

The State of Eritrea
Ministry of Education

Mathematics

for

Grade 12

Textbook

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Introduction

Dear Learners,

Welcome to the grade 12 Mathematics text book!

This textbook is written based on the new national curriculum; which is learner centred and interactive pedagogy approach. It aims to improve the understanding of mathematics which you have developed in your previous grades and to lay solid foundation for your continued use of mathematics.

The textbook contains four chapters: Sequence and Series, Descriptive Statistics and Probability, Coordinate Geometry, and Matrices and Determinants. Each chapter is further divided into units and subunits. You are already familiar with some of the chapters but they are presented in more detail and with new additional concepts.

By the end of Grade 12, it is expected that you can use and work on the basic concepts of sequence and series. Differentiate arithmetic and geometric progressions. Develop formulas of a variety of sequences. Construct a frequency distribution table from a given data to represent in graphs, calculate the measures of central tendency and measures of dispersion of a given data and use the elementary concept of probability. Furthermore, you will be able to find the distance between any two given points, divide a segment into a given ratios and determine the equation of a line. Finally, you will be equipped with some elementary concept of Matrices and Determinant in order to solve system of linear equations.

This book has been designed and prepared to make mathematics attractive by making the teaching–learning process activity oriented. Every unit and subunit begins with activities that will be attempted by you in class. These activities are intended to develop the skills and creative abilities necessary for learner centred approach in mathematics. It is to be hoped that the activities will help you to grasp and assimilate better the formal and precise definitions of concepts; hence you are expected to do them by your own.

In addition to activities, the book includes examples and exercises. These are given after the basic concepts are introduced with sufficient explanations and formal definitions are stated. The presentation of a stated principles and definition followed by examples, make the concept more understandable and more meaningful. Throughout the text, many examples are worked out in great detail. It may not be possible in the time allocated to discuss all these in class. You should be encouraged to do them on your own, in order to develop your computational skill. Each unit ends with problems to be assigned for class work or homework. These are carefully selected exercises in different varieties to enhance understanding of the basic concepts and principles, and to illustrate their applications in real problems. It is essential that you attempt to solve all of these. No one can master mathematics without a great deal of practice in problem solving. Mathematics is a doing subject. That is, to learn mathematics is to do mathematics. Besides, there are also review questions at the end of each chapter. Review questions are questions that are meant for revision. Both exercises and review questions will help you to understand what is being taught and to what extent you have understood the topic.

Theorems are stated carefully, without mentioning the word theorem and the proofs are provided and are written in a simplified manner intended to be within the reach of the average student. Theorems are mathematical statements that need proofs. In this text they are given as examples or part of exercises to be proved by you. Important ideas, such as definitions, rules, and principles, are given in boxes. This book is written with the intent that you can read, understand, and enjoy and at the same time gain confidence in your ability to use mathematics.

This book is accompanied by a teacher's guide. The guide includes the level of expected learning outcomes, teaching methodology, and information on mechanism and instruments of assessment. Assessment is part and parcel of teaching-learning process. It is a very essential tool for you to know the strength and weakness and improve accordingly. Therefore, as the learning goes on you will be continuously assessed using various methods of assessment.

Parents or guardians and community need to understand and lay their role in the teaching-learning process. The involvement and follow up of parents motivates students to develop positive attitude toward learning and improve their achievement.

Finally the Ministry of Education Department of General Education would like to remind teachers, students, parents, and others that their comments, criticisms and feedbacks for the improvement of the book will be received with thanks.

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CHAPTER 1

Sequence and Series

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- Find the terms of a sequence and derive the general formula for the terms;
- Calculate the terms of arithmetic, geometric and harmonic progressions;
- Apply methods of progression to solve and analyze real situation problems ;
- Formulate equations and inequalities based on the concept of sequence and series; and
- Apply methods of series to solve real situation problems.

1.1 Sequences

Activity 1.1

1. Find the word sequence in a dictionary and discuss its meaning.
2. Consider the following towns: Afabet, Adkieh, Dekemhare and Nakfa.
 - a) Which one librated first? Second? Third and fourth?
 - b) Find all the ordered pairs of the function f given by $f(n) = \text{the } n^{\text{th}} \text{ librated town, where } n = 1, 2, 3 \text{ and } 4.$
 - c) Find the domain of the function in (b).

In mathematics, the word sequence is used in much the same way as in ordinary English. A given set is listed in sequence means that its elements are ordered so that it has a first element, a second element, a third element, and so on.

Example 1.1

Suppose a person has 1025 T cells before infected with HIV. The average rate at which the infection spreads corresponds to a loss

of 75 T cells per year. Once the T cells are less than or equal to 200, then the person begins to suffer the AIDS disease.

- a) What is the number of T cells left after one year of the infection? After 3 years? After 5 years?
- b) After how many years could this person begin to be AIDS patient?

Solution:

- a) The original T cells before the infection are 1025.

Hence, one year after the infection the number of T cells is
 $1025 - 75 = 950$.

After 2 years = $(1025 - 75) - 75 = 1025 - (2)75 = 875$

After 3 years = $(1025 - (2)75) - 75 = 1025 - (3)75 = 800$

After 4 years = $(1025 - (3)75) - 75 = 1025 - (4)75 = 725$

After 5 years = $(1025 - (4)75) - 75 = 1025 - (5)75 = 650$

- b) Let us assume that after n years of the infection the number of T cells is less than or equal to 200.

Then,

$$1025 - (n)75 \leq 200$$

$$-75n \leq 200 - 1025 = -825$$

$$n \geq 825/75 = 11$$

Hence, after 11 years, the number of T cells will be less than or equal to 200, which means after 11 years he will begin to suffer from AIDS disease.

Activity 1.2

1. Consider the set of positive even integers.
 - a) List the positive even integers in ascending order.
 - b) What is the first number in the list?
 - c) What is the third number in the list?
 - d) What is the seventh number in the list?
 - e) What is the twentieth number in the list?
2. Consider the function f defined by:
 $f(n)$ = the n^{th} positive even integer, where n is positive integer.
 - a) Find $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$.
 - b) Try to find a general formula that describes the function f .
 - c) Find the domain and range of the function f .

3. For each of the following sequence of numbers, find the next three terms.
- a) 1, 5, 9, 13, ...
- b) 1, 2, 5, 10, 17, 26, ...
- c) $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \dots$
4. For each of the sequences in task (3) try to find a general formula that describes the given sequence.

A **sequence** in mathematics is a function whose domain is the set of positive integers. The function values are labelled

$$a_1, a_2, a_3, a_4, a_5, \dots, a_n, \dots,$$

and are the **terms** of the sequence.

If the domain consists of the first “n” positive integers only, then the sequence is a **finite sequence**.

Example 1.2

Consider the set A of all positive integers such that when divided by 4 the remainder is 2.

- a) List the elements of set A in ascending order.
- b) What is the first number in the list? The second number? The third number? The fourth number?
- c) Find a general formula that describes the given sequence.

Solution:

- a) The elements of A listed in ascending order is:
6, 10, 14, 18, 22, ...
- b) The first is 6. The second is 10. The third is 14 and the fourth is 18.
- c) To find the general formula we have to see the pattern. That is,

$$a_1 = 6$$

$$a_2 = 10 = 6 + 4 = 6 + (1)4$$

$$a_3 = 14 = 10 + 4 = 6 + 4 + 4 = 6 + 2(4)$$

$$a_4 = 18 = 14 + 4 = 6 + 4 + 4 + 4 = 6 + 3(4)$$

$$a_5 = 22 = 18 + 4 = 6 + 4 + 4 + 4 + 4 = 6 + 4(4)$$

From the above we have the following:

$$a_6 = 6 + 5(4), a_7 = 6 + 6(4) \text{ and } a_8 = 6 + 7(4)$$

Hence, the general formula of the above sequence is:

$$\begin{aligned} a_n &= 6 + (n-1)(4) \\ &= 6 + 4n - 4 \\ &= 4n + 2. \end{aligned}$$

Example 1.3

For the sequence given by:

$$a_n = \frac{8}{2n-1};$$

- Find the first 4 terms of the sequence.
- Draw the graph of the sequence.

Solution:

$$\text{a) First term} = a_1 = \frac{8}{2(1)-1} = 8$$

$$\text{Second term} = a_2 = \frac{8}{2(2)-1} = \frac{8}{3}$$

$$\text{Third term} = a_3 = \frac{8}{2(3)-1} = \frac{8}{5}$$

$$\text{Fourth term} = a_4 = \frac{8}{2(4)-1} = \frac{8}{7}$$

- In order to draw the graph, we have to construct a table of values.

n	1	2	3	4	5	6	7	8
$a_n = \frac{8}{2n-1}$	8	2.67	1.6	1.14	0.89	0.73	0.6	0.5

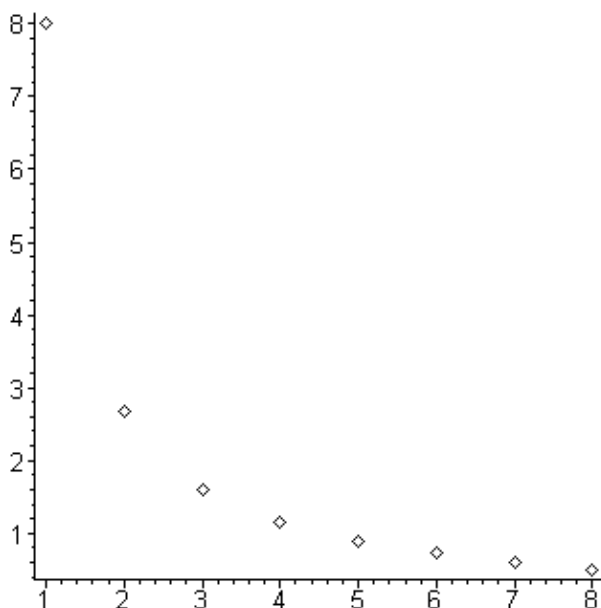


Fig. 1.1

Remark: The general formula of a sequence is also known as the n^{th} term formula of the sequence.

Activity 1.3

- In given sequence $a_1 = 2$ and $a_n = a_{n-1} + 3$, for $n > 1$. Find:
 - The first 5 terms of the given sequence.
 - The tenth, fifteenth and twentieth terms of the sequence.
 - Try to find a general formula for the given sequence.
- For the sequence given by $a_1 = 4$ and $a_n = 2(a_{n-1}) + 1$, where $n > 1$, find:
 - The first 4 terms of the given sequence.
 - The seventh, eleventh and fifteenth terms of the sequence.

Given the first few terms of a sequence, if all the rest terms of the sequence are then defined using previous terms, we say the sequence is defined recursively.

For instance, sequences discussed in Activity 1.3 are recursive sequences.

Example 1.4

Find the first 5 terms of the recursively defined sequence.

a) $a_1 = 3$ and $a_n = n + a_{n-1}$, for $n > 1$.

b) $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n > 2$.

Solution:

a) Given $a_1 = 3$ and $a_n = n + a_{n-1}$, for $n > 1$, then

$$a_2 = 2 + a_{2-1} = 2 + a_1 = 2 + 3 = 5$$

$$a_3 = 3 + a_{3-1} = 3 + a_2 = 3 + 5 = 8$$

$$a_4 = 4 + a_{4-1} = 4 + a_3 = 4 + 8 = 12$$

$$a_5 = 5 + a_{5-1} = 5 + a_4 = 5 + 12 = 17$$

b) $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n > 2$, then

$$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 1 = 2$$

$$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 1 + 2 = 3$$

$$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 2 + 3 = 5$$

Factorials

Consider the sequence defined as follows:

$$a_1 = 1$$

$$a_2 = 1 \times 2 = 2$$

$$a_3 = 1 \times 2 \times 3 = 6$$

$$a_4 = 1 \times 2 \times 3 \times 4 = 24$$

$$a_5 = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

The general formula of the above sequence is given as:

$$a_n = 1 \times 2 \times 3 \times \cdots \times (n-2)(n-1)n$$

For n a natural number n , **n -factorial**, denoted by $n!$ is the product of the first n natural numbers. That is,

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-2)(n-1)n.$$

As a special case, zero factorial is defined as $0! = 1$.

For instance, $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

Exercise 1.1

1. For each of the following sequences, write the first five terms.

a) $a_n = 4n - 1$

d) $a_n = \sqrt{n}$

b) $a_n = n^2 - 3n + 1$

e) $a_n = \frac{2n+1}{n}$

c) $a_n = 3^n + 1$

2. For each of the following sequences, find the indicated term of the sequence.

a) $a_n = (-1)^n (2n + 3)$, a_7

b) $a_n = \frac{4n}{2n^2 - 3}$, a_{11}

c) $a_n = \frac{n}{n!}$, a_6

d) $a_n = 1 + \log 10^n$, a_{25}

3. For each of the following sequence of numbers, find the next three terms.

a) 8, 5, 2, -1, -4, ...

b) 3, -6, 12, -24, 48, ...

c) -1, 2, 7, 14, 23, ...

d) $\frac{1}{3}, -\frac{2}{9}, \frac{1}{9}, -\frac{4}{81}, \dots$

4. For each of the sequences in problem 3 of the above, find the general formula that describes the sequence.
5. For each of the following sequences defined recursively, find the first five terms.
 - a) $a_1 = 2$ and $a_n = 5 + a_{n-1}$, for $n > 1$
 - b) $a_1 = 60$ and $a_n = 0.5a_{n-1} - 2$, for $n > 1$
 - c) $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-2} + 2a_{n-1} - 3$, for $n > 2$
6. Simplify the following, where n is a positive integer.

a) $\frac{8!}{4!}$	c) $\frac{(n+1)!}{n!}$
b) $\frac{9!}{7!2!}$	d) $\frac{(n+1)!}{(n+3)!}$

7. For the years 1981 to 2010, the resident population of a town is approximated by the following sequence.

$$a_n = n^2 + 2n + 204, \quad n = 1, 2, 3, \dots, 30$$

where a_n is the population in thousands and n represents the year, with $n = 1$ corresponding to 1981.

- a) Find the population of the town in the years 1981, 1982 and 1983.
 - b) Find the last three terms of the sequence.
8. For the sequence given by $a_n = \frac{4^n}{n!}$
 - a) Find the first 5 terms of the sequence.
 - b) Draw the graph of the sequence.

1.2 Arithmetic Progression

Activity 1.4

- Consider the following sequence:
2, 7, 12, 17, 22, ...
 - What is the next term of the sequence?
 - Find the seventh, eighth, ninth and the tenth terms of the sequence.
 - Do you observe that every next term of the sequence is obtained by adding a fixed number to the previous term?
 - What is the fixed number added?
 - Find a general formula for the sequence.
- Repeat (a), (b), (c), (d) and (e) of task 1 for the sequence
18, 15, 12, 9, ...
- Consider a sequence with first term equal to A_1 , the second term $A_1 + d$, the third term is $A_1 + d + d = A_1 + 2d$, and so on. That is, every next term is obtained by adding “d” to the previous term, where d is a real number.
 - What is the fourth term? The fifth term? The sixth term?
 - Find a general formula for the sequence.

In a sequence where the first term is given and every next term is obtained by adding a fixed number called common difference d is an arithmetic sequence or an **arithmetic progression**. That is,

The first term is A_1 ,

The second term is $A_2 = A_1 + d$

The third term is $A_3 = A_2 + d$

The fourth term is $A_4 = A_3 + d$, and so on.

Consider an arithmetic progression with first term A_1 and common difference d. Then;

$$\text{The first term} = A_1 = A_1 + 0(d) = a_1 + (1-1)d$$

$$\text{The second term} = A_2 = A_1 + 1(d) = A_1 + (2-1)d$$

$$\text{The third term} = A_3 = A_2 + d = A_1 + 2d = A_1 + (3-1)d$$

$$\text{The fourth term} = A_4 = A_3 + d = A_1 + 3d = A_1 + (4-1)d$$

$$\text{The fifth term} = A_5 = A_4 + d = A_1 + 4d = A_1 + (5-1)d$$

The general formula for the arithmetic progression with first term A_1 and common difference “d” is equal to:

$$A_n = A_1 + (n - 1)d.$$

Example 1.5

Given $A_1 = 5$ and $d = 8$, then find:

- The first four terms of the arithmetic progression.
- The general formula of the arithmetic progression.

Solution:

- Given $A_1 = 5$ and $d = 8$:

$$\text{The second term} = A_2 = A_1 + 8 = 5 + 8 = 13$$

$$\text{The third term} = A_3 = A_2 + 8 = 13 + 8 = 21$$

$$\text{The fourth term} = A_4 = A_3 + 8 = 21 + 8 = 29$$

- The general formula A_n of the arithmetic progression is given by:

$$\begin{aligned} A_n &= A_1 + (n - 1)d = 5 + (n - 1)8 \\ &= 5 + 8n - 8 \\ &= 8n - 3 \end{aligned}$$

Example 1.6

The first and the fourth terms of an arithmetic progression are 4 and 19. Then

- Find the common difference of the arithmetic progression.
- Find the n^{th} term of the arithmetic progression,
- Find the 8^{th} , 10^{th} and 14^{th} terms of the sequence.

Solution:

- Given $A_1 = 4$ and $A_4 = 19$, then we have

$$A_4 = A_1 + 3d = 19, \text{ where } d \text{ is the common difference.}$$

$$A_4 = 4 + 3d = 19$$

$$3d = 19 - 4 = 15$$

$$d = 5$$

Hence, the common difference is 5.

Given the first term A_1 and the common difference d , then the general formula of the arithmetic progression is equal to:

$$A_n = A_1 + (n-1)d.$$

Hence,

$$\begin{aligned} A_n &= 4 + (n-1)5 = 4 + 5n - 5 \\ &= 5n - 1. \end{aligned}$$

b) The eighth term $A_8 = 5(8) - 1 = 40 - 1$
 $= 39$

The tenth term $A_{10} = 5(10) - 1 = 10 - 1$
 $= 49$

The fourteenth term $A_{14} = 5(14) - 1 = 70 - 1$
 $= 69$

Example 1.7

A business makes a profit of 10,000 Nakfa during its first year. If the yearly profit increases by 2,500 Nakfa in each subsequent year, then:

- a) What will be the profit in the third year, in the fifth year?
- b) What will be the profit during the n^{th} year?

Solution:

The profit during the first year is 10,000 Nakfa. Then,

a) The profit in 2nd year $= 10,000 + 2500$
 $= 12,500$

The profit in 3rd year $= 10,000 + 2500 + 2500$
 $= 15000$

The profit in 4th year $= 10,000 + 2500 + 2500 + 2500$
 $= 17500$

The profit in 5th year $= 10,000 + 2500 + 2500 + 2500 + 2500$
 $= 20,000$

- b) From (a) in the above we observe that the profits in the subsequent years form an arithmetic progression with

$$A_1 = 10,000 \text{ and } d = 2500.$$

Hence, the profit during the n^{th} is equal to:

$$\begin{aligned} A_n &= A_1 + (n-1)d = 10,000 + (n-1) 2500 \\ &= 10,000 + 2500n - 2500 \\ &= 2500n + 7500 \end{aligned}$$

Arithmetic Mean

If three numbers a , b and c are in arithmetic progression, then b is the arithmetic mean of a and c .

Let a and b be any two numbers. Then, x is arithmetic mean of a and b if a , x and b are in arithmetic progression. That is,

$$\begin{array}{ll} x - a = b - x & \text{Why?} \\ 2x = a + b & \text{Why?} \end{array}$$

$$x = \frac{a+b}{2} \quad \text{Why?}$$

Example 1.8

- Find the arithmetic mean of 5 and 13.
- For what value of x will -3 be the arithmetic mean of $2x$ and $6 - 5x$?

Solution:

- Let x be the arithmetic mean of 5 and 13. Then

$$\begin{aligned} x &= \frac{5+13}{2} \\ &= 9 \end{aligned}$$

Hence, the arithmetic mean of 5 and 13 is 9.

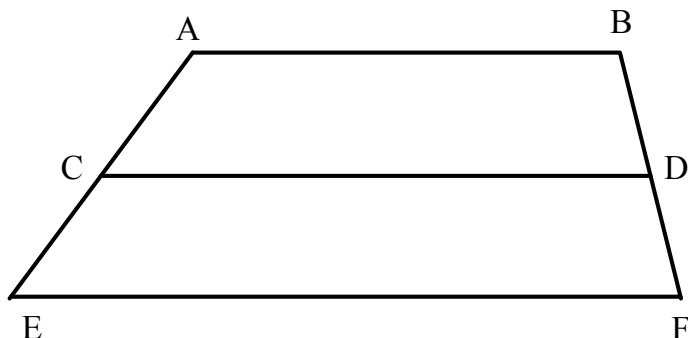
- If -3 the arithmetic mean of $2x$ and $6 - 5x$, then

$$\begin{aligned} \frac{2x + (-5x + 6)}{2} &= -3 \\ 2x - 5x + 6 &= -6 \\ -3x &= -12 \\ x &= 4 \end{aligned}$$

Exercise 1.2

1. For each of the following sequences determine if the sequence is an arithmetic progression. If it is find the common difference.
 - a) 3, 7, 11, 15, 19, ...
 - b) 1, 4, 8, 13, 19, ...
 - c) 0.5, 3, 5.5, 8, 10.5, ...
 - d) $\pi, \frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \dots$
 - e) 1, 5, 9, 14, 18, ...
 2. Determine if the given sequence is arithmetic progression. If it is find the common difference.
 - a) $a_n = -2n + 5$
 - b) $a_n = n^2$
 - c) $a_n = 5n - 3$
 - d) $a_n = 3(2^n)$
 3. Find the general formula of the following arithmetic progressions.
 - a) $A_1 = 7, d = 8$
 - b) $A_1 = 4, A_2 = -1$
 - c) $A_3 = 11, A_7 = 3$
 - d) $A_5 = 2, d = 3$
 4. The temperature at the top of a mountain at 5 PM is 36° F. If the temperature is decreased by 3°F every half-hour for the next 7 hours, at what time did the temperature become 0°F?
 5. A new school has 250 students at the beginning. If the school receives 32 new students every year, what will be the number students after 4 years?
 6. Find the arithmetic mean for each of the following:
 - a) 3.5 and 7
 - b) $2 - \sqrt{2}$ and $2 + \sqrt{2}$
 - c) $9x$ and $5x$
 - d) $(a - b)^2$ and $(a + b)^2$
 7. For what values of x will 2 be the arithmetic mean of $x^2 - 1$ and $2x + 2$?
-

8. In the figure below ABFE is trapezium with $\overline{AB} \parallel \overline{EF}$ and points C and D are mid points of $\overline{AE}$ and $\overline{BF}$ respectively. Then show that CD is arithmetic mean of AB and EF.



Activity 1.5

- Consider the arithmetic progression given by;
 $1, 2, 3, 4, 5, \dots$
 - Find the common difference and the formula of the general term of the sequence.
 - Obtain a new sequence from the above given sequence as follows:
 The sum of the first term, $S_1 = 1$
 The sum of the first 2 terms $S_2 = 1 + 2 = 3$
 The sum of the first 3 terms $S_3 = 1 + 2 + 3 = 6$
 The sum of the first 4 terms $S_4 = 1 + 2 + 3 + 4 = 10$

 Find the 10^{th} , 15^{th} , 20^{th} and 50^{th} terms of the above new sequence.
 - Try to find a general formula for the new sequence.
- For the arithmetic progression given by;
 $2, 4, 6, 8, 10, \dots$
 Repeat (a), (b) and (c) of task 1.
- Consider an arithmetic progression, with first term a_1 , and common difference d .
 - Find the first 5 terms of the sequence.

- b) Obtain a new sequence by adding the terms similar to task 1 and 2 of the above. That is,

$$S_1 = A_1,$$

$$S_2 = A_1 + A_2 = A_1 + (A_1 + d) = 2A_1 + d$$

$$S_3 = A_1 + A_2 + A_3 = A_1 + (A_1 + d) + (2A_1 + d)$$

$$S_3 = (A_1 + 2d) + (A_1 + d) + A_1$$

$$S_3 = (A_1 + 2d) + (A_1 + d) + A_1$$

$$2S_3 = (2A_1 + 2d) + (2A_1 + 2d) + (2A_1 + 2d) = 3(2A_1 + 2d)$$

Hence,

$$S_3 = \frac{3(2A_1 + 2d)}{2} = \frac{3}{2}(A_1 + (A_1 + 2d)) = \frac{3}{2}(A_1 + A_3)$$

$$S_4 = A_1 + A_2 + A_3 + A_4 = A_1 + (A_1 + d) + (A_1 + 2d) + (A_1 + 3d)$$

$$S_4 = A_1 + (A_1 + d) + (A_1 + 2d) + A_1 + 3d$$

$$S_4 = (a_1 + 3d) + (a_1 + 2d) + (a_1 + d) + a_1$$

$$\begin{aligned} 2S_4 &= (2A_1 + 3d) + (2A_1 + 3d) + (2A_1 + 3d) + (2A_1 + 3d) \\ &= 4(2a_1 + 3d) \end{aligned}$$

Hence,

$$S_4 = \frac{4(2A_1 + 3d)}{2} = \frac{4}{2}(A_1 + (A_1 + 3d)) = \frac{4}{2}(A_1 + A_4).$$

- c) Using the same method like in (b) of the above find S_5 , S_6 , S_7 , S_8 , S_9 , and S_{10} .
- d) Find a formula for S_n (the sum of the first n terms of the sequence).

Given an arithmetic progression with first term a_1 and common difference “d”, the sum of the first “n” terms S_n is equal to:

$$\begin{aligned} S_n &= \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(A_1 + A_1 + (n-1)d) \\ &= \frac{n}{2}(2A_1 + (n-1)d) \end{aligned}$$

Example 1.9

Find the sum of the first 10 terms of an arithmetic progression if $A_1 = 4$ and $d = 6$.

Solution:

Given $A_1 = 4$ and $d = 6$, then we have

$$\begin{aligned} S_{10} &= \frac{10(2(4) + (10-1)6)}{2} \\ &= 5(8 + 54) \\ &= 310. \end{aligned}$$

In order to find the sum of the first 10 terms we can use the formula:

$$S_{10} = \frac{10}{2}(A_1 + A_{10}), \text{ by first finding } A_{10}$$

Example 1.10

In arithmetic progression the fifth term is 44 and the seventeenth term 152. Find the sum of the first 20 terms of the progression.

Solution:

Let the first term of the arithmetic progression be A_1 and the common difference be d .

Then, $A_5 = A_1 + (5-1)d = A_1 + 4d = 44$, and

$$A_{17} = A_1 + (17-1)d = A_1 + 16d = 152.$$

From the above we get the following:

$$\begin{cases} A_1 + 4d = 44 \\ A_1 + 16d = 152 \end{cases}$$

Solving the simultaneous equation we have $A_1 = 8$ and $d = 9$.

Hence,

$$\begin{aligned} S_{20} &= \frac{n(2A_1 + (n-1)d)}{2} = \frac{20(2(8) + (20-1)9)}{2} \\ &= 10(16 + 19(9)) \\ &= 1870 \end{aligned}$$

Example 1.11

Show that the sum of the first n positive odd integers is n^2 .

Solution:

Given the set of positive odd integers, we can form an arithmetic progression with $A_1 = 1$ and the common difference $d = 2$.

Using the above, the n^{th} term is given by:

$$A_n = A_1 + (n-1)d = 1 + 2(n-1) = 1 + 2n - 2 = 2n - 1$$

Hence, the sum of the first n positive even integers is equal to:

$$S_n = \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(1 + 2n - 1) = \frac{2n^2}{2} = n^2$$

Example 1.12

An object with negligible air resistance is dropped from a plane. During the first second the object falls 4.9 m; during the 2nd second, it falls 14.7 m; during the 3rd second, it falls 24.5 m; during the 4th second, it falls 34.3 m. If this trend continues, how many meters will the object fall in ten seconds?

Solution:

It is given:

In the first second the object falls 4.9m.

In the 2nd second the object falls 14.7m.

In the 3rd second the object falls 24.5m.

In the 4th second the object falls 34.3m.

