

The State of Eritrea
Ministry of Education

Mathematics

for

Grade 10

Textbook

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Introduction

Dear learners, geometry is one of the branches of mathematics which is used extensively in our daily life. For example, we use the basic concept of geometry such as area and perimeter in making wooden furniture, and volume and surface area for a variety of purposes, from measuring liquids in cooking and baking to building different types of reservoir. Some of the properties of points, lines, surfaces and solids are used a stronomy.

This textbook comprises five chapters. The foundation of functions, geometric structures, geometric patterns, dimensionality and the geometry of location, and congruence and the geometry of size.

By the end of Grade 10, you should be able to apply basic concepts, attributes and relationships within functions and use linear expressions of functions to solve situational problems, identify geometric structures, the attributes of geometric figures and relationships within axiomatic systems. You should also be able able use mathematical proofs to solve geometric problems using geometric diagrams and to; identify, describe and analyse geometric patterns that emerge from two-and three-dimensional spaces. You should be able to understand and solve the situational problems relating to perimeter, area and volume in geometric figures and apply the concept of congruency.

This textbook is based on the new national curriculum, that is, a learner centered and an interactive pedagogical approach. The basic concepts are introduced with sufficient activities and illustrations before formal definitions or results are stated. We hope that this manner of introduction will help you to grasp the precise definitions of concepts and the basic ideas of mathematical statements. The activities will enable you to use the relevant learning materials in this textbook. You are therefore encouraged to read them on your own in addition to discussing them in groups in the classroom.

The main terms are highlighted in bold font. Important definitions, results and procedures that require focus are placed in identifiable boxes.

In order to help you think critically and relate the knowledge, skills and attitudes learned to daily life, many situational problems are included in the form of activities, examples and exercises.

Moreover, there is a need for continuous assessment in order to help you with your over-all progress in each chapter. This is the purpose of the review questions that are included at the end of each chapter. You should work through the review questions regularly. Remember that, while solving problems, it is always advisable to begin with a simple problem and to proceed step by step through the challenging ones.

We recommend that parents and interested individuals also play their role in facilitating the process of teaching-learning so that the intended targets is can be mets.

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Chapter 1

Foundations for Functions

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- define relations and functions;
- identify functions from relations given as set of ordered pairs, equations and graphs;
- state the domain and range of functions;
- apply the basic concepts, attributes and relationships within functions;
- find the formula for the inverse of a function
- apply functional relationships to problem situations.

1.1 Relations, Functions and their Applications

1.1.1 Relations

Activity 1.1

There are many phrases that describe relationships between pairs of persons in a family. Consider the two families in Fig. 1.1, where the names and ages of the members are given.



Fig. 1.1

- a) Find all ordered pairs (x, y) of the families associated with the relation “ x is a husband of y .”
- b) Find all ordered pairs (x, y) of the families associated with the relation “ x is a son of y .”
- c) Find all ordered pairs (x, y) of the families associated with the relation “ x is a father of y .”
- d) Find any other three relations that exist between members of the families and list all the ordered pairs associated with each relation.

Activity 1.2

In mathematics, there are different symbols that represent relationships between ordered pairs of objects. Consider the symbol “ $<$ ”.

- a) What does the symbol “ $<$ ” stand for in the expression “ $x < y$ ”.
- b) What does the expression “ $x < y$ ” tell us about the ordered pair (x, y) ?
- c) What is the phrase that describes the relationship between x and y in the ordered pair (x, y) in b)?
- d) Consider the ordered pairs $(1, 1)$, $(3, 5)$, $(1, 4)$, $(3, 6)$ and $(6, 3)$. Which of these ordered pairs belong to the set of ordered pairs that satisfy the relationship in b)?
- e) Can you mention any other five different ordered pairs of real numbers which belong to the set of ordered pairs that satisfy the relationship in b)?

In Activity 1.1 and Activity 1.2, we observe that, for any given relation, ordered pairs are associated with it. In mathematics, these ordered pairs are described in a set form and given a special name.

A **relation** is a set of ordered pairs and is named using capital letters such as R , S , T , etc. The set of all first elements of the ordered pairs is called the **domain** of the relation. The set of all second elements of the ordered pairs is called the **range** of the relation.

Example 1.1

Let R be a relation among the numbers 1, 2, 3, 5 and 6 such that “a given number is assigned to a number larger than it by one.”

Then,

- a) Using a set of ordered pairs, R can be described as

$$R = \{(x, y): y \text{ is one larger than } x\}$$

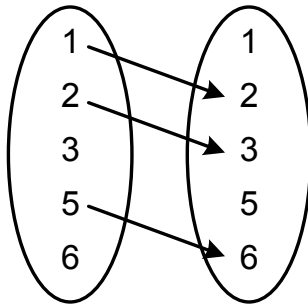
or

$$R = \{(1, 2), (2, 3), (5, 6)\}$$

- b) Using a table of values, R can be described as

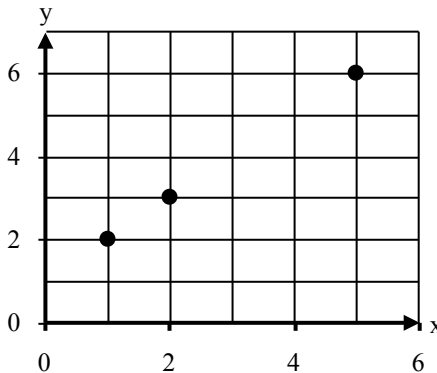
x	1	2	5
y	2	3	6

- c) Using an arrow diagram, R can be described as



The arrows indicate the pairings between the elements of the domain and the range.

- d) Using a graph, R can be described as



The three points on the coordinate plane show the graph of the relation R.

- e) Using an algebraic rule, R can be described as

$$R = \{(x, y): y = x + 1\}$$

An ordered pair of the numbers 1, 2, 3, 5 and 6 that satisfies the equation $y = x + 1$ belongs to the relation R.

Besides, note that

Domain of $R = \{1, 2, 5\}$, and Range of $R = \{2, 3, 6\}$.

Exercise 1.1

1. Consider the following list of members of a family.

Member	Age (in years)	Relationship
Hagos	45	Father
Dehab	40	Mother
Yohana	15	Daughter
Selam	13	Daughter
Haben	10	Son

Describe each of the following relations defined on the family using a set of ordered pairs that lists all of its elements.

- $R = \{(x, y): x \text{ is a child of } y\}$
 - $S = \{(x, y): x \text{ is } y \text{ years old}\}$
 - $T = \{(x, y): x \text{ is a mother of } y\}$
2. Let $R = \{(x, y): y = 2x - 1\}$ with domain $\{-2, -1, 0, 1, 2\}$.
- Describe R using words.
 - Describe R using a table of values.
 - Describe R using a set of ordered pairs that lists all of its elements.
 - Describe R using an arrow diagram.
 - Draw the graph of R and determine whether the graph is discrete or continuous.
 - Find the range of R.
3. Let $R = \{(x, y): y = x^2 - 1\}$ with domain the set of all real numbers.
- Identify any five ordered pairs which belong to the relation R.

- b) Identify any five ordered pairs which do not belong to the relation R.
 - c) Draw the graph of R and determine whether the graph is discrete or continuous.
 - d) Find the range of R.
4. Find the domain and range of each of the following relations.
- a) $R = \{(-1, 2), (2, 51), (1, 3), (8, 22), (9, 51)\}$
 - b) $T = \{(-5, 6), (21, -51), (11, 93), (81, 202), (19, 51)\}$
 - c) $S = \{(1, 2), (4, 7), (5, 8)\}$

1.1.2 Functions

Activity 1.3

The relation R described by the arrow diagram in Fig. 1.2 represents “x is a Zoba in Eritrea where town y is located.”

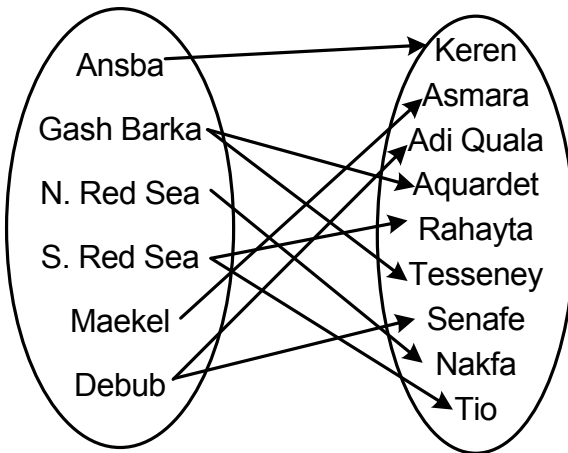


Fig. 1.2

- a) Describe the relation R using a set of ordered pairs.
- b) What is the domain of R?
- c) What is the range of R?
- d) Are there more than one arrow coming from the same element of the domain? Which ones?
- e) Are there more than one ordered pair in a) with the same first element? Which ones?

- f) Is every element of the domain related to exactly one element of the range?

Activity 1.4

Consider the relation S given by

$$S = \{(-1, 0), (1, 2), (0, 3), (4, 2), (5, 0)\}$$

- Describe the relation S using an arrow diagram.
- What is the domain of S ?
- What is the range of S ?
- Are there more than one ordered pair with the same first element? Which ones?
- Are there more than one arrow in a) coming from the same element of the domain? Which ones?
- Is every element of the domain related to exactly one element of the range?

In Activity 1.3, we have observed that there are elements of the domain that are related to two different elements of the range. However, in Activity 1.4, we have observed that every element of the domain is related with exactly one element of the range. This later relation is of special interest in mathematics and is called a **function**.

A **function** is a relation in which every element of the domain is related to exactly one element of the range.

Example 1.3

Which of the following relations are functions?

- $R = \{(-1, 2), (-4, 51), (1, 2), (8, -51)\}$
- $S = \{(13, 14), (16, 7), (13, 5), (18, 13)\}$

Solution:

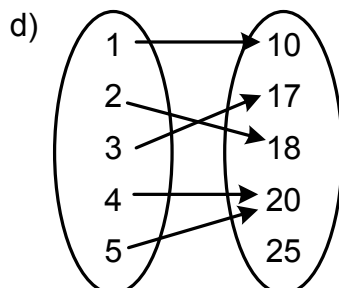
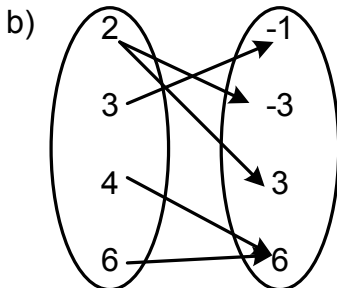
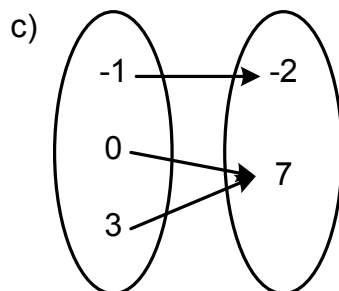
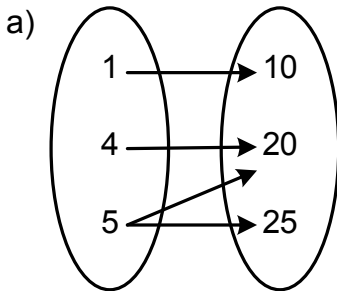
- Since there are no two different ordered pairs in R that have the same first element, every element of the domain is related to exactly one element of the range. Hence, the relation R is a function.

- b) Since the ordered pairs $(13, 14)$ and $(13, 5)$ in S have the same first element, there is an element of the domain which is related to more than one element of the range. Hence, the relation S is not a function.

In a relation, if there are two different ordered pairs with the same first element, then the relation is not a function. In an arrow diagram, if there are two arrows coming from the same element of the domain, then the relation is not a function.

Exercise 1.2

1. Find the domain and range of each of the following relations. Which of the relations are functions?
 - a) $R = \{(3, 90), (4, 54), (6, 71), (8, 90)\}$
 - b) $S = \{(3, 4), (4, 5), (6, 7), (8, 9)\}$
 - c) $T = \{(3, 4), (4, 5), (6, 7), (3, 9)\}$
 - d) $U = \{(-3, 4), (4, -5), (0, 0), (8, 9)\}$
 - e) $V = \{(8, 11), (34, 5), (6, 17), (8, 19)\}$
 - f) $W = \{(0, 0), (2, 2), (2, -2), (6, 8), (6, -8)\}$
2. Which of the following relations are functions?



3. Each of the following describes a rule that relates the elements of two sets. Which of these rules are functions?
 - a) People assigned to their birthdays.
 - b) Pencils assigned to their lengths.
 - c) People assigned to their telephone numbers.
 - d) Fathers assigned to their children.
 - e) Children assigned to their blood mothers.

4. For each of the following relations to be a function, what are the numbers that could not be assigned to x ?
 - a) $R = \{(8, 1), (3, 4), (6, 7), (x, 22)\}$
 - b) $S = \{(12, 13), (-11, 22), (33, 101), (x, 22)\}$
 - c) $T = \{(12, 14), (13, 5), (-2, 7), (x, 13)\}$

1.1.3 Graphs of Functions

Activity 1.5

Consider the relation R described by the graph in Fig. 1.3.

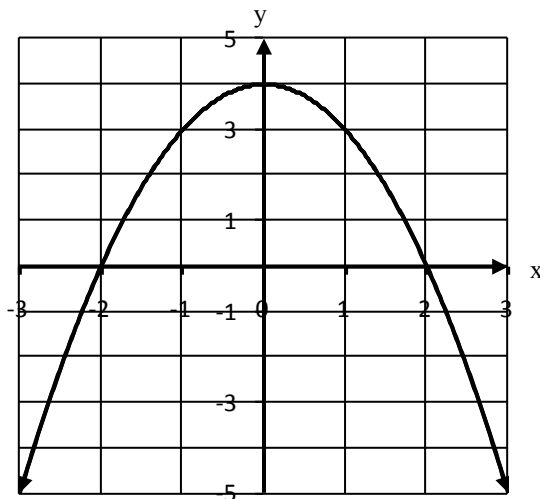


Fig. 1.3

- a) Find any five ordered pairs that belong to R .
- b) Find the domain and range of R .
- c) Can you draw a vertical line that crosses the graph of the relation R at more than one point?

- d) Can you find two ordered pairs in the graph of R that have the same first element?
- e) Is the relation R a function?

Activity 1.6

Consider the relation S described by the graph in Fig. 1.4.

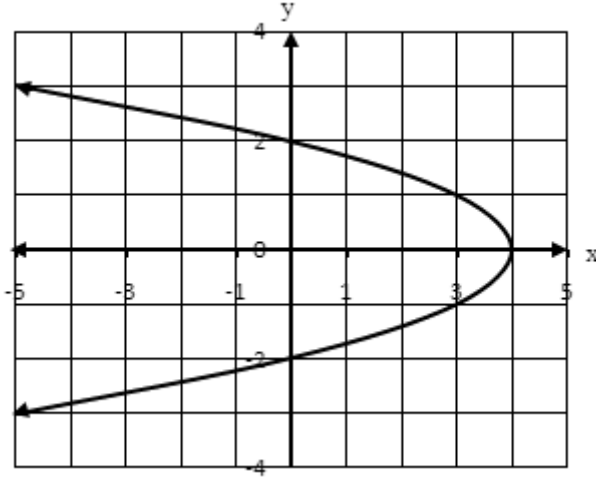


Fig. 1.4

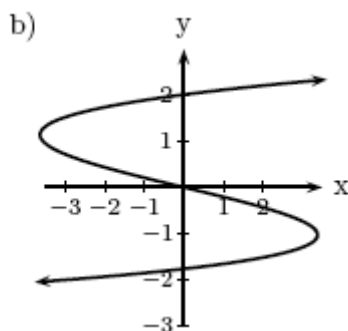
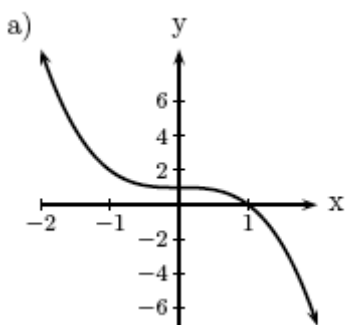
- Find any five ordered pairs that belong to S .
- Find the domain and range of S .
- Can you draw a vertical line that crosses the graph of the relation S at more than one point?
- If you can, what are the ordered pairs of the points where the line crosses the graph of S ?
- Do these ordered pairs belong to the relation S ?
- Do these ordered pairs have the same first element?
- Is the relation S a function? Why?
- From this activity and Activity 1.5, when does a graph of a relation represent a function?

If there exists a vertical line that crosses the graph of a relation at more than one point, then the relation is not a function. This is called the **vertical line test**.

Graphically, the domain of a relation consists of the x -values which lie on the graph of the relation. The range of the relation consists of the y -values which lie on the graph of the relation.

Example 1.4

In each of the following, find the domain and range of the relation described by the graph and determine whether it is a function or not.

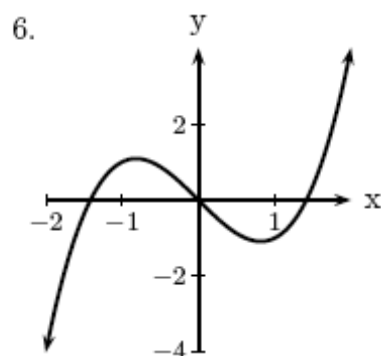
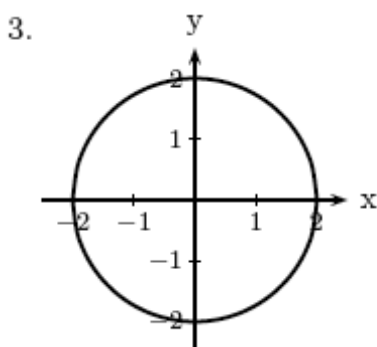
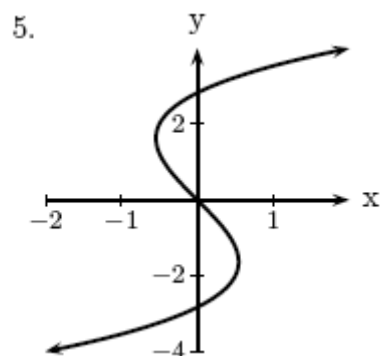
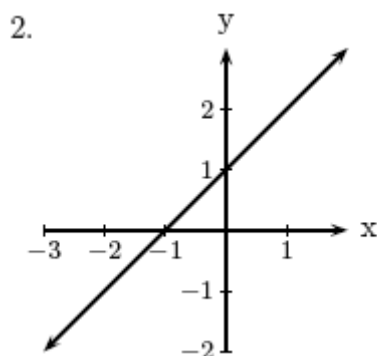
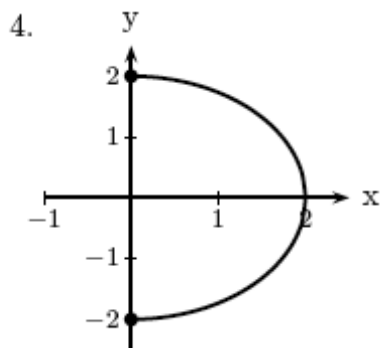
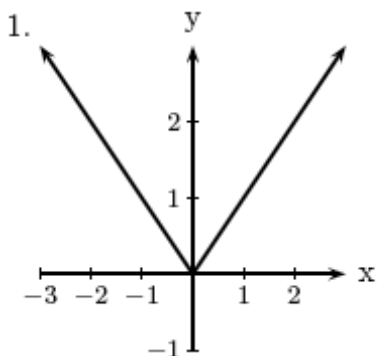


Solution:

- a) Note:** The graph is extending horizontally to the left and to the right without bounds. Hence, the domain is the set of all real numbers. Similarly, the graph is extending vertically up and down without bounds. Hence, the range is the set of all real numbers. Besides, note that there is no vertical line that crosses the graph at more than one point. Hence, the relation is a function.
- b) Note:** The graph is extending horizontally far to the left and to the right. Hence, the domain is the set of all real numbers. Similarly, the graph is extending vertically far up and down. Hence, the range is the set of all real numbers. There is a vertical line (for instance the y -axis) that crosses the graph at more than one point. Hence, the relation is not a function.

Exercise 1.3

Find the domain and range of the relation described by each of the following graphs, and determine whether the graph represents a function or not.



1.1.4 Function Notation

Activity 1.7

Let $R = \{(x, y): y = x + 1\}$ with domain $\{1, 2, 3, 5, 6\}$.

- Describe R using a set of ordered pairs that lists all of its elements.
- Find the range of R .
- Is R a function? Why?

Activity 1.8

Let $S = \{(x, y): y^2 = x\}$ with domain $\{0, 1, 4, 9, 16\}$.

- Describe S using a set of ordered pairs that lists all of its elements.
- Find the range of S .
- Is S a function? Why?

Example 1.4

Let $R = \{(x, y): y + x^2 = 1\}$ with domain $\{-2, -1, 0, 1, 2\}$.

Determine whether R is a function or not.

Solution:

First, Kemal rewrites the equation as $y = 1 - x^2$. Then, by assigning values for x from the domain, he determined the corresponding values of y . Then, he described R using a set of ordered pairs as follows:

$$R = \{(-2, -3), (-1, 0), (0, 1), (1, 0), (2, -3)\}$$

Since for every value of the independent variable x , there is only one value of the dependent variable y , he concluded that the relation is a function.

When an equation in x and y represents a function, we say **y is a function of x** . This means, if we know the value of x , then we can determine the value of y . Functions are usually named using small letters such as f , g , h , etc.

Example 1.5

Determine whether the relation $R = \{(x, y): y^2 - x^2 = 1\}$ with domain the set of all real numbers represents y as a function of x .

Solution:

Hadas rewrites the equation as $y^2 = 1 + x^2$. Then, by assigning values for x from the domain, she tried to determine the corresponding values of y . She noted that for each value of x there are two different values of y . For instance, if $x = 0$, then y could be either -1 or 1 . Hence, she concluded that y is not a function of x .

If f is a function with independent variable x and dependent variable y , it is common to replace y by $f(x)$. The expression $f(x)$ is read as “ f of x .” This latter way of describing functions is called the **function notation**.

For example, if $f = \{(x, y): y = 4x + 3\}$, then using the function notation we simply write

$$f(x) = 4x + 3$$

The $f(x)$ notation can be thought of as another way of representing the y -values of the function f . That is, $f(1)$ means the number which the function f pairs with 1 . To find $f(1)$, find the number which $4x + 3$ represents when x is replaced by 1 . Therefore,

$$\begin{aligned}f(x) &= 4x + 3 \\f(1) &= 4(1) + 3 = 7\end{aligned}$$

That is, 7 is the y -value assigned to the x -value 1 by the function f . Hence, the ordered pair $(1, 7)$ belongs to the function f .

Example 1.6

Let f be a function defined by $f = \{(x, y): 2x^2 + 2y - 4 = 0\}$. Describe f using the function notation.

Solution:

First, the equation needs to be rewritten so that the independent variable y is isolated on the left side of the equation. This can be done as follows.

$2x^2 + 2y - 4 = 0$	Given equation
$2y - 4 = -2x^2$	Subtracting $2x^2$ from both sides
$2y = -2x^2 + 4$	Adding 4 to both sides
$y = -x^2 + 2$	Dividing both sides by 2

Then, letting $y = f(x)$, the function f can be described using the function notation as

$$f(x) = -x^2 + 2.$$

Example 1.7

Let $g(x) = \sqrt{x^2 - 4}$. Find the values of the function at $x = -2$, 3 and $\sqrt{5}$.

Solution:

Replacing the given values of x in the formula one by one, the values of the dependent variable can be obtained as follows:

$g(x) = \sqrt{x^2 - 4}$	Given function
$g(-2) = \sqrt{(-2)^2 - 4}$	Replacing x by -2
$\quad = 0$	Simplifying
$g(3) = \sqrt{3^2 - 4}$	Replacing x by 3
$\quad = \sqrt{5}$	Simplifying
$g(\sqrt{5}) = \sqrt{(\sqrt{5})^2 - 4}$	Replacing x by $\sqrt{5}$
$\quad = 1$	Simplifying

That is, the ordered pairs $(-2, 0)$, $(3, \sqrt{5})$ and $(\sqrt{5}, 1)$ belong to the function g .

Example 1.8

Find the domain of each of the following functions.

a) $f(x) = 2x + 1$	c) $h(x) = \frac{x}{x+1}$
b) $g(x) = \sqrt{x-4}$	

Solution:

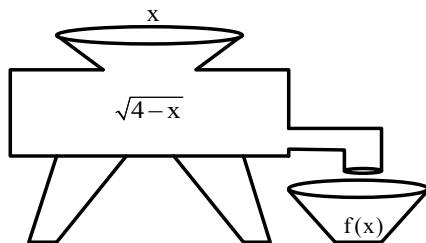
- a) Note that for any real number x , the expression $2x + 1$ is defined and is a real number. Hence, the domain of $f(x)=2x+1$ is the set of all real numbers.
- b) Recall that the square root of a number is a real number provided the number is nonnegative. Hence, $\sqrt{x-4}$ is a real number provided the expression $x-4$ is nonnegative. That is, $x-4 \geq 0$ which is equivalent to $x \geq 4$. Therefore, the domain of the function $g(x) = \sqrt{x-4}$ is $\{x: x \geq 4\}$.
- c) Note that the ratio of two numbers is defined provided the denominator is different from zero. Hence, $\frac{x}{x+1}$ is defined provided $x+1 \neq 0$. That is, $x \neq -1$. Therefore, the domain of
- d) the function $h(x) = \frac{x}{x+1}$ is $\{x: x \neq -1\}$.

If a function is defined by a formula and if its domain is not stated, then it is understood that the domain consists of all real numbers for which the function is defined in the set of real numbers.

Another way to think about a function is as a machine. Functions are machines which process an **input (a domain) value** and produce an **output (a range) value**.

Activity 1.9

Consider the function machine f in Fig. 1.5. Now, the machine operates as follows: It takes in an input value x , subtract it from 4, and produce the square root of the difference as an output value.

**Fig. 1.5**

- a) What would be the output value if the input value is 0? -5? 3? 1? -2?
- b) Are there any values which could not be put into the machine because they would cause the function to be undefined?
- c) What is the domain of the function?
- d) What are the possible values which can be produced by the function machine?
- e) What is the range of the function?

To determine the domain of a function, think about the values which might cause the function machine to be undefined and eliminate those from the domain. To determine the range, think about the values which can be produced by the function machine.

Exercise 1.4

1. Determine whether each of the following equations represents y as a function of x . If it does, describe the function using the function notation.
 - a) $y^2 = x + 1$
 - b) $x^2 + y = 4$
 - c) $y - 2x + 3 = 0$
 - d) $y^2 = x^2 - 1$
 - e) $y = \sqrt{x}$
2. Determine the value of each of the following functions for the given values of x .
 - a) $f(x) = x + 1$ for $x = 2, 3, -5$ and 0
 - b) $g(x) = 4 - x^2$ for $x = -3, -\frac{1}{2}, 0$ and $\sqrt{2}$
 - c) $h(x) = x^3 + 1$ for $x = -5, -1, 1, 3$ and 6
3. A tea shop sells a cup of tea for 2 Nakfa. It costs the tea shop 1.60 Nakfa to make a cup of tea. Besides, it pays a monthly rent of 70 Nakfa. Hence, the profit p per month from selling c cups of tea is given by the equation $p = 0.4c - 70$.
 - a) Is p a function of c ?
 - b) Use a function notation that uses the letter f and the independent variable c to describe the profit function of the tea shop.

- c) What would the profit or loss be if the tea shop sells 100 cups of tea? 200 cups of tea? 300 cups of tea? 500 cups of tea?
- d) Can c take the value -50 ?
- e) What is the domain of the profit function?
4. The area A of a circle with radius r is given by $A = \pi r^2$.
- a) Is A a function of r ?
- b) Use a function notation that uses the letter g and the independent variable r to describe the area function of the circle.
- c) What would the area of the circle be if the radius is 1 cm? 2 cm? 4 cm? 5 cm?
- d) Can r take the value -2 ?
- e) What is the domain of the area function?
5. Determine the domain of each of the following functions.
- a) $f(x) = -2x + 3$
- b) $f(x) = 2x^2 - 5x + 1$
- c) $f(x) = \sqrt{x+1}$
- d) $f(x) = \frac{1}{x^2 - 1}$
- e) $f(x) = 2x^3 + 1$
- f) $f(x) = \frac{1-5x}{2}$
- g) $f(x) = \sqrt{4-2x}$

1.2 Inverse Relationships of Functions

1.2.1 One-to-one Function

Activity 1.10

Consider the function f defined by the set of ordered pairs

$$f = \{(2, 1), (4, 5), (-2, -1), (6, 7), (8, 9), (3, 5)\}$$

- a) What is the domain of f ?
- b) What is the range of f ?
- c) Are there two elements of the domain that are assigned to the same element of the range? Which ones?

Activity 1.11

Consider the function f defined by the set of ordered pairs

$$g = \{(0, 1), (5, 2), (6, 4), (-1, -1), (1, -2)\}$$

- What is the domain of g ?
- What is the range of g ?
- Are there two elements of the domain that are assigned to the same element of the range? Which ones?

A **one-to-one function** is a function in which no two elements in the domain of the function are assigned to the same element in the range of the function.

Thus, in a function described by a set of ordered pairs, if there are two different ordered pairs with the same second element, then the function is not one-to-one.

Activity 1.12

Consider the function f described by the graph in Fig. 1.6.

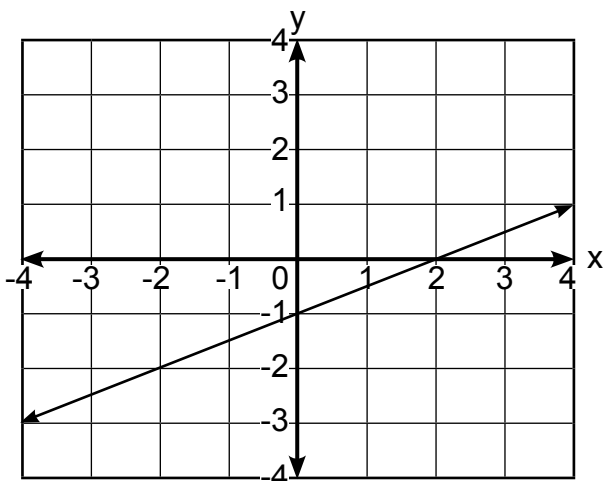


Fig. 1.6

- Find any five ordered pairs that belong to f .
- Find the domain and range of f .
- Can you draw a horizontal line that crosses the graph of the function f at more than one point?
- Can you find two ordered pairs in the graph of f that have the same second element?
- Is the function f one-to-one?

