3	4	5
Aron's age	7	8	9	10	11

- Write a verbal expression to describe how the ages are related.
 - Let x represent Dahlak's age and y represent Aron's age. Then write an equation for the verbal expression.
 - Graph the equation.
 - From your equation, find Aron's age when Dahlak is 10 .
- The formula for the perimeter of a square is $P = 4s$, where P is the perimeter and s is the length of a side.
 - Write a verbal description of the equation.
 - Make a table of values and then graph the function.
 - The mass ' m ' of pure gold is given by $m = 19v$, where m is measured in gram and v in cm^3 .

- a) Find m when $v = 0.1\text{cm}^3$.
- b) Find v when $m = 38\text{gm}$.

6. Fill in the missing numbers in the table?

x	1	2	4		16
$y = 2x - 5$	-3		3	11	27

7. Which of the following statements represent linear functions?

- a) The circumference (C) of a circle is π times its diameter (d).
- b) The perimeter (P) of an equilateral triangle of side (s) is three times s .
- c) The side s of a square is the square root of its area A .
- d) A father's age y is 20 more than twice his son's age x .

3.2 Graphing Linear Functions and Inequalities

3.2.1 Graphing Linear Functions

Activity 3.2

1. Graph the function $y = 7x - 2$.
 - a) Let x be an integer from -2 to 2. Complete the table below.

x	-2	-1	0	1	2
y					

- b) Plot the points. Are the points collinear?
 - c) Draw the graph.
2. Graph the functions $y = -x$; $y = x$; and $y = 3x$ on the same coordinate plane.
 - a) Discuss what they have in common.
 - b) Ahmed said that their graphs all pass through the origin $(0, 0)$. Do you agree with him?
 - c) What generalisation could you make about the graph of $y = mx$ for a constant real number m from this?
3. Graph the functions $y = x$; $y = x + 2$ and $y = x - 2$ on the same coordinate plane.
 - a) What do you observe about the two graphs in relation to the graph of $y = x$?

- b) Given the graph of $y = mx$, explain how you would draw the graphs of $y = mx + 3$ and $y = mx - 3$?

The tasks in question 3 above suggest that

- a) The graphs of $y = mx$ and $y = mx + b$ are parallel.
 b) The graph of $y = mx + b$ is just the graph of $y = mx$ **shifted vertically upward** by b units when b is **positive**, and **shifted vertically downward** by b units when b is **negative**.

Example 3.2

Draw the graph of $y = x - 2$ by using the shifting system on the graph of $y = x$.

Solution:

- a) The effect of -2 on $y = x - 2$ is to shift the graph of $y = x$ vertically down by 2 units. To do this, we subtract 2 from each x -coordinate of the points of $y = x$.
 b) Select any two points of $y = x$ such as $(0, 0)$ and $(1, 1)$, to get the points $(0, -2)$ and $(1, -1)$.
 c) Therefore, joining the points $(0, -2)$ and $(1, -1)$ with a straight line gives the graph of $y = x - 2$. See the solid line of Fig. 3.2.

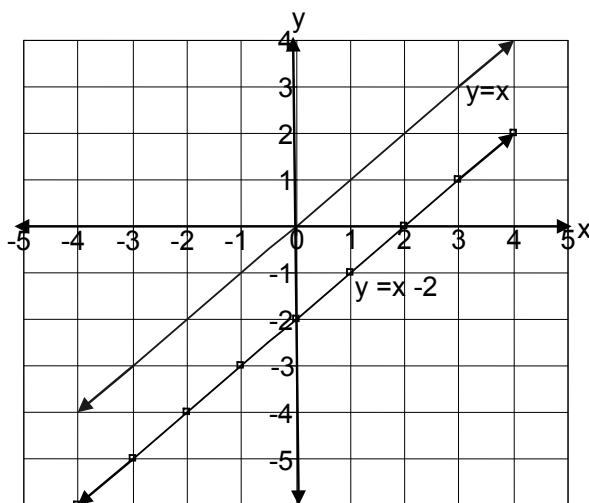


Fig. 3.2

Exercise 3.2

1. Graph each of the following functions.
 - a) $y = 2x - 3$
 - b) $2x - 3y = 5$
 - c) $3x + 2y = 7$
 - d) $y = 0.5x + 1$
2. If the point (a, b) lies on the graph of $y = mx$, then which of the following points will definitely lie on the graph of $y = mx - 7$?
 - a) $(a + 7, b)$
 - b) $(a - 7, b)$
 - c) $(a, b - 7)$
 - d) $(a, b + 7)$
3. Draw the graphs of $y = 2x$, $y = 2x + 1$ and $y = 2x - 1$ on the same coordinate plane.

3.2.2 Graphing Linear Inequalities

Activity 3.3

To draw the graph of the inequality $x > 0$, follow the instructions below.

- a) Do the x -coordinates of the points of the line $x = 0$ such as $(0, 0)$, $(0, -1)$ or $(0, 1)$ satisfy the inequality $x > 0$? If no, draw the line $x = 0$ in dashed form to indicate that the points of the line do not belong to the graph of $x > 0$. If yes, draw it as a solid line.
- b) Do the x -coordinates of the points to the right of the line $x = 0$ such as $(1, 1)$, $(0.5, 0.1)$, or $(2, -1)$ make true the inequality $x > 0$? If yes, shade in the entire region to the right side of the line $x = 0$.
- c) Do the x -coordinates of the points to the left of the line $x = 0$ such as $(-1, 1)$, $(-0.5, 0.1)$, or $(-2, -1)$ make true the inequality $x > 0$? If yes, shade in the entire region to the right side of the line $x = 0$. Otherwise, leave it unshaded.
- d) What can you say about the graph of $x > 0$? Is it a line? Is it the region to the right of the line $x = 0$? Is it the region to the left of the line $x = 0$?
- e) Sitel's graph is shown in Fig. 3.3 below. Is your graph the same as hers?

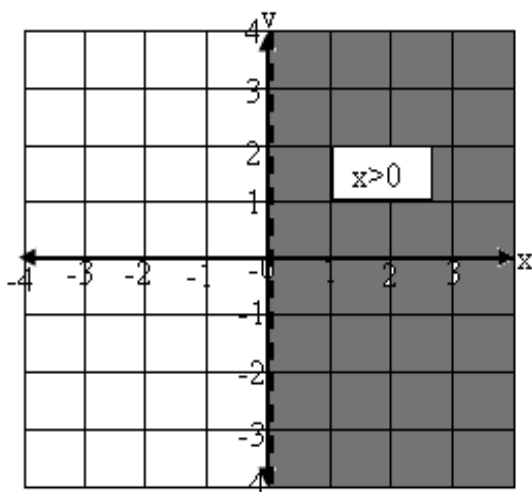


Fig. 3.3

The **graph of an inequality in two variables** is the set of all points whose coordinates make the inequality true.

Activity 3.4

- Draw the graphs of:

a) $x < 0$	d) $x \geq 0$
b) $y > 0$	e) $y < -3$
c) $y \geq 0$	f) $x \leq -2$
- In what ways does the graph of $x > 0$ differ from that of $x \geq 0$?
 - In what ways does the graph of $y > 0$ differ from that of $y \geq 0$?

Activity 3.5

- To draw the graph of the inequality $y > -x$, follow the instructions below.
 - Do the coordinates of the points of the line $y = -x$ such as $(0, 0)$, $(1, -1)$ or $(2, -2)$ satisfy the inequality $y > -x$? If no, draw the line $y = -x$ in dashed form to indicate that the points of the line do not belong to the graph of $y > -x$. If yes, draw it as a solid line.

- b) Do the coordinates of the points above the line $y = -x$ such as $(0, 1)$, $(-1, 2)$, or $(2, -1)$ make true the inequality $y > -x$? If yes, shade in the entire region above the line $y = -x$.
- c) Do the coordinates of the points below the line $y = -x$ such as $(0, -1)$, $(-2, 1)$, or $(2, -3)$ make true the inequality $y > -x$? If yes, shade in the entire region below the line $y = -x$.
- d) What can you say about the graph of $y > -x$? Is it a line? Is it the region above the line $y = -x$ or the region below the line?
- e) Merhawit's graph of $y > -x$ is shown in Fig. 3.4 below. Is your graph the same as hers?

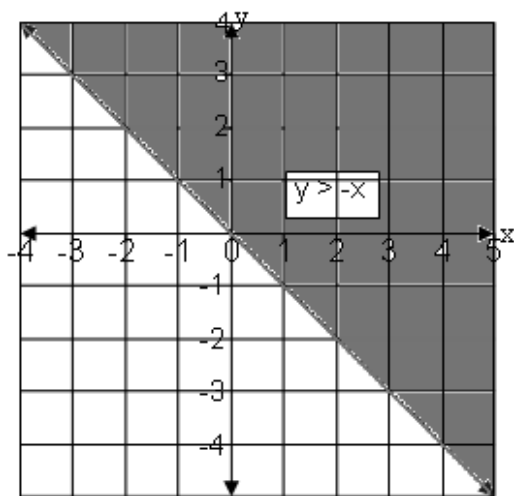


Fig. 3. 4

2. Draw the graphs of

- | | |
|------------|----------------|
| a) $y < x$ | c) $y < -x$ |
| b) $y > x$ | d) $y \geq -x$ |

You have already seen that equations in the form $Ax + By = C$ are straight lines when graphed. If we replace the $=$ sign by $>$, $<$, $\geq$, or $\leq$, we get a **linear inequality in two variables**. Some examples of linear inequalities in two variables are $x + y < 2$ and $3x - 5y \geq 15$.

The tasks of the above activities suggest that:

- The graph of an inequality in the coordinate plane is the set of all points **on one side** of the **boundary line**, where the boundary line is the graph of the linear equation obtained by replacing the inequality by equality.
- The boundary line itself would not be part of the graph when the inequality is either $<$ or $>$ and we indicate this by drawing the boundary line as a **dashed line**.
- The boundary line would be part of the graph when the inequality is of the type $\leq$ or $\geq$ and we indicate this by making the boundary line a **solid line**.

Example 3.3

Draw the graph of $2x - 3y > 5$.

Solution:

Step 1: The equation of the boundary line is $2x - 3y = 5$. Since the points $(1, -1)$ and $(-2, -3)$ lie on the line $2x - 3y = 5$, we draw a dashed line passing through them because the inequality is " $>$ ". See Fig. 3.5 below.

Step 2: Let us choose $(0, 0)$ a point above the boundary line as our test point. Notice that if the point $(0, 0)$ does not lie on the boundary line of an inequality, it is best to use it as a test point for the inequality because the computation becomes easy when the 2 variables are both zero.

Now,

$$2x - 3y > 5$$

$$2(0) - 3(0) > 5$$

$$0 - 0 > 5$$

$$0 > 5$$

This is the given inequality.

Testing $(0, 0)$.

Multiplying by 0.

This statement is false.

Step 3: Because 0 is not greater than 5, the test point $(0, 0)$, is not part of the solution set. Therefore, shading the region below the dashed line gives the graph of $2x - 3y > 5$ as shown in Fig. 3.5.

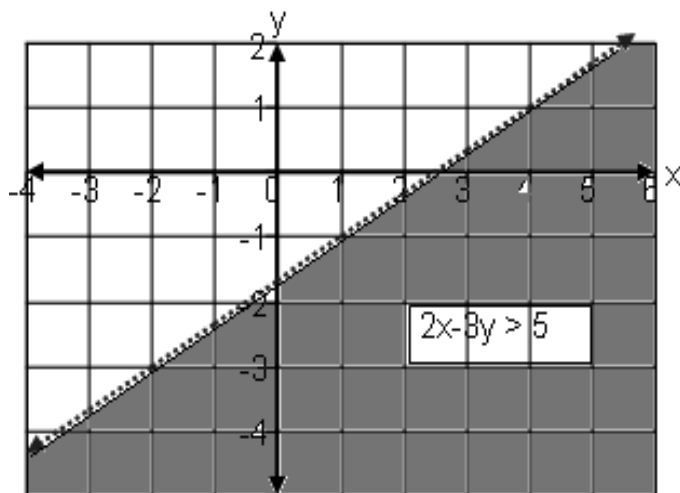


Fig. 3. 5

Exercise 3.3

1. Graph the linear inequalities given below.

- $y > -3$
- $x \leq -2$
- $y > 2x + 3$
- $y < -3x + 1$

2. Do the points (0, -1) and (0, -2) both satisfy the inequality $y < -3x + 1$? If yes, then can $y < -3x + 1$ represent a function?

3. Which of the following inequalities does the shaded region of Fig. 3.5 represent?

- $x - 2y > 3$
- $-x + 2y < 3$
- $-x + 2y > 3$
- $-x + 2y \leq 3$
- $-x + 2y \geq 3$

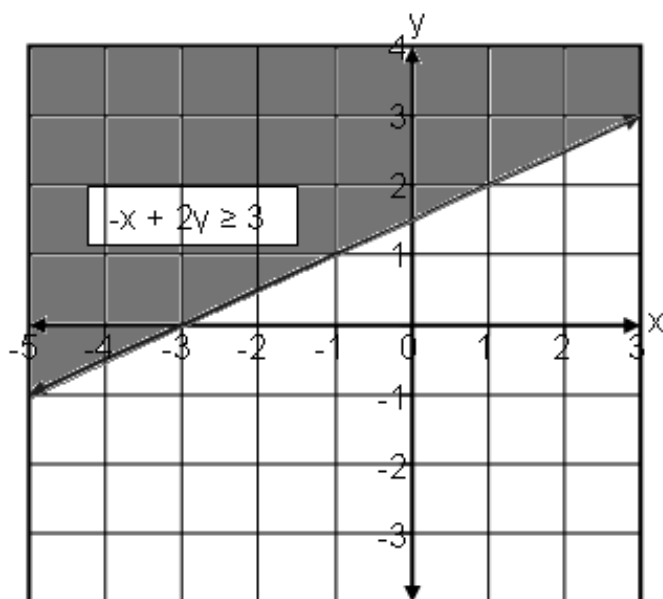


Fig. 3.6

3.3 Slopes of Linear Functions

Activity 3.6

1. A ladder is leaning against a vertical wall as shown in Fig. 3.7 below.

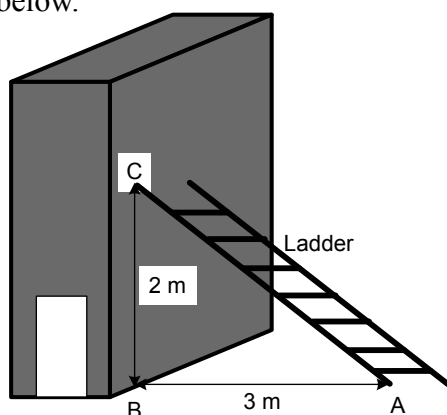


Fig. 3.7

- a) What do we call the distance AB? What do we call the distance BC?
- b) If $AB = 3\text{m}$ and $BC = 2\text{m}$, what does the ratio $\frac{BC}{AB}$ represent?

The ratio $\frac{BC}{AB} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x}$
is the **slope** of the ladder.

Activity 3.7

1. In Fig. 3.8 below, a line passing through A and C is given.

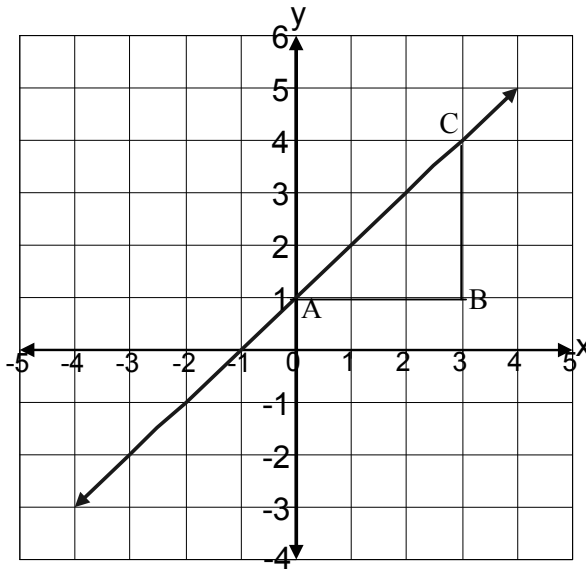


Fig. 3.8

- a) How many units is the length of BC?
- b) How many units is the length of AB?
- c) Calculate the ratio of vertical change to horizontal change of the line $\overleftrightarrow{AC}$.
- d) Calculate the slope of the line $\overleftrightarrow{AC}$
- e) Are your answers to d) and c) the same?

From this activity, you observe that

The slope m of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$\text{Slope}(m) = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 - x_1 \neq 0$$

Activity 3.8

1. In each of the following, find the slope of the line passing through the pair of points.
 - a) (1, 6) and (3, 5)
 - b) (-2, 1) and (2, 2)
 - c) (5, 1) and (-2, 1)
 - d) (3, -4) and (3, 7)
 - e) (6, -4) and (4, -2)
 - f) (-5, -2) and (-6, -2)
2. A teacher gave a task to calculate the slope of the line that passes through the points $A(0, -3)$ and $B(1, -1)$.
 - a) Letting $(x_1, y_1) = A(0, -3)$ and $(x_2, y_2) = B(1, -1)$, find the slope of the line $\overline{AB}$.
 - b) Letting $(x_1, y_1) = B(1, -1)$ and $(x_2, y_2) = A(0, -3)$, find the slope of the line $\overline{AB}$.
 - c) Are the slopes you obtained in a) and b) the same?
 - d) What can you generalise from the results of a) and b)?

The slope of a line is always the same no matter which point you call (x_1, y_1) and which point you call (x_2, y_2) .

Activity 3.9

Points P (0, -3), Q (1, -1) and R (2, 1) lie on the graph of $y = 2x - 3$ as shown in Fig. 3.9 below.

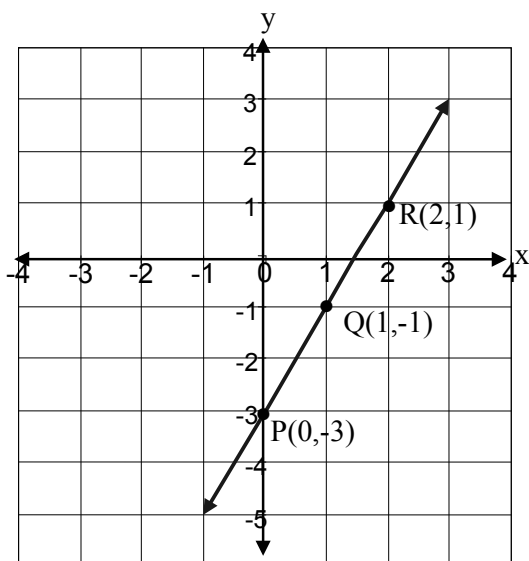


Fig. 3.9

- Find the slope using the coordinates of points P and Q.
- Find the slope using the coordinates of points P and R.
- Find the slope using the coordinates of points Q and R.
- Compare the slopes you got in a), b), and c). Are they the same or different?

This activity suggests that the slope of a line is always the same no matter what two points of the line are considered.

Activity 3.10

- Find the slope of each line given below by finding any two points of the line.
 $y = 3x - 2$; $y = -2x + 1$; $y = 5x - 7$; $y = -4x + 3$
- Compare the slope of each line with the coefficient of x in its equation. What do you observe?

This activity suggests that when the equation of a line is written in the form of $y = mx + b$ its slope is m .

Example 3.4

Find the slope of the line whose equation is $-12x + 3y = 5$.

Solution:

$$-12x + 3y = 5$$

$$3y = 12x + 5. \quad \text{Why?}$$

$$y = 4x + \frac{5}{3}. \quad \text{Why?}$$

Therefore; the slope of $-12x + 3y = 5$ is 4.

Activity 3.11

1. Draw the graph of each of the following and write its slope.

a) $y = 3x - 2$ b) $y = -3x - 2$ c) $y = 2$ d) $x = 3$

2. Which of the lines in question 1 has a

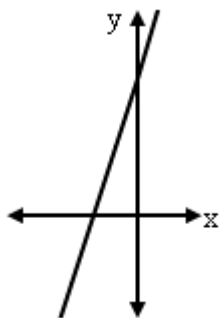
- | | |
|--------------------|---------------------|
| a) positive slope? | c) zero slope? |
| b) negative slope? | d) undefined slope? |

3. When you trace each line from left to right, whose graph

- | | |
|----------------|--------------------------|
| a) falls down? | c) is a horizontal line? |
| b) rises up? | d) is a vertical line? |

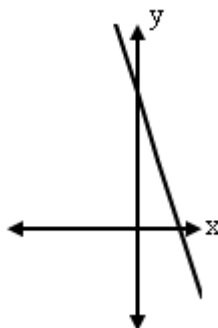
4. For each of the following graphs, identify its slope as positive, negative, zero, or undefined.

a)

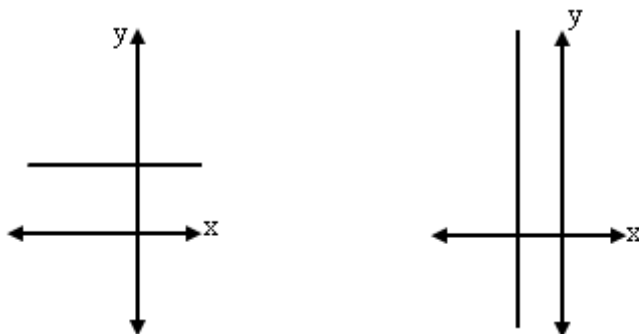


c)

b)



d)



This activity suggests that it is possible to determine the sign of the slope of a line by just looking at its graph.

- a) A line with a positive slope **risers up** when traced from left to right.
- b) A line with a negative slope **falls down** when traced from left to right.
- c) A line with 0 slope is **horizontal**.
- d) The slopes of **vertical lines** are **undefined** because we cannot divide by zero.

We have seen that the slope of a line is vertical change divided by horizontal change but what does it mean in real-life situations? The activity below illustrates this.

Exercise 3.4

1. Look at the table below. It shows the temperature of the water in a hot water tank recorded at 15-second intervals after the heater is turned on.

Time (s)	0	15	30	45	60	75	90
Temperature ($^{\circ}\text{C}$)	30	35	40	45	50	55	60

- a) Let the vertical-axis represent temperature and the horizontal-axis time. Use a scale of one side of a small square to represent 15 seconds on the horizontal axis and 10°C on the vertical axis. Then plot the ordered pairs (time, temperature) and draw the graph.

- b) At what rate is the temperature increasing?
c) What is the slope of the graph?
d) What connection do you see between the slope and the rate of temperature increase?
2. In exercises a) – f) below, calculate the slope of the line passing through the given points. Then indicate whether the line falls, rises, is horizontal, or vertical without drawing the lines.
- | | |
|-----------------------|------------------------|
| a) (2, 6) and (3, 5) | d) (3, -4) and (3, 7) |
| b) (-2, 1) and (2, 2) | e) (6, -4) and (4, -2) |
| c) (5, 1) and (-2, 1) | f) (0, 1) and (1, -2) |
3. Find the slope of each line just from its equation.
- | | |
|----------------------------|-------------------|
| a) $y = -\frac{9}{5}x + 7$ | d) $2y = -3$ |
| b) $5x + 3y = 15$ | e) $-2x - 4y = 9$ |
| c) $3y = -4x$ | f) $5x = 7y - 6$ |
4. Graph each line and determine its slope.
- | |
|----------------|
| a) $y = 2$ |
| b) $x = 3$ |
| c) $x + 5 = 0$ |
| d) $y + 2 = 0$ |
5. The distance S travelled by a car in a given time T , where S is in km and T in hours (h) is given by the equation $S = 50T$.
- | |
|--|
| a) What is its speed in km per h? |
| b) How many kilometres does it travel in 3hrs? |
6. A ladder is leaning against a vertical wall with its foot 1.5m away from the wall on a level ground. If it makes a slope of $5/3$, how far up the wall does it reach?

3.4 Intercepts of Linear Functions

Activity 3.12

Consider the linear function $y = x - 2$.

- Draw its graph.
- What is the x -coordinate of the point where the graph crosses the y -axis?
- What are the coordinates of the point where the graph crosses the y -axis?
- What is the y -coordinate of the point where the graph crosses the x -axis?
- What are the coordinates of the point where the graph crosses the x -axis?
- Is it possible to answer (c) and (e) without drawing each graph? If yes, explain how.

The x -coordinate of the point where the graph crosses the x -axis is called the **x -intercept**. The y -coordinate of the point where the graph crosses the y -axis is called the **y -intercept**.

Example 3.5

Find the x - and y -intercepts of $y = -2x + 4$.

Solution:

- To find the x -intercept, let $y = 0$ and solve for x .

$$y = -2x + 3$$

$$0 = -2x + 3$$

$$2x = 3$$

$$x = 3/2 \text{ or } 1.5$$

The x -intercept is 1.5, so the line passes through the point (1.5, 0).

- To find the y -intercept, we let $x = 0$ and solve for y .

$$y = -2x + 3$$

$$y = -2(0) + 3 \text{ or } y = 3$$

The y -intercept is 3, so the line passes through the point (0, 3).

In Activity 3.13, we saw that the y -intercept of the graph of $y = x - 2$ is -2 and in Example 3.4 the y -intercept of the graph of $y = -2x + 3$ is 3.

- What do you think is the y -intercept of the graph of $y = mx + b$? Can you show it?

Activity 3.13

- a) Mark any two points on a piece of paper. How many different lines can you draw passing through them?
- b) Can you draw the graph of a linear function if only 2 of its points are given?
- c) Would it be possible to draw the graph of a linear function by just finding its x-and y-intercepts?

Example 3.6

Graph the line whose equation is $3x + 2y = 6$ by finding its x-and y-intercepts.

Solution:

To find the x-intercept, let $y = 0$ and solve for x .

$$3x + 2y = 6$$

$$3x + 2(0) =$$

$$3x = 6 \text{ or } x = 2$$

To find the y-intercept, we let $x = 0$ and solve for y .

$$3x + 2y = 6$$

$$3(0) + 2y = 6$$

$$2y = 6 \text{ or } y = 3$$

The x-intercept is 2, so the line must pass through the point (2, 0).
The y-intercept is 3, so the line must pass through the point (0, 3).
The graph is shown in Fig. 3.10.

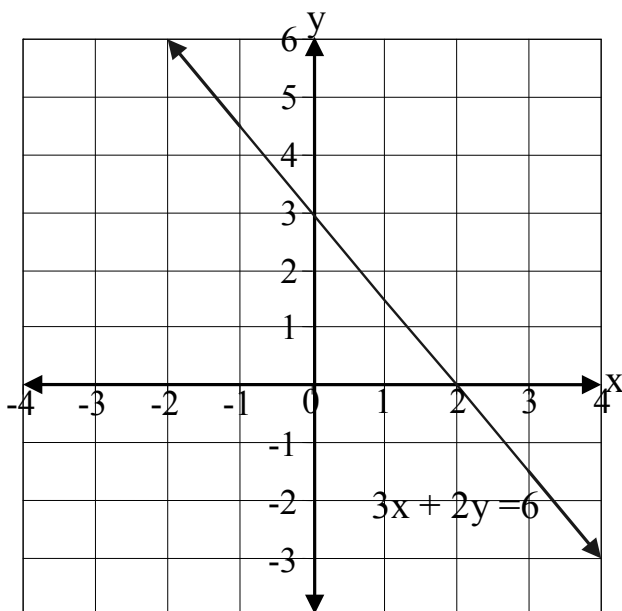


Fig. 3.10

Example 3.7

Write the equation of the line passing through $(-1, 3)$ and $(2, 9)$.