If the trend continues like the above, then we have an arithmetic progression with

$$A_1 = 4.9 \quad \text{and} \quad d = 9.8\text{m}$$

Hence, in ten seconds the object falls:

$$\begin{aligned} S_{10} &= \frac{n(2A_1 + (n-1)d)}{2} = \frac{10(2(4.9) + (10-1)9.8)}{2} \\ &= 5(9.8 + 9(9.8)) \\ &= 490\text{m}. \end{aligned}$$

Exercise 1.3

1. For each of the following arithmetic progressions find the indicated sum.
 - a) $A_1 = 7, d = 2, S_{10}$
 - b) $A_3 = 10, A_7 = -2, S_8$
 - c) $A_n = 3n + 8, S_{11}$
 - d) $A_{11} = -3, A_{31} = -2, S_5$
 - e) $A_n = -5n + 2, S_{12}$
2. Find the sum of the integers from -20 to 80 .
3. Find the sum of all positive multiples of 7 up to 700 .
4. The sum of the first 20 terms of arithmetic progression with a common difference of 3 is 650 . Find the first term.
5. There are 16 school football teams in a certain tournament. The school teams are promised to be awarded a total of $8,000$ Nakfa. If the last-place team is awarded 275 Nakfa, and the

award increases by the same amount for each successive finishing place, how much will the first-place team receive?

6. A school hall has 20 rows of seats. There are 10 seats in the first row, 12 seats in the second row and 14 seats in the third row, and so on. How many seats are there in all 20 rows?

1.3 Geometric Progression

Activity 1.6

1. Consider the following sequence:
1, 2, 4, 8, 16, ...
 - a) What is the next term of the sequence?
 - b) Find the seventh, eighth, ninth and the tenth terms of the sequence.
 - c) Do you observe that every next term of the sequence is obtained by multiplying a fixed number to the previous term?
 - d) What is the fixed number used in the multiplication?
 - e) Find a general formula for the sequence.
2. Repeat (a), (b), (c), (d) and (e) of task 1 for the sequence

$$27, 9, 3, 1, \frac{1}{3}, \dots$$

3. Consider a sequence with first term equal to G_1 , the second term G_1r , the third term is $G_1r \times r = G_1r^2$, and so on. That is, every next term is obtained by multiplying “ r ” to the previous term, where r is a real number.
 - a) What is the fourth? The fifth term? The sixth term?
 - b) Find a general formula for the sequence.

In a sequence, where the first term is given and every next term is obtained by multiplying by a fixed number called common ratio r is a geometric sequence or a **geometric progression**. That is

The first term is G_1 ,
 The second term is $G_2 = G_1 r$
 The third term is $G_3 = G_2 r$
 The fourth term is $G_4 = G_3 r$, and so on.

Consider a geometric progression with first term G_1 and common ratio r . Then;

$$\text{The first term} = G_1 = G_1 r^0 = G_1 r^{1-1}$$

$$\text{The second term} = G_2 = G_1 r^1 = G_1 r^{2-1}$$

$$\text{The third term} = G_3 = G_2 r = (G_1 r) r = G_1 r^2 = G_1 r^{3-1}$$

$$\text{The fourth term} = G_4 = G_3 r = (G_1 r^2) r = G_1 r^3 = G_1 r^{4-1}$$

$$\text{The fifth term} = G_5 = G_4 r = (G_1 r^3) r = G_1 r^4 = G_1 r^{5-1}$$

The general formula for the geometric progression with first term G_1 and common difference “ r ” is equal to:

$$G_n = G_{n-1} r = G_1 r^{n-1}.$$

Example 1.13

For the geometric progression whose first two terms are 2 and $\frac{-2}{5}$:

- Find the common ratio.
- Find a formula for the n^{th} term.

Solution:

- Given a geometric progression with $G_1 = 2$ and $G_2 = \frac{-2}{5}$, then

$$G_2 = G_1 r = \frac{-2}{5}, \text{ where } r \text{ is the common ratio.}$$

$$\text{Hence, } 2r = \frac{-2}{5}$$

$$r = \frac{-2}{5} \times \frac{1}{2} = \frac{-1}{5}$$

- b) Given the first term and the common ratio of geometric progression the n^{th} term is:

$$G_n = G_1 r^{n-1} = 2 \left(\frac{-1}{5} \right)^{n-1}$$

Example 1.14

The number of students per computer in a certain school in year n (with $n=1$ corresponding to 2008) can be approximated by a geometric progression whose first two terms are 128 and 64.

- a) Find a formula for the n^{th} term.
b) What is the number of students per computer in 2011?
c) In what year will there be a computer for each student?

Solution:

The number of students per computer in the year n , where $n=1$ corresponds 2008 ($n=1$) follows a geometric progression with:

$$G_1 = 128 \text{ and } G_2 = 64$$

- a) Then $G_2 = G_1 r = 64$, where r is the common ratio.

$$128r = 64$$

$$r = \frac{1}{2}$$

$$\begin{aligned} \text{Hence, the } n^{\text{th}} \text{ term } G_n &= G_1 r^{n-1} = 128 \left(\frac{1}{2} \right)^{n-1} = 2^7 (2)^{1-n} \\ &= 2^{8-n} \end{aligned}$$

- b) The year 2011 corresponds to $n = 4$

$$\text{That is, } G_4 = 2^{8-4} = 2^4 = 16$$

Hence, the number of students per computer in the year 2011 is 16.

- c) A computer for each student means, the number of students per computer is 1

Let n be a positive integer such that $G_n = 1$

That is, $G_n = 2^{8-n} = 1 = 2^0$

$$8 - n = 0$$

$$n = 8$$

The year that corresponds to $n = 8$ is 2015

Geometric Mean

If three numbers a , b and c are in geometric progression, then b is the geometric mean of a and c .

Let a and b be any two numbers. Then, x is geometric mean of a and b if a , x and b are in a geometric progression. That is,

$$\frac{x}{a} = \frac{b}{x} \quad \text{Why?}$$

$$x^2 = ab \quad \text{Why?}$$

$$x = \sqrt{ab} \quad \text{Why?}$$

Note that the geometric mean is usually considered to be a positive number.

Example 1.15

- a) Find the geometric mean of 8 and 12.
 b) If the geometric mean of two numbers is 10 and one of the terms is $10\sqrt{2}$, then what is the other term?

Solution:

- a) Let x be the geometric mean of 6 and 72. Then

$$x = \sqrt{(8)(12)}$$

$$= \sqrt{96}$$

$$= 4\sqrt{6}$$

Hence, the geometric mean of 8 and 12 is $4\sqrt{6}$

- b) Let x be the other term. Then x , 10 and $10\sqrt{2}$ are in geometric progression.

Hence,

$$\begin{aligned}\sqrt{x(10\sqrt{2})} &= 10 \\ x(10\sqrt{2}) &= 100 \\ x &= 5\sqrt{2}\end{aligned}$$

Exercise 1.4

1. Determine if the given sequence is geometric progression. If it is find the common ratio.

a) $-11, -16, -21, \dots$

b) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \dots$

c) $\frac{1}{25}, \frac{-1}{10}, \frac{1}{4}, \frac{-5}{8}, \dots$

d) $3, 12, 21, 30, \dots$

e) $800, 1360, 2312, 3930.4, \dots$

2. Determine if the given sequence is geometric progression. If it is find the common ratio.

a) $a_n = 3(2^n)$

c) $a_n = (4^n)\left(\frac{2}{3}\right)^n$

b) $a_n = 3 + (2^n)$

d) $a_n = 5^n + 10^n$

3. Find the general formula of the following geometric progressions.

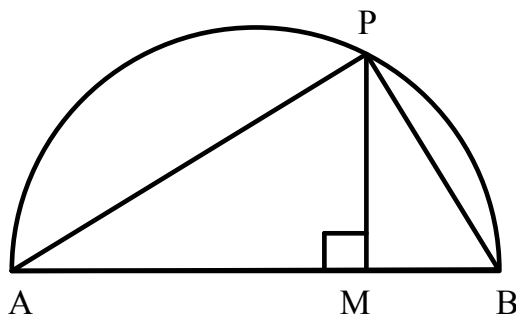
a) $G_1 = 1, r = 4$

c) $G_1 = 5, G_2 = -1$

b) $G_1 = -81, r = \frac{-1}{3}$

d) $G_5 = 6, G_7 = \frac{27}{8}$

4. A principal of 1,500 Nakfa is invested at 4% compound interest calculated semi-annually.
 - a) Find the amount after one year.
 - b) Find the amount after 3 years.
 - c) Find a general formula that describes the amount after n years.
 - d) Find the amount after 10 years.
5. The population of a country starting from 2007 were, 3600000, 4320000, 5184000 and 6220800, $\dots$. If this rate continues, what will be the population of the country in 2012?
6. A single cell of bacteria reproduces through the process called binary fission. Escherichia coli cells divide into two every 20 minutes. Suppose the same rate of division is maintained for 24 hours after the original cell enters the body. How many bacteria cells would be in the body 12 hours later? 24 hours later?
7. Find the geometric mean for each of the following:
 - a) 3 and 12
 - b) 0.6 and 0.8
 - c) $3 - \sqrt{5}$ and $3 + \sqrt{5}$
 - d) ab^2 and ab^4
8. If M is the foot perpendicular drawn from a point P of a semi-circle to the diameter $\overline{AB}$ (see the figure below). Show that MP is the geometric mean of MB and MA .



Activity 1.7

1. Consider the geometric progression given by;
 $1, 3, 9, 27, 81, \dots$
 - a) Find the common ratio and the formula of the general term of the sequence.
 - b) Obtain a new sequence from the above given sequence as follows:
 The sum of the first term, $S_1 = 1$
 The sum of the first 2 terms $S_2 = 1 + 3 = 4$
 The sum of the first 3 terms $S_3 = 1 + 3 + 9 = 13$
 The sum of the first 4 terms $S_4 = 1 + 3 + 9 + 27 = 40$

 Find the 6th, 8th, 10th and 15th terms of the above new sequence.
 - c) Try to find a general formula for the new sequence.

2. For the geometric progression given by;
 $32, 16, 8, 4, \dots$
 Repeat (a), (b) and (c) of task 1.
3. Consider a geometric progression, with first term G_1 , and common ratio r ($r \neq 1$).
 - a) Find the first 5 terms of the geometric progression.
 - b) Obtain a new sequence by adding the terms similar to task 1 and 2 of the above. That is,

$$S_1 = G_1,$$

$$S_2 = G_1 + G_2 = G_1 + G_1 r = G_1(1 + r)$$

$$S_3 = G_1 + G_2 + G_3 = G_1 + G_1 r + G_1 r^2$$

Then, we have:

$$-S_3 = -G_1 - G_1 r - G_1 r^2$$

$$rS_3 = G_1 r + G_1 r^2 + G_1 r^3$$

$$rS_3 - S_3 = -G_1 + 0 + 0 + G_1 r^3 = -G_1 + G_1 r^3 = G_1(-1 + r^3)$$

$$S_3(r - 1) = G_1(-1 + r^3)$$

Hence,

$$S_3 = \frac{G_1(-1+r^3)}{r-1} = \frac{G_1(1-r^3)}{1-r}.$$

$$S_4 = G_1 + G_2 + G_3 + G_4 = G_1 + G_1r + G_1r^2 + G_1r^3$$

Then,

$$-S_4 = -G_1 - G_1r - G_1r^2 - G_1r^3$$

$$rS_4 = G_1r + G_1r^2 + G_1r^3 + G_1r^4$$

$$rS_4 - S_4 = -G_1 + 0 + 0 + 0 + G_1r^4 = -G_1 + G_1r^4 = G_1(-1+r^4)$$

Hence,

$$S_4(r-1) = G_1(-1+r^4)$$

$$S_4 = \frac{G_1(-1+r^4)}{r-1} = \frac{G_1(1-r^4)}{1-r}$$

- c) Using the same method like in (b) of the above find S_5 , S_6 , S_7 , S_8 , S_9 , and S_{10}
- d) Find a formula for S_n (the sum of the first n terms of the sequence).
- e) Find a formula for S_n , when $r = 1$

Given geometric progression with first term g_1 and common ratio “ r ”, the sum of the first “ n ” terms S_n is equal to:

$$S_n = \frac{G_1(-1+r^n)}{r-1} = \frac{G_1(1-r^n)}{1-r}, \text{ for } r \neq 1$$

If $r = 1$, then every term in the geometric progression is equal to the first term. That is, the formula for the n^{th} term is simply G_1

Hence,

$$\begin{aligned} S_n &= G_1 + G_1 + G_1 + \cdots G_1 \text{ (n terms)} \\ &= nG_1, \text{ for } r = 1 \end{aligned}$$

Example 1.16

In geometric progression the third term and the fifth term are 6 and $\frac{3}{2}$ respectively.

- Find the first term and the common ratio if all the terms are positive. If the terms are positive and negative alternately.
- Find the sum of the first 10 terms of the progression when the terms are positive. When the terms are positive and negative alternately.

Solution:

- a) Let the first term be g_1 and the common ratio is r .

$$\text{Then, } G_3 = G_1 r^2 = 6$$

$$G_5 = G_1 r^4 = \frac{3}{2}$$

$$= (G_1 r^2) r^2 = \frac{3}{2}$$

$$= 6r^2 = \frac{3}{2}, \text{ since } G_1 r^2 = 6$$

Hence,

$$6r^2 = \frac{3}{2}$$

$$r^2 = \frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$$

$$r = \pm \frac{1}{2}$$

If $r = \frac{1}{2}$, then the terms are all positive and if $r = -\frac{1}{2}$, then the terms are positive and negative alternately.

In any case the first term $G_1 = 24$

- b) The sum of the first ten terms when the terms are all positive is equal to:

$$S_{10} = \frac{G_1(1-r^n)}{1-r} = \frac{24(1-(0.5)^{10})}{1-0.5}$$

$$\begin{aligned}
 &= 48(1 - (0.5)^{10}) \\
 &= 48\left(1 - \frac{1}{2^{10}}\right) = 48\left(\frac{2^{10} - 1}{2^{10}}\right) \\
 &= 48\left(\frac{1023}{1024}\right) = \frac{3069}{64}
 \end{aligned}$$

The sum of the first ten terms when the terms are positive and negative alternately is equal to:

$$\begin{aligned}
 S_{10} &= \frac{G_1(1-r^n)}{1-r} = \frac{24(1-(-0.5)^{10})}{1-(-0.5)} = \frac{24(1-(-0.5)^{10})}{\frac{3}{2}} \\
 &= 16(1 - (-0.5)^{10}) \\
 &= 16\left(1 - \left(-\frac{1}{2}\right)^{10}\right) = 16\left(\frac{2^{10} - 1}{2^{10}}\right) \\
 &= 16\left(\frac{1023}{1024}\right) = \frac{1023}{64}
 \end{aligned}$$

Example 1.17

A ball is dropped from a height of 90 cm. On each bounce, it raises to two-third of its previous height. How far has the ball travelled (both up and down) when it hits the ground for the seventh time?

Solution:

If the ball is dropped from a height of 90 cm and each bounce rises to two-third of its previous height, then we have a geometric

progression with $G_1 = 90$ and the common ratio is $r = \frac{2}{3}$

The total distance travelled by the ball when it falls to the ground the seventh time is equal to:

$$\begin{aligned}
 S_7 &= \frac{G_1(1-r^n)}{1-r} = \frac{90\left(1-\left(\frac{2}{3}\right)^7\right)}{1-\frac{2}{3}} = \frac{90\left(1-\left(\frac{2}{3}\right)^7\right)}{\frac{1}{3}} \\
 &= 270\left(1-\left(\frac{2}{3}\right)^7\right) = 270\left(\frac{3^7-2^7}{3^7}\right) \\
 &= \frac{270(2059)}{2187} \\
 &= 254.2\text{cm.}
 \end{aligned}$$

On the other hand the distance that the ball rebounds is given by

$$\begin{aligned}
 S_6 &= \frac{G_1(1-r^n)}{1-r} = \frac{60\left(1-\left(\frac{2}{3}\right)^7\right)}{1-\frac{2}{3}} = \frac{60\left(1-\left(\frac{2}{3}\right)^7\right)}{\frac{1}{3}} \\
 &= 164.2
 \end{aligned}$$

Hence, the total distance travelled by the ball is
 $254.2 + 164.2 = 418.4$ cm

Exercise 1.5

1. For each of the following geometric progressions find the indicated sum.

a) $G_1 = 3, r = 2, S_{10}$

b) $G_2 = 6, r = \frac{1}{4}, S_9$

c) $G_3 = 4, G_5 = \frac{16}{25}, S_7$

d) $G_n = 5(2^{n-1}), S_{11}$

e) $G_n = -6(0.9^{n-1}), S_{14}$

2. In the geometric progression 1, 2, 4, 8, 16, $\dots$, show that each term is one plus the sum of all the preceding terms.

3. Using the concept of geometric progression show that

$$\frac{1-x^n}{1-x} = 1+x+x^2+x^3+\cdots+x^{n-1}, \text{ where } x \neq 1, \text{ and } n \geq 1$$

4. If a person is paid a salary of 1 cent on the first day of November and 2 cents in the second, and the salary continues to double each day, how much will earn in the month of October?
5. A company has a profit of 30,000 Nakfa for the first year. Suppose that during the next 39 years, there is a 5% rise each year from the previous year. Find the total profit over the 40 year period.

1.4 Harmonic Progression

Activity 1.8

1. Consider the following sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

- What is the 4th term?
- What is the sixth term?
- What is the 10th term?
- What is the n th term?
- Write the reciprocals of the numbers in the sequence
- What type of a sequence is formed by taking the reciprocals of the numbers?
- What is the sum of the first 5 terms?
- Can we find a general formula for the sum of the first n terms?

1. Consider the following sequence

$$\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \dots$$

- What is the fourth term?
- What is the tenth term?
- Find the sequence of the reciprocals of the numbers

A sequence of numbers is said to form a **harmonic progression** provided that the reciprocals of the numbers form an arithmetic progression.

Example 1.18

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ is harmonic progression, since 2, 4, 8, .. is an arithmetic progression.

Example 1.19

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}$ form a harmonic progression provided that none of the denominators is zero.

Example 1.20

Find the harmonic progression with first term $\frac{1}{2}$ and the fifth term is $\frac{1}{22}$

Solution

We first find the arithmetic progression with first term 2 and fifth term 22.

$$A_1 = 2, A_5 = 22$$

$$A_5 = A_1 + 4d$$

$$22 = 2 + 4d$$

$$4d = 20$$

$$d = 5$$

Therefore, the arithmetic progression is 2, 7, 12, 17, 22, ...

Hence, the harmonic progression is $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22}, \dots$

Example 1.21

If the numbers a, m, b form a harmonic progression, then m is called the **harmonic mean** between a and b.

Since a, m, b are in harmonic progression $\frac{1}{a}, \frac{1}{m}, \frac{1}{b}$ are in arithmetic progression and hence

$$\frac{1}{m} = \frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{a+b}{2ab}, \text{ solving for } m$$

$$m = \frac{2ab}{a+b}$$

Exercise 1.6

1. Which of the following sequences are in harmonic progression?

- a) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \dots$ d) 2, 4, 6, 8, ...
- b) 0.1, 0.01, 0.001, ... e) 0.1, 0.2, 0.3, ...
- c) 20, 18, 16, 14, 12, 10, ...

2. Find the harmonic mean of

- a) $\frac{1}{3}$ and $\frac{1}{7}$ b) $\frac{1}{4}$ and $\frac{1}{16}$

3. Insert three arithmetic means between

- a) $\frac{1}{4}$ and $\frac{1}{24}$ b) $\frac{1}{14}$ and $-\frac{1}{2}$

1.5 Infinite Series

Activity 1.9

- Find the n th term of the sequence 2, 4, 6, 8, 10, ...
 - Find the sum of the first 5 terms of the sequence
 - Find the sum of the first n terms of the sequence
- What is the difference between the following series?
 - $2 + 4 + 6 + 8 + 10$
 - $2 + 4 + 6 + 8 + 10 + \dots + 2n + \dots$
 - Write the above series given in (2a) and (2b) using sigma notation.
- Write the general term for the infinite sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots$

b) Find the following sum $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}$

c) Find the following sum $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

Let $a_1, a_2, a_3, \dots, a_n$ be a given sequence. Then, the expression $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called the series associated with the given sequence.

The series is finite or infinite according as the given sequence is finite or infinite.

Series are often written in a compact form using sigma notation Σ

For instance $a_1 + a_2 + a_3 + \dots + a_n$ is abbreviated as $\sum_{k=1}^n a_k$

For an infinite series the index of summation k takes the values 1, 2, 3, and so on, without end. To indicate that the values for k keep increasing without bound, we say k takes the values from 1 to ∞ (infinity)

That is, $a_1 + a_2 + a_3 + \dots + a_k + \dots = \sum_{k=1}^{\infty} a_k$

Example 1.22

1. $2 + 4 + 6 + 8 + \dots + 20 = \sum_{n=1}^{10} 2n$

2. $2 + 4 + 6 + 8 + \dots + 20 + \dots = \sum_{n=1}^{\infty} 2n$

3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{64} = \sum_{n=1}^6 \left(\frac{1}{2}\right)^n$

4. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{64} + \dots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Exercise 1.7

a. Write the following using sigma notation

b) $1 + 3 + 9 + \dots + 243$

c) $3 + 2 + \frac{4}{3} + \dots$

b) $1 + 3 + 9 + \dots$

d) $5 + 10 + 15 + \dots$

- b. Find the sum of the series given in (1). If the sum can not be found, give reason

Infinite Geometric Series

Activity 1.10

1. Consider the following geometric series

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$$

- What is the common ratio?
- Find the sum of the first 10 terms
- Find the sum of the first 20 terms
- Find the sum of the first 100 terms
- What happens to the sum as the terms to be added gets larger and larger?

2. Consider the following Geometric series

$$2 + 4 + 8 + 16 + \dots$$

- What is the common ratio?
 - Find the sum of the first 10 terms
 - Find the sum of the first 20 terms
 - Find the sum of the first 100 terms
 - What happens to the sum as the terms to be added gets larger and larger?
3. Discuss the difference between the above two given series.

An infinite series whose sum approaches a finite number is called a **convergent series**. An infinite series whose sum does not tend to a finite number is called a **divergent series**

Example 1.23

$2 + 4 + 8 + 16 + \dots$ is a divergent series

$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \dots$ is a convergent series

For a convergent geometric series we have the following.

If $a_1 + a_1r + a_1r^2 + \dots$ is an infinite geometric series, with $|r| < 1$, then the sum S of all of the terms of the series is given by

$$S = \frac{a_1}{1 - r}$$

Example 1.24

Find the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Solution

The series is an infinite geometric series with $a_1 = \frac{1}{2}$, $r = \frac{1}{2}$

Because $r < 1$, we have $S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

Example 1.25

Find the value of the sum $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$

Solution

This series is an infinite geometric series with first term 2 and

ratio $\frac{3}{4}$ Therefore

$$S = \frac{2}{1 - \frac{3}{4}} = 8$$

Example 1.26

Suppose a ball rebounds $\frac{2}{3}$ of the height from which it falls and

the ball is dropped from a height of 6 meters.

Find the total distance that the ball travels.

Solution:

The ball falls 6 meters and rebounds 4 meters, then falls 4 meters and rebounds $\frac{8}{3}$ meters.

The following series gives the total distance that the ball falls.

$$F = 6 + 4 + \frac{8}{3} + \frac{16}{9} + \dots$$

The distance that the ball rebounds is given by the series

$$R = 4 + \frac{8}{3} + \frac{16}{9} + \dots$$

Each of these series is an infinite geometric series with ratio $\frac{2}{3}$. Therefore each sum is given by

$$F = \frac{6}{1 - \frac{2}{3}} = 18, \quad R = \frac{4}{1 - \frac{2}{3}} = 12$$

The total distance travelled by the ball is the sum of F and R, 30 meters.

Exercise 1.8

- Which of the following is infinitely decreasing geometric series
 - $45 + 4.5 + .45 + \dots$
 - $2 + 6 + 18 + \dots$
 - $6 + 2 + \frac{2}{3} + \dots$
- Determine whether the following geometric series converge or diverge. If the series converges, find its sum.
 - $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
 - $16 + 8 + 4 + \dots$
 - $0.7 + 0.07 + 0.007 + \dots$
 - $1 + 1 + 1 + 1 + \dots$
 - $1 - 1 + 1 - 1 + 1 + \dots$

3. Find the sum of the following infinite series.

a) $1 + \frac{1}{9} + \frac{1}{81} + \dots$

b) $\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots$

c) $\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \dots$

4. Express the repeating decimal 1.81818... as fraction

Review Exercise

1. For each of the following sequences, find the indicated terms of the sequence.

a) $a_n = 0.7$, a_1 , a_5 , a_{15} , a_{100}

b) $a_n = \frac{1 + (-1)^n}{n}$, a_3 , a_6 , a_{17} , a_{20}

c) $a_n = \frac{(n-1)!}{(n+1)!}$, a_4 , a_8 , a_{10} .

d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \dots$, a_6 , a_7 , a_8 , a_9 , a_{10}

e) $a_1 = 1$ and $a_n = (-1)^n 3a_{n-1} - 4$, for $n > 1$, a_2 , a_3 , a_4 , a_5

$a_n = a_{n-3} + a_{n-2} + a_{n-1}$, for $n > 3$, a_4 , a_5 , a_6 , a_7

2. Find the first 8 numbers in an arithmetic progression in which the sum of the first and the seventh term is 40 and the product of the first and the fourth terms is 160.

3. A monument is constructed by first laying a row of 60 bricks at ground level. A second row with two fewer bricks is centred on that; a third row, with two fewer bricks, is centred on the second; and so on. The top row contains 10 bricks. How many bricks are there in the monument?

4. In the geometric progression 2, 6, 18, 54, $\dots$, show that each term is twice the sum of one and all the preceding terms.
5. Suppose the first term of a geometric progression is g_1 and the common ratio is r , where g_1 and r are both positive. Show that the sequence given by:

$$\log(g_1), \log(g_2), \log(g_3), \log(g_4), \dots,$$

is an arithmetic progression, and find its common difference.

6. Suppose the first term of an arithmetic progression is a_1 and the common difference is d . For C any positive number, show that:

$$C^{a_1}, C^{a_2}, C^{a_3}, C^{a_4}, \dots,$$

is a geometric progression, and find its common ratio.

7. A ball is dropped from a height of 5m. On each bounce, it rises to 45% of its previous height. How far has the ball travelled (both up and down) when it hits the ground for the sixth time?
8. A car that sold 80,000 Nakfa depreciates in value 25% each year. What is it worth after eight years?
9. Find two numbers whose arithmetic is 25 and whose geometric mean is 15.
10. For x and y two positive numbers show that the arithmetic mean of x and y is greater than the geometric mean of x and y .
11. The repeating decimal number $0.444444\dots$ can be written as $\frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \dots$ an infinite geometric series. Find the sum of this geometric series.

12. Write the repeating decimal number $0.24242424\dots$ as an infinite geometric series and find the sum of the geometric series

13. Is the following statement true?

$$2 + 4 + 8 + 16 + \dots = \frac{2}{1-2}$$

14. Find the sum of the following infinite geometric series

a) $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

b) $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

c) $4 - 2 + 1 - \frac{1}{2} + \dots$

d) $\sum_{i=1}^{\infty} (0.3)^i$

e) $\sum_{i=1}^{\infty} 12(0.01)^i$

15. Inside a square whose side is 2m, a second square is formed by joining the midpoints of the sides of the 2 m square. A third square is formed inside the second one in the same way. If this process is continued indefinitely, what is the sum of the areas of the squares?

CHAPTER 2

Descriptive Statistics and Probability

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- Construct graphs from given statistical and grouped data;
- Calculate the mean, median and mode from a given data set;
- Calculate range and standard deviation from a given data set;
- Apply elementary probability to daily activities; and
- Estimate the certainty of unpredicted futures.

2.1. Population and sampling

What is Statistics?

Senait, a mother of three sons has a shop.

- a) One of her sons has been recording the weekly profit from the shop. His records show:

251	198	260	205	174	199	225	280	213	290
235	239	175	160	312	246	182	242	219	145

- b) The second son tried to organize the record as follows.

Profit (Nakfa)	Number of times
below 150	1
150-199	6
200-249	8
above 250	5
Total	20

- c) The third son observed that the profit was above 250 on five out of 20 weeks. He also noted that the chance of getting a profit above 250 was 20%.
- d) How do you describe the tasks performed by the three sons?