Activity 1.13

Consider the function g described by the graph in Fig 1.7:

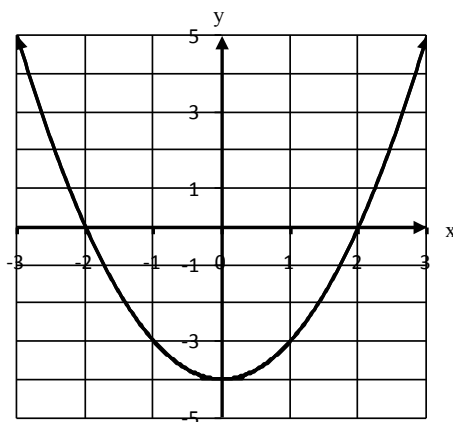


Fig. 1.7

- Find any five ordered pairs that belong to g .
- Find the domain and range of g .
- Can you draw a horizontal line that crosses the graph of the function g at more than one point?
- If you can, what are the ordered pairs of the points where the line crosses the graph of g ?
- Do these ordered pairs belong to the function g ?
- Do these ordered pairs have the same second element?
- Is the function f one-to-one? Why?
- From this activity and Activity 1.12, when does a graph of a function represent the graph of a one-to-one function?

If there exists a horizontal line that crosses the graph of a function at more than one point, then the function is not a one-to-one function. This is called the **horizontal line test**.

Activity 1.14

Consider the function $f(x) = x^2$.

- Find the domain and range of f .
- Are there two elements of the domain that are assigned to the same element of the range? Which ones?
- Is f a one-to-one function? Why?

Example 1.9

Determine whether the function $f(x) = \frac{3}{x^2 + 1}$ is one-to-one function.

Solution:

Studying carefully the formula for the function, we can note that

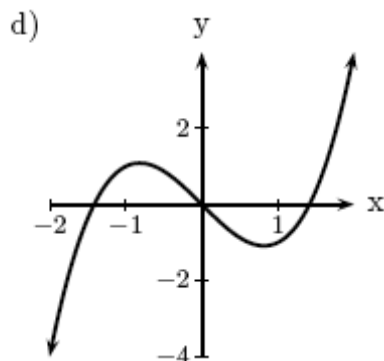
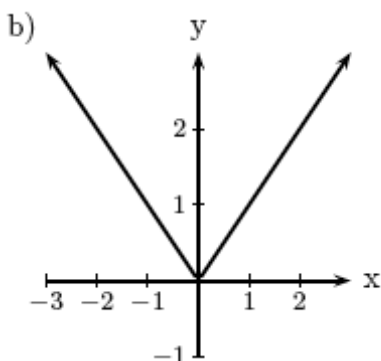
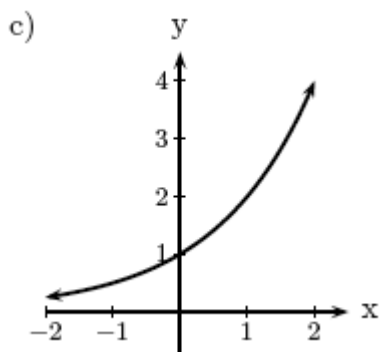
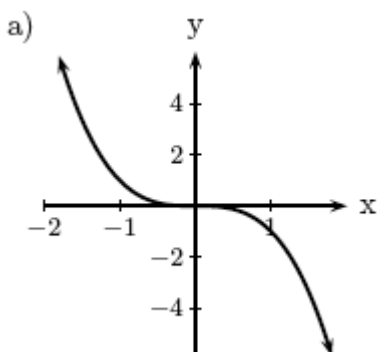
$$f(1) = \frac{3}{1^2 + 1} = \frac{3}{2}, \text{ and } f(-1) = \frac{3}{(-1)^2 + 1} = \frac{3}{2}.$$

That is, there are two values of the independent variable x , namely 1 and -1, that are assigned to the same value of the dependent variable y , namely $3/2$. Hence, f is not one-to-one function.

Exercise 1.5

- Which of the following functions are one-to-one?
 - $f = \{(3, -4), (8, 5), (6, 7), (22, 4)\}$
 - $g = \{(-3, 4), (21, -5), (0, 0), (8, 9)\}$
 - $h = \{(9, 19), (34, 5), (6, 17), (8, 19)\}$
 - $k = \{(2, 27), (3, 28), (4, 29), (5, 30)\}$
 - $m = \{(11, 14), (12, 14), (16, 7), (18, 13)\}$
 - $n = \{(3, 12), (4, 13), (6, 14), (8, 1)\}$

2. Which of the following graphs of functions represent a one-to-one function?



3. Which of the following functions are one-to-one?

a) $f(x) = x - 4$

e) $g(x) = |x - 4|$

b) $g(x) = x^2 + 2$

c) $h(x) = \sqrt{4 - x^2}$

f) $h(x) = \frac{1}{2x - 4}$

d) $f(x) = -5x + \frac{1}{2}$

1.2.2 Inverse of a Function

Activity 1.15

Consider the relation R defined as

$$R = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$$

- a) Find the domain and range of R.

- b) Let S be a relation obtained from R by interchanging the first and second entries of each ordered pair in R . Describe S using a set of ordered pairs.
- c) Find the domain and range of S .
- d) Are there any relationships among the domains and ranges of the relations R and S ?

The **inverse of a relation R** , denoted by R^{-1} , is a relation obtained from R by interchanging the first and second entries of each ordered pair in R . Consequently, the domain of R^{-1} is the range of R , and the range of R^{-1} is the domain of R .

Activity 1.16

Consider the function f defined as

$$f = \{(-1, 2), (-3, 1), (0, -2), (5, 6)\}$$

- a) Is f a one-to-one function?
- b) Find f^{-1} .
- c) Is f^{-1} a function? Why?

Activity 1.17

Consider the function g defined as

$$g = \{(1, -2), (3, 6), (5, 4), (8, 6), (10, -1)\}$$

- a) Is g a one-to-one function?
- b) Find g^{-1} .
- c) Is g^{-1} a function? Why?
- d) Is the inverse of a function always a function?
- e) When is the inverse of a function itself a function?

The inverse of a function is not always a function. However, the inverse of a one-to-one function is always a function.

For some functions described using an equation, the formula for the inverse of a function can be easily obtained following the steps given below.

Suppose the inverse of a function f in function notation is a function. To find the formula for the inverse of f in function notation:

Step 1. Replace $f(x)$ by y .

Step 2. Interchange x and y .

Step 3. Solve for y in terms of x .

Step 4. Replace y by $f^{-1}(x)$.

Example 1.10

Find the formula for the inverse of the function $f(x) = 3x - 6$.

Solution:

The inverse of the function can be found as follows.

$$f(x) = 3x - 6$$

Given function

$$y = 3x - 6$$

Replacing $f(x)$ by y

$$x = 3y - 6$$

Interchanging x and y

$$y = \frac{1}{3}x + 2$$

Solving for y

$$f^{-1}(x) = \frac{1}{3}x + 2$$

Replacing y by $f^{-1}(x)$

Example 1.11

Find the formula for the inverse of the function $g(x) = \frac{x-2}{x+3}$.

Solution:

The inverse of the function can be found as follows.

$$g(x) = \frac{x-2}{x+3}$$

Given function

$$y = \frac{x-2}{x+3}$$

Replacing $g(x)$ by y

$$x = \frac{y-2}{y+3}$$

Interchanging x and y

$$xy + 3x = y - 2$$

Multiplying both sides by $y + 3$

$$y = \frac{3x+2}{1-x}$$

Solving for y

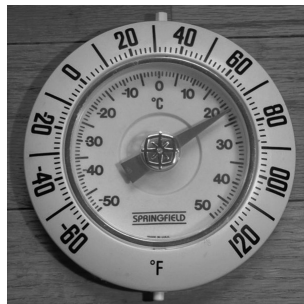
$$g^{-1}(x) = \frac{3x+2}{1-x}$$

Replacing y by $g^{-1}(x)$

Exercise 1.6

- Find the inverse of each of the following relations.
 - $R = \{(2, -3), (4, 6), (3, -1), (6, 6), (2, 3)\}$
 - $S = \{(-2, -4), (-1, -3), (0, -1), (1, 1), (2, 3), (3, 5)\}$
 - $T = \{(4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$
 - $V = \{(\text{Asmara, Eritrea}), (\text{Cairo, Egypt}), (\text{Nairobi, Kenya}), (\text{Djibouti, Djibouti}), (\text{Khartoum, Sudan})\}$
- Find the inverse of each of the following functions and determine whether the inverse is a function or not.
 - $f = \{(-1, 2), (0, 4), (2, -4), (5, 6), (10, 0)\}$
 - $g = \{(-3, 0), (-1, 1), (0, 5), (2, 6)\}$
 - $h = \{(2, 2), (3, 1), (6, 5), (7, 1)\}$
 - $m = \{(-3, 5), (-2, 3), (-1, 1), (0, -1), (1, -3), (2, -5), (3, -7)\}$
- Find the formula for the inverse of each of the following functions, if the inverse is a function.

a) $f(x) = x - 1$	f) $g(x) = -x$
b) $g(x) = 3x + 2$	g) $h(x) = \frac{x-1}{x+1}$
c) $h(x) = x^3 + 3$	h) $k(x) = \frac{1}{x}$
d) $k(x) = \sqrt{x-1}$	
e) $f(x) = x^2 - 4$	
- In measuring temperature, the Celsius scale (y) is a linear function of Fahrenheit scale (x). On the Celsius scale, the freezing point of water is 0 °C (degrees Celsius) and the boiling point is 100 °C. Whereas on the Fahrenheit scale, the freezing point of



water is 32°F (degrees Fahrenheit) and the boiling point is 212°F .

- Find the formula that describes the Celsius scale (y) as a function of the Fahrenheit scale (x).
- Find the temperature in Celsius scale when the temperature in the Fahrenheit scale is -40°F ? -30°F ? 14°F ? 50°F ? 60°F ?
- Find the formula for the inverse of the function in (a).
- Find the temperature in Fahrenheit scale when the temperature in the Celsius scale is -40°C ? -5°C ? 0°C ? 20°C ? 80°C ?

1.2.3 Graph of the Inverse of a Function

Activity 1.18

Take a square paper and draw a rectangular coordinate axes using the same scale on both axes as shown in Fig 1.8.

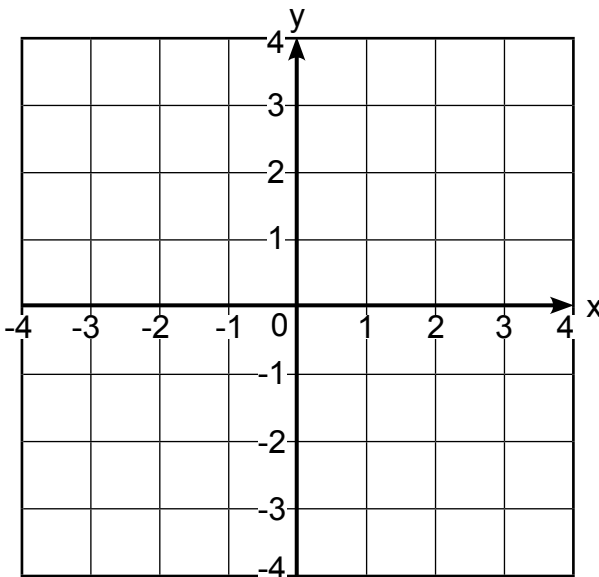


Fig. 1.8

- a) Draw the graph of the line $y = x$.
- b) Plot the following pairs of points marking them heavily so that they are visible from the backside of the paper: (2, 1) and (1, 2); (3, 2) and (2, 3); (-2, 0) and (0, -2); (-1, 1) and (1, -1).
- c) What do you observe about these pairs of points?
- d) If you fold your paper along the line $y = x$, what happened to these pairs of points?
- e) What is the point that pairs with the point (-2, 1)? (3, 0)?

From Activity 1.18, you should be able to see that the effect of reversing the elements of an ordered pair in the number plane is to move the point about the line $y = x$ as if the line were a mirror. This change in position of a point on a number plane is called **reflection about the line $y = x$** .

Activity 1.19

Consider the function f given by

$$f = \{(2, 1), (3, -1), (-2, -2), (-1, 0), (1, 2)\}$$

- a) Draw the graph of f on a coordinate plane drawn on a square paper.
- b) Draw the graph of the line $y = x$ on the same coordinate plane.
- c) Find f^{-1} and draw the graph of f^{-1} on the same coordinate plane.
- d) If you fold the paper along the line $y = x$, what do you observe about the graphs of f and f^{-1} ?
- e) What do you conclude about the graphs of f and f^{-1} ?

The graph of the inverse f^{-1} of a function f is a reflection of the graph of f about the line $y = x$.

Thus, to draw the graph f^{-1} from the graph of f , first draw the graph of f and then reflect the resulting graph about the line $y = x$.

Example 1.12

Draw the graph of the inverse of the function $f(x) = \sqrt{x-1}$.

Solution:

First, we draw the graph of the function $f(x) = \sqrt{x-1}$ and the line $y = x$ which are shown in Fig. 1.9. Then, we reflect every point in the graph of the function $f(x) = \sqrt{x-1}$ about the line $y = x$ and obtain the graph in Fig. 1.10 which is the graph of f^{-1} .

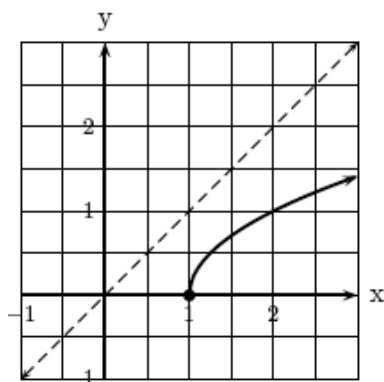


Fig. 1.9

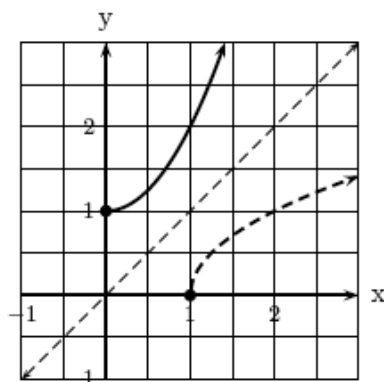
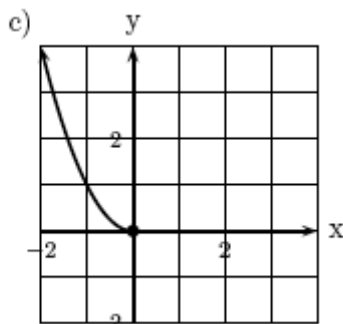
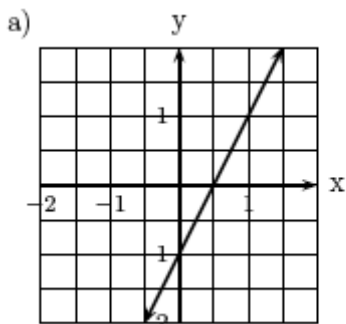
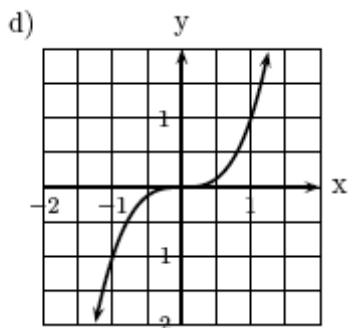
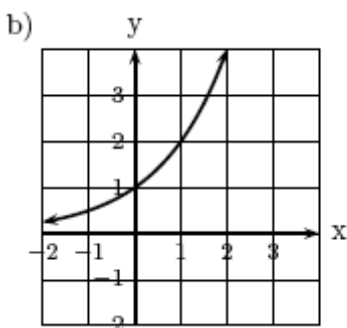


Fig. 1.10

Exercise 1.7

- Find the image of the reflection of each of the following points about the line $y = x$.
 - $(-1, 3)$
 - $(3, 2)$
 - $(0, 1)$
 - (a, b)
 - $(-2, -2)$
 - $(3, 0)$
 - $(4, -5)$
 - (x, y)
- For each of the following functions, draw the graph of its inverse.



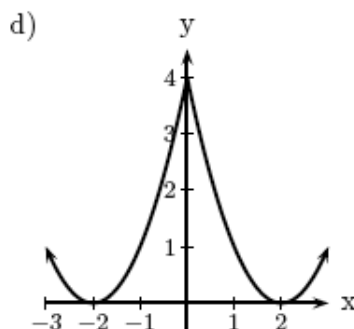
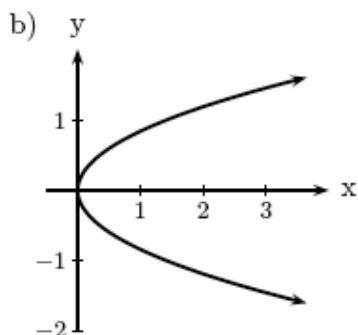
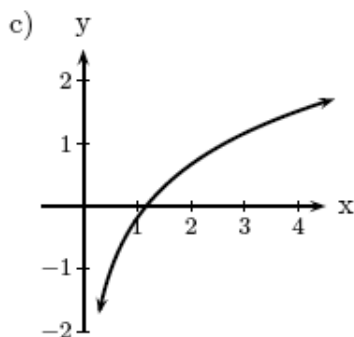
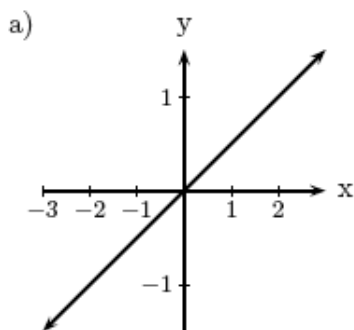


3. For each of the following functions, draw the graph of its inverse from the graph of the function.
- $f(x) = 2x - 2$
 - $f(x) = \sqrt{x}$
 - $f(x) = x^2 - 2$, for $x \leq 0$
 - $f(x) = (x - 1)^3$

Review Exercise 1

- Find the domain and range of each relation, and determine whether it is a function.
 - $R = \{(1, 2), (3, 4), (5, 6), (8, 6), (10, -1)\}$
 - $S = \{(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)\}$
 - $T = \{(2, 1), (3, 1), (2, 4), (-1, 0)\}$
 - $U = \{(A, a), (B, b), (C, c), (D, d), (E, e)\}$
- Determine whether each of the following equations describe y as a function of x ? If it describes a function, write it using the function notation.
 - $2x + 3y - 5 = 0$
 - $y^2 - x + 1 = 0$
 - $y = x^2 - 1$
 - $|x| + |y| = 2$
- Which of the following functions are one-to-one?
 - $f = \{(12, 2), (15, 4), (19, -4), (25, 6), (78, 0)\}$
 - $g = \{(-2, -2), (-1, -1), (0, 0), (1, -1), (2, -2)\}$
 - $h(x) = x^2 - 4x + 4$
 - $k(x) = \sqrt{x - 1}$

4. Which of the following graphs represent one-to-one functions?



5. Find the inverse of each of the following functions and determine whether the inverse is a function.

a) $f = \{(0, 1), (5, 6), (-2, -3)\}$

b) $g = \{(0, 2), (1, 1), (3, 0), (4, -1), (5, 0), (2, 2)\}$

c) $h = \{(0, 0), (1, 2), (2, 4), (3, 9)\}$

d) $k = \{(-3, 1), (0.5, 3), (4, 2), (-0.5, 3), (3, -1)\}$

6. Find the formula for the inverse of each of the following functions.

a) $f(x) = 2x - 3$

c) $h(x) = (x - 2)^3 + 1$

b) $g(x) = \frac{6}{x} + 1$

d) $k(x) = \frac{3 - x}{2x + 1}$

Chapter 2

Geometric Structure

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- explain what an axiomatic system is;
- identify the structure and the relationships within an axiomatic system;
- analyse the relationships of structure in order to make and verify conjectures;
- apply logical reasoning and justify arguments in mathematical proofs;
- select appropriate representations (concrete, pictorial graphical or symbolic) in solving problems.

2.1 Axiomatic Systems

Activity 2.1

1. Draw a square with a side of length 4 cm.
2. Which of the following is a definition of a square?
 - a) A square is a quadrilateral with all the four sides equal.
 - b) A square is a plane figure having four sides equal.
 - c) A square is a quadrilateral with all the four angles equal.
 - d) A square is a parallelogram with all the four sides equal.
 - e) A square is a quadrilateral whose four angles are right angles, and four sides are congruent.
3. Awet and Kedija were asked to define a square. Comment on two learners' the discussion.

Awet: A square is a quadrilateral, where all its sides and angles are equal.

Kedija: What is a side?

Awet: A side is a line segment.

Kedija: What is a line segment?

Awet: A line segment with endpoints A and B is the set of two points A and B and all points between the two points A and B.

Kedija: What is a point?

Awet: Is that possible to define every word?

Kedija: I don't know.

4. What happens if we try to define every word?
5. As a bases for use in geometry, which words do you think should be defined first?

We cannot define every word we use, because we would be talking in circles. In order to avoid this type of circular definition, we choose some special simple terms that are not defined.

Therefore words such as:

points, lines, and planes are some of the words used in geometry as undefined terms.

Activity 2.2

1. Describe points, lines and planes using your own words.
2. Which of the following can describe a point, a line or a plane?
 - a) The floor of a class room.
 - b) The edge of a table.
 - c) the corner of a box.
 - d) The longest side of a foot ball field.
 - e) The sharp end of a needle.
 - f) The flat screen on a television.

A point has no dimension (actual size). Therefore, even though a point is represented by a dot, it has no length, width, or thickness.

A point is usually labelled with a capital letter.

In the coordinate plane, a point is named by an ordered pair, (x,y) .

A line has no thickness but its length extends in one dimension and goes on forever in both directions.

A line is depicted as a straight line with two arrowheads indicating that the line extends endlessly in two directions.

A line is named by a single small letter ℓ, h, k or by two points on the line, $\overleftrightarrow{AB}$.

A plane has no thickness but extends indefinitely in all directions.

Planes are usually represented by a shape that looks like a tabletop or wall.

Even though the diagram of a plane has edges, you must remember that the plane has no boundaries.

In figure 2.1, m is a plane, the point C is intersection of the two lines and A and B are points on the plane m .

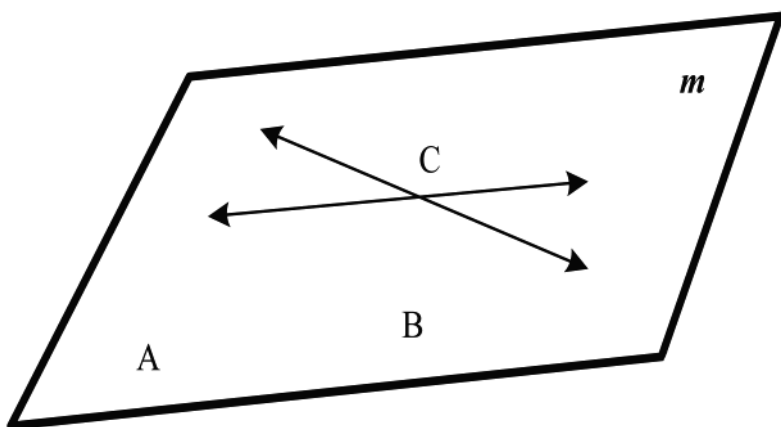
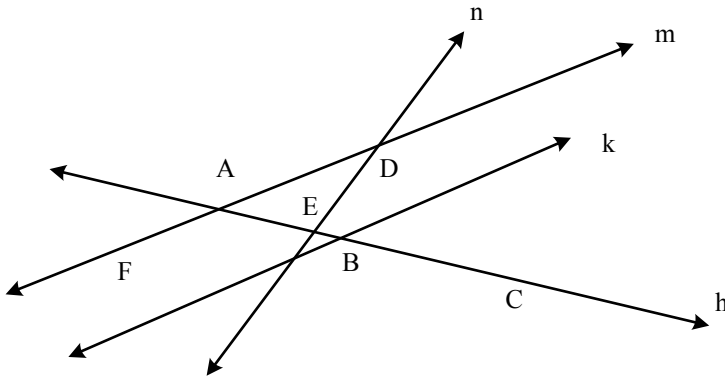


Fig. 2.1

Activity 2.3**Fig. 2.2**

Using the figure above, answer the following:

- Which lines contain the point A?
- Which points lie on the same line?
- Is it not better to give a name for points that lie on the same line than to say points that lie on the same line all the time?
- If a particular name is given for points that lie on the same line, that name must be understood and used by everybody. Why?

From Activity 2.3 we observe the need for definitions in order to have clear understanding of mathematical concepts and notions.

The undefined terms are then used in the definition of other terms.

For example, “Collinear points are points that lie on the same line.” is a definition.

Activity 2.4

1. Mention some of the theorems you have learnt before.
2. Which of the following statements do you think are theorems and which of them are not? Give your reasons.
 - a) In an isosceles triangle, the line drawn from the vertex to the mid-point of the opposite side is perpendicular to the opposite side.
 - b) The whole is greater than the part.
 - c) An exterior angle of a triangle is equal to the sum of the two opposite interior angles.
 - d) The area of a rectangle with sides x and y is equal to xy sq. units.
 - e) The shortest distance between two points is the straight-line segment joining them.
 - f) Things equal to the same thing are equal to each other.
3. Discuss the difference between statements that are theorems and statements that are not theorems.
4. Is it possible to prove all statements?
5. What happens if we try to prove every statement?

It is not possible to prove every statement in geometry (mathematics). We choose some statements to be true without any proof, and use these statements to prove others.

Axioms or postulates are statements that are accepted to be true without a proof.

The following postulates describe the relationships of points, lines and planes.

1. A line contains at least two points. A plane contains at least three non-collinear points.
2. Through any two points there is exactly one line.
3. Through any three non-collinear points there is exactly one plane.
4. If two planes intersect, their intersection is a line.

Statements that can be proved using other statements (axioms or earlier proved statements) are known as **theorems**.

Terms are defined using undefined terms and previously defined terms, and theorems are proved using axioms (postulates) and previously proved theorems.

An axiomatic system (structure) in mathematics contains some undefined terms and terms that can be defined; and some unproved statements (axioms or postulates) and statements that can be proved (theorems).

Exercise 2.1

1. Find the meaning of the word “similar” in a dictionary. Could you learn from your dictionary what “a similar” is if you don’t know what “a resemblance” or “alike” is?
2. What is wrong with the definition “a line is a set of points”?
3. What is the name given to points that lie in the same line?
4. Draw two rays that lie on the same line, with a common endpoint and no other points in common. What is the measure of the angle formed by the two rays?

2.2 Attributes of Geometric Figures

Activity 2.5

1. Draw the following line segments using a ruler.
a) 3 cm b) 2.5 cm c) 4 cm
2. Copy the above line segments onto plain white paper (without any grid lines), using a compass and a straightedge (an unmarked ruler).

Example 2.1

Construct (copy) the line segment $\overline{PQ}$ using a compass and straightedge (a ruler without a scale).

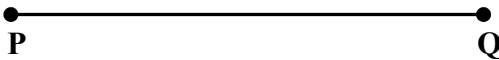


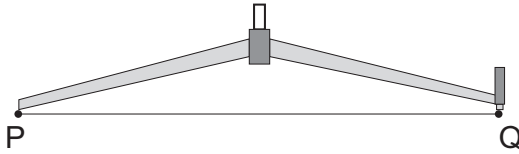
Fig. 2.3

Solution

1. Mark a point R that will be one endpoint of the new line segment.

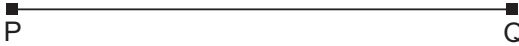


2. Adjust the compass width to the length PQ. The compass width is now equal to the length of the line segment PQ.



•
R

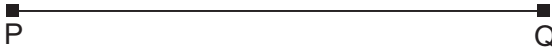
3. Without changing the compass width, draw an arc with centre at point R roughly where the other endpoint will be.



■
R



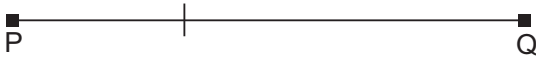
4. Pick a point S on the arc that will be the other endpoint of the new line segment.



■
R



5. The line segment RS is equal in length (congruent to) the line segment PQ. Why?



Drawings made with only a straightedge (an unmarked ruler) and compasses are known as **constructions**.

Exercise 2.2

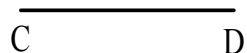
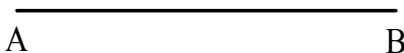
1. Construct a segment equal to the segment given below.

a) _____

b) _____

c) _____

2. Given $\overline{AB}$ and $\overline{CD}$,



construct line segments with the following lengths:

a) $AB + CD$

b) $2AB + CD$

Activity 2.6

1. Draw line segments with the following measurements, and find the midpoint of each one.
a) 2 cm b) 4.5 cm c) 3.6 cm
2. Construct each of the above segments, and try to find the midpoint without measurement.
3. The teacher asked Osman to find the midpoint of the segment $\overline{AB}$. That is, to bisect $\overline{AB}$. Discuss Osman's steps.

Osman used the following steps.

Step 1: With points A and B as centres and radius equal to $\overline{AB}$ construct pairs of intersecting arcs.

Step 2: Name the points of intersection of the arcs P and Q.

Step 3: Join the points P and Q using a straightedge. The point of intersection R of $\overline{AB}$ and $\overline{PQ}$ is the midpoint of $\overline{AB}$.

4. Using the steps discussed in task (3) find the midpoint of the following line segments.
a) 3 cm b) 4 cm c) 2.6 cm

Activity 2.7

1. Using a protractor and a ruler draw angles with the following measurements.
a) 60° b) 40° c) 50°
2. Divide the above angles into two equal angles (i.e. bisect the angles).

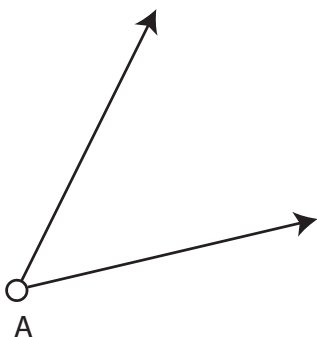
Example 2.2

Construct an angle congruent to a given angle.

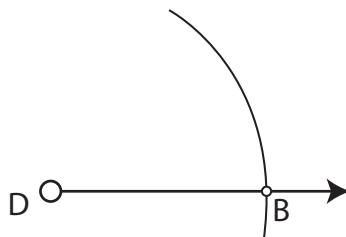
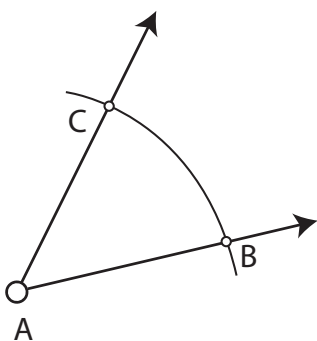
Construct (copy) the angle $\angle A$ using a compass and straightedge.