Solution:

At first, find the slope using the two given points.

$$\text{The slope } m = \frac{9-3}{2-(-1)} = \frac{6}{3} = 2$$

Now, taking the point $(-1, 3)$ and substituting the coordinates in $y = mx + b$, gives

$$y = mx + b$$

$$3 = 2(-1) + b$$

$$3 = -2 + b$$

$$b = 5$$

Therefore, the equation of the line is $y = 2x + 5$.

You may also take the slope 2 and the point (2, 9) to find b.

$$y = mx + b$$

$$9 = 2(2) + b$$

$$9 = 4 + b$$

$$b = 5$$

Therefore, the equation of the line is $y = 2x + 5$.

A way of finding the equation of a line when two points are given is stated below.

Step 1: Find the slope m using the two given points.

Step 2: Take any one of the given points and substitute its coordinates in $y = mx + b$ to find b .

Exercise 3.5

- Write the equation of each line passing through
 - the points $(-4, 7)$ and $(6, -3)$.
 - the point $(5, 2)$ having the slope of -2 .
- Find the x -intercept and the y -intercept for each line passing through
 - $(3, 5)$ and $(2, 5)$.
 - $(-4, 3)$ and $(-4, 7)$.
 - $(-2, -6)$ and $(5, 8)$.
 - $(1, -2)$ and $(-2, 1)$.
- For each equation given below, determine the slope of the line and the y -intercept.
 - $y - 3x = 1$
 - $3y - x = 1$
 - $7y - 3x - 10 = 0$
 - $5y + x = 15$
- Find the equation of the line with
 - x -intercept -1 and y -intercept -3 .
 - x -intercept 4 and slope -3 .
 - y -intercept 2 and slope 1 .
 - x -intercept 1 and contains the point $(2, -1)$.
 - y -intercept -5 and passing through $(1, 3)$.
- In (a) and (b) below, find the coefficients of x and y that must be filled in each blank so that the equation's graph will be a line with the specified intercepts.

- a) $\underline{\hspace{1cm}}x + \underline{\hspace{1cm}}y = 10$; x-intercept = 5; y-intercept = 2
 b) $\underline{\hspace{1cm}}x + \underline{\hspace{1cm}}y = 12$; x-intercept = -2; y-intercept =

3.5 Solving Systems of Linear Equations

3.5.1 Systems of Linear Equations

Activity 3.14

1. Do you remember that equations of the form

are called systems of linear equations with two variables or just simultaneous equations?

2. The sum of two numbers x and y is 7, and their difference is 3.
 What are the numbers?

- a) What mathematical equation in terms of x and y do you get from the first sentence of the problem?
 b) What mathematical equation in terms of x and y do you get from the second sentence of the problem?
 c) Yosuf's two equations are given below. Are your equations also the same as his?

- d) We want to find values for x and y which can make both equations true at the same time. Try the ordered pairs (1, 6), (4, 1), (3, 4), (4, 3), (5, 2), (2, 5), (10, -3).
 Yosuf said that x = 5 and y = 2 is the answer. Do you agree?

3.5.2 Elimination Method

Activity 3.15

1. a) In each of the following systems of two equations, add the left sides together and the right sides together. Then compare the sums. Do you get a true equation in each case?

$$\begin{cases} 12 = 12 \\ 7 = 7 \end{cases} \quad \begin{cases} -13 = -13 \\ 9 = 9 \end{cases} \quad \begin{cases} a = b \\ c = d \end{cases}$$

- b) In each of the following systems of two equations, subtract the left sides and the right sides. Then compare the differences. Do you get a true equation in each case?

$$\begin{cases} 12 = 12 \\ 7 = 7 \end{cases} \quad \begin{cases} -13 = -13 \\ 9 = 9 \end{cases} \quad \begin{cases} a = b \\ c = d \end{cases}$$

The above activity suggests that given two equalities

$$\begin{cases} a = b \\ c = d \end{cases} \\ \hline a + c = b + d$$

$$\begin{cases} a = b \\ c = d \end{cases} \\ \hline a - c = b - d$$

2. a) The sum of the length and width of a rectangle is 7 and the difference is 3. Find the length and width of the rectangle.
b) Ahmed let the length be L and the width W and wrote the two equations given below. Do you agree with his equations?

$$L + W = 7 \quad \text{--- (1)}$$

$$\underline{L - W = 3} \quad \text{--- (2)}$$

- c) What would you get when you add the left sides? What would you get when you add the right sides?
d) Ahmed got the equation $2L = 10$ when he added the left sides and the right sides of the original system of equations.. What do you think happened to W ?
e) Can you solve now L from the equation $2L = 10$?
f) How would you get the value of W if you know that $L = 5$?

Example 3.8

Below is given a system of linear equations.

$$\begin{cases} 5x - 3y = 1 \\ 3x + 4y = 18 \end{cases}$$

- a) It is not possible to eliminate x or y just by adding or subtracting the two equations as they are. Neither x nor y can be eliminated when we add or subtract the two equations.

b) Adem multiplied the upper equation by 4 and the lower equation by 3 and got

$$\begin{cases} 20x - 12y = 4 \\ 9x + 12y = 54 \end{cases}$$

Adem continued the solution as follows.

$$20x - 12y = 4$$

$$\underline{9x + 12y = 54}$$

$$29x = 58 \quad (\text{Adding the two equations})$$

$$x = 2 \quad (\text{Simplifying})$$

To solve for y , Adem substituted 2 for x in the equation

$$3x + 4y = 18 \text{ and proceeded as follows}$$

$$3x + 4y = 18$$

$$3(2) + 4y = 18$$

$$4y = 18 - 6$$

$$4y = 12 \text{ or } y = 3$$

Finally put $x = 2$ and $y = 3$, or $(2, 3)$ as the solution of the system.

If you decide to eliminate y , multiply the upper equation by 4 and the lower equation by 3, and then adding will eliminate y .

Other numbers are also possible. All what you need to do is, to find any common multiple of 3 and 4, say, 24. Now you multiply the upper equation by $24 \div 3$ or 8, and the lower equation by $24 \div 4$ or 6.

Multiplying the upper equation by -3 and the lower equation by 5, and then adding will eliminate x as shown below

$$\begin{cases} 5x - 3y = 1 \\ 3x + 4y = 18 \end{cases}$$

$$\begin{cases} -15x + 9y = -3 \\ 15x + 20y = 90 \end{cases}$$

$$29y = 87$$

$$y = 3$$

Now, substituting 3 for y , for example, in $5x - 3y = 1$, we get $5x - 9 = 1$ or $x = 2$. Therefore, the solution is $x = 2$ and $y = 3$, which is the same as Adem's solution.

Example 3.9

Solve by elimination method:
$$\begin{cases} 7x = 5 - 2y \\ 3y = 16 - 2x \end{cases}$$

Solution:

Omer argues that at first we arrange the system so that the variable terms appear on the left and constant terms appear on the right.

When he did this he obtained

$$\begin{cases} 7x + 2y = 5 \\ 2x + 3y = 16 \end{cases}$$

Then, to find the solution, first by eliminating y , we get $x = -1$.

Next, by eliminating x , we get $y = 6$. Thus, the solution is $x = -1$ and $y = 6$.

3.5.3 Substitution Method**Activity 3.16**

To solve the simultaneous equations below, answer the questions in (a) to (d) below.

$$\begin{cases} 5x - 4y = 9 & (1) \\ x = 2y - 3 & (2) \end{cases}$$

- In which equation is one of the unknowns already expressed in terms of the other?
- Haile's answer to (a) is equation (2). Do you agree?
- From equation (2) we have $x = 2y - 3$. When you substitute x by $2y - 3$ in equation (1), what do you get?
- Haile's answer to (c) is $5(2y - 3) - 4y = 9$. Is he correct?
- Can you now solve the equation $5(2y - 3) - 4y = 9$ for y ? Explain how.
- How would you now get the value of x ?

Example 3.10

Solve by substitution method
$$\begin{cases} 3a - 2b = 5 \\ a + 5b = -4 \end{cases}$$

Solution:

- a) Choose the equation $a + 5b = -4$ for substitution and rewrote it as $a = -5b - 4$.
- b) Take the first equation of the original system and solved for b as shown below.

$$\begin{aligned} 3a - 2b &= 5 \\ 3(-5b - 4) - 2b &= 5 \\ -17b &= 17 \\ b &= -1 \end{aligned}$$

- c) Substitute -1 for b in $a = -5b - 4$ and solved for a as follows.

$$\begin{aligned} a &= -5b - 4 \\ a &= -5(-1) - 4 \\ a &= 5 - 4 \\ a &= 1 \end{aligned}$$

Then put $a = 1$ and $b = -1$ as the solution.

- d) Alternatively, you may also choose $3a - 2b = 5$ and get .

$$a = \frac{2b+5}{3}$$

- e) Use $a = \frac{2b+5}{3}$ for substitution and solve for b in the equation $a + 5b = -4$.

- f) Did you also get $a = 1$ and $b = -1$ as your solution?

Exercise 3.6

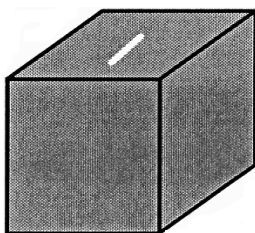
1. Solve each system by the substitution method.

$$\begin{array}{lll} \text{a) } \begin{cases} 3x+4y=2 \\ x-y=17 \end{cases} & \text{b) } \begin{cases} 4x+y=8 \\ 5x-2y=23 \end{cases} & \text{c) } \begin{cases} f+2g=-2 \\ g-f=-7 \end{cases} \end{array}$$

2. Solve each system by the elimination method.

$$\begin{array}{lll} \text{a) } \begin{cases} 2x+3y=6 \\ 8x=3y+9 \end{cases} & \text{c) } \begin{cases} 5m+2n=3 \\ 3m-4n=6 \end{cases} & \text{e) } \begin{cases} x-y=1 \\ 2x=2y-3 \end{cases} \\ \text{b) } \begin{cases} 5a+2b=12 \\ 6a+5b=20 \end{cases} & \text{d) } \begin{cases} 3a=2b+5 \\ 6a-4b=10 \end{cases} & \text{f) } \begin{cases} 4w=-2z \\ 5z-3w+26=0 \end{cases} \end{array}$$

3. The sum of two numbers is 17 and the difference between them is 5. What are the numbers?
4. The perimeter of a rectangle is 16, and its length is three times its breadth. Find the length and breadth of the rectangle.
5. The present age of a father is 5 years more than twice his daughter's. After 5 years he will be twice as old as his daughter. How old is the daughter now?
6. Ahmed's saving box contains 27 coins. If all the coins are only 25-cent and 50-cent coins and the value of the coins is 9.25 Nakfa, how many of each kind of coin are there?



3.5.4 Graphical Method

Activity 3.17

To Solve $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$ by graphical method, follow the instructions

and answer the questions below.

- a) Draw the graphs of $y = -x + 5$ and $y = x - 1$ on the same coordinate plane.
- b) Do your graphs intersect? If yes, what are then the coordinates of the intersection point?
- c) Nadya's graphs look like the ones given in Fig. 3.11. Are yours also the same as hers?

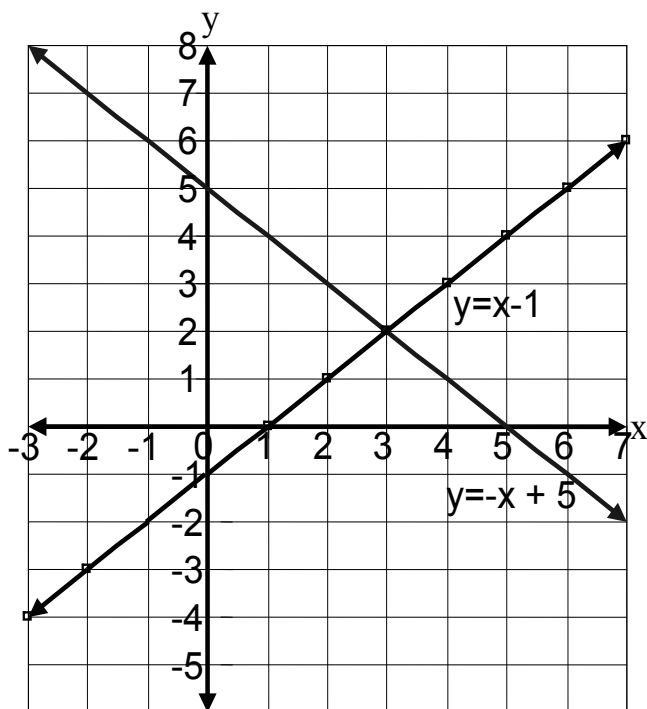


Fig. 3.11

- d) From Nadya's graphs, we see that the two lines intersect only at the point $(3, 2)$, which means that $x = 3$ and $y = 2$. Do these values of x and y satisfy both equations of the system?

Example 3.11

Solve the system $\begin{cases} y = x + 2 \\ x + y = 4 \end{cases}$ by graphing. **Solution:** The graphs

of the two equations are given in Fig. 3.12. From the graph, the answer is $(1, 3)$. Do the coordinates of $(1, 3)$ satisfy both equations in the system?

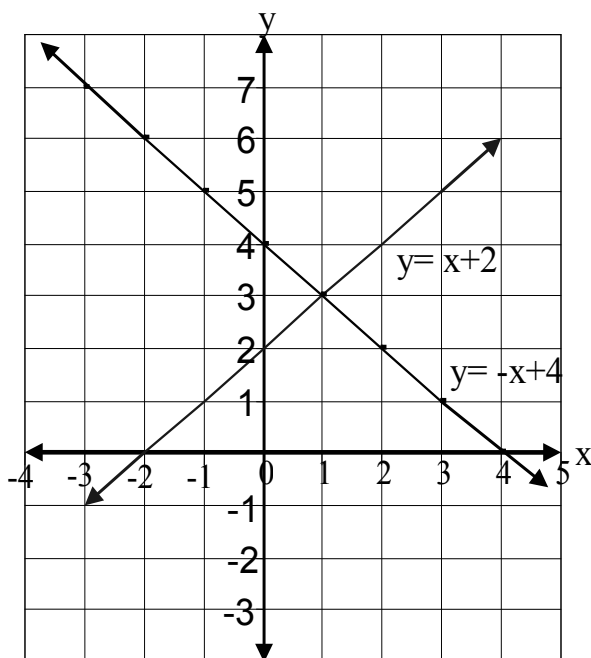


Fig. 3.12

Activity 3.18

1. Solve the system:
$$\begin{cases} 2x - y = 1 \\ -4x + 2y = 3 \end{cases}$$
 - a) Find the solution (if any) of the following system by graphing.
 - b) Find the solution of the system using either the elimination or the substitution method.
 - c) Did you get something like $0 = 5$? Is the statement $0 = 5$ true? What does getting $0 = 5$ tell about the solution of the system?
 - d) What do you think the solution of the system is?
2. Solve the system:
$$\begin{cases} y + x = 2 \\ 4x = -4y + 8 \end{cases}$$
 - a) Find the solution (if any) of the system by graphing.
 - b) Do you think the ordered pairs $(0, 2)$, $(1, 1)$, $(-1, 3)$, and $(2, 0)$ are all solutions of the simultaneous equations?

- c) Find the solution of the system either by substitution or elimination method.
- d) Did you finally get something like $0 = 0$ or $8 = 8$? Are the statements $0 = 0$ or $8 = 8$ true?
- e) What does getting $0 = 0$ or $2 = 2$ indicate about the solution of the system?

The tasks in activities 3.19 and 3.20 demonstrate that in a system of two linear equations:

- a) If the graphs of the equations intersect in a single point, then the system will have only one solution.
- b) If the graphs of the equations do not intersect (are parallel), then the system will have no solution.
- c) If the graphs of the equations are the same line, then the system will have infinitely many solutions.

Exercise 3.7

1. Determine the number of solutions of each system.

a) $\begin{cases} y = 3x - 1 \\ y = x + 3 \end{cases}$

d) $\begin{cases} 4x - 6y = 1 \\ 6x - 9y = 1.5 \end{cases}$

b) $\begin{cases} 2x - 6y = 7 \\ 3x - 9y = 10 \end{cases}$

e) $\begin{cases} x + 2y = 3 \\ 2x - y = 9 \end{cases}$

c) $\begin{cases} 3x + 4y = -17 \\ 2x + 3y = -13 \end{cases}$

f) $\begin{cases} m - 2n = 1 \\ 4n - 2m = -2 \end{cases}$

2. Solve each of the following simultaneous equations by graphing.

a) $\begin{cases} 2x - 3y = 6 \\ 3y - 2x = 3 \end{cases}$

c) $\begin{cases} y = x - 3 \\ y = -x + 1 \end{cases}$

b) $\begin{cases} x - 2y = 3 \\ 2x - 4y = 7 \end{cases}$

d) $\begin{cases} y = 2x \\ y = -x + 3 \end{cases}$

Review Exercise 3

- Graph each of the following functions.
a) $y = -x + 2$ b) $y = 4 - 2x$
- Draw the following triplets in the same coordinate plane.
 $y = -3x$; $y = -3x + 2$; $y = -3x - 2$
- Which of the following represent linear functions?
a) $t = (u+1)(u-1)$ d) $r = \frac{1}{2}s - 11$
b) $-x + 1 = 5y$ e) $d = -10t^2 + 40$
c) $p = 3(5v - 5)$ f) $s = 40t$
- Draw the graphs of the following.
a) $y \leq -2x + 1$ b) $3x + 2y > 0$ c) $y - 2x \geq 1$
- Determine the slope of the line that passes through
a) $(-5, 3)$ and $(7, 3)$ c) $(4, 3)$ and $(4, -5)$
b) $(-2, 2)$ and $(3, -3)$ d) $(-1, -3)$ and $(2, 1)$
- Rewrite each of the following in the form of $y = mx + b$ and determine its slope and y-intercept.
a) $3x - 2y = 6$ b) $4y + 8x = 12$
- Solve each of the following systems of linear equations.
a) $\begin{cases} m = 2n - 1 \\ 3m + 2n - 21 = 0 \end{cases}$ b) $\begin{cases} k + 2m = 6 \\ 3k - m = 18 \end{cases}$ c) $\begin{cases} 3t + s = 6 \\ 2t + s = 8 \end{cases}$
- Semira bought 32 stamps, each of which is either a 20-cent or a 50-cent stamp. If the total value of the stamps is 10.90 Nakfa, find the number of each kind of stamp.
- Tickets for a musical show cost 10 Nakfa for adults and 5 Nakfa for children. If in one show tickets worth 3000 Nakfa were sold to 400 people, how many adults and how many children attended the show?
- Awet is 3 years older than Danait. If the sum of their ages is 27, how old are they?
- The length of a rectangular swimming pool is twice its width. If its perimeter is 42 meters, find its length and width.

CHAPTER 4

Quadratic Functions

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- recognize situations modelled by quadratic functions;
- sketch graphs of quadratic functions;
- interpret the effects of changes in “a” and “c” of the quadratic function $y = ax^2 + c$;
- solve quadratic equations by factorisation and completing the square
- apply quadratic equations to solve real life problems.

4.1 Representing Quadratic Function

Activity 4.1

1. Have you heard about quadratic expressions, quadratic equations or quadratic functions before in Mathematics lessons? Give some examples.
2. Find the area of an isosceles right-angled triangle:
 - a) If the legs are 4cm each;

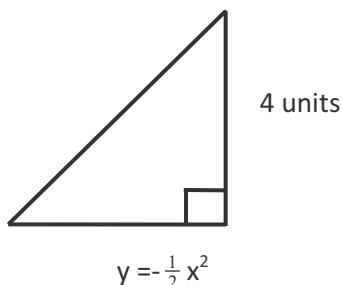


Fig. 4.1

- b) What is the area of an isosceles right triangle whose legs are x units?

3. Using the answer of 2 b), write an expression for the following:
- Two times of the area.
 - Three times of the area.
 - One-half of the area.
 - Two-third of the area.
 - Increasing the area by 2 square units.
 - Decreasing the area by 6 square units.

Example 4.1

Think of a number; write an expression for the square of the number plus 4 times the number minus 7.

Solution:

First let us represent the number by a symbol say n . Then, the square of the number is equal to:

$$n \cdot n = n^2$$

Now, the square of a number plus 4 times the number is equal to:

$$n^2 + 4n$$

Finally, the square of the number plus 4 times the number minus 7 is equal to:

$$n^2 + 4n - 7$$

An expression that can be written in the form of

$$ax^2 + bx + c,$$

Where a , b and c are real numbers, $a \neq 0$, is known as a **quadratic expression**.

Why should the coefficient of x^2 (" a ") be different to 0?

If $a = 0$ in the above expression, then we have the following:

$$\begin{aligned} ax^2 + bx + c &= 0x^2 + bx + c \\ &= 0 + bx + c \\ &= bx + c \end{aligned}$$

But $bx + c$ is linear not quadratic.

Exercise 4.1

1. Find a quadratic expression that describes the following situation.
 - a) The area of a rectangle where its width is x metres and its length is 5 metres more than its width.
 - b) The product of two non-zero consecutive integers.
 - c) Give 2 or 3 physical situations that can be described by quadratic functions.
2. The distance S in metres travelled by a freely falling body from rest (such as a fruit falling from a tree) after t seconds can be approximated by the function given by $S = 5t^2$.
 - a) An orange is dropped from an orange tree and takes half a second to hit the ground. How far was the orange from the ground before falling?
 - b) A piece of wood falls from a roof of a house and takes 3 seconds to hit the ground. How high is the roof of the house from the ground?
 - c) If an object is dropped down to the ground from a height of 80m, calculate the time taken to reach the ground.
3. A rocket is shot vertically with an initial velocity of 96 m/sec (see figure 4.2). The distance (vertical height) travelled by the rocket after t seconds is given by

$$h = -10t^2 + 100t$$



Fig. 4.2

Find the distance travelled by the rocket at the following given times.

a) $t = 0$ s.

c) $t = 2$ s.

b) $t = 5$ s.

d) $t = 6$ s.

4. Find a quadratic expression that describes the following situations.
- The area of a rectangle where its width is x metres and its length is $2x + 2$ metres.
 - The product of two odd consecutive integers.
 - The total surface area of a cube with side x units.

4.2 Graphing Quadratic Functions

Activity 4.2

- Discuss the methods and the steps used to draw the graph of linear functions.
- For the following quadratic functions find the value of the function (the value of y) at the given values of x .
 - $y = x^2$, $x = -3, -2, -1, 0, 1, 2, 3$.
 - $y = x^2 + 1$, $x = -4, -2, -1, 0, 1, 2, 3$.
- Fill the missing values in the following tables.

a)

x	-3	-2	-1	0	1	2	3
$y = 2x^2 - 1$		7		-1			17

b)

x	-3	-2	-1	0	1	2	3
$y = x^2 + x$	6		0			6	

The function defined as $y = ax^2 + bx + c$, $a \neq 0$ is a quadratic function.

Example 4.2

A teacher gave a task to his students to draw the graph of $y = x^2$.

Bereket: Two points are enough to draw the graph of $y = x^2$.

Kedija: It is not possible to draw the graph of $y = x^2$ using only 2 points only, because it is not linear. It is necessary to make a table of values and find more than 2 points to draw the graph.

a) They made a table of values to draw the graph of $y = x^2$.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

b) They plotted the points $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$ and $(3, 9)$ on the coordinate plane (x-y plane) to get a scattered plot.

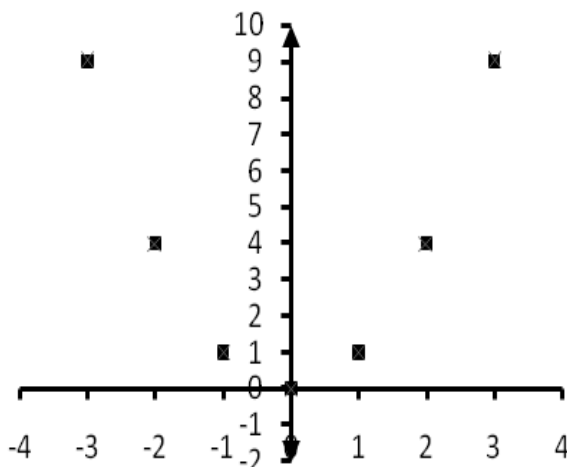


Fig. 4.3

c) They connected the points with a curve to produce the following graph:

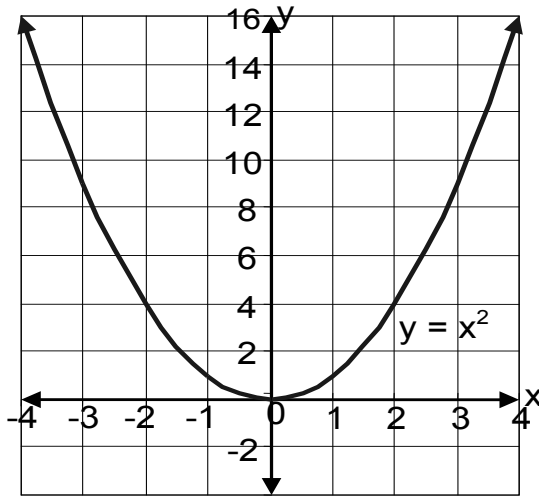


Fig. 4.4

Example 4.3

Given the quadratic function $y = x^2 + 4x$, for $-6 \leq x \leq 2$.

- Draw the graph of the function.
- Find the range.

Solution:

a) $y = x^2 + 4x$

x	-6	-5	-4	-3	-2	-1	0	1	2
$y = x^2 + 4x$	12	4	0	-3	-4	-3	0	4	12

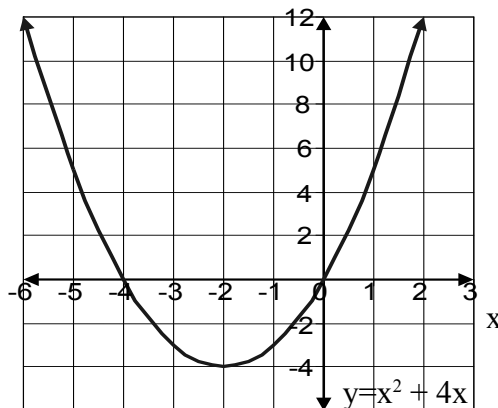


Fig. 4.5

- b) From the table and the graph of the function, the range is:
 $-4 \leq y \leq 12$.

Exercise 4.2

1. Draw the graph of the following quadratic functions, and determine the range of the function from its graph.
 - a) $y = x^2 + 2x$, for $-4 \leq x \leq 2$
 - b) $y = -x^2 - 2x + 3$ for $-5 \leq x \leq 3$
2. Consider the distance function discussed in Exercise 4.1 given by $S = 5t^2$.
 - a) What is the domain of the function?
 - b) Draw the graph.
 - c) Find the range of the function.
 - d) From the graph What is the distance travelled by the object at $t = 0.5s$? $t = 2.5s$? $t = 3.5s$?
3. Draw the graph of the following quadratic functions:
 - a) $y = x^2 + 2x + 1$
 - b) $y = -x^2 + 4x + 12$
 - c) $y = x^2 - 9$
 - d) $y = -2x^2 + 8$

4.3 Effects of Parametric Changes of Quadratic Functions

Activity 4.3

1. Draw the graph of the following functions:
 - a) $y = x^2$
 - b) $y = 2x^2$
 - c) $y = 3x^2$
 - d) $y = \frac{1}{2}x^2$
 - e) $y = \frac{1}{3}x^2$

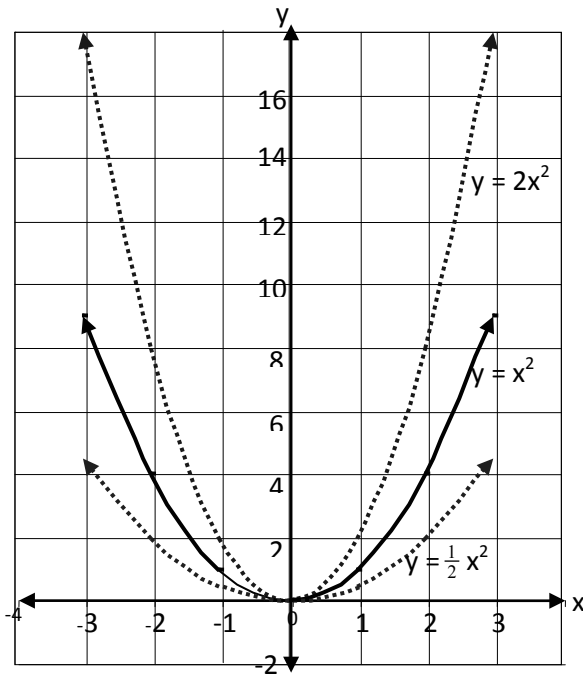
Example 4.4

Draw the graph of the following functions in the same coordinate system.