Statistics is the collection, presentation , analysis and interpretation of the numerical data.

2.1.1 Population and Sample

The collection of data has been introduced in earlier grades as a method of obtaining answers to questions about the world around us. This work will be briefly reviewed.

Activity 2.1

- 1. Do you remember the methods (ways) of data collection?
- 2. The class teacher of Grade 12 section B (consisting of 60) students assigned two students, Zeineb and Ticabo from his class to study the number of times that each of the student is late in the previous academic year.
 - a) Zeineb suggested to collect data from every student and recorded the number of times that each of the students is late from the record office of the school for the whole and reported the following to the class teacher.

0	7	6	4	10	1	8	9	10	9
3	9	6	2	8	1	2	1	2	3
7	8	1	8	7	1	10	6	2	5
7	1	8	1	3	4	1	1	4	5
7	1	0	10	4	7	2	5	3	7
0	5	0	7	7	2	9	1	7	8

- b) Ticabo suggested to collect data from some of the population recorded the number of times that each of the students is late for 20 of the students from the recorded school and reported the following to the class teacher.

9	3	9	9	10	9	8	1	2	1
1	9	1	10	3	5	8	1	7	7

- What is the difference between the techniques applied by each student?
- What is the advantage and disadvantage of the method followed by Zeineb?
- What is the advantage of the method followed by Zeineb?
- What is the advantage of the method followed by Ticabo?
- What name do you suggest to Zeineb's method?
- What name do you suggest to Ticabo's method?

Population is the collection of all elements (individuals, objects, areas, etc.) under consideration in a study. Population describes the entire group under consideration in a study.

Sample is part (or subset) of the population from which we actually collect data and draw some conclusions about the population. Sample describes a group chosen to represent the population under consideration in a study.

Example 2.1.

- a) Consider the library at your school. What do you call the group of all books in the library? What do you call the group of books that contain only the mathematics books?

Solution:

Population: All books

Sample: Mathematics books

- b) A study published in 2009 attempted to estimate the proportion of Eritreans who read a news paper every morning. 963 persons were interviewed. Describe the population and sample?

Solution:

Population: Eritrean Population

Sample: 963 persons

- c) A quality controller at the Red Sea bottlers share company daily checks the quality of Coca cola produced. He daily selects 150 Coca cola bottles at random. Determine the population and the sample.

Solution:

Population: The set of all Coca cola drinks produced daily by the Red Sea bottlers share company

Sample: 150 Coca cola bottles

2.1.2 Sampling

Activity 2.2

Yassin and Goitom were in estimating the average income of Eritrean farmers. Yassin decided to collect the information from each Eritrean farmer. But, Goitom preferred to collect information from some of the farmers in order to get some idea about the average income of all farmers.

- a) Is it easy to collect income information from each Eritrean farmer? Explain.
- b) What are the advantages of Goitom method?

The procedure by which the characteristics of all the members of a defined population are measured is called a **census**. **Census (complete enumeration)** is a collection of data from every member of a population.

The procedure by which the characteristics of some the members of a defined population are measured is called a **sampling**.

Activity 2.3

This semester the Head of the English Department of a school with 2000 students wanted to get student feedback on teaching styles of his staff, but he did not have time to get feedback from every student in the school. He wants the data to be collected in a short period of time.

a) What type of advice do you provide to the Head?

Selam suggested to the Head to collect the information from those which are easiest to contact (in the first row desks) and who are excellent in their studies.

Ahmed suggested the teacher to assign a number to each individual student (starting at 1 to 2000), write these numbers in 2000 pieces of paper, put them into a hat and draw names from the hat to take a sample of $n = 50$ students from the school by selecting students (lottery method) to give every student a chance of participation.

b) What is the advantage of Selam's suggestion?

c) What is the problem with Selam's suggestion?

d) What is the advantage of Ahmed's suggestion?

Simple random sampling is a probability sampling process in which, every possible grouping of a particular number of items will all have the same chances of being chosen as the sample.

Example 2.2.

1. Suppose that a warehouse contains 50 tires of different brands and you want to select a simple random sample of only 4 tires. Suggest a way to do that.
2. Suppose that the Ministry of Health is interested in knowing the health status of grade 10 students of your school and wants to interview 50 students.
 - a) Identify the population of interest?
 - b) Can you suggest a method of sampling to get a sample of 50 students?

Solution:

1. List the tires from 1 to 50. Select 4 tires from the list by lottery procedure.
2.
 - a) The population of interest is all grade 10 students of your school.
 - b) Number all grade 10 students of your school from 1 to N, where N indicates the number of all grade 10 students of your school and select 50 by means of a lottery.

Exercise 2.1

1. Mr. Ahmed, a candidate for Baito membership in a small town, wants to determine if he should campaign harder against his opponent. To determine this, he will poll 300 of the 1800 registered voters. If the results indicate that he has 30% more votes than his opponent, he will not intensify his campaign efforts against the opponent. Identify the population and sample.
2. A mathematics teacher thinks that 95% of the students taking the course pass. To check this assumption, he takes a random sample of 37 students who have taken the course in the past three years. Determine population and sample
3. Identify the population and the sample of interest to the researcher.

- a) Selecting the proper diet for cows or other animals is an important aspect of Livestock production. A researcher wishes to estimate the mean weight of Barka cows maintained on a specific diet for a period of six months. One hundred Barka cows are randomly selected from Zoba Gash Barka and each is weighed.
 - b) A social worker in a particular city wishes to obtain information on the number of disabled children in households that receive support from government. A random sample of 400 households was selected from the welfare rolls of the city. A check on government support data provided the number of children in each household.
4. The Director's office in your school is interested in studying the average Mathematics mark of grade 10 students.
- a) Identify the population of interest?
 - b) Suggest a method of sampling to get a sample of, say, 30 students?

2.2. Organizing Data

2.2.1 Types of Data

Activity 2.4

1. The following table provides the first 5 records of a survey of grade 12 students in a school.

A	B	C	D	E
Hana	Tigrigna	female	160.0 cm	53.5 kg
Kedir	Tigre	male	171.3 cm	65.0 kg
Adongeni	Kunama	male	182.9 cm	74.2 kg
Meriem	Bilen	female	164.1 cm	55.6 Kg
Hawa	Saho	female	157.0 cm	51.2 Kg

Suggest a name (definition) terms (titles) to A, B, C, D and E?

2. You are given an assignment to study a group of mathematics class students. You ask about their Sex, Place of residence, Religion, Age, Weight and their Height

Each of the above characteristics varies from student to student.

- Do you remember the name given to these characteristics?
- Can you suggest a name to the value we collect (A, B, C, D and E) from each student about the above characteristics?

The above activities suggest to the following terms.

Variable is the characteristic of interest of an element that can assume different values for the different elements in the population or sample.

Datum: The values that a variable can assume are called **data** (single measured value (singular of data))

Data or Data Set: collection of numerical / non-numerical values (plural of datum). Data are facts or figures from which conclusions can be drawn.

Example 2.3.

Determine the possible values of the following variables.

- Height of a child
- Weight of a child
- Ethnic group of an Eritrean
- Income of a daily worker
- Agricultural land size of a farmer
- Zoba

Solution:

- It can take any value greater than 0 m.
- It can take any value greater than 0 Kg.
- Ethnic group, Afar, Billen, Hdareb, Nara, Kunama Rashaida, Tigre, Tigrigna. Saho,
- It can take any value greater than 0 Nakfa
- Agricultural land size of a farmer (Small, medium large) or it can take any value greater than 0 Hectare
- Zoba (Maekel, NRS, SRS, Debub, Anseba, Gash-Baraka)

Activity 2.5

1. Think about the possible values of the following variables (characteristics) of a student can take:

Group 1	Group 2
Sex	age
Health	weight
Ethnic group	Number of brothers and sisters
Blood type	height
Place of residence	Number of exercise book

- a) Do you observe any similarities in values assigned to each variable in each group?
- b) Do you observe any differences in values assigned to each variable between the two groups of variables?

Qualitative (or categorical) variables allow for classification of individuals based on some attribute or characteristic.

Quantitative variable represents a counted or measured quantity.

Example 2.4.

Classify the following data into qualitative and quantitative data

- a) Nationality
- b) Yield per hectare
- c) Disease status
- d) Distance travelled
- e) Temperature
- f) Place of Birth
- g) House ownership
- h) Waiting time for a bus to arrive at a stop

Solution:

Nationality, disease status, place of birth and house ownership are qualitative variables, but yield per hectare, distance travelled, temperature and waiting time for a bus to arrive at a stop are quantitative variables.

Activity 2.6

1. Consider the following quantitative variables in group 1 and group 2.

Group 1	Group 2
Number of brothers and sisters	Age
Number of exercise book	Weight
Number of visits to a doctor	Time of wake up
Number of pen bought in a year	Height
Number of students absent from class	Body temperature

Think about the possible values of the following variables (characteristics) of a student can take:

- Do you observe any similarities in values assigned to each variable in each group?
- Do you observe any differences in values assigned to each variable between the two groups of variables?

Quantitative data is also called numerical data, and may either be discrete or continuous.

A **discrete variable** is a quantitative variable that has either a finite number of possible values or a countable number of possible values

A **continuous variable** is a quantitative variable that has an infinite number of possible values that are not countable.

Example 2.5.

For each of the following quantitative variables, indicate whether each is a discrete or continuous variable.

- Number of children in a family. Discrete
- Amount of income tax paid in by business man during 2009.
Continuous
- Height of a student in your class. Continuous
- Yearly amount of rainfall in Asmara. Continuous
- Number of bicycles parked at your school compound. Discrete

Solution:

- a) and e) are discrete variables
- b) , c) and d) are continuous variables

Exercise 2.2

1. Identify the variable of interest to the researcher.
 - a) Selecting the proper diet for cows or other animals is an important aspect of livestock production. A researcher wishes to estimate the mean weight of Barka cows maintained on a specific diet for a period of six months. One hundred Barka cows are randomly selected from one Sub-Zoba and each is weighed.
 - b) A social worker in a particular city wishes to obtain information on the number of disabled children in households that receive support from government. A random sample of 400 households was selected from the welfare rolls of the city. A check on government support data provided the number of children in each household.
 - c) Suppose that the Ministry of Health is interested in knowing the average weight (in kg) of the children in the age group from 6 to 8 years.
2. Classify the following data into qualitative and quantitative
 - a) Smoking status
 - b) Number of children
 - c) Annual income
 - d) HIV Status
 - e) Cups of tea consumed by a student per day
 - f) Vaccination status
 - g) Body temperature
 - h) Ethnic group
 - i) Sex
 - j) Rank in class
 - k) Visits to a doctor
 - l) Time
 - m) number of students who turn in an assignment on time

3. Identify each of the following as an example of a qualitative or quantitative variable.
- a) The breaking strength of a piece of string.
 - b) The number of stop signs in towns of less than 10,000 people.
 - c) The hair colour of children auditioning for a play.
 - d) Whether or not a valve is defective.
 - e) The number of questions answered correctly on a test.
 - f) The length of time spent on hold to have a question answered by the help desk via telephone.
4. Identify each of the following as an example of a discrete or continuous variable.
- a) A survey of registered voters about which candidate they will support.
 - b) the length of time required for a wound to heal when a new medicine is used
 - c) The number of telephone calls received by a help desk in a 10-minute period.
 - d) The distance freshmen can kick a football.
 - e) The number of pages in a Mathematics textbook.
 - f) the total income of a family
 - g) the number of suitcases lost per day by an airline company
 - h) the miles driven by a person commuting to work

2.2.2 Frequency Distribution of Qualitative Data

Activity 2.7

A nurse from a local clinic wanted to recruit blood donors from a local high school. The blood type of a group of students is given below.

AB	B	A	O	B	O	B	O	A	O
B	O	B	O	B	O	O	AB	AB	A
O	A	B	AB	O	A	A	B	O	A
B	B	O	AB	O	O	A	O	A	O

The nurse wants the following information from the above list. Determine,

- (a). Which blood type is the most frequent in this class?
 (b). Which blood is the least frequent in this class?

Raw data or unclassified data is the set of data in its original form. It has not been organized in any manner and is recorded in the order observed.

Activity 2.8

Issa, one of the students in this class who is a volunteer of the National Blood Bank of Eritrea volunteered to help the nurse organize the data and started to arrange the above data in Table 1 using tally marks. Complete the incomplete cells.

Blood Type	Tally marks	Frequency	Percent
A		9	22.5
AB			12.5
B		10	
O			
Total		40	100.0

Answer the following questions based on the table given above.

- Which blood type is the most common in this class?
- Which blood type is the least common in this class?
- What is the number of students with blood type B?
- What is the proportion of students whose blood type is O?
- What is the percentage of students whose blood type is B?

The number of tallies before each subject constructed by Issa gives the number of students with a specific blood type.

Frequency gives the number of times that a particular entry occurs.

The table constructed by Issa is known as **frequency distribution table** as it gives the number of times an entry occurs.

Frequency distribution table is a summary table that shows the number of observations that belong in the different classes (groups).

The **relative frequency** is the proportion of observations within a category and is found using the formula

$$\text{relative frequency} = \frac{\text{frequency}}{\text{sum of all frequencies}}$$

The **percentage relative frequency** is the proportion of observations within a category and is found using the formula
Percentage

$$\text{percentage of relative frequency} = \frac{\text{frequency}}{\text{sum of all frequencies}} \times 100\%$$

Example 2.6.

1. A group of children from one village were asked to say which animal they would like most to have as a domestic animal. The results are given below:

Dog	cow	cow	sheep	cow	goat	dog	cow	goat	dog
sheep	goat	dog	cow	dog	cow	cow	dog	goat	cow
Cow	cat	dog	dog	dog	sheep	dog	dog	cow	cat

Make a frequency distribution table for the data with frequencies and relative frequencies. And answer the following questions

- a) Which animal the most liked?
- b) Which animal is the least liked?
- c) Find the number of students who like dogs and cats?
- d) Determine the proportion of students who like sheep?

Solution:

Animal	Number of children	Relative frequency
Cow	10	0.33
Cat	2	0.07
Dog	11	0.37
Goat	4	0.13
Sheep	3	0.10
Total	30	1.00

- a) The most liked animal is a dog.
- b) The least liked animal a cow.
- c) Number of students who like dogs and cats= $10+11=21$
- d) The proportion of students who like sheep= $1/10$?

2.2.3 Frequency Distribution of Quantitative Data**Activity 2.9**

Consider the marks obtained by 10 students in a mathematics test as given below:

55 36 95 73 60 42 25 78 75 62

The data in this form is called raw data.

- a) Can you find the highest and the lowest marks?
- b) Did it take you some time to search for the maximum and minimum scores?
- c) Arrange the scores in ascending?

25 36 42 55 60 62 73 75 78 95

- d) Now determine the maximum and minimum score from the sorted data

Activity 2.10

Consider the marks obtained (out of 100 marks) by 30 students of Grade 12 of one high school:

10 20 36 92 95 40 50 60 60 70
92 88 80 70 70 70 36 40 36 40
92 40 60 50 70 60 70 60 60 88

- a) Sort the data in ascending order?

- b) Find the highest and lowest marks? Can you find the highest and the lowest marks easily?
- c) Is it easy to find any pattern or be able to answer a question such “how many students scored 50?”
- d) Can you find the number of students whose mark is 60?
- e) Let’s arrange the marks in ascending order. Recall that the number of students who have obtained a certain number of marks is called the **frequency** of those marks. Complete the following table by tallying the marks.

Marks	Tally marks	Number of students (frequency)
10		1
20		
36		3
40		
50		2
60		
70		6
80		1
88		
92		3
95		1
Total		

- f) What can you generalize from the above activity?

Example 2.7.

A nurse from a local clinic collected data on the number of children of 50 currently married women who are active members of a local community.

0	0	1	2	2	2	3	3	4	4	0	1	1	2	2	3	3
0	0	1	2	2	3	3	3	4	4	3	4	5	0	1	1	2
0	1	1	2	2	3	3	3	4	4	2	3	3	3	4	5	

- a) Construct summary table for the data set?
- b) Find the proportion of married women with no child.

- c) Determine the number of married women with less than 3 children?
- d) Find the proportion of married women with more than 1 child

Solution:**a)**

Number of Children	No. of Married Women	%
0	7	14
1	8	16
2	11	22
3	14	28
4	8	16
5	2	4
Total	50	100

- b) The proportion of married women with no child is 14%.
- c) the number of married women with less than 3 children is 52%.
- d) The proportion of married women with more than 1 child is 70%.

Grouped Frequency Distribution**Activity 2.11**

100 plants each were planted in 100 plantation sites by a group of “summer work program” participant students. After one month, the number of plants that survived was recorded by the group as follows.

95	67	28	32	65	65	69	33	98	96
76	42	32	38	42	40	40	69	95	92
75	83	76	83	85	62	37	65	63	42
89	65	73	81	49	52	64	76	83	92
93	68	52	79	81	83	59	82	75	82
86	90	44	62	31	36	38	42	39	83
87	56	58	23	35	76	83	85	30	68
69	83	86	43	45	39	83	75	66	83
92	75	89	66	91	27	88	89	93	42
53	69	90	55	66	49	52	83	34	36

To judge the success of their work the group started to organize the data as follows by using a frequency distribution.

Number of planted survived	Frequency (count)
23	1
27	1
28	1
30	1
31	1
32	2
33	1
⋮	⋮
92	3
93	2
95	2
96	1
98	1

- Is this a useful summary?
- Find the number of sites with more than 50 survived plants?
- To present such a large amount of data so that the group can make sense of it easily, suggest a better alternative?
- One of the group members suggested to condense it into groups like 20-29, 30-39, . . . , 90-99 (since the data is from 23 to 98). Complete the following Table.

Number of plants survived	Tally Marks	Number of sites (frequency)
20 - 29		3
30 - 39		
40 - 49		12
50 - 59		
60 - 69		18
70 - 79		
80 - 89		23
90 - 99		12
Total		100

- e) Find the number of plantations sites with 80-89 survived plants?
- f) Determine the total number of sites number of sites that has at most 69 survived plants?
- g) Find the number of plantation sites that has at least 70 survived plants?
- h) What can you generalize from the activity?

A **Frequency distribution table** is a list (summary table) which pairs of ranges of values of quantitative variable with their frequencies (counts). Each range is called a class.

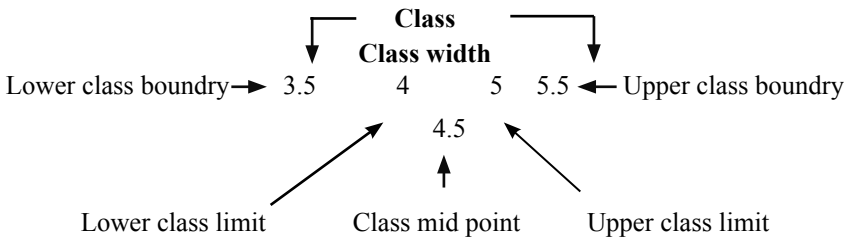
The **Cumulative frequency** for a specific value in a frequency distribution table is the sum of frequencies for values at or below the given value.

Important terms

- **Class interval** is the range of values that belong in the category.
- **Class frequency** is the number of observations that belong in a class interval.
- **Class limits** are the end numbers of a class interval. The lower class limit (LCL) is the lower end of the class interval and the upper class limit (UCL) is the upper end of the class interval
- The **lower class limit** of a class is the smallest value within the class.
- The **upper class limit** of a class is the largest value within the class.
- The **class width** is the difference between upper class boundary and lower class boundary.
- **Class boundaries** are the true class limits. For rounded figures, the **lower class boundary (LCB)** is defined as halfway between the lower class limit of the class and the upper class limit of the preceding class while the **upper**

class boundary (UCB) is defined as halfway between the upper class limit of the class and the lower class limit of the next class.

- **Class mark** is the midpoint of a class interval. It is the average of the lower class limit and the upper class limit or the average of the lower class boundary and upper class boundary of a class interval.



Example 2.8.

Consider the grouped frequency distribution of the marks obtained by 60 students in Mathematics test. Study the frequency distribution table and answer the questions given below.

Class Interval	Frequency (Number of students)
0-10	2
10-20	10
20-30	21
30-40	19
40-50	7
50-60	1
Total	60

- What is the size (width) of the class intervals?
- Which class has the highest frequency?
- Which class has the lowest frequency?
- What is the upper limit of the class interval 30-40?
- Determine the number of students whose mark is less than or 40?

Solution:

- width of the class intervals
 $10-0=20-10=\dots=60-50=10$
- The third class (20-30) has the highest frequency.
- The last class (50-60) has lowest frequency.
- The upper limit of the class interval 30-40 is 40.
- The number of students whose mark is less than or 60 is the cumulative frequency of the class 30-40.
 Cumulative frequency = $2+10+21+19=52$

Example 2.9.

The Monthly wages (in Nakfa) of 30 manual workers in a factory are.

830, 835, 890, 810, 835, 836, 869, 845, 898, 890, 820, 860, 832, 833, 855, 845, 804, 808, 812, 840, 885, 835, 835, 836, 878, 840, 868, 890, 806, 840

- Using tally marks make a frequency distribution table with intervals as 800–810, 810–820 and so on.
- Which group has the maximum number of workers?
- How many workers earn Nakfa 850 and more?

Solution:

a)

Interval (wages)	Tally Marks	Frequency (Number of workers)
800-810		3
810-820		2
820-830		1
830-840		9
840-850		5
850-860		1
860-870		3
870-880		1
880-890		1
890-900		4
Total		30

- b) The group 830-840 has the highest number of workers.
- c) 10 workers earn 850 Nakfa and more.

Exercise 2.3

- 1 As part of an evaluation scheme by the school an instructor was rated by his students in a previous semester as A= excellent, B= Very good, C=good, D=satisfactory, E=not good. A sample of ratings of students is given below.

A A A A B A B B A A
E C A C D A B C D B

- a) Construct a frequency distribution of the ratings with frequencies a relative frequencies and percentages for each
 - b) Find the number of students who rated the instructor B?
- 2 Hospitals keep records for injury incidence arriving at the emergency room. The following is a data set that contains 66 injury incidences in a given day.

Sp	Co	Co	Sp	Fr	Co	Co	Sp	Fr	Sp	Fr
Fr	Sp	Fr	Co	St	Sp	Fr	Co	Sp	Co	Co
Co	St	Sp	Fr	Sp	Fr	St	St	St	Sp	Co
St	Sp	St	Sp	Co	Co	Co	Fr	Sp	Fr	St
Co	Co	Sp	Fr	Fr	Fr	Fr	Sp	Co	Sp	Fr
Co	Fr	Sp	St	Sp	Sp	St	Fr	Sp	St	Co

Where,

Sp = sprain, Co = contusion, Fr =fracture, and St = strain.

- a) Construct a summary table for the type of injury incident.
 - b) Which injury type is the common injury incidence?
 - c) Which injury types is the least common?
 - d) Find the proportion of patients with fraction injury
- 3 Construct a frequency distribution for the final exam scores of students in an earlier semester of Mathematics.

68	75	76	75	72	71	75	75	76	72	69	72
73	75	74	77	71	76	72	73	68	67	77	73

Which score is the most common score?

- 4 A student rolled fair die 200 times and the values on the top face were recorded. The data are recorded below.

3	5	3	6	2	6	6	5	5	6	6	4	2	1	5	3	2	4	5	4
1	4	3	2	6	6	4	6	2	6	5	1	5	1	2	4	4	2	4	4
4	2	6	4	5	4	3	5	5	4	6	1	1	4	6	6	4	5	3	5
2	6	3	2	4	5	3	2	2	6	3	4	3	2	6	4	5	2	1	5
5	4	1	3	1	3	5	1	3	6	5	3	4	3	4	5	1	2	1	2
1	3	2	3	6	3	1	6	3	6	6	1	4	5	2	2	6	3	5	3
1	1	6	4	5	1	6	5	3	2	6	2	3	2	5	6	3	5	5	6
2	6	6	3	5	4	1	4	5	1	4	1	3	4	3	6	2	4	3	6
6	1	1	2	4	5	2	5	3	4	3	4	5	3	3	3	1	1	4	3
5	2	1	4	2	5	2	2	1	5	4	5	1	5	3	2	2	5	1	1

- Organize the data by using a frequency distribution?
 - Which numbers are the least and the most occurring?
 - Determine the proportion of occurrence of the number 3?
 - Find the proportion of occurrence of even numbers?
 - What can you conclude about the process of the frequency distribution table?
- 5 The following is the distribution of the ages of new employees joined at a factory.

Age	No. of employees
20 – 29	7
30 – 39	21
40 – 49	4
50 – 59	2
60 – 69	1

- Obtain the class intervals and class marks of the class boundaries.
 - What is the upper class limit of the class 30 – 39?
 - What is the lower class boundary of the class 50 – 59?
 - What is the class mark of the class 40 – 49?
- 6 Consider the following data representing the length in minutes of 24 students spent working their homework.

50.4	78.2	72.8	56.3	73.1	67.2	89.1	41.7
87.1	77.3	40.1	56.6	66.0	74.1	67.9	53.8
89.3	84.6	68.4	53.7	78.	80.9	78.9	78.1

- Sort the data
- Construct a frequency and relative frequency table for the data with five categories.

2.3. Statistical Graphs

Activity 2.12

Mesfin, the owner of clothing shop constructed the following table after he recorded and summarized colour of shirt bought by his 500 customers last year by a frequency distribution table.

He constructed the following chart.

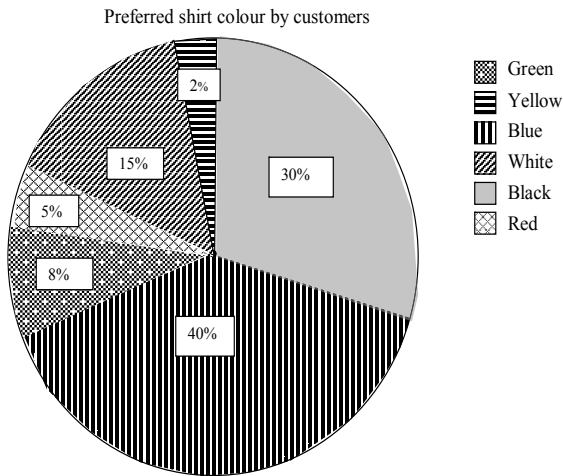


Fig 2. 1

Answer the following.

- Which colour is the most preferred?
- Determine the number of customers whose preference is black colour?
- Do you remember the name of the chart displayed above?

A **pie chart** is a circle divided into sectors. Each sector represents a category of data. The area of each sector is proportional to the frequency of the category.

Example 2.10.

A family spends its monthly income in the following manner:
Rent 25%, Food 35%, Education 5%, Miscellaneous 15%, and Savings 20%.

- a) Represent this data by a pie chart.
- b) Suppose the monthly income of a family is 1500 Nakfa.
 - i. Find the amount of money spent on food monthly?
 - ii. Find the amount of money spent on education monthly?
 - iii. Find the amount of money spent on rent and savings monthly?

Solution:

a)

Expenditure category	Percent
Rent	25%,
Food	35%
Education	5%
Miscellaneous	15%
Savings	20%.