Solution

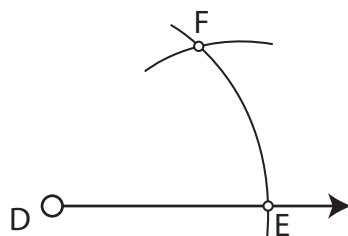
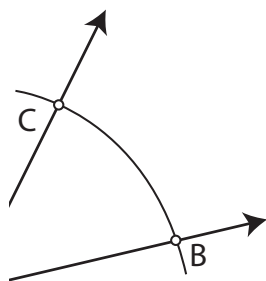
1. To draw an angle congruent to $\angle A$ begin by drawing a ray with endpoint D.



2. Place the compass on point A and draw an arc across both sides of the angle. Without changing the compass radius, place the compass on point D and draw a long arc crossing the ray. Label the three intersection points as shown.

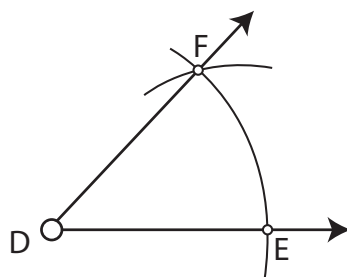
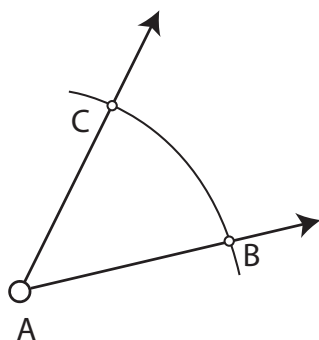


3. Set the compass so that its radius is BC. Place the compass on point E and draw an arc intersecting the one drawn in the previous step. Label the intersection point F.



4. Use the straight edge to draw ray DF.

$$\angle EDF \equiv \angle BAC$$



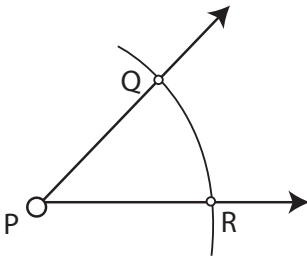
Example 2.3

Construct the bisector of an angle

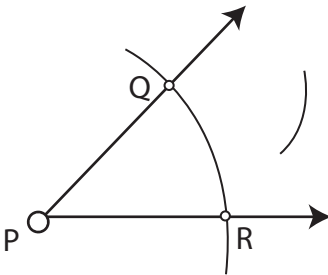
Construct the bisector of the $\angle P$ using a compass and straightedge. That is, divide the given angle into two congruent angles.

Solution: Construct the bisector of an angle

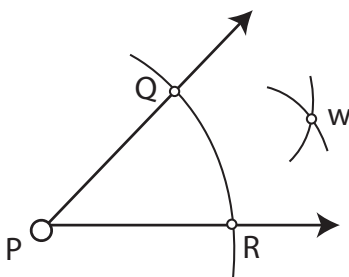
1. Let point P be the vertex of the angle. Place the compass on point P and draw an arc across both sides of the angle. Label the intersection points Q and R.



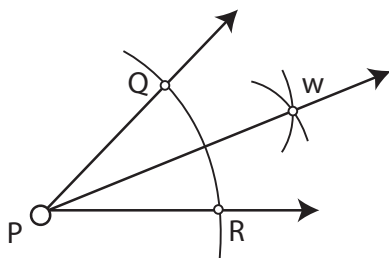
2. Place the compass on point Q and draw an arc across the interior of the angle.



3. Without changing the radius of the compass, place it on point R and draw an arc intersecting the one drawn in the previous step. Label the intersection point W.



4. Using the straightedge, draw ray PW. This is the bisector of $\angle QPR$.



Exercise 2.3

1.
 - a) Draw $\overline{AB}$ of length 5 cm.
 - b) Construct a line segment equal to segment AB using an unmarked ruler (straightedge) and compass.
 - c) Find the midpoint of the constructed line segment.
2. Repeat a), b) and c) for $\overline{CD}$ of length 6 cm.
3. Construction and bisecting of an angle.
 - a) Draw a ray OA (a ray with initial point O).
 - b) Draw an angle of 60° with vertex at O and one side OA.
 - c) Construct an angle equal to the angle given in b) using a straightedge and a compass only (without measuring the angle with a protractor).
 - d) Divide (bisect) the constructed angle into angles.
4. Repeat a), b), c) and d) for the following angles.

a) 90°	b) 45°	c) 30°
---------------	---------------	---------------

Activity 2.8

1.
 - a) Draw a circle with radius of 2 cm and center at a point O.
 - b) Choose any four points on the above circle. Name the points A, B, C, and D respectively.
 - c) Using a ruler draw the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.
 - d) What is the name of the figure formed in c)?
2. Draw a circle of radius 3 cm. Draw a square ABCD, where all the vertices A, B, C and D lie on the circle. What is the measure of one side of the square?

A polygon is **inscribed** in a circle when each vertex of the polygon lies on the circle.

Example 2.4

Consider the following circle:

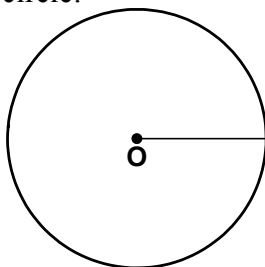
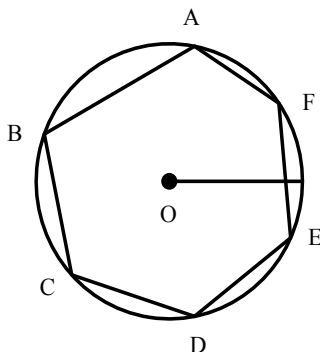


Fig. 2.4

- a) Draw an inscribed hexagon.
- b) Draw an inscribed hexagon in the circle, where all the sides of the hexagon are equal.

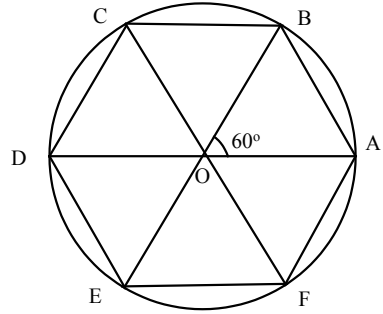
Solution

- a) You can choose any six points on the circle say A, B, C, D, E and F. Next join the consecutive points by line segments using a ruler as shown in the figure. Then the resulting figure is a hexagon.



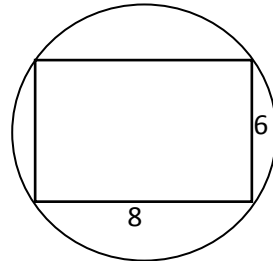
- b) 1. First draw a circle of a radius r , then draw any radius say $\overline{OA}$. Then:

2. Divide 360° into 6. Why?
3. Measure 60° (from the radius $\overline{OA}$) and draw $\angle AOB$, where B is on the circle.
4. $\overline{AB}$ is one side of the hexagon.
5. Using the same method (step 3 and 4) we get $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$.



Exercise 2.4

1. The rectangle inscribed in the circle has sides of lengths 8 and 6 units.
 - a) Determine the centre.
 - b) Find the radius of the circle.



2. Draw a circle with radius 4 cm.
 - c) Draw an inscribed octagon in the circle by choosing any eight points on the circle.
 - d) Draw an inscribed octagon in the circle, where all the sides of the octagon are equal.

2.3 Conjectures

Activities 2.9

1. Find the meaning of the word “conjecture” from a dictionary.
2. Draw two intersecting lines on white transparent paper. Let the point of intersection of the two lines be P.

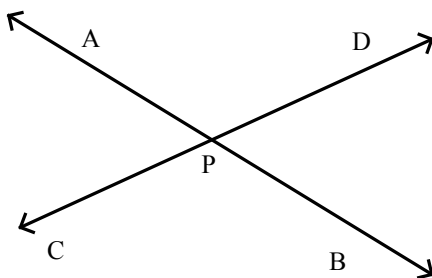


Fig. 2.5

- a) How many angles are formed by the two intersecting lines at the point of intersection P?
 - b) Bisect $\angle APD$ using the method of construction used in the Example 2.3.
 - c) Fold your paper along the line that bisects $\angle APD$.
 - d) After folding your paper, were you able to observe that $\angle APC$ and $\angle BPC$ fit each other?
 - e) Repeat a), b), c) and d) for any two intersecting lines.
3. Consider figure 2.6.

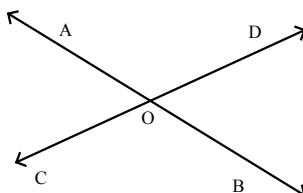


Fig. 2.6

- Which of the four angles have equal measures?
- If angles $\angle AOC$ and $\angle BOD$, $\angle AOD$ and $\angle BOC$ are known as vertical opposite angles. What can you say about vertical opposite angles?
- Do vertical opposite angles have equal measures?

Activity 2.10

- On squared paper draw parallel lines h and k and a transversal m as shown in figure 2.7.

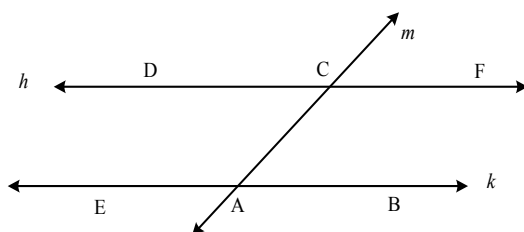


Fig. 2.7

- Using a protractor measure the four angles $\angle BAC$, $\angle DCA$, $\angle EAC$ and $\angle FCA$.
 - Which of the four angles have equal measure?
 - The angles $\angle BAC$ and $\angle DCA$ as well as $\angle EAC$ and $\angle FCA$ are known as pair of alternate interior angles. What can we say about the measure of the two pair of alternate angles?
- Draw parallel lines and a transversal to the parallel lines, and repeat (a), (b) and (c) of task (1).
 - Try to make a general statement about alternate angles formed by parallel lines and a transversal.

Statements that appear to be true based on observations and experiments (such as drawing) are **conjectures**.

Making conjectures is an important way to discover new concepts of mathematics.

Exercise 2.5

1.
 - a) Draw a triangle, say $\triangle ABC$
 - b) Using a protractor, measure the three angles of the triangle.
 - c) Find the sum of the three angles.
2.
 - a) Draw another triangle and repeat a), b), and c) of task (1).
 - b) What conjecture can you make about the sum of the angle measures of any triangle?
3. Make conjectures about the following:
 - a) Diagonals of a rectangle.
 - b) The base angles of an isosceles triangle.
 - c) Faces of a rectangular prism.

2.4 Understanding Mathematical Proofs

Activity 2.11

1. Consider the following two statements.
All mammals are warm blooded.
Cows are mammals.
 - a) Based on the given pair of statements, what can we say about Cows?
 - b) Can we say cows are cold blooded?
2. Given: An obtuse triangle is a triangle with one obtuse angle.
In $\triangle ABC$, $m(\angle ABC) = 125^\circ$.
What can you conclude about $\triangle ABC$?
3. All human beings need to breathe to live.
Yohana is a human being.
What can we conclude from the above pair of statements?

The method of reasoning used to make a valid conclusion from a given set of accepted statements (such as definitions, axioms and earlier proved statements) is a **deductive reasoning**.

Activity 2.12

Consider the figures A, B and C below.

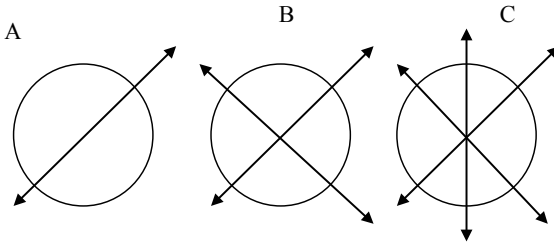


Fig. 2.8

- Into how many regions is the circle in A divided by the one line?
- Into how many regions is the circle in B divided by the two lines?
- Into how many regions is the circle in C divided by the three lines?
- Draw a circle and four lines that pass through the centre of the circle (like A, B and C in the above example). Into how many regions is the circle divided by the four lines?
- Into how many regions is a circle divided by 5, 6 and 8 lines?
- Make a table like the following:

Number of lines	1	2	3	4
Number of regions (parts)	2	4	6	?

- Try to find a general formula (that is, make a conjecture) for the regions (parts) formed by n lines.

Inductive reasoning is a method of forming a conclusion (conjecture) based on several specific cases (usually experiments).

Activity 2.13

1. Discuss Pythagoras' Theorem using an illustration of a right triangle.
2. Draw $\triangle ABC$ with legs $AC = 3$ cm and $BC = 4$ cm respectively and $m(\angle ACB) = 90^\circ$.
 - a) Find the length of the hypotenuse AB .
 - b) Show that $AB^2 = CA^2 + CB^2$.
 - c) Can we consider b) as a proof of the Pythagoras' Theorem? Give a reason for your answer.

Activity 2.14

A teacher asked learners to prove that the sum of two positive odd integers is even.

Ande said: since $3 + 5 = 8$, and then we can conclude that the sum of two odd integers is even.

Nafisa said: $3 + 5 = 8$ helps to verify that the sum of two odd integers is even, but it is not a proof because 3 and 5 are particular odd numbers.

In order to prove the above statement, the teacher told the learners to do the following:

- a) Define even and odd integers.
- b) If w is a positive even integers, can we write $w = 2n$ for some integer n ?
- c) If x and y are any two positive odd integers, can we write $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n ?
- d) If x and y are positive odd integers, can we write $x + y$ as $2k$ or $2k + 1$ for some integer k ? Given your reason.
- e) Can you say the discussion in b) and c) is a proof of the statement; the sum of two odd positive integers is even?
- f) Discuss the difference between **verification** and **proof**.

Example 2.4

Prove the conjecture we discussed in Activity 2.9.

If two lines intersect at a point then vertically opposite angles are equal.

Proof:

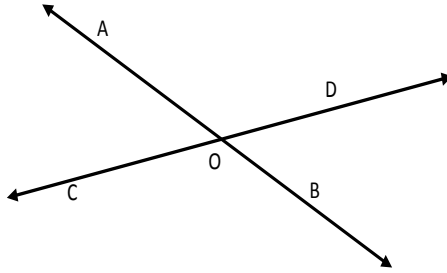


Fig. 2.9

Steps

Reasons

- | | |
|---|------------------------------|
| 1. $m(\angle AOD) + m(\angle DOB) = 180^\circ$ | Straight angle = 180° |
| 2. $m(\angle COB) + m(\angle BOD) = 180^\circ$ | Straight angle = 180° |
| 3. $m(\angle AOD) + m(\angle DOB)$
$m(\angle COB) + m(\angle DOB) = 180^\circ$ | Why? |
| 4. $m(\angle AOD) = m(\angle COB)$ | Why? |

The method of the proof used in Example 2.4 is a **direct proof**.

When a conjecture is proved it becomes a **theorem**.

Statements such as “If two lines intersect at a point then vertically opposite angles are equal” are conditional statements.

The assumption that “Two lines meet at a point and form vertical opposite angles” is the **hypothesis** of the theorem; and “Vertically opposite angles are equal” is the **conclusion** of the theorem.

Activity 2.15

Consider the conditional statements:

Statement A: If a figure is a triangle, then it is a polygon.

Statement B: If a figure is a polygon, then it is a triangle.

- What are the hypothesis and the conclusion of statement A?
- Is the conditional statement A true or false? Give your reason.
- What are the hypothesis and the conclusion of the statement B?
- Is the conditional statement B true or false? Give your reason.
- Did you observe that the hypothesis of statement A is the conclusion of statement B?
- Did you observe that the conclusion of statement A is the hypothesis of statement B?

When the hypothesis and conclusion of a conditional statement are interchanged, the conditional statement formed is the **converse** of the original statement.

Exercise 2.6

- For the following situations given by a pair of statements, make a correct deductions (i.e. state the conclusion that can be made).
 - Two angles $\angle A$ and $\angle B$ are said to be complimentary if $m(\angle A) + m(\angle B) = 90^\circ$.
 $m(\angle A) = x^\circ$ and $m(\angle B) = 90^\circ - x^\circ$, for any positive number x .
 - All rectangles are parallelograms.
 $\square ABCD$ is a rectangle.
 - All human beings are mortal.
Socrates is a human being.

2. Draw the graph of the following on the same coordinate axes.

i) $y = 2x - 1$ ii) $y = 2x + 3$, iii) $y = 2x + 5$ iv) $y = 2x$.

Using inductive reasoning, make a conclusion about the graphs of $y = mx + b$ and $y = nx + d$, where m , b , n and d are real numbers with $m = n$.

3. Given $a = b$, where a is real number. What is wrong with the following proof?

$$a = b$$

$$a^2 = ab$$

$$a^2 - b^2 = ab - b^2$$

$$(a - b)(a + b) = b(a - b)$$

$$a + b = b$$

$$b + b = b$$

$$2b = b$$

$$2 = 1$$

Given

Multiplying both sides by a .

Subtracting from both sides b^2

Factorising

Dividing both sides by $a - b$

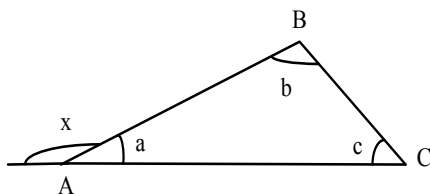
Since $a = b$

Simplify

Dividing both sides by b

4. Consider the following theorem.

An exterior angle of a triangle equals the sum of the two opposite angles.



That is, $m(\angle x) = m(\angle b) + m(\angle c)$.

- a) Verify the theorem by measuring the exterior angle and the two interior angles of an equilateral triangle.

Complete the following statements to prove the theorem.

- b) $\angle x$ is _____ angle and $\angle b$ and $\angle c$ are its _____ angles.

- c) $m(\angle a) + m(\angle b) + m(\angle c) = \underline{\hspace{2cm}}$.

Reason: _____.

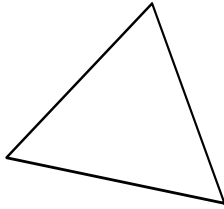
- d) $m(\angle x) + m(\angle a) = \underline{\hspace{2cm}}$. Reason: _____.

- e) Therefore, from the above we have:
 $m(\angle x) + m(\angle a) = \underline{\hspace{2cm}}$.
- f) Hence, $m(\angle x) = \underline{\hspace{2cm}}$.
5. Find the hypothesis and the conclusion for the following sentences and form the converse of each statement.
- If opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.
 - If a triangle is isosceles then two of its sides are equal.
 - If the radius of a circle is 5 units then its area is 25π sq. units.

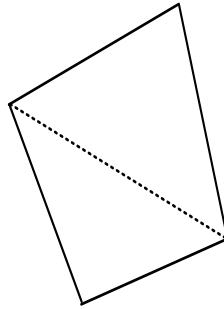
Review Exercise 2

- If we define a “plane” as a “flat surface,” we would then have to either accept “flat” and “surface” as undefined terms or else provide definition for these words. Try to write a short definition for each.
- Draw the following line segments using a ruler.
a) 2 cm b) 5 cm c) 4 cm
- Using a compass and a straightedge (an unmarked ruler) construct (copy) the line segments in problem (2) above.
- Using a compass and a straightedge (an unmarked ruler) bisect (divide into two equal parts) the line segments constructed in problem (3) above.
- Draw a circle with radius 3 cm.
 - Draw an inscribed pentagon in the above circle by choosing any five points on the circle.
 - Draw an inscribed pentagon in the above circle, where all the sides of the pentagon are equal.

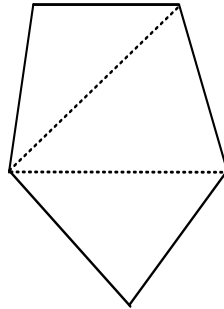
6. Consider the following figures:



One triangle



Two triangles



Three triangles

a) Complete the following table.

Name of the polygon	Number of sides	Number of triangles	Number of diagonals from one vertex
Triangle	3	1	0
Quadrilateral	4	2	1
Pentagon	5	3	2
Hexagon	6	?	?
Heptagon	7	?	?
Octagon	8	?	?

- How does the number of triangles relate to the number of diagonals?
- Using inductive reasoning (based on the above table) make a conjecture about the relationship between the number of triangles and the number of diagonals from one vertex.
- Using inductive reasoning (based on the above table) make a conjecture about the relationship between the number of sides and the number of diagonals from one vertex.

Chapter 3

Geometric Patterns

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- identify, analyse, and describe patterns that emerge from two dimensional figures
- use numeric and geometric patterns to make generalisations about geometric properties of polygons and angle relationships in polygons and circles.

3.1 Angle and Side Properties of Polygons

3.1.1 Polygons

Activity 3.1

- a) Do you remember what is meant by a triangle and a quadrilateral?

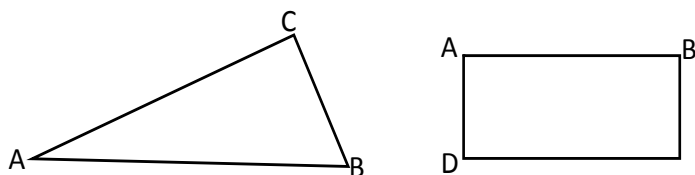


Fig.3.1

- b) Name the sides of $\triangle ACB$ and the rectangle ABCD.
c) Name the vertices of $\triangle ACB$ and the rectangle ABCD.
d) Do you remember what a polygon is?

Example 3.1

- a) Which of the following figures are polygons?

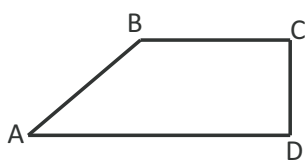


Fig.3.2a

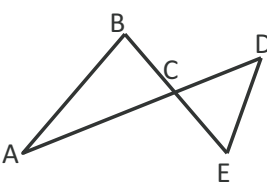


Fig.3.2b

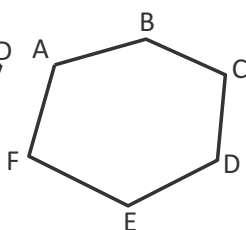


Fig.3.2c

In Figs 3.2 a) and 3.2 c), the sides do not cross each other. Exactly two sides meet at a vertex. These figures are examples of polygons.

In Fig. 3.2 b), the line segments AD and BE intersect at C, which is not an endpoint of any one of the line segments. Therefore, the figure is not a polygon.

- b) What is the name given to polygons of sides 7, 8, 9, 10, and 12?
c) What is the name given to an n-sided polygon?

Polygons could be classified according to the number of sides they have. Some examples are given below

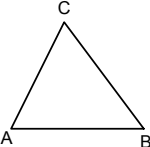
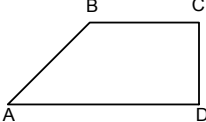
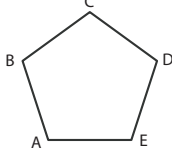
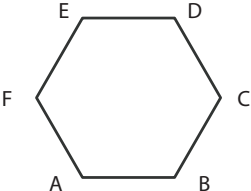
Figure	Number of sides	Name of polygon
	3	Triangle
	4	Quadrilateral
	5	Pentagon
	6	Hexagon

Fig. 3.3

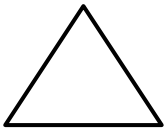
Polygons of sides 7, 8, 9, 10 and 12 could also be given special names.

Types of Polygon

Regular polygons are polygons with **all angles equal and all sides of the same length**. Regular polygons are both **equiangular and equilateral**.

Example 3.2

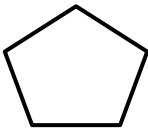
The following figures are examples of regular polygons.



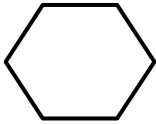
Equilateral Triangle



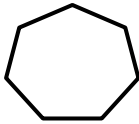
Square



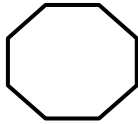
Pentagon



Hexagon



Heptagon



Octagon

Exercise 3.1

1. Complete the following table.

Name of polygon	Number of sides
Heptagon	
	8
Nonagon	
	10

2. Draw a triangle with all sides of length 4 units and measure each of its angles.
3. Draw a regular hexagon.
4. An equilateral triangle is always equiangular. Give an example of a polygon to show that an equilateral polygon of more than three sides is not necessarily equiangular.
5. Classify each of the following polygons as equilateral, equiangular or regular.

- a) Square
- b) Rectangle
- c) Rhombus

3.1.2 Sum of the Interior Angles of Polygons

Activity 3.2

1. a) Draw a triangle of any size and shape on a piece of paper.
- b) Cut it out and label the angles A, B and C on both sides of the paper.
- c) Draw dotted lines as in Fig 3.5 a) and cut along these lines to get three pieces of a paper.
- d) Place them along a ruler as shown in Fig 3.5 b) to see that

$$m\angle A + m\angle B + m\angle C = 180^\circ.$$

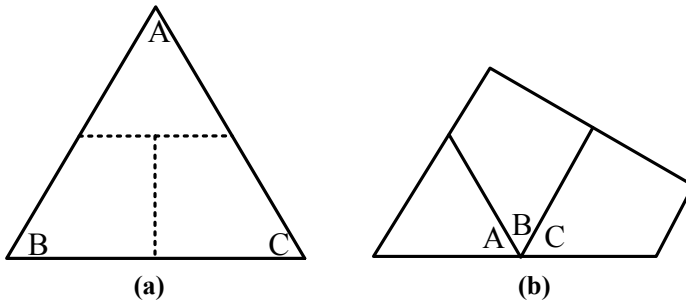


Fig. 3.5

Theorem 3.1: The sum of the measures of the angles in any triangle is 180° .

Proof:

Construct $\triangle ABC$ as shown in Fig 3.6. Then, we have to show that $a + b + c = 180^\circ$

Construct a straight line that passes through A parallel to $\overline{BC}$. There is a unique line passing through the point A and parallel to $\overline{BC}$

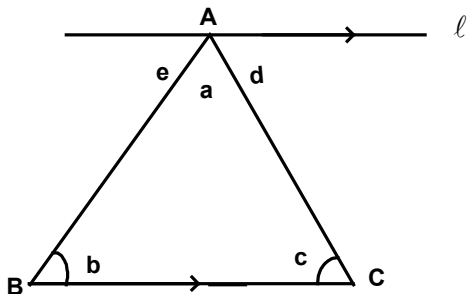
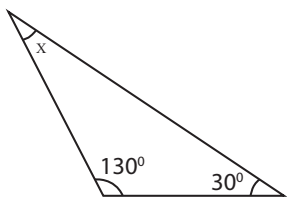


Fig. 3.6

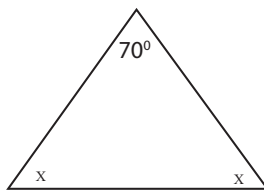
Statement	Reason
1. $e + a + d = 180^\circ$	Straight line
2. $d = c$	Alternate interior angles
3. $e = b$	Alternate interior angles
4. $\therefore a + b + c = 180^\circ$	Substituting $\angle d$ by $\angle c$ and $\angle e$ by $\angle b$.

Example 3.3

Find the measure of each indicated angle.



(a)



(b)

Solution:

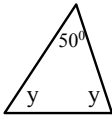
$$\begin{aligned} \text{a) } x + 30^\circ + 130^\circ &= 180^\circ \\ x &= 180^\circ - 30^\circ - 130^\circ \\ x &= 20^\circ \end{aligned}$$

$$\begin{aligned} \text{b) } x + x + 70^\circ &= 180^\circ \\ 2x &= 180^\circ - 70^\circ \\ 2x &= 110^\circ \\ x &= 55^\circ \end{aligned}$$

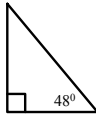
Exercise 3.2

1. Find the value of x or y in each of the following.

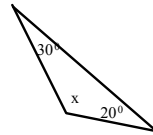
a)



b)



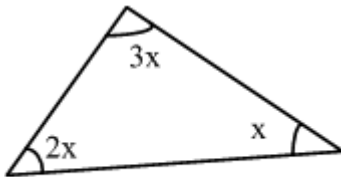
c)



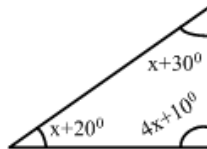
2. Find the measure of the third angle of the triangle given these two angles.

a) $43^\circ, 54^\circ$ d) $90^\circ, 42^\circ$ b) $88^\circ, 29^\circ$ e) $5x, 2x$ c) $120^\circ, 20^\circ$ f) $x + 40^\circ, x - 20^\circ$

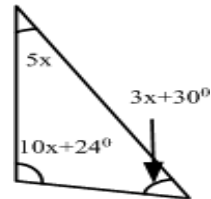
3. Calculate the measure of each angle.



(a)



(b)



(c)

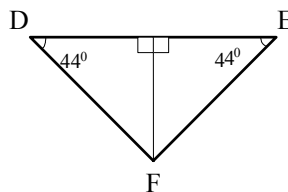
4. In a certain triangle the measure of one angle is 34° . What is the measure of the other acute angle?

5. In $\triangle ABC$, $m(\angle ABC) = m(\angle BCA)$ and =.

$$m(\angle BCA) = \frac{1}{2}m(\angle BAC).$$

Find the measure of the three angles.

6. Calculate $m(\angle DFE)$



Activity 3.3

1. a) Draw a quadrilateral ABCD as shown in Fig. 3.7.

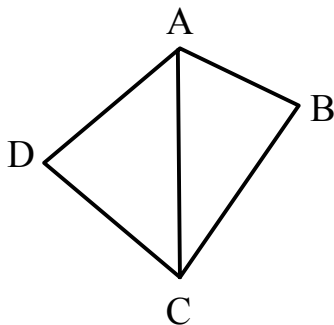


Fig. 3.7

- b) Choose vertex A and join $\overline{AC}$.
c) What is the sum of the measures of all angles of a triangle?
d) What is the sum of the measures of all the angles of the quadrilateral ABCD?

2. a) Draw a pentagon ABCDE as shown in Fig. 3.12.

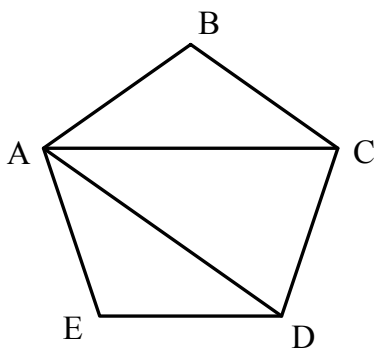


Fig. 3.8

- b) Choose vertex A and join $\overline{AC}$ and $\overline{AD}$.
c) What is the sum of the measures of the angles of the three triangles formed?
d) What is the sum of the measures of the angles of the pentagon ABCDE?