- a) $y = x^2$
- b) $y = 2x^2$
- c) $y = \frac{1}{2}x^2$

Solution:

x	$y = x^2$	$y = 2x^2$	$y = \frac{1}{2}x^2$
-3	9	18	4.5
-2	4	8	2
-1	1	2	1
0	0	0	0
1	1	2	1
2	4	8	2
3	9	18	4.5

**Fig. 4.6****Example 4.5**

Draw the graph of the following functions in the same coordinate system.

a) $y = -x^2$

b) $y = -2x^2$

c) $y = -\frac{1}{2}x^2$

Solution:

x	$y = -x^2$	$y = -2x^2$	$y = -\frac{1}{2}x^2$
-3	-9	-18	-4.5
-2	-4	-8	-2
-1	-1	-2	-1
0	0	0	0
1	-1	-2	-1
2	-4	-8	-2
3	-9	-18	-4.5

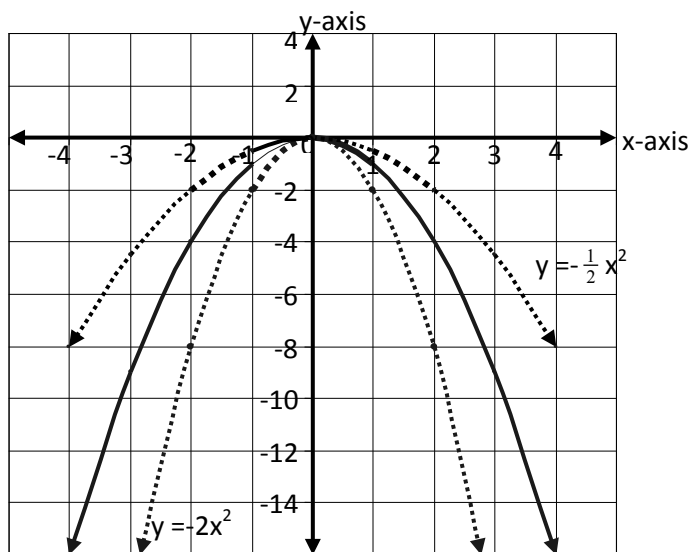


Fig. 4.7

Activity 4.4

Discuss the following:

- Describe the domain and range of the functions in Example 4.4 and 4.5 of the above.
- Describe the vertices (the turning points) of the graphs in Example 4.4 and 4.5 of the above.
- How do we get the graph of $y = ax^2$ from the graph of $y = x^2$, where “a” is any real number different from zero?

From the above activity, we can conclude the following:

If $a > 0$, the graph opens upward and the range is all real numbers greater than or equal to zero.

If $a < 0$, the graph opens downward and the range is all real numbers less than or equal to zero.

Example 4.6

Draw the graph of the following functions in the same coordinate system.

a) $y = x^2$

b) $y = x^2 + 1$

c) $y = x^2 - 2$

Solution

x	$y = x^2$	$y = x^2 + 1$	$y = x^2 - 2$
-3	9	10	7
-2	4	5	0
-1	1	2	-1
0	0	1	-2
1	1	2	-1
2	4	5	0
3	9	10	7

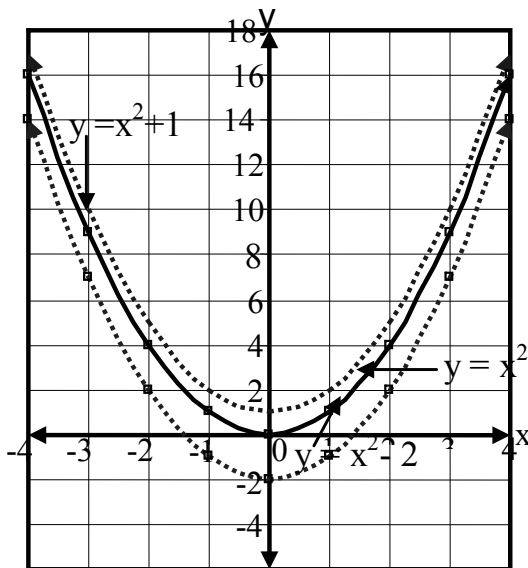


Fig. 4.8

Activity 4.5

Discuss the following.

- Describe the domain and range of the three functions in Example 4.6.
- Describe the vertices (the turning points) of the three graphs in Example 4.6.
- Make a general statement about the effects of “c” on the graph of $y = x^2 + c$, where “c” is any real number.

If $c > 0$, then the graph shifts c units upward from the origin and the turning point (vertex) becomes $(0, c)$, and the range is all real numbers greater than or equal to c .

If $c < 0$, then the graph shifts c units downward from the origin and the turning point (vertex) becomes $(0, c)$, and the range is all real numbers greater than or equal to c .

Exercise 4.3

Sketch the graph of the following pair of functions in the same coordinate system.

- $y = 2x^2$ and $y = 2x^2 + 2$
- $y = -2x^2$ and $y = -2x^2 + 2$
- $y = 2(x^2 + 2)$ and $y = -2(x^2 + 2)$

4.4 Solving Quadratic Equations

4.4.1 Solving Quadratic Equations by Factorisation

Activity 4.6

- List the different type of equation you have discussed so far, and discuss the different methods you used to solve the equations you listed.
- If the area of a square is 49 sq units and the side is x units, then find x .

3. Yohana and Beshir were asked to solve the following quadratic equation: $x^2 + 4x = 0$

Beshir worked it as follows:

$x^2 + 4x = 0$	Original equation
$x^2 + 4x - 4x = 0 - 4x$	Subtracting $4x$ from both sides
$x^2 = -4x$	Simplify
$\frac{x^2}{x} = \frac{-4x}{x}$,	Dividing both sides by x
$x = -4$	Simplify

Hence, the solution is -4 .

Yohana solved as follows:

$x^2 + 4x = 0$	Original equation
$x(x + 4) = 0$	Factorization (x is common)
$x = 0$ or $x + 4 = 0$	Using the Product Property of Zero.
$x = 0$ or $x = -4$	Solving the linear equations.

Hence, the solutions are 0 and -4 .

Who has got the correct solution(s)? Discuss the two methods.

Beshir: Said; my solution is correct, and let me simply show you.

The equation is: $x^2 + 4x = 0$. Then, substituting $x = -4$, we have;
 $(-4)^2 + 4(-4)$

$$= 16 - 16$$

$$= 0$$

Therefore, -4 is also a solution.

Yohana: Using the same way I can also show you 0 is a solution.

$$(0)^2 + 4(0) = 0 + 0 = 0, \text{ which shows that } 0 \text{ is indeed a solution.}$$

4. Discuss the product of zero using examples.

5. Investigate why Beshir's method didn't give all the solutions.

A quadratic equation is an equation that can be written in the form of: $ax^2 + bx + c = 0$

where a , b and c are real numbers and $a \neq 0$.

Exercise 4.4

Expand each expression by removing parenthesis.

a) $5(x + 2)$

f) $-x(-4 + 10x)$

b) $7(y - 1)$

g) $(x + 2)(x + 3)$

c) $x(2x + 3)$

h) $(x - 4)(x + 1)$

d) $a(b + 6)$

i) $(x + 7)(x - 1)$

e) $-x(8 - 5x)$

j) $(x - 2)(x - 3)$

Activity 4.7

Factorise $4x^2 - 6x$.

Fatima says the common factor of $4x^2$ and $-6x$ is 2. Therefore the factorisation of $4x^2 - 6x$ is $2(2x^2 - 3x)$.

Andom says the common factor of $4x^2$ and $-6x$ is x . Therefore the factorisation of $4x^2 - 6x$ is $x(4x - 6)$.

Ali says the *greatest common factor* of $4x^2$ and $-6x$ is $2x$. Therefore the factorisation of $4x^2 - 6x$ is $2x(2x - 3)$.

Discuss the three answers. Which one do you think more appropriate factorisation of $4x^2 - 6x$?

Although, the three are possible answers, we consider Ali's answer a more appropriate factorisation.

In factorisation we take the greatest common factor among the common factors.

Example 4.7Factorise $3x^2 - 12x$.**Solution:** $3x^2 - 12x$ Original expression $3x(x - 4)$ $3x$ is the greatest common factor of $3x^2$ and $12x$.Hence, the factorisation of $3x^2 - 12x$ is $3x(x - 4)$.

Difference of two squares:

$$x^2 - y^2 = (x + y)(x - y)$$

Exercise 4.5

1. Factorise the following

a) $x^2 + x$

f) $-0.2x^2 - 0.4x$

b) $x - x^2$

g) $7x^2 - 28$

c) $x^2 - 16$

h) $9x - 6x^2$

d) $4x^2 + 5x$

i) $12 - 8x^2$

e) $0.5x^2 + 2x$

j) $2x^2 - 10$

2. Solve the following quadratic equations.

a) $x^2 + x = 0$

e) $3x^2 + 12x = 0$

b) $x^2 - 7x = 0$

f) $4x + 3x^2 = 0$

c) $x + 5x^2 = 0$

g) $6x^2 - 18x = 0$

d) $x - x^2 = 0$

h) $10x - 5x^2 = 0$

Activity 4.81. Try to solve the equation $x^2 + 7x + 12 = 0$, by splitting $7x$ as follows:

a) $7x = x + 6x$, ($7 = 1 + 6$)

b) $7x = 2x + 5x$ ($7 = 2 + 5$)

c) $7x = 3x + 4x$ ($7 = 3 + 4$)

d) Which one of the above splitting is the correct pair that gives the solution?

e) How is the (splitting of 7) pair of numbers related to 12?

2. For the equation $2x^2 + 7x + 5 = 0$, repeat a), b), c), d) and e) of problem 1.
3. Solve $3x^2 + 8x + 5 = 0$, by splitting $8x$ or simply 8 into suitable pair.

Example 4.8

Solve $x^2 + 4x + 2x + 8 = 0$

Solution:

$x^2 + 4x + 2x + 8 = 0$	Original equation
$(x^2 + 4x) + (2x + 8) = 0$	Making two groups
$x(x + 4) + 2(x + 4) = 0$	Factorise each group

Here, we found another common factor, which is $x + 4$. So, let us make second factorisation as follows:

$(x + 4)(x + 2) = 0$	Factorise using factor $x + 4$
$x + 4 = 0$ or $x + 2 = 0$	Product property of zero.
$x = -4$ or $x = -2$	Solving linear equations.

Hence, the solutions are -4 and -2 .

Check whether the solutions are correct or not.

$(-4)^2 + 4(-4) + 2(-4) + 8 = 16 - 16 - 8 + 8 = 0$, then -4 is a solution.
 $(-2)^2 + 4(-2) + 2(-2) + 8 = 4 - 8 - 4 + 8 = 0$, -2 is also a solution.

Hence, the solutions are -4 and -2 .

Example 4.9

Solve $x^2 + 5x + 6 = 0$.

Solution:

Let us try splitting $5x$; as $2x$ and $3x$.

$x^2 + 2x + 3x + 6 = 0$	Splitting $5x$ ($5x = 2x + 3x$)
$x(x + 2) + 3(x + 2) = 0$	Factorise the 2 groups.
$(x + 2)(x + 3) = 0$	Factorise (common factor $x + 2$)
$x + 2 = 0$ or $x + 3 = 0$	Product property of zero
$x = -2$ or $x = -3$	Solving linear equation.

Hence, -2 and -3 are the 2 solutions, which implies the splitting of $5x$ into $2x$ and $3x$ is perfect.

Exercise 4.6

Solve the following quadratic equations.

a) $x^2 + x + 4x + 4 = 0$

d) $3x^2 - 2x + 6x - 4 = 0$

b) $x^2 - 5x + 2x - 10 = 0$

e) $8 + 6x + 4x + 3x^2 = 0$

c) $2x^2 + 4x + 7x + 14 = 0$

f) $4 - 2x - 2x + x^2 = 0$

Given $ax^2 + bx + c = 0$, ($a \neq 0$), if p and q are the splitting pair of b that produce the factorisation, then $p + q = b$ and $pq = ac$.
i.e. Sum $p + q = b$ and product $pq = ac$.

Example 4.10

The sum of the squares of 2 positive consecutive integers is 25.
Find the 2 numbers.

Solution:

Let the smallest of the 2 consecutive integers be x .

The next consecutive positive integer would be $x + 1$.

The sum of the squares of the two consecutive integers is $x^2 + (x + 1)^2$, which is equal to 25.

Therefore:

$$x^2 + (x + 1)^2 = 25$$

$$x^2 + x^2 + 2x + 1 = 25$$

$$2x^2 + 2x + 1 - 25 = 25 - 25$$

$$2x^2 + 2x - 24 = 0$$

$$x^2 + x - 12 = 0$$

$$x^2 + 4x - 3x - 12 = 0$$

$$x(x + 4) - 3(x + 4) = 0$$

$$(x + 4)(x - 3) = 0$$

$$x + 4 = 0 \text{ or } x - 3 = 0$$

$$x = -4 \text{ or } x = 3$$

Expanding $(x + 1)^2$

Collecting like terms

Simplify

Divide all terms by 2

Split x into $4x$ and $-3x$

Common factor

Factorize

Product rule of zero

Hence, $x = 3$ and $x + 1 = 3 + 1 = 4$. The 2 consecutive numbers are 3 and 4. Note that x cannot be equal to -4. Why?

Exercise 4.7

- Solve the following using factorisation.

a) $x^2 - 6x + 9 = 0$	d) $x^2 - 4x - 12 = 0$
b) $2x^2 + 5x + 3 = 0$	e) $-2x^2 + 8x - 6 = 0$
c) $-x^2 - 5x + 14 = 0$	f) $4x^2 + 8x - 5 = 0$
- A rectangle has a perimeter 28 cm and an area of 33 cm^2 . What are the dimensions?
- Find 2 positive numbers that differ by 4 and whose product is 21.

4.4.2 Solving Quadratic Equations Using Completing the Square Method

Activity 4.9

Read and comment on the discussion by three learners students to solve the quadratic equation $x^2 = 16$.

Berhe, Meriem and Fatima are discussing how to find the solution.

Berhe: The solution is simply 4, since $4^2 = 16$.

Meriem: There is another solution, and that is -4, since

$$(-4)^2 = (-4)(-4) = 16.$$

Hence, the solutions are 4 and -4.

Fatima: You are right. The following method can be applied to get the solutions.

$x^2 = 16$	Original equation
$x^2 - 16 = 16 - 16$	Subtracting 16 from both sides
$x^2 - 16 = 0$	Simplifying
$x^2 + 0x - 16 = 0$	Adding $0x$ doesn't affect the equation
$x^2 + 4x - 4x - 16 = 0$	Splitting of 0 into 4 and -4
$(x + 4)(x - 4) = 0$	Factorising
$x + 4 = 0$ or $x - 4 = 0$	Product property of zero
$x = -4$ or $x = 4$	Solving linear equations.

Meriem: It is also possible to use the following short method.

$x^2 = 16$	Original equation
------------	-------------------

$$\sqrt{x^2} = \sqrt{16} \quad \text{Taking the square root of both sides}$$

$$|x| = 4 \quad \text{Property of absolute values}$$

$$x = \pm 4 \quad \text{Solving absolute value equations}$$

Hence, we have got the same solutions.

Exercise 4.8

1. Solve the following quadratic equations.

- | | |
|----------------|---------------------|
| a) $x^2 = 25$ | d) $-x^2 = -100$ |
| b) $x^2 = 121$ | e) $(x-2)^2 = 9$ |
| c) $x^2 = 169$ | f) $-2(x-4)^2 = -8$ |

2. Find the expanded form of the following.

- | | |
|--------------|---------------|
| a) $(x+2)^2$ | d) $(x-5)^2$ |
| b) $(x+5)^2$ | e) $(3-x)^2$ |
| c) $(x-2)^2$ | f) $(-x-3)^2$ |

3. Find the solution of the following using factorization.

- | | |
|-------------------------|-------------------------|
| a) $x^2 + 8x + 16 = 0$ | c) $x^2 - 8x + 16 = 0$ |
| b) $x^2 + 10x + 25 = 0$ | d) $x^2 - 10x + 25 = 0$ |

4. How many solutions have each question in problem 3?

Perfect squares

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

5. Find the solution of the following.

- | | |
|-------------------|---------------------|
| a) $(x+4)^2 = 36$ | c) $-(x+7)^2 = -81$ |
| b) $(x-1)^2 = 9$ | d) $-(x-6)^2 = -4$ |

6. What makes each of the following expressions a perfect square?

- | | |
|--|--|
| a) $x^2 + 6x + \underline{\hspace{1cm}}$ | c) $x^2 + \underline{\hspace{1cm}} + 25$ |
| b) $x^2 - 2x + \underline{\hspace{1cm}}$ | d) $x^2 + 5x + \underline{\hspace{1cm}}$ |

Example 4.11

Solve $x^2 + 6x + 8 = 0$

Solution:

$x^2 + 6x + 8 = 0$	Original equation
$x^2 + 6x + 8 - 8 = 0 - 8$	Subtracting 8 from both sides
$x^2 + 6x = -8$	Simplifying
$x^2 + 6x + 9 = -8 + 9$	Add 9 to both sides to make the left side a perfect square
$(x + 3)^2 = 1$	Factorize the left side
$ x + 3 = 1$	Property of absolute values
$x + 3 = 1$ or $x + 3 = -1$	Property of absolute value equations
$x = -2$ or $x = -4$	Solving absolute value equations

The above method of solving quadratic equations is known as the **completing the square**.

Exercise 4.9

1. Solve the following using the completing square method.

- | | |
|-------------------------|----------------------------------|
| a) $x^2 + 14x + 49 = 0$ | c) $x^2 + 5x + \frac{25}{4} = 0$ |
| b) $x^2 - 4x + 4 = 0$ | d) $x^2 - 3x + \frac{9}{4} = 0$ |

2. Solve the following using the completing square method.

- | | |
|------------------------|-----------------------|
| a) $x^2 + 4x - 10 = 0$ | d) $x^2 + 5x - 8 = 0$ |
| b) $x^2 + 3x - 8 = 0$ | e) $x^2 - 4x - 6 = 0$ |
| c) $x^2 - 6x - 5 = 0$ | f) $x^2 - 8x - 5 = 0$ |

3. Is it possible to construct a rectangle with a perimeter of 36 cm and an area of 36 sq. cm? If yes, what are its dimensions?

Review Exercise 4

1. If the area of an isosceles right triangle is 8 sq units and the legs are x units, then find x .

2. The length of a basketball court is 2m less than 2 times the width. If the total area is 420 m^2 , find the dimensions of the court.
3. Find a quadratic expression that describes the following situation.
 - a) The total number of football games played by n teams, when each team plays every other team twice.
 - b) The area of a right triangle where its base is x m and its height is $3x - 4$ m.
 - c) The product of 2 even consecutive integers.
 - d) The total surface area of a box with side x , $x+1$ and $x+3$ units.

4. The profit gained by selling n hens is given by the function

$$P = -n^2 + 8n, \text{ for } 0 \leq n \leq 10.$$

- a) Make a table of values on the given domain.
 - b) How many hens should be sold to get a profit of 12 Nakfa?
 - c) How many hens should be sold to get a maximum profit?
 - d) How much will the profit be if 8 hens are sold?
 - e) Will there be a profit or lose if 10 hens are sold?
5. Sketch the graph of the following quadratic functions.
 - a) $y = x^2 - 5x$
 - b) $y = 2x^2 + x + 1$
 - c) $y = -x^2 + 16$
 - d) $y = -x^2 - 6x + 8$
6. Solve the following quadratic equations by factorisation.
 - a) $x^2 + 3x = 0$
 - b) $3x^2 - 7x - 6x + 14 = 0$
 - c) $6x^2 + 11x + 4 = 0$
 - d) $3y^2 - 8y - 3 = 0$
 - e) $6a^2 + 5a - 6 = 0$
 - f) $1 + 3x - 10x^2 = 0$

7. Find a quadratic equation having roots (solution):

a) 3 and 2

c) -5 and -3

b) 4 and -1

d) -2 and 7

8. Solve the following by completing the square.

a) $x^2 - 5x = 0$

b) $x^2 - 2x + 2 = 0$

c) $x^2 + 8x - 9 = 0$

d) $-y^2 - 4y - 18 = 0$

CHAPTER 5

Exponential and Logarithmic Expressions

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- describe properties of power and logarithmic notations;
- identify relationships between logarithmic and exponential notations;
- simplify exponential and logarithmic expressions;
- solve exponential and logarithmic equations
- apply logarithms in mathematical calculations.

5.1 Exponential Expressions

5.1.1 Exponential Notations

Activity 5.1

1. a) Do you remember that 5^3 means $5 \times 5 \times 5$?
b) What is the power, the exponent and the base?
2. How do you express $\underbrace{a \times a \times a \times \dots \times a}_{n \text{ terms}}$ using a power notation if there are n terms?

$$\underbrace{a \times a \times a \times \dots \times a}_{n \text{ terms}} = a^n$$

where a is a real number and n is any positive integer.

Example 5.1

The number of insects in a laboratory doubles every month. There are currently about 500 insects in the laboratory.

- a) How many insects will there be after 1 year?
- b) Write the expression to represent the number of insects after n months.

Solution:

- a) Number of insects after 1 month $= 500 \times 2 = 500 \times 2^1$
 Number of insects after 2 months $= 500 \times 2 \times 2 = 500 \times 2^2$
 Number of insects after 3 months $= 500 \times 2 \times 2 \times 2$
 $= 500 \times 2^3$

Number of insects after 12 months
 $= 500 \times 2 \times 2 \times 2 \times \dots \times 2$ (2 multiplied by itself 12 times)
 $= 500 \times 2^{12}$
 $= 500 \times 4096$
 $= 2,048,000$

- b) From the pattern in part a) above, it can be concluded that the number of insects after n months is 500×2^n .

Exercise 5.1

1. Write the power, exponent and the base of the following.

a) 2^3

b) $(-5)^6$

c) a^7

d) 6^n

e) $\left(-\frac{1}{2}\right)^8$

f) $(3t)^5$

2. Write the following expressions in a power notation.

a. $3 \times 3 \times 3 \times 3 \times 3 \times 3$

b) $d \times d \times d \times d \times d \times d \times d$

c) $\left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$

d) $\left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right)$

e) $(-3) \times (-3) \times (-3) \times (-3) \times (-3)$

f) $(2k) \times (2k) \times (2k)$

3. What can be said if 6 is multiplied by itself n times?

4. Simplify $3(-2)^3 + 5(3)^2$.

5. Kiflom was saving money in a saving box. He started with 5 cents and doubled the amount each day. How much did he save after 2 days? 3 days?

6. A hive contains 1000 bees. If the population doubles every month, how many bees are there after 2 months? 3 months? 5 months?
7. The number of insects in a colony reduces by half every month. There are currently 20,000 insects in the colony. How many insects will there be after 2 months? 5 months?
8. Simplify:
- a) $5^2 + 7^2$ b) $(5 + 7)^2$
- c) $(-1)^4 + (-3)^3$ d) $3(-2)^3 + 4(3)^2$
9. Find:
- a) 3^9 b) 6^5 c) $(-5)^4$
- d) $(2.1)^2$ e) $(1.3)^4$ f) -7^4
10. Evaluate:
- a) $(-2)^4$ b) -2^4 c) $(-2)^5$ d) $-(-2)^5$
- e) Discuss the effect of the brackets in the expressions given in exercise 10 (a) – (d).
11. Nichomachus who lived around 100 CE discovered an interesting number pattern involving cubes and sums of odd numbers as follows.
- $1 = 1^3$
- $3 + 5 = 8 = 2^3$
- $7 + 9 + 11 = 27 = 3^3$, etc.
- Can you find the next 2 terms using this pattern?

5.1.2 Product of Exponents

Activity 5.2

A teacher gave the learners the following task;

$$2^3 \times 2^2.$$

Hawa multiplied it as follows.

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 2^5 = 32$$

$$2^3 \times 2^2 = 8 \times 4 = 32 \text{ and } 2^5 = 32.$$

She noted that $2^3 \times 2^2 = 2^{(3+2)}$

Aster also tried to multiply 3^2 by 3^4

$$3^2 \times 3^4 =$$

$$(3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6$$

Aster also discovered that $3^2 \times 3^4 = 3^{(2+4)} = 3^6$

a) What do you observe from the learners' two examples?

b) How do you relate the examples of Hawa and Aster?

By using the definition of a power, the activities of Hawa and Aster suggest one of the properties of exponents.

$$\begin{aligned} a^m \times a^n &= \underbrace{a \times a \times \dots \times a}_m \times \underbrace{a \times a \times \dots \times a}_n \\ &= \underbrace{a \times a \times \dots \times a}_{(m+n)} \\ &= a^{(m+n)} \end{aligned}$$

$$a^m \times a^n = a^{(m+n)},$$

where m and n are natural numbers and a is any real number.

Exercise 5.2

Multiply the following.

1. $3^2 \times 3^5$

2. $a^{12} \times a^6$

3. $m^8 \times m^{24}$

4. $c^3 \times c^{14} \times c^6$

5. $5y^3 \times 2y^6$

6. $a^6 b^3 \times a^8 b^6$

5.1.3 Quotient of Exponents

Activity 5.3

Motivated by the results of Hawa and Aster, Tesfay and Adem also tried to investigate some relations in the process of division.

Tesfay divided 2^6 by 2^2 as follows.

$$\frac{2^6}{2^2} = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} = 2^4 = 2^{(6-2)}$$

Adem also divided 5^7 by 5^4 as follows.

$$\frac{5^7}{5^4} = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5 \times 5} = 5^3 = 5^{(7-4)}$$

What do you observe from this activity?

$$\frac{a^m}{a^n} = a^{(m-n)},$$

where m and n are natural numbers, $m > n$, a is any real number and $a \neq 0$.

Exercise 5.3

Simplify:

a) $\frac{7^6}{7^2}$

b) $\frac{a^3 \times a^6}{a^2}$

c) $\frac{m^3 \times n^6}{m^2 \times n^2}$

d) $36a^{12}b^6 \div 9a^3b^2$

e) $125k^{18} \div 25k^6$

f) $(a^{28} \div a^7) \div a^4$

5.1.4 Power of a Product of Exponents

Activity 5.4

Mehari and Seid are classmates and they always study together. While practising power notations, they noted that:

$$\begin{aligned}(2 \times 3)^3 &= (2 \times 3) \times (2 \times 3) \times (2 \times 3) \\ &= (2 \times 2 \times 2) \times (3 \times 3 \times 3) \\ &= 2^3 \times 3^3\end{aligned}$$

Similarly, they expressed $(5 \times 7)^4$ as follows.

$$\begin{aligned}(5 \times 7)^4 &= (5 \times 7) \times (5 \times 7) \times (5 \times 7) \times (5 \times 7) \\ &= (5 \times 5 \times 5 \times 5) \times (7 \times 7 \times 7 \times 7) \\ &= 5^4 \times 7^4\end{aligned}$$

- What do you observe from their results?
- Can you suggest a law of this type?

$$(ab)^n = a^n \times b^n,$$

where n is a natural number and a and b are real numbers.