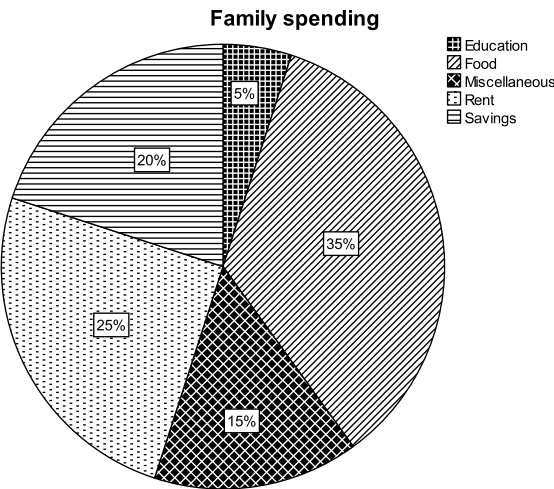


Fig 2. 2

b)

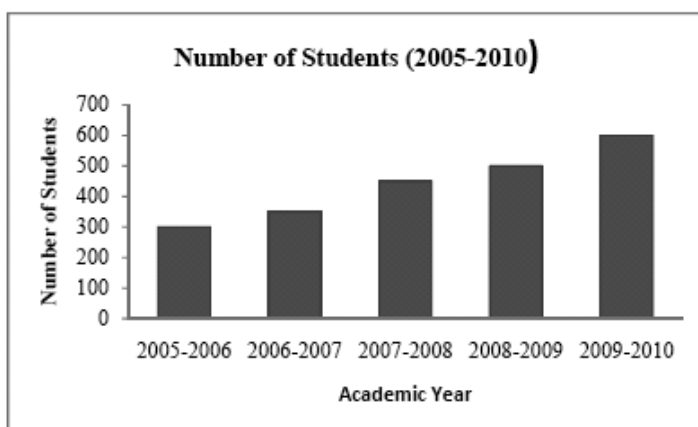
i. $0.35 \times 1500 \text{ Nakfa} = 525 \text{ Nakfa}$

ii. $0.05 \times 1500 \text{ Nakfa} = 75 \text{ Nakfa}$

iii. $0.25 \times 1500 \text{ Nakfa} = 75 \text{ Nakfa} + 0.2 \times 1500 \text{ Nakfa} = 675 \text{ Nakfa}$

Activity 2.13

Consider the following graphical display?

**Fig 2. 3**

- What is the information given by the bar graph?
- In which academic year is the increase in the number of students maximum? In which year is the number of students minimum?
- Do you remember the name given to this graph?

The graph described above is a bar graph.

A **bar graph** is constructed by labelling each category of data on a horizontal axis and the frequency or relative frequency of the category on the vertical axis. Rectangles of equal width are drawn for each category. The height of each rectangle is the category's frequency or relative frequency.

Types of Bar Graph

Simple Bar Graph: is used to show a single characteristic of different groups, categories or years.

Example 2.11.

Grade	Six	Seven	Eight
Number of Students	45	42	48

Construct a simple bar graph?

Solution:

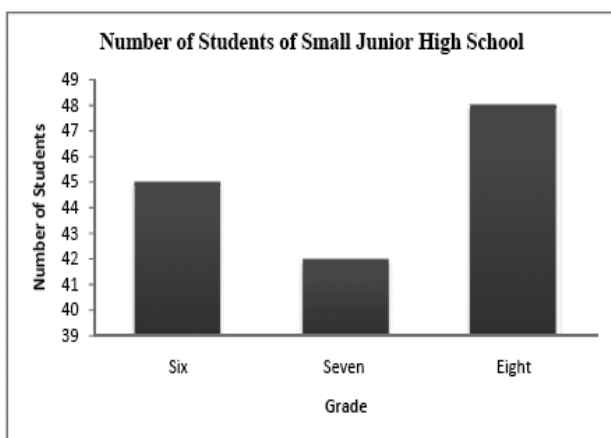


Fig 2. 4

Activity 2.14

Consider the following graphical display

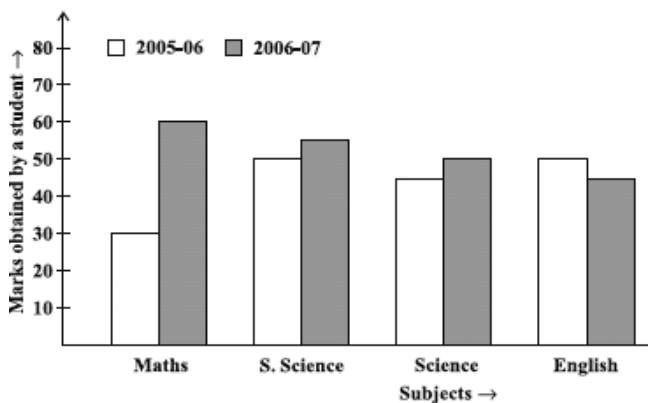


Fig 2. 5

- What is the information given by the graph?
- In which subject has the performance improved the most?
- In which subject has the performance deteriorated?
- In which subject is the performance approximately at par?

Double Bar Graph: A bar graph showing two sets of data simultaneously. It is useful for the comparison of the data.

Multiple Bar Graphs: Multiple bar graphs visualize the different objects of study under one particular limiting condition.

Example 2.12.

Construct a double bar chart for the following distribution.

Year	Grade 9	Grade 10
2007-2008	1,200	1,000
2008-2009	1,300	1,000
2009-2010	1,500	1,100

Solution

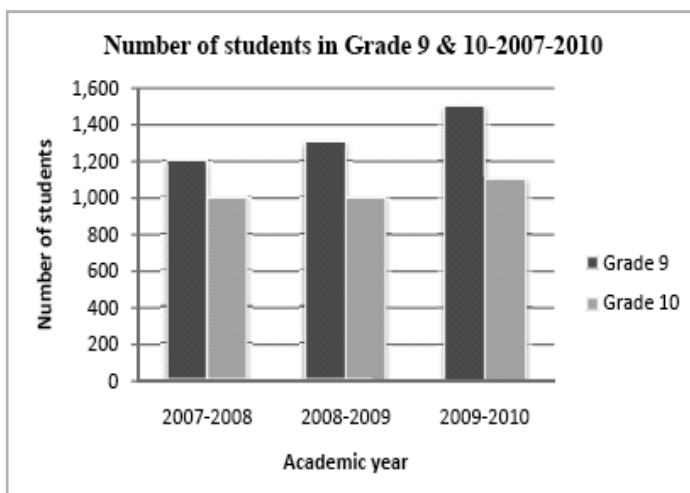


Fig 2. 6

Activity 2.15

Consider the following graphical display.

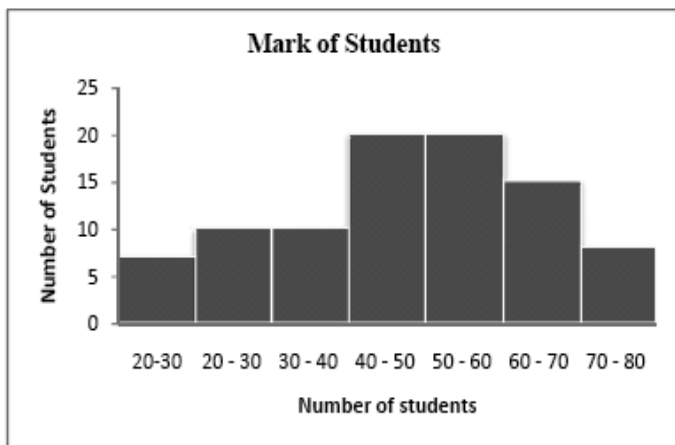


Fig 2. 7

- Do you remember the name given to above graph?
- Find the number of students whose score is between 60-70?
- Determine the number of students whose score between 20-60?
- Find the number of students who scored above 60?
- Find the number of students whose score is less than or equal to 40?
- What observation can you deduce from the graphic display?

A **histogram** is a graph of the frequency distribution. It is a vertical bar chart in which the rectangular bars are constructed at the limits of each class.

Some observations about histograms that must be noted are:

- There are no gaps between bars.
- The height of each bar is equal to the frequency of the class the bar represents.

- The width of each bar extends from the lower limit to the upper limit of the class the bar represents.
- Each axis of the histogram must be labelled; given a name, and the whole graph must be given a title.
- Histograms are a great way to show results of continuous data, such as weight, height, time, etc...

Example 2.13.

The following marks (out of 50) are obtained in by 60 students of grade 10 students.

21	10	30	22	33	5	37	12	25	42	15	39	26
32	18	27	28	19	29	35	31	24	36	18	20	38
22	44	16	24	10	27	39	28	49	29	32	23	31
21	34	22	23	36	24	36	33	47	48	50	39	20
7	16	36	45	47	30	22	17					

- Construct a frequency distribution by using the class intervals $0 \leq c < 10$
- Construct a histogram

Solution:

a)

Marks	Frequency
0-10	2
10-20	10
20-30	21
30-40	19
40-50	7
50-60	1
Total	60

b)

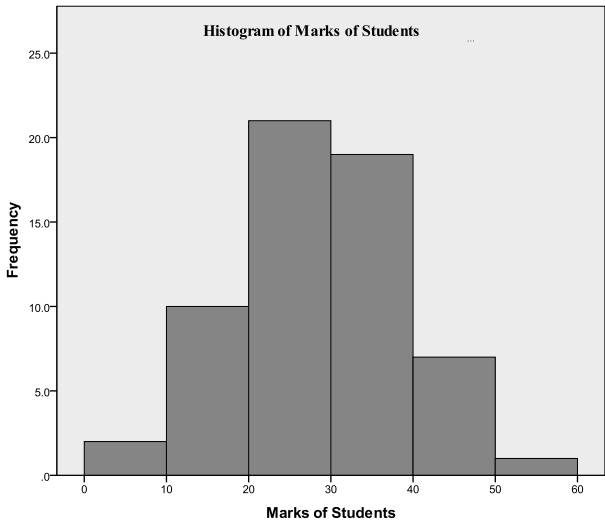


Fig 2. 8

Activity 2.16

Abdulwasie: Greeting Yohannes. Do you have time for discussion regarding the assignment given to us by our math teacher on Friday class?

Yohannes: why not.

Abdulwasie: our teacher said that there is another visual way of representing quantitative data and its frequencies. I have constructed the frequency distribution of the data as follows.

Weights (in kg)	Class Mark	Number of students
30.5 - 35.5	32.5	9
35.5 - 40.5	37.5	6
40.5 - 45.5	42.5	15
45.5 - 50.5	47.5	3
50.5 - 55.5	52.5	1
55.5 - 60.5	57.5	2
Total		36

Yohannes: so far we have explored the histogram, which uses the height of rectangles to represent frequencies and the width to represent the width of a class interval of a grouped frequency distribution. I didn't see any other method other than the histogram.

Abdulwasie: I have an idea. Let's try to plot the frequency or the relative frequency of a class interval versus the middle value (the class mark) of each class and connect these points with a straight line.

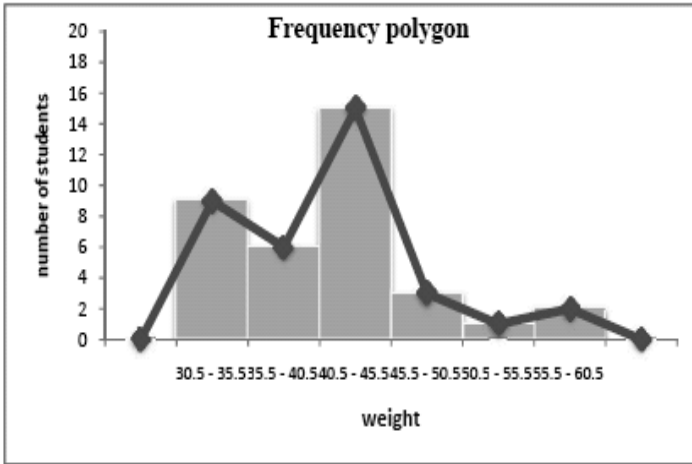


Fig 2. 9

Advantage

- The frequency polygon is simpler than its histogram counterpart.
- It sketches an outline of the data pattern more clearly.
- The polygon becomes increasingly smooth and curve like as we increase the number of classes and the number of observations.

Activity 2.17

The table below shows daily temperatures for Massawa city, recorded for 6 days, in degrees Celsius.

Day	Temperature ($^{\circ}$ C)
1	23
2	33
3	30
4	37
5	39
6	42

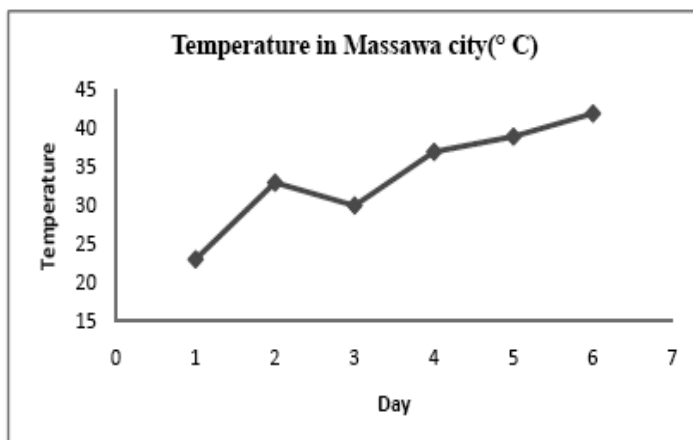
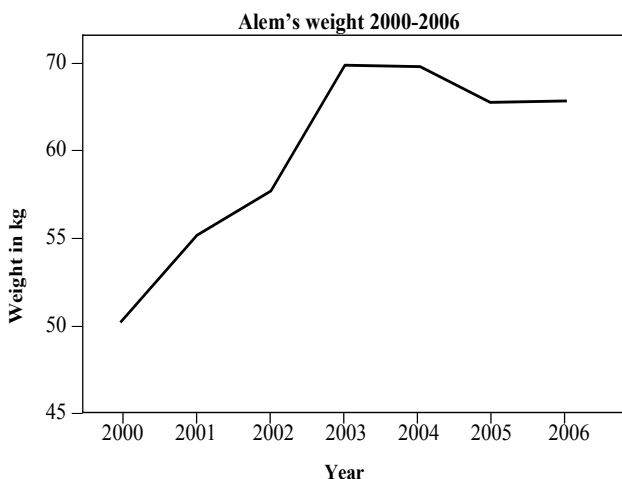


Fig 2. 10

- What is the range of values on the horizontal scale?
- What is the range of values on the vertical scale?
- How many points are in the graph?
- What was the lowest temperature recorded?
- What was the highest temperature recorded?

Activity 2.18



The above graph shows how Alem's weight varied from the beginning of 2000 to the beginning of 2006. Answer the following using the graph?

- Determine Alem's weight during the beginning of 2000?
- What was Alem's weight during 2006?

- c) Do you observe weight increase?
 d) Do you observe weight decrease?

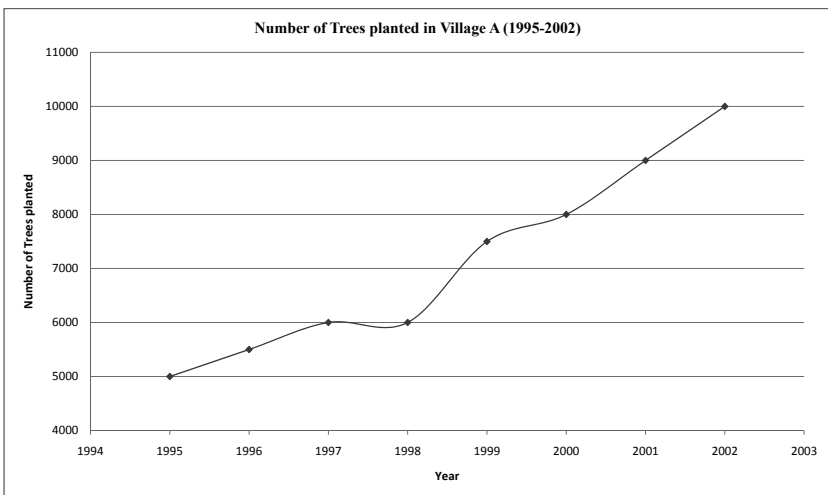
A **line graph** is graph that uses points connected by lines to show how something changes in value (as time goes by, or as something else happens). A line graph is a way to summarize how two pieces of information are related and how they vary depending on one another.

Example 2.14.

The following table provides information on the number of trees planted in Village of summer Students Campaign members from 1995-2002

Year	1995	1996	1997	1998	1999	2000	2001	2002
Number of planted	5000	5500	6000	6000	7500	8000	9000	10000

Draw a graph and plot the points on it



Exercise 2.4

1. A survey was made to find the type of music that a certain group of young people liked in a city. The following pie chart shows the findings of this survey.

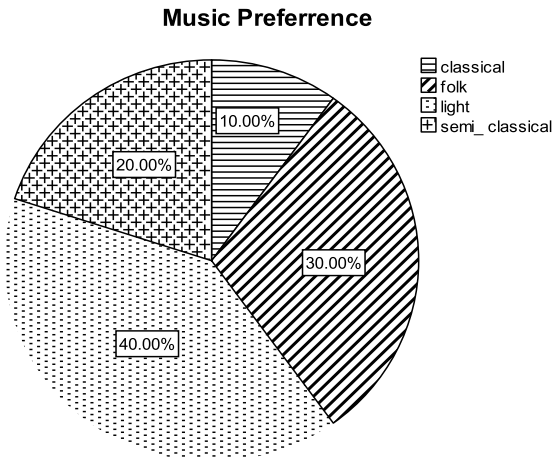


Fig 2. 11

- From this pie chart answer the following:
- a) If 20 people liked classical music, how many young people were surveyed?
 - b) Which type of music is liked by the maximum number of people?
 - c) If a recording company were to make 1000 CD's, how many of each type would they make?

2. The table shows the colours of shirt preferred by a group of people.

Colours	Number of people
Blue	18
Green	9
Red	6
Yellow	3
Total	36

- a) Draw a pie chart showing the following information.
 - b) Which coloured shirt is the most preferred shirt?
 - c) Which coloured shirt is the least preferred shirt?
3. Hospitals keep records for injury incidence arriving at the emergency room. The data set contains 66 injury incidences in a given day. The data was given in **Exercise 2.3, question Number 1.**

- a) Draw a pie chart. Remember, for pie chart you need relative frequency distribution.
- b) Draw a bar chart.

4. The number of students in a sport festival gathering, speaking different languages is given below. Display the data in a bar chart.

Language	Kunama	Arabic	Tigrigna	Tigre	Hidareb	Saho	Afar	Nara	Bilen	Total
Number	6	3	41	18	6	10	7	5	4	100

5. For the following distribution plot a bar chart

Children who prefer	School A	School B	School C
Walking	40	55	15
Cycling	45	25	35

6. Represent the above frequency distribution by a multiple bar chart?

Year	Grade 9	Grade 10	Grade 11	Total
2007-2008	1,200	1,000	800	3,000
2008-2009	1,300	1,000	700	3,000
2009-2010	1,500	1,100	750	3,350

7. The frequency distribution of the favourite sport of a sample of 100 students from one high school is given below.

Favourite Sport	Boys	Girls
Football	25	15
Basketball	20	15
Volleyball	15	10
Total	60	40

Construct an appropriate bar graph to compare groups and sport types

8. The following data represents the mileage of a car and its corresponding price.

Mileage (Km)	0	20000	40000	60000	80000	100000	120000
Car value (Nakfa)	100000	80000	60000	50000	40000	30000	20000

- Construct an appropriate graph
- Describe the relationship between car mileage and car value.

9. The lengths of a certain plant leaves were measured by a biologist and the information grouped as shown. Measurements were taken correct to the nearest cm. draw a histogram to illustrate the data

Length of leaf (cm)	0-4	5-9	10-14	15-19
Frequency	3	8	12	7

10. The daily income of a group of skilled workers is given below in a frequency distribution table

Daily Income (in Nakaf)	Number of workers (in a factory)
75-100	45
100-125	35
125-150	55
150-175	30
175-200	50
200-225	125
225-250	140
Total	480

- Construct a histogram?
- Construct a frequency polygon?

11. The table shows the ages (in years) of women who gave birth at a local maternity hospital during a particular year?

Age (Years)	16-19	20-24	25-29	30-34	35-39	40-45
Number of births	70	470	535	280	118	0

Draw a frequency polygon to illustrate this information

12. The following data is collected by weighing a sample of 30 children from middle school

39.12	61.74	37.29	44.35	57.29	51.5	40.15	60.29	41.26	49.32
64.1	48.25	67.25	58.95	39.95	36.07	46.01	41.13	67.29	45.68
38.42	55.8	44.35	38.75	63.91	63.55	62.12	36.85	45.97	42.89

- Construct a frequency distribution table.
- Construct a histogram from the frequency distribution table?

13. The daily income of a group of skilled workers is given below in a frequency distribution table

Daily Income (in Nakaf)	Number of workers (in a factory)
75-100	45
100-125	35
125-150	55
150-175	30
175-200	50
200-225	125
225-250	140
Total	480

Construct a frequency polygon?

14. Thirty women were examined in a hospital by a doctor and the number of heart beats per minute were recorded and summarised as follows.

Number of heart beats per minute	Number of women
65 - 68	2
68 - 71	4
71 - 74	3
74 - 77	8
77 - 80	7
80 - 83	4
83 - 86	2

Construct a histogram and answer the following?

- a) A woman with heart beat more than 80 beats per minute is considered as a woman at risk. Find the number of women who are at risk
- b) Find the proportion of women whose heart beat per minute is between 74 and 83?

2.4. Statistical Analysis of Ungrouped Data

2.4.1 Measures of Central Tendency

Activity 2.19

Consider a situation when three students Mary, Saliha and Metkel received their test copies. The test had five questions, each carrying ten marks. Their scores were as follows:

Question Numbers	1	2	3	4	5	Total Score
Mary's score	10	8	9	8	7	
Saliha's Score	6	7	10	9	9	
Metkel's score	5	7	10	9	10	

- a) Find the total score of each student? Who has the highest total mark?
- b) Find the average of each student? Who has the highest average score?
- c) Sort the test scores of each student and find the middle score. Which student has the highest middle score?
- d) Find out the student with maximum number of repeated score/scores?
- e) Upon getting the test copies, each student claimed to have performed better than the other. What statistical summary method do you suggest?

The above activity explored lead to the following result.

A measure of central tendency is a single value that can be used to represent all the other values in the collection.

The **mean (Arithmetic Mean)** of a variable is computed by the sum of all of the values of the variable in the data set divided by the number of observations. The **mean** is denoted $\bar{X}$.

If $\{X_1, X_2, \dots, X_N\}$ are the N observations, Then

$$\text{Mean} = \bar{X} = \frac{X_1 + X_2 + \dots + X_N}{N} = \frac{\sum_{i=1}^N X_i}{N}$$

Example 2.15.

The number of children of 10 currently married women of small community is given :3, 4, 2, 5, 1, 3, 4, 2, 3, 3

Find the mean number of children of the currently married women in the population?

Solution:

We compute for the population mean as follows.

$$\text{Mean} = \bar{X} = \frac{3 + 4 + 2 + 5 + 1 + 3 + 4 + 2 + 3 + 3}{10} = \frac{30}{10} = 3$$

The average number of children of the currently married women in the population is 3 children.

Example 2.16.

In a sample of 4 days, the temperature (in $^{\circ}\text{C}$) 28.5, 27.0, 31.2, 29.4

Solution:

$$\text{Mean} = \bar{X} = \frac{28.5 + 27.0 + 31.2 + 29.4}{4} = \frac{116.1}{4} = 29.025^{\circ}\text{C}$$

Example 2.17.

Consider the data in the table below which represent the lives of 40 cell phone (mobile phone) batteries measured the nearest of a year.

1.6 2.6 3.1 3.2 3.4 3.7 3.9 4.3
 1.8 2.9 3.1 3.3 3.4 3.7 3.9 4.4
 2.2 3.0 3.1 3.3 3.5 3.7 4.1 4.5
 2.5 3.0 3.2 3.3 3.5 3.8 4.1 4.7
 2.6 3.1 3.2 3.4 3.6 3.8 4.2 4.7

Find mean lives of the batteries?

Solution:

$$a) \text{ Mean} = \bar{x} = \frac{1.6 + 2.6 + 3.1 + \dots + 4.2 + 4.7}{40} = \frac{136.4}{40} = 3.41$$

Activity 2.20

Consider again the test score of 3 students

Question Numbers	1	2	3	4	5	Total Score
Mary's score	10	8	9	8	7	42
Saliha's Score	6	7	10	9	9	41
Metkel's score	5	7	10	9	10	41

- Sort each student's grade from smallest to largest?
- Identify the score of each student positioned at the centre of the observation?
- Whose central score is higher?

The **median** divides the array into two equal parts. If the number of observations is even, the median is simply the middle observation in the array while if the number of observations is odd, the median is the average of the two middle observations in the array.

The **median** of a variable is the value that lies in the middle of the data when arranged in ascending order. Half the data are below the median and half are above the median.

Finding the Median

Step 1: Arrange the observations in an array. We let $X_{(i)}$ be the i^{th} observation in the array, where $i = 1, 2, \dots, n$. Thus, $X_{(1)}$ is the smallest observation while $X_{(n)}$ is the largest observation.

Step 2: Determine the median.

Case 1: If n is odd, the median is the observation in the

$\left(\frac{n+1}{2}\right)^{\text{th}}$ position. If n is even, the median is the mean of the two middle observations.

$$\text{Median} = X_{\left(\frac{n+1}{2}\right)}$$

Case 2: If the number of observations n is even, the median is the mean of the two middle observations.

$$\text{Median} = \frac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2}$$

Example 2.18.

The following numbers represent the weight of packed fruits measured in grams.

0.89 0.902 0.902 0.911 0.93 0.949 1.002

Find the median weight.

Solution:

Since $n=7$, $\text{Median} = X_{\left(\frac{7+1}{2}\right)} = X_{(4)} = 0.911$

Example 2.19.

Consider the data given automobile battery lifespan data from a previous example.

1.6 2.6 3.1 3.2 3.4 3.7 3.9 4.3 2.6 3.1
 1.8 2.9 3.1 3.3 3.4 3.7 3.9 4.4 3.6 3.8
 2.2 3.0 3.1 3.3 3.5 3.7 4.1 4.5 3.2 3.4
 2.5 3.0 3.2 3.3 3.5 3.8 4.1 4.7 4.2 4.7

Find the median lifespan?

Solution:

Step 1: sort the data

1.6	1.8	2.2	2.5	2.6	2.6	2.9	3	3	3.1	3.1
3.1	3.1	3.2	3.2	3.2	3.3	3.3	3.3	3.4	3.4	3.4
3.5	3.5	3.6	3.7	3.7	3.7	3.8	3.8	3.9	3.9	4.1
4.1	4.2	4.3	4.4	4.5	4.7	4.7				

Step 2

Since $N=40$ is even we have two middle values, positioned at

$$\left(\frac{40}{2}\right)^{\text{th}} \text{ and } \left(\frac{40}{2} + 1\right)^{\text{th}}$$

$$\begin{aligned} \text{Median} &= \frac{X_{\left(\frac{40}{2}\right)} + X_{\left(\frac{40}{2} + 1\right)}}{2} \\ &= \frac{X_{(20)} + X_{(21)}}{2} \\ &= \frac{3.4 + 3.4}{2} = 3.4 \end{aligned}$$

Activity 2.21

Consider again the test score of 3 students

Question Numbers	1	2	3	4	5	Total Score
Mary's score	10	8	9	8	7	42
Saliha's Score	6	7	10	9	9	41
Metkel's score	5	7	10	9	10	41

- Find the most repeated score of each student?
- Who has the repeated highest score?

The **mode** of a variable is the most frequent observation that occurs in the data set.

Example 2.20.

Given the data on number of children of 12 currently married women: Find the mode?

2, 2, 1, 1, 1, 3, 3, 4, 4, 2, 2, 2

Solution:

The mode is 2. The most frequent number of children among the 12 currently married women is 2.

Example 2.21.

Data on number of maternal deaths in 10 health centres per year is 5, 4, 1, 1, 3, 3, 2, 1, 3, 0, Find the mode?

Solution:

The modes are 1 and 3. Sometimes the mode is not unique. The most frequent number of maternal deaths are 1 and 3.

Example 2.22.

Given the following data on grain consumption (in metric tons) in 14 locations:

629, 645, 356, 656, 231, 455, 412, 289, 444, 452, 642, 225, 335, 411, Find the mode?

Solution:

There is no mode. Sometimes the mode does not exist.

Activity 2.22

A student collects age data from two classes to calculate average age for both classes

R.No	Data from Class A	Data from Class B
1	16	16
2	18	18
3	17	17
4	18	18
5	19	19
6	38	20

- a) Find the mean, median and mode of both data?
- b) The age of student number 6 on both classes is 38 and 20 years respectively? Comment on the effect of this value on the mean, median and mode of data?
- c) What can you conclude from (b)

Exercise 2.5

1. A high school teacher prepared mathematics problems to be worked in class by each student. At the end of the period, the teacher examines the amount of time each student spent working the problems. The data, in minutes, is shown below.