3. a) Draw hexagon ABCDEF as shown in Fig. 3.9.

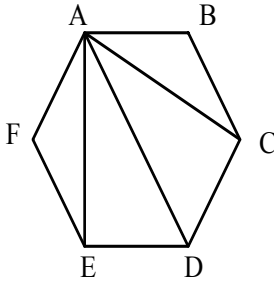


Fig. 3.9

- b) Choose vertex A and join $\overline{AC}$, $\overline{AD}$ and $\overline{AE}$.
 c) What is the sum of the measures of all the angles of the triangles formed?
 d) What is the sum of the measures of the angles of the hexagon ABCDEF?

Note: The results from Activity 3.3 can be summarised as follows.

Polygon	No. of Sides	No. of triangle	Sum of the interior angles of the polygon
Triangle	3	1	$180^0 = 1 \times 180^0 = (3 - 2) \times 180^0$
Quadrilateral	4	2	$180^0 = 2 \times 180^0 = (4 - 2) \times 180^0$
Pentagon	5	3	$180^0 = 3 \times 180^0 = (5 - 2) \times 180^0$
Hexagon	6	4	
Heptagon	7	5	
Octagon	8	6	
Nonagon	9	7	
n-gon			

- e) What is the sum of the measures of the angles of a heptagon, octagon and nonagon?
 f) Write a conjecture for the sum of the measures of the interior angles of a polygon with n sides?

Activities 3.2 and 3.3 suggest the following rule.

The sum of the measures of the interior angles of an n -sided polygon is given by $(n - 2) \times 180^\circ$

Example 3.4

Find the sum of the measures of the interior angles of a decagon.

Solution:

A decagon has 10 sides.

Hence, the sum of the measures of the interior angles

$$\begin{aligned} &= (10 - 2) \times 180^\circ \\ &= 8 \times 180^\circ \\ &= 1440^\circ \end{aligned}$$

If a polygon is regular, all its interior angles have equal measure. Hence, the measure of each interior angle of an

n -sided regular polygon is given by $\frac{(n - 2) \times 180^\circ}{n}$

Example 3.5

Find the measure of each interior angle of a 12 sided regular polygon.

Solution:

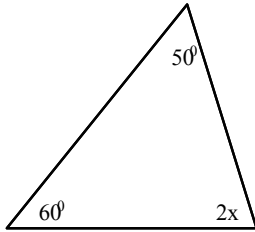
The measure of each interior angle of an n -sided regular polygon

is given by $\frac{(n - 2) \times 180^\circ}{n}$.

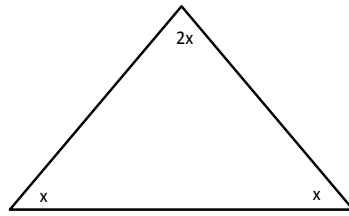
$$\begin{aligned} \text{If } n = 12, \text{ the measure of each interior angle} &= \frac{(12 - 2) \times 180^\circ}{12} \\ &= 150^\circ \end{aligned}$$

Exercise 3.3

1. What is the number of sides of a polygon if the sum of the measures of its interior angles is 2160° ?
2. If the sum of the measures of six interior angles of a heptagon is 820° , find the measure of the remaining angle?
3. Find x :

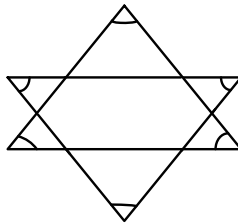


a)



b)

4. What is the sum of the marked angles in the following figure?



5. What is the sum of the measures of the interior angles of a:

a) Quadrilateral	c) Hexagon
b) Pentagon	d) Octagon
6. Find the measure of each interior angle of a regular:

a) Pentagon	c) Hexagon
b) Octagon	d) Decagon

7. How many sides has a regular polygon if each interior angle is:
a) 108° b) 140° c) 156°
8. The measures of the four angles of a pentagon are 110° , 70° , 100° and 110° . What is the measure of the fifth angle?
9. How many sides have a regular polygon, if the measure of each interior angle is $1\frac{3}{4}$ right angles?
10. How many sides have a polygon, if the sum of the measures of its interior angles is 9 straight angles?
11. The sum of two angles of a triangle is 100° and their difference is 40° . Find the measure of each of the three interior angles of the triangle.
12. Is there a regular polygon such that:
a) each interior angle is 1000° ?
b) the sum of the interior angles is 8200° ?

3.1.3 Exterior Angles of a Triangle

Activity 3.4

1. a) What do we mean by an exterior angle?

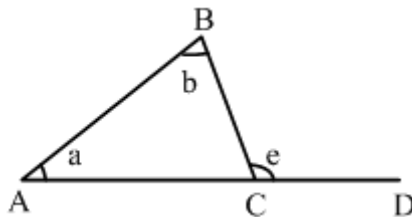


Fig. 3.10

In Fig. 3.10, $\angle BCD$ is an example of an exterior angle.

- b) Do you remember what opposite (remote) interior angles mean?

In Fig. 3.10, $\angle BAC$ and $\angle ABC$ are called **the opposite (remote) interior angles** of the exterior angle $\angle BCD$.

2. a) Draw $\triangle ABC$ on a sheet of paper.
- b) Draw dotted lines as shown in Fig. 3.11 a) and cut along these lines to get the pieces of the paper.
- c) Extend $\overline{BC}$ to draw an exterior angle at C and place $\angle A$ and $\angle B$ over the exterior angle as shown in Fig. 3.11 b).
- d) How do you relate the sum of measures of $\angle A$ and $\angle B$, and the exterior angle at C?

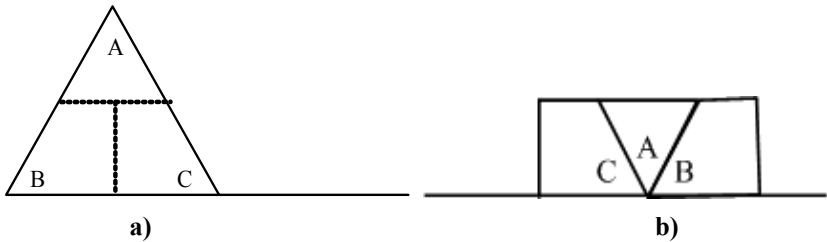


Fig. 3.11

3. a) Draw $\triangle ABC$ on a sheet of paper.
- b) Extend $\overline{BC}$ to draw an exterior angle at C.
- c) Using a protractor, measure $\angle A$ and $\angle B$ and find their sum.
- d) Measure the exterior angle at C.
- e) What is the relationship of the results obtained in steps c) and d)?
- f) Repeat the process for different types of triangles.
- g) What do you observe from these results?

These activities suggest the following theorem.

Theorem 3.2: (Exterior Angle Theorem)

The measure of the exterior angle of a triangle is equal to the sum of the opposite (remote) interior angles.

Proof:

Draw $\triangle ABC$ and extend $\overline{AC}$ to D as shown in Fig. 3.12.

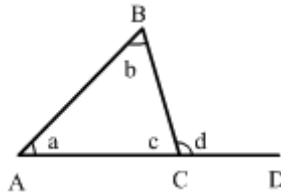


Fig. 3.12

Statement

Reason

- | | |
|----------------------------|-------------------|
| 1. $a + b + c = 180^\circ$ | Angle Sum Theorem |
| 2. $c + d = 180^\circ$ | Straight Line |
| 3. $a + b + c = c + d$ | Transitive |
| 4. $\therefore a + b = d$ | Subtraction |

Example 3.6

In the figure below, find a and b.

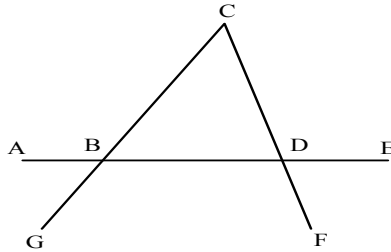


Solution:

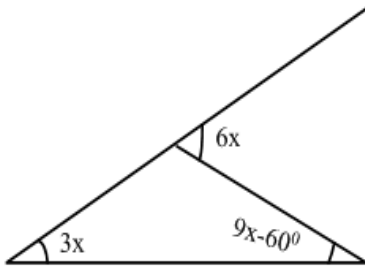
- $a + 110^\circ = 180^\circ$, straight line.
- $a = 180^\circ - 110^\circ = 70^\circ$
- $a + 60^\circ = b$, Exterior Angle Theorem
- $\therefore b = 70^\circ + 60^\circ = 130^\circ$

Exercise 3.4

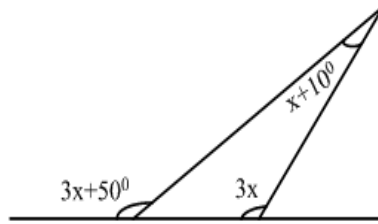
1. Name the exterior angle and the corresponding opposite interior angles in the following figure.



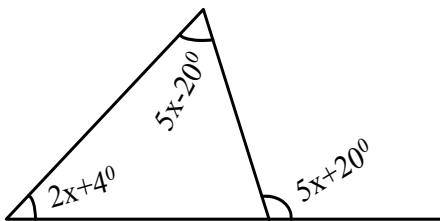
2. Calculate the measure of each marked angle in the following triangles.



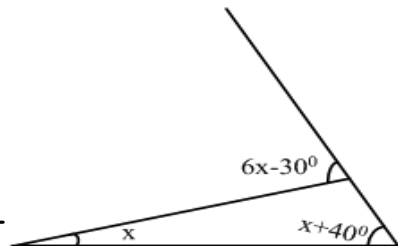
(a)



(b)

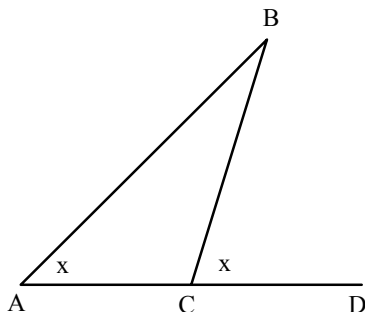


(c)



(d)

3. Explain why the following is an impossible situation.



3.1.4 Sum of the Exterior Angles of Polygons

Activity 3.5

Draw $\triangle ABC$ as shown in Fig. 3.13.

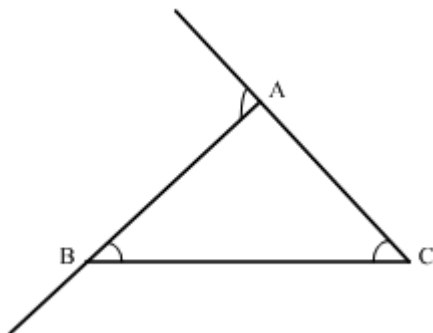


Fig. 3.13

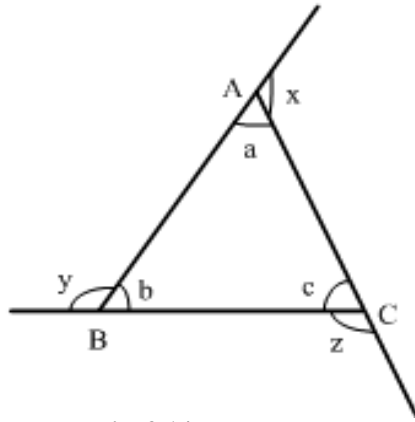
- Extend the sides to form exterior angles of the triangle.
- Using a protractor measure the marked exterior angles. (Use only one exterior angle at each vertex).
- Repeat the process for other triangles.
- Write a conjecture about the sum of the exterior angles of the triangle?

This activity suggests the following theorem.

Theorem 3.3: The sum of the measures of the exterior angles of a triangle is 2 straight angles (360°).

Proof:

Draw $\triangle ABC$ and produce the sides to form exterior angle at each vertex. Look at the angles marked

**Fig. 3.14**

$$(a + x) + (y + b) + (c + z) = 3 \times 180^\circ \quad \text{Why?}$$

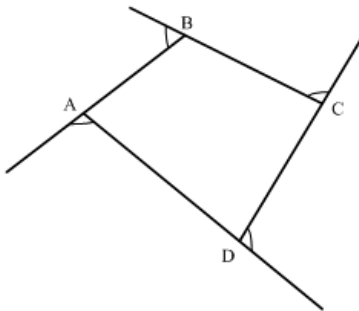
$$(a + b + c) + (x + y + z) = 3 \times 180^\circ \quad \text{Why?}$$

$$a + b + c = 1 \times 180^\circ \quad \text{Why?}$$

$$\text{Therefore } x + y + z = 3 \times 180^\circ - 1 \times 180^\circ = 2 \times 180^\circ$$

Activity 3.6

- a) Draw a quadrilateral ABCD as shown in Fig. 3.15

**Fig. 3.15**

- b) Extend the sides to form exterior angles of the quadrilateral.
c) Using a protractor, measure the marked exterior angles.

- d) What is the sum of the exterior angles of the quadrilateral?
(Use one exterior angle only at each vertex).
- e) Repeat the above processes for a pentagon, hexagon and heptagon, and complete the following table.

Polygon	Sum of exterior angles
Triangle	360°
Qadriateral	
Pentagon	
Hexagon	
Heptagon	

- f) What do you think will be the sum of the exterior angles of a polygon with n sides?

Theorem 3.4: The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is equal to 360° .

Proof:

Let $A_1, A_2, A_3, A_4, A_5 \dots$ be a polygon with n sides.

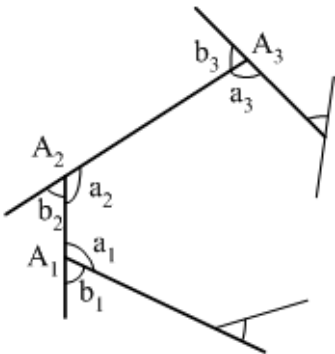


Fig. 3.16

Let the interior angles be denoted by $a_1, a_2, a_3, a_4, a_5, \dots$ and the exterior angles by $b_1, b_2, b_3, b_4, b_5, \dots$

$a_1 + b_1 = 180^{\circ}$, straight angle.

Similarly, $a_2 + b_2 = a_3 + b_3 = \dots = a_n + b_n = 180^\circ$.

Since the polygon has n sides, the sum of the interior and exterior angles $= n \times 180^\circ$.

But the sum of the interior angles $= (n - 2) \times 180^\circ$.

Therefore, the sum of the exterior angles

$$= n \times 180^\circ - (n - 2) \times 180^\circ$$

$$= 2 \times 180^\circ$$

$$= 360^\circ.$$

If a polygon with n sides is regular, then the measure of each

$$\text{exterior angle} = \frac{360^\circ}{n}.$$

Example 3.7

Each exterior angle of a regular polygon is 60° . How many sides does the polygon have?

Solution: Each exterior angle $= \frac{360}{n}$

$$60^\circ = \frac{360}{n}$$

$$n \cdot 60^\circ = 360^\circ$$

$$n = 6$$

$\therefore$ The polygon has 6 sides

Example 3.8

What is the degree measure of an exterior angle of a regular hexagon?

Solution:

Note that a regular hexagon has six equal sides and congruent exterior angles.

$$\text{Measure of an exterior angle} = \frac{360^\circ}{6} = 60^\circ.$$

Exercise 3.5

1. What is the measure of each exterior angle of a regular:
 - a) Quadrilateral
 - b) Pentagon
 - c) Hexagon
 - d) Octagon
2. How many sides has a regular polygon if each exterior angle is:
 - a) 72°
 - b) 120°
 - c) 60°
3. An exterior angle of a triangle is 120° and one of the opposite interior angles 50° . Find the measure of each interior angle of the triangle?
4. One exterior angle of a parallelogram is $\frac{1}{8}$ of the sum of the four exterior angles. Find the measures of each angle of the parallelogram.
5. In a right-angled triangle, one acute angle is twice the other. Find the measure of each of the exterior angles.
6. Is there a regular polygon such that each exterior angles is:
 - a) 90°
 - b) 50°
 - c) 72°
7. The measure of each exterior angle of a regular polygon is twice the measure of each interior angle. How many sides does the polygon have?

3.2 Angles and Arc Relationship of Circles

3.2.1 Angles in a Circle

Activity 3.7

- Do you remember what a circle, chord, diameter and radius mean?
- Name the chords, radii, diameter and centre from Fig. 3.17

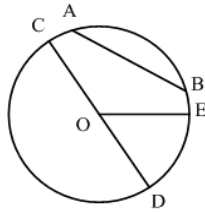
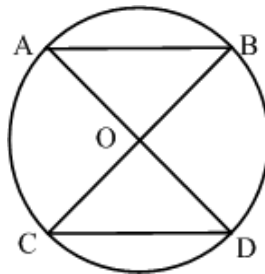


Fig. 3.17

Chord is a line segment joining any two points of a circle. A chord that passes through the centre of a circle is called a **diameter**. Any segment whose endpoints are the centre and a point on the circle is called a **radius**.

Example 3.8

Consider the Fig. below.



- The radii of the circle are $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ and $\overline{OD}$.
- The diameters are $\overline{AD}$ and $\overline{BC}$.
- The chords are $\overline{AB}$, $\overline{DC}$, $\overline{AC}$ and $\overline{BD}$.

Activity 3.8

- Draw a circle with centre O.
- Take any two points A and B on the circle.
- What is the name given to a curve joining point A and point B on the circle?

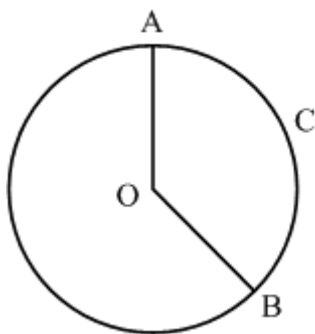


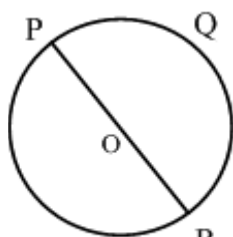
Fig. 3.18

Arc ACB is denoted by $\widehat{ACB}$

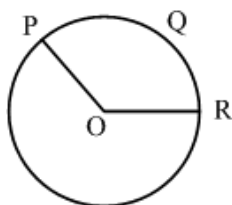
An arc is any portion of a circle

Example 3.10

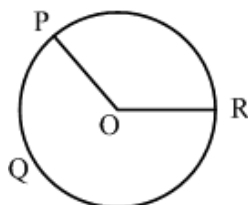
Three different types of arc are illustrated below.



Semi-circle



Minor arc PQR



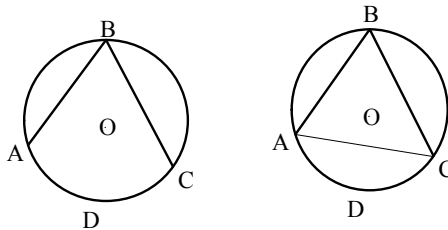
Major arc PQR

Exercise 3.6

- Tell whether the following points are inside, outside or on the circle with diameter 8 cm.
 - A point 3 cm from the centre.
 - A point 4 cm from the centre.
 - A point 5 cm from the centre.
- Sketch a circle with radius 5 cm.
 - Is it possible to draw a chord of length 12 cm?
 - What is the length of the longest possible chord?
 - What is the name of this longest possible chord?
- In a circle with radius 5 cm, two chords $\overline{AB}$ and $\overline{CD}$ have lengths 4 cm and 6 cm, respectively. Which chord is nearer to the centre?

Inscribed and Central Angles**Activity 3.9**

- Draw circles similar to those in Fig. 3.19.
- Take any points A, B, and C on the circles as shown in Fig. 3.19.

**Fig. 3.19**

Then, $\angle ABC$ is called an **inscribed angle** of the circle subtended by the **arc ADC** or the **chord $\overline{AC}$** . **Note:** That the vertex B of $\angle ABC$ is on the circle.

Activity 3.10

1. a) Draw circles similar to those in Fig. 3.20.
- b) Take any points A, D, and C on the circles as shown in Fig. 3.20.

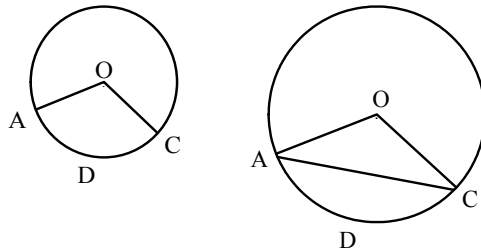


Fig. 3.20

In Fig. 3.20, $\angle AOC$ is called a **central angle** of the circle, subtended by the **arc ADC** or the **chord AC**. The vertex “O” is the centre of the circle.

Activity 3.11

1. a) Draw a circle with centre O
- b) Plot any three different points A, B and C on the circle.
- c) Join the chords $\overline{AB}$ and $\overline{BC}$ as shown in Fig. 3.21.

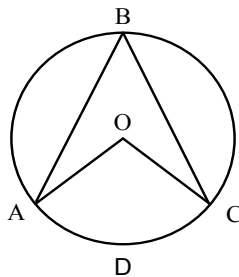


Fig. 3.21

- d) Use a protractor and measure $\angle ABC$ and $\angle AOC$
- e) Measure the central angle subtended by arc ADC.
- f) What relationship do you observe between $\angle ABC$ and the central angle subtended by the measure of arc ADC?

2. a) Draw another circle with centre O.
- b) Take any three points D, E and F on the circle.
- c) Make an angle EDF as shown in Fig. 3.22.
- d) Use a protractor and measure $\angle EDF$.
- e) Measure the central angle subtended by arc EGF.

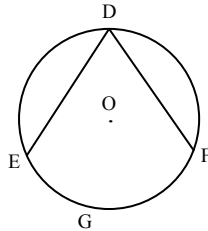


Fig. 3.22

3. What relationship do you observe between the central angle and the inscribed angle subtended by the major arc EGF?
4. Based on Activity 3.11, what do you observe?

Theorem 3.5: If a central angle and an inscribed angle of a circle are subtended on the same arc, then the inscribed angle is half of the central angle.

Proof:

Given $\angle ABC$ inscribed in a circle with centre O and O lies on BC as in Fig. 3.23.

Prove $m(\angle ABC) = \frac{1}{2} m(\angle AOC)$.

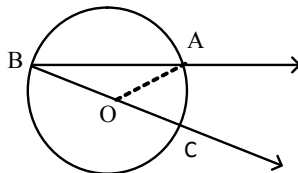


Fig. 3.23

Statement

Reason

1. Draw $\overline{OA}$

Two points determine a line.

2. $OA = OB$

radii of the same circle

3. $m(\angle ABC) = m(\angle BAO)$

If the two sides of a triangle are congruent, then the angles opposite these sides are congruent.

4. $m\angle ABC + m\angle BAO = m(\angle AOC)$

The measure of an exterior angle is the sum of the measures of the two remote interior angles of the triangle.

5. $2 m\angle ABC = m\angle AOC$

Substitution

6. $m\angle ABC = \frac{1}{2} m(\angle AOC)$

Division property.

7. $\angle AOC$ is the central angle subtended on arc AC.

Central angle

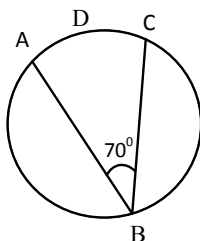
8. $\angle ABC =$ half the central angle subtended on arc AC

Substitution property.

Example 3.11

Find measure of the central angle subtended by the minor arc AC.

Solution:



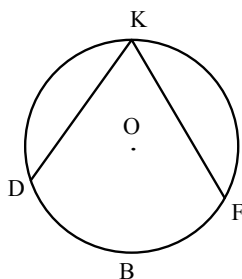
$$m(\angle B) = \frac{1}{2} m(\text{arc ADC})$$

$$m(\text{arc ADC}) = 2 \times 70^\circ$$

$$m(\text{arc ADC}) = 140^\circ$$

Example 3.12

If $m(\text{arc DBF}) = 130^\circ$, find $m(\angle DKF)$.

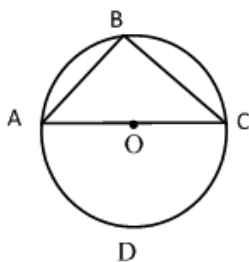


Solution:

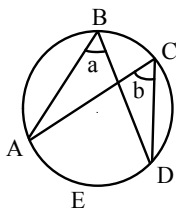
$$m \angle DKF = \frac{m(\text{arc DBF})}{2} = \frac{130^\circ}{2} = 65^\circ$$

Exercise 3.7

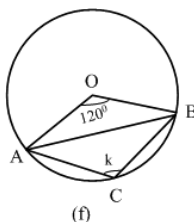
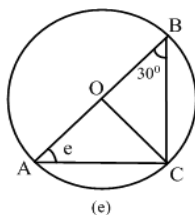
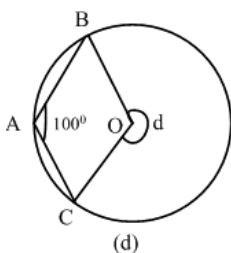
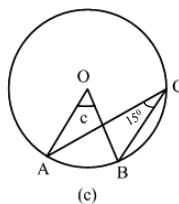
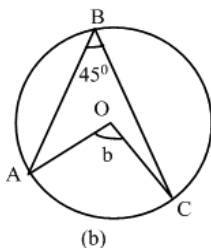
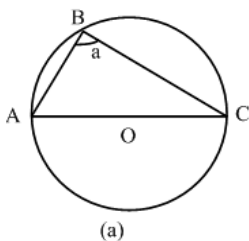
- a. Prove that an angle inscribed in a semi-circle is a right angle.
That is, prove that $m\angle ABC = 90^\circ$.



- b. In the figure below, prove that $a = b$. That is, angles inscribed on the same arc or chord are equal.



c. Find the angles indicated in each of the following figures.



3.2.2 Cyclic Quadrilateral

Activity 3.12

1. a) Draw a circle with centre O of any radius.
 b) Plot any four different points A, B, C, and D on the circle.
 c) Join the chords $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.
 d) Measure each angle of the quadrilateral.
 e) Find $m(\angle A) + (m\angle C)$.
 f) Find $m(\angle B) + m(\angle D)$
2. Repeat the process for other circles with different radii.
3. What relationship do you observe?

A quadrilateral whose vertices lie on a circle or which is inscribed in a circle is called a **cyclic quadrilateral**.

The above activity suggests that:

Theorem 3.6: The opposite angles of a cyclic quadrilateral are supplementary.

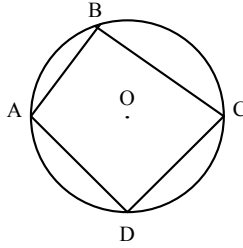


Fig.3.24

Prove: $m(\angle A) + m(\angle C) = 180^\circ$
 $m(\angle B) + m(\angle D) = 180^\circ$

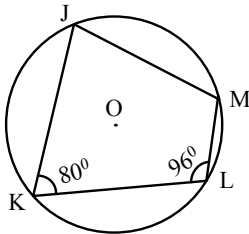
Proof:

Statement	Reason
1. $m(\text{arc } BCD) + m(\text{arc } BAD) = 360^\circ$	Complete circle
2. $m(\angle A) = \frac{1}{2} m(\angle BCD)$	Inscribed angle
3. $m(\angle C) = \frac{1}{2} m(\angle BAD)$	Inscribed angle
4. $m(\angle A) + m(\angle C) = \frac{1}{2} [m(\angle BCD) + m(\angle BAD)]$	Steps 2 and 3
5. $m(\angle A) + m(\angle C) = \frac{1}{2} (360^\circ) = 180^\circ$	Step 1

Similarly, it can be shown that $m(\angle B) + m(\angle D) = 180^\circ$.

Example 3.13

Find $m(\angle J)$ and $m(\angle M)$



Solution:

$$m(\angle J) + m(\angle L) = 180^\circ$$

$$m(\angle J) + 96^\circ = 180^\circ$$

$$m(\angle J) = 180^\circ - 96^\circ$$

$$m(\angle J) = 84^\circ$$

$$m(\angle K) + m(\angle M) = 180^\circ$$

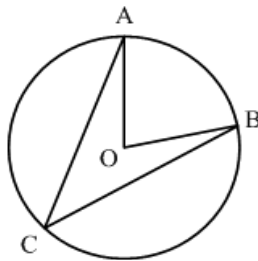
$$80^\circ + m(\angle M) = 180^\circ$$

$$m(\angle M) = 180^\circ - 80^\circ$$

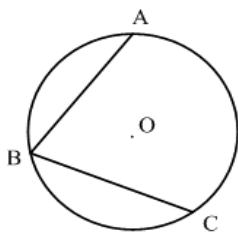
$$m\angle M = 100^\circ$$

Exercise 3.8

1. Draw an arc of 70° in a circle of radius 5cm?
2. In a circle with centre O, $\angle POQ$ is a 70° central angle. How big is the minor arc PQ?
3. In a circle with centre O, $m(\text{minor arc AB}) = 80^\circ$. Find
 - a) $m(\angle AOB)$
 - b) $m(\angle ACB)$



4. If $m(\angle ABC) = 65^\circ$, what is the measure of the central angle subtended by arc ABC?



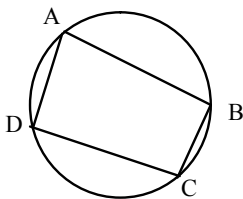
5. Let A, B, C and D be points on a circle. If the diameter of the circle is $\overline{AB}$ and $m(\angle ADC) = 127^\circ$, find $m(\angle BAC)$.

6. In a circle, if $\overline{AC}$ is a diameter of a circle, C and D are points on the circle such that $m(\angle BDC) = 25^\circ$, find $m(\angle ACB)$.

7. Quadrilateral ABCD is inscribed in a circle.

If $m(\angle ADC) = 70^\circ$ and $m(\angle ACD) = 50^\circ$, find $m(\angle CBD)$.

8. In the figure below, $m(\angle DAB) = 75^\circ$, find $m(\angle BCD)$.



Activity 3.13

1. a) Draw a circle with centre O.
- b) Plot any four different points on the circle.
- c) Construct any two chords using the points on the circle intersecting at a point, as shown in Fig. 3.25.

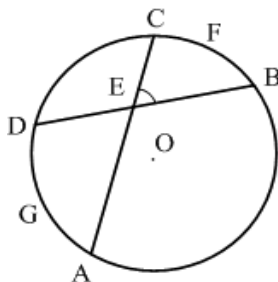


Fig. 3.25

- d) Use a protractor and measure $\angle COB$.
 - e) Measure the central angles subtended by arc CFB and DGA.
2. What relationship do you observe between angle COB and the arc CFB?
 3. Consider the circle below.
 - a) Use a protractor and measure of $\angle AEB$.
 - b) Measure the arcs AKB and DMC.