Exercise 5.4

Express the following powers as a product of 2 powers.

- | | | |
|---------------------|---------------------|---------------------|
| a) $(9 \times 4)^2$ | b) $(3 \times 8)^5$ | c) $(5 \times 7)^4$ |
| d) $(5 \times 2)^3$ | e) $(3 \times 7)^2$ | f) $(a \times b)^3$ |

5.1.5 Power of a Quotient of Exponents

Activity 5.5

Extending their investigations, Mehari and Seid observed that:

$$\begin{aligned}\text{a) } \left(\frac{2}{3}\right)^2 &= \frac{2}{3} \times \frac{2}{3} = \frac{2^2}{3^2} \\ \text{b) } \left(\frac{3}{5}\right)^3 &= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{3^3}{5^3}\end{aligned}$$

- Can you give similar examples of your own?
- What can be generalised from your answers?

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \text{ where } b \neq 0, \text{ and } n \text{ a natural number.}$$

Exercise 5.5

Write in the form of $\left(\frac{a}{b}\right)^n$.

a) $\frac{2^6}{5^6}$

b) $\frac{3^4}{2^4}$

c) $\frac{2^2}{3^2}$

d) $\frac{7^5}{6^5}$

e) $\frac{8^4}{9^4}$

f) $\frac{10^7}{12^7}$

5.1.6 Power of a Power of Exponents**Activity 5.6**

The previous activities of Hawa and Aster encouraged Helen and Aziza to investigate properties related to $(a^m)^n$. They took the power $(2^3)^4$. They expressed it as follows.

$$\begin{aligned}(2^3)^4 &= 2^3 \times 2^3 \times 2^3 \times 2^3 \\ &= (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2^{12} \\ &= 2^{3 \times 4}\end{aligned}$$

- a) Give 3 similar examples of your own?
b) What can be generalised from your answers?

Exercise 5.6

Express the following in the form of a^{mn}

a) $(3^2)^3$

b) $(a^5)^3$

c) $((-2)^3)^6$

d) $(a^3)^4$

e) $\left(\left(\frac{-3}{5}\right)^2\right)^2$

f) $\left(\left(\frac{2}{3}\right)^3\right)^2$

- g) What do you observe from these results?

$$(a^m)^n = a^{mn},$$

where m and n are natural numbers and a is a real number.

5.1.7 Integral Exponents

Activity 5.7

1. A teacher gave a task to the class to simplify $\frac{7^3}{7^3}$

Haile simplified the expression $\frac{7^3}{7^3}$ as follows.

$$\frac{7^3}{7^3} = \frac{7 \times 7 \times 7}{7 \times 7 \times 7} = 1$$

Extending the previous laws, Halima simplified the expression as follows.

$$\frac{7^3}{7^3} = 7^{3-3} = 7^0$$

- Give other examples of your own.
- What do you observe from their activities?
- What can be generalised from the activity?

This activity suggests the following property.

$a^0 = 1$, where a is a real number and different from zero.

- c) Can you evaluate 6^0 , 2^0 , $(-4)^0$ and $(1.5446)^0$?

2. Kelil and Antore were discussing to simplify $\frac{5^4}{5^7}$
Kelil did it as follows.

$$\frac{5^4}{5^7} = \frac{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5}}{\cancel{5} \times \cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5 \times 5} = \frac{1}{5^3}$$

Extending the previous law to negative integral exponents, Antore did it as follows;

$$\frac{5^4}{5^7} = 5^{(4-7)} = 5^{-3}$$

- a) What could be concluded from the activities of Kelil and Antore?

$$a^{-n} = \frac{1}{a^n}, n \text{ is an integer and } a \neq 0.$$

- b) What happens if n is equal to zero?

Example 5.2

The number of insects in a laboratory doubles every month. If there are now 80 insects in the laboratory, how many were in the laboratory 3 months ago?

Solution:

The number of insects after n months is 80×2^n .

Letting $n = -3$, we get

$$= 80 \times 2^n = 80 \times 2^{-3} = 80 \times \frac{1}{2^3} = 80 \times \frac{1}{8} = 10. \text{ Do you know why we took } n = -3?$$

Therefore, there were about 10 insects in the laboratory three months ago.

Exercise 5.7

1. Find the value:

a) 2^{-4}

b) $(0.1)^{-2}$

c) $\left(\frac{2}{3}\right)^{-2}$

d) -5^{-3}

e) $(3)^2$

f) $(-3)^2$

g) $(3)^{-2}$

h) $\left(\frac{1}{2}\right)^{-4}$

i) $\left(\frac{-3}{2}\right)^{-2}$

j) $5^2 - 5^{-2}$

k) $(-2)^{-3} + 3^{-2}$

l) $7^0 + (-3)^3$

2. The number of insects in a laboratory triples every month. If there are now 135 insects in the laboratory, how many were in the laboratory 2 months ago?
3. A colony of 5,000 bees doubles in number every month.
 - a) How many bees were there 2 months ago?
 - b) How many bees were there 5 months ago?
4. The amount of a certain chemical reduces by half every month. There is currently 600 g of the chemical.
 - a) How much will there be in 3 months?
 - b) How much was there 4 months ago?

5.1.8 Expressing Exponentials as Radicals

Activity 5.8

1. a) Expressing $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ as one power, we get

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\left(\frac{1}{2} + \frac{1}{2}\right)} = 2^1 = 2.$$

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = \sqrt{2} \times \sqrt{2} = \sqrt{2 \times 2} = \sqrt{4} = 2$$

From the two steps we can conclude that $2^{\frac{1}{2}} = \sqrt{2}$

- b) Can you apply the same method for $7^{\frac{1}{3}} \times 7^{\frac{1}{3}} \times 7^{\frac{1}{3}}$?

- c) Repeat the process for $5^{-\frac{1}{3}} \times 5^{-\frac{1}{3}} \times 5^{-\frac{1}{3}}$.

- d) What do you conclude from these activities?

$$a^{\frac{1}{n}} = \sqrt[n]{a},$$

where n is a natural number greater or equal to 2.

$a \geq 0$ if n is even.

$$a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}}, \text{ where } n \text{ is a natural number and } a \neq 0,$$

$a > 0$ if n is even.

2. Do you observe that $16^{\frac{3}{2}}$ could be simplified as follows?

$$16^{\frac{3}{2}} = \left(16^{\frac{1}{2}}\right)^3 = (\sqrt{16})^3 = 4^3 = 64$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m, \text{ } m \text{ and } n \text{ are integers, } n \geq 2, \text{ and}$$

$a \geq 0$ if n is even.

Exercise 5.8

1. Simplify the following:

a) $25^{\frac{1}{2}}$

b) $81^{-\frac{1}{4}}$

c) $(-27)^{-\frac{1}{3}}$

d) $16^{\frac{1}{4}}$

e) $(-125)^{\frac{1}{3}}$

f) $(-64)^{-\frac{1}{3}}$

g) $64^{0.5}$

h) $27^{\frac{1}{3}}$

i) $16^{0.25}$

j) $\left(\frac{25}{16}\right)^{\frac{1}{2}}$

k) $\left(\frac{27}{64}\right)^{\frac{1}{3}}$

l) $(0.04)^{0.5}$

2. a) $(-27)^{\frac{2}{3}}$ b) $16^{0.75}$ c) $\left(-\frac{1}{27}\right)^{-\frac{2}{3}}$
 d) $125^{\frac{2}{3}}$ e) $4^{\frac{3}{2}}$ f) $\left(-\frac{8}{27}\right)^{\frac{4}{3}}$
3. Express as a power.
 a) $\sqrt{34}$ b) $\sqrt[3]{-91}$ c) $(\sqrt[5]{6})^2$ d) $\frac{1}{(\sqrt{28})^5}$
4. Simplify to its lowest term.
 a) $(\sqrt[5]{36})^4$ b) $(\sqrt[7]{81})^5$ c) $(\sqrt[3]{49^5})$
 d) $(\sqrt[3]{27})^{-\frac{1}{2}}$ e) $\left(-3^{\frac{2}{3}} \times 3^{-\frac{1}{2}}\right)^6$ f) $\frac{a^3b^{\frac{1}{2}}}{a^{-2}b}$
 g) $\left(ab^{\frac{1}{2}}\right)^{-3} (a^{-2}b)^{\frac{1}{2}}$ h) $\sqrt{(8a^3b^{-3})(2ab^0)}$
5. The volume of a cube is $v \text{ cm}^3$. Write the length of the edge (side) as a power of v .

5.1.9 Solving Exponential Equations

Activity 5.9

Halai Secondary School arranged a table tennis competition among 32 learners selected from various classes. The competition was based on a knockout competition. That is, each student continued competing until he or she lost. How many rounds of competitions were organised before a winner was declared?

They considered the equation $2^n = 32$, where n is the number of rounds.

How do you find the number of rounds of the competitions?

Example 5.3

The number of insects in a certain area triples every month. If there are 100 insects, about how long will it take for the number to grow to 8100?

Solution:

Let n be the required number of months.

Then,

$$100 \times 3^n = 8100$$

$$3^n = \frac{8100}{100} = 81 = 3^4$$

Therefore $n = 4$

The number of insects will be 8100 in 4 months.

Expressions such as $2^n = 2^5$ are called exponential equations.

Note: if $a^x = a^y$, then $x = y$.

Exercise 5.9

1. Solve: a) $5^x = 125$ b) $4^{(x-1)} = 8$
2. The number of ants in a nest doubles every month. If there are 300 ants now, about how long will it take for the number to grow to 4800?
3. Solve:
a) $9^x = 729$ b) $4^{(x+2)} = 16$
c) $10^x = 0.01$ d) $16^{(x-1)} = 64$
e) $9^{(1-2x)} = 81$ f) $3 \times 6^x = 108$
4. Solve:
a) $3^x + 3^{(x+1)} = 36$ b) $\left(\frac{3}{5}\right)^x = \frac{27}{125}$ c) $5^x = \frac{1}{625}$
5. The number of bugs in a house triples every month. If there are 20 bugs now, how long does it take for their number to grow to 540?

6. The number of insects in a colony doubles every month. If there are 50 insects now, about how long will it take for the colony to grow to 1600?

5.2 Logarithmic Expressions

5.2.1 Logarithmic Notation

Activity 5.10

1. a) Do you remember how to express $5^x = 125$ in logarithmic form?
b) Do you remember how to express $y = \log_3 27$ in exponential form?
2. a) Write $x = a^y$ in logarithmic form.

<u>Exponential form</u>	<u>Logarithmic form</u>
$x = a^y$ We say, a raised to the exponent y is equal to x.	$y = \log_a x$, $a > 0$, and $a \neq 1$ We say, y is the exponent to which you raise the base a to get the answer x.

Exercise 5.10

1. Express each logarithmic form in exponential form.

a) $\log_3 27 = 3$
b) $\log_{64} 8 = \frac{1}{2}$
c) $\log_{10} 0.001 = -3$
2. Express each of the following in logarithmic form.

a) $5^3 = 125$
b) $64^{\frac{1}{2}} = 8$
c) $a^6 = k$
3. Applying the definition, find the value of:

a) $\log_6 36$
b) $\log_3 81$
c) $\log_{10} 1000$

d) $\log_a(a^3)$

e) $\log_5 625$

f) $\log_9 81$

g) $\log_7(7^7)$

h) $\log_{25} 625$

i) $\log_7 343$

4. Write each of the following in logarithmic form.

a) $16 = 2^4$

b) $a = 10^k$

c) $125 = 5^3$

d) $0.001 = 10^{-3}$

e) $\frac{1}{27} = 3^{-3}$

f) $0.04 = 5^{-2}$

5. Evaluate each of the following:

a) $y = \log_7 2401$

b) $y = \log_a a^a$

c) $y = \log_{10} 0.001$

d) $y = \log_7 49$

e) $y = \log_5 0.04$

f) $y = \log_2 0.25$

5.2.2 Laws of Logarithms

Activity 5.11

1. Find:

a) $\log_{10} 100 + \log_{10} 1000$ and $\log_{10} 100 \times 1000$

b) $\log_2 16 + \log_2 8$ and $\log_2 16 \times 8$

c) $\log_3 27 + \log_3 9$ and $\log_3 27 \times 9$

2. From your answers in the above (1(a)–(c)), suggest a possible simplification for $\log_a m + \log_a n$?

For any positive numbers m and n ,

$$\log_a (m \times n) = \log_a m + \log_a n$$

Exercise 5.11

Find the value of:

a) $\log_3 54 + \log_3 \left(\frac{3}{2}\right)$

b) $\log_{10} 40 + \log_{10} \left(\frac{10}{4}\right)$

c) $\log_2 \left(\frac{4}{5}\right) + \log_2 \left(\frac{5}{2}\right)$

d) $\log_5 20 + \log_5 \left(\frac{25}{4}\right)$

e) $\log_5 50 + \log_5 2.5$

f) $\log_2 \left(\frac{8}{3}\right) + \log_2 12$

Activity 5.12

A teacher gave his learners the following activity that dealt with division so that they could investigate the law of division of logarithms. Find the value of:

1. a) $\log_3 \left(\frac{81}{9}\right)$ and $\log_3 81 - \log_3 9$

b) $\log_{10} \left(\frac{1000}{100}\right)$ and $\log_{10} 1000 - \log_{10} 100$

c) $\log_2 \left(\frac{16}{2}\right)$ and $\log_2 16 - \log_2 2$

2. What relation do you observe from these activities?

For any positive numbers m and n ,

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Exercise 5.12

Find the value of:

a) $\log_2 144 - \log_2 9$

b) $\log_7 98 - \log_7 2$

c) $\log_5 200 - \log_5 40$

d) $\log_3 135 - \log_3 5$

e) $\log_2 40 - \log_2 10$

f) $\log_3 324 - \log_3 2$

g) $\log_7\left(\frac{49}{2}\right) - \log_7\left(\frac{7}{2}\right)$

h) $\log_6\left(\frac{17}{2}\right) - \log_6\left(\frac{17}{2}\right)$

i) $\log_3\left(\frac{243}{2}\right) - \log_3\left(\frac{3}{2}\right)$

j) $\log_8\left(\frac{128}{5}\right) - \log_8\left(\frac{2}{5}\right)$

Activity 5.13

Compare the values of the following pair of expressions:

1. a) $3 \times \log_2 4$ and $\log_2 4^3$

b) $2 \times \log_3 9$ and $\log_3 9^2$

c) $2 \times \log_5 25$ and $\log_5 25^2$

2. What did you notice from these activities?

If m is any positive number and n is any real number,
then: $\log_a(m^n) = n \log_a m$

Exercise 5.13

Find the value of:

a) $\log_3 \sqrt{27}$

b) $\log_{10} \sqrt[3]{100}$

c) $\log_{13} \sqrt[5]{13}$

d) $\log_2 \sqrt[4]{8}$

e) $\log_2 \sqrt[5]{16}$

f) $\log_3 \sqrt[5]{81}$

g) $\log_3 (27)^{\frac{1}{2}}$

h) $\log_5 \left(\frac{1}{25}\right)$

i) $\log_3 9^3$

5.2.3 Solving Simple Logarithmic Equations

Activity 5.14

- a) How do you solve for y if $y = \log_5 25$?
- b) Is it a useful approach to express it using the exponential form?

Example 5.4

1. Solve: $\log_2 k = 5$

2. Solve: $\log_9 3 = m$

Solution:

1. $\log_2 k = 5$

Then, $k = 2^5$

$k = 32$

2. $\log_9 3 = m$

Then, $(3^2)^m = 3$

$3^{2m} = 3^1$

$2m = 1$

$m = \frac{1}{2}$

Exercise 5.14

1. Solve:

a) $\log_3 x = \log_3 12 + \log_3 4$

b) $\log_6 (x - 3) = 2$

2. Solve:

a) $\log_5 x = 6$

b) $\log_2 72 = \log_2 x + \log_2 12$

5.2.4 Common Logarithms

Activity 5.15

- a) Consider the number 276,000. Do you remember that this number can be written as $2.76 \times 10,000 = 2.76 \times 10^5$? We have expressed 276,000 as a product of 2 numbers.
- b) Do you also observe that $0.000521 = 5.21 \times 10^{-4}$?
- c) If the number of molecules in a certain quantity of gas is 602,500,000,000,000,000,000,000, can you see that this number could be written as 6.025×10^{26} ?
- d) Consider other similar numbers. Write their scientific notations.

The method of writing a given number as the product of 2 factors, 1 of which is greater than or equal to 1 but less than 10, and the other is a power of 10, is called the **scientific notation**.

Exercise 5.15

1. Write in scientific notations:
 - a) The distance between the sun and the Earth, which is 149, 600, 000,000m.
 - b) The speed of light, which is 300,000,000m per second.
2. Write the following in scientific notations;
 - a) 296
 - b) 0.002356
 - c) 3451
 - d) 0.572
 - e) 0.0008793

Do you observe that any decimal numeral N can be written as $N = n \times 10^m$, where $1 \leq n < 10$ and m is an integer?
3. Express the following using scientific notations;
 - a) 5854 b) 27 c) 0.986 d) 0.0000382

4. From the definition of logarithm;

- a) What is $\log_{10} 1$?
- b) What is $\log_{10} 10$?

Activity 5.16

- a) Do you know that $\log_{10} x$ is called a common logarithm?
- b) How can we find the logarithm to the base 10 of a given number?

Logarithms to the base 10 have a nice property that makes it possible to obtain all the desired logarithms from a fairly small list. $\log_{10} x$ is denoted by $\log x$.

Example 5.5

- 1. Find: $\log_{10} 359$
- 2. Find: $\log_{10} 0.0643$

Solution:

$$\begin{aligned}
 1. \log_{10} 359 &= \log_{10} (3.59 \times 10^2) \\
 &= \log_{10} 3.59 + \log_{10} 10^2 \\
 &\approx 0.5551 + 2
 \end{aligned}$$



Mantissa



Characteristic

The integral part is called the **characteristic** and the decimal part is called the **mantissa**.

The common logarithm of a number = Mantissa + Characteristic.

$$\begin{aligned}
 2. \log_{10} 0.0643 &= \log_{10} (6.43 \times 10^{-2}) \\
 &= \log_{10} 6.43 + \log_{10} (10)^{-2} \\
 &\approx 0.8082 + (-2)
 \end{aligned}$$

The characteristic is -2 and the mantissa is 0.8082.

Exercise 5.16

1. From the table of logarithms, find the following logarithms.

a) $\log_{10} 7.85$

b) $\log_{10} 6$

c) $\log_{10} 2.06$

d) $\log_{10} 2.25$

e) $\log_{10} 3.01$

f) $\log_{10} 5.69$

2. Find the following logarithms.

a) $\log_{10} 238$

b) $\log_{10} 2500$

c) $\log_{10} 0.0086$

d) $\log_{10} 0.00000873$

5.2.5 Antilogarithms**Activity 5.17**

1. If, $\log_{10} x = 0.4409$ then $x = 10^{0.4409}$.

Find 4409 inside the table of logarithms as shown below.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
...										
26	4150	4166	4182	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757

In the table of logarithms, 2.76 corresponds to 0.4409.

Therefore, $\log 2.76 \approx 0.4409$.

Can you find the antilogarithm of 0.4594 from the table?

If the logarithm of a number is given, the process of finding the number is called **antilogarithm**.

2. Amina and Rahwa wanted to calculate using logarithms.

They assumed that $x = 2680 \times 81.6$. Then they applied logarithm on both sides.

$$\begin{aligned}
 \text{That is: } \log x &= \log (2680 \times 81.6) \\
 &= \log (2.680 \times 10^3 \times 8.16 \times 10^1) \\
 &= \log 2.680 + \log 10^3 + \log 8.16 + \log 10^1 \\
 &\approx 0.4281 + 3 + 0.9117 + 1
 \end{aligned}$$

For ease of calculation, they found it convenient to arrange the calculations vertically.

$$\begin{array}{rcl}
 \log 2860 & \approx & 0.4281 + 3 \\
 + \log 81.6 & \approx & 0.9117 + 1 \\
 \hline
 \log x & \approx & 1.3398 + 4 \\
 \log x & \approx & 0.3398 + 5
 \end{array}$$

$$\text{Thus } x = 10^{0.3398} \times 10^5$$

The second factor is 100,000. Check that the antilogarithm of 0.3398 is approximately 2.19. In the table of logarithms, 3404 is closer to 3398 than it is to 3385. Hence, we take 2.19 which corresponds to 3404.

$$\text{Therefore, } 2680 \times 81.6 \approx 219,000.$$

Apply the same method to calculate 157×234.5

Note: that if $\log_{10} N = x$, then $N = \text{anti log } x$.

Example 5.6

1. If, $\log_{10} x = 2.4969$ find x .

Solution

$$\begin{aligned}
 \text{If, } \log_{10} x &= 2.4969 \text{ then } x = 10^{2.4969} \\
 &= 10^{(2+0.4969)} \\
 &= 10^2 \times 10^{0.4969}
 \end{aligned}$$

$$\text{Therefore, } x = 100 \times 3.14 = 314.$$

What is $\log_{10} 314$? Check.

2. Evaluate: $864 \div 0.745$

Solution

$$\text{Let } x = \frac{864}{0.745}$$

$$\log x = \log \left(\frac{864}{0.745} \right)$$

$$= \log 864 - \log 0.745$$

$$= \log 8.64 \times 10^2 - \log 7.45 \times 10^{-1}$$

$$= \log 8.64 + 2 - [\log 7.45 + (-1)]$$

$$= 0.9365 + 2 - [0.8722 + (-1)]$$

$$= 0.0643 + 3$$

$$x \approx \text{antilog } (0.0643 + 3)$$

$$\approx 1.16 \times 1000$$

$$\text{Therefore, } x \approx 1160$$

Exercise 5.17

1. Find the antilogarithm of the following.

a) 7

b) 0.7126

c) -3

d) 0

2. Calculate using logarithms.

a) 2760×84.5

b) $(54.2)^5$

c) $123 \div 45.2$

d) $243 \div 0.618$

3. Using logarithms calculate $\sqrt{7}$

Review Exercise 5

1. If a ball is dropped from a height of 4m and allowed to bounce freely, the height h in metres, to which it bounces is given by the formula $h = 4 \times (0.9)^n$, where n is the number of bounces. What height does the ball reach after 2 bounces?

After 3 bounces?

1. Express the following as a power.

a) $\sqrt[10]{15^7}$

b) $\sqrt[3]{(-5)^7}$

c) $\sqrt[4]{26^5}$

2. Simplify:

a) $7^{\frac{3}{4}} \times 9^{-\frac{1}{3}}$

b) $25^{\frac{3}{5}} \times 50^{-\frac{3}{5}}$

3. Solve:

a) $3^x = 243$

b) $2 \times 3^x = 162$

c) $7^{(2x-1)} = 1$

d) $3 \times 5^x = 75$

e) $5 \times 2^x = 80$

f) $9^x = \frac{1}{3}$

4. Evaluate:

a) $\log_5 125$

b) $\log_3 3^{15}$

c) $\log_{27} \sqrt{3}$

5. If $\log_5 m = 3$, find $\log_5 (25 \times m^2)$.

6. Calculate using logarithms:

a) 371×0.678

b) $\frac{265}{568}$

c) $(3.81)^9$

d) $\sqrt{13}$

7. Express each of the following as a sum of logarithms.

a) $\log_5 (7 \times 16)$

b) $\log_k xy$

8. Express each of the following as the logarithm of a product.

a) $\log_7 19 + \log_7 20$

b) $\log_a a^2 + \log_a k$

9. Express each of the following as the logarithm of a quotient.

a) $\log_7 19 - \log_7 20$

b) $\log_a mn - \log_a b$

10. Express each of the following as a difference of logarithms.

a) $\log_8 \left(\frac{27}{16} \right)$

b) $\log_5 (23 \div 6)$

11. Evaluate:

a) $\log_{10} 20 + \log_{10} \left(\frac{10}{2} \right)$

b) $\log_2 40 - \log_2 5$

c) $\log_3 \sqrt{27}$

d) $\log_2 8^3$

12. Solve:

a) $\log_2 k = 5$

b) $\log_9 3 = m$

13. Solve:

a) $\log_8 5 = \log_8 x - \log_8 55$

b) $4 \times \log_8 x = \log_8 16$

c) $\log_{10} x = 4 \times \log_{10} 2$

Logarithms of numbers

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

CHAPTER 6

Right Angled Triangle and Trigonometric Ratios

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- identify a right-angled triangle given the measures of the sides of a triangle;
- use the Pythagoras' Theorem to solve real-world problems;
- read trigonometric ratios of different angles from a table of trigonometric ratios;
- use trigonometric ratios to solve many real-world problems; and
- apply the laws of sines and cosines in different fields of study.

6.1 Pythagoras' Theorem

Activity 6.1

Consider the right-angled triangle in Fig. 6.1.

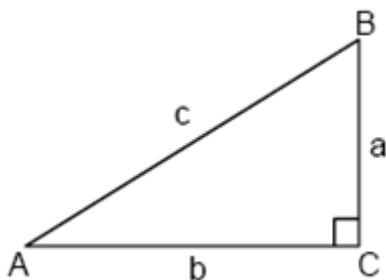


Fig. 6.1

- Identify the hypotenuse and the legs of $\triangle ABC$.
- What does the Pythagoras' Theorem state about the sides of $\triangle ABC$?

- c) For each triangle in Fig. 6.2, find the length of the side labelled x .

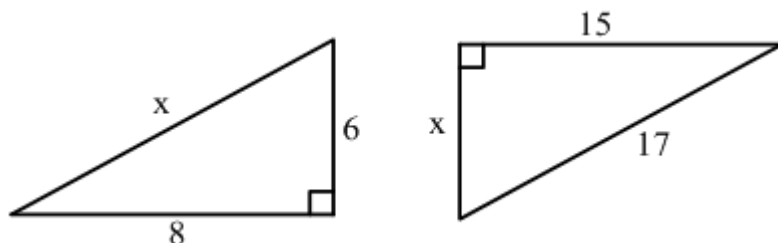


Fig. 6.2

Pythagoras' Theorem

In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs.

That is, for the triangle in Fig. 6.1, we have $c^2 = a^2 + b^2$.

Example 6.1

In a triangle, the lengths of the sides are 5, 12 and 13. Is this a right-angled triangle?

Solution:

Let $c = 13$, the longest side. Let $a = 5$ and $b = 12$. Then,

$$c^2 = 13^2 = 169, \text{ and}$$

$$a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$$

Since $c^2 = a^2 + b^2$, by the Pythagoras' Theorem, this triangle is a right-angled triangle.

Example 6.2

In a triangle, the lengths of the sides are 7, 8 and 12. Is this a right-angled triangle?

Solution:

Let $c = 12$, the longest side. Let $a = 8$ and $b = 7$. Then,

$$c^2 = 12^2 = 144, \text{ and}$$

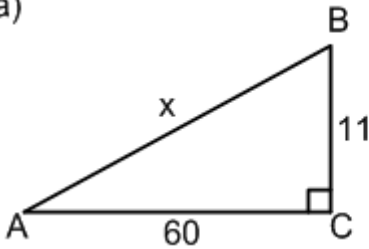
$$a^2 + b^2 = 8^2 + 7^2 = 64 + 49 = 113$$

Since $c^2 \neq a^2 + b^2$, by the Pythagoras' Theorem, this triangle is not a right-angled triangle.