15	28	25	48	22	43	49	32	22	39	31
43	49	32	22	33	27	25	22	20	24	

- a) Find the mean, median, and mode for the above data.
 - b) What does this information tell the teacher about students' length of time on solving the problems?
2. Find the mean, median, and mode for each of the following sets of data.
 - a) 18, 23, 7, 33, 25, 26, 23, 42, 18, 23, 11
 - b) 25, 26, 27, 28, 25, 28, 29, 30, 31, 30, 26, 27, 28
 - c) 103, 99, 114, 22, 99, 119, 117, 105, 100, 119, 108, 96
 - d) 2.3, 5.6, 6.3, 4.7, 8.5, 2.3, 6.5, 2.9, 7.7, 8.1, 7.3, 4.6
3. Find the mean and median for each of the following sets of data.
 - a) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
 - b) 1, 2, 3, 4, 5, 6, 7, 8, 9, 100
 - c) 1, 2, 3, 4, 5, 6, 7, 8, 90, 100

What happens to the values of the mean and median?

4. The monthly expenditure of ten families in Nakfa are given below:

Family	Expenditure (Nkfa)
Afrah	800
Berhe	700
Caltum	1000
Daniel	750
Eden	1500
Feruz	1800
Gebremedhin	1200
Haymanot	250
Issak	2100
Jemal	370

Calculate the Arithmetic mean.

5. The population of 36 villages in one sub-zoba is as follows:
468, 591, 437, 20, 213, 1430, 1490, 407, 284, 176, 263, 19, 181,
777, 387, 302, 213, 204, 153, 733, 391, 176, 178, 122, 532, 360,
65, 260, 193, 92, 672, 259, 239, 160, 147, 151. Calculate
- The median Population size
 - The mode Population size
6. Suppose a firm has 500 employees. The average salary of male employs in a firm was 1520 Nakfas, and that of females was 1420Nakafa. The mean of all employees was 1500Nakfa. Find the percentage of male and female employees.
7. Calculate the median. 2468, 591, 437, 20, 213, 143, 1490, 407, 284, 176, 263, 19, 181, 777, 387, 302, 213, 204, 153, 733, 391, 176, 178, 122, 532, 360, 65, 260, 193, 92, 672, 258, 239, 160, 147, 151

1.4.2 Measures of Dispersion

Activity 2.23

A mathematics teacher wants to choose one student from three top students of the school to participate in the 12th grade national science computation representing the school at the end of the semester. The teacher arranged an exam on 5 main subjects and the result is given below.

Student's Name	Mathematics score	Chemistry score	Biology score	Physics Score	English Score
Kahsay	90	90	90	90	90
Dawd	85	90	95	90	90
Gebrekal	75	90	95	90	100

- What is the mean of each student?
- What is the median of each student?
- What is the mode of each student?
- What difference do you observe among the three measures of central tendency?
- Who will represent the school?

Recall that a measure of central tendency is a number which is “typical” of all the values of a variable in a data set. The activity explored suggests that a measure of central tendency is not sufficient for data description. Most importantly these measures do not describe the spread (variation) of data.

A measure of dispersion is a number which describes the spread of the data.

Activity 2.24

- Calculate the difference between the largest and smallest subject score of each student?
- Which student has the highest difference and which student has the lowest difference?
- What can conclude from the above result?

The activity shows that dispersion (spread of data values can be investigated by using the difference between the maximum and minimum value in the data.

The **range**, denoted R , is the difference between the largest and smallest data values.

$R = \text{largest data value} - \text{smallest data value}.$

Example 2.23.

Consider the following sets of numbers and find the range of each data?

- a) 8, 8, 8, 8, 8
- b) 6, 7, 8, 9, 10
- c) -194, -94, 8, 108, 208

Solution:

- a) $\text{Range} = 8 - 8 = 0$
- b) $\text{Range} = 10 - 6 = 4$
- c) $\text{Range} = 208 - (-194) = 400$

Example 2.24.

The amount of the times (in minutes) spent waiting in line at two different banks is shown below.

Bank A	Bank B
6.5	4.2
6.6	5.4
6.7	5.8
6.8	6.2
7.1	6.7
7.3	7.7
7.4	7.7
7.7	8.5
7.7	9.3
7.7	10.0

Find

- a) The mean, median, mode.
- b) The range in waiting times for each of the banks and comment on the value of the range

Solution:

a)

Measure of Central Tendency	Bank A	Bank B
Mean	7.15	7.15
Median	7.2	7.2
Mode	7.7	7.7

b)

Range of Bank A = 7.7 minutes - 6.5 minutes = 1.2 minutes

Range of Bank B = 10.0 minutes - 4.2 minutes = 5.8 minutes

Waiting time of Bank B shows more variation than waiting time of Bank A.

Activity 2.25

Consider the following two datasets (ages (in years) of 3rd graders from a two small communities, A and B).

A: 8, 9, 10, 11, 12

B: 6, 8, 10, 12, 14,

a) Compute the mean age of each group

b) Complete the following tables

A

x_i	Deviations from mean ($x_i - \bar{x}$)	$(x_i - \bar{x})^2$
8	-2	4
9		
10	0	
11		1
12	2	4
Total		

B

x_i	Deviations from mean ($x_i - \bar{x}$)	$(x_i - \bar{x})^2$
6	-4	
8		4
10		0
12	2	
14		16
Total		40

- c) For each group, find the ratio of the sum of squared deviations of each value from the mean and the number of observations.
- d) What can you say about the ratios calculated for each group?

The above activity leads us to the following important measure of dispersion.

Let the data consist of N elements $\{x_1, x_2, \dots, x_N\}$, with mean $\bar{x}$. The **variance** of a variable, denoted by σ^2 , is the sum of the squared deviations about the population mean divided by the number of observations in the population.

$$\begin{aligned}\text{Variance} = \sigma^2 &= \frac{(x_1 - \bar{X})^2 + (x_2 - \bar{X})^2 + \dots + (x_N - \bar{X})^2}{N} \\ &= \frac{\sum_{i=1}^N (x_i - \bar{X})^2}{N}\end{aligned}$$

Example 2.25.

Consider the following sets of numbers

- a) 8, 8, 8, 8, 8
- b) 6, 7, 8, 9, 10
- c) -194, -94, 8, 108, 208

Find the variance of each data?

Solution

For each data, the mean ($\bar{x}$) equals 8

$$\begin{aligned}\text{a) Variance} = \sigma^2 &= \frac{\sum_{i=1}^5 (x_i - \bar{x})^2}{5} = \frac{\sum_{i=1}^5 (x_i - 8)^2}{5} \\ &= \frac{(8-8)^2 + (8-8)^2 + (8-8)^2 + (8-8)^2 + (8-8)^2}{5} = 0\end{aligned}$$

Therefore, Variance = 0

$$\begin{aligned}
 \text{b) } \frac{\sum_{i=1}^N (x_i - \bar{X})^2}{N} &= \frac{\sum_{i=1}^5 (x_i - 8)^2}{5} \\
 &= \frac{(6-8)^2 + (7-8)^2 + (8-8)^2 + (9-8)^2 + (10-8)^2}{5} \\
 &= \frac{10}{5} \\
 &= 2
 \end{aligned}$$

Therefore, variance = 2

$$\begin{aligned}
 \text{c) } \text{Variance} = \sigma^2 &= \frac{\sum_{i=1}^N (x_i - \bar{X})^2}{N} = \frac{\sum_{i=1}^5 (x_i - 8)^2}{5} \\
 &= \frac{(-194-8)^2 + (-94-8)^2 + (8-8)^2 + (108-8)^2 + (208-8)^2}{5} \\
 &= \frac{40,000 + 10,000 + 0 + 10,000 + 40,000}{5} \\
 &= \frac{100,000}{5} = 20,000
 \end{aligned}$$

Activity 2.26

What can you conclude from the above example?

Example 2.26.

Suppose the following table contains the total number of hours students spent studying per week for their subjects.

28 44 36 25 34 30 32 19

Find the variance.

Solution:

To find the variance, first find the mean.

$$\text{Mean} = \bar{x} = \frac{\sum_{i=1}^8 X_i}{8} = \frac{248 \text{ hours}}{8} = 31 \text{ hours}$$

	Hours(x_i)	Deviation($x_i - \bar{x}$)	Squared Deviation ($(x_i - \bar{x})^2$)
	28	-3	9
	44	13	169
	36	5	25
	25	-6	36
	34	3	9
	30	-1	1
	32	1	1
	19	-12	144
Total	248	0	394

$$\sigma^2 = \frac{\sum_{i=1}^8 (x_i \text{ hour} - 31 \text{ hour})^2}{8} = \frac{394}{8} (\text{hour})^2 = 49.25 (\text{hour})^2$$

Activity 2.27

A school Basket ball Coach recorded the height of a sample of children from his school as follows.

1.4m	1.3m	1.5m	1.4m	1.6m	1.4m	1.6m	1.7m
------	------	------	------	------	------	------	------

- Find the mean, median and mode?
- Find the variance?
- What is the unit of measurement of the variance?
- Does the variance convey correct information? , if no, what is the problem with the variance?

The above activity leads to the following result.

The **population standard deviation**, denoted by σ is the square root of the population variance.

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^N (x_i - \bar{X})^2}{N}}$$

Example 2.27.

Compute the sample Standard deviation of the total number of hours students spent studying per week for their subjects for the data given in **Example 2.25**.

Solution:

The variance of the total number of hours students spent studying per week = 49.25 (hour)².

Hence

Standard deviation of the total number of hours students spent

studying is given by

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^N (x_i - \bar{X})^2}{N}}$$

$$= \sqrt{\frac{\sum_{i=1}^8 (x_i - \bar{X})^2}{8}} = \sqrt{\frac{394}{8}} = \sqrt{49.25} = 7.018$$

Exercise 2.6

1. For the data set 8, 18, 10, 20, 24, 12, 14, 16, 6, and 22.

Find

- the range
- variance
- standard deviation

2. Find the mean and variance for each of the following data.

- 1, 3, 6, 7, 8, 9
- 11, 14, 18, 23, 29
- 4.6, 2.7, 3.1, 0.5, 6.2
- 200, 203, 206, 207, 209
- 1, 6, 7, 10, 12, 13, 4, 8, 12

3.

- Find the mean and variance of the set of numbers 4, 6, 9, 3, 5, 6, 9
- Deduce the mean and variance of the set of numbers 514, 516, 519, 513, 515, 516, 519.
- Deduce the mean and variance of the set of numbers 52, 78, 117, 39, 65, 78, 117

4. Show that the standard deviation of {1, 2, 3, 4, 5, 6, 7} is 2.

Using this result find the standard deviation of the following sets.

- {101, 102, 103, 104, 105, 106, 107}
- {100, 200, 300, 400, 500, 600, 700}
- {2.01, 3.02, 4.03, 5.04, 6.05, 7.06, 8.07}

5. A municipality of Asmara officer routinely controls the weights of each bread produced by private bakeries. The officer takes ten breads from each bakery produced daily

Bakery 1 (mass in g)	96	98	98	99	100	100	101	101	102	105
Bakery 2 (mass in g)	92	94	95	98	100	101	103	104	106	107

- Compute the mean bread weight of each bakery?
- Calculate the variance and standard deviation of bread weights of the two Bakeries?
- The municipality rule stipulates that the weight of single bread must be 100 g. Which bakery is following the regulations and which is deviating from the rule? Justify your answer?

2.5. Probability

2.5.1 Basic Concepts

Activity 2.28

- Toss a coin.
 - Can you be certain that the outcome is a head?
 - Can you be certain that the outcome is a tail?
 - Can you be certain that the outcome is either a head or a tail?
 - Can you be certain that neither of the two outcomes will occur?
- Roll a die.
 - Can you be certain that the outcome is 3?
 - What are the possible outcomes?
 - Can you be certain that the outcome is one of the numbers: 1, 2, 3, 4, 5 and 6?
 - Is it possible to get 8 as an outcome?

In statistics, the word experiment is used to describe any process that generates a set of data. For instance, tossing of a coin which has two possible outcomes head or tail is a statistical experiment. Rolling a die has 6 possible outcomes, namely, 1, 2, 3, 4, 5 and 6. This is also an example of statistical experiment.

A phenomenon is called **random** if individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions.

Any process which generates well defined outcomes is called an **experiment**.

In Activity 2.29, we have been observing some statistical outcomes. These outcomes indicate that different outcomes have different chances of occurring.

In general, events from daily life can be classified as either:

- a) certain that they will happen; or
- b) certain that they will not happen; or
- c) Uncertain.

In Activity 2.29, we cannot be certain that a given toss will result in a head.

In experiments involving chance, we usually use the following terms.

The number of **trials** is the total number of times the experiment is repeated. The **outcomes** are the different results possible for one trial of the experiment.

Probability theory is concerned with predicting statistical outcomes.

Let us now introduce the definitions of some terms that are relevant to the study of probability.

The set of all possible outcomes of a statistical experiment is called a **sample space** and it is denoted by S .

An **element** or a **member** of a sample space is sometimes called as a **sample point**.

Example 2.28.

1. Suppose a coin is tossed. The set of all possible outcomes is {Head, Tail}. Therefore, the sample space is $S = \{\text{Head, Tail}\}$.
2. When a die is rolled, there are 6 possible outcomes. The sample space = $\{1, 2, 3, 4, 5, 6\}$.
3. When two coins are tossed, the sample space = $\{HH, HT, TH, TT\}$.

Example 2.29.

1. Sample spaces could be best displayed visually using two dimensional grids.

Toss coins A and B and plot the outcomes as in Figure 2. 12

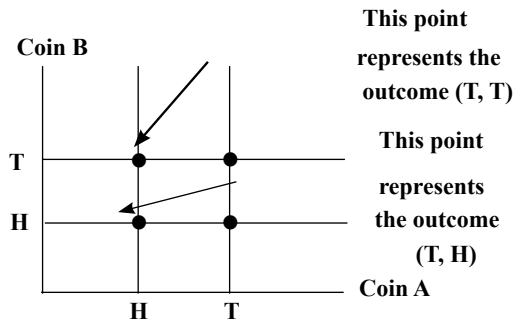


Fig 2. 12

Sample Space = $\{HH, HT, TH, TT\}$

2. It is also possible to use tree diagrams to determine a sample space.

Three items are selected at random from a manufacturing process. Each item is inspected and classified defective (D) or non-defective (N). List the elements of the sample space.

Construct a tree diagram as in Figure 2. 13. Start at the top with the first path and you get DDD. Proceed with the other paths to list the elements of the sample space.

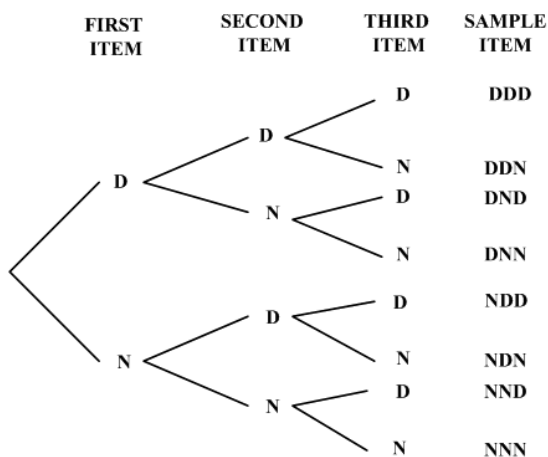


Fig 2. 13

Sample space = {DDD, DDN, DND, DNN, NDD, NDN, NND, NNN}

In a given experiment, we may be interested in the occurrence of certain events rather than in the outcome of a specific element in the sample space. For example, we may be interested in the event that “the outcome is even” when a die is rolled once.

An **event** is a subset of a sample space.

If an event is a set containing only one element of the sample space, then it is called a **simple event**. A **compound event** is one that can be expressed as the union of simple events.

Null space:

A subset of the sample space which contains no elements is called the **null space** or **empty space**.

Example 2.30.

Let the event A be defined by $A = \{x | x \text{ is an even positive factor of } 9\}$

There is no even factor of 9. Therefore, $A = \emptyset$. Hence, A is a null space.

Exercise 2.7

1. Using a tree diagram, illustrate the sample space for:
 - a) Tossing two coins.
 - b) Rolling two dice.
2. Draw two balls from a bag containing a red, a green and a yellow ball. Find the sample space using a tree diagram.
3. List the sample space for the following.
 - a) The sexes of two children of a family.
 - b) The sexes of three children of a family.

2.5.2 Operations on Events

Activity 2.29

Suppose a dice is tossed, let A be the event “an even number turns up” B is the event “an odd’ number turns up” and C is the event “a number less than 3” turns up

Determine

- a) $A \cap B$
- b) $A \cup B$

We now consider new events based on A and B which are subsets of the same sample space S.

Intersection of Events: The intersection of two events A and B, denoted by $A \cap B$, is the event containing all elements that are common to A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

If two events A and B cannot both occur simultaneously or if $A \cap B = \emptyset$, then they are called **mutually exclusive**. That means events A and B have no points in common.

Union of Events: The union of the two events A and B, denoted by $A \cup B$, is the event containing all the elements that belong to A or to B or to both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Activity 2.30

Suppose that a school director wants to consider the performance of students in an examination as a sample space S. Let A be a subset of S which correspond to the event that “a student scores greater than 80”. Then all students scoring less or equal to 80 correspond to some event which is a subset of S. This called the **complement** of A.

Complement of an Event: The complement of an event A with respect to a sample space S is the set of all elements of S that are not in A. We denote the complement of A by the symbol A' .

$$A' = \{x \mid x \in S \text{ and } x \notin A\}$$

2.5.3 Experimental Probability

Activity 2.31

- Prepare a solid cone using hard paper. Then toss the cone into the air 50 times and count the number of times it falls on its side and on its base.

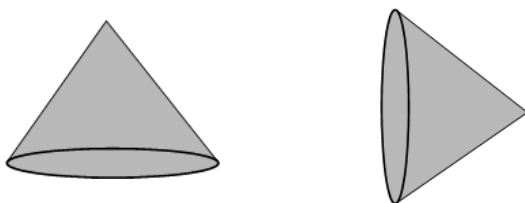


Fig 2. 14

- b) Count the number of times it falls on its side.
- c) Count the number of times it falls on its base.
- d) Determine the relative frequencies of side and base.
- e) What is the probability of each event occurring.

Activity 2.32

If two coins are tossed simultaneously, the possible outcomes are: HH, HT, TH and TT.

- a) Toss the pair of coins 40 times and record the outcomes in a table.
- b) Obtain relative frequencies (experimental probability) for each of the four events.
- c) Combine your results with four other students and so obtain experimental probabilities from 200 tosses, using the same coins.
- d) Which gives the most reliable estimates? Your results or the groups'? Why?

The **frequency** of a particular outcome is the number of times that this outcome is observed. The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

$$\text{Relative Frequency} = \frac{\text{The frequency of an outcome}}{\text{Total number of trials}}$$

In the absence of other alternatives, the relative frequency is the best estimate of each event occurring.

Experimental probability = Relative frequency

$$= \frac{\text{The frequency of an outcome}}{\text{Total number of trials}}$$

Example 2.31.

A survey was conducted to determine students' favourite colour. Each student chose only one colour

colour	yellow	green	blue	Red	black	white
Number of students	10	25	35	8	5	17

What is the probability that a student's favorite colour is blue?

Solution:

Let C=student's favorite colour is blue.

35 out of the 100 students chose blue. Then,

$$P(C) = P(\text{student's favorite colour is blue}) = \frac{35}{100} \\ = 0.35$$

2.5.4 Theoretical Probability

Activity 2.33

Consider the orthogonal spinner in Figure 2.15 below.

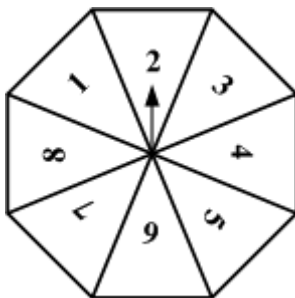


Fig 2. 15

A game of chance consists of spinning an arrow that comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8. The numbers are equally likely outcomes. That means the arrowed marker could finish in one of the numbers 1 to 8. What is the chance that it will point at:

- 5?
- An even number?
- A number less than 5?
- A number greater than 6?

The **theoretical probability** of a particular event (E), denoted by $P(E)$, is a measure of the chance of that event occurring in any trial of the experiment.

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

Example 2.32.

1. A coin is tossed once. Find the probability of getting:
a) head b) tail

Solution:

The number of possible outcomes is 2. That is head (H) or tail (T).

- a) Let E_1 be the event 'getting a head'. The number of outcomes favourable to E_1 is 1.

Therefore,

$$P(E_1) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

$$P(E_1) = \frac{1}{2}$$

b) Similarly, $P(E_2) = \frac{1}{2}$

2. A bag contains three balls, red, blue and yellow, all of the same size. Kibrom takes out a ball from the bag without looking into it. What is the probability that he takes out :
a) Red ball?
b) Blue ball?
c) Yellow ball?

Solution:

Since Kibrom takes out a ball from the bag without looking into it, it is equally likely that he takes out any one of them.

Let R be the event ‘the ball taken out is Red’, B the event ‘the ball taken out is blue’, and Y be the event ‘the ball taken out is yellow’.

The number of possible outcomes = 3.

- a) The number of outcomes favourable to the event R = 1.

$$\text{Therefore, } P(R) = \frac{1}{3}.$$

Similarly,

b) $P(B) = \frac{1}{3}$

c) $P(Y) = \frac{1}{3}$

3. A card is randomly selected from a bag containing three green, four yellow and five blue cards. What is the probability of getting:
- a) A yellow card?
- b) A yellow or blue?
- c) A green, yellow or blue card?
- d) A black card?

Solution:

The sample space is {G, G, G, Y, Y, Y, Y, B, B, B, B, B}

a) $P(\text{yellow}) = \frac{4}{12} = \frac{1}{3}$

b) $P(\text{yellow or blue}) = \frac{9}{12} = \frac{3}{4}$

c) $P(\text{green, yellow blue}) = \frac{12}{12} = 1$. The chance of occurring green, yellow or blue is 100% or 1.

d) $P(\text{black}) = \frac{0}{12} = 0$. Note that there is no chance of occurring a black card.

Comparing Experimental and Theoretical Probabilities

Activity 2.34

Kidane and Jemal roll two dice 50 times and record their results in the accompanying table.

Sum of the rolls of two dice
3, 5, 5, 4, 6, 7, 7, 5, 9, 10, 12, 9, 6, 5, 7, 8, 7, 4, 11, 6, 8, 8, 10, 6, 7, 4, 4, 5, 7, 9, 9, 7, 8, 11, 6, 5, 4, 7, 7, 4, 3, 6, 7, 7, 7, 8, 6, 7, 8, 9

- What is experimental probability of rolling a 7?
- What is the theoretical probability of rolling a 7?
- How do the experimental and theoretical probabilities compare?

In question number 3 of Example 2.31, observe that if there is no chance of occurring, then the probability is 0, whereas, if an event is certain to occur, then the probability is equal to 1.

This leads us to the following conclusion.

If an event is denoted by E , then $0 \leq P(E) \leq 1$.

Exercise 2.8

- A coin is tossed twice. What is the probability that at least one head occurs?
- A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number greater than 3 occurs on a single toss of the die, find $P(E)$.
- If a letter is chosen at random from the English alphabet, find the probability the letter is:
 - Consonant.
 - Follows the letter j.
 - Precedes the letter g.

4. A carton of a dozen eggs contains 8 brown eggs. The rest are white.
 - a) How many eggs are there in the box?
 - b) What is the probability that an egg selected at random is:
 - a) a brown egg? b) a white egg?
5. In a class of 40 students, the first letter of their names begins with A, B, and D. The numbers of students are 23, 12 and 5 respectively. A student is selected at random. Determine the probability that:
 - a) The first names begin with B.
 - b) The first name does not begin with D.
 - c) The first names with A, B or D.
 - d) The first name begins with T.

2.5.5 Laws of Probability

Addition Law of Probability

Activity 2.35

1. Let $A = \{HHT, HTH, THH\}$ and $B = \{HTH, THH, HHH\}$ be two events associated with 'tossing of a coin thrice'

$$A \cup B = \{HHT, HTH, THH, HHH\}$$

$$\text{Now } P(A \cup B) = P(HHT) + P(HTH) + P(THH) + P(HHH)$$

$$\text{Since all are equally likely, } P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\text{Also } P(A) = P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \text{ and}$$

$$P(B) = P(HTH) + P(THH) + P(HHH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$\text{Therefore } P(A) + P(B) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$

It is clear that $P(A \cup B) \neq P(A) + P(B)$

- a) Find $A \cap B$.
- b) Determine $P(A \cap B)$.
- c) How do you relate $P(A \cup B)$, $P(A)$, $P(B)$ and $P(A \cap B)$?

2. A die is rolled once. Let $A = \{2, 5\}$ and $B = \{1, 5, 6\}$ be two events.
- Determine: $P(A \cup B)$, $P(A)$, $P(B)$ and $P(A \cap B)$.
 - What is the relationship between $P(A \cup B)$, $P(A)$, $P(B)$ and $P(A \cap B)$?

Activity 2.36 suggests that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Addition Law of Probability

For two events A and B:

$P(\text{either A or B}) = P(A) + P(B) - P(\text{A and B})$, or

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example 2.33.

If $P(B) = 0.4$, $P(A \cup B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(A)$.

Solution:

By the addition law of probability,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Therefore, $0.7 = P(A) + 0.4 - 0.3$

$$P(A) = 0.7 + 0.3 - 0.4$$

$$P(A) = 0.6$$

Remark:

If events A and B are mutually exclusive, (i.e. they have no common elements, $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B)$.

Complementary Event

Activity 2.36

A die is rolled once.

- What is the probability of g getting a 5? not getting a 5? getting 3 or 4? not getting 3 or 4?
- What is $P(\text{getting a 5}) + P(\text{not getting a 5})$?

- c) What is $P(\text{getting 3 or 4}) + P(\text{not getting 3 or 4})$?
- d) What relationship do you get from conclusions of (a) and (b)?

In general, it is true that an event may or may not occur. For instance, either “getting a 5” or “not getting a 5” have to occur. These events are called **complementary events**. If E is an event, its complementary event is denoted by E' .

This activity suggests that the following.

$$\text{If } E \text{ is an event, then } P(E) + P(E') = 1$$

Remark:

The statement, $P(E) + P(E') = 1$ could be expressed as:

$$P(E \text{ not occurring}) = 1 - P(E \text{ occurring})$$

Example 2.34.

Two players, Senait and Rahma, play a tennis match. If the probability of Senait winning the match is 0.7, what is the probability of Rahma winning the match?

Solution:

Let S denote the event that Senait wins the match.

Let R denote that Rahma wins the match.

Then, $P(S) = 0.7$.

$P(R) = 1 - P(S)$, since the events R and S are complementary.

Therefore, $P(R) = 1 - 0.7 = 0.3$

Multiplication Rule

Activity 2.37

1. Roll one die followed by another. Find the probability of rolling
 - a) a 4 on the first die
 - b) An even number on the second die.
 - c) a 4 on the first die and rolling an even number on the second die

2. A coin and a dice are thrown simultaneously. What is the probability
- a) Of getting an head
 - b) getting three
 - c) that he will get head & three simultaneously

What can you conclude from the above activity?

If A and B are events, then the probability of both occurring is

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Example 2.35.

A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and a yellow marble?

Solution:

$$P(\text{green}) = \frac{5}{16}$$

$$P(\text{yellow}) = \frac{6}{16}$$

$$P(\text{green and yellow}) = P(\text{green}) \cdot P(\text{yellow})$$

$$= \frac{5}{16} \cdot \frac{6}{16}$$

$$= \frac{30}{256}$$

$$= \frac{15}{128}$$

Exercise 2.9

1. The probability that a student passes chemistry is $\frac{2}{3}$, and the probability that he passes physics is $\frac{4}{9}$. If the probability that he passes both courses is $\frac{2}{9}$. What is the probability that he will pass at least one course?
2. What is the probability of getting a total of 7 or 11 when a pair of dice is tossed?
3. A coin is tossed 6 times in succession. What is the probability that at least one tail occurs?
4. Suppose that 10% of the units produced in a certain bottle production company are defective. What is the probability of:
 - a) One bottle picked out at random is not defective?
 - b) One bottle picked at random is defective.
5. If A and B are mutually exclusive events and $P(A) = 0.3$ and $P(B) = 0.5$, find
 - a) $P(A \cup B)$
 - b) $P(A')$
 - c) $P(A' \cap B)$(Hint: Use Venn diagrams)
6. A jar contains 3 green balls, 5 white balls and 7 yellow balls. Two balls are chosen from the jar, with replacement. What is the probability that both balls chosen are green?