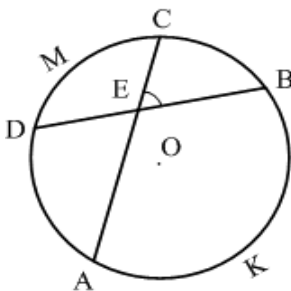


Fig. 3.26

4. What relationship do you observe between $\angle AEB$ and Measures of arcs AKB and DMC?
5. What is your conclusion?

Theorem 3.7: The measure of the angle formed by two lines intersecting inside a circle is half the sum of the measures of the arcs subtending the angle and its vertical opposite.

Proof:

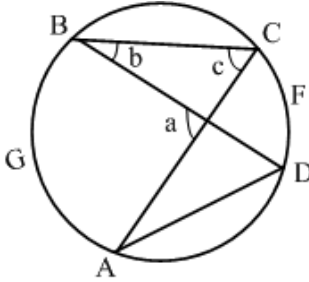


Fig. 3.27

Given: Secants BD and AC.

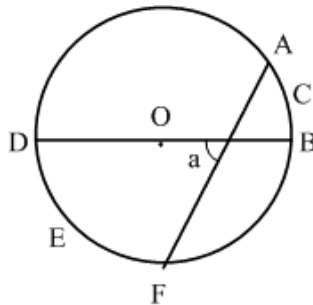
Prove that $a = \frac{1}{2} [m(\text{arc CFD}) + m(\text{arc AGB})]$.

Draw $\overline{BC}$

Statement	Reason
1. $a = b + c$	Exterior angle theorem
2. $b = \frac{1}{2} m(\text{arc CD})$, and $c = \frac{1}{2} m(\text{arc AGB})$	The measure of inscribed angle = Half the measure of the intercepted arc.
3. $a = \frac{1}{2} m(\text{arc CFD}) + \frac{1}{2} m(\text{arc AGB})$	Substitution
4. $a = \frac{1}{2} [(m(\text{arc CFD}) + m(\text{arc AGB}))]$	Distributive property

Example 3.14

In the following figure, measure of arc $ACB = 35^\circ$ and measure of arc $DEF = 65^\circ$. Find a .



$$\begin{aligned} \text{Solution: } a &= \frac{65^\circ + 35^\circ}{2} \\ &= 50^\circ \end{aligned}$$

3.2.3 Properties of Tangents in Circles

Activity 3.14

1. a) Draw a circle with centre O.
- b) Plot a point C on the circle.
- c) Construct a line AB through C as shown in Fig. 3.28.

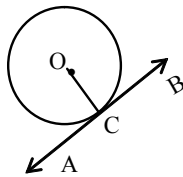


Fig. 3.28

- d) Draw any radius of the circle.
- e) Draw a radius $\overline{OC}$.

2. What relationship do you observe between the radius and $\overline{AB}$?
3. Measure angle OCB and angle OCA
4. What relationship do you observe between angle OCB and the line AB?

A tangent line to a circle is a line in the plane of the circle, which intersects the circle at one and only one point. The point at which the tangent intersects is called the **point of tangency** or the point of contact.

A tangent line is perpendicular to the radius at the point of contact.

Tangent Chord Property

Theorem 3.8: The angle between a tangent and a chord is equal to the angle subtended on the opposite sides of the chord.

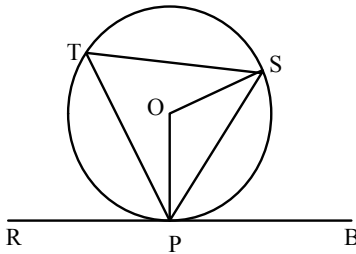


Fig. 3.29

Proof

Statement

Join OS, OP

Let $m(\angle OPS) = x$

1. Since OP is radius, OP is perpendicular to RB
2. Therefore $m(\angle SPB) = 90^\circ - x$
3. In $\triangle OPS$, $OP = OS$
4. Then $m(\angle OPS) = m(\angle OSP)$
5. $m(\angle POS) = 180^\circ - m(\angle OPS) - m(\angle OSP)$
6. $m(\angle POS) = 180^\circ - (90^\circ - x) - (90^\circ - x)$
7. $m(\angle POS) = 2x$
8. $\angle PTS$ is an inscribed angle, and
9. $\angle POS$ is a central angle on the same chord PS
10. Therefore, $m(\angle PTS) = x$

Reasons

Property of tangent

$m(\angle OPB) = 90^\circ$
radii

Isosceles triangle

Sum of the angles of a triangle

Property of inscribed angle

Therefore, $m(\angle PTS) = m(\angle SPB)$. This is what was required to be proved.

Activity 3.15

1. a) Draw a circle with centre O.
- b) Plot any four different points on the circle.
- c) Construct any two chords using the points on the circle intersecting at a point P outside the circle as shown in Fig. 3.30.

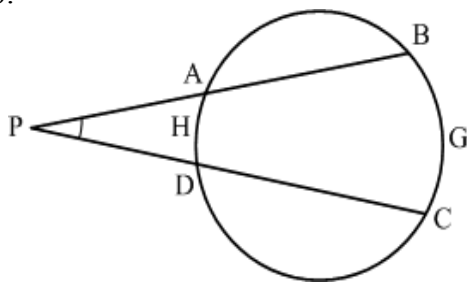


Fig. 3.30

- d) Measure arc BGC and arc AHD
 - e) Find the difference of the measures of arc BGC and arc AHD
 - f) Measure $\angle BPC$.
2. What relationship do you observe between angle $\angle BPC$, and the difference of the measures of arcs BGC and AHD?

Activity 3.16

1. Draw a circle with centre O.
2. Take two parallel chords on the circle as shown in Fig 3.31.

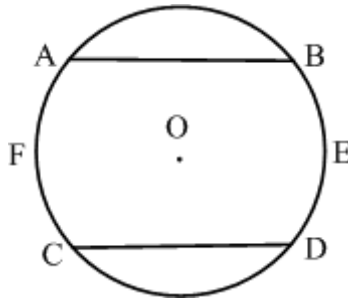


Fig. 3.31

3. Measure arc AFC and BED.
4. What is your conclusion?

Arcs intercepted by parallel chords are congruent.

Activity 3.17

1. a) Draw a circle with centre O.
- b) Take any two segments intersecting outside the circle as in figure 3.32.
- c) Construct a segment AE parallel to DC

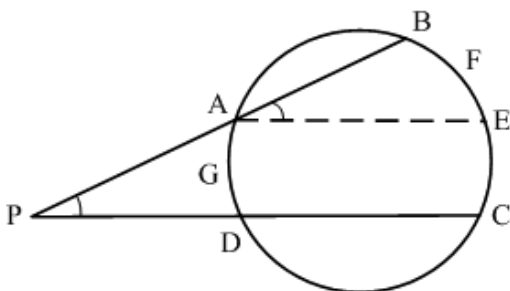


Fig. 3.32

2. What relationship do you observe between $\angle BAE$ and measure of arc BFE?
3. What relationship do you observe between $m(\angle BPC)$ and (arc BFE)?

Theorem 3.9: The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measures of the arcs they intercept.

Proof:

Construct a chord $\overline{AE}$ parallel to $\overline{DC}$

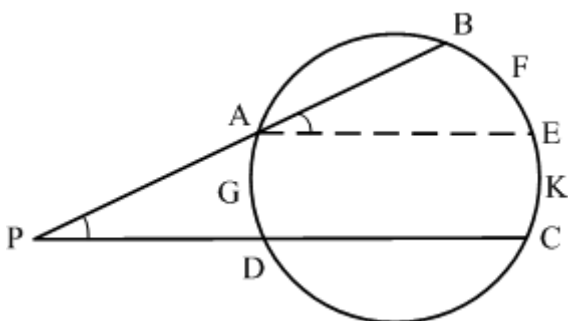


Fig. 3.34

Statement

1. $m(\angle BAE) = \frac{1}{2}m(\text{arc BFE})$,
2. $m(\text{arc BFE}) = m(\text{arc BEC}) - m(\text{arc CKE})$
3. $m(\text{arc BFE}) = m(\text{arc BEC}) - m(\text{arc AGD})$
4. $m(\angle BAE) = m(\angle BPC)$.

Reason

$\angle BAE$ is subtended by arc BFE.

arc CKE and arc AGD are subtended by parallel chords.

$\overline{AE}$ is parallel to $\overline{DC}$ corresponding angles

$$5. m(\angle BPC) = \frac{1}{2}m(\text{arc BFE})$$

Step 1 and 4

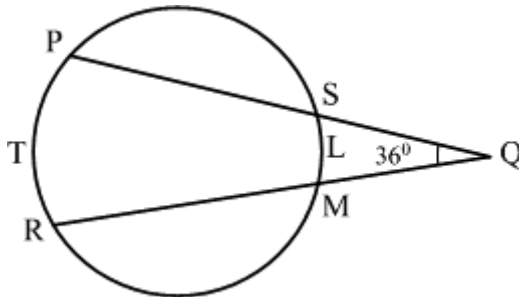
$$6. m(\angle BPC) = \frac{1}{2}(\text{Measure of arc BEC} \\ - \text{Measure of arc AGD})$$

Step 3 and 5

Example 3.15

In the figure below, $m(\text{arc PTR}) = 120^\circ$ and $m(\angle PQR) = 36^\circ$. Find $m(\text{arc SLM})$.

Solution:



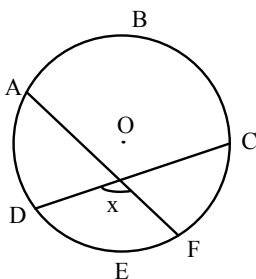
$$m(\angle PQR) = \frac{m(\text{arc PTR}) - m(\text{arc SLM})}{2}$$

$$= \frac{120^\circ - m(\text{arc SLM})}{2}$$

$$\begin{aligned} m(\text{arc SLM}) &= m(\text{arc SLM}) = 120^\circ - 2 \times 36^\circ \\ &= 48^\circ \end{aligned}$$

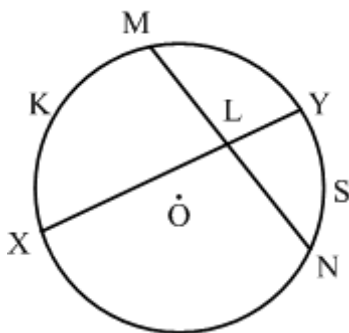
Exercise 3.9

1. If $m(\text{arc ABC}) = 150^\circ$ and $m(\text{arc DEF}) = 120^\circ$, find x .

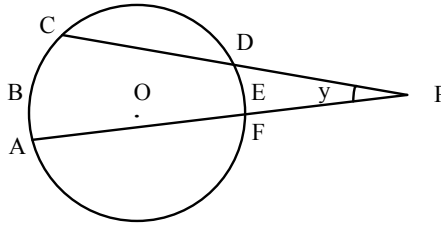


2. Two chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at right angle at a point inside the circle. If $m \angle BAC = 35^\circ$, find $m \angle ABD$.

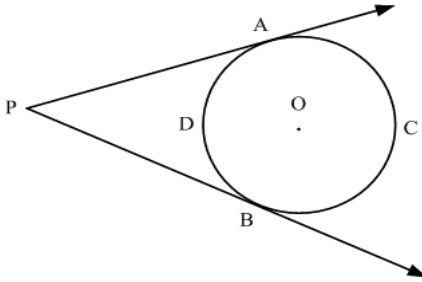
3. If MKX is an arc of 43° and YSN is an arc of 25° , what is the degree measure of: a) $\angle \text{XLM}$? b) $\angle \text{XLN}$



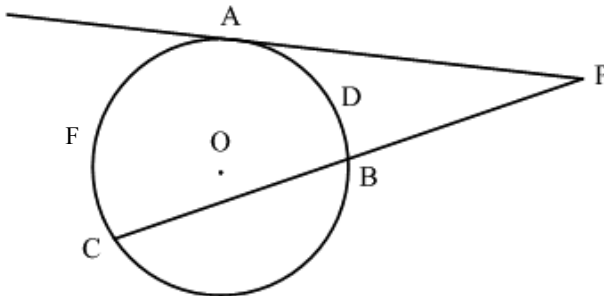
4. If $m(\text{arc } ABC) = 82^\circ$ and $m(\text{arc } DEF) = 40^\circ$, Find y .



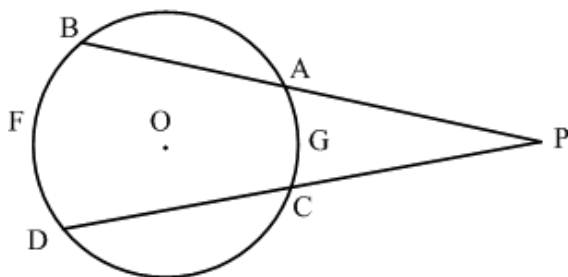
5. If $m(\text{arc } ADB) = 128^\circ$, find $m(\angle APB)$.



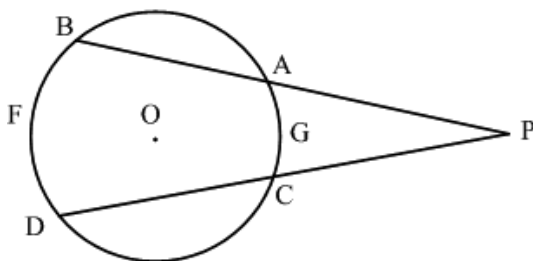
6. In the figure below, $m(\angle APB) = 40^\circ$, $m(\text{arc } ADB) = 35^\circ$. Find the $m(\text{arc } AFC)$.



7. In the figure below, $m(\text{arc } BFD) = 160^\circ$, $m(\angle APC) = 25^\circ$. Find $m(\text{arc } AGC)$.

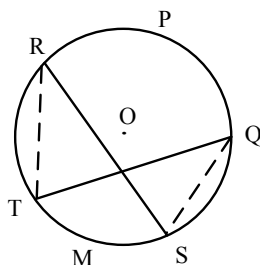


8. In the figure below, $\angle APC = 60^\circ$, $m(\text{arc } BFD) = 3x + 80$ and $m(\text{arc } AGC) = x + 20$. Calculate the value of x .



9. The angle formed by two tangents drawn to a circle from the same external point measures 80° . Find the measure of the smaller of the intercepted arcs.

10. In the figure below, which measures of angles are equal?



3.3 Secants, Tangent and Chords: Properties

3.3.1 Lines intersecting Inside the Circle

Activity 3.18

1. a) Draw a circle.
- b) Plot any four different points on the circle.
- c) Construct two chords $\overline{RS}$ and $\overline{TQ}$ using the points on the circle intersecting at a point E inside the circle as shown in Fig. 3.35.

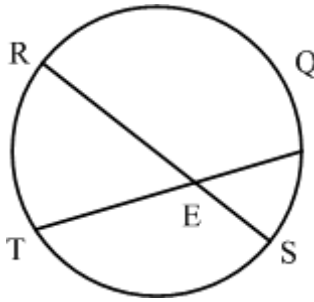


Fig. 3.35

- d) Use your ruler and measure the length of $\overline{RE}$, $\overline{ES}$, $\overline{TE}$ and $\overline{EQ}$.
- e) Calculate the product of (RE) (ES) and (TE)(EQ).
2. Repeat task (1) of Activity 3.18.
3. What relationship do you observe?

If two chords TQ and RS, intersect at E, then
 $TE \times EQ = RE \times ES$.

3.3.2 Intersection of a Tangent Line and a Chord

Activity 3.19

1. a) Draw a circle.
- b) Take any two points on the circle.
- c) Construct a line through these points.
- d) What is the name given to the line?

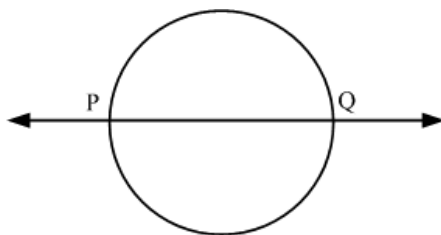


Fig. 3.36

A line that intersects a circle in two points is called a secant of the circle.

2. Consider the figure below.

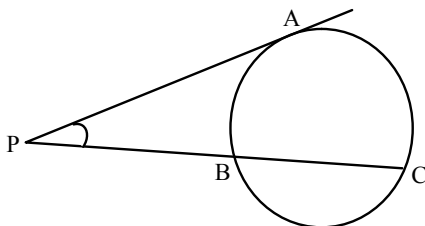


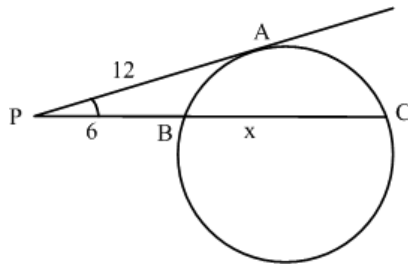
Fig. 3.37

- Use a ruler and measure the length of $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$.
- Calculate the product of $(PA)(PA)$ and $(PB)(PC)$.

If a tangent from an external point P meets a circle at A, and a secant from the external point P meets the circle at points B and C, then $(PA)^2 = PB \times PC$.

Example 3.16

Find the value of 'x' in the following diagram.



Solution:

$$(PA)^2 = (PB)(PC)$$

$$12^2 = 6(x + 6)$$

$$6x + 36 = 144$$

$$6x = 144 - 36$$

$$6x = 108$$

$$x = 18$$

3.3.3 Chords Intersecting Outside of a Circle

Activity 3.20

Consider the figure below.

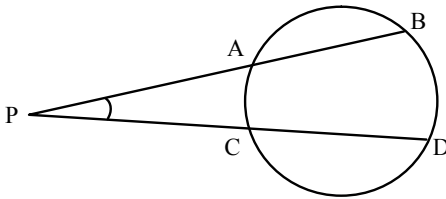


Fig. 3.38

- Use a ruler and measure the lengths of PA, PB, PC and PD
- Calculate the product of (PA) (PB) and (PC)(PD).
- What is your conclusion?

Theorem 3.10: If two chords $\overline{PB}$ and $\overline{PD}$ intersect the circle at points A and C respectively, then $(PA)(PB) = (PC)(PD)$.

Proof:

Consider the figure below

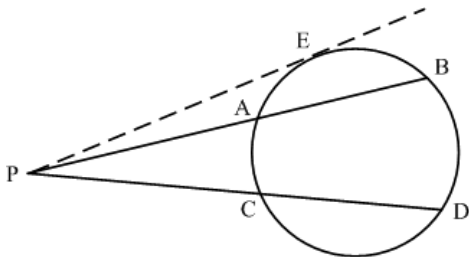


Fig. 3.39

Let $\overline{PE}$ be a tangent line at a point E.

Statement

Reasons

$$(PE)^2 = PA \times PB$$

Tangent secant theorem

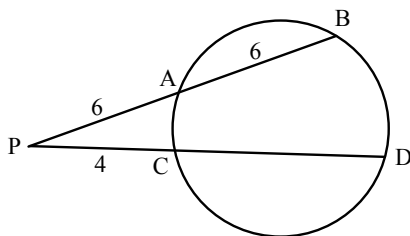
$$(PE)^2 = PC \times PD$$

Tangent secant theorem

$$PA \times PB = PC \times PD$$

Example 3.17

In the figure below, $PA = 6$, $AB = 6$ and $PC = 4$. Find CD.



Solution:

$$(PA)(PB) = (PC)(PD)$$

$$6 \times 12 = 4(x + 4)$$

$$4x + 16 = 72$$

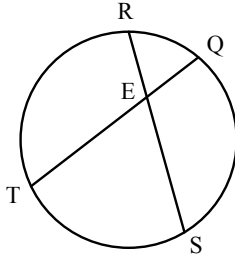
$$4x = 72 - 16$$

$$4x = 56$$

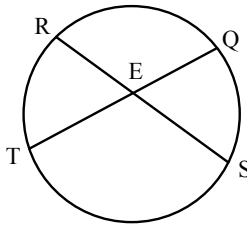
$$x = 14$$

Exercise 3.10

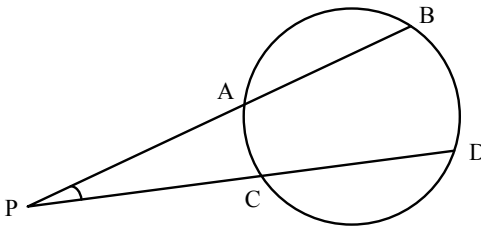
1. Find TE, if $RE = 7$, $ES = 8$ and $EQ = 4$.



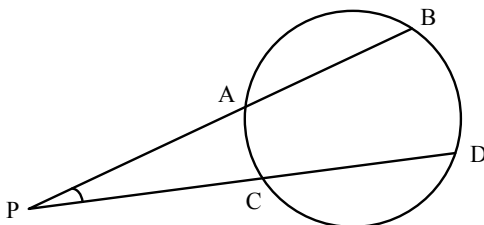
2. Find TE, if $RE = 4$, $ES = 6$ and $EQ = 3$.



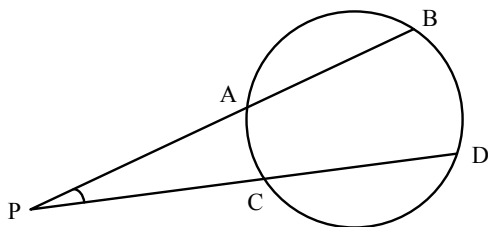
3. In the figure below, find PB, if $PA = 5$, $PC = 6$ and $DC = 4$.



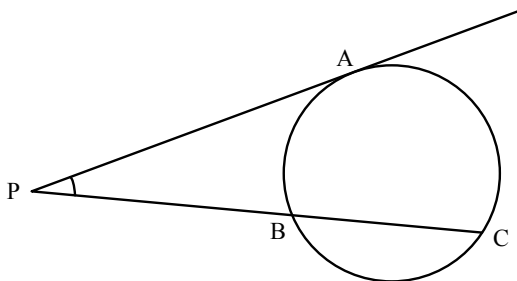
4. In the figure below, find PA, if $PB = 10$, $PC = 5$ and $DC = 3$.



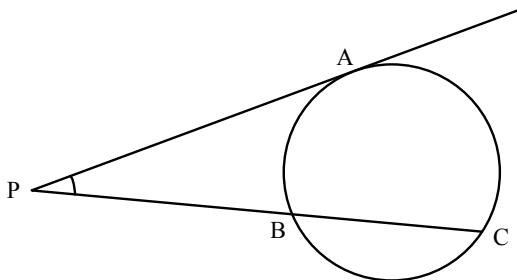
5. In the figure below, find AB, if $PB = 20$, $PC = 10$ and $DC = 6$.



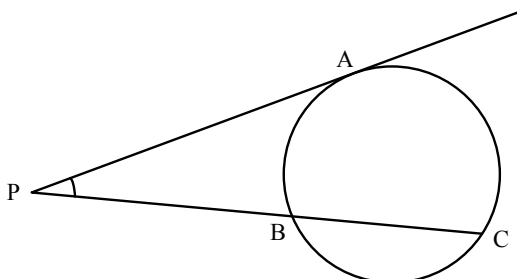
6. In the figure below, find PB if $PA = 6$ and $PC = 9$.



7. In the figure below, find PA, if $PB = 4$ and $BC = 5$.



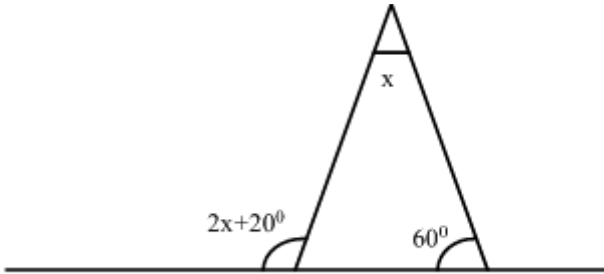
8. In the figure below, find PB, if $PA = 6$ and $BC = 5$.



Review Exercise 3

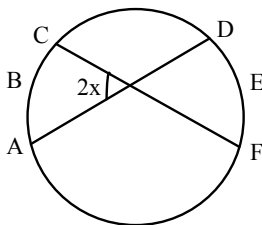
- Two angles of a triangle measure 40° and 65° . What is the measure of the third angle? What type of triangle is it?

- In the figure, find the value of x .



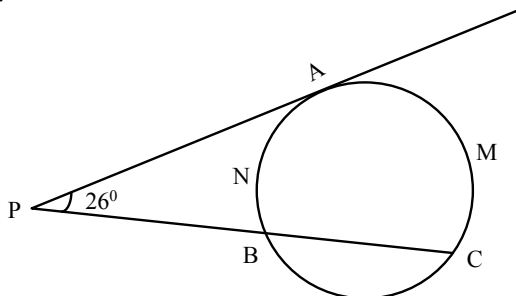
- Find the sum of the interior angles of a 15- sided polygon.
- The angles of a quadrilateral are related so that the second is twice the first, the third is three times the first, and the fourth is four times the first. Find the measures of each interior angle?
- Find the smallest measure of degrees that an interior angle of a regular polygon may contain.
- Find the number of sides of a polygon whose sum of its interior angles is 1800° ?
- Each interior angle of a regular polygon is 140° . How many sides does the polygon have?
- If the sum of the measures of the interior angles of a polygon is equal to the sum of the measures of its exterior angles, what is the number of sides of the polygon?
- If the sum of the measures of the interior angles of a polygon is three to the sum of the measures of its exterior angles, what is the number of sides of the polygon?
- If the sum of the measures of the interior angles of a polygon is twice the sum of the measures of its interior angles of a hexagon, what is the number of sides of the polygon?

11. If $m(\text{arc } ABC) = 2x + 11^\circ$ and $m(\text{arc } DEF) = x + 10^\circ$, find the value of x .



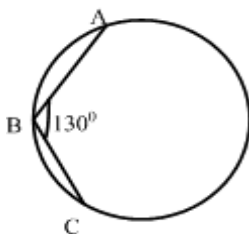
12. Two secants intersect outside a circle to form an angle whose measure is 40° . If the measure of the larger intercepted arc is 110° , find the measure of the smaller intercepted arc.

13. In the figure below, a tangent and a secant to a circle intersect outside the circle to form an angle whose measure is 26° . If $m(\text{arc } ABC) = 80^\circ$, find the measure of the other intercepted arc.

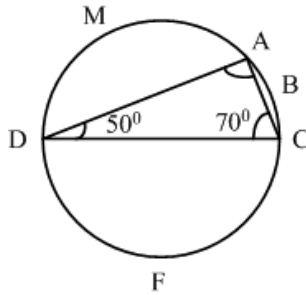


14. Chords $\overline{PQ}$ and $\overline{RS}$ of a circle intersect at A. If $PA = 12$, $QA = x$, $RA = x + 1$, and $SA = x + 6$, find PQ and RS .

15. If $m(\angle ABC) = 130^\circ$, what is the measure of arc AC ?



16. Two secants $\overline{AB}$ and $\overline{CD}$ of a circle with centre O intersect outside of the circle at K. If $m(\angle KAD) = 31^\circ$ and $m(\angle AKC) = 42^\circ$, find $m(\angle AOC)$.
17. In the figure below, quadrilateral ABCD is inscribed in a circle. If $m(\angle ADC) = 50^\circ$ and $m(\angle ACD) = 70^\circ$, find $m(\text{arc DFC})$.



Chapter 4

Dimensionality and Geometry of Location

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- analyse the relationship between three-dimensional objects and related two-dimensional representations;
- use the representations to solve problems;
- recognise that the coordinate systems provide convenient and efficient ways of representing figures;
- solve geometrical problems analytically
- use properties of transformations and their compositions to make connections between mathematics and the real world in applications.

4.1 Plane Figures and Solid Figures

4.1.1 Polyhedrons

Activity 4.1

a) Identify the following figures.

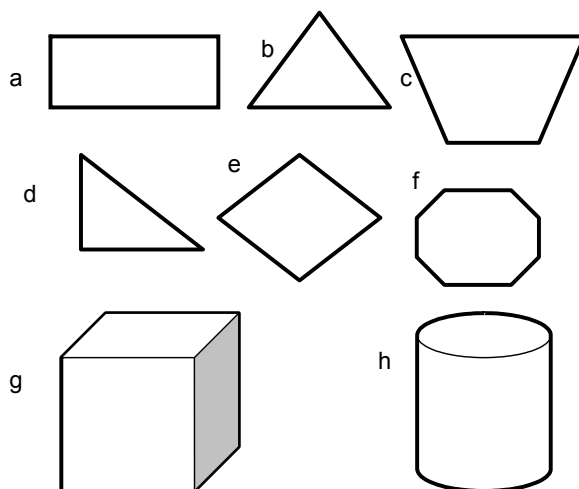


Fig. 4.1

Plane figures have length and width and therefore they are called **two-dimensional** figures. A solid object has length, width, and height. They are called **three-dimensional** figures. Also a solid object occupies some space.

A solid is an enclosed portion of space bounded by plane or curved surfaces.

Each of the solids in Fig. 4.2 is made up of polygonal regions which are called its faces, these faces meet at edges which are line segments, and these edges meet at vertices which are points. Such solids are called **polyhedrons**.

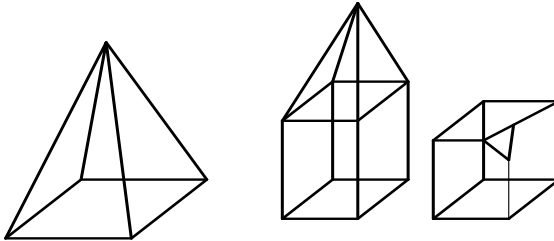


Fig. 4.2

Activity 4.2

1. Which of the following solid figures are polyhedrons?

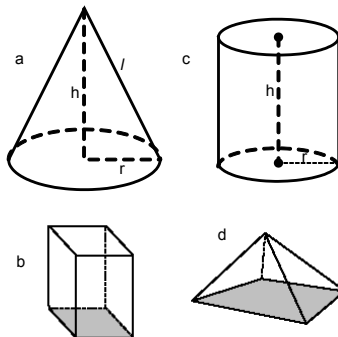


Fig. 4.3

2. In what way are polyhedrons different from non polyhedrons?
3. Is a prism a polyhedron?
4. Is a pyramid a polyhedron?
5. Which of the figures in Fig. 4.4 is a regular polyhedron?
Assume the faces are congruent.

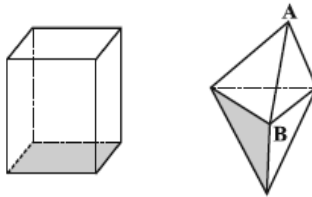


Fig. 4.4

A polyhedron is a solid bounded by plane (flat) surfaces only.

Example 4.1

The following four solid figures are polyhedrons.

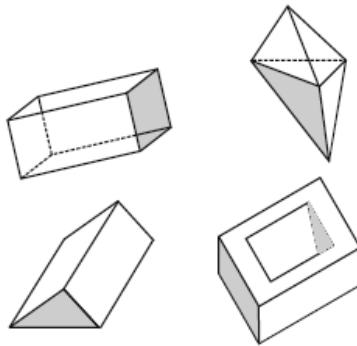


Fig. 4.5

A polyhedron is regular if its faces are made up of regular polygons and the same number of faces meet at each vertex.