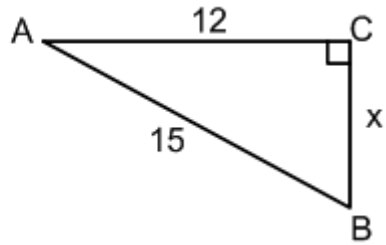
Exercise 6.1

1. In each of the following right-angled triangles, determine the length of the side labelled x .

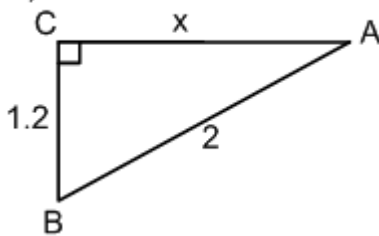
a)



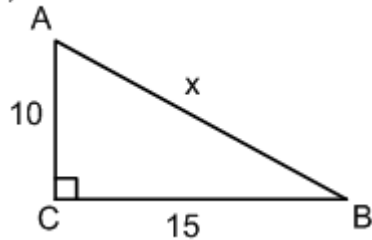
c)



b)



d)



2. The lengths of the three sides of a triangle are given. Determine whether the triangle is a right-angled triangle or not.

a) 7, 24, 25

e) 4, 8, $4\sqrt{3}$

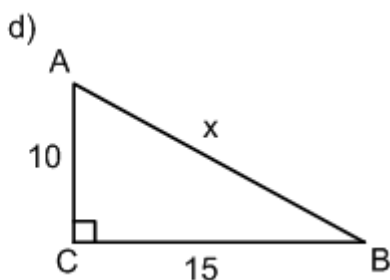
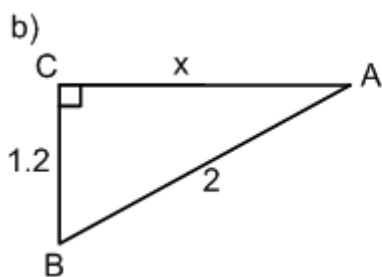
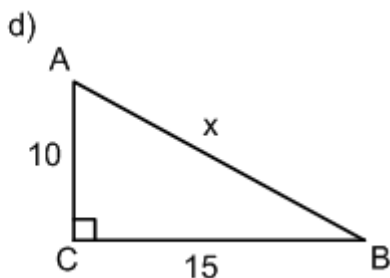
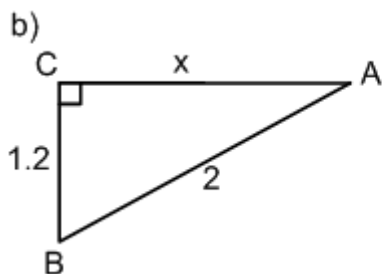
b) 5, 7, 8

f) $\frac{5}{3}$, $\frac{8}{3}$, $\frac{10}{3}$

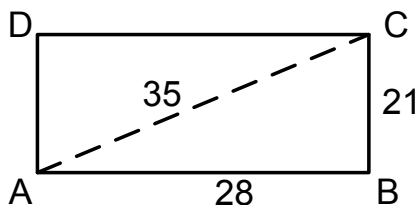
c) 10, 15, 20

d) 36, 48, 60

3. Determine whether each of the following triangles is a right-angled triangle.



4. The lengths of two adjacent sides of a parallelogram are 21 cm and 28 cm (See figure). If the length of a diagonal of the parallelogram is 35 cm, is the parallelogram a rectangle?



6.2 Solving Problems using Pythagoras' Theorem

Activity 6.2

In a plot of land, make a garden area in the shape of a right-angled triangle of legs 6m and 8m. How long is the third side of the triangular plot of land?

Example 6.3

A small aeroplane takes off from an airport and rises uniformly. After it has travelled over a horizontal distance of 400m, its altitude was found to be 300m. How far did the aeroplane fly?

Solution:

Kedija solved the problem as follows. First, she represented the positions of the airport by the letter A, the plane by P, and the foot of the altitude from the aeroplane to the ground by G. See Fig. 6.3.

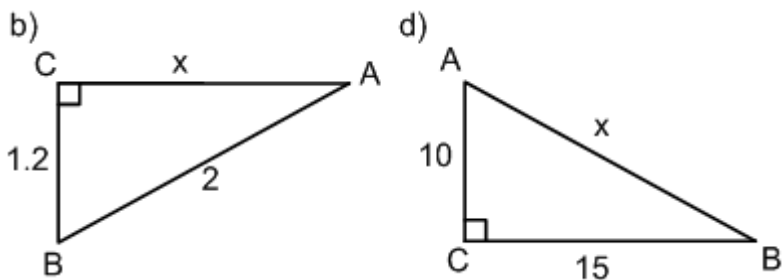


Fig. 6.3

Then she joined the points by straight lines and obtained the right-angled triangle APG. Using the Pythagoras' Theorem, she determined the distance g the plane flew as follows.

$$g^2 = p^2 + a^2 \quad \text{Pythagoras' Theorem on } \triangle APG$$

$$g^2 = 400^2 + 300^2 \quad \text{Substituting known values}$$

$$g^2 = 250000 \quad \text{Simplifying the RHS}$$

$$g = 500 \quad \text{Solving for g}$$

Then, she concluded that the aeroplane flew 500m.

Example 6.4

Dehab wants to purchase a 30-inch television. She knows that television screens are measured across the diagonal. She also knows that this screen is 18 inches high. How wide is the screen horizontally?

Solution:

Her son Berhane solved the problem as follows. He draws a television and levels its edges as shown in Fig. 6.4. He thought that the screen of a television is rectangular. Hence, the diagonal, the base and the left side of the screen form a right-angled triangle as indicated in the figure.

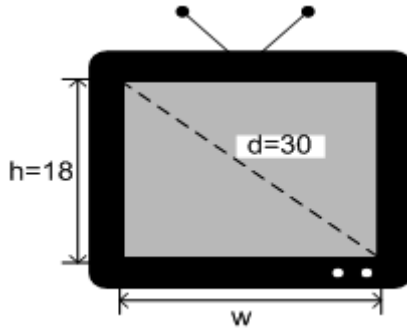


Fig. 6.4

Then, using the Pythagoras' Theorem, he determined the width w of the television as follows.

$d^2 = h^2 + w^2$	Pythagoras' Theorem
$30^2 = 18^2 + w^2$	Substituting known values
$900 = 324 + w^2$	Simplifying both sides
$w = 24$	Solving for w

That is, the television is 24 inches wide.

Exercise 6.2

1. Find the length of a rectangle that has a diagonal of 25cm and a width of 15cm.
2. A football field is a rectangle 90m wide and 120m long. During a training session the coach asks players to run from one corner to the corner diagonally opposite. What is this distance?
3. There is a building with a 12m high window. Yemane wants to use a ladder to climb up to the window. He decides to keep the ladder 5 metres away from the building so that it is at a safe angle. How long should the ladder be?
4. How far up a wall will an 11m ladder reach, if the foot of the ladder is 4m from the base of the wall?
5. Ghenet rides her bicycle 9 km south and then 12 km east. How far is she from her starting point?

6.3 Trigonometric Ratios and Their Applications

6.3.1 Table of Trigonometric Ratios

Activity 6.3

Consider the right-angled triangle in Fig. 6.5.

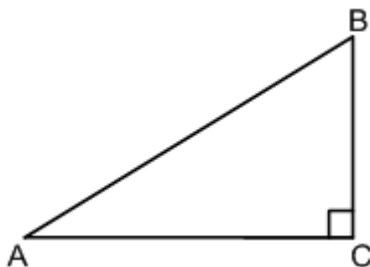


Fig. 6.5

- Which side is the hypotenuse of $\triangle ABC$?
- Which side is opposite to angle A and which one is adjacent to angle A?
- Which side is opposite to angle B and which one is adjacent to angle B?
- What is sine of angle A ($\sin A$)?
- What is cosine of angle A ($\cos A$)?
- What is tangent of angle A ($\tan A$)?

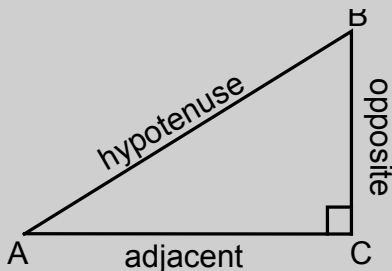
Trigonometric ratios provide relationships between the sides and angles of a right-angled triangle.

If $\triangle ABC$ is a right-angled triangle with $\angle C = 90^\circ$, then we define the trigonometric ratios of angle A as :

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}$$

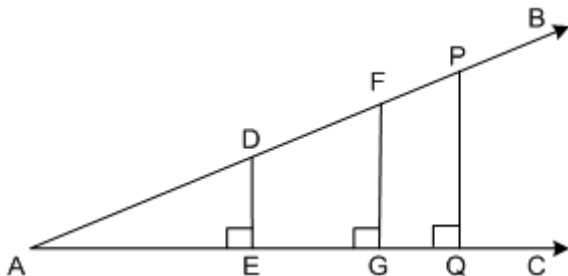
$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}$$



Activity 6.4

In Fig. 6.6, $\overline{AB}$ and $\overline{AC}$ are rays that make an acute angle .
 $\overline{DE}$, $\overline{FG}$ and $\overline{PQ}$ are any 3 line segments perpendicular to $\overline{AC}$ such that $\triangle ADE$, $\triangle AFG$ and $\triangle APQ$ are right-angled triangles.

**Fig. 6.6**

- Are $\triangle ADE$, $\triangle AFG$ and $\triangle APQ$ similar? Why?
- Is ? Why?
- What is $\sin A$ in $\triangle ADE$? What about in $\triangle AFG$? What about in $\triangle APQ$?
- Are the values of $\sin A$ the same for all three triangles? Why?
- What can you conclude about $\sin A$?

Each trigonometric ratio depends upon the measure of an angle, not upon the size of the right-angled triangle that contains the angle. As the measure of the angle changes, each trigonometric ratio of the angle also changes.

Activity 6.5

In Grade 8 Mathematics, you calculated the trigonometric ratios of 30° , 45° and 60° angles. In this activity, you are going to calculate the trigonometric ratios of 25° .

- What are the trigonometric ratios of 30° , 45° and 60° ?
- On your notebook, draw a horizontal line segment $\overline{AC}$ and measure its length using a ruler.
- Draw a vertical ray $\overline{CD}$.
- Use your protractor to draw $\angle CAE$ such that $m(\angle CAE) = 25^\circ$.
- Extend rays $\overline{CD}$ and $\overline{AE}$ until they intersect each other. Let B be the point of intersection.
- Is $\triangle ABC$ a right-angled triangle?

- g) Measure the lengths of $\overline{AB}$ and $\overline{BC}$ using a ruler.
- h) Find $\sin A = \sin 25^\circ$, $\cos A = \cos 25^\circ$ and $\tan A = \tan 25^\circ$ approximated to 4 decimal places.
- i) Compare your results with the results obtained by others. Did you get the same ratios?

As demonstrated in Activity 6.5, trigonometric ratios can be calculated for any acute angle. Mathematicians have found it useful to list these trigonometric ratios in tabular form. This list is called **a table of trigonometric ratios**. The a table is at the end of this chapter. This table approximates the trigonometric ratios of acute angles to four decimal places.

Example 6.5

Use the table of trigonometric ratios to find $\sin 12^\circ$, $\cos 45^\circ$ and $\tan 70^\circ$.

Solution:

Berhe found the trigonometric ratios as follows. To find $\sin 12^\circ$:

- a) First, he found 12° under the Angle column of the table of trigonometric ratios.
- b) Next, he moved horizontally to the right until he reached the sin column as shown in Fig. 6.7.
- c) Then, he concluded that $\sin 12^\circ = 0.2079$.

Angle	sin	cos	tan
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9963	0.0874
6	0.1042	0.9945	0.1045
7	0.1217	0.9923	0.1220
8	0.1392	0.9896	0.1396
9	0.1564	0.9864	0.1570
10	0.1736	0.9828	0.1763
11	0.1908	0.9789	0.1944
12	0.2079	0.9748	0.2126
13	0.2250	0.9704	0.2309

Fig. 6.7

Similarly, he found that $\cos 45^\circ = 0.7071$ and $\tan 70^\circ = 2.7475$.

Example 6.6

Use the table of trigonometric ratios to find the angles x , y and z such that $\tan x = 28.6363$, $\cos y = 0.8090$ and $\sin y = 0.7660$.

Solution:

Saedia found the angles as follows. To find x such that $\tan x = 28.6363$:

- First, she found 28.6363 under the tan column of the table of trigonometric ratios.
- Next, she moved horizontally to the left until she reached the Angle column as shown in Fig. 6.8.
- Then, she concluded that $x = 88^\circ$.

Angle	sin	cos	tan
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1096
49	0.7547	0.6561	1.1470
50	0.7660	0.6428	1.1844
51	0.7771	0.6293	1.2219
52	0.7880	0.6157	1.2596
53	0.7986	0.6020	1.2974
54	0.8090	0.5882	1.3354
55	0.8191	0.5743	1.3735
56	0.8290	0.5603	1.4117
57	0.8387	0.5462	1.4500
58	0.8482	0.5320	1.4884
59	0.8575	0.5177	1.5269
60	0.8667	0.5033	1.5656
61	0.8757	0.4888	1.6044
62	0.8845	0.4742	1.6433
63	0.8932	0.4595	1.6824
64	0.9018	0.4447	1.7216
65	0.9102	0.4298	1.7610
66	0.9184	0.4148	1.8006
67	0.9265	0.3997	1.8403
68	0.9344	0.3845	1.8802
69	0.9421	0.3692	1.9203
70	0.9497	0.3537	1.9605
71	0.9572	0.3381	2.0009
72	0.9645	0.3224	2.0415
73	0.9717	0.3066	2.0823
74	0.9788	0.2907	2.1232
75	0.9857	0.2747	2.1643
76	0.9925	0.2586	2.2056
77	0.9991	0.2424	2.2470
78	1.0000	0.2262	2.2887
79	1.0000	0.2099	2.3305
80	1.0000	0.1936	2.3726
81	1.0000	0.1772	2.4148
82	1.0000	0.1607	2.4572
83	1.0000	0.1442	2.5000
84	1.0000	0.1276	2.5429
85	1.0000	0.1109	2.5861
86	1.0000	0.0942	2.6295
87	1.0000	0.0774	2.6732
88	1.0000	0.0606	28.6363
89	1.0000	0.0438	57.2900
90	1.0000	0.0270	114.5800
Angle	sin	cos	tan

Fig. 6.8

Similarly, she found that $y = 36^\circ$ and $z = 50^\circ$.

Example 6.7

An 8m ladder leans against a vertical wall. It makes an angle of 60° with the ground. How high up the wall does the ladder reach?

Solution:

Note: That the ladder, the wall and the ground form a right-angled triangle. Label the sides and vertices of this triangle as shown in Fig. 6.9.

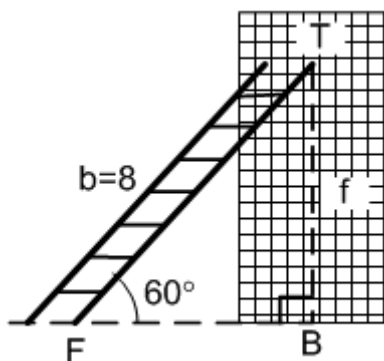


Fig. 6.9

Then, the problem is solved as follows:

$$\sin F = \frac{f}{b} \quad \text{Definition of sine ratio}$$

$$\sin 60^\circ = \frac{f}{8} \quad \text{Substituting known values}$$

$$f = 8 \sin 60^\circ \quad \text{Isolating the unknown } f$$

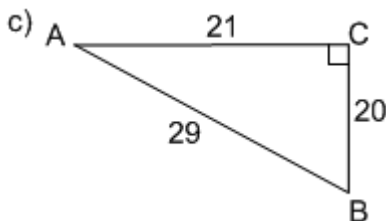
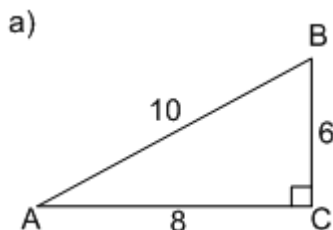
$$f = 8 \times 0.8660 \quad \text{From the table of trigonometric ratios}$$

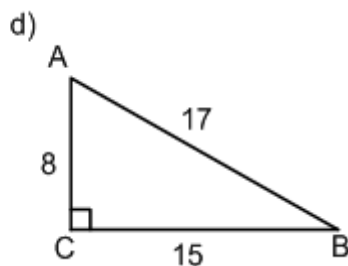
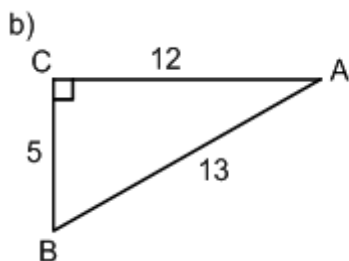
$$f \approx 6.9 \quad \text{Simplifying the RHS}$$

Thus, the ladder will reach about 6.9m up the wall.

Exercise 6.3

- In each of the following triangles, find the trigonometric ratios of angles A and B.





2. Use the table of trigonometric ratios to find each of the following.
3.

a) $\sin 18^\circ$	f) $\tan 45^\circ$
b) $\sin 1^\circ$	g) x if $\sin x = 0.9703$
c) $\cos 21^\circ$	h) y if $\cos y = 0.5150$
d) $\cos 75^\circ$	i) z if $\tan z = 0.2679$
e) $\tan 58^\circ$	j) w if $\sin w = \cos w$
4. The diagonal of a rectangle makes an angle of 39° with its longest side whose length is 30cm. Find the width of the rectangle.
5. A hill slopes upwards at an angle of 12° to the horizontal ground. How high does a person rise when the person travels 90m up the hill?

6.3.2 Angle of Elevation and Angle of Depression

Activity 6.6

Suppose Fatima is standing some metres away from the flagpole of her school on the horizontal ground looking straight at the pole.

- a) If she wants to see the top of the flagpole, what does she need to do?
- b) If she wants to see the foot of the flagpole, what does she need to do?
- c) She is wondering how to determine the height of the flagpole. What suggestions do you have for her?

If an object is above the horizontal plane of an eye, then we have to raise our head to look at it. In this process, our eyes move upwards

at an angle. This angle is called the **angle of elevation**. In Fig. 6.10, angle E is the angle of elevation.

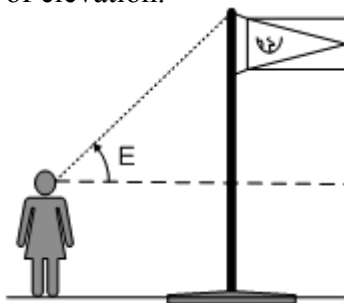


Fig. 6.10

If the object is below the horizontal plane of our eyes, we have to move our head downwards. Even by doing this, our eyes move making an angle called the **angle of depression**. In Fig. 6.11, angle D is the angle of depression. Both angles of elevation and depression are measured from the horizontal as shown in the figures.

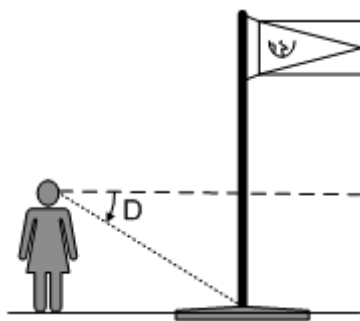


Fig. 6.11

Example 6.8

A rope supporting a flagpole is fixed to the ground 8m away from the pole forming an angle of elevation of 60° to the ground. Find the length of the rope.

Solution:

Note that the flagpole, the rope and the ground form a right-angled triangle. Label the sides and vertices of this triangle as shown in Fig. 6.12.

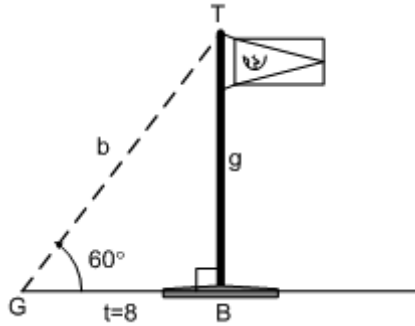


Fig. 6.12

Then, the problem is solved as follows:

$$\cos G = \frac{t}{b} \quad \text{Definition of cosine ratio}$$

$$\cos 60^\circ = \frac{8}{b} \quad \text{Substituting known values}$$

$$b = \frac{8}{\cos 60^\circ} \quad \text{Isolating the unknown } b$$

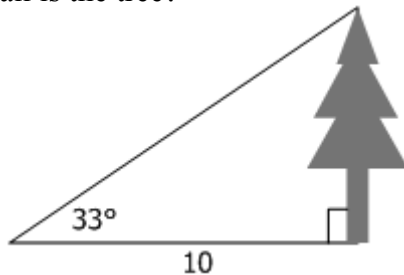
$$b = \frac{8}{0.5} \quad \text{From the table of trigonometric ratios}$$

$$b = 16 \quad \text{Simplifying the RHS}$$

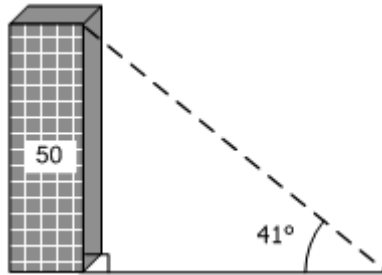
That is, the rope is 16 metres long.

Exercise 6.4

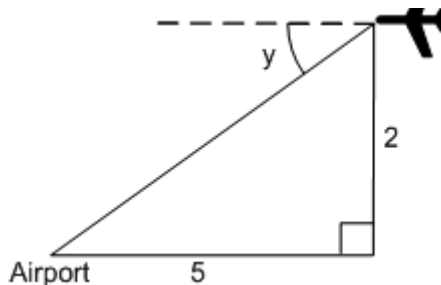
1. Tekle wants to measure the height of a tree. He walks exactly 10m away from the base of the tree and looks up. The angle of elevation from the ground to the top of the tree is 33° . See figure. How tall is the tree?



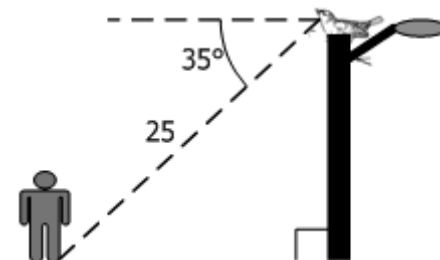
2. A building is 50m high. At a certain distance away from the building, an observer notices that the angle of elevation to the top of the building from the ground is 41° . See figure. How far is the observer from the base of the building?



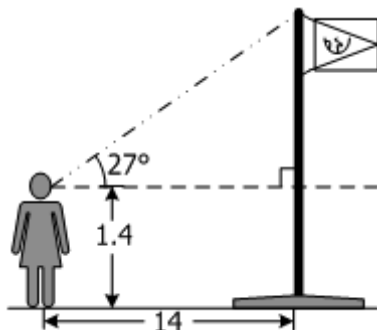
3. An aeroplane is flying at a height of 2km above the ground. The distance along the ground from the aeroplane to the airport is 5 km. See figure. What is the angle of depression from the aeroplane to the airport?



4. A bird sits on top of a lamppost. The angle of depression from the bird to the feet of an observer standing away from the lamppost is 35° . The distance from the bird to the observer is 25m. See figure. How high is the lamppost?



5. Jemila knows that when she stands 14m from the base of the flagpole of her school, the angle of elevation to the top of the flagpole is 27° (See figure). If her eyes are 1.4m from the ground, find the height of the flagpole.



6.3.3 The Law of Sines

Activity 6.7

Consider triangle ABC in Fig. 6.13.

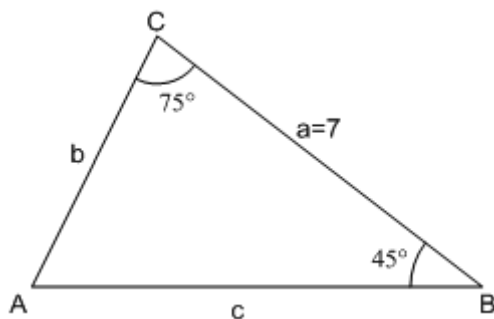


Fig. 6.13

- What is the measure of angle A?
- Is $\triangle ABC$ a right-angled triangle? Why?
- How would you determine the lengths of sides b and c?

Activity 6.8

Draw any acute triangle and name the vertices using the capital letters A, B and C, and the sides using the small letters a, b and c as shown in Fig. 6.14.

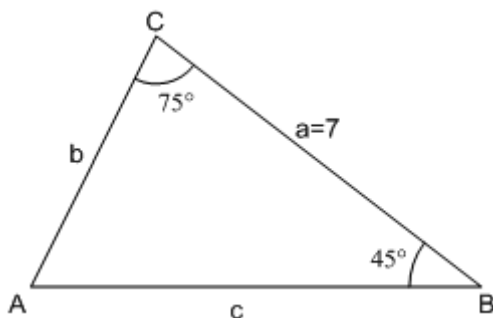


Fig. 6.14

- Use your protractor to measure the angles A, B and C of $\triangle ABC$.
- Find $\sin A$, $\sin B$ and $\sin C$ using the table of trigonometric ratios.
- Use your ruler to find the lengths of the sides a, b and c of $\triangle ABC$.
- Find $\frac{\sin A}{a}$, $\frac{\sin B}{b}$ and $\frac{\sin C}{c}$, and compare them.
- Repeat activities a) to d) for several triangles.
- What conclusion can you make about the values of $\frac{\sin A}{a}$, $\frac{\sin B}{b}$ and $\frac{\sin C}{c}$?

For any $\triangle ABC$, we have $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

This relationship is known as the **law of sines**.

The law of sines can be used to determine the measures of a triangle when the measure of one angle, its opposite side and any other measure of the triangle are known.

Example 6.9

Consider triangle ABC in Fig. 6.15. Find the lengths of sides b and c.

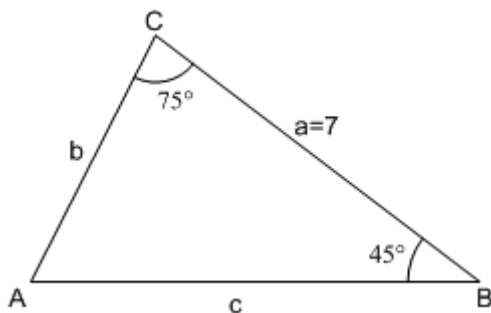


Fig. 6.15

Solution:

From Activity 6.7 a), recall that the measure of angle A is 60°. Then using the law of sines, sides b and c are found as follows.

To find b:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{The law of sines}$$

$$\frac{\sin 60^\circ}{7} = \frac{\sin 45^\circ}{b} \quad \text{Substituting known values}$$

$$b = \frac{7 \sin 45^\circ}{\sin 60^\circ} \quad \text{Isolating } b$$

$$b = \frac{7 \times 0.7071}{0.8660} \quad \text{From table of trigonometric ratios}$$

$$b \approx 5.7 \quad \text{Simplifying the RHS}$$

To find c:

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{The law of sines}$$

$$\frac{\sin 60^\circ}{7} = \frac{\sin 75^\circ}{c} \quad \text{Substituting known values}$$

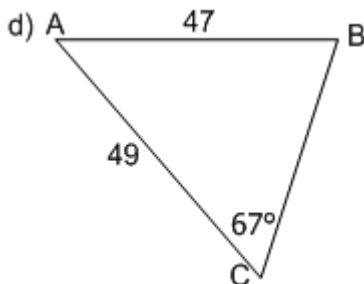
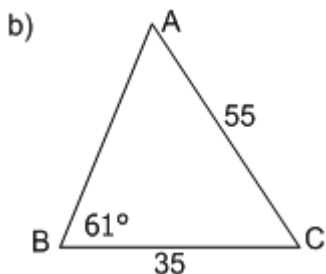
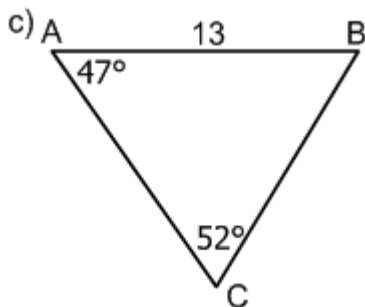
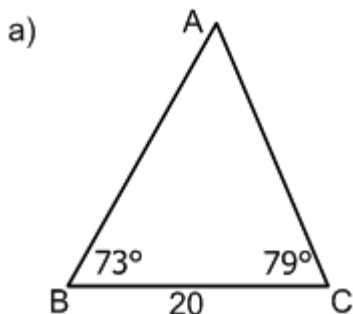
$$c = \frac{7 \sin 75^\circ}{\sin 60^\circ} \quad \text{Isolating c}$$

$$c = \frac{7 \times 0.9659}{0.8660} \quad \text{From table of trigonometric ratios}$$

$$b \approx 7.8 \quad \text{Simplifying the RHS}$$

Exercise 6.5

1. Find the missing parts of each of the following triangles.



- Suppose triangle ABC has angle $A = 71^\circ$, angle $B = 39^\circ$ and a side $a = 14$. Find the length of side b.
- Suppose triangle ABC has angle $A = 73^\circ$, angle $B = 28^\circ$, and a side $c = 14$. Find the length of side b.