2.5.6 Permutations and Combinations

Fundamental Principle of Counting

Activity 2.38

Mohammed has 3 trousers and 3 shirts.

- In how many ways can he choose the trousers?
- In how many ways can he choose the shirts?
- For every choice of a trouser, how many choices of a shirt does he have?
- How many pairs of trousers and shirts does he have?

In order to visualize the above questions, you may consider the following Figure 2.16. The trousers are represented by T_1 , T_2 and T_3 . The shirts are named S_1 , S_2 , and S_3 .

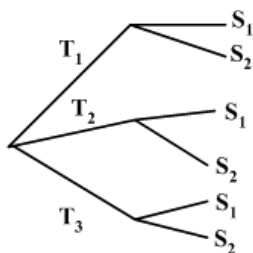


Fig 2. 16

Example 2.36.

Three students are selected at random from a class. Each student is given an oral test and classified as “Passed (P)” or “Failed (F)”. In how many ways can you arrange their academic status?

Solution:

Construct a tree diagram as in the Figure 2.17 below. List all possible elements of the sample space.

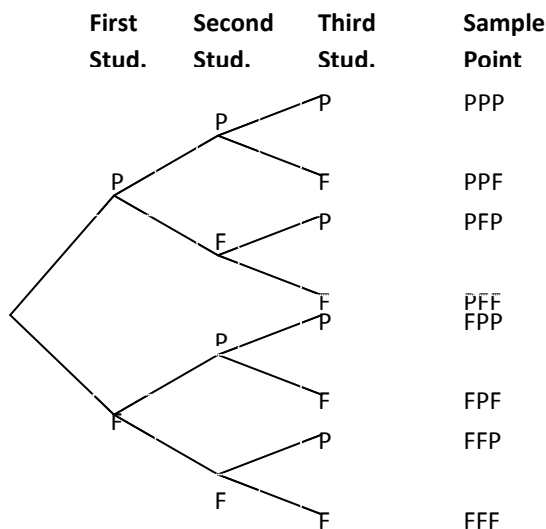


Fig 2. 17

Starting with the first path, we get various sample points from the tree diagram. The sample space is $S = \{PPP, PPF, PFP, PFF, FPP, FPF, FFP, FFF\}$. There are 8 possible choices for their academic status.

Example 2.37.

Seghid has 2 trousers, 3 shirts and 2 coats. In how many ways can he choose the trousers, shirts and coats? (Choosing one each)

Solution:

You may sketch a tree diagram to visualize the problem.

A trouser can be chosen in 2 different ways. After a trouser is chosen, a shirt can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of choosing a trouser and a shirt.

For each of these pairs a coat can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Seghid can choose the clothes.

The problems in Activity 2.38 and Example 2.36 can be solved by applying the following principle, known as the fundamental principle of counting, or, simply, the multiplication principle.

Fundamental Principle of Counting (Multiplication Principle)

If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.

The above principle can be generalized for any finite number of events.

Generalized Multiplication Rule

If an operation can be performed in n_1 ways, if for each of these a second operation can be performed in n_2 ways, if for each of these a third operation can be performed in n_3 ways, and so on, then the sequence of k operations can be performed in $n_1 \times n_2 \times n_3 \times \cdots \times n_k$ ways.

Example 2.38.

How many choices of breakfasts are possible consisting of cake, sandwich and drinks if one can select from 4 cakes, 3 sandwiches and 5 drinks?

Solution:

The total number of choices of breakfasts is $4 \times 3 \times 5 = 60$

Example 2.39.

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word "FAST" where the repetition of the letters is not allowed.

Solution:

The first place can be filled in 4 different ways by anyone of the 4 letters F, A, S, T.

The second place can be filled in by anyone of the remaining 3 letters in 3 different ways,

The third place can be filled in 2 different ways.

The fourth place can be filled in 1 way.

Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is

$$4 \times 3 \times 2 \times 1 = 24.$$

Hence, the number of words is 24.

2.5.7 Permutations and Combinations

Permutations

Activity 2.39

Consider sets $A = \{a\}$, $B = \{a, b\}$, $C = \{a, b, c\}$ and

$$D = \{a, b, c, d\}.$$

Complete the missing information in the following table.

Sets	Number of Elements of the Sets	Distinct Arrangements of the Elements	Number of Arrangements of the Elements
$\{a\}$	1	a	$1 = 1$
$\{a, b\}$	2	ab, ba	$2 = 2 \times 1$
$\{a, b, c\}$	3	abc, acb, bac, bca, cab, cba,	$3 = 3 \times 2 \times 1$
$\{a, b, c, d\}$	4		$24 = 4 \times 3 \times 2 \times 1$

- How do you relate the number of elements of each set with the corresponding number of arrangements of the elements?
- How many distinct arrangements are possible for a set with 6 elements?, 7 elements?, 8 elements? n elements?

Example 2.40.

Consider the letters A, B, C, D, E.

The first letter (one of the letters A, B, C, D, E) can be chosen in 5 ways.

The second letter can be chosen in 4 ways.

The third letter can be chosen in 3 ways.

The fourth letter can be chosen in 2 ways.

The fifth letter can be chosen in only one way.

Therefore, the number of ways of arranging the 5 letters is

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

Factorial Notation:

- 1) If n is a positive integer, then $n(n-1)(n-2)\cdots(3)(2)1$ is denoted by the symbol $n!$, which is read as “ n factorial.”
- 2) $1! = 1$
- 3) $0! = 1$.

This activity suggests the following definition and rule.

The arrangement of all or part of a set of objects is called **permutation**. The number of permutations of n distinct objects is $n!$. In other words, the number of ways of arranging n unlike objects in a line is $n!$.

Example 2.41.

1. $6! = 6(6-1)(6-2)\cdots(3)(2)1$
 $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$
 $= 720$
2. How many number of different plate numbers if each is to contain the three letters M, S and T followed by the three digits 1, 2 and 3?

Solution:

There are $3!$ ways of arranging the letters M, S and T.

There are $3!$ Ways of arranging the numbers 1, 2 and 3.

Therefore, the total number of different identification numbers =
 $(3!)(3!) = 36$

In Activity 2.39 above, we considered n distinct objects. What could be said if some of the objects are repeated?

Activity 2.40

For the letters A, B, C, D and E, consider the number of ways of placing 3 of the letters in three empty spaces.

- You can fill the first space with 5 different ways. Why?
- You can fill the second space with 4 different ways. Why?
- You can fill the third space with 3 different ways. Why?

Therefore, there are $(5)(4)(3)$ ways of arranging three letters taken from 5 letters. This is the number of permutations of 3 objects taken from 5.

This is denoted by ${}_5P_3$

Observe that ${}_5P_3 = (5)(4)(3) = 60$

Now, $(5)(4)(3)$ could be written as $\frac{(5)(4)(3)(2)(1)}{(2)(1)} = \frac{5!}{(5-3)!}$

Remark:

In permutations, the order in which the objects are arranged is important. For instance, ABCD is different from BADC.

Activity 2.5 could be generalized as follows.

The number of permutations, or ordered arrangements, of r objects taken from n unlike objects, denoted by ${}_nP_r$, is defined

$$\text{by } {}_nP_r = \frac{n!}{(n-r)!}$$

Remark: the above can be written as

$$P_r = \frac{n!}{(n-r)!} = n! \times (n-1)! \times (n-2)! \times \dots \times (n-r+1)!$$

Example 2.42.

Find the number of ways in which 6 teachers can be assigned to 4 classes of a mathematics course if no teacher is assigned to more than one class.

Solution:

The number of permutations of 4 teachers taken from

$$6 \text{ teachers} = {}_6P_4$$

$$= \frac{6!}{(6-4)!}$$

$$= \frac{6!}{2!} = 360$$

Activity 2.41

- Consider 3 students A, B and C, who are to be seated in a round table. Fix and find the number of arrangements for B and C.
- Consider 4 students A, B, C and D who are again to be seated in round table. Fix A and find the number of possible arrangements of B, C and D.
- What is the number of arrangements if the number of students is n ?

Activity 2.41 leads us to the following rule.

The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are different is $(n-1)!$.

Example 2.43.

Six students are seating in a circular table and two of them are best friends.

- What is the total number of arrangements?
- What is the probability that the two students who are best friends are seating next to each other?

Solution:

- If the sample space is S , then $n(S) = 5!$
- Let the event that “the students are seating to each other” be denoted by E .

The two students who are best friends could be arranged in $2! = 2$ ways.

Treating the two students as one item, the remaining five items could be arranged in $(5 - 1)! = 4!$.

Therefore, $n(E) = 2!4!$

$$\begin{aligned}\text{Hence, } P(E) &= \frac{n(E)}{n(S)} \\ &= \frac{2!4!}{5!} \\ &= \frac{2}{5}\end{aligned}$$

The probability that the students who are best friends seat next to each other is $\frac{2}{5} = 360$

Activity 2.42

Consider the sets $\{a, a, b\}$, $\{a, a, a, b\}$, $\{a, a, a, a, d\}$ and $\{a, a, a, b, c\}$:

- For the elements indicated in the table, list all possible arrangements and find the number of arrangements.
- Let n represent the number of elements of each set. Let r denote the number of elements of repeated letters in each set.

Determine $\frac{n!}{r!}$ for each of the sets.

- How do you relate the results of (a) and (b)?

Activity 2.40 suggests the following rule.

The number of ways of arranging in n objects in a line, of which r are alike, is $\frac{n!}{r!}$.

This result could be extended as follows.

The number of ways of arranging n objects in a line (distinguishable permutations), of which n_1 are of one kind, n_2 are of another kind, ... n_k are of the k_{th} kind is $\frac{n!}{n_1!n_2!\dots n_k!}$.

Example 2.44.

- a) In how many ways can the letters of the word “STATISTICS” be arranged?
- b) If the letters of the word “STATISTICS” are arranged in a line at random, what is the probability that the arrangement begins with SSS?

Solution:

- a) There are 10 letters. S occurs 3 times, T occurs three times and I occurs twice.

$$\begin{aligned}\text{Therefore, the number of ways of arrangement} &= \frac{10!}{3!3!2!} \\ &= \frac{(10)(9)(8)}{3!3!2!} \\ &= 50400\end{aligned}$$

- b) From (a), the number of arrangements of the word “STATISTICS” is 50400.

The arrangement will be of the form SSSxxxxxxx. Since SSS could be arranged only once, the remaining 7 letters could be

$$\text{arranged in } \frac{7!}{3!2!} = 84 \text{ ways.}$$

$\therefore$ The probability that the arrangement begins with SSS

$$= \frac{84}{50400}$$

$$\approx 0017$$

In many instances, we may be interested in the number of ways of selecting r objects from n without regard to order. These types of selections are called **combinations**. For instance the outcomes ABC, ACB, BAC, BCA, CBA and CAB are six different permutations, but they are the same combination.

Activity 2.43

Consider the letters A, B, C and D representing students. Let us write all possible arrangements of these 4 students taking 3 at a time.

a) How many arrangements do you expect?

The arrangements are given below.

ABC	ACB	BCA	BAC	CAB	CBA
ABD	ADB	BDA	BAD	DAB	DBA
ACD	ADC	CDA	CAD	DAC	DCA
BCD	BDC	CDB	CBD	DBC	DCB

Observe that there are 24 arrangements. In each arrangement of each row the students are the same. That means, there is only one committee in each row. The number of committees is given by the number of rows. Therefore, there are 4 possible committees. Each row contains 3! permutations.

b) Find the number of committees using the formula:

$$\frac{\text{Total number of arrangements}}{3!}$$

c) Find $\frac{{}_4P_3}{3!}$

d) Determine $\frac{4!}{3!(4-3)!}$

e) What is the relationship between the results of (b), (c) and (d)?

The number of combinations of the committees consisting of 3

students from the 4 students is $\frac{{}_4P_3}{3!}$. It is denoted by ${}_4C_3$

Combinations of r Objects from n Objects

The number of combinations of r objects from n unlike objects,

denoted by ${}_nC_r$, is defined by, ${}_nC_r = \frac{n!}{r!(n-r)!}$.

Example 2.45.

From 4 teachers and 3 students find the number of committees of 3 that can be formed with 2 teachers and 1 student.

Solution:

The number of ways of selecting two teachers from

$$4 = {}_nC_r = \frac{4!}{2!(4-2)!} = 6$$

The number of ways of selecting 1 student from

$$3 = {}_3C_1 = \frac{3!}{1!(3-1)!} = 3$$

Therefore, the number of committees that could be formed with 2 teachers and 1 student is $6 \times 3 = 18$

Example 2.46.

Two letters are chosen at random from the word “MONDAY”. What is the probability that 2 of the all two letters chosen are consonants?

Solution:

Let S be the sample space.

Let E represent the event “two consonants are chosen”

$$\begin{aligned} n(S) &= {}_6C_2 \\ &= \frac{6!}{2!(6-2)!} \\ &= \frac{6!}{2!4!} \\ &= 15 \end{aligned}$$

$n(E) = {}_4C_2$ since there are 4 consonants (M, N, D, Y).

$$\begin{aligned}
 &= \frac{4!}{2!(4-2)!} \\
 &= \frac{4!}{2!2!} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{Therefore, } P(E) &= \frac{n(E)}{n(S)} \\
 &= \frac{6}{15} \\
 &= \frac{2}{5}
 \end{aligned}$$

Exercise 2.11

1. Suppose there are 3 roads from city Asmara to city Barentu and 2 roads from Barentu to Tessenei. Determine the total number of ways (possibilities) you can get from Asmara city to Tessenei
2. Suppose you have 6 shirts of different colours and 4 pants of different colors. How many possibilities (ways) are there to pair one shirt with 1 pant?
3. How many license plates consisting of 2 letters followed by 4 numbers are possible?
4. In How many ways can 3 boys and 2 girls form a line?
5. In How many ways can 3 boys and 2 girls form a line if the two girls must be together?
6. In How many ways can 3 boys and 2 girls form a line if the boys must be together and the girls must be together?
7. In How many ways can a librarian arrange 4 English books, 3 mathematics books, 2 biology books, and 1 Chemistry book on a bookshelf so that all the English books are together, all the mathematics books are together, all the biology books are together.

8. How many ways can you choose four summer courses from a set of ten offered next summer and order is important?
9. Calculate the number of ways if we want to choose three books from a set of seven distinct books, without replacement and order is not important.

Review Exercises 2

1. One journal reported on a study that “frequently Heading the ball in soccer lowers’ IQs”. To verify this claim, a psychologist tested 60 male soccer players selected from all First division foot ball players’ ages between 15 and 30 years old, who played up to five times in a week. Players who averaged 10 or more headers a game had an average IQ of 103, players who headed between 2 and 10 times per game had an average IQ of 107 and players who headed two or fewer times per game had an average IQ of 110.
 - a) Describe the population of interest to the psychologist.
 - b) Identify the variable of interest. Is the variable qualitative or quantitative?
 - c) Describe the sample.
 - d) Is the sample representative? Why or why not?
2. In a medical study, researchers would like to compare two drugs (A and B) for patients with a specific type of disease. There were 600 patients used in the study (they were recruited from clinics all over Eritrea). The patients were randomly assigned to receive one of the drugs.
 - a) What is the population of interest here?
 - b) What is the sample?
3. At the beginning of second semester the Sub-Zoba office of the ministry of education conducted a one day educational festival. The next day a sample of those attended were asked to rate the disco on a five point scale A, B, C, D, E, where A represents maximum enjoyment, , B= good quality enjoyment, C= quality enjoyment, D= satisfactory enjoyment and E=minimum enjoyment. Their ratings are shown below.

A	C	E	C	C	E	A	C	B	C
E	C	B	B	A	B	C	B	D	D
D	B	D	B	B	B	C	B	C	C
E	B	B	B	D	A	B	A	B	B
C	D	C	E	B	D	C	C	D	D

- (a) Construct a frequency distribution for these ratings with columns tally and frequency.
 - (b) Find the number of festival attendant who rated the festival as A?
4. You have asked 20 persons whether they think they are financially better off than their parents. The following are the responses of these 20 persons. (B indicates that the person is better off, W represents that the person is worse off, and S stands for the same.)

B	W	B	S	B	B	B	S	W	S
W	S	B	B	B	W	W	B	S	W

- a) Identify the variable of interest.
 - b) State the type of variable
 - c) Construct a frequency distribution table.
 - d) Calculate the relative frequencies and percentages for all categories.
 - e) Draw a pie chart for the percentage distribution.
5. 38 children solved a mathematical problem and the time taken by each was noted.

Time (seconds)	0-4	5-9	10-14	15-19	20-24	25-29
Frequency	2	12	7	15	2	0

- i. Draw a histogram
 - ii. Comment on the graph
6. The following table shows the imports/exports of a country in millions of dollars.

Year	1995-96	1996-97	1997-98	1998-99
Imports	40	60	80	100
Exports	50	50	60	50

Represent this data by a line graph.

7. Given below is the data of college going students:

Mode of transport	Bus	Walking	Bicycle	Motor cycle	Car
Number of boys	60	30	40	50	20
Number of girls	60	20	30	10	10

Represent this data by a

(a) Multiple bar charts.

8. A Hotel wants to introduce a new food type. The management of the hotel conducted a study to determine the average of number of customers ordering the food type day. The management reordered the number of daily orders for ten days and are listed in the table below.

34 39 40 46 33 31 34 14 15 45

Determine each of the following measures of the sample.

- Mean
 - Median
 - Mode
 - Range
 - Variance
 - Standard deviation
9. The mean of a certain number of observations is 40. If two items with values 50 and 64 are added to this data, the mean rises to 42. Find the number of items in the original data.
10. The data in the following table represents the number of days customers take to pay their electric bills.

13 13 13 20 26 27 31 34 34 34
 35 35 36 37 38 41 41 41 45 47
 47 47 50 51 53 54 56 62 67 92

Find

- Mean
- Median
- Mode
- Range
- Variance
- Standard deviation

11. Illustrate on a tree diagram the sample space for:
 - a) Tossing a 10-cent coin and 25-cent coin simultaneously.
 - b) Drawing two tickets from a box containing a number of blue, green and pink tickets.

12. A box contains five balls, a black (b), white (w), red (r), orange (o), and green (g). Three balls are to be selected at random.
 - c) Find the sample space S
 - d) Find the probability of selecting a black ball.
 - e) Find the probability of selecting one black and one red ball.

13. A pair of dice is tossed. Find the probability of getting
 - a) A total of 8.
 - b) At most a total of 5.

14. If 3 books are picked at random from a box containing 5 books of mathematics, 3 books of history and one dictionary, what is the probability that:
 - a) $P(\text{The dictionary is selected}) =$.
 - b) Two books of mathematics and one book of history are selected.

15. If A, B and C are mutually exclusive events and $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.2$, find:
 - a) $P(A \cup B)$
 - b) $P[A' \cap (B \cup C)]$
 - c) $P(B \cup C')$

16. The probability that a student A fail in the exam is 0.2, and a student B fail in the exam is 0.3. find the probability that.
 - a) Both the students fail in the exam
 - b) One of them only fails.
 - c) At least one of them fails
 - d) None of them fail.

17. Suppose you write the digits from 1 to 10 on separate pieces of paper fold up the pieces and put them in a hat. Then you ask a friend of yours to draw, while blindfolded, one piece from the hat. What is the probability your friend will draw
- a) An even number?
 - b) A number divisible by 3?
 - c) An odd number or a number divisible by 3?
 - d) A number both even and divisible by 3?
 - e) A number larger than 6?
 - f) A number larger than 6 or less than 3?
18. Suppose that a weather forecaster estimates the probability of rain tomorrow as 60%, the probability of lightning as 30%, and the probability of both rain and lightning as 20%. Determine the probability that tomorrow there will be
- a) no rain,
 - b) no lightning,
 - c) rain or lightning,
 - d) rain but no lightning,
 - e) lightning but no rain,
 - f) Neither rain nor lightning.
19. Haymanot a student at the vocational school plans to take two courses next summer, a department requirement course and a school requirement course (S). The probability she will succeed in department requirement course is 0.80 and in school requirement course is 0.60. Assume the two courses are independent. What is the probability Haymanot succeeds in
- a) Both courses?
 - b) In at least one of the two courses? (This means that Haymanot may succeed in either one or in both courses.)

CHAPTER 3

Analytic Geometry

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- Write equation of a line;
- Distinguish parallel and perpendicular lines in a plane;
- Recognize the coordinate systems provide convenient and efficient ways of representing figures;
- Solve geometrical problems analytically; and
- Connect algebraic and geometric representation to solve equations involving circles.

3.1 Division of a Line Segment

Activity 3.1

1. A line segment has endpoints A(3, 2) and B(13, 10).
 - a) Find the distance between A and B.
 - b) Find the coordinates of the midpoint of $\overline{AB}$.
 - c) Find the coordinates of the points that divides $\overline{AB}$ into four equal parts.
 - d) Find the coordinate of the point that divides $\overline{AB}$ in the ratio 1: 4.
2. Can you find the coordinates of points that divide a given line segment $\overline{P_1P_2}$ in a given ratio?

Consider that a line segment $\overline{P_1P_2}$ is given as shown in Fig.3.1

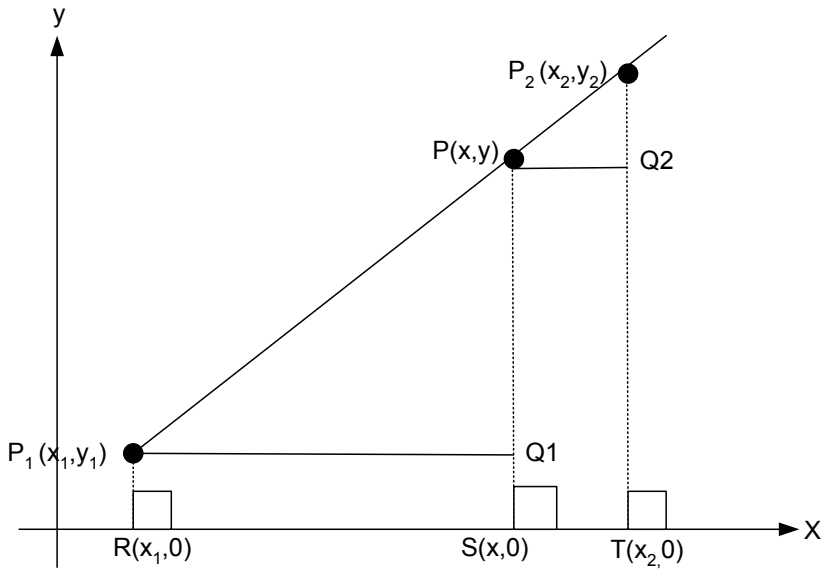


Fig. 3. 1

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ define the segment. Let other point in the line, call it $P(x, y)$, is said to divide the segment $\overline{P_1P_2}$ in the ratio r given by

$$r = \frac{P_1P}{PP_2}$$

r is called the **ratio of division**. We consider only when P lies between P_1 and P_2 . That is r divides $\overline{P_1P_2}$ internally.

The formula for the coordinates of P in terms of r and of the coordinates of P_1 and P_2 are found as follows. Through P_1 , P , and P_2 draw vertical lines which are perpendiculars to x -axis as shown in the Fig. 3.7. Then by AA similarity $\triangle PP_1Q_1 \sim \triangle PP_2Q_2$. Therefore,

$$\frac{P_1P}{PP_2} = \frac{P_1Q_1}{PQ_2} = \frac{PQ_1}{P_2Q_2}$$

But $P_1Q_1 = x - x_1$, $PQ_2 = x_2 - x$. Hence,

$$r = \frac{x - x_1}{x_2 - x}$$

Clearing of fractions and solving for x,

$$x = \frac{x_1 + rx_2}{1 + r}$$

Similarly,

$$y = \frac{y_1 + ry_2}{1 + r}$$

Note that if the ratio $r = \frac{r_1}{r_2}$, then the formula can be written as follows:

The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, in the ratio $r_1 : r_2$ are

$$\left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2} \right)$$

Example 3.1

Find the point which divides the segment from $P_1(-1, 7)$ to $P_2(4, -3)$ in the ratio $2 : 3$.

Solution:

Let the point $P(x, y)$ be the point that divides the segment $\overline{P_1 P_2}$ in the ratio $2 : 3$.

Using the formula,

$$x = \frac{r_1 x_2 + r_2 x_1}{r_1 + r_2} = \frac{2(-1) + 3(4)}{2 + 3} = 1$$

$$y = \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2} = \frac{2(-3) + 3(7)}{2 + 3} = 3$$

The required point is $P(1, 3)$

Example 3.2

Find the coordinates of the point which divides the line segment joining the points (4,-3) and (8,5) in the ratio 3:1.

Solution:

Let P(x, y) be the required point. Using the formula, we get

$$x = \frac{3(8) + 1(4)}{3 + 1} = 7, \quad y = \frac{3(5) + 1(-3)}{3 + 1} = 3$$

Therefore, (7, 3) is the required point.

Exercise 3.1

1. A line segment has endpoints (5,-3) and (14, 18). Find the coordinates of points that trisect the segment.
2. In what ratio does the point (-4, 6) divide the line segment joining the points A(-6,10) and B(3, -8)?
3. Find the coordinates of the points that trisect the segment joining the points A(2, -2) and B(-7, 4).
4. Find the ratio in which the y-axis divides the line segment joining the points (5, -6) and (-1, -4). Find also the point of intersection.
5. If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken in order, find the value of p.
6. Find the area of a rhombus if its vertices are (3, 0), (4, 5), (-1, 4) and (-2, -1) taken in order.
7. Does the formula for division of a line segment into a given ratio coincides with mid-point formula if $r = 1$?
8. A line segment has endpoints (-9, -2) and (6, 2). Find the coordinates of the point that divides it in the ratio 3:4 internally.

9. A is a point on the x-axis and B is a point on the y-axis. If the point P (6, -2) divides segment AB internally in the ratio $\frac{1}{2}$, find the coordinates of A and B.
10. One endpoint of a line segment is the origin and the other endpoint is (12, 10). Find the coordinates of the point that divide it internally in the ratio 4:7.

3.2 Straight Lines

3.2.1 Slope of a Line

Activity 3.2

1. The three lines $l_1 : y = x + 1$, $l_2 : y = 2x + 1$, $l_3 : y = -\frac{1}{2}x + 1$ are drawn on the same system of axes in Fig. 3.2. What do the lines have in common and how do they differ?

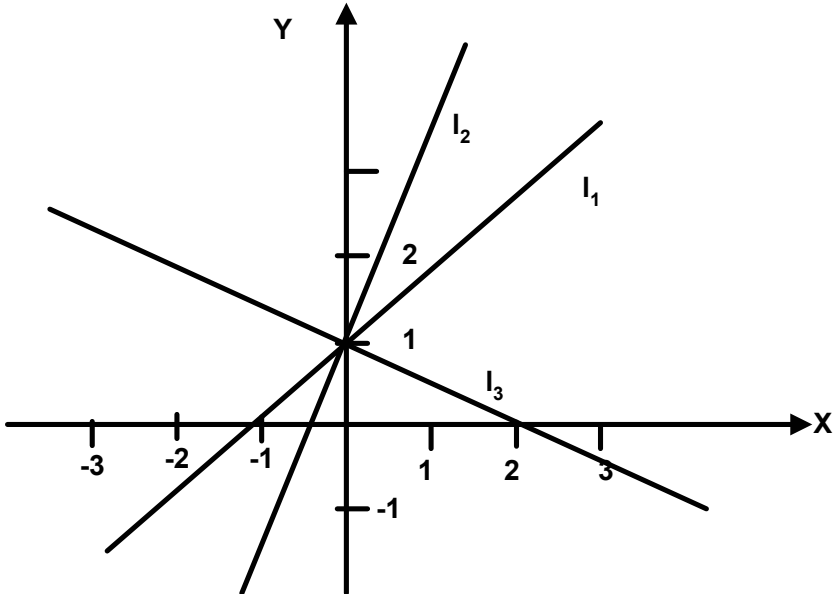


Fig. 3. 2

2. What do we mean by slope of 1 in 100?
3. Which one is steeper? A hill with a slope of 1 in 30 or a hill with a slope of 1 in 100?
4. Consider the line l , defined by $y = 2x + 1$ drawn in Fig. 3.3. Points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the line and M is the point (x_2, y_1) .