Activity 4.3

- What is a prism?
- What are the bases of a prism?
- What is a right prism?

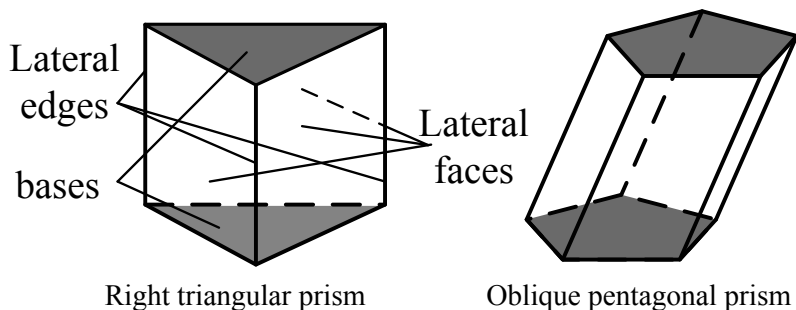


Fig. 4.6

A **rectangular solid** is a prism bounded by six rectangles. A **cube** is a rectangular solid bounded by six squares.

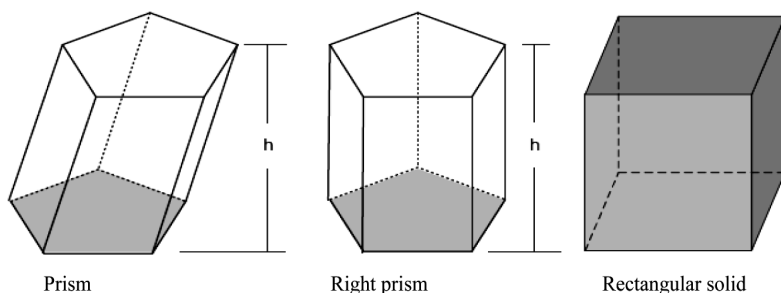


Fig. 4.7

A **cross section** of a prism is the intersection of the prism with a plane parallel to the plane of the base.

Theorem 4.1

All the cross sections of a triangular prism are congruent to the base.

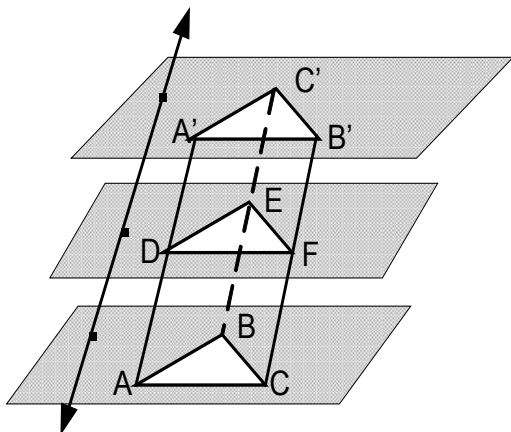


Fig. 4.8

Theorem 4.2

The upper and lower bases of a triangular prism are congruent.

Theorem 4.3

All cross sections of a prism have the same area.

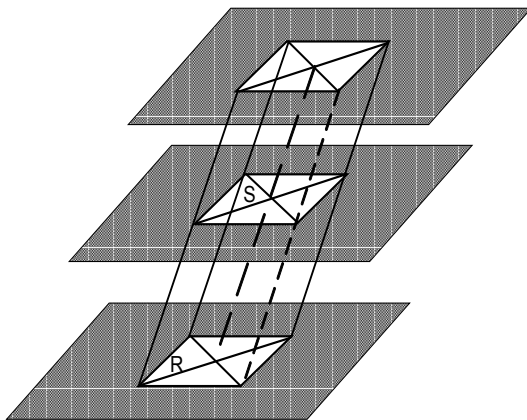


Fig. 4.9

Note: The bases of a prism have the same area. This is because the upper base is a cross section.

The union of the lateral faces of a prism is called its **lateral surface**.

The union of the lateral faces of a prism and its two bases is called **total surface** of the prism.

Activity 4.4

- What is a triangular pyramid?
- What is a square pyramid?
- What is a pentagonal pyramid?
- Can we say a pyramid is named after its base?
- What is a regular pyramid?
- What is a frustum of a pyramid?

A pyramid has a **base** and triangular sides having a common vertex. The base may be any polygon such as a square, rectangle, triangle, etc.

The common vertex is called the **apex** of the pyramid.

The triangular faces that make up the sides are called the **lateral faces** of the pyramid. These lateral faces intersect in line segments called the **lateral edges**.

The line segment from the apex perpendicular to the base is the **altitude** and its length is the **height** of the pyramid.

The lengths of the altitudes of the lateral faces drawn from the apex are called the **slant heights**.

Fig. 4.10 shows a rectangular pyramid.

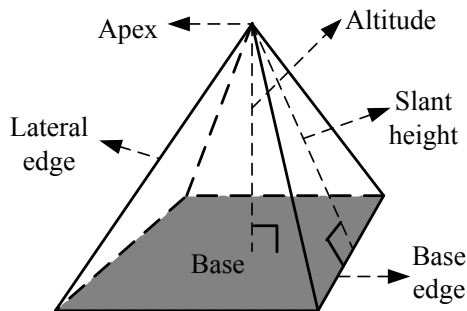


Fig. 4.10

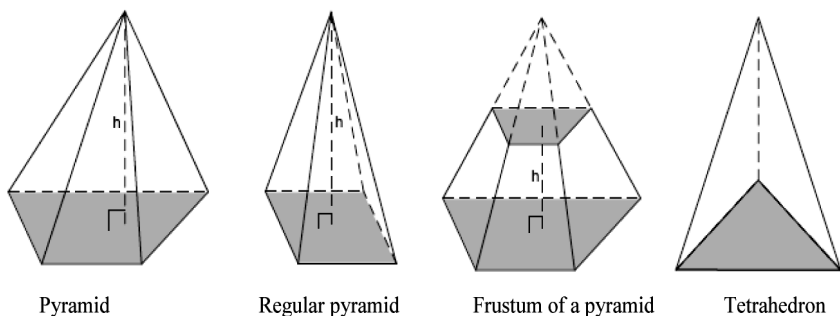


Fig. 4.11

A **frustum** of a pyramid is the part of a pyramid which remains if the top of the pyramid is cut off by a plane parallel to the base. Its lateral faces are trapezoids.

Fig. 4.12 shows a frustum of a rectangular pyramid

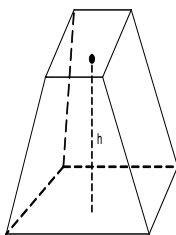


Fig. 4.12

A **tetrahedron** is a pyramid whose base is a triangle. That is a tetrahedron is a pyramid composed of four triangular faces three of which meet at each vertex.

Exercise 4.1

1. Can a polyhedron have for its faces:
 - a) 3 triangles?
 - b) 4 triangles?
 - c) A square and four triangles?
2. Is it possible to have a polyhedron of any given number of faces?

3. How are prisms and cylinders alike?
4. How are pyramids and cones alike?
5. Is a square prism the same as a cube? Explain

4.1.2 Cylinders, Cones and Spheres

Activity 4.5

1. What is a cylinder?
2. Why is it incorrect to call the solid in Fig 4.13 a cylinder?

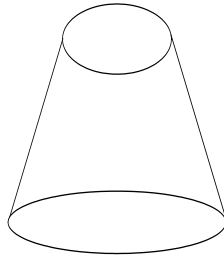


Fig. 4.13

3. How do prisms and cylinders relate to one another?
4. Consider a cylinder standing vertically on its base.
Take a horizontal cross-section of the cylinder.
 - a) What is the shape of the cross section?
 - b) How does the shape relate to the base of the cylinder?

Activity 4.6

1. What is a cone?
2. How do pyramids and cones relate to one another?
3. Take a cross section of a cone.
Describe the two resulting figures.

A cylinder has two parallel circular **bases** of the same size and a **lateral surface** that rises from one base to the other as shown in Fig. 4.14 If the line segment that joins the centres of the bases is perpendicular to the diameter of each base, then the cylinder is called a **right cylinder**.

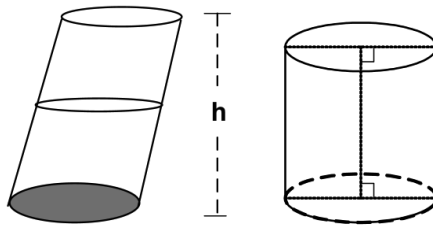


Fig. 4.14

A **circular cone** is a solid whose base is a circle and whose lateral surface come to a point.

Note that a circular cone is referred as a cone.

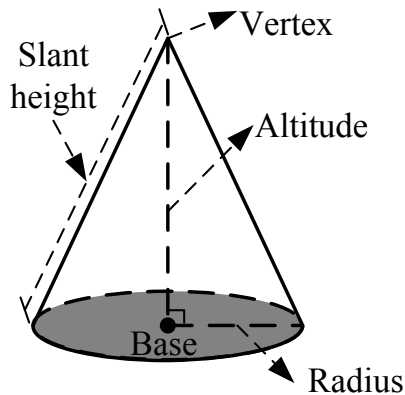


Fig. 4.15

If a plane parallel to the base intersects a cone, not at the vertex, the part of the cone bounded by the base, the cross-section, and the portion of the lateral surface between the plane is called a frustum of the cone.

Fig. 4.16 shows a frustum of a cone

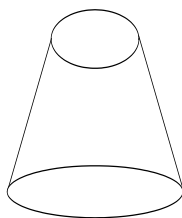


Fig. 4.16

Sphere

Activity 4.7

- What is a sphere?
- Is it the same as circle?
- Can you draw a circle on a paper?
- If you paste a string along a diameter of a disc and rotate it you see a new solid. What does it resemble?
- Can you guess what happens to the centre of the circle, when it forms a sphere on rotation?

A sphere is a solid such that every point on its surface is at an equal distance from the same point its centre.

A sphere is formed by revolving a semicircle about its diameter as an axis. The outer end of the radius perpendicular to the axis generates a great circle while the outer ends of the other chords perpendicular to the diameter generate small circles.

Therefore, in Fig.4.17, the sphere O is formed by the rotation of

semicircle $\overline{ACB}$ about $\overline{AB}$ as an axis. In the process, point C generates a great circle while points E and G generate small circles.

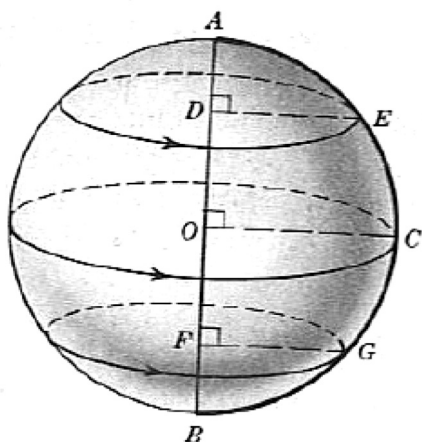


Fig. 4.17

Given a sphere of radius r ; the horizontal cross sections are circular regions. If the cross section is at a distance s from the center, and its radius is t , then we know from Pythagoras' Theorem that $t^2 = r^2 - s^2$.

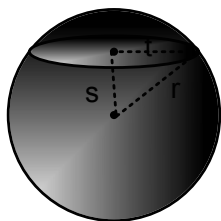


Fig. 4.18

Therefore the area of the cross section at distance s unit is $\pi(r^2 - s^2)$. This formula has a geometric meaning. It is the area of the ring shaped lying inside a circle of radius r and outside a circle of radius s , as shown in Fig. 4.19. A figure such as this is called an **annulus**.

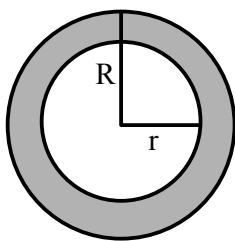


Fig. 4.19

4.1.3 Nets and Views of Solids

Activity 4.8

1. What is a net of a solid?
2. How many faces does a cube have?
3. Can you find a real-life example of a cube?
4. The easiest way to find a net is to think of a cube as four sides, a top and a bottom. Arrange four squares in a line. These are the sides. Now put the top square on one side of this line, and the bottom on the other. It does not really matter where on each side, they all work. How many arrangements are there apart from rotations and reflections? **Note:** That here are other layouts which work.

A net is a two dimensional figure which can be turned into a solid figure by folding it.

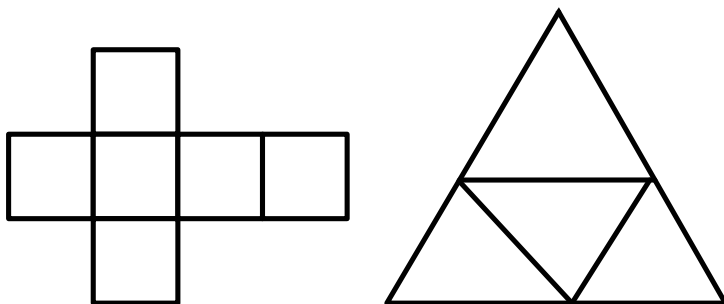


Fig. 4.20

The illustrations in Fig. 4.20 show nets for the cube and tetrahedron. Polyhedron nets are not unique. For example, there are a total of 11 distinct nets for cube.

Activity 4.9

1. The following are nets of a cube. Eight of them are given. Draw the other three.

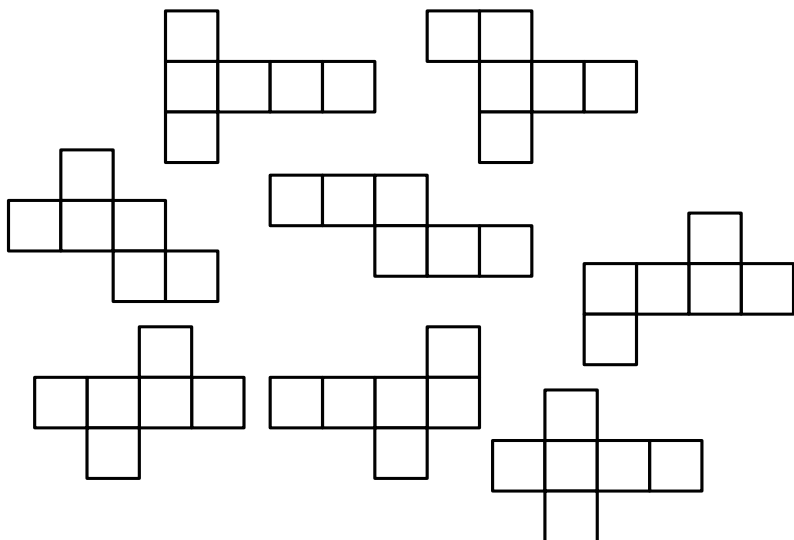


Fig. 4.21

2. How many nets are there for tetrahedron?

Example 4.2

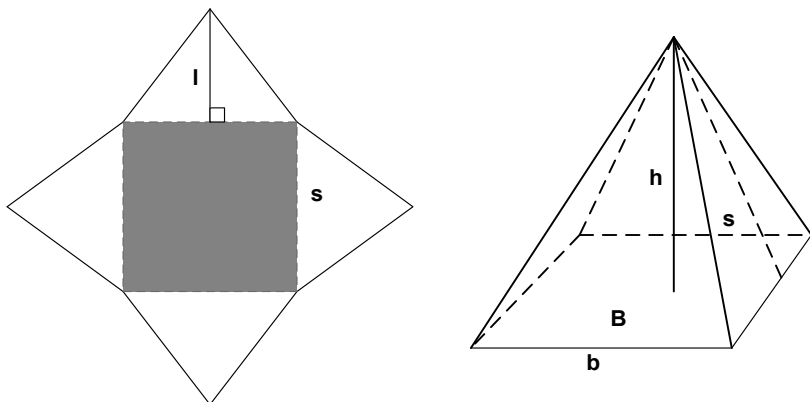


Fig. 4.22

Activity: 4.10

1. Complete the following table.

Solids	Number of faces (F)	Number of edges (E)	Number of vertices (V)
Rectangular solid			
Square pyramid			
Hexagonal prism			
Triangular pyramid			

1. What do you infer from the table?
2. Do you find $F+V-E = 2$?

The relationship between the number of faces, edges, and vertices is given by $F+V-E = 2$. This is called **Euler’s Formula**.

Activity 4.11

1. Draw the top, front, and side view of the hut given in Fig. 4.23

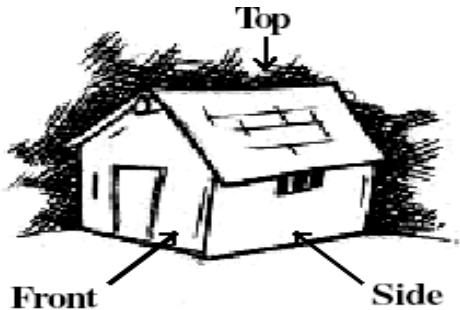


Fig. 4.23

2. Identify the solid figure by considering the different views given in Fig. 4.24 and Fig 4.25

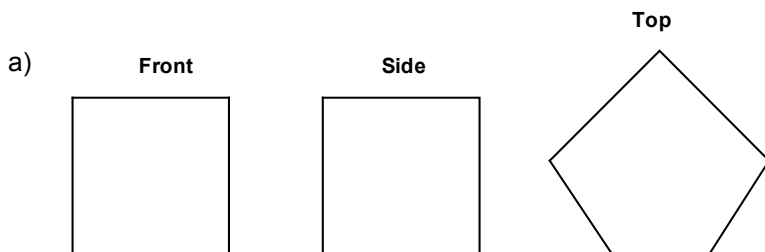


Fig. 4.24

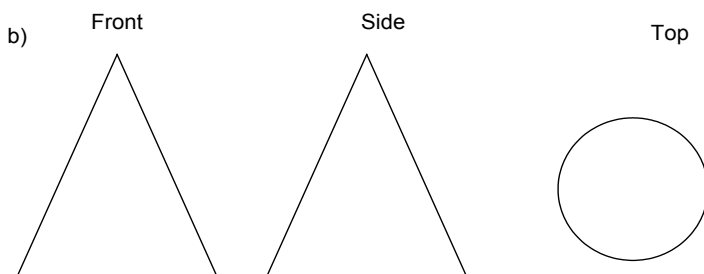


Fig. 4.25

3. What do you think the solid figure to be, if its top is a point, and sides are triangles and base is a square?
4. What do you think the solid figure to be, if its top is a circle, and side a rectangle and base is a circle?

A 3 dimensional object can look different from different positions so that can be drawn from different perspectives. We can also get different views of figures made by joining cubes.

Example 4.3

Consider the following solids made of three cubes in Fig. 4. 26 and four cubes in Fig.4.27.

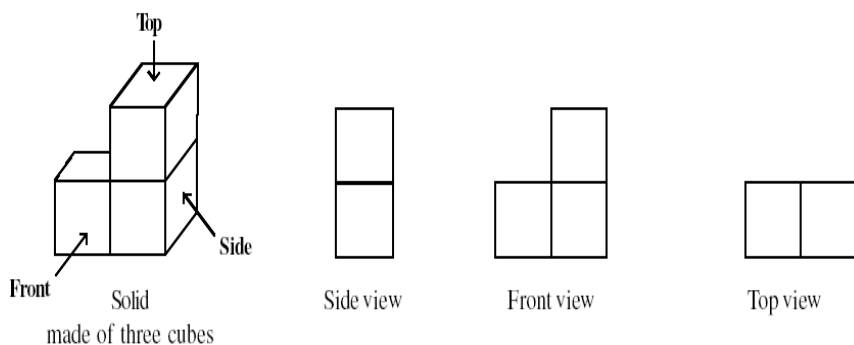


Fig. 4.26

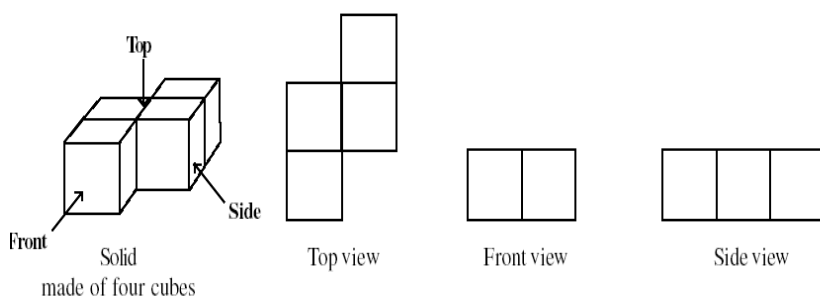
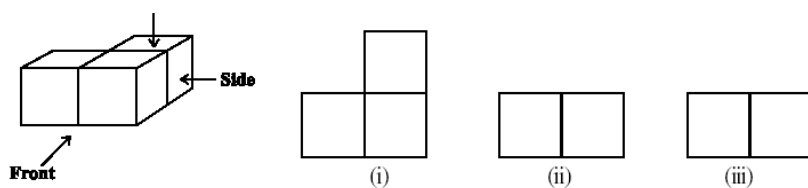


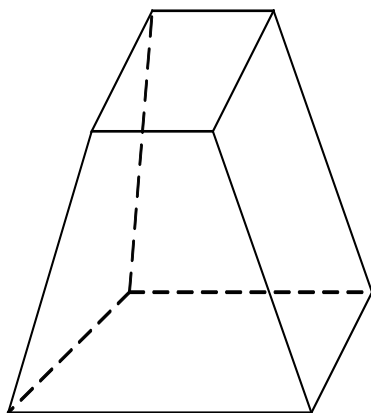
Fig. 4.27

Exercise 4.3

1. Identify the front view, the top view and the side view of the following



2. Verify Euler's Formula for the following figure.



4.2 Coordinate geometry

4.2.1 Coordinates in a plane

Activity 4.12

1. The point in the positive direction at a distance r from the origin represents the number r . The point in the negative direction at a distance r from the origin represents the number $-r$. Location of different numbers are shown in Fig.4. 28. How many units are there between:
 - a) 3 and 7
 - b) -5 and -1
 - c) -3 and 9
 - d) -2 and 5

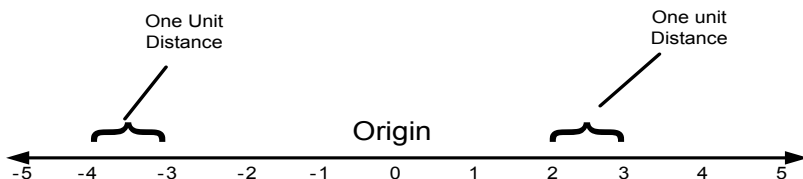


Fig. 4.28

2. What is the name given to the point where the x-axis and the y-axis intersect?
3. If the axes are called the x-axis and the y-axis, the plane is often referred to as _____
4. It should be known that the length of the intervals on the x-axis is the same and on the y-axis need also be the same. Is it must have the same scale on both axes?
5. The axes divide the plane into four regions. Each region is called a quadrant. Do you agree the naming of the quadrants is as in Fig.4. 29?

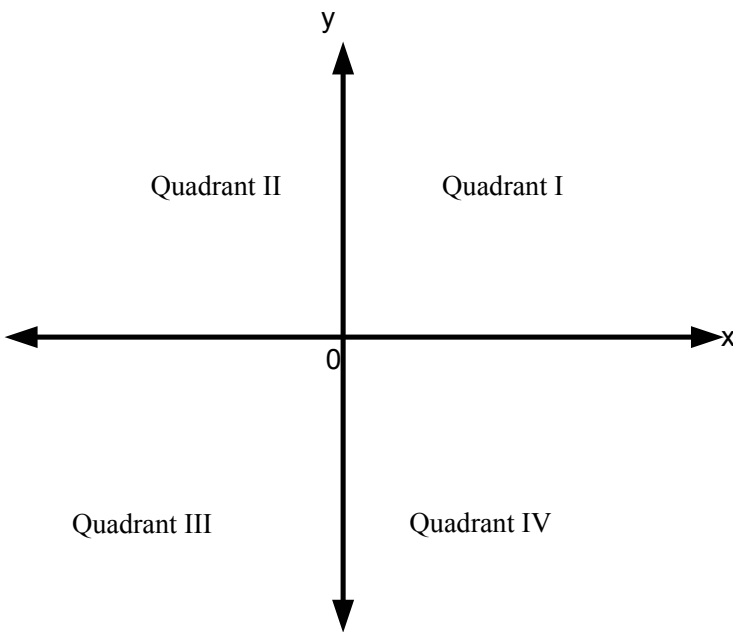


Fig. 4.29

Suppose now P is any point in the plane. Let M be the foot of the perpendicular from P to the x-axis, and a , the coordinate of M on the x-axis. Let N be the foot of the perpendicular from P to the y-axis, and b the coordinate of N on the y-axis. We assign the ordered pair (a, b) to the point P and refer to the pair as the coordinate of P . The first entry of the pair (a, b) is called the **x-coordinate** or the **abscissa of P** , and the second entry is called the **y-coordinate** or the **ordinate of P** .

Note: We call (a, b) an **ordered pair** since the order of the numbers is important. We say P is the point with coordinate (a, b) and we write as $P(a, b)$.

The point O where the two axes meet is called the **origin**.

Example 4.4

Plot the points $P(0, 0)$, $Q(5, 0)$, and $R(0, 12)$. Find the area and the perimeter of $\triangle PQR$

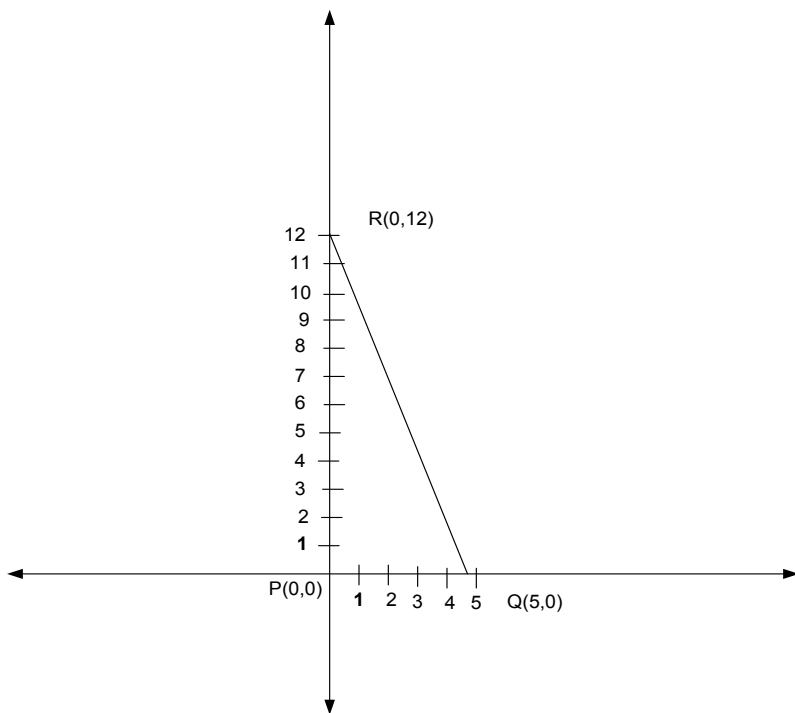


Fig. 4.30

Exercise 4. 4

1. Draw a system of axes and plot the following points.
 $(2, 3)$, $(-4, 3)$, $((2,-1)$
2. In which quadrants do the following points lie?
 $(-1, 2)$, $(2, 4)$, $(-1,-1)$, $(2, -1)$
3. Which of the following points lie on the x-axis?
 $(2, 3)$, $(0, 1)$, $(4, 0)$, $(0,0)$, $(0,9)$

4. Which of the following points lie on the y-axis?
 $(-1, 0)$, $(0, 2)$, $(9, 0)$, $(11, 2)$
5. If $a < 0$ and $b > 0$ where does $(-a, -b)$ lie?

4.2.2 Distance Between Two Points

Activity 4.13

1. Do you know how to find distance between two points on a number line?
2. What is the distance between the points $P(2, 4)$ and $Q(2, 10)$?
3. What is the distance between the points $P(2, 4)$ and $Q(-8, 4)$?
4. A town M is located 36 km east and 15 km north of town N (see Fig. 4.31). How would you find the distance from town N to town M without actually measuring it?

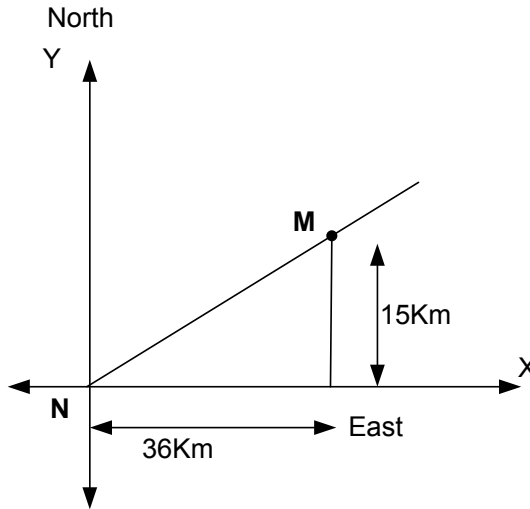


Fig. 4.31

Consider any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ that are neither on the same vertical line nor horizontal line. We can arbitrarily plot the points. Construct a right-angled triangle as indicated in Fig.4.32.

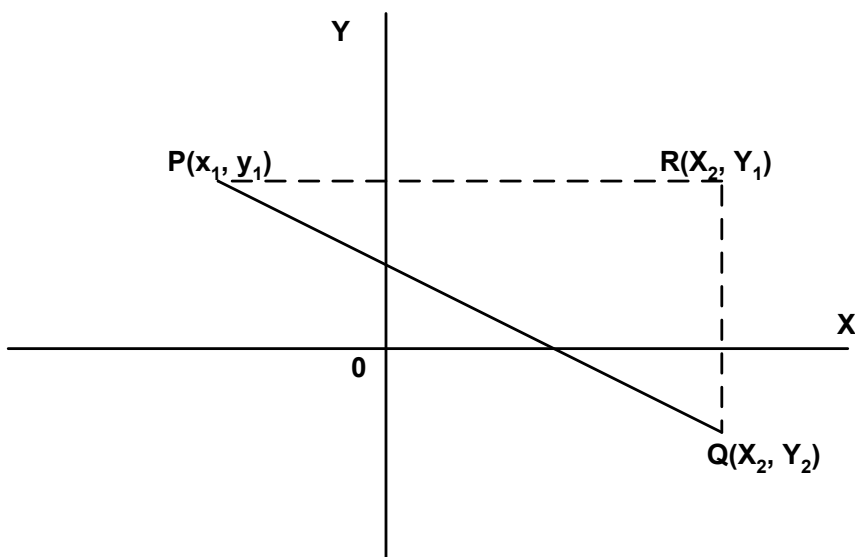


Fig. 4.32

We are interested to find the length of $\overline{PQ}$.

$$PR = |x_2 - x_1|$$

$$RQ = |y_2 - y_1|$$

$$\begin{aligned} \text{Using Pythagoras' Theorem, } PQ &= \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

Note: that, Since distance is always non-negative, we take the positive square root only. Therefore; we have the following:

The distance d between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is called the **distance formula** and can be used to find the distance between any two points on the coordinate plane.