6.3.4 The Law of Cosines

Activity 6.9

Draw any acute triangle and name the vertices using the capital letters A, B and C, and the sides using the small letters a, b and c as shown in Fig. 6.16.

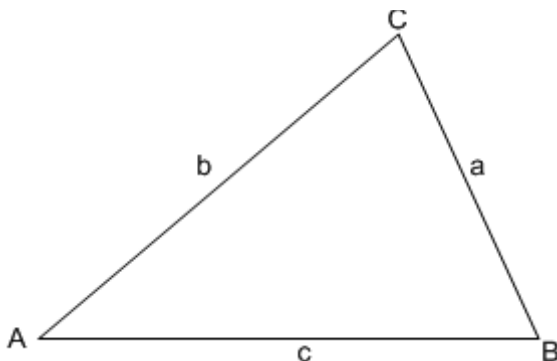


Fig. 6.16

- Use your protractor to measure the angles A, B and C of $\triangle ABC$.
- Find $\cos A$, $\cos B$ and $\cos C$ using the table of trigonometric ratios.
- Use your ruler to find the lengths of the sides a, b and c of $\triangle ABC$.
- Find a^2 , $b^2 + c^2$ and $2bc \cos A$. Can you find any equation relating these three values?
- Can you write a^2 in term of $b^2 + c^2$ and $2bc \cos A$?
- Repeat activities a) to e) for several triangles.

For any $\triangle ABC$, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

These relationships are known as the **law of cosines**.

The law of cosines can be used to determine the measures of a triangle when the lengths of two sides and the included angle, or the lengths of the three sides are known.

Example 6.10

Consider triangle ABC in Fig. 6.17. Find the length of side b.

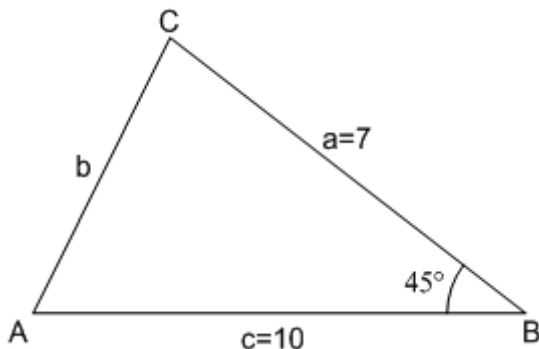


Fig. 6.17

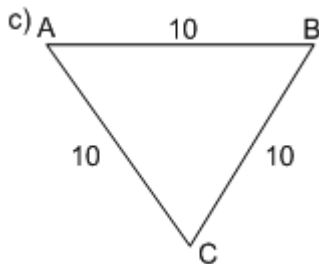
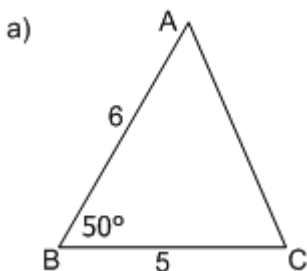
Solution:

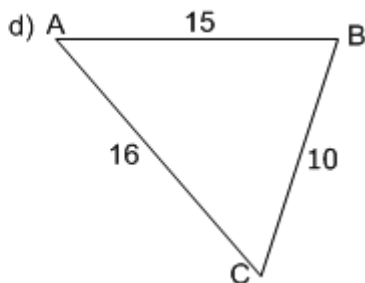
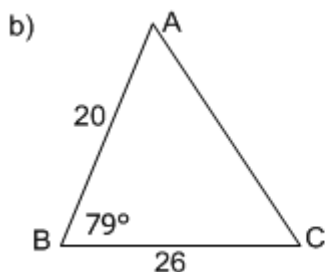
Using the law of cosines, the length of side b can be found as follows:

$b^2 = a^2 + c^2 - 2ac \cos B$	The law of cosines
$b^2 = 7^2 + 10^2 - 2(7)(10) \cos 45^\circ$	Substituting known values
$b^2 = 7^2 + 10^2 - 2(7)(10)(0.7071)$	From table of trig. ratios
$b^2 \approx 50$	Simplifying the RHS
$b \approx 7.1$	Solving for b

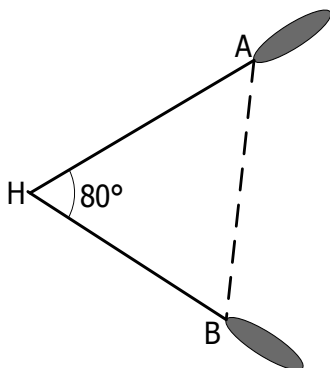
Exercise 6.6

1. Find the missing parts of each of the following triangles.





- Suppose triangle ABC has angle $A = 60^\circ$ and sides $b = 8$ cm and $c = 10$ cm. Find the length of side a .
- Suppose triangle PQR has angle $R = 14^\circ$ and sides $p = 5$ cm and $q = 10$ cm. Find the length of side r .
- Two ships leave a harbour (H) at the same time, travelling on routes that have an angle of 80° between them. See figure. The first ship (A) travels at 34 kilometres per hour and the second ship (B) travels at 26 kilometres per hour. How far apart are the two ships after 3 hours?

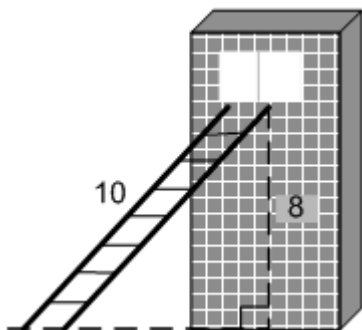


Review Exercise 6

- The lengths of the three sides of a triangle are given. Determine whether the triangle is a right-angled triangle.
 - 10, 24, 27
 - $4, \sqrt{20}, 6$
 - 0.75, 1, 1.25
 - 20, 24, 31
- In each of the following some parts of $\triangle ABC$ are given. Find the missing parts of the triangle.
 - $B = 12^\circ$, $C = 80^\circ$ and $b = 13$.
 - $A = 24^\circ$, $C = 90^\circ$ and $a = 8$.
 - $C = 90^\circ$, $a = 12$ and $b = 5$.
 - $a = 5$, $b = 8$ and $c = 7$.
 - $a = 8$, $b = 6$ and $c = 10$.
- The length of a rectangle is 7cm longer than its width. A

diagonal of the rectangle measures 17cm. Find the length and width of the rectangle.

4. A window-cleaner wishes to clean a window of a building. The window is 8m high off the ground. He has a ladder which is 10m long. See Figure.



- a) How far from the building he should place the foot of the ladder so that he can just to reach the window?
 - b) Later he learned that the safest position to put the foot of the ladder is 7m from the building. Can he still use the same ladder to reach the window?
 - c) If not, what should be the length of the new ladder he should look for?
5. An aeroplane is flying at an altitude of 5150m, directly above a straight highway. Two motorists are driving cars on the highway on opposite sides of the plane. The angle of depression to one car is 37° and to the other is 53° (See figure). How far apart are the cars?

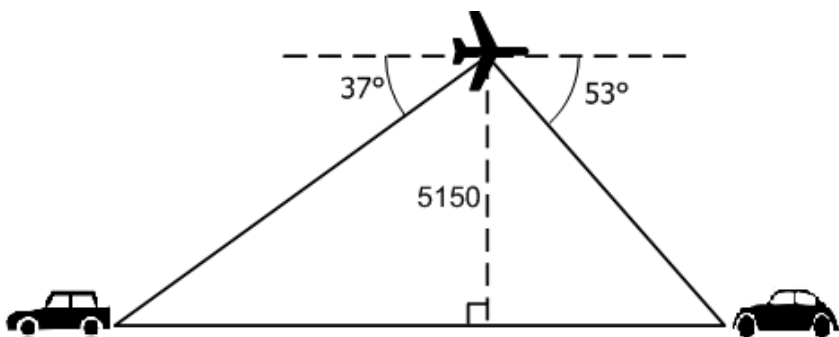


Table of Trigonometric Ratios

Angle	sin	cos	tan
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.3640
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.4040
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000
Angle	sin	cos	tan

Angle	sin	cos	tan
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900
Angle	sin	cos	tan

CHAPTER 7

Business Mathematics

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- express fractions and ratios in terms of percentages;
- apply fractions and ratios in businesses;
- Calculate profit and loss, commission and brokerage and simple interest;
- apply the concept of profit and loss in business transactions and investments
- investigate the concept of commission and brokerage in your daily life.

7.1 Percentages and Proportions

7.1.1 Percentages

Activity 7.1

1. Have you heard about percentages before? Give some examples of activities where you use percentages.
2. In a class of 50 learners, 40% of the learners are boys.
 - a) How many of the students are boys?
 - b) What fraction of the learners are boys?
 - c) If 90% of the students promoted to the next grade, how many learners didn't pass to the next grade?

Per cent (%) is a method of expressing parts of a whole, where the whole is divided into 100 parts.

For instance, 40% of learners are boys means, $\frac{40}{100}$ of the learners are boys.

The fraction $\frac{40}{100}$ can be reduced to its lowest form.

That is, $\frac{40}{100} = \frac{2 \times \cancel{2} \times \cancel{2} \times \cancel{5}}{\cancel{2} \times \cancel{2} \times 5 \times \cancel{5}} = \frac{2}{5}$.

The fraction $\frac{40}{100} = \frac{2 \times \cancel{2} \times \cancel{2} \times \cancel{5}}{\cancel{2} \times \cancel{2} \times 5 \times \cancel{5}} = \frac{2}{5} = 0.4$.

Hence, 40% of the learners are boys; is the same as to say two-fifths of the learners are boys or 0.4 are boys.

Exercise 7.1

- Change the following percentages (%) into simplest form of fraction and decimal forms.
 - 25%
 - 68%
 - 85%
 - 5%
 - 120%
 - 0.8%
- Change the following fractions and decimals to percentages.
 - $\frac{1}{5}$
 - 0.12
 - $\frac{22}{25}$
 - 0.74
 - $\frac{9}{20}$
 - 0.025
- Ibrahim's monthly salary is 2500 Nakfa. If his monthly salary increases by 10%, what is his new salary?
- Last year 1000 people were infected with malaria. If the infection rate decreases by 35% this year, how many people were infected by malaria this year?
- If a person invests (48,000.00 Nakfa) and expects a return (gain) of 9% at the end of the year, how much money will the person have by the end of the year?
- W/ro Abrhet received 10% increase in her salary. If her previous salary was 2000.00 Nakfa per month, what is her salary after the increase?

7. This year, the net sales of the Himbol Corporation were 11,565,000.00 Nakfa, while the net sales of last year were 10,452,000.00 Nakfa. What is the percentage increase of this year's sales relative to last year?
8. In a community library, 24% of the books are classified as fiction. If the total book collection comprises 34,600 books, how many of them are fiction books?
9. This month's sales of a company are 40,420.00 Nakfa. If there was a decrease of 6% from last month, what were last month's sales?
10. A car's selling price was originally 178,740 Nakfa. If the seller gave a discount of 8%, what is its current selling price?
11. Find the original price if a child's raincoat is offered at a reduced price of 125 Nakfa after discount of 20% off the original price.
12. A bicycle set originally priced at 1725 Nakfa is reduced to 1425 Nakfa. Find the rate of discount of the original price.

7.1.2 Ratio

Activity 7.2

Write the following using ratios, and express the ratio in lowest terms;

- | | |
|-------------------------------|-------------------------|
| a) 18 km to 64 km | c) 10 Nakfa to 50 cents |
| b) 216 students to 4 teachers | d) 20 hours to 5 days |

Example 7.1

A small shop owner made a profit of 8,000.00 Nakfa one year and 24,000.00 Nakfa the next year. Compare the two profits.

Solution:

a) A comparison using division is:

$$\frac{8000}{24000} = \frac{8 \times 1000}{8 \times 3 \times 1000} = \frac{1}{3}.$$

Then we can conclude that the shop owner gained 3 times as large as the first year in the second year.

b) Can you make a comparison using a different method to a)?

A **ratio** is a comparison between quantities that are usually measured in the same units and describes the number of times that one quantity contains another quantity.

For instance, in the above example, the gain in the second year is 3 times of the gain made in the first year; or the gain made in the first year is onethird of the gain made in the second year.

That is, 3:1 or 1:3.

Example 7.2

Three business partners want to divide their annual profit in the ratio of 3: 4 : 5. If their annual profit is 408,000.00 Nakfa, then find partner's share.

Solution:

The money to be divided in the ratio of 3: 4: 5 is 408,000, the sum of the ratios is $3 + 4 + 5 = 12$.

Hence;

$$\text{First share} = \frac{3}{12} \text{ of } 408,000 = \frac{3}{12} \times 408,000 = 102,000.00$$

$$\text{Second share} = \frac{4}{12} \text{ of } 408,000 = \frac{4}{12} \times 408,000 = 136,000.00$$

$$\text{Third share} = \frac{5}{12} \text{ of } 408,000 = \frac{5}{12} \times 408,000 = 170,000.00$$

The common ways to write the ratio $\frac{m}{n}$

where m and n are positive integers are:

$$m : n, \quad m \text{ to } n \quad \text{or} \quad \frac{m}{n}.$$

The sum of the ratios is $m + n$. $\frac{m}{n}$

Exercise 7.2

1. In a class of 80 learners, 30% of them are girls. Find:
 - a) The number of girls in the class.
 - b) The number of boys in the class.
 - c) The ratio of boys to girls in the class.
 - d) The ratio of girls to boys in the class.
 - e) The ratio of girls to whole learners in the class.
 - f) The ratio of boys to whole learners in the class.
2. Divide 670 in the ratio of 3:7.
3. Find the ratio of Genet's salary to Berhane's salary, if Genet's salary is 2750.00 Nakfa and Berhane's salary is 2250.00 Nakfa.
4. The Ministry of Agriculture wants to distribute 550 quintals of fertiliser to 3 villages in Zoba Debub in proportion to the number of farmers in each village. If the farmers of the 3 villages are 200, 500 and 600, respectively, find the number of quintals allotted to each village.
5. The scale of a plan is given in the ratio of 1: 2000. Find the actual length in metres of a road which measures 9cm on the plan.
6. Telephone and water bills together cost a company 5500 Nakfa for a month. If the telephone cost 4 times as much as the water, how much was the cost of each bill?

7.1.3 Proportions

Activity 7.3

In a class of 70 learners the ratio of boys to girls is 2 : 5.

In each group of 7 learners there can be 2 boys. The ratio is 2:5.

In each group of 14 learners there can be 4 boys. The ratio is 4:10.

In each group of 21 learners there can be 6 boys. The ratio is 6:15.

What can you say about the ratios, 2:5, 4:10 and 6:15?

It can be seen that 2:5, 4:10 and 6:15 describe the same ratio.

The ratios $a : b$ and $c : d$ form a proportion if and only if, $\frac{a}{b} = \frac{c}{d}$ which is the same to say that $a \cdot d = b \cdot c$, where $b, d \neq 0$.
Hence, proportion is equality of two ratios.

Example 7.3

Suppose 8 books (of the same copy) cost 560.00 Nakfa. What will be the cost of 17 books?

Solution:

We know that the cost of the 8 books = 560.

Let the cost of the 17 books = x .

Then, 8 : 560 should be the same as the ratio 17 : x

(8:560 = 17: x). Why?

Therefore, $\frac{8}{560} = \frac{17}{x}$ or $\frac{x}{17} = \frac{560}{8}$. Why?

$$x = \frac{560}{8} \times 17 = \frac{\cancel{8} \times 70}{\cancel{8}} \times 17 = 1190.00$$

That is, the price of 17 books is equal to 1190.00 Nakfa.

Exercise 7.3

1. Find all possible values of x and y , such that $3:5$, and $x:y$ are proportional ($3:5 = x:y$).
2. If $25:N$ and $5:3$ are proportional, then find N .
3. If a company with income of 42,000.00 Nakfa pays a tax of 6000.00 Nakfa, how much tax a company pay if its income is 63,000.00 Nakfa?
4. A car can travel 540 km with 45ℓ of benzene. How far can it travel with:
 - a) 60ℓ of benzene?
 - b) 20ℓ of benzene?
5. Seven quintals of fertiliser is needed for 3325m² of farm land. Find the number of quintals needed for 7125m² of farm land.

7.2 Commission and Brokerage

7.2.1 Commission

Activity 7.4

Try to find business agents in your town (such as lottery agents or news paper agents) and discuss with them how they work as agents.

Example 7.4

Gabriel sells instant lottery tickets, each ticket worth 5.00 Nakfa. For every ticket he sell he gets 50 cents commission. If he sells on average of 20 tickets a day, find

- a) His total sales in a month (one month = 30 days).
- b) His commission for a month.
- c) The rate of commission he receives in a month.

Solution:

- a) His daily sales = $20 \times 5 = 100$
His monthly sales = $30 \times 100 = 3000$

- b) His daily commission = $20 \times 0.5 = 10$
 His monthly commission = $30 \times 10 = 300$

- c) His monthly commission is 300 out of 3000.

Hence, the rate of commission = $\frac{300}{3000} \times 100 = 10\%$

The payment that an agent receives for the service he or she delivers is called commission.

The commission is generally fixed on a percentage basis; usually known as the **rate of commission**, and is calculated on the total value of the sales made by the agent.

$$\text{Rate of commission} = \frac{\text{Commission}}{\text{Total sales}} \times 100$$

Exercise 7.4

- Nasser receives a 4% commission on all sales. If he sells 120,000 Nakfa worth goods, how much commission will he receive?
- A salesperson received 540 Nakfa commission for selling 10,800 Nakfa. Find the rate of commission of his sale.
- A saleswoman is paid a 3400 Nakfa monthly salary plus an 8% commission on goods sold. If she received 4076 Nakfa last month, then find
 - The commission she received.
 - The amount of sales made.
- Semere is paid 225 Nakfa per week by a company, plus 3% commission on all sales. Find his total income for a week in which his sales were 7250 Nakfa.
- Trhas sells videotapes to rental shops (shops that rent videotapes), and is paid as follows :

Sales	Rate
Up to 10,000 Nakfa	6%
10,001 – 20,000	8%
20,001 and more	9%

Calculate her total income, if she makes video sales of 32,768 Nakfa.

7.2.2 Brokerage

Activity 7.5

1. Have you or your family ever looked for a house to rent?
2. Did you ask someone to help you find a house to rent?
3. Did he or she charge you for the service they offered you?
4. Did you contact a person when you wanted to buy or sell a property?

A **broker** is a person who facilitates the purchasing or the selling of properties or supplies the information needed in businesses. **Brokerage** is the amount of money paid to the broker. Brokerages are given usually in percentages.

Example 7.5

Kaltum sold her house to Senay for 652,000.00 Nakfa through a broker paying 1% brokerage. In addition, the broker charged Senay 0.5% for assisting him in the purchase of the house.

- a) Find the total amount of money the broker receives.
- b) How much did the buyer (Senay) pay in total for the house?
- c) How much did the seller receive (the net amount of money) for her house?

Solution:

$$\begin{aligned} \text{a) The broker gets } 1\% \text{ from Kaltum} &= 1\% \text{ of } 652,000 \\ &= \frac{1}{100} \times 652,000 = 6520 \end{aligned}$$

The broker gets 0.5% from Senay = 0.5% of 652,000

$$= \frac{0.5}{100} \times 652,000 = 3260$$

Hence, the total amount the broker receives gets = 6520 + 3260
= 9780

$$\begin{aligned} \text{b) Senay's payment} &= \text{Cost of the house} + \text{Broker's fee} \\ &= 652,000 + 3260 = 655,260 \text{ Nakfa} \end{aligned}$$

$$\begin{aligned} \text{c) Kaltum gets} &= \text{Cost price of the house} - \text{Broker's fee} \\ &= 652,000 - 6520 = 645,480 \text{ Nakfa} \end{aligned}$$

Exercise 7.5

1. Salih rented his shop to Naib through a broker. The rent agreed by both is 20000 Nakfa per year. The broker charged each of them 10% commission.
 - a) Find the total brokerage in this transaction (process).
 - b) Find the total money paid by Naib.
 - c) Find the net amount received by Salih.
2. Yemane wants to sell his bicycle through a broker who demands 3% brokerage. At what price should Yemane sell his bicycle so that he will get 1500 Nakfa net?
3. Awet bought a car through a broker paying 4% brokerage. In addition, he paid 2.5% of the estimated value of the car in order for it to be registered in his name. If he paid 120,000 Nakfa in total, what is the estimated value of the car?
4. A company bought 2500 kg of tomatoes for 4 Nakfa per kg through a broker who charges 5% brokerage. In addition, the company paid 250 Nakfa for transport. Find the total amount paid by the company.

7.3 Profit and Loss

Activity 7.6

Harnet buys hens in Tserona and sells them in Dekemhare. On a particular day she bought 20 hens from Tserona at a cost of 55.00 Nakfa each and sold all of them in Dekemhare for 65.00 Nakfa each.

- a) How much did she pay for the 20 hens?
- b) How much did she receive for selling the 20 hens?
- c) What is the difference between the total amount she received by selling and buying the 20 hens?

d)

If selling price is greater than the cost price, then there is a profit.

$$\text{Profit} = \text{Selling price} - \text{Cost price} = \text{SP} - \text{CP}$$

If cost price is greater than the selling price, then there is a loss.

$$\text{Loss} = \text{Cost Price} - \text{Selling Price} = \text{CP} - \text{SP}$$

Profits or losses are usually expressed in percentages relative to the cost price.

$$\text{Profit percent} = \frac{\text{Profit}}{\text{CP}} \times 100 = \frac{\text{SP} - \text{CP}}{\text{CP}} \times 100$$

$$\text{Loss percent} = \frac{\text{Loss}}{\text{CP}} \times 100 = \frac{\text{CP} - \text{SP}}{\text{CP}} \times 100$$

Exercise 7.6

1. A trader sold a radio for 550 Nakfa which costs 500 Nakfa and a tape recorder for 1050 Nakfa which costs 1000 Nakfa.
 - a) Find the profit and the profit percentage of each item.
 - b) Determine which item is the more profitable for the trader.
2. A man bought a motor cycle for 24,000 Nakfa. Two years later, he sold it for 20,000 Nakfa. Find his loss and percentage loss.
3. An item was sold for 82 Nakfa at a profit of 6%. What was the cost price of the item?
4. A fruit vendor bought bananas for 5 Nakfa per kg. If she wants to make a profit of 20%, how much per kg should she sell the bananas for?
5. Ali sold an article to Tesfu at a profit of 12.5% and Tesfu sold it to Halima at a loss of 15% of what he paid for it. If the article cost Ali 740 Nakfa, find out how much Halima bought it for.

7.4 Simple Interest

Activity 7.7

1. **Teacher:** Where do people keep their money?

Dawit: Many people keep their money in banks.

Romodan: Some people keep their money in their houses

Teacher: Where do you want to keep your money, and why?

Zeineb: I would like to keep my money in a bank since it is safe and I can receive an additional amount of money after some time.

2. Adhanom deposited 800 Nakfa in a bank. The bank pays 5% interest after a year. How much interest does Adhanom receive from the bank after 5 years?

Azeb and Nejat tried to solve the problem as follows.

$$\text{Interest after 1 year} = 800 \times \frac{5}{100} = 40 = 800 \times \frac{5}{100} \times 1$$

$$\text{Interest after 2 years} = 40 + 800 \times \frac{5}{100} = 80 = 800 \times \frac{5}{100} \times 2$$

$$\text{Interest after 3 years} = 80 + 800 \times \frac{5}{100} = 120 = 800 \times \frac{5}{100} \times 3$$

$$\text{Interest after 4 years} = 120 + 800 \times \frac{5}{100} = 160 = 800 \times \frac{5}{100} \times 4$$

What do you observe from the above pattern?

What will be the interest after 5 years?

What will be the interest after T years?

This activity suggests that if a sum of money (principal P) is invested for a period of time (T), then the **simple interest (I)** is given by:

$$\text{Simple interest} = \text{Principal} \times \text{Rate} \times \text{Time}$$

$$I = PRT$$

The initial amount of money invested in the bank is called the **Principal (P)**. The rate or percent that is stated on a specified period of time is called the **simple interest rate (R)**. This is the amount one will pay for the use of someone else's money. When percentages are written as fractions, the denominator is 100. The rate is, therefore, always represented as a decimal. For instance 6% means $\frac{6}{100} = 0.06$. The amount of money after a certain period of time is the sum of the principal and interest. That is,

$$\text{Amount} = \text{Principal} + \text{Interest}$$

$$A = P + I$$

The amount of time that will be taken to pay down the debt is called **time (T)**. Amount = Principal + Interest. ($A = P + I$)

Example 7.6

Makda deposited 500 Nakfa in a bank. The bank pays 6% interest after a year. How much interest does Makda receive from the bank after a year?

Solution

$$\begin{aligned}\text{Interest} &= 500 \times \frac{6\%}{100} \text{ Nakfa} \\ &= 500 \times \frac{6}{100} \text{ Nakfa} \\ &= 500 \times 0.06 \text{ Nakfa} \\ &= 30 \text{ Nakfa}\end{aligned}$$

Therefore, Makda receives an additional amount of 30 Nakfa after a year.

Exercise 7.7

1. Ismael borrowed 2000 Nakfa from a shop. He agreed that he would pay him back at a rate of 4% simple interest. Find the interest after 4 years.
2. Ato Fatinga borrowed some amount of money from his friend on the condition that he would repay the money at a rate of 6% simple interest after 3 years. If the interest after 3 years is 300 Nakfa, what is the amount of money he borrowed from the shop?

3. What annual rate of simple interest is needed for 1500 Nakfa to earn an interest of 180 Nakfa in 4 years?
4. If Rigbe deposits 4500 Nakfa into a bank which pays 6%, for how many years must Rigbe invest this amount to receive 5310 Nakfa?
5. A certain amount of money was invested in a bank at 5.5% simple interest. Find the principal if the accumulated amount after 9 months is 770 Nakfa.
6. A person deposited 2400 Nakfa in a bank with a simple interest rate of 5%. Find the accumulated amount after
 - a) 4 months
 - b) 9 months
7. Mohammed opens a daily interest account in a bank with a principal of 150 Nakfa at a rate of 7%. What will be the interest after 20 days? (Hint: Note that time $(T) = \frac{20}{365}$)

Review Exercise 7

1. Calculate the percentage increase of the following.
 - a) From 80 to 84
 - b) From 9 to 12
2. Calculate the percentage decrease of the following.
 - a) From 300 to 250
 - b) From 20 to 18
3. A 10% service charge is added to the cost of food in a restaurant. If the food costs 70 Nakfa, what would be the total charge including the service charge?
4. Given the ratios $x : y = 2 : 7$ and $y : z = 4 : 3$, find the ratio of $x : z$.
5. If the price 3 of articles A, B and C are such that the price of A is one-fifth of B and the price of C is twice of B, then find the ratio of the prices of A, B and C.