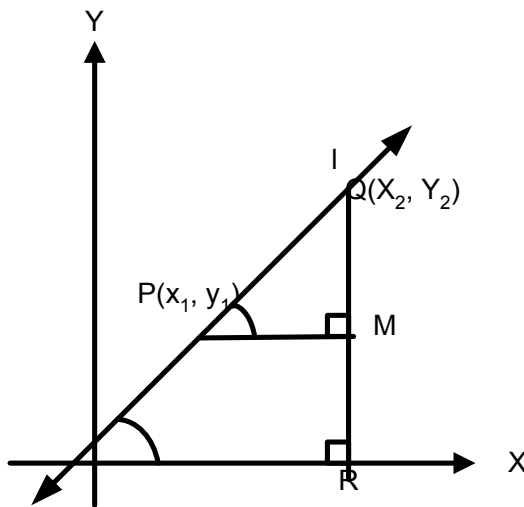


Fig. 3. 3

- a) Calculate the lengths of QM and PM if $x_1 = 1$ and $x_2 = 3$.
- b) Calculate $\frac{y_2 - y_1}{x_2 - x_1}$ using the values determined in (a).
- c) Compare your answer to (b) with the coefficient of x in $y = 2x + 1$.
- d) Find $\tan \theta$.
- e) What is the relationship between your answers for task 4(b) and task 4(d)?

The angle θ made by the line l with positive direction of x-axis and measured anti clockwise is called **the inclination of the line**.

We observe that:

1. $0^\circ \leq \theta \leq 180^\circ$.
2. Lines parallel to x-axis or coinciding with x-axis have inclination of 0° .
3. The inclination of a vertical (parallel to or coinciding with y-axis) is 90° .

If θ is the inclination of a line l , the **slope** of l is $\tan \theta$.

Example 3.3

Find the slope of the line that makes an angle of inclination of 60° with positive direction of the x-axis.

Solution:

Here the angle of inclination $\theta = 60^\circ$ and $\tan 60^\circ = \sqrt{3}$.

Therefore, the slope of the line is $\sqrt{3}$

Exercise 3.2

1. Find the inclination of a line with slope.

a) $\sqrt{3}$

c) 0

b) $\frac{1}{\sqrt{3}}$

2. Find the slope of a line whose inclination is:

a) 30°

b) 45°

The slope of a non-vertical line l is the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}$$

where $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two distinct points on l .

Note that if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two different points on the line then

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}$$

In other words, the answer is the same regardless of which point we use as the starting point, provided we subtract in the same order in the numerator and denominator.

Note that if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are distinct points of a

vertical line, then $x_1 = x_2$. In this case the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is not defined. Therefore a vertical line has no slope.

Remarks

1. A line with positive slope slants upwards from left to right
2. A line with negative slope slants downwards from left to right.
3. A horizontal line has a slope of zero.
4. The slope of a vertical line is undefined
5. For positive slopes, the bigger the slope, the steeper the line.
6. For negative slopes, the smaller the slope, the steeper the line.

Exercise 3.3

1. Find the slope of the lines passing through the given pair of points.

a) (2, 3), (9, 12)	d) (3, -2), (-1, 4)
b) (-7, 10), (8, -2)	e) ((a, b), (-a, -b))
c) (a, b), (b, a)	f) (0, 4), (-2, 0)
2. Determine whether the three points are collinear or not.

a) (8, 5), (3, 2), (-3, -1)
b) (-3, 3), (2, 1), (12, -3)
c) (-4, 6), (-4, -1), (-4, 20)

3.2.2 Parallel and Perpendicular lines

Activity 3.3

- Draw the graphs of $y = 2x$ and the graph of $y = 2x + 5$ on the same set of axes.
 - What are the slopes of the graphs you have drawn?
 - What did you observe about the two lines?
- In a coordinate plane, suppose that non vertical lines l_1 and l_2 have slopes m_1 and m_2 respectively. If the two lines l_1 and l_2 are parallel, what can you say about their slopes m_1 and m_2 ? (Use Fig. 3.4 to prove your assertion)

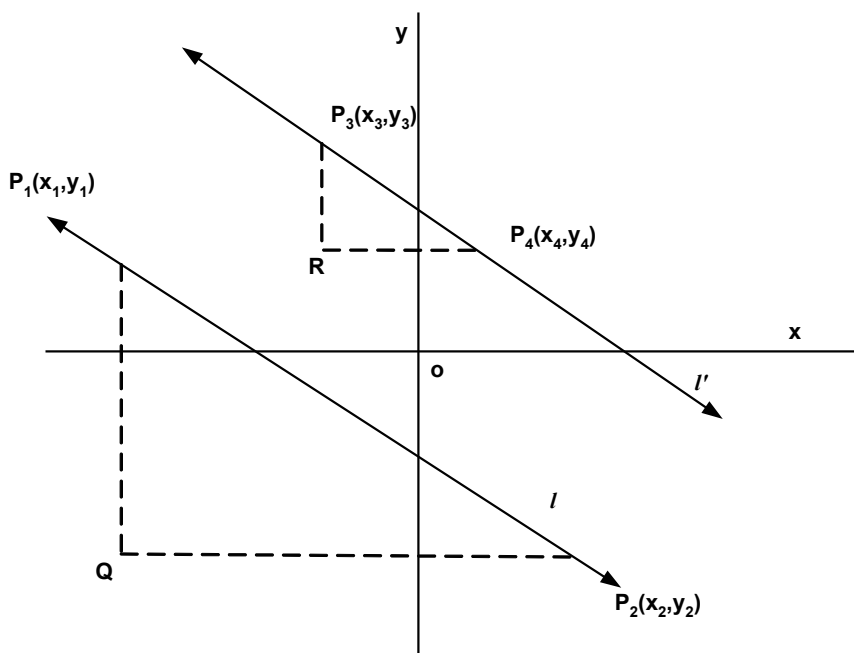


Fig. 3. 4

- Draw the graphs of $y = 2x$ and the graph of $y = -\frac{1}{2}x + 5$ on the same set of axes.
 - What are the slopes of the graphs you have drawn?
 - What did you observe about the two lines?

4. In a coordinate plane, suppose that non vertical lines l_1 and l_2 have slopes m_1 and m_2 respectively. If the two lines l_1 and l_2 are perpendicular, what can you say about their slopes m_1 and m_2 ? (Use Fig. 3.5 to prove your assertion)

Sketch of the proof

- a) Choose points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, R, and Q as shown in Fig.3.5. α , β , and γ are used to represent the degree measures of the angles indicated. What is the sum of the degree measures of α and β ? That is $\alpha + \beta$
- b) What is the sum of the degree measures of α and γ ? That is $\alpha + \gamma$
- c) What can you conclude about β and γ ?
- d) Are triangles P_2QP_3 and P_1RP_2 similar? Give reason
- e) Can we conclude that $\frac{x_3 - x_2}{y_3 - y_2} = \frac{y_1 - y_2}{x_2 - x_1} = -\frac{y_1 - y_2}{x_1 - x_2}$? Why?
- f) Find the slopes of l_1 and l_2 . That is express m_1 and m_2 in terms of x_1, y_1, x_2, y_2, x_3 , and y_3
- g) What can you conclude from (f)?

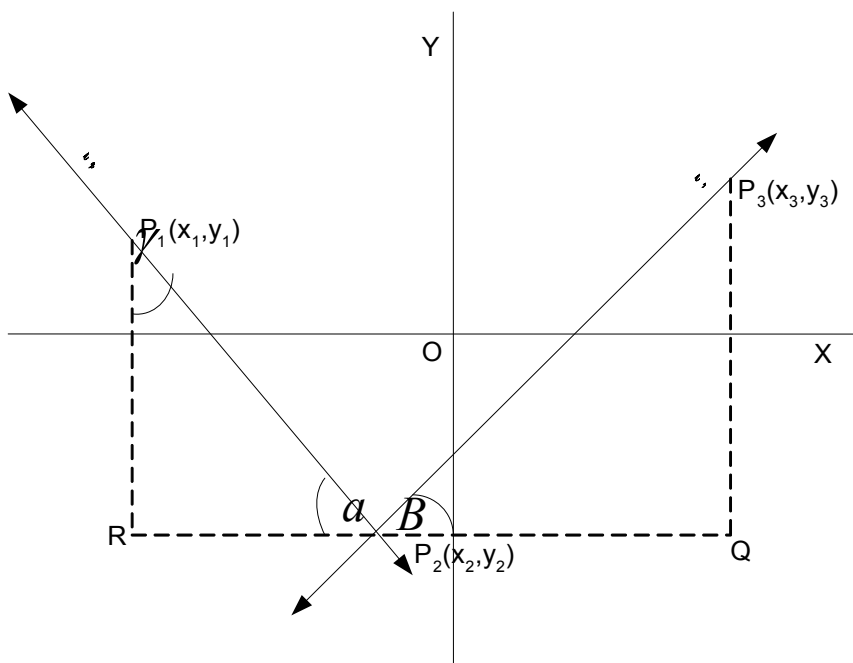


Fig. 3. 5

Two non vertical lines are parallel if they have the same slope. Conversely, if two distinct lines have the same slope they are parallel.

Example 3.4

Let $l_1 : y = \frac{1}{2}x + 1$ and $l_2 : y = \frac{1}{2}x + 3$, then the two lines are parallel since they have the same slope.

Two non-vertical lines which have slopes m_1 and m_2 are **perpendicular** if $m_1 m_2 = -1$. Conversely, if the two non-vertical lines are perpendicular, then $m_1 m_2 = -1$

Note that we say when two lines are perpendicular their slopes are negative reciprocals of each other, i.e. $m_2 = -\frac{1}{m_1}$

Note that horizontal lines have slope 0 and vertical lines have no slope, and we deal with them separately. Horizontal and vertical lines are perpendicular to each other.

Example 3.5

Suppose $l_1 : y = \frac{2}{3}x + 1$ and $l_2 : y = -\frac{3}{2}x + 5$. Are the lines perpendicular?

Solution:

The two lines are perpendicular, since their slopes are $\frac{2}{3}$ and $-\frac{3}{2}$ whose product is -1.

Example 3.6

Prove that the quadrilateral with vertices at P(0,-2), Q(4,2), R(0,6), S(-4,2), taken in order is a square.

Solution:

Recall that a square is a quadrilateral with all angle and all sides equal. Each measure of its angles is 90.

$$\begin{aligned}PQ &= \sqrt{(4-0)^2 + (2-(-2))^2} \\ &= \sqrt{32}\end{aligned}$$

$$\begin{aligned}QR &= \sqrt{(4-0)^2 + (2-6)^2} \\ &= \sqrt{32}\end{aligned}$$

$$\begin{aligned}RS &= \sqrt{(0-(-4))^2 + (6-2)^2} \\ &= \sqrt{32}\end{aligned}$$

$$\begin{aligned}PS &= \sqrt{(0-(-4))^2 + (-2-2)^2} \\ &= \sqrt{32}\end{aligned}$$

Observe all sides are equal. It suffices to show one of the angles is right angle. To do this we find the slopes of any two consecutive sides.

$$\text{The slope of } \overline{PQ} = \frac{2 - (-2)}{4} = 1$$

$$\text{The slope of } \overline{QR} = \frac{6 - 2}{-4} = -1$$

Therefore, $\overline{PQ}$ is perpendicular to $\overline{QR}$. That is the angle between these lines is 90.

Hence the quadrilateral is a square.

Example 3.7

Show that the quadrilateral with vertices A(-1,1), B(1, -1), C(2,1), and D(1,2) is a trapezium with perpendicular diagonals

Proof

The proof may be done by obtaining one and only one pair of opposite sides parallel and show that the diagonals are perpendicular.

The slopes of $\overline{AB}$ and $\overline{CD}$ are equal, while the slopes of the other opposite sides are not. We conclude that the quadrilateral is a trapezium.

We observe that the diagonals are vertical and horizontal lines which are perpendicular.

Exercise 3.4

1. What values of a will make the line through $(a, 5)$ and $(-2, 3)$ parallel to the line containing $(6, a)$ and $(2, 0)$? What values of a will make the lines perpendicular?
2. The line l_1 passes through the points $(-3, 8)$ and $(-1, 3)$. The line l_2 passes through $(-1, 8)$ and $(3, -2)$. The line l_3 passes through $(-1, -1)$ and $(4, 1)$.
 - a) Show that l_2 is parallel to l_1
 - b) Show that l_3 is perpendicular to l_2

3. Find p if the line through $(1, 2)$ and $(2, 5)$, is parallel to the line which passes through $(0, -2)$ and $(2, p)$.
4. Find q if the line through $(-1, -1)$ and $(2, 3)$, is perpendicular to the line through $(2, 4)$ and $(q, 1)$.
5. What type of triangle has vertices with the following coordinates?
 - a) $(3, 0)$, $(-4, 0)$, $(0, 5)$
 - b) $(-1, 0)$, $(1, 0)$, $(0, 1)$
6. Prove that the line segment determined by the midpoints of two sides of a triangle is parallel to the third side and its length is equal to half the length of the third side.
7. Prove that the diagonals of a parallelogram bisect each other.
8. Prove that $(4, 7)$, $(0, 4)$, $(3, 0)$, and $(7, 3)$ taken in order are the vertices of a square.
9. Find the value of x for which the points $(x, -1)$, $(2, 1)$, and $(4, 5)$ are collinear.

3.2.3 Equation of a line

Activity 3.4

- a) We have seen that the slope $\frac{y_2 - y_1}{x_2 - x_1}$ of a non vertical line does not depend on the particular choice of points $P(x_1, y_1)$ and $Q(x_2, y_2)$ used to compute it. Therefore if $R(x, y)$ is any point of the line other than the two given points, then can we say that
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} ?$$
- b) What do we get, if we multiply both sides of the above equation by $(x - x_1)(x_2 - x_1)$?

- c) Can we determine the equation of a line if we are given the slope and y-intercept of the line?
- d) How about if we are given the slope and any point on the line.
- e) What is the equation of any horizontal line?
- f) What is the equation of any vertical line?
- g) What is the general equation of a line?

The solution set of the equation

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

is the line l determined by the two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$. This equation is called a **two-point equation** of the line.

Example 3.8

Find an equation for the line l passing through the two points $(-4, 2)$ and $(-3, 1)$.

Solution:

From the equation for slopes, we know that any point (x, y) on l other than $(-4, 2)$ must satisfy the equation

$$\frac{y - 2}{x - (-4)} = \frac{1 - (2)}{-3 - (-4)} = 1$$

i.e., $\frac{y - 2}{x + 4} = 1$, simplifying we get

$$y - 2 = x + 4 \text{ which is equivalent to } y = x + 6$$

Every non-vertical line has an equation of the form

$$y = mx + b,$$

where m is the slope of the line and $(0, b)$ is the point where the line intersects the y-axis. This equation is called a **slope-intercept equation** of the line.

The equation of a line that passes through the point $P(x_1, y_1)$ and has slope m is

$$y - y_1 = m(x - x_1)$$

This equation is called **point – slope form** of the equation of a line.

Example 3.9

What is the equation of the line that passes through $(0, 7)$ and having slope -3 ?

Solution:

Since -3 is the slope and the point $(0, 7)$ is the y -intercept, this equation must be $y = -3x + 7$.

Example 3.10

Find the equation of the line through $(2, 7)$ having slope 2 .

Solution:

It has an equation of the form $\frac{y - y_1}{x - x_1} = m$. Letting $(x_1, y_1) = (2, 7)$

we obtain the equation $\frac{y - 7}{x - 2} = 2$. The slope intercept form is thus $y = 2x + 3$.

Example 3.11

Show that $3y + 5x - 11 = 0$ is the equation of a straight line.

Solution:

$3y + 5x - 11 = 0$ is equivalent to $y = -\frac{5}{3}x + \frac{11}{3}$ which is an

equation of a straight line with slope $-\frac{5}{3}$ and y -intercept $\frac{11}{3}$

Consider the slope – intercept form of the equation of a line, namely $y = mx + b$.

Recall that the slope of a horizontal line is $m = 0$. Hence if l is a horizontal line (i.e. l is a line parallel to the x -axis) that passes through the point (p, q) , then its equation is $y = q$.

A vertical line does not have a slope and thus it does not have an equation of the form $y = mx + b$. However, if l is a vertical line (i.e. l is a line parallel to the y -axis) that passes through the point (p, q) , then its equation is $x = p$.

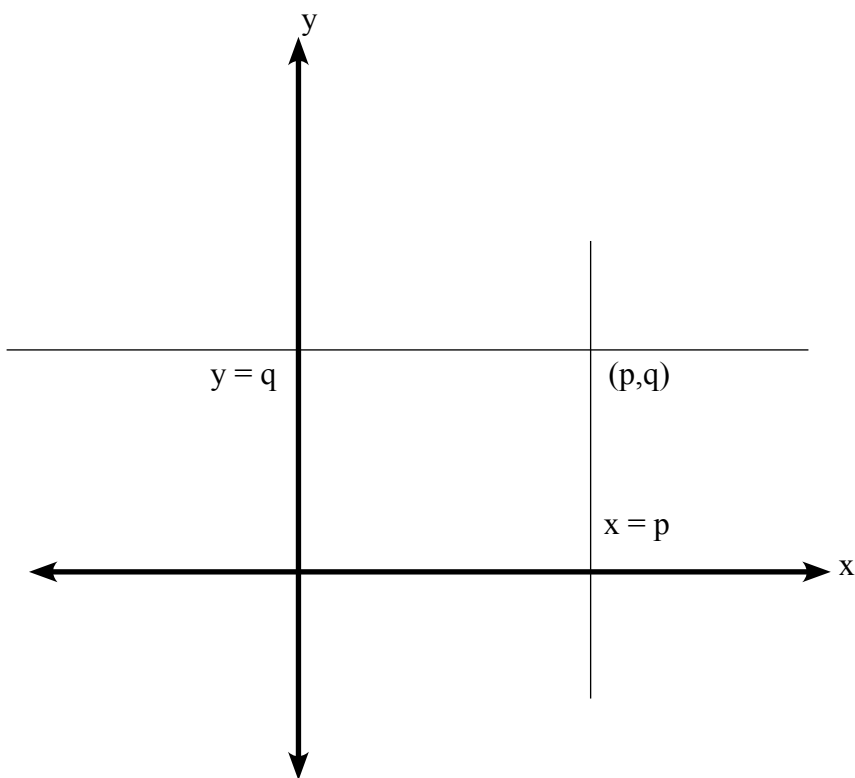


Fig. 3. 6

The equation of the horizontal line through the point (p, q) is

$$y = q$$

The equation of the vertical line through the point (p, q) is

$$x = p$$

Activity 3.5

- Find the equation of the line whose x-intercept is the point (5, 0) and y intercept is the point (0,-5).
- Can we devise an equation that is applicable for a line whose x and y intercepts are given? In other words, if (a, 0) and (0, b) are x and y intercepts respectively, what would be the equation?

Suppose the line l intersects the x-axis at the point (a, 0) and the y-axis at the point (0, b) as it shown in Fig. 3.6.

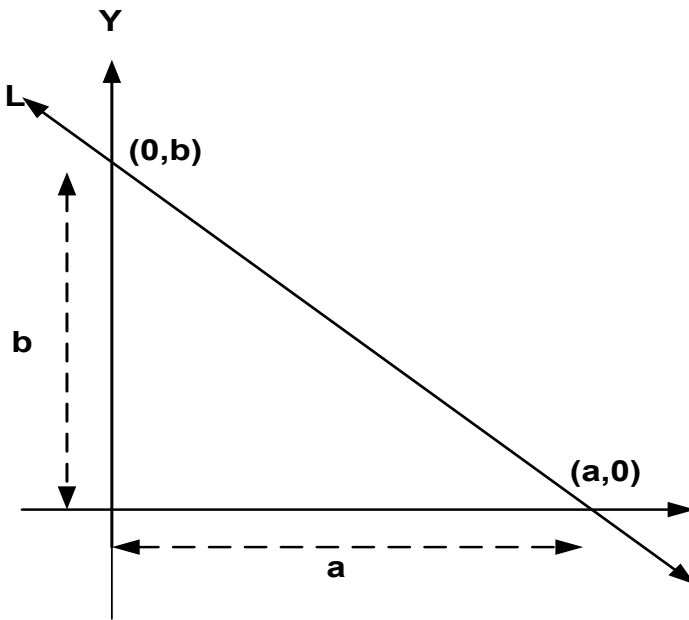


Fig. 3. 7

Let $P(x, y)$ be any point on the line, using the two- point form

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

If we let $(x_1, y_1) = (a, 0)$ and $(x_2, y_2) = (0, b)$, then

$$(y - 0)(0 - a) = (x - a)(b - 0)$$

$$-ay = bx - ab$$

$$bx + ay = ab$$

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{dividing each term by } ab$$

The equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the x and y intercepts respectively, is called the **double –intercept form equation of a line**.

Example 3.12

Find the equation of the line whose x -intercept is the point $(9, 0)$ and y -intercept is the point $(0, -5)$.

Solution:

Since $a = 9$ and $b = -3$, the double intercept form of the line is

$$\frac{x}{9} + \frac{y}{-5} = 1. \text{ Simplifying we get: } -5x + 9y + 45 = 0$$

It would be useful if we could find some general equation which would describe any line, including vertical lines.

Every line has an equation of the form

$$ax + by + c = 0$$

Where a , b , and c are constants and a and b are not both zero.

Exercise 3.5

1. Find equations for the lines determined by the following pairs of points. Determine their slopes and their intercepts.

a) $(2, 4), (-1, -5)$

d) $(1, 5), (-10, 5)$

b) $(3, -3), (-3, 1)$

e) $(-1, -1), (4, 4)$

c) $\left(\frac{1}{2}, \frac{3}{4}\right), \left(2, \frac{1}{3}\right)$

f) $(3, -9), (3, 7)$

a) $3, (0,-1)$

b) $-5, (0, \frac{1}{2})$

c) $\frac{1}{2}, (0,4)$

d) 15, (0,-7)

3. Determine slope –intercept equations for the lines having the following slopes and passing through the given points.

a) $-5, (5, -5)$

c) $3, (-7, 2)$

b) $\frac{1}{2}, (4, -3)$

d) $-6, (4, 2)$

4. Show the following are equations of lines. In each case indicate the slope and y-intercept of the line.

a) $3y + 2x - 5 = 0$

b) $-4x + 5y + 12 = 0$

c) $4x - 6y = 12$

d) $5y - 6x = 30$

5. Find the equation of the line which satisfies the given conditions.

a) Passing through $(0, 0)$ with slope m .

b) Passing through $(4, 3)$ with slope -4 .

c) Intersecting x-axis at a distance of 3 units to the left of origin with slope -2.

d) The vertices of $\triangle PQR$ are P(2, 1), Q(-2, 3) and R(4, 5). Find equation of the median through the vertex R.

e) Find the equation of the line passing through $(-3, 5)$ and perpendicular to the line through the points $(2, 5)$ and $(-3, 6)$.

f) Show that $(-2, -2)$, $(8, 2)$, and $(3, 0)$ are collinear.

3.2.3 Further use of Analytic methods

The advantage in introducing a coordinate system into a plane is that we can use algebraic methods to prove geometrical concepts. We have given simple proofs by using slopes and distance formula. We will work out more such analytic proofs of geometric theorems.

Activity 3.6

1.
 - a) What is an altitude of a triangle?
 - b) How many altitudes have a triangle?
 - c) Draw a triangle and its altitudes

2. The following activity proves that the altitudes of a triangle meet at a point.
 - a) Choose the coordinates of the vertices of the triangle to be as indicated in the Fig. 3.8. We are free to choose any triangle. But for convenience $A(0,0)$, $B(b,0)$, and $C(c, d)$. Can d be 0? Why?
 - b) We can also assume c is not zero. What happens if c is zero?
 - c) The altitude l_1 has equation $x = c$. Why? Note it is also perpendicular to $\overline{AB}$.
 - d) Calculate the slope of $\overline{AC}$. Then what would be the slope of l_3 ?
 - e) Derive the equation of l_3 and l_2 .
 - f) Find the intersection point of l_1 and l_3 .
 - g) Does this point satisfy the equation for the line l_2 ?
 - h) What is your conclusion from this activity?

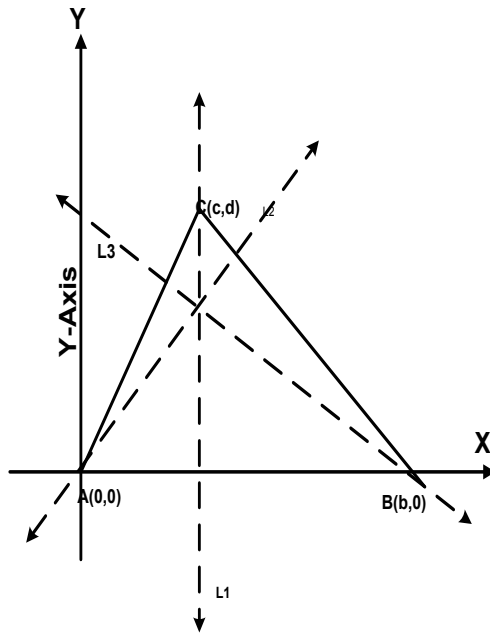


Fig. 3. 8

Two or more lines are **concurrent** if there is a single point which lies on all of them

Activity 3.7

1.
 - a) What is a median of a triangle?
 - b) How many medians have a triangle?
 - c) What is a centroid of a triangle?

2.
 - a) What is an angle bisector of a triangle?
 - b) How many angle bisectors have a triangle?
 - c) The angle bisectors are concurrent at a point which is equidistant from _____ of the triangle.

3.
 - a) What is a perpendicular bisector of a triangle?
 - b) How many perpendicular bisectors have a triangle?
 - c) What is the name given to the point where the perpendicular bisectors meet?

Example 3.13

Given a triangle with vertices $A(-3, 4)$, $B(3, -3)$, and $C(5, 8)$, find the equation of the median of the triangle through vertex A .

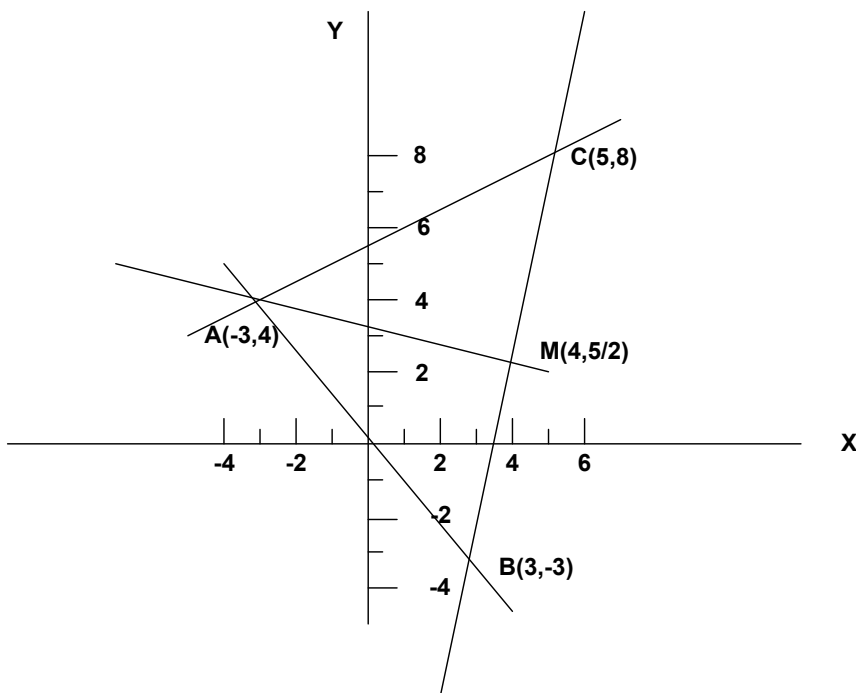


Fig. 3. 9

Solution

The median through A must pass through the midpoint M of B and C .