Example 4.4

1. Determine the distance between the points P(-1, 2) and Q(2, -5).
2. Do the points (3, 2), (-2, -3) and (2, 3) form a triangle? If so, name the type of triangle formed.
3. Show that the triangle with vertices P(-2,3), Q(3,8), and R(4,1) is isosceles.

Solution

$$\begin{aligned}
 1. \quad PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
 &= \sqrt{(2 - (-1))^2 + (-5 - 2)^2} \\
 &= \sqrt{3^2 + (-7)^2} \\
 &= \sqrt{58}
 \end{aligned}$$

2. Let us apply the distance formula to find the distances PQ, QR and PR, where P(3,2), Q(-2,-3) and R(2,3) are the given points. We have:

$$PQ = \sqrt{(3 - (-2))^2 + (2 - (-3))^2} = \sqrt{50}$$

$$QR = \sqrt{(-2 - 2)^2 + (-3 - 3)^2} = \sqrt{52}$$

$$PR = \sqrt{(3 - 2)^2 + (2 - 3)^2} = \sqrt{2}$$

Since the sum of any two sides is greater than the third side, the points P, Q, and R form a triangle. Moreover,

$PQ^2 + PR^2 = QR^2$, by the converse of Pythagoras Theorem, therefore, $\triangle PQR$ is a right-angled triangle.

$$\begin{aligned}
 3. \quad PQ &= \sqrt{(3 - (-2))^2 + (8 - 3)^2} \\
 &= \sqrt{25 + 25} \\
 &= \sqrt{50} \\
 QR &= \sqrt{(4 - 3)^2 + (1 - 8)^2}
 \end{aligned}$$

$$= \sqrt{1+49}$$

$$= \sqrt{50}$$

$$PR = \sqrt{(4-(-2))^2 + (1-3)^2}$$

$$= \sqrt{36+4}$$

$$= \sqrt{40}$$

Observe $\overline{PQ} = \overline{QR}$. Therefore $\triangle PQR$ is an isosceles triangle.

Exercise 4.5

- Find the distance between each pair of points.

a) $(-3,2), (6,2)$	d) $(-1,-1), (-2,-3)$
b) $(5,4), (5,-9)$	e) $(1,-2), (-3,4)$
c) $(1,7), (3,4)$	f) $(5,4), (-1,2)$
- Find the point on the x-axis which is equidistant from $(2, -5)$ and $(-2,9)$.
- Show that $(5, -2)$, $(6, 4)$ and $(7,-2)$ are the vertices of an isosceles triangle.
- Show that the triangle with vertices $A(1,1)$, $B(4,3)$, and $C(4,1)$ is a right-angled triangle.

4.2.3 Midpoint of a line segment

Activity 4.14

- Plot the following pairs of points and find the distance between them:
 - $(0,0), (4,0)$
 - $(2,3), (7,3)$
 - $(-5,-5), (-9,-5)$
 - $\left(-\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right)$

2. Plot the following pairs of points and find the distance between them.
 - a) $(0,3), (0,5)$
 - b) $(2,5), (2,-11)$
 - c) $(-3,11), (-3, -15)$
 - d) $\left(\frac{3}{4}, -12\right), \left(\frac{3}{4}, \frac{5}{2}\right)$
3. Find the coordinates of the midpoints of the line segments determined by the pairs of points in (1) and (2).
4. What do you think the formula for finding the midpoint of
 - a) a vertical line segment?
 - b) a horizontal line segment?

The midpoint M of the line segment $\overline{P_1P_2}$

where $P_1(x_1, y)$ and $P_2(x_2, y)$ is $M\left(\frac{x_1 + x_2}{2}, y\right)$

The midpoint M of the line segment $\overline{P_1P_2}$ where $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $M\left(x, \frac{y_1 + y_2}{2}\right)$

The following activity refers to Fig. 4.35

Activity 4.15

We now derive a formula for finding the coordinates of the midpoint of an arbitrary line segment $\overline{P_1P_2}$ where $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

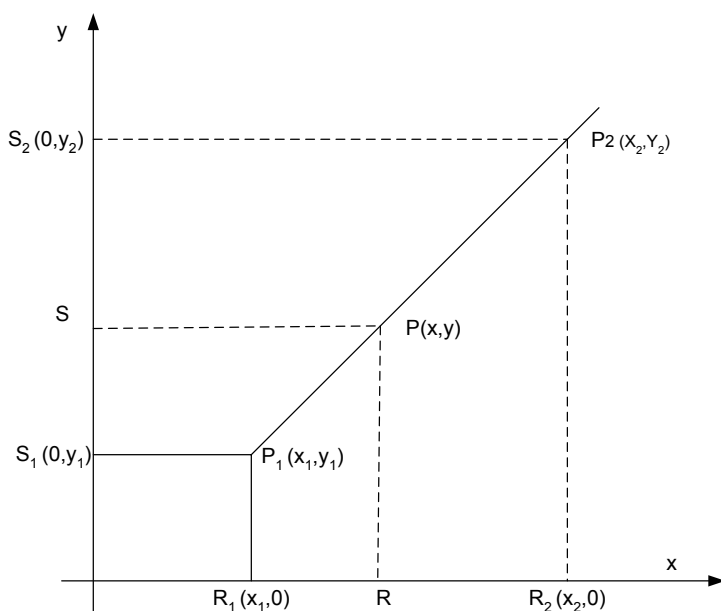


Fig. 4.33

- The vertical lines through P_1 and P_2 meet the x-axis in the points $R_1(x_1, 0)$ and $R_2(x_2, 0)$ respectively. Find the coordinates of the midpoint R of $\overline{R_1R_2}$?
- Let the vertical line through R meet $\overline{P_1P_2}$ in P . Then P is the midpoint of $\overline{P_1P_2}$. Why?
- The horizontal lines through P_1 and P_2 meet the y-axis in the points $S_1(0, y_1)$ and $S_2(0, y_2)$ respectively. Find the coordinates of the midpoint S of $\overline{S_1S_2}$?
- Find the coordinates of P .
- Write a formula for the midpoint of a line segment.
- Does the formula for the midpoint of a line segment hold if the segment is horizontal or vertical?

The midpoint of a line segment $\overline{P_1P_2}$ with endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 4.5

- a) The coordinates of the midpoints of the line segments with endpoints (0, 3) and (0, 8) is $\left(0, \frac{3+8}{2}\right) = \left(0, \frac{11}{2}\right)$
- b) The coordinates of the midpoints of the line segments with endpoints (-4, 4) and (3, -2) is $\left(\frac{-4+3}{2}, \frac{4+(-2)}{2}\right) = \left(\frac{-1}{2}, 1\right)$
- c) The coordinates of the midpoints of the line segments with endpoints (-4, 0) and (3, 0) is $\left(\frac{-4+3}{2}, 0\right) = \left(\frac{-1}{2}, 0\right)$

Exercise 4.6

1. Find the coordinates of the midpoints of the line segments with end points:
 - a) (-6,5), (7,2)
 - b) (6,7), (0,0)
 - c) $\left(\frac{-1}{2}, 3\right), \left(\frac{1}{4}, 3\right)$
 - d) $\left(\frac{-1}{2}, 3\right), \left(\frac{1}{4}, 7\right)$
 - e) (7,11), (7,-13)
2. The midpoint of a line segment is (3, 2). One endpoint of the line segment is (-5, 12). Find the other endpoint.

4.3 Transformations

Activity 4.16

1. What do you understand by transformation of a plane?
2. What can you say about triangles DEF and D'E'F' by observing Fig. 4.34?

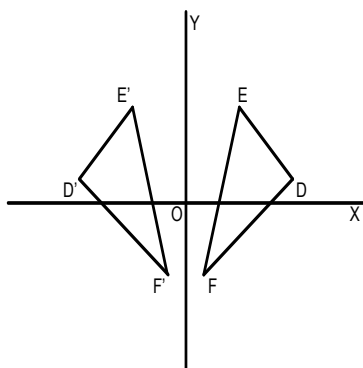


Fig. 4.34

3. What do you observe if the rectangle ABCD in Fig. 4.35 is folded along the line GH where G and H are midpoints of AD and BC respectively?

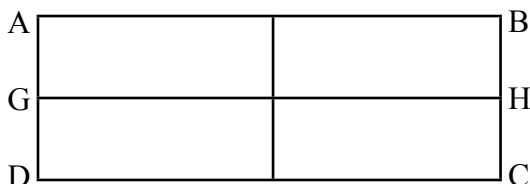


Fig. 4.35

4. When you stand before a mirror, your image appears to be as far behind the mirror as you are in front of it. Is the distance between you and the mirror equal to the distance from the mirror to your image?
5. Suppose the line l in Fig.4.36 represents a mirror. The point A is in front of the mirror. Its image is the point A' . What do you think is the relationship between the line l and the line segment $\overline{AA'}$? The point C is on the mirror. What is its image? How would you find the image of $\overline{PQ}$? How would you find the pre-image of $\triangle EFG$?

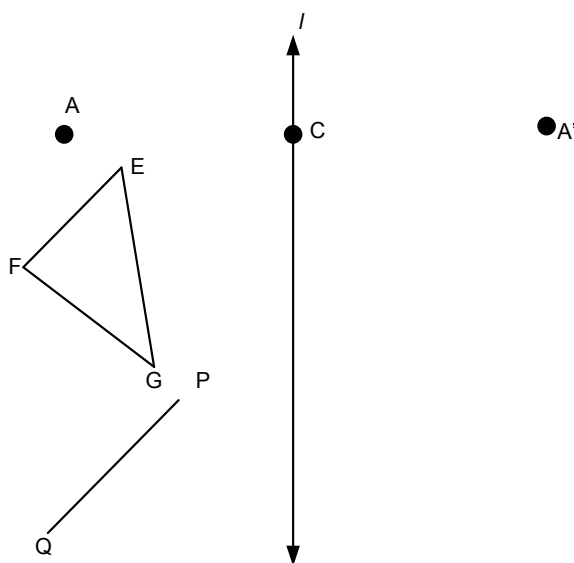


Fig. 4.36

A **transformation** is a one to one mapping from the whole plane to the whole plane

4.3.1 Reflection

Activity 4.17

1. Draw the reflection of the following objects in the given line l .

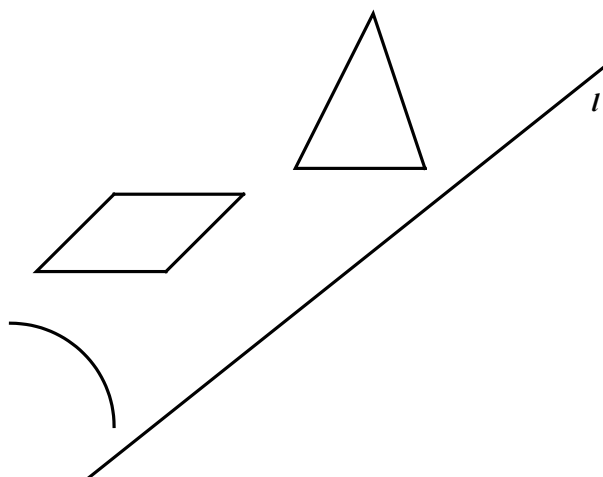


Fig. 4.37

2. If the figures in Fig. 4.38 are an isosceles triangle and a regular hexagon, what are the lines of symmetries of each?

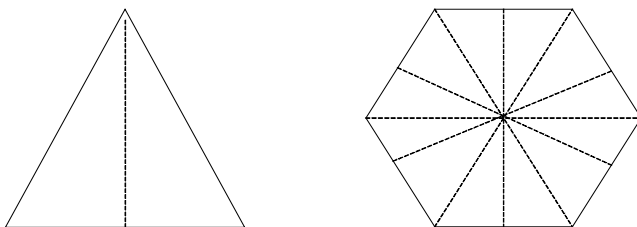


Fig. 4.38

3. If A' is the image of A and B' is the image of B under reflection, then what can you say about the lengths AB and $A'B'$? Prove your assertion.

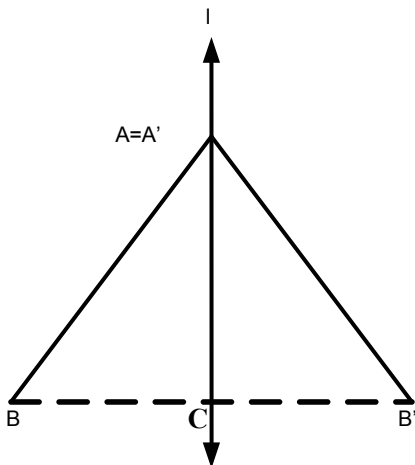


Fig. 4.39

4. If you reflect a circle about a line through its centre (see Fig.4.40), what is the image?

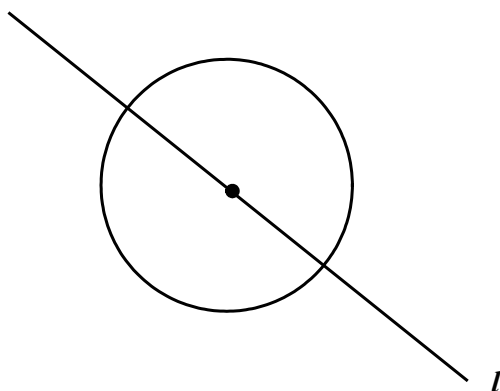


Fig. 4.40

A point is the image of another point, across a line l , if the line l is the perpendicular bisector of the segment joining the two points. If the point lies on the line, the reflection is the point itself.

In other words, a reflection in line l maps every point p to a point p' such that:

1. If p is not on the line l , then l is the perpendicular bisector of $\overline{PQ}$.

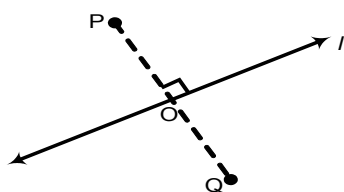


Fig. 4.41

2. If p is on line l , then $Q = p$.

Here we say p is mapped onto p' by the reflection on the line l .

Let C be a set of points, and let L be a line. If C contains the reflection across L of each point of C , then C is **symmetric** about L . The line L is called the **axis of symmetry**.

Example 4.7

In Fig. 4.42, l is a line of symmetry

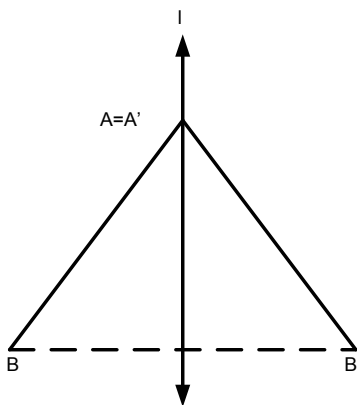


Fig. 4.42

Activity 4.18

Consider a point $P(3, 2)$ is given on a coordinate plane. What will be the coordinates of the image of the point by reflecting:

- a) across the x -axis?
- b) across the y -axis?
- c) across the line $y = x$?

Fig. 4.43 represents a reflection of a point $P(x, y)$ across the x -axis.

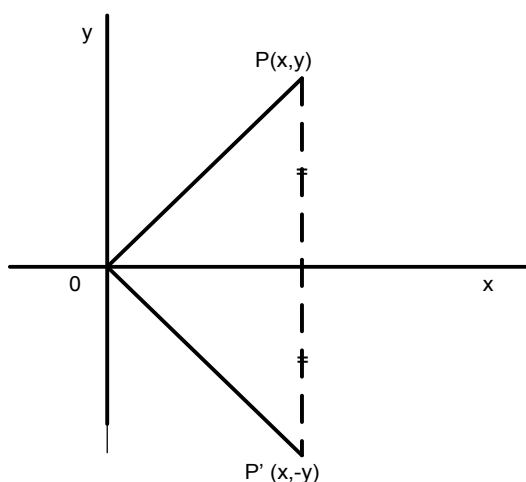


Fig. 4.43

Under reflection in the x -axis, the image p' of the point $p(x, y)$ will be $p'(x, -y)$.

In a similar way, under reflection in the y -axis, the image p' of the point $p(x, y)$ will be $p'(-x, y)$.

The images under reflection in the different four lines can be summarised in the following table.

Line	x -axis	y -axis	$y = x$	$y = -x$
$P(x, y)$	$p'(x, -y)$	$p'(-x, y)$	$p'(y, x)$	$p'(-y, -x)$

Activity 4.19

We can find the image $p'(x', y')$ of a point $p(x, y)$ under the reflection L in the line $y = x$ as follows.

- Find the coordinates of C , the midpoint of $\overline{pp'}$.
- Does C lie on the line $y = x$? Use your answer to write a true statement involving x, y, x', y' .
- Find the slopes of $\overline{OC}$ and $\overline{pp'}$. How are these slopes related? Use your answer to write another true statement involving x, y, x', y' .
- Use the statements you found in (b) and (c) to deduce that $L(x, y) = (y, x)$

Point reflection

Activity 4.20

- If a point P is the image of Q under point reflection through O, then what is the relation between O and $\overline{PQ}$?
- Determine the coordinates of the image of a point P whose coordinates are (2, -5), when it is reflected in the origin.

Under reflection in the **origin** a point P(x, y) is mapped into the point $P'(-x, -y)$.

Exercise 4.7

- Find the coordinates of the images of the following points under the reflection in
 - x-axis
 - y-axis
 - line $y = x$
 - line $y = -x$
 - origin
 - (4, 5)
 - (6, -1)
 - (-11, 2)
 - (0, 0)
- The triangle ABC with vertices A(-3, 4), B(-2, 1), C(-6, -4) is reflected in the x-axis. Sketch $\triangle ABC$ and its image.
- Reflect the point (6, -3) in the x-axis and then reflect the image in the x-axis again. What are the coordinates of the final image?
- Reflect (2, -3) in the line $y = -x$ and then in the x-axis. What difference would there be, if any, if the point had been reflected first in the x-axis and then in the line $y = -x$?

4.3.2 Translations

Activity 4.21

In Fig. 4.44 the arrow $\overline{AA'}$ is 3 cm long. Suppose every figure in the plane moves 3 cm in the same direction as the arrow. Would you say that the given images of the point L and the triangle PQR are correct?

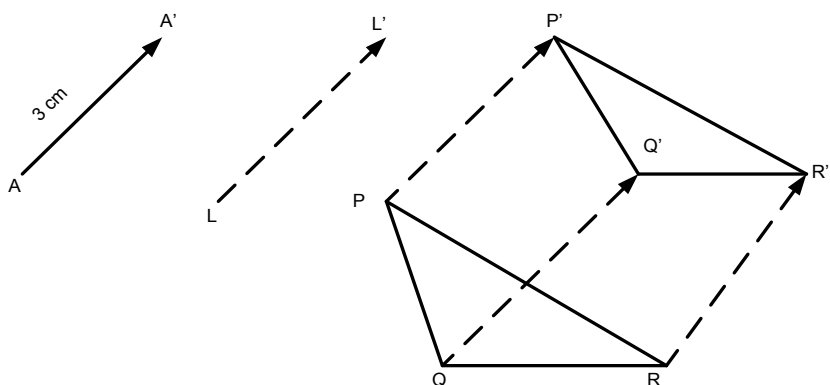


Fig. 4.44

A translation moves every point the same distance in the same direction. i.e., a translation is a correspondence between points and their image points so that each image is the same distance in the same direction from the original point.

If a translation takes the origin to the point (m, n) , then for each point (x, y)

$$T: (x, y) \rightarrow (x + m, y + n).$$

If a translation takes the point (a, b) to the point (c, d) , then

$$T: (0, 0) \rightarrow (c - a, d - b).$$

Exercise 4.8

1. A translation takes the origin to $(-2, 1)$. Find where it takes $(2, 1)$, $(-2, 1)$, $(0, 1)$
2. A translation takes every point a distance 1 to the left and 2 downward. Find where it takes $(0, 0)$, $(0, 6)$, $(1, 1)$
3. A translation takes the point $(2, 3)$ to $(3, -4)$. Find where it takes $(0, 0)$, $(2, -1)$, $(1, 0)$.
4. Find the pre-image of each of the following points under the translation by $(3, 7)$.
 - a) $(4, 9)$
 - b) $(3, 2)$
 - c) $(-3, 4)$
 - d) $(-2, -1)$

4.3.3 Dilation

Activity 4.22

1. Does the shape change when a photograph is enlarged or reduced?
2. Is there a relation between the distances of two points in the original photograph compared to the distances between their images in the enlargement or reduction?
3. Do the following.
Begin with $\triangle ABC$ and point O.

Let A' lie on $\overline{OA}$ such that $OA' = 2OA$.

Let B' lie on $\overline{OB}$ such that $OB' = 2OB$.

Let C' lie on $\overline{OC}$ such that $OC' = 2OC$,

What do you observe between $\triangle ABC$ and $\triangle A'B'C'$?

Reflection and translation are transformations which preserve the size and the shape of an object. These are called **rigid motion**.

There are other transformations which enlarge, shrink the shape of an object. When a shape is resized it gets bigger or smaller but it still looks similar that is, all angles stay the same and the relative sizes are the same.

Let $P \leftrightarrow Q$ be a one-to-one correspondence between the points of a plane and themselves. Suppose that there is a number $r \neq 0$ such that $Q_1Q_2 = rP_1P_2$

For every pair of points P_1, P_2 . Then the correspondence is called **dilation** and r is called a **scale factor**.

- a) The use of the term **dilation** is more natural in the case $r > 1$, for all the distance are increased under the transformation, and so the whole plane is stretched or dilated. The dilation is called an **expansion**.
- b) For $r < 1$ the image of any figure in the plane is a scale model of the original figure, having exactly the same shape but a smaller size. The dilation is called a **contraction**.
- c) In the case $r = 1$, all distances are preserved, and so the dilation is a **rigid motion**.

Note: For simplicity, negative scale factors are not included.

Example 4.8

Given a point O and $\triangle RST$ in Fig. 4.53, find the image of $\triangle RST$ under the contraction with centre O and scale factor $\frac{2}{3}$.

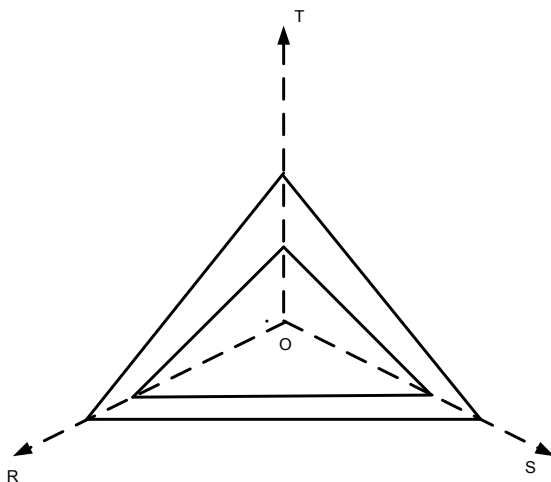


Fig. 4.45

Review Exercise 4

1. In which quadrant or on which axis do each of the points (2, 4), (-2, -3), (1, -9), (0, 1), (4, 0) lie?
2. Determine if the points (1, 5), (2, 3), and (-2, 11) are collinear?
3. Check whether the points (5, -2), (6, 4), (7, -2) are the vertices of an isosceles triangle.
4. Find the point on the x-axis which is equidistant from (2, -5) and (-2, 9).
5. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.
6. Show that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices. Use coordinate geometry.

7. Find the image of each of the following points under the translation by $(2, -3)$
- a) $(4, 7)$
 - b) $(-2, 1)$
 - c) $(-5, -4)$
8. Given a coordinate plane. For each point named below, give its reflection across i) the x-axis ii) the y-axis iii) the origin.
- a) $(5, -3)$
 - b) $(-8, 14)$
 - c) $(6, -6)$
 - d) $(3, 11)$
9. How many line symmetries do each of the following figures have?
- a) An equilateral triangle
 - b) A scalene triangle
 - c) A regular polygon of n sides
 - d) A circle.

Chapter 5

Congruency and the Geometry of Size

Expected Learning Outcomes

By the end of this chapter, you will be able to:

- extend measurement concepts to find area, perimeter and volume;
- analyse properties and describe relationships in geometric figures;
- apply concepts of congruence to justify properties of figures
- apply the concept of congruence in solving real life problems.

5.1 Congruency

5.1.1 Congruent Figures

Activity 5.1

1. What are congruent line segments?
2. What are congruent angles?
3. What are congruent triangles?
4. What are congruent polygons?
5. Draw two circles of the same radius and place one on the other. What do you observe?
6. Draw two equilateral triangles of equal sides and place one on the other. What do you observe?
7. Which of the following figures in Fig. 5.1 are congruent?

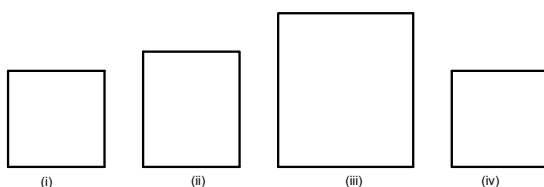


Fig. 5.1

8. Is a quadrilateral with consecutive sides of 4, 5, 6, and 7 units congruent to a quadrilateral of consecutive side of 4, 6, 5, and 7?

Congruent figures are figures with the same shape and same size.

Two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of the other.

If $\triangle ABC$ is congruent to $\triangle DEF$, we write as $\triangle ABC \equiv \triangle DEF$.

If $\triangle ABC \equiv \triangle DEF$, there is a one to one correspondence between their vertices. That is, A corresponds to D, B to E and C to F which is written

$$A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F.$$

Note: Under this correspondence, $\triangle ABC \equiv \triangle DEF$; but it is not correct to write $\triangle BAC \equiv \triangle DEF$.

Therefore, $\triangle ABC \equiv \triangle DEF$, if the following six statements are true for $\triangle ABC$ and $\triangle DEF$:

- | | |
|---|-------------------------------|
| a) $\overline{AB} \equiv \overline{DE}$ | c) $\angle A \equiv \angle D$ |
| b) $\overline{AC} \equiv \overline{DF}$ | d) $\angle B \equiv \angle E$ |
| c) $\overline{BC} \equiv \overline{EF}$ | e) $\angle C \equiv \angle F$ |

In a figure, it is more convenient to indicate congruence between segments and angles by using check marks as shown in Fig. 5.2..

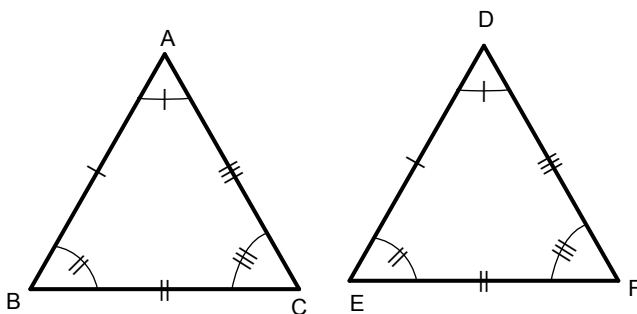


Fig. 5.2

Exercise 5.1

1. Given that $\triangle MQP \equiv \triangle NOR$. List the six pairs of corresponding congruent parts of these two triangles.
2. Write the congruence for the two triangles which is determined by these six pairs of congruent parts:

$$\overline{AK} \equiv \overline{BW}$$

$$\overline{KT} \equiv \overline{WR}$$

$$\overline{AT} \equiv \overline{BR}$$

$$\angle A \equiv \angle B$$

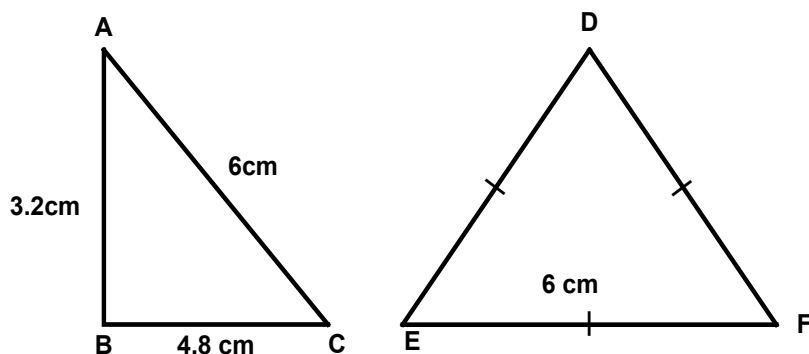
$$\angle K \equiv \angle W$$

$$\angle T \equiv \angle R$$

3. Is a figure congruent to itself?
4. If each of two figures is congruent to a third, are they congruent to each other?
5. $\triangle ABC$ is equilateral. Write all congruence between the triangle and itself.

5.1.2 Methods for Congruence of Triangles**Activity 5.2**

1. Are the pair of triangles in Fig. 5.3 congruent? Why?

**Fig. 5.3**

2. Are the pair of triangles in Fig. 5.4 congruent? Why?

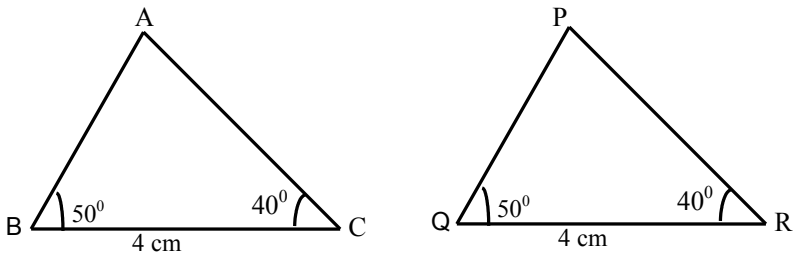


Fig. 5.4

3. Is congruence of one pair of sides and one pair of angles enough to prove congruence of triangles?
4. Are the following triangles in Fig.5.5 congruent? Justify your answer.