6. Which of the following are correct proportions? Give your reasons.

a) 5, 9, 15 and 27

c) 2, 1, 6 and 18

b) 1, 1.5, 3 and 4.5

d) 30, 45, 2 and 5

7. If the cost of 12 pens is 20 Nakfa, then what is the cost of 20 pens?

8. Nafisa is a saleswoman of an automobile company. The company offered her two options;

i) A monthly salary of 4600.00 Nakfa without a commission.

ii) A monthly salary of 3200.00 Nakfa plus a commission of 1% on her total monthly sales.

Determine which offer is better for her, if the average monthly automobile sales are predicted to be 150,000 Nakfa.

9. Calculate the missing information

Principal (P)	Interest rate (R)	Time (T)	Simple interest (I)
3000		4	360
1250	3%		150
850	6%	5	
	2%	7	70

10. How long does it take 500 Nakfa to grow to 510 Nakfa at 9% simple interest?

11. What amount invested now will grow to 3500 Nakfa in 8 months at 9% simple interest?

CHAPTER 8

Polynomials

Expected Learning Outcomes

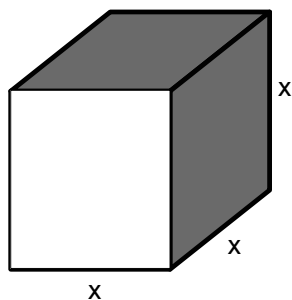
By the end of this chapter, you will be able to:

- distinguish polynomials from non-polynomials;
- determine the degree, the coefficient of any term, and the constant term of a polynomial;
- perform the operations of addition, subtraction, multiplication and division on polynomials;
- draw graphs of polynomial functions of degree three of the form $y = ax^3 + b$ and explain their properties
- given the graph of a polynomial function of degree less than or equal to three, determine its possible degree and the sign of its leading coefficient.

8.1 Representing Polynomials

Activity 8.1

1. a) Consider the expression $mx + b$. What do we call such an expression?
b) Consider the expression $ax^2 + bx + c$ for $a \neq 0$. What do we call such an expression?
2. Consider the box in the shape of a cube of side x units shown below.
 - a) What expression, in terms of x will give the perimeter of the base?
 - b) What expression, in terms of x , will give the area of the base?
 - c) What expression, in terms of x , will give the volume of the cube?
 - d) Haben wrote the expressions $4x$; x^2 ; and x^3 for a), b), and c) respectively. Do you agree with her?



- e) Which one of them is a linear expression? A quadratic expression? Neither a linear nor a quadratic expression?

The three expressions belong to a large family of expressions called **polynomial expressions**. The following are all examples of polynomial expressions.

- a) 7 b) $-3x + 11$ c) $7x^2 - 5x + 9$ d) $0.7x^3 + 2x - 1$

For whole number **n** and real number **a**, an expression of the form **ax^n** is called a **term** of a polynomial in x . The number **a** preceding the variable **x^n** is called the **coefficient** of **x^n** . Thus $-5x^3$, for example, is a term and -5 is the coefficient of x^3 .

$11x^5 - 15x^3 + 0.5x - 2$ is also an example of a polynomial expression with four **terms**, namely, $11x^5$; $-15x^3$; $0.5x$; and -2 . The term -2 which is free of the variable x is called the **constant term**. Moreover, the highest power of x in the expression, which in this case is 5, is called the **degree** of the polynomial.

In general, the definition of a polynomial expression and the terms associated with it are as is given below.

A polynomial expression is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers, but $a_n \neq 0$, and the exponents of x are all whole numbers. For this expression, its **degree** is n .

a_n is the **leading coefficient**.

a_0 is the **constant term**.

When the coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ are all zeros, we get 0, which is also considered to be a polynomial, but is not assigned a degree.

Terms that contain the same variables with the same powers are called **like terms**. Conversely, when the variables and powers are not the same, they are called **unlike terms**.

For example, $3x^2$ and $-5x^2$ are **like terms**, but $3x^2$ and $3x^4$ are **unlike terms**.

When a polynomial in one variable is written with the exponents in **decreasing order** from left to right, such as $4x^3 - 15x^2 + 7x - 2$, the polynomial is said to be written in **standard form** and the coefficient of its first term is called the **leading coefficient**.

Activity 8.2

In each of the following polynomials, count the number of terms.

a) $4x^3$

e) $4x^4 - 5x^2 + 2x + 3$

b) $5x - 9$

c) $-\frac{2}{3}x^5 + x^2 + \frac{3}{4}$

f) $x^3 + 2x + 7$

g) $7 + 8x^3$

d) $x^2 - 9$

h) $8x - 30x^3$

Certain polynomials have special names depending on their number of terms. A **monomial** is a polynomial that has one term.

A **binomial** is a polynomial that has two unlike terms. A **trinomial** is a polynomial that has three unlike terms.

Exercise 8.1

- In $4x^3 - 7x - 5$ what is the:
 - Degree of the polynomial expression?
 - Coefficient of x^3 ?
 - Coefficient of x^2 ?
 - Coefficient of x ?
 - Constant term?
- Identify each pair of terms as like terms or unlike terms.
 - $3x$ and $3y$
 - $5x^2$ and $-2x^2$
 - $5x^7$ and $5x^2$
- Consider the polynomial $-2x^6 + 0.5x^3 - 11x^7 + 9 - 12x$.
 - Is the expression in standard form? If no, rewrite it in standard form.

- b) What is the leading coefficient?
 - c) What is the degree of the polynomial?
 - d) What is the coefficient of x^6 ?
 - e) What are the coefficients of x^5 , x^4 , and x^2 ?
 - f) What is the constant term?
4. For the polynomial, $3m^7 - \frac{2}{3}m^2 + 5m^{10} - 2m + 1$ find the
 - a) degree
 - b) leading coefficient
 - c) coefficient of m^9
 - d) constant term
5. Determine whether each of the following is a polynomial.
 - a) $3x^{-2} + 4$
 - b) $3x + 2^{-1} - x^2$
 - c) $5x^{\frac{2}{3}} + x^2 - 9$
 - d) $\sqrt{6x^2 + 2}$
 - e) $\log_2(x^2 - 4)$
 - f) $\frac{1}{x} + \frac{1}{x^2}$
 - g) $-8x^2 + 11x + x^{40} - 3$
 - h) $|5x|$
 - i) $\sqrt{3}x^5 - 2x^2 + 0.3x^7 - 6$
 - j) 2^x
6. For each of the following polynomials, write the constant term, leading coefficient, and its degree.
 - a) $-8x^2 + 11x + x^{40} - 3$
 - b) $7x^3 + 9$
 - c) $-5x^3 + 2x + 2$
 - d) $3x^4 - 2x$
 - e) $0.6x^2 + 3x^7 - 11$
 - f) $\sqrt{3}x^5 - 2x^2 + 0.3x^7 - 6$
 - g) $x^2 - 5x - 7$
 - h) $x - 1$
 - i) $\frac{3x^2 - 6x + 12}{6}$
7. Determine whether each polynomial is a monomial, binomial, or trinomial.
 - a) $x^4 - 8x^3$
 - b) $7x^5 - 0.4x^3 - \sqrt{5}x + 3$
 - c) $\frac{x^7}{15}$
 - d) -8
 - e) $x^5 + 32$
 - f) $\frac{x^4}{2} + \frac{5x}{3} - 7$

8. For each given polynomial evaluate its value at the indicated value of the variable.

a) $x^2 - 3$; $x = 5$

d) $-2a^7 + a^{20}$; $a = 1$

b) $-t^3 + 3t + 5$; $t = -2$

e) $3x^2 - 5x + 7$; $x = -1$

c) $a^{17} + a^{20}$; $a = -1$

f) $2x^4 - 5$; $x = -2$

8.2 Operations on Polynomials

8.2.1 Addition of Polynomials

Activity 8.3

- Which of the following can be simplified by adding? Give reasons for your answers.
 - 8 goats + 4 goats
 - 8 goats + 4 dogs
 - 6 cows + 2 cows
- Suppose x is the length of a side of a square in cm.
 - What would $4x$ represent?
 - What would x^2 represent?
 - Is it possible to simplify $4x + x^2$? Give your reason.
- To get the sum of $(x^2 - 5x - 7) + (7x^2 - 4x + 10)$, Zahra worked as shown below;

$$\begin{aligned}
 &(x^2 - 5x - 7) + (7x^2 - 4x + 10) \\
 &= (x^2 + 7x^2) + (-5x - 4x) + (-7 + 10) \\
 &= (1 + 7)x^2 + (-5 - 4)x + 3 \\
 &= 8x^2 - 9x + 3
 \end{aligned}$$

Fortuna got the same sum using a vertical format as shown below:

$$\begin{array}{r}
 x^2 - 5x - 7 \\
 + 7x^2 - 4x + 10 \\
 \hline
 8x^2 - 9x + 3
 \end{array}$$

Whose format is more efficient?

4. To find the sum in $(3x^3 - 5x^2 - 7) + (4x^2 - 2x + 3)$, Zahra worked it out using the vertical format, as follows;

$$\begin{array}{r} 3x^3 - 5x^2 - 7 \\ + 4x^2 - 2x + 3 \\ \hline 7x^5 - 7x^3 - 4 \end{array}$$

Is she correct? If not, how would you help her correct her mistake?

From Activity 8.3 we can suggest that only like objects or like terms can be added.

To add polynomials, add the coefficients of like terms.

Example 8.1

Simplify the expression

$$(-9x + 5x^2 - 3 + 4x^3) + (-5x^3 + 2x + 2) + (2x^4 - 3x^2 - 7x - 1)$$

Solution:

Lining up like terms vertically, we get

$$\begin{array}{r} 4x^3 + 5x^2 - 9x - 3 \\ - 5x^3 \quad \quad + 2x + 2 \\ + 2x^4 \quad \quad - 3x^2 - 7x - 1 \\ \hline 2x^4 - x^3 + 2x^2 - 14x - 2 \end{array} \quad \text{or} \quad \begin{array}{r} 0x^4 + 4x^3 + 5x^2 - 9x - 3 \\ 0x^4 - 5x^3 + 0x^2 + 2x + 2 \\ + 2x^4 + 0x^3 - 3x^2 - 7x - 1 \\ \hline 2x^4 - x^3 + 2x^2 - 14x - 2 \end{array}$$

Exercise 8.2

- Find the sum of each of the following.
 - $(-5x^3 - 11x^2 + 8x - 6) + (8x^3 + 9x^2 - 12x - 5)$
 - $(-x^5 + 3x^4 - 2x^2 + 4x) + (x^5 - 2x^2 - 5x - 1)$
 - $(7x^2 + 12x - 9x^3 + 12) + (9x^3 - 7x^2 - 13x - 15)$
 - $5x^4 - 4x^2 + 10x - 3 + (1.5x^4 - 5x^3 + 4) + (5x^3 + 4x^2 - 10)$
 - $(x^2 + 7) + (-x^2 - 13 + 11x) + (-11x + 3x^2 + 2x - 1)$
 - $(3n^2 - 4n + 13) + (n^3 - 2n^2 + 5n) + (-3n^3 + 4n^2 + 6n - 14)$
- Find the degree of the sum of the polynomials
 - $-5x^3 + 2x + 2$ and $2x^4 - 3x^2 - 7x - 1$.
 - $5x^7 - 8$ and $-4x^6 - x^4 + 9$.
- Is the sum of two polynomials always a polynomial?

8.2.2 Subtraction of Polynomials

Activity 8.4

1. Find the difference in $(2x^3 + x^2 - 7x - 2) - (5x^2 + 6x - 4)$.

a) Simret subtracted the polynomials as follows:

$$\begin{aligned}(2x^3 + x^2 - 7x - 2) - (5x^2 + 6x - 4) &= 2x^3 + x^2 - 7x - 2 - 5x^2 - 6x + 4 \\ &= 2x^3 + (x^2 - 5x^2) + (-7x - 6x) + (-2 + 4) \\ &= 2x^3 - 4x^2 - 13x + 2\end{aligned}$$

Do you agree with her answer?

b) Aron found the difference as shown below.

$$\begin{array}{r} 2x^3 + x^2 - 7x - 2 \\ -5x^2 - 6x + 4 \\ \hline 2x^3 - 4x^2 - 13x + 2 \end{array}$$

- c) Did you understand Aron's approach?
 d) Why do you think Aron changed the signs of all terms in $5x^2 + 6x - 4$ and then added?
 e) Comparing Simret's and Aron's approach, whose approach do you find easier to work with?
2. A teacher asked her learners to subtract $-3m^2 + 5m + 9$ from $7m^3 - 4m + 11$. Three learners called Hamid, Tirhas, and Rehana tried to solve the problem, but they all got it wrong. Their workings out are shown below. Identify each learner's error and help him or her avoid the error.

Hamid's work

$$\begin{array}{r} -3m^2 + 5m + 9 \\ -7m^3 \quad + 4m - 11 \\ \hline -7m^3 - 3m^2 + 9m - 2 \end{array}$$

Tirhas's work

$$\begin{array}{r} 7m^3 - 4m + 11 \\ -3m^2 + 5m + 9 \\ \hline 7m^3 - 3m^2 + m + 20 \end{array}$$

Rehana's work

$$\begin{array}{r} 7m^3 - 4m + 11 \\ 3m^2 - 5m - 9 \\ \hline 10m^3 - 9m + 2 \end{array}$$

A common mistake is to forget to change signs correctly when subtracting one polynomial from another. See the example below.

$$\begin{array}{r} \text{wrong signs} \\ \downarrow \quad \downarrow \\ (x^2 - 4x + 4) - (x^2 + 5x - 1) \neq x^2 - 4x + 4 - x^2 + 5x - 1 \end{array}$$

Exercise 8.3

1. Perform the indicated operations.

- $(x - 7) + (2x - 3) + (5 - x)$
- $(5x - 3) + (x^3 + 3x - 2) + (-2x^2 - 3)$
- $(2y^2 - 3y - 8) - (y^2 + 4y - 1)$
- $(2x^3 - 4x - 3) - (x^2 - 2x + 5)$
- $\left(\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{8}\right) - \left(x^2 - \frac{2}{3}x - \frac{1}{5}\right)$
- $\left(\frac{2}{3}x^2 - \frac{1}{3}x + \frac{1}{6}\right) - \left(-\frac{1}{3}x^2 + x + 1\right)$

2. Perform the indicated operations.

- | | | |
|--|----------------------------------|------------------------------------|
| a) Add | b) Subtract | c) Add |
| $x^3 + 3x^2 - 5x - 2$ | $-3x^2 + 5x - 2$ | $x^2 - 3x + 7$ |
| <u>$-x^3 + 8x^2 + 5x - 7$</u> | <u>$x^2 - 5x - 6$</u> | <u>$-2x^2 - 5x + 2$</u> |

3. Is the difference of two polynomials always a polynomial? Why?

4. Simplify the following expression.

- $(3x^2 - 2x) + (5x^3 - 2x)$
- $4(y^2 + 5y) - 2(y - 3)$
- $7x^2 - 11 + 8x - 9x^2 + 4x^3 + 6 - 10x$

8.2.3 Multiplication of Polynomials

Activity 8.5

1. To simplify the expression $(-2x^3)(4x^2)$, Zahra multiplied $-2x^3$ and $4x^2$ as follows:

$$\begin{aligned} (-2x^3)(4x^2) &= (-2 \times 4)(x^3 \cdot x^2) \\ &= -8x^5. \end{aligned}$$

Is she correct? Multiply each of the following expressions.

- | | |
|------------------------|---|
| a) $(2ab^2)(3a^2b)$ | d) $(-4xy^3)(-0.5xyz^2)$ |
| b) $(5x^4)(4x^3)$ | e) $\left(\frac{2}{3}x^2\right)\left(\frac{3}{4}x\right)$ |
| c) $(-y^2x^4)(-3xy^3)$ | f) $(7x^4)(6x^7)$ |

2. Halima multiplied $-7x(-3 + 2x)$ as shown below.

$$\begin{aligned} -7x(-3 + 2x) &= -7x(-3) + (-7x)(2x) \\ &= 21x - 14x^2 \end{aligned}$$

Do you agree with her? What property of multiplication is she using? Multiply each of the following expressions.

- $c(d + e)$
 - $4x(2x - 3)$
 - $0.5m(-2n + 4)$
 - $(3x^3 - 7x^2 - 5x + 9)8x^2$
 - $\sqrt{2} x^3 y^2 \left(\frac{\sqrt{2}}{2} x^2 y^3 \right)$
3. Study the dialogue between Zahra and Senait carefully, which took place when they were asked to multiply $(x - 3)(5x - 1)$.

Senait: Zahra, I am now quite clear how to multiply a monomial by any polynomial. But I have no idea how to multiply $(x - 3)(5x - 1)$.

Zahra: All what we need to do is to apply the distributive property repeatedly. Thus

$$\begin{aligned} (x - 3)(5x - 1) &= x(5x - 1) - 3(5x - 1) \\ &= 5x^2 - x - 15x + 3 \\ &= 5x^2 - 16x + 3 \end{aligned}$$

Do you agree with Zahra's method? Multiply each of the following expressions.

- | | |
|----------------------------------|----------------------------|
| a) $(x + 2)(-x + 7)$ | e) $(2y - 3)(2y - 6)$ |
| b) $(3x - 5)(2x + 8)$ | f) $(1 - 9a)(7a + 5)$ |
| c) $(-10m - 9)(2m + 11)$ | g) $(n^2 - 4)(2n^3 - 4)$ |
| d) $(-5b^2a + 6ba^3)(5ba^2 - 1)$ | h) $5xy^2(2xy - 5y^3 + 7)$ |

Some binomial products occur so frequently that it is worth remembering their special products. Activities 8.6 to 8.8 below, will enable you discover these special products.

Activity 8.6

1. Find the product of $(x - 2)(x + 2)$.

- a) Beletse multiplied $(x - 2)(x + 2)$ as follows

$$\begin{aligned}(x - 2)(x + 2) &= x(x + 2) - 2(x + 2) \\ &= x^2 + 2x - 2x - 4 \\ &= x^2 - 4\end{aligned}$$

Is he correct?

- b) What relation do you observe between $(x - 2)$ and the product $x^2 - 4$?

2. Find the product of $(2x + 3y)(2x - 3y)$.

- a) Nura found the product as follows.

$$\begin{aligned}(2x + 3y)(2x - 3y) &= 2x(2x - 3y) + 3y(2x - 3y) \\ &= (2x \cdot 2x) - (2x \cdot 3y) + (3y \cdot 2x) - (3y \cdot 3y) \\ &= 4x^2 - 6xy + 6xy - 9y^2 \\ &= 4x^2 - 9y^2\end{aligned}$$

Did you follow her process of finding the product?

- b) What relationship do you observe between the terms of $(2x - 3y)$ and the terms of $4x^2 - 9y^2$?
- c) What general rule can you formulate for $(a - b)(a + b)$?

Product of a sum and a difference

$$(a - b)(a + b) = a^2 - b^2$$

In words, this means that, the product is just the **square of the first minus the square of the second**.

Exercise 8.4

Find the product of each of the following.

1. $(5x + 3)(5x - 3)$
2. $(\sqrt{2}a + \sqrt{3})(\sqrt{2}a - \sqrt{3})$
3. $(7 - 4y)(7 + 4y)$
4. $(3x^2 - 2)(3x^2 + 2)$
5. $(7y - 3x)(7y + 3x)$
6. $(10m - 9n)(10m + 9n)$

Activity 8.7

To find the product of $(x + 3)^2$, Huria worked it out as shown below:

$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3) \\ &= x(x + 3) + 3(x + 3) \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9\end{aligned}$$

- Is she correct?
- What relationships do you observe between the terms of $(x + 3)$ and the terms of $x^2 + 6x + 9$?
- Can you think of a short way of getting the answer?
- What general rule can you formulate about $(a + b)^2$?

Square of the sum of a binomial:

$$(a + b)^2 = a^2 + 2ab + b^2$$

To square a sum, we square the first term (a^2), add twice the product of the two terms ($2ab$), and then add the square of the last term (b^2).

Exercise 8.5

Find the product in each of the following.

- $(a + 2)^2$
- $(7w + 4)^2$
- $(6 + 5v)^2$
- $(2m + 5)^2$
- $(2x + 3y)^2$
- $(10y + 2x)^2$
- $(3x + 4y)^2$
- $(\sqrt{5}n + 2)^2$

Activity 8.8

To Find the product of $(x - 3)^2$, Semhar worked it out as follows:

$$\begin{aligned}(x - 3)^2 &= (x - 3)(x - 3) \\ &= x(x - 3) - 3(x - 3) \\ &= x^2 - 3x - 3x + 9 \\ &= x^2 - 6x + 9\end{aligned}$$

- Do you agree with her solution?
- What relation do you observe between the terms of $(x - 3)$ and the terms of $x^2 - 6x + 9$?
- Can you think of getting the answer in a very short way?
- What general rule can you formulate about $(a - b)^2$?

Square of the difference of a binomial

$$(a - b)^2 = a^2 - 2ab + b^2$$

To square a difference, we square the first term (a^2), subtract twice the product of the two terms ($2ab$), and then add the square of the last term (b^2).

Exercise 8.6

Find the product in each of the following.

1. $(a - 2)^2$
2. $(7w - 4)^2$
3. $(6 - 5v)^2$
5. $(2x - 3y)^2$
6. $(10y - 2x)^2$
7. $(3x - 4y)^2 (2m - 5)^2$
8. $(\sqrt{5}n - 2)^2$
9. $(\sqrt{2} - \sqrt{3})^2$

Having seen how to multiply two binomials, let us now see how to multiply any two polynomials.

Activity 8.9

To find the product of $(3y + 7)(2y^2 - 5y - 2)$, Senait found the product as follows.

$$\begin{aligned} (3y + 7)(2y^2 - 5y - 2) &= 3y(2y^2 - 5y - 2) + 7(2y^2 - 5y - 2) \\ &= 6y^3 - 15y^2 - 6y + 14y^2 - 35y - 14 \\ &= 6y^3 - y^2 - 41y - 14 \end{aligned}$$

To find the product of $(2x^2 + 5x - 3)$ and $(4 + 2x - x^2)$, Shifa found the product as follows:

$$\begin{aligned} (2x^2 + 5x - 3)(4 + 2x - x^2) &= 2x^2(4 + 2x - x^2) + 5x(4 + 2x - x^2) - 3(4 + 2x - x^2) \\ &= 8x^2 + 4x^3 - 2x^4 + 20x + 10x^2 - 5x^3 - 12 - 6x + 3x^2 \\ &= -2x^4 + (4x^3 - 5x^3) + (8x^2 + 10x^2 + 3x^2) + (20x - 6x) - 12 \\ &= -2x^4 - x^3 + 21x^2 + 14x - 12 \end{aligned}$$

- a) Do you understand their procedures?
- b) What property of numbers with respect to multiplication and addition have they applied in their first step?

Example 8.2

Find the product of $(2x^2 + 5x - 3)$ and $(4 + 2x - x^2)$.

Solution:

$$\begin{array}{r}
 2x^2 + 5x - 3 \\
 \times \quad -x^2 + 2x + 4 \\
 \hline
 -2x^4 - 5x^3 + 3x^2 \dots \quad \text{This is } -x^2(2x^2 + 5x - 3) \\
 + 4x^3 + 10x^2 - 6x \dots \quad \text{This is } 2x(2x^2 + 5x - 3) \\
 \hline
 8x^2 + 20x - 12 \dots \quad \text{This is } 4(2x^2 + 5x - 3) \\
 -2x^4 - x^3 + 21x^2 + 14x - 12 \dots \quad \text{Simplified form}
 \end{array}$$

Exercise 8.7

Find the product for each of the following:

1. $(-5y^3 + 3y - 2)(4y - 1)$
2. $(a - 3)(a^2 + 3a + 9)$
3. $(x - y)(x^2 + xy + y^2)$
4. $(3a - 7 + 2a^3)(2a^2 - 5)$
5. $(x^2 - 5x + 6)(2x - 3)$
6. $(2x + 3)(4x^2 - 6x + 9)$

8.2.4 Division of Polynomials

Remember that when we divide in the set of whole numbers, we stop dividing when we get a remainder less than the divisor.

For example, to divide 37 by 17, we do it as shown below.

$$\begin{array}{r}
 2 \rightarrow \text{quotient} \\
 \text{divisor} \rightarrow 17 \overline{) 37 \rightarrow \text{dividend}} \\
 \underline{34} \\
 3 \rightarrow \text{remainder}
 \end{array}$$

How would you check whether the division process is correct?

When we divide a polynomial by a polynomial, however, we stop dividing when the degree of the remainder is less than the degree of the divisor. This means that division of polynomials will make sense only when the degree of the dividend is either equal to or less than the degree of the divisor. For example,

$$(x - 1) \div (x^2 + 1)$$

would not make sense in the set of polynomials.

Activity 8.10

Dividing a polynomial by a monomial.

1. Divide $-10x^5$ by $2x^2$.
 - a) What do you do with the coefficients?
 - b) What do you do with the exponents?
 - c) What is the quotient, that is, the simplest form of $\frac{-10x^5}{2x^2}$?
2. Divide $6x^3 + 15x^2 - 12x$ by $3x$ when $x \neq 0$.
 - a) Why $x \neq 0$?
 - b) Hamid rewrote the question as $\frac{6x^3 + 15x^2 - 12x}{3x}$ and then worked it out as shown below.

$$\frac{6x^3 + 15x^2 - 12x}{3x} = \frac{6x^3}{3x} + \frac{15x^2}{3x} - \frac{12x}{3x} = 2x^2 + 5x - 4$$

Do you agree with his answer?

Activity 8.11

In mathematics, it is common practice to solve an unfamiliar problem by translating the problem into a familiar one. Hamid's procedure in the above division is based on the following facts:

$$(1) \frac{a}{d} + \frac{b}{d} - \frac{c}{d} = \frac{a + b - c}{d} \text{ (This process is combining)}$$

But this also means that: (2) $\frac{a + b - c}{d} = \frac{a}{d} + \frac{b}{d} - \frac{c}{d}$ (This process is **splitting**)

- a) Which of these two is useful in dividing a polynomial by a monomial, (1) or (2)?
- b) Divide $12x^3 + 9x^2 - 21x$ by $3x$.

Exercise 8.8

1. Divide

a) $3x$ by x	c) $15x^3$ by $-3x^2$	e) $-12a^4$ by $-4a^2$
b) $-9x^2$ by $6x$	d) $10w^4$ by $7w^3$	f) $48x^5y^2$ by $16x^3y^2$

2. Divide

- a) $-6x^4 - 4x^2$ by $2x$
- b) $-7m^6 - 3m^5 + 5m^3$ by $-5m^2$
- c) $10x^7 - 5x^5 + 15x^2$ by $5x^2$
- d) $-14x^4 - 6x^2$ by $4x^2$
- e) $2y^8 - 3y^7 + y^5 - y^4$ by $-0.5y^3$

Activity 8.12

Division of a polynomial by a polynomial.

1. Divide $x^2 + 7x + 10$ by $(x + 5)$.

Berhe at first factorised $x^2 + 7x + 10$ as shown below:

$$\begin{aligned}x^2 + 7x + 10 &= x^2 + 5x + 2x + 10 \\&= x(x + 5) + 2(x + 5) \\&= (x + 5)(x + 2)\end{aligned}$$

- a) Is he correct? How would you verify this?
- b) Berhe said that:

$$(x^2 + 7x + 10) \div (x + 5)$$

is the same as

$$(x + 5)(x + 2) \div (x + 5)$$

and therefore

$$(x + 5)(x + 2) \div (x + 5) = (x + 2).$$

Do you agree with Berhe's argument?