By the mid-point formulas we find $M\left(4, \frac{5}{2}\right)$ The equation of $\overline{AM}$,

$$\text{the required median is } \frac{y-4}{x-(-3)} = \frac{\frac{5}{2}-4}{4-(-3)}$$

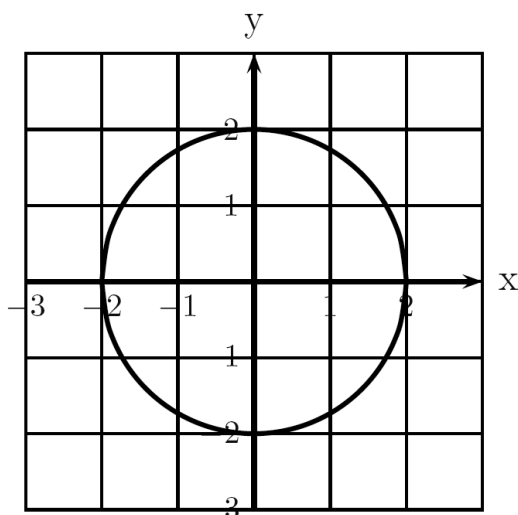
Reducing to general form, $3x + 14y - 47 = 0$

Exercise 3.6

1. Prove that the perpendicular bisectors of the sides of a triangle are concurrent.
2. Prove that the midpoints of the sides of any quadrilateral form the vertices of a parallelogram.
3. Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices.
4. The points A(2, 2), B(0, 6), and C(8, 10) are the vertices of a triangle. Determine whether the medians are concurrent by finding the point on each median which is $\frac{2}{3}$ of the way from the vertex to the opposite midpoint.
5. Find the equation of the perpendicular bisector of the line segment with endpoint P (-2, 3) and Q (8, -1).

3.3 Circle**Activity 3.8**

1. Fig 3.9 shows a circle of radius 2 cm on a coordinate plane with centre at the origin.

**Fig. 3.10**

- a) Take any four points on the circle and find their coordinates.
- b) Find the distance between each of the five points on the circle and the centre of the circle.
- c) What do you observe?
- d) Determine whether the points $(2, 0)$, $(-1, 1)$ and $(1, -3)$ are on the circle or not. Give reasons.

2. How do you define a circle?

A **circle** is the set of all points in a plane that are equidistant from a fixed point. The fixed point is called the **centre** of the circle and the fixed distance is called the **radius** of the circle.

Activity 3.9

1. Fig. 3.10 shows a circle with radius 3 cm and centre at the origin.

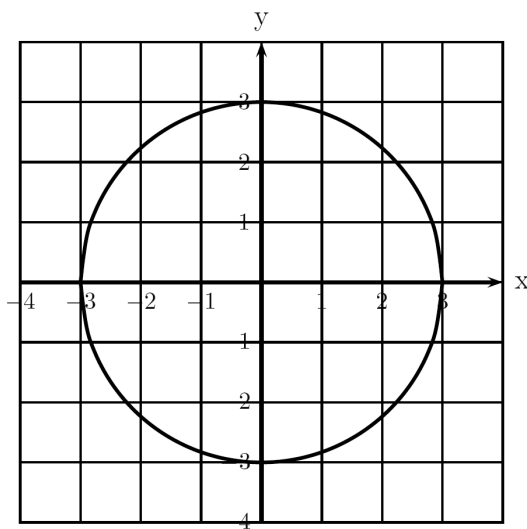
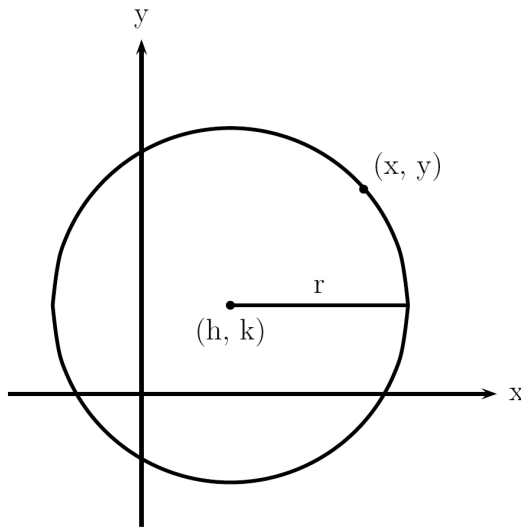


Fig. 3.11

- a) What is the distance between any point on the circle and the centre of the circle?
- b) Suppose (x, y) is the coordinates of a point on the circle as shown in Fig. 3.10. What is the formula for the distance between (x, y) and the centre of the circle?

- c) Find an equation that relates the formula in Task 1(b) and the radius of the circle.
 - d) What would be the resulting equation if you square both sides of the equation in Task 1(c) to remove a radical?
2. Fig. 3.11 shows a circle with radius r units and centre at a point with coordinates (h, k) .

**Fig. 3.12**

- a) Suppose (x, y) is the coordinates of a point on the circle as shown in Fig. 3.11. What is the formula for the distance between (x, y) and the centre of the circle?
- b) Find an equation that relates the formula in Task 2(a) and the radius of the circle.
- c) What would be the resulting equation if you square both sides of the equation in Task 2(b) to remove a radical?

The **standard form** of the equation of a circle with radius r and centre at (h, k) is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

Example 3.13

Find the standard form of the equation of a circle with radius 5 and centre at (2, -1) and draw its graph.

Solution:

We are given $r = 5$, $h = 2$ and $k = -1$. Thus, we have

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Standard form equation}$$

$$(x - 2)^2 + (y - (-1))^2 = 5^2 \quad \text{Substituting given numbers}$$

$$(x - 2)^2 + (y + 1)^2 = 25 \quad \text{Simplifying}$$

That is, the standard form of the equation of the circle is given by

$$(x - 2)^2 + (y + 1)^2 = 25$$

To draw the circle, plot the centre (2, -1) on a coordinate plane. Find a point on the circle by assigning value to one of the variables and using the equation to determine the corresponding value of the other variable. In our case, let $x = 2$ in the equation. Then,

$$(x - 2)^2 + (y + 1)^2 = 25 \quad \text{Equation of the circle}$$

$$(2 - 2)^2 + (y + 1)^2 = 25 \quad \text{Substituting } x = 2$$

$$(y + 1)^2 = 25 \quad \text{Simplifying}$$

$$y = -6 \text{ or } y = 4 \quad \text{Solving for } y$$

That is, the circle passes through the points (2, -6) and (2, 4). Plot one of the points, say (2, 4), on the coordinate plane. See Fig. 3.12. Then, use your compasses to draw the circle with centre at (2, -1) and passing through (2, 4) as shown in Fig. 3.13.

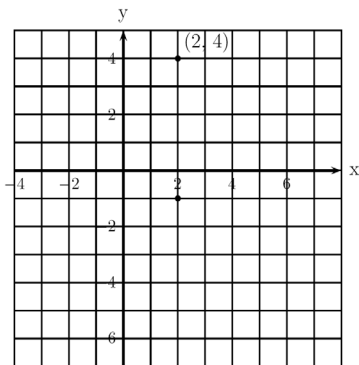


Fig. 3.13

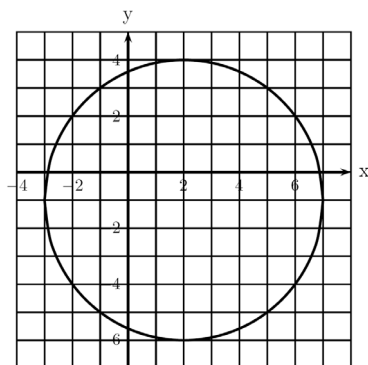


Fig. 3.14

Example 3.14

Find the standard form of the equation of a circle with radius 4 and centre at the origin.

Solution:

Note that the coordinates of the origin are (0, 0). Hence, we are given $r = 4$, $h = 0$ and $k = 0$. Thus, we have

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Standard form equation}$$

$$(x - 0)^2 + (y - 0)^2 = 4^2 \quad \text{Substituting given numbers}$$

$$x^2 + y^2 = 16 \quad \text{Simplifying}$$

Activity 3.10

Consider a circle with radius 9 and centre at (-3, 1).

- Find the standard form of the equation of the circle.
- Rewrite the equation in Task 1(a) in the form of

$$x^2 + y^2 + Dx + Ey + F = 0$$

where D, E and F are constant real numbers.

- What are the values of D, E and F?

The **general form** of the equation of a circle is given by

$$x^2 + y^2 + Dx + Ey + F = 0$$

where D, E and F are constant real numbers.

Example 3.15

Find the general form of the equation of a circle with radius $\sqrt{5}$ and centre at (-2, 4).

Solution:

We are given $r = \sqrt{5}$, $h = -2$ and $k = 4$. Then,

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Standard form}$$

$$(x + 2)^2 + (y - 4)^2 = 5 \quad \text{Substituting given values}$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 = 5 \quad \text{Eliminating the parentheses}$$

$$x^2 + y^2 + 4x - 8y + 15 = 0 \quad \text{Simplifying and rewriting}$$

Thus, the general form equation of the circle is given by

$$x^2 + y^2 + 4x - 8y + 15 = 0$$

Activity 3.11

Consider a circle whose general form equation is given by

$$x^2 + y^2 + 16x + 8y + 44 = 0$$

We want to rewrite the equation in standard form, and then identify the radius and centre of the circle.

- Group the x and y terms, respectively, on the left side of the equation and move the constant term to the right side. What is the resulting equation?
- If you complete the squares on both the x and y terms of the equation in Task (a), what will be the resulting equation?
- If you factor the perfect squares on the left and simplify the right side of the equation in Task (b), what will be the resulting equation?
- Comparing the equation in Task (c) with the standard form

$$(x - h)^2 + (y - k)^2 = r^2$$

what are the values of h , k and r ?

- What are the radius and centre of the circle?

Example 3.16

Find the radius and centre of the circle whose general equation is given by

$$x^2 + y^2 - 2x + 8y - 8 = 0$$

Solution:

We transform the equation into standard form by completing squares in the x and y terms as follows.

$$x^2 + y^2 - 2x + 8y - 8 = 0 \quad \text{Given equation}$$

$$(x^2 - 2x) + (y^2 + 8y) = 8 \quad \text{Grouping terms}$$

$$(x^2 - 2x + 1) + (y^2 + 8y + 16) = 8 + 1 + 16 \quad \text{Completing the squares}$$

$$(x - 1)^2 + (y + 4)^2 = 25 \quad \text{Factorizing}$$

Now, comparing with the standard form equation

$$(x - h)^2 + (y - k)^2 = r^2$$

We see that $h = 1$, $k = -4$ and $r = 5$. Thus, the circle has radius 5 and the centre at $(1, -4)$.

Exercise 3.7

- Find the standard form of the equation of each of the following circles and draw the circle.
 - Radius 4 and centre at $(-3, -4)$.
 - Radius 2 and centre at $(2, 1)$.
 - Radius $\sqrt{5}$ and centre at $(1, -1)$.
 - Radius 3 and centre at $(0, 0)$.
- A circle with radius 1 and centre at the origin is called the **unit circle**. Find the equation of the unit circle and draw the circle.
- If the point $(10, -4)$ lies on a circle with centre at $(7, -8)$, find the standard form of the equation of the circle.
- Find the radius and centre of the circle with the given equations.
 - $(x - 1)^2 + (y - 3)^2 = 25$
 - $(x + 3)^2 + (y - 7)^2 = 81$
 - $(x + 1)^2 + (y + 3)^2 = 11$
 - $x^2 + y^2 = 12$
 - $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{9}{25}$
- Find the general form of the equation of each of the following circles.
 - $(x + 2)^2 + (y - 3)^2 = 5$
 - $(x - 3)^2 + (y + 3)^2 = 20$
 - Radius 3 and centre at $(4, -10)$.

- d) Radius 9 and centre at (5, 7).
 - e) Radius $\sqrt{2}$ and centre at the origin.
5. If the points (1, 7) and (-5, -1) are the endpoints of a diameter of a circle, find the general form of the equation of the circle.
6. Transform the following equations into standard form by completing the squares and state the radius and centre of each circle.
- a) $x^2 + y^2 + 6x + 8y - 75 = 0$
 - b) $x^2 + y^2 + 16x - 14y + 88 = 0$
 - c) $x^2 + y^2 - 4x - 16y + 32 = 0$
 - d) $x^2 + y^2 - 10x + 6y + 22 = 0$
 - e) $x^2 + y^2 - x + y + \frac{1}{4} = 0$
7. A mobile phone tower has a reception radius of 100 km. If you live 97 km north and 33 km east of the tower, can you use your mobile phone while at home?
8. What does the equation $x^2 + y^2 - 2x + 6y + 10 = 0$ represent?
9. What does the equation $x^2 + y^2 - 4x + 10y + 49 = 0$ represent?

Review Exercise 3

- 1. What is the equation of the line which passes through the point (3, 4) and whose slope is 2.
- 2. Find the equation of the line that contains the points (-2, 3) and (1, 5).
- 3. Find the equation of a line perpendicular to $x - 2y + 3 = 0$ and passing through the point (1, -2).

4. For what values of k is the line $2x - ky + 5 = 0$ parallel to the line $8x - 14y + 3 = 0$. For what value of k are the lines perpendicular?
5. Find the equation of the line whose x - intercept is 4 and y – intercept -5.
6. Prove that the line segment joining the midpoint of two sides of a triangle is parallel to the third side and half of it.
7. Find the standard and general forms of the equation of each of the following circles.
 - a) Radius 8 and centre at $(7, -3)$.
 - b) Radius $\sqrt{7}$ and centre at $(-1, 2)$.
 - c) Passing through the point $(2, 5)$ and centre at $(4, 9)$.
 - d) A diameter whose endpoints are $(-2, 6)$ and $(4, 2)$.
8. State the radius and centre of each of the following circles.
 - a) $(x + 3)^2 + (y + 2)^2 = 36$
 - b) $x^2 + y^2 - 4x + 2y - 45 = 0$
 - c) $x^2 + y^2 - 10x + 4y = 0$
 - d) $x^2 + y^2 - 10x + 21 = 0$
9. A mobile phone tower has a reception radius of 100 km. If you live 45 km south and 87 km west of the tower, can you use your mobile phone while at home?

CHAPTER 4

Matrices and Determinant

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- Add, subtract and multiply matrices
- Find the inverse of 2×2 matrix
- Calculate the determinant of a matrix
- Solve system of linear equations using determinants.

1.1 Matrices

4.1.1 Introduction

Activity 4.1

1. Suppose we wish to express the possession of notebooks and pencils by two friends Ali and Abraham which is given as follows.

Ali has 3 notebooks and 2 pencils, while Abraham has 10 notebooks and 13 pencils. Arrange this in tabular form.

2. Is it descriptive if we arrange as follows without making tables?

$$\begin{pmatrix} 3 & 2 \\ 10 & 13 \end{pmatrix}$$

3. What do you think the entries in the first and second row represent?

4. How about the entries in the first and second column represent?

A matrix is an ordered rectangular array of numbers. The numbers are called the elements or the entries of the matrix.

We denote a matrix by any capital letter.

Example 4.1

$$1. \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$2. \quad C = \begin{pmatrix} 2 & -3 \\ 5 & 13 \end{pmatrix}$$

$$3. \quad M = \begin{pmatrix} 3 & -5 & 11 \\ 6 & 7 & 4 \\ -12 & 10 & 1 \end{pmatrix}$$

$$4. \quad B = \begin{pmatrix} 2 & 5 \\ 3 & 9 \\ 4 & -7 \end{pmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, **rows** of the matrix and the vertical lines of elements are said to constitute, **columns** of the matrix.

A matrix having m rows and n columns is called a matrix of order $m \times n$.

So referring to the above example A is a 2×2 matrix, M is a 3×3 , and B is a 3×2 matrix.

In general, an $m \times n$ matrix has the following rectangular array

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

or $[a_{ij}]_{m \times n} \quad 1 \leq i \leq m, 1 \leq j \leq n, i, j \in \mathbb{N}$

Note that a_{ij} is an element lying in the i^{th} row and the j^{th} column.

4.1.2 Types of Matrices

A matrix is said to be a **column** matrix if it has only one column

Example 4.2 $A = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix}$

A matrix is said to be a **row** matrix if it has only one row

Example 4.3

$$B = (12 \quad 3 \quad -6)$$

A matrix in which the number of rows is equal to the number of columns is called a **square matrix**.

Example 4.4

$$C = \begin{pmatrix} 2 & -3 \\ 5 & 13 \end{pmatrix} \text{ is a square matrix} \quad D = \begin{pmatrix} 1 & 5 & 4 \\ 3 & 6 & 8 \\ 7 & 2 & 9 \end{pmatrix} \text{ is a square matrix.}$$

A square matrix is said to be a **diagonal** matrix if all its non diagonal elements are zero

Example 4.5

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

A square matrix in which elements in the diagonal are all 1 and rest are all zeros is called an **identity matrix**.

Example 4.6

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ are identities of order 2 and of order 3 respectively.}$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if

- They are of the same order and
- Each element of A is equal to the corresponding element of B. That is $a_{ij} = b_{ij}$ for all i and j

Example 4.7

$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ are equal matrices

$A = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ are not equal matrices.

Exercise 4.1

1. Write the order of the following matrices

a) $\begin{pmatrix} -2 & 2 \\ 5 & 9 \end{pmatrix}$ b) $\begin{bmatrix} 1 & 17 \\ -17 & 7 \\ 3 & 2 \end{bmatrix}$ c) $\begin{pmatrix} 1 & 5 & 4 \\ 3 & 6 & 8 \\ 7 & 2 & 9 \end{pmatrix}$

2. Which of the following pair of matrices are equal?

a) $A = \begin{pmatrix} -2 & 2 \\ 5 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 5 \\ 2 & 9 \end{pmatrix}$

b) $A = \begin{pmatrix} -2 & 2 \\ 5 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 2 \\ 5 & 9 \end{pmatrix}$

4.1.3 Operation on Matrices**Activity 4.2**

1. If $A = \begin{pmatrix} -2 & 5 \\ 2 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 9 \\ 1 & 4 \end{pmatrix}$, Find

a) $A + B$ b) $A - B$ c) $3A$ c) $A \times B$

2. Can we find the sum of A and B where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$? Why?

If A and B are both $n \times m$ matrices then the sum of A and B denoted by $A + B$ is the $n \times m$ whose entries are $a_{ij} + b_{ij}$ for each $i=1, 2, \dots, n$ and $j = 1, 2, \dots, m$

Example 4.8

$$\text{If } A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{bmatrix}, \text{ then}$$

$$A + B = \begin{bmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{bmatrix} \quad A - B = \begin{bmatrix} -2 & -3 & 15 \\ 3 & 0 & -6 \end{bmatrix}$$

In particular for 2×2 matrices, we have

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \text{then the sum}$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Example 4.9

$$\text{Given } A = \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 11 & 23 \\ 0 & -6 \end{pmatrix} \quad A + B = \begin{pmatrix} 12 & 27 \\ -2 & -5 \end{pmatrix}$$

Observe the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is a } 2 \times 2 \text{ matrix then } \lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

Example 4.10

$$\text{If } A = \begin{pmatrix} 6 & -7 \\ 1 & 2 \end{pmatrix} \text{ then } 5A = \begin{pmatrix} 30 & -35 \\ 5 & 10 \end{pmatrix} \text{ and } \frac{1}{2}A = \begin{pmatrix} 3 & \frac{-7}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Example 4.11

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } 7A = \begin{pmatrix} 7a & 7b \\ 7c & 7d \end{pmatrix}$$

The negative of a matrix is denoted by $-A$.

Thus, $-A = -1(A)$

Example 4.12

$$\text{If } A = \begin{pmatrix} 1 & 6 \\ -11 & 2 \end{pmatrix} \text{ then } -A = (-1)A = -\begin{pmatrix} 1 & 6 \\ -11 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -6 \\ 11 & -2 \end{pmatrix}$$

The product of two matrices A and B is defined if the numbers of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ is a $m \times n$ matrix and $B = [b_{jk}]$ is $n \times p$ matrix.

Then the product of the two matrices is the matrix C of order

$m \times p$. To get the $(i, k)^{\text{th}}$ element C_{ik} of the matrix C, we take the i^{th} element of A and the k^{th} element of B, multiply them element wise and take the sum of all these products.

Example 4.13

$$\text{If } A = \begin{pmatrix} 1 & -3 & 6 \\ 2 & 5 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 3 \\ 2 & 5 \\ 6 & -9 \end{pmatrix} \text{ then the product } AB \text{ is defined and given by}$$

$$AB = \begin{pmatrix} 1 & -3 & 6 \\ 2 & 5 & 4 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 5 \\ 6 & -9 \end{pmatrix}. \text{ This is } 2 \times 2 \text{ matrix.}$$

Four computations are involved. These are

$$(1 \ -3 \ 6) \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} = (1 \times (-1) + (-3) \times 2 + 6 \times 6) = 29$$

$$(1 \ -3 \ 6) \begin{pmatrix} 3 \\ 5 \\ -9 \end{pmatrix} = (1 \times 3 + (-3) \times 5 + (-9) \times 6) = -63$$

$$(2 \ 5 \ 4) \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} = (2 \times (-1) + 5 \times 2 + 4 \times 6) = 32$$

$$(2 \ 5 \ 4) \begin{pmatrix} 3 \\ 5 \\ -9 \end{pmatrix} = (2 \times 3 + 5 \times 5 + 4 \times (-9)) = -15$$

Thus $AB = \begin{pmatrix} 29 & -63 \\ 32 & -15 \end{pmatrix}$

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then the product

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Example 4.14

If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ then $AB = \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix}$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then $AB =$

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Example 4.15

$$\text{If } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \text{ then } A^2 = A \times A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{pmatrix}$$

Exercise 4.2

1. Find $A + B$, AB , and BA for the following matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix}$$

2. If $M = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, show that $M^2 - 3M + 8I = 0$

3. Compute the indicated operations

$$\text{a) } \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{b) } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (2 \ 3 \ 4) \quad \text{c) } \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix}$$

4.1.5 Invertible Matrices**Activity 4.3**

Consider the following matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

a) Find AB b) Find BA c) Is $AB = BA$

If A is a square matrix of order m , and if there is another matrix of the same order m , such that $AB = BA = I$, then B is called the inverse matrix of A and is denoted by A^{-1} . In that case A is said to be invertible.

Example 4.16

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

We want to find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

which is equivalent to the following system

$$2a + c = 1 \quad 2b + d = 0 \quad a + c = 0 \quad b + d = 1$$

Solving these, we get $a = 1$, $b = -1$, $c = -1$, and $d = 2$

Therefore, $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Exercise 4.3

Compute the inverse of the following matrices.

a) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$ d) $\begin{pmatrix} 7 & -10 \\ -2 & 3 \end{pmatrix}$

4.1.5 The Determinant of a Matrix**Activity 4.4**

Consider the following system of equation

$$a_1x + b_1y = c_1 \quad , \quad a_2x + b_2y = c_2$$

- When does the given equation have a unique solution?
- Write the matrix of the coefficients of x and y .
- What is the relation between the matrix of the coefficients and the condition that tells the uniqueness?

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix of order 2×2 , then the determinant of A is denoted by $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$ is defined as **$\det(A) = ad - bc$**

Remark

- Determinant is a number associated to a square matrix.
- The symbol for determinant is a pair of vertical lines similar to the absolute value symbol, while a matrix is enclosed in brackets.

Example 4.17

1. Evaluate the determinant of the following matrices

$$A = \begin{pmatrix} 3 & 6 \\ 5 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 5 \\ 9 & -2 \end{pmatrix}$$

Solution

$$|A| = \det(A) = 3 \times 12 - 5 \times 6 = 36 - 30 = 6$$

$$|B| = \det(B) = (-3 \times -2) - (5 \times 9) = 6 - 45 = -39$$

Determinant of 3×3 matrix can be expressed in terms of the determinant of 2×2 matrix.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the determinant of A denoted by $|A|$ is given by $a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Example 4.18

Find the determinant of $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & -2 \end{pmatrix}$

Solution

$$|A| = 2 \begin{vmatrix} -1 & 1 \\ 6 & -2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} + 0 \begin{vmatrix} 3 & -1 \\ 4 & 6 \end{vmatrix} = 2(-8) - 1(-10) + 0 = -6$$

Exercise 4.4

1. Evaluate the determinant of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 5 \\ 9 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$$

2. Find the value of x if

$$\text{a) } \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}, \quad \text{b) } \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & -5 \end{vmatrix}$$

3. Find the determinant of $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{pmatrix}$

4.2 Solving Systems of Equations Using Determinants

Activity 4.5

- What is the name of the equation given by $a_1x + b_1y = c_1$
 $a_2x + b_2y = c_2$
- What is an independent system? Give an example.
- What is inconsistent system? Give an example.
- When do you say the system given in (1) has a unique solution?

5. Solve by Elimination method. $\begin{cases} 3x + 4y = 12 \\ x - y = -1 \end{cases}$

Activity 4.6

Consider the following 2×2 system of equation. $\begin{cases} x - 2y = 3 \\ 2x + y = 4 \end{cases}$

a) Write the coefficient matrix and denote it by A.

b) Find $\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$ This is called determinant of the coefficient matrix. That is $|A|$.

c) Solve for x as follows $x = \frac{\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix}}{|A|}$

d) Solve for y, $y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}}{|A|}$

e) Write the solution set.

The steps given in Activity 4.5 are methods of solving system of equation by **Cramer's rule (Matrix Method)**.

Example 4.19

Solve the following system of equation by using the Matrix Method.

$$ax + by = e$$

$$cx + dy = f$$

Solution

The first step is to write the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Evaluate $|A|$, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Let us call this D for simplicity.

$$\text{Find } D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} = ed - bf, \quad \text{Find } D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ec$$

Solve for x and y .

$$x = \frac{D_x}{|A|} = \frac{D_x}{D} = \frac{ed - bf}{ad - bc}, \text{ provided } ad - bc \neq 0$$

$$y = \frac{D_y}{|A|} = \frac{D_y}{D} = \frac{af - ec}{ad - bc}, \text{ provided } ad - bc \neq 0$$

Note that D is the denominator made up of the original coefficients of x and y . D_x is obtained by replacing the first column of D by the constants e and f . D_y is found by replacing the second column of D by the constants e and f .

Example 4.20

$$\text{Solve } \begin{cases} 3x + 2y = 8 \\ 4x - 3y = 5 \end{cases}$$

Solution

The coefficient matrix of the system is $A = \begin{pmatrix} 3 & 2 \\ 4 & -3 \end{pmatrix}$.

$$\text{Its determinant is } A = |A| = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = -17.$$

$$D_x = \begin{vmatrix} 8 & 2 \\ 5 & -3 \end{vmatrix} = -34, \quad D_y = \begin{vmatrix} 3 & 8 \\ 4 & 5 \end{vmatrix} = -17$$

Therefore $x = \frac{-34}{-17} = 2$, $y = \frac{-17}{-17} = 1$

Therefore, the solution set is $\{(2, 1)\}$.

Exercise 4.5

Solve the following using the Matrix Method

1. $\begin{cases} 9x - 2y = 3 \\ 3x - y = 6 \end{cases}$
2. $\begin{cases} x + y = 3 \\ y - x = 1 \end{cases}$
3. $\begin{cases} 0.8x - 0.6y = 1 \\ 0.6x + 0.8y = 2 \end{cases}$
4. $\begin{cases} 3x - 2y + 3z = 8 \\ 2x + y - z = 1 \\ 4x - 3y + 2z = 4 \end{cases}$

Review exercises

1. Find the sum, difference and product of the following matrices

$$A = \begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -11 & 9 \\ 4 & -6 \end{pmatrix}$$

2. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 3 \\ 4 & 3 \end{pmatrix}$ then show that $AB = AC$

3. Use Cramer's rule (Matrix Method) to solve the system

$$\begin{array}{ll} \text{a)} & \begin{cases} 3x - 2y = 4 \\ 2x + y = -3 \end{cases} & \text{b)} & \begin{cases} 2x - 3(y + 1) = -3 \\ 2y = 3x - 5 \end{cases} \end{array}$$

$$\begin{array}{l} \text{c)} \quad \begin{cases} 2x - y = 5 \\ 3x + 2y = -3 \end{cases} \end{array}$$

4. The sum of the digits of a two-digit number is 10. If the digits are reversed, the new number is one less than twice the original number. Find the original number.