2. Find the quotient in each of the following by factoring the dividend.

- a) $(x^2 - 25) \div (x + 5)$
- b) $(2y^2 - 3y + 1) \div (2y - 1)$
- c) $(4y^2 - 9) \div (2y - 3)$
- d) $(6u^2 + u - 1) \div (3u - 1)$

3. Divide $3x^2 - 7x + 4$ by $x - 1$.

Yosief worked it out as shown below.

$$\begin{aligned}3x^2 - 7x + 4 &= 3x^2 - 3x - 4x + 4 \\&= 3x(x - 1) - 4(x - 1) \\&= (x - 1)(3x - 4)\end{aligned}$$

Then he concluded that $(3x^2 - 7x + 4) \div (x - 1) = 3x - 4$.

- a) Do you think this factoring method will always work?
- b) Can you find what $(x^2 - 7x + 4) \div (x + 1)$ will be by factoring method? If no, what is the reason?

Activity 8.13

Sitel who often refers other mathematics books found the quotient in $(3x^2 - 7x + 4) \div (x - 1)$, using another method. Her method is similar to the long division of whole numbers. She explained her process as shown below. Study it carefully.

To get the first term of the quotient, divide the first term ($3x^2$) of the dividend by the first term (x) of the divisor. That is, $3x^2 \div x = 3x$.

$$\begin{array}{r} 3x^2 - 7x + 4 \div \underline{x - 1} \\ 3x \end{array}$$

Next, multiply $3x$ by the divisor $x - 1$ and subtract the product from the dividend. Then bring down $+ 4$

$$\begin{array}{r} 3x^2 - 7x + 4 \div \underline{x - 1} \\ \underline{3x^2 - 3x} \\ -4x + 4 \end{array}$$

To get the second term of the quotient, divide the first term of $-4x + 4$ by the first term of the divisor $x - 1$ to get $-4x \div x = -4$.

$$\begin{array}{r} 3x^2 - 7x + 4 \div \underline{x - 1} \\ \underline{3x^2 - 3x} \\ -4x + 4 \\ \underline{-4x + 4} \\ 0 \end{array}$$

Multiply -4 by the divisor $x - 1$ and subtract the product from $-4x + 4$ to get 0.

- Do you understand Sitel's method?
- Does her method always work?
- Compare Yosief's and Sitel's methods. Whose method is easier?

In the above division process, $3x^2 - 7x + 4$ is the **dividend**, $x - 1$ the **divisor**, $3x - 4$ the **quotient**, and the **remainder** is 0. You can check that the answer is correct by substituting these expressions in the formula

$$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}.$$

There are two more things you need to do when dividing polynomials using the long division. They are:

- Before starting to divide, write the **dividend** and the **divisor** in **standard form**.
- If any terms are **missing** in the dividend, **insert** terms with a **coefficient 0**.

The following example illustrates these two important points.

Example 8.3

Find the quotient and remainder for

$$(-x^2 - 1 + 2x^3) \div (x^2 - 3)$$

Solution:

The divisor is already in standard form, but the dividend is not.

Therefore we should rearrange

$-x^2 - 1 + 2x^3$ as $2x^3 - x^2 - 1$ and insert the term $0x$.

$$\begin{array}{r} 2x^3 - x^2 + 0x - 1 \div x^2 - 3 \\ \underline{2x^3} - 6x \\ - x^2 + 6x - 1 \\ \underline{-x^2} + 3 \\ 6x - 4 \end{array}$$

Exercise 8.9

- Find the quotient and the remainder for each of the following:
 - $(4x^3 - x - 9) \div (2x^2 + 4 + 3x)$
 - $(6x^2 + x - 15) \div (3x + 5)$
 - $(x^4 - 1) \div (x + 1)$
 - $(x^3 + 8) \div (x + 2)$
 - $(5x^4 - 3x^2 - 9) \div (-x^2 + 2)$
 - $(x^2 + 7x + 13) \div (x + 5)$
 - $(x^3 - 8) \div (x^2 + 2x + 4)$
 - $(15x^2 + x - 7) \div (5x + 2)$
- Find the quotient and remainder in each of the following:
 - $(10x^4 - 8x^3 + 6x^2) \div -2x^2$
 - $(2x^3 + 4x^2 - 3x + 1) \div (x^2 - 2)$
 - $(3x^4 + 4x^2 + 2) \div (x^2 + 1)$
 - $(x^3 - 5) \div (x^2 + x + 1)$
 - $(x^2 - 5x + 1) \div (3x + 6)$
 - $(2x^2 - x + 6) \div (3x - 2)$
- Suppose a rectangular room has a length of $x + 3m$ and its area is $x^2 + 4x + 3m^2$. Find its width in terms of x .
- Find the quotient and remainder when the first polynomial is divided by the second.
 - $x^3 - 5x^2 + 6x - 3 ; x - 2$
 - $5x^2 - 13x - 37 ; x + 1$
 - $x^4 - 5 ; x + 1$
 - $-2x^3 + 3x + 15 ; x^2 + 3x - 1$
- A certain polynomial when divided by $3x^2 - 2x + 1$ gives a quotient of $5x - 4$ and a remainder of $-7x + 9$. Find the polynomial.

8.3 Graphs of Polynomial Functions

Activity 8.14

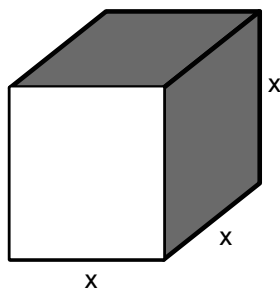
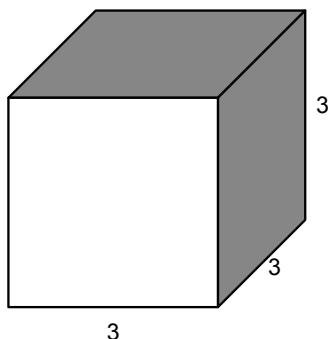
1. Consider the function represented by $y = mx + b$.
 - a) What is the name of the function?
 - b) What does its graph look like?
 - c) What is its y-intercept?
 - d) What is the slope of the line?

2. Consider the function $y = ax^2 + bx + c$ for $a \neq 0$.
 - a) What is the name of the function?
 - b) What does its graph look like?
 - c) What is its y-intercept?

3. A closed box has the shape of a cube with each edge 3 units long.
 - a) Calculate the perimeter of the base.
 - b) Calculate the base area.
 - c) Calculate the volume of the cube.

4. A closed box has the shape of a cube with each edge being x units long. Let the perimeter and area of one face be denoted by P and A respectively, and the volume of the box be V . Now, answer the following questions.
 - a) Calculate the perimeter of the base.
 - b) Calculate the area of the base.
 - c) Calculate the volume of the box.

- d) Write the perimeter, area of the base, and the volume of the box as a function of x .
- e) Which one of them is a linear function?
- f) Which one of them is a quadratic function?



- g) Which one of them is neither linear nor quadratic?
 h) Haben wrote the functions as $P = 4x$; $A = x^2$ and $V = x^3$. Do you agree with her? Why?

The three functions P , A and V belong to a set of functions called polynomial functions. In general, a **polynomial function** is a function which is expressed in terms of a polynomial.

A **polynomial function** is a function which can be written in the form:

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers with $a_n \neq 0$, and the exponents of x are all whole numbers.

The whole number n is called the **degree** of the polynomial function, a_n is the **leading coefficient** and a_0 is the **constant term**.

When the coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ are all zeros, we get $y = 0$, which is also considered to be a polynomial function but is not assigned a degree.

Activity 8.15

Draw the graph of each of the following.

- a) $y = 2x + 1$
 b) $y = -3x + 2$
 c) $y = x^2 + 4x + 6$
 d) $y = -x^2 + 4x + 5$

Example 8.4

Draw the graph of $f(x) = x^3$.

Solution:

First, we construct the following table of values for the function:

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-8	-3.4	-1	-0.13	0	0.13	1	3.4	8

Then, plotting the coordinates, we get the scattered points shown in Fig. 8.1. Note that the y-coordinates of the points $(-0.5, -0.13)$ and $(0.5, 0.13)$ are very close to zero. That is why these points appear to lie on the x-axis.

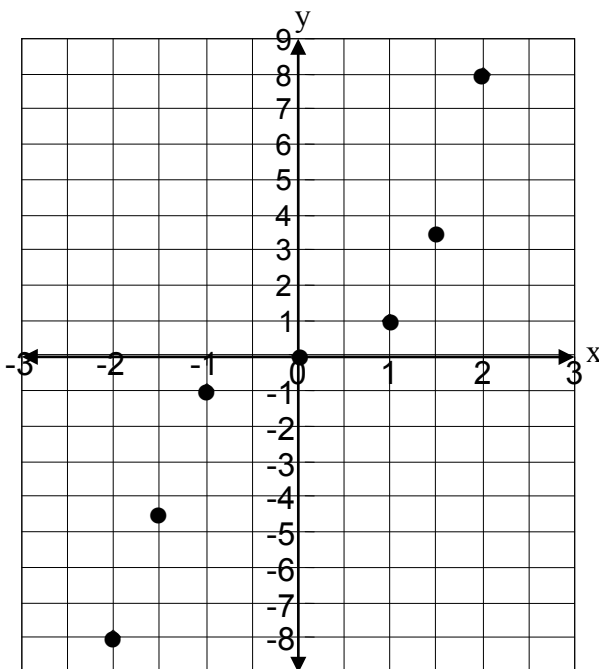


Fig. 8.1

Now, since the domain is the set of all real numbers, we join these scattered points by a smooth curve and get the graph shown in Fig 8.2 This is the graph of the function $y = x^3$.

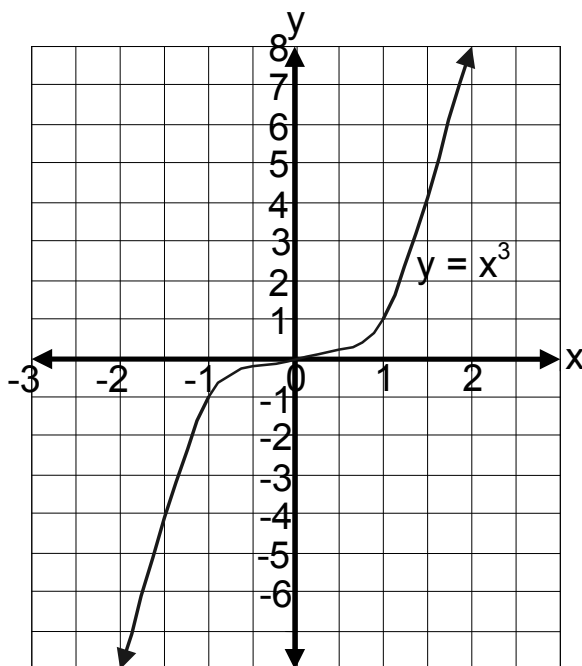


Fig. 8.2

Observe again that for values of x from -0.5 to 0.5 the graph is flattened and seems to overlap with the x -axis. But in reality, the graph cuts the x -axis only at $(0, 0)$. Why do you think is the graph flattened there?

Activity 8.16

1. Study the graph of $y = x^3$ in Fig 8.2 carefully, and then answer the following questions.
 - a) What is the domain of the function?
 - b) What is the range of the function?
 - c) When $x > 0$, does the graph go up or down?
 - d) When $x < 0$, does the graph go up or down?
 - e) Find the x - and y -intercepts of the graph.
2. Draw the graph of each of the following functions.
 - a) $y = -x^3$
 - b) $y = x^3 + 1$
 - c) $y = x^3 - 1$
 - d) $y = 2x^3 - 2$

3. The graph of $y = x^3 - 2x^2 - 5x + 6$ is drawn for you as shown in Fig. 8.3 below. Study carefully the graph, and then answer the questions that follow.

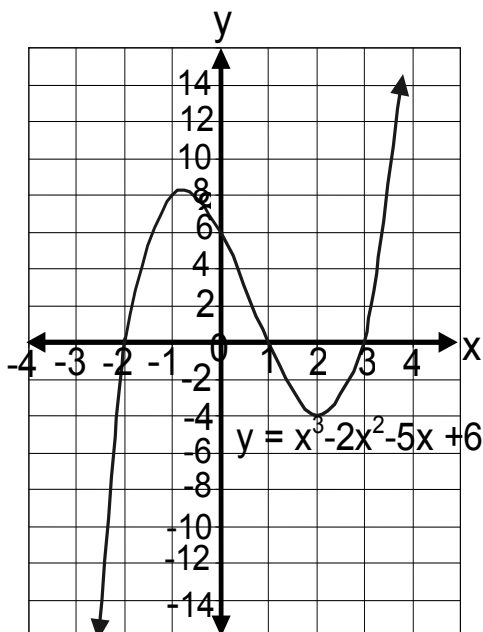


Fig. 8.3

How many x-intercepts does the graph have? And what are they?

- How many turning points does the graph have? And what are the approximate values of x at the turning points?
- What are the domain and range of the function?
- What is the sign of the leading coefficient?
- For large positive values of x , does the graph go up or down?
- For large negative values of x , does the graph go up or down?

Activity 8.17

1. Consider the functions $y = 2x^2 - 4x + 4$ and $y = -x^2 + 4x - 4$.
 - a) Draw their graphs.
 - b) The graph of which function opens upward and which one opens downward?
 - c) What are the signs of the leading coefficients of the functions?
 - d) In general, when would the graph of $y = ax^2 + bx + c$ open upward?
 - e) When would the graph of $y = ax^2 + bx + c$ open downward?

2. Consider the functions $y = 2x^3 + 2$ and $y = -2x^3 - 2$.
 - a) Draw their graphs.
 - b) For large positive values of x , the graph of which function goes up and which one goes down?
 - c) For large negative values of x , the graph of which function goes up and which one goes down?
 - d) What are the signs of the leading coefficients of the functions?
 - e) In general, when would the graph of $y = ax^3 + b$ go down for $x < 0$ and go up for $x > 0$?
 - f) When would the graph of $y = ax^3 + b$ go up for $x < 0$ and go down for $x > 0$?

- I. When $a_n > 0$ and **n is even**, its graph increases without bound as x decreases to the left and as x increases to the right. That is the graph will have lowest point but no highest point. See for example the graph of $y = x^2 - 4$ shown in Fig. 8.4.

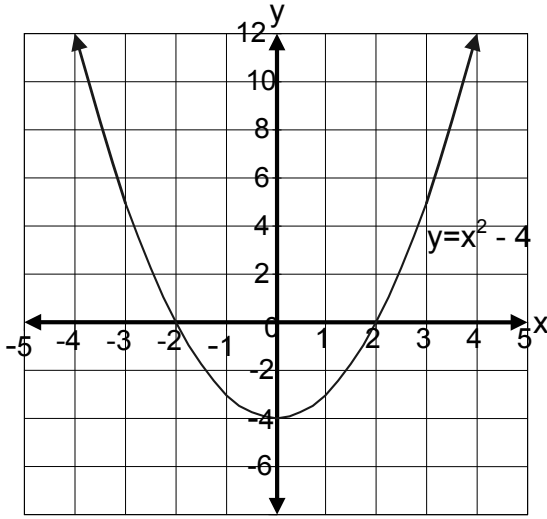


Fig 8.4

- II. When $a_n > 0$ and **n is odd**, its graph decreases without bound as x decreases to the left and increases without bound as x increases to the right. For example, see the graph of the function $y = 0.5x^3 + 2$ shown in Fig. 8.5.

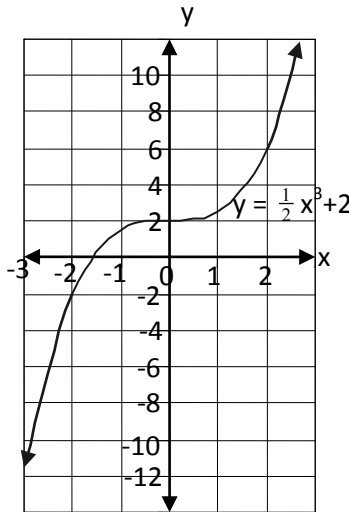


Fig. 8.5

- III. When $a_n < 0$ and **n is even**, its graph decreases without bound as x decreases to the left and as x increases to the right. That is the graph will have highest point but no lowest point such as the graph of $f(x) = -x^2 + 4$ shown in Fig. 8.6.

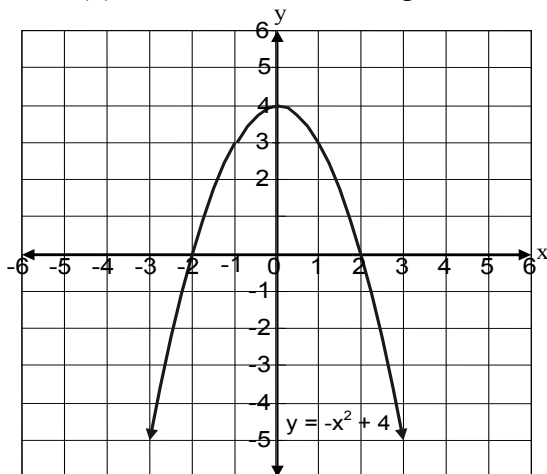


Fig.8.6

- IV. When $a_n < 0$ and **n is odd**, its graph increases without bound as x decreases to the left and decreases without bound as x increases to the right. Look at the graph of $y = -0.5x^3$ shown in Fig. 8.7.

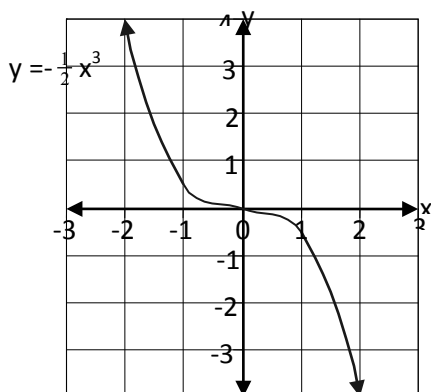
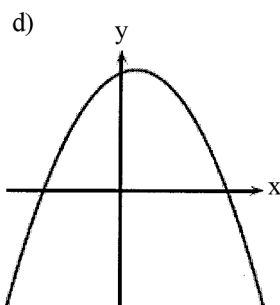
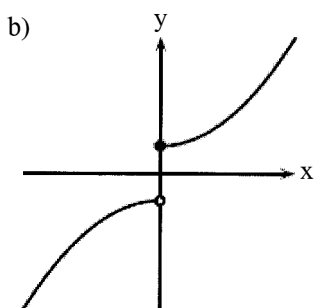
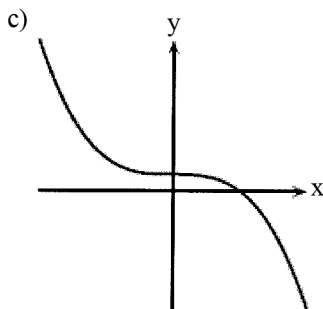
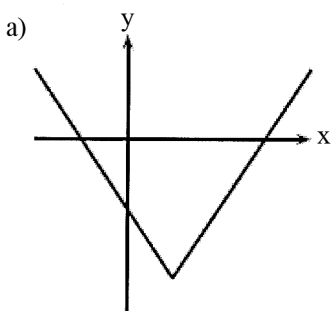


Fig. 8.7

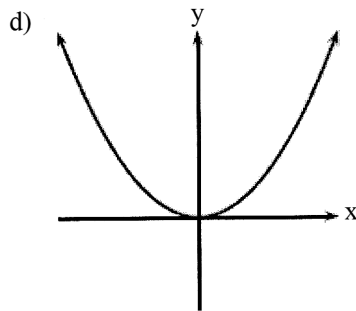
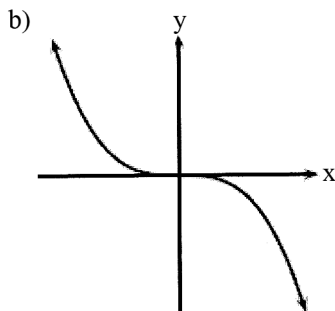
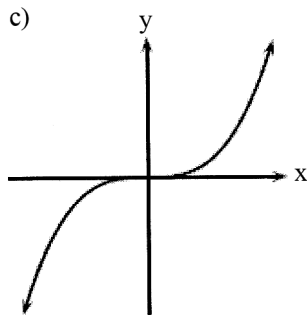
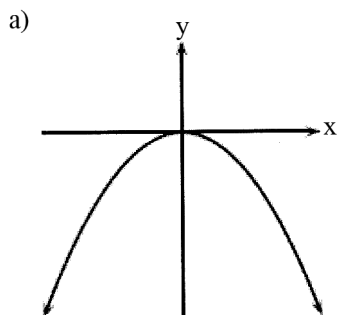
Exercise 8.10

1. In problems a) - d), either give an example of a polynomial that satisfies the given conditions or explain why such a polynomial cannot exist.
 - a) A third-degree polynomial with one x -intercept.
 - b) A fourth-degree polynomial with no x -intercept.
 - c) A third-degree polynomial with no x -intercept.
 - d) A third-degree polynomial with two x -intercepts.
 - e) A third-degree polynomial with three x -intercepts
2. Referring to the graphs of functions shown below, answer questions i) - iii).



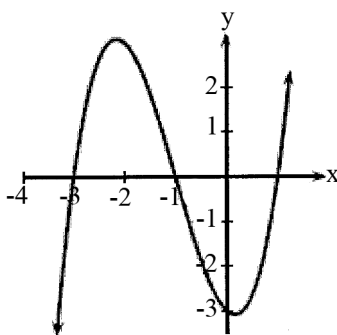
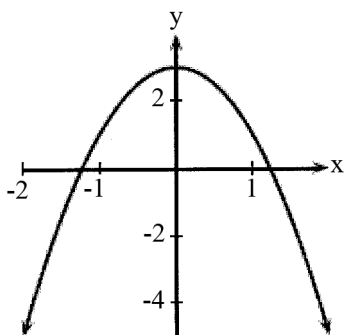
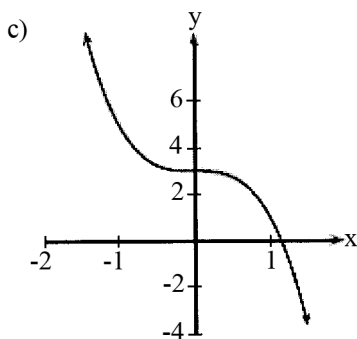
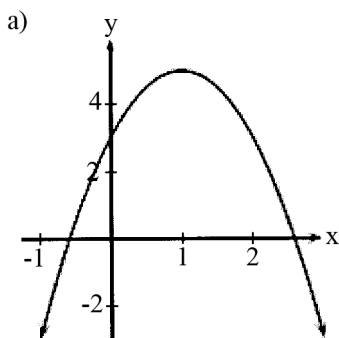
- i. Which of these functions could be second degree polynomial?
- ii. Which of these functions could be a third degree polynomial?
- iii. Which of these functions are not polynomials? Explain why they are not polynomials.

3. For a positive real number a , match each of the following graphs with one of the functions $y = ax^3$, $y = -ax^2$, $y = ax^2$ or $y = -ax^3$.



4. Write true or false. Explain the reason for your answer.
- There is only one x -intercept for the graph of $y = x^2 - 4x + 4$.
 - The graph of $y = (x - 2)^2(x + 3)$ has x -intercepts at -2 and 3 .
 - The y -intercept for the graph of $y = 3x^3 - 7x^2 + 8x - 9$ is 9 .
 - The graph of $y = x^3 - 8$ has three x -intercepts.
5. Graph the following polynomial functions: $y = x^3 - 2x^2$ and $y = -x^3 + 1$. From your graph, determine
- The x -intercept(s) of each function.
 - The y -intercept of each function.
 - The domain and range of each function.

6. The graph of the polynomial function $y = ax^3 + b$ contains the points $(-1, 5)$ and $(1, 1)$. Find a and b , and then sketch the graph.
7. Given two polynomial functions of degree 3 and 2, respectively. Find the degree of
 - a) their sum b) their difference c) their product
8. Match each of the following polynomial functions $y = -2x^2 + 4x + 3$, $y = x^3 + 3x^2 - x - 3$, $y = -2x^3 + 3$, and $y = -2x^2 + 3$ with its graph from a) - d).



Review Exercise 8

1. Say whether the following are true or false. Explain your answers.
 - a) The expression $3x^{-2} - 5x + 2$ is a trinomial.
 - b) $-4r^2$ and $5t^2$ are like terms.
 - c) The degree of the polynomial $x^2 + 3x - 5x^3 + 4$ is 2.
 - d) $4mv^2$ and $-5mv^2$ are unlike terms
 - e) In the polynomial $3x^2 - 5x + 3$ the coefficient of x is 5.
 - f) $-k^3$ and $7k^3$ are like terms.
 - g) $(x + 2)(x + 5) = x^2 + 7x + 10$ for any value of x .
 - h) $(2x - 3)(3x + 5) = 6x^2 + x - 15$ for any value of x .
 - i) $(2 + 3)^2 = 2^2 + 3^2$
 - j) $(8 - 3)^2 = 64 - 9$
 - k) $(x - 7)^2 = x^2 - 49$
 - l) $(5x + 4)^2 = 25x^2 + 16$
 - m) $(60 - 1)(60 + 1) = 3600 - 1$
2. Give an example of two polynomials each of degree three such that their sum has
 - a) degree three
 - b) degree two
 - c) degree one
 - d) no degree
3. Give an example of two polynomials each of degree three such that their difference has
 - a) degree three
 - b) degree two
 - c) degree one
 - d) no degree
4. Use the special product rules:
$$(a + b)(a - b) = a^2 - b^2$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

to find each product.

- | | |
|-----------------------------------|-----------------------|
| a) $(5x + 3)(5x - 3)$ | e) $(3m + 4)^2$ |
| b) $[(2x + 3) - y][(2x + 3) + y]$ | f) $(4y^2 + 1)^2$ |
| c) $(3t - 5u)^2$ | g) $[(2y - 1) + 3]^2$ |
| d) $(3x^2 - 2)(3x^2 + 2)$ | h) $(5a^4 - 2b)^2$ |

5. Perform the indicated operation.

- $(-5x^4 + 2x^3 - 7x - 9) + (3x^3 + x^2 - 9x - 12)$
- $(7x^4 - 10x^3 + 4x^2 - 6x + 11) - (8x^4 - 11x^3 - x^2 - 7x + 10)$
- $(-5x^4 + 2x^3 - 7x - 9) \div (x^2 - 6x + 7)$
- $(-x^2 - 3x - 4)(3x^3 - 2)$

6. For the polynomial $-3x^2 + 7x^5 - 2x - x^3$, give

- the degree
- the constant term
- the leading coefficient
- the coefficient of

7. Give an example of each of the following:

- A monomial of degree seven and leading coefficient negative.
- A binomial of degree four and constant term 0.
- A degree 2 polynomial which cannot be 0 for any x .
- A degree 3 polynomial function with only two x -intercepts.

8. What is the quotient and remainder in:

- $(5x^3 + 3x - 9) \div (x^2 + x - 2)$
- $(3x^3 + 7x^2 - 7x - 3) \div (x^2 + 2x - 3)$

9. When a polynomial is divided by $5x^2 + 3x - 2$, we obtain the quotient $-3x + 1$ and remainder 10. What is the polynomial?

10. The graph of a degree two polynomial of the form $y = ax^2 + bx + c$ has -5 as its y -intercept and passes through the points $(-1, 0)$ and $(5, 0)$. Find the equation of the function, and then draw its graph.