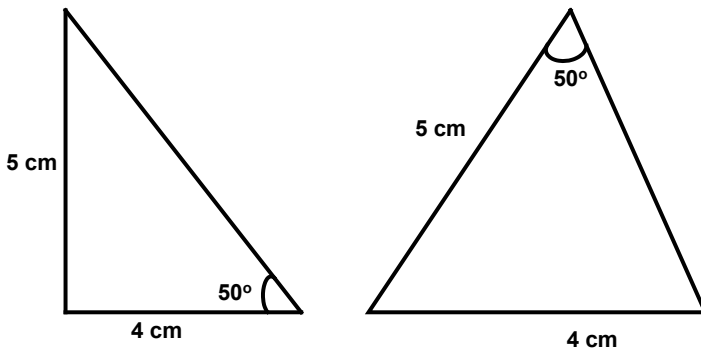


Fig. 5.5

5. Are the following triangles in Fig.5.6 congruent?

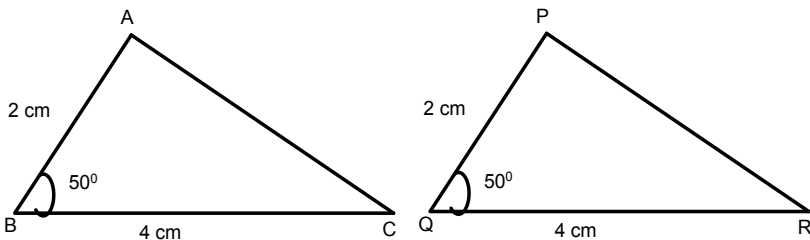


Fig. 5.6

6. Do you observe that congruence of two pair of sides and a pair of included angles is enough for congruence of triangles?
7. What do you understand by included angle?
8. a) In $\triangle ABC$ of Fig.5.7, which is the included angle of side $\overline{BC}$ and $\overline{AB}$?
 b) Which is the included side of $\angle A$ and $\angle C$?
 c) Which sides include $\angle C$?
 d) Which sides include $\angle A$?

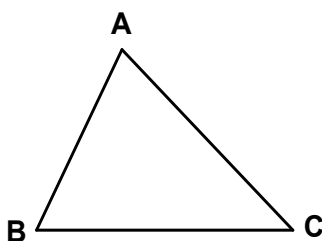


Fig. 5.7

Postulate 1 (SAS Congruence)

If two sides and the included angle of one triangle are equal to two sides and included angle of the other, then the two triangles are congruent.

Restatement

If in Fig 5.8 $\overline{AB} \equiv \overline{DE}$ $\overline{BC} \equiv \overline{EF}$ and $\angle B \equiv \angle E$ then $\triangle ABC \equiv \triangle DEF$

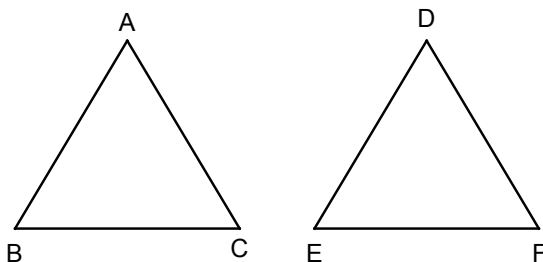


Fig. 5.8

Example 5.1

In the figure $\overline{AO} \equiv \overline{BO}$, $\overline{CO} \equiv \overline{DO}$. Prove that $\triangle AOD \equiv \triangle BOC$

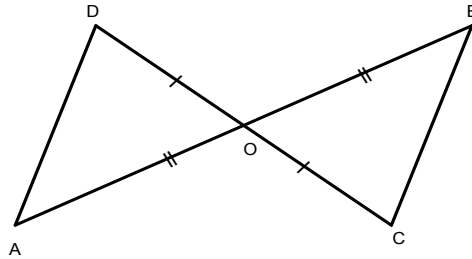


Fig. 5.9

Proof:

Statement	Reason
1. $\overline{AO} \equiv \overline{BO}$	Given
2. $\overline{CO} \equiv \overline{DO}$	Given
3. $\angle AOD \equiv \angle BOC$	Vertical opposite angles
4. $\therefore \triangle AOD \equiv \triangle BOC$	SAS

Example 5.2

l is a perpendicular bisector of the segment $\overline{AB}$. If a point P lies on l , show that P is equidistant from A and B .

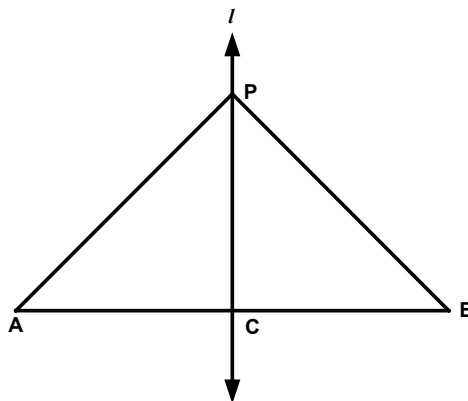


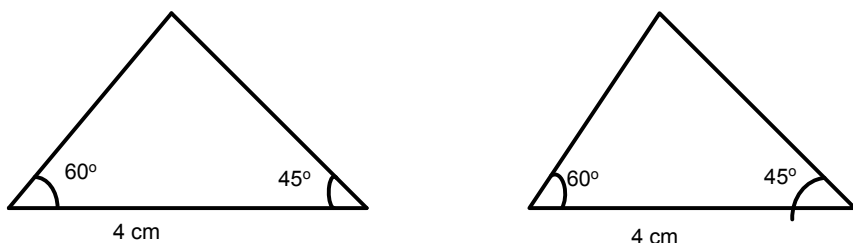
Fig. 5.10

Proof:

Statement	Reason
1. $\overline{AC} \equiv \overline{BC}$	C is the midpoint given
2. $\angle ACP \equiv \angle BCP$	common side
3. $\overline{PC} \equiv \overline{PC}$	SAS
4. $\therefore \triangle APC \equiv \triangle BPC$	sides of congruent triangles
5. Hence, $\overline{PA} = \overline{PB}$	

Activity 5.3

- a) Construct two triangles on a piece of paper as shown in Fig. 5.11, whose measure of two angles are 45° and 60° and the included side of 4 cm.

**Fig. 5.11**

- b) Cut out these two triangles and place one on the other. What do you observe?

Postulate 2(ASA Congruence)

If two angles and the included side of one triangle are equal to two angles and the included side of other triangle, then the two triangles are congruent.

Restatement:

If in Fig 5.12, $\angle B \equiv \angle E$, $\angle C \equiv \angle F$ and $\overline{BC} \equiv \overline{EF}$, then $\triangle ABC \equiv \triangle DEF$

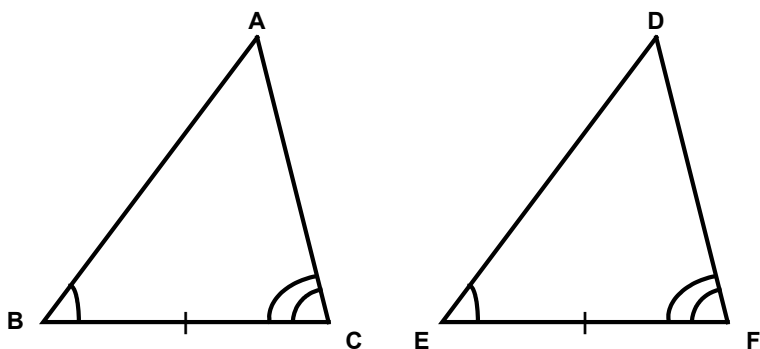


Fig. 5.12

Example 5.3

$\overline{AB}$ is a line segment and P is its midpoint. D and E are on the same side of AB such that $\angle BAD \equiv \angle ABE$ and $\angle APE \equiv \angle BPD$ (See Fig.5.13). Show that $\triangle ADP \equiv \triangle BEP$

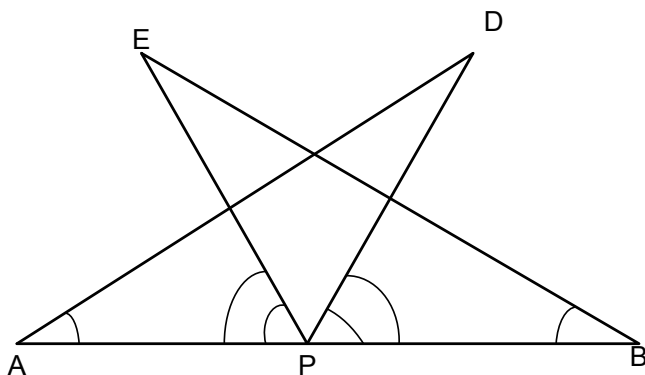


Fig. 5.13

Proof:

In $\triangle ADP$ and $\triangle BEP$

Statement

Reason

1. $\overline{AP} \equiv \overline{BP}$
2. $\angle BAD \equiv \angle ABE$
3. $\angle APE \equiv \angle BPD$
4. $m(\angle APD) = m(\angle APE) + m(\angle EPD)$
5. $m(\angle BPE) = m(\angle BPD) + m(\angle EPD)$
6. Therefore, $\angle APD \equiv \angle BPE$
7. Hence, $\triangle ADP \equiv \triangle BEP$

Given
Given
Given

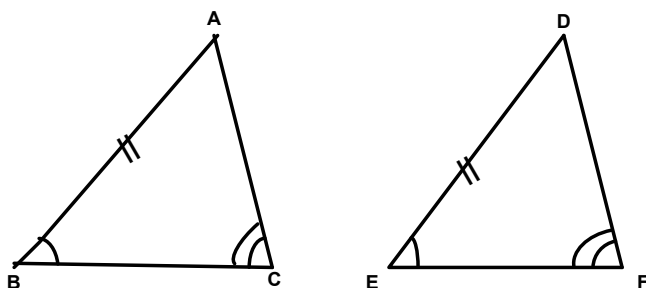
ASA

Activity 5.4

1. If in two triangles, two angles and a side of one triangle are congruent to two angles and a corresponding side of the second, but the side may not be the included, can we say that the two triangles are congruent?
2. Is equality of the three angles in two triangles enough to say that the triangles are congruent? Why?
To answer this, draw triangles with angles 30° , 60° , and 90° . How many such triangles can you draw?

Theorem 5.1 (AAS congruence)

If two angles and a side of one triangle are congruent to two angles and a corresponding side of another triangle, then the two triangles are congruent.

**Fig. 5.14****Proof:**

Given $\angle B \equiv \angle E$, $\angle C \equiv \angle F$, $\overline{AB} \equiv \overline{DE}$

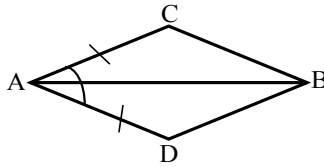
We want to prove that $\triangle ABC \equiv \triangle DEF$

Statement**Reason**

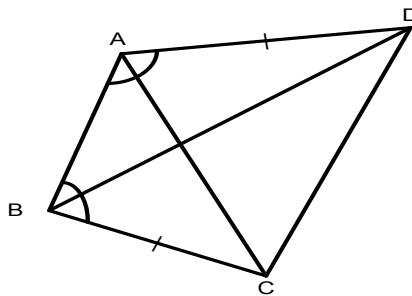
- | | |
|---|--|
| 1. $\angle B \equiv \angle E$ | Given |
| 2. $\angle C \equiv \angle F$ | Given |
| 3. $\therefore \angle A \equiv \angle D$ | Sum of measures of angles of Δ is 180° |
| 4. $\overline{AB} \equiv \overline{DE}$ | Given |
| 5. Therefore $\triangle ABC \equiv \triangle DEF$ | ASA |

Exercise 5.2

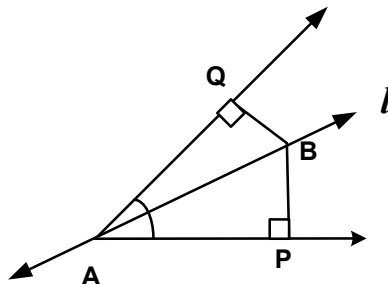
1. In quadrilateral $ACBD$, $\overline{AC} \equiv \overline{AD}$ and AB bisects $\angle A$. Show that $\triangle ABC \equiv \triangle ABD$. What can you say about $\overline{BC}$ and $\overline{BD}$?



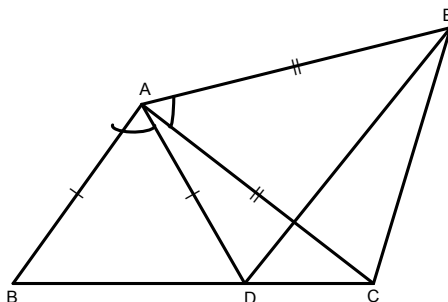
2. In quadrilateral $ABCD$, $\overline{AD} \equiv \overline{BC}$ and $\angle BAD \equiv \angle CBA$.
Prove that $\overline{AC} \equiv \overline{BD}$ and $\angle ABD \equiv \angle BAC$



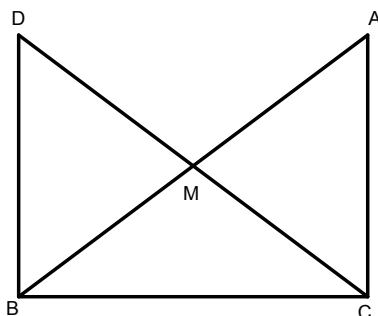
3. Line l is the bisector of an angle $\angle A$ and B is any point on l .
 $\overline{BP}$ and $\overline{BQ}$ are perpendiculars from B to the arms of $\angle A$.
Show that $\overline{BP} \equiv \overline{BQ}$.



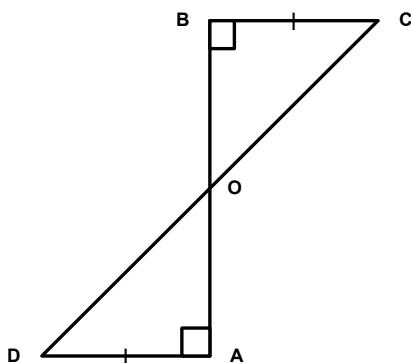
4. In the following figure, $\overline{AE} \equiv \overline{AC}$, $\overline{AB} \equiv \overline{AD}$, and $\angle BAD \equiv \angle CAE$. Prove that $\overline{BC} \equiv \overline{ED}$.



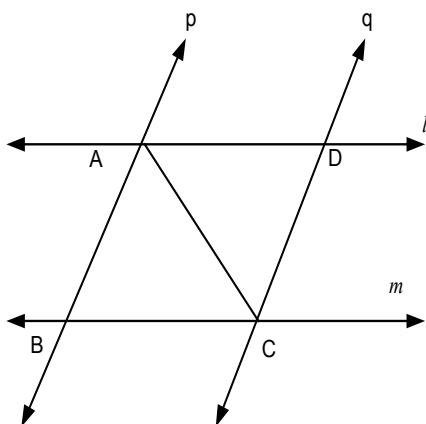
5. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $\overline{DM} \equiv \overline{CM}$. Point D is joined to point B. Show that:
- $\triangle AMC \equiv \triangle BMD$
 - $\angle DBC$ is a right angle.
 - $\triangle DBC \equiv \triangle ACB$
 - $CM = \frac{1}{2} AB$



6. $\overline{AD} \equiv \overline{BC}$ and both are perpendiculars to $\overline{AB}$. Show that $\overline{CD}$ bisects $\overline{AB}$.



7. l and m are two parallel lines intersected by another two parallel lines p and q . Prove that $\triangle ABC \cong \triangle CDA$



Activity 5.5

- Construct a triangle in which two sides are equal. You may take a triangle with sides 3cm, 3cm, and 5cm or a triangle with sides 7, 7, 9.
- What is the name given to a triangle with two sides equal?
- Measure the angles opposite to the equal sides
- What do you observe?

Activity 5.5 leads to the following theorem

Theorem 5.2

If two sides of a triangle are equal, the angles opposite these sides are equal.

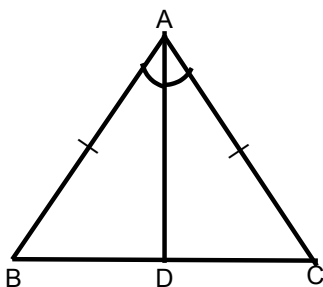


Fig. 5.15

Proof

Draw the angle bisector of $\angle A$.

Let D be the point where this angle bisector intersects $\overline{BC}$

Consider $\triangle ABD$ and $\triangle ACD$

Statement
Reason

- | | |
|--|--------------------------------|
| 1. $\overline{AB} \equiv \overline{AC}$ | Given |
| 2. $\angle BAD \equiv \angle CAD$ | Construction |
| 3. $\overline{AD} \equiv \overline{AD}$ | Common side |
| 4. Therefore, $\triangle ABD \equiv \triangle ACD$ | SAS |
| 5. Hence, $\angle ABD \equiv \angle ACD$ | Angles of congruent triangles. |

The converse of Theorem 5.2 is also true. That is, if two angles of a triangle are equal, then the sides opposite these angles are equal.

Example 5.4

E and F are respectively midpoints of the equal sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$. Show that $\overline{BF} \equiv \overline{CE}$.

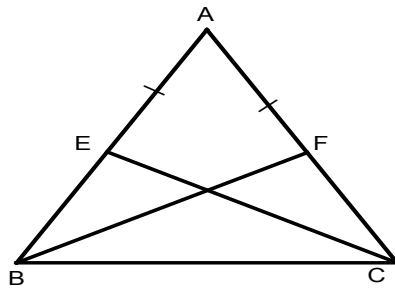


Fig. 5.16

Proof:

Consider $\triangle ABF$ and $\triangle ACE$

Statement

Reason

- | | |
|--|-------|
| 1. $\overline{AB} \equiv \overline{AC}$ | Given |
| 2. $\overline{AF} \equiv \overline{AE}$ | Given |
| 3. $\angle A$ is common angle | |
| 4. $\therefore \triangle ABF \equiv \triangle ACE$ | SAS |
| 5. Hence, $\overline{BF} \equiv \overline{CE}$ | |

Example 5.5

In an isosceles triangle ABC with $\overline{AB} \equiv \overline{AC}$, D and E are points on $\overline{BC}$ such that $\overline{BE} \equiv \overline{DC}$. Show that $\overline{AD} \equiv \overline{AE}$.

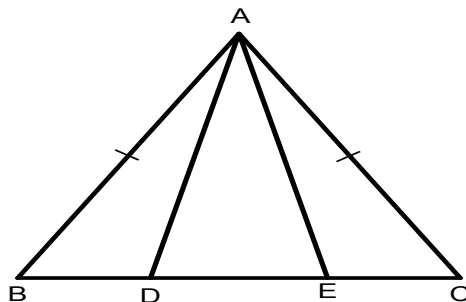


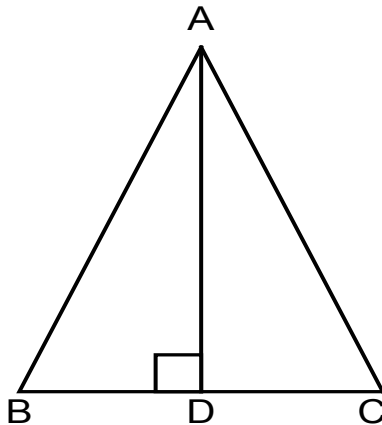
Fig. 5.17

Proof:Consider $\triangle ABD$ and $\triangle ACE$ **Statement****Reason**

- | | |
|--|-----------------------------------|
| 1. $\overline{AB} \equiv \overline{AC}$ | Given |
| 2. $\angle B \equiv \angle C$ | Base angles of isosceles triangle |
| 3. $\overline{BE} \equiv \overline{CD}$ | Given |
| 4. $BD = BE - DE$ | |
| 5. $CE = CD - DE$ | |
| 6. $\therefore \overline{BD} = \overline{CE}$ | |
| 7. $\triangle ABD \equiv \triangle ACE$ SAS | |
| 8. $\therefore \overline{AD} \equiv \overline{AE}$ | |

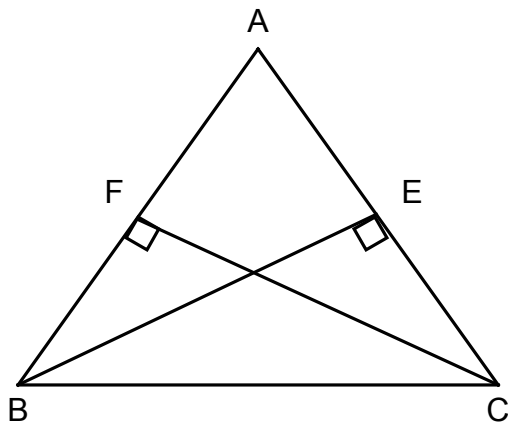
Exercise 5.3

- In an isosceles triangle ABC , with $\overline{AB} \equiv \overline{AC}$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O . Join A to O . Show that
 - $\overline{OB} \equiv \overline{OC}$
 - $\overline{AO}$ bisects $\angle A$
- $\overline{AD}$ is a perpendicular bisector of $\overline{BC}$. Show that $\triangle ABC$ is an isosceles triangle.



3. In the figure, $\overline{AB} \equiv \overline{AC}$. $\overline{BE}$ and $\overline{CF}$ are altitudes of $\triangle ABC$ drawn to sides $\overline{AC}$ and $\overline{AB}$ respectively.

Prove that the altitudes $\overline{BE}$ and $\overline{CF}$ are equal.



4. Given a right-angled triangle ABC in which $\angle A = 90^\circ$ and $\overline{AB} \equiv \overline{AC}$. Find $\angle B$ and $\angle C$.
5. Show that the angles of an equilateral triangle are 60° each.
6. Prove that the median to the base of an isosceles triangle is perpendicular to the base and bisects the angle opposite the base.
7. Quadrilateral MOPQ is a square, with R the midpoint of $\overline{MQ}$. Prove that $\triangle ROP$ is isosceles.

Activity 5.6

1. Construct two triangles with sides 3.5cm, 4cm, and 4.5cm on a piece of paper as shown in Fig. 5.18. Cut them out and place them on each other. What do you observe?

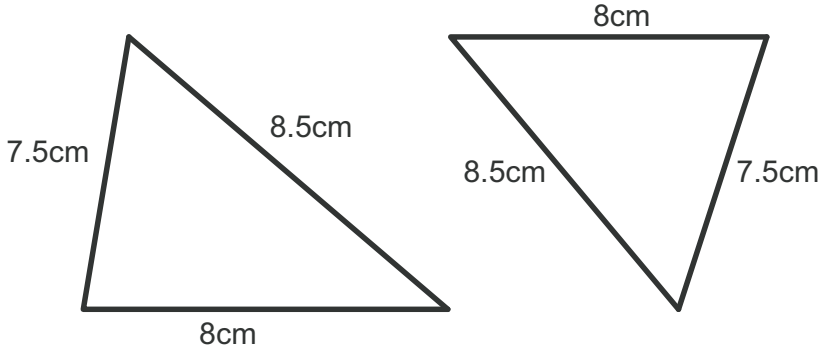


Fig. 5.18

2. Is congruence of three sides of one triangle to the three sides of another triangle enough for congruence of the two triangles?
3. Repeat task (1) by taking the measures of the sides to be 5cm, 5.5cm, and 4.5cm. What do you observe?
4. Generalise your finding.

Postulate 3 (SSS Congruence)

If three sides of one triangle are congruent to the three sides of another triangle, then the two triangles are congruent.

Activity 5.7

- a) Construct two right triangles with hypotenuse 5cm and one side of 3cm.
- b) Cut them out and place one of the triangles over the other with equal sides placed on top of each other.
- c) Repeat this with other pairs of right triangles. What do you observe?

Note: The right angle may not be the included angle.

Theorem 5.3(RHS)

If in two right angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and one side of the second triangle, then the two triangles are congruent.

Proof

Given $\triangle ABC$ and $\triangle DEF$, such that $\angle A \equiv \angle D = 90^\circ$, $\overline{AB} \equiv \overline{DE}$, and $\overline{BC} \equiv \overline{EF}$. We want to show that $\triangle ABC \equiv \triangle DEF$.

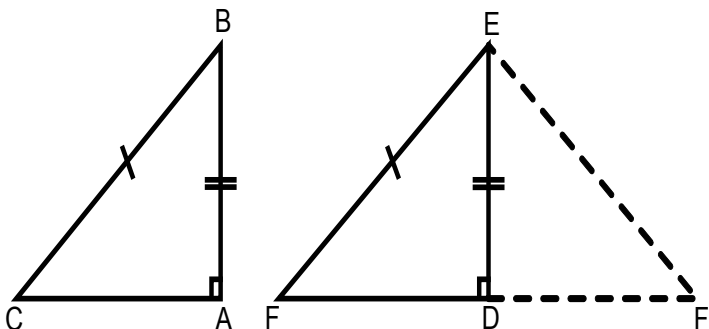
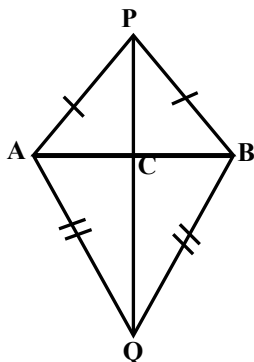


Fig. 5.19

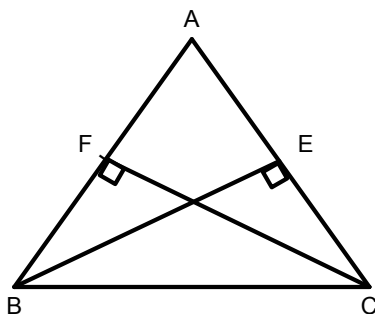
1. On the ray opposite to $\overline{DF}$ take F' such that $\overline{DF'} \equiv \overline{AC}$. Draw $\overline{EF'}$.
2. $\overline{AB} \equiv \overline{DE}$. Given
3. $\angle A \equiv \angle FDE \equiv \angle F'DE = 90^\circ$ Given and construction
4. $\therefore \triangle DEF' \equiv \triangle ABC$ SAS
5. $\therefore \overline{EF'} = \overline{BC}$ corresponding parts
6. $\overline{BC} \equiv \overline{EF}$ Given
7. $\therefore \overline{EF'} = \overline{EF}$ Transitivity of equality
8. $\therefore \angle F \equiv \angle F'$ Base angles of isosceles $\triangle$
9. $\therefore \triangle DEF \equiv \triangle DEF'$ AAS
10. $\therefore \triangle ABC \equiv \triangle DEF$ Transitivity of congruence

Exercise 5.4

1. In the following figure, $\overline{AP} \equiv \overline{PB}$ and $\overline{AQ} \equiv \overline{BQ}$. Prove that $\overline{PQ}$ is the perpendicular bisector of $\overline{AB}$.



2. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base $\overline{BC}$. If $\overline{AD}$ is extended to intersect $\overline{BC}$ at P, show that
- $\triangle ABD \equiv \triangle ACD$.
 - $\triangle ABP \equiv \triangle ACP$.
 - $\overline{AP}$ bisects $\angle A$ and $\angle D$.
 - $\overline{AP}$ is the perpendicular bisector of BC .
3. In the figure, $\overline{BE}$ and $\overline{CF}$ are two equal altitudes of triangle ABC . Using RHS congruence, prove that the triangle ABC is isosceles.



5.1.3 Quadrilaterals and Congruency

Activity 5.8

- What is a quadrilateral?
- Name some of quadrilaterals.
- What is a parallelogram?
- Name all the parallelograms
- Is a trapezium a parallelogram?
- Is a square a rhombus?
- Is a square a rectangle?
- List some properties of a parallelogram

Theorem 5.4

A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Proof:

Draw parallelogram ABCD, and then show that $\triangle ABC \equiv \triangle CDA$.

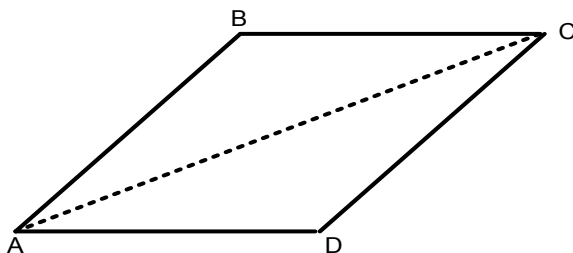


Fig. 5.20

Statement	Reason
1. $BC \parallel AD$	Definition of a parallelogram
2. $\angle ACB \equiv \angle CAD$	alternate interior angles
3. $\overline{AB} \parallel \overline{DC}$	Definition of a parallelogram
4. $\angle CAB \equiv \angle ACD$	alternate interior angles
5. $\overline{AC} \equiv \overline{CA}$	common side
6. $\therefore \triangle ABC \equiv \triangle CDA$	ASA

From the above theorem, we have the following:

Theorem 5.5

In a parallelogram, opposite sides are congruent.

Is the converse of Theorem 5.5 true?

Theorem 5.6

If opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

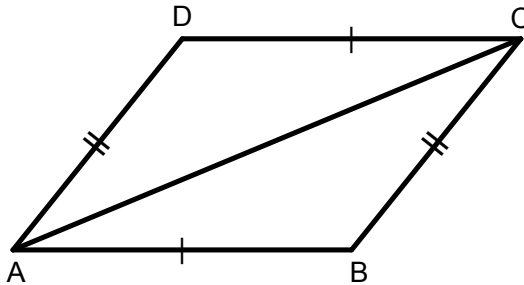


Fig. 5.21

Proof:

Statement	Reason
1. Draw the diagonal $\overline{AC}$	
2. $\triangle ACD \cong \triangle CAB$	SSS
3. $\angle ACD \cong \angle CAB$	Corresponding parts
4. $\angle ACB \cong \angle DAC$	Corresponding parts
5. Therefore ABCD is a parallelogram.	Why?

Theorem 5.7

In a parallelogram, opposite angles are congruent

Proof

It is already proved that a diagonal divides a parallelogram into two congruent triangles. Then the theorem follows from the definition of congruent triangles.

Example 5.6

Show that if one angle of a parallelogram is a right angle, then it is a rectangle.

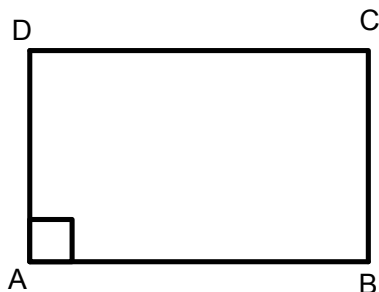


Fig. 5.22

Proof:

Let ABCD be a parallelogram in which $m(\angle A) = 90^\circ$. We have to show that $\angle B \equiv \angle C \equiv \angle D = 90^\circ$.

1. $\overline{AD} \parallel \overline{BC}$ Definition of a parallelogram
2. $m(\angle A) + m(\angle B) = 180^\circ$ Interior angles on the same side of $\overline{AB}$
3. But $m(\angle A) = 90^\circ$
4. So $m(\angle B) = 180 - m(\angle A) = 180^\circ - 90^\circ = 90^\circ$
5. Now, $\angle C \equiv \angle A$ and $\angle D \equiv \angle B$
6. So, $m(\angle C) = 90^\circ$ and $m(\angle D) = 90^\circ$
7. Therefore, each angle of the parallelogram is 90° .
8. Hence, ABCD is a rectangle

Theorem 5.8

If two opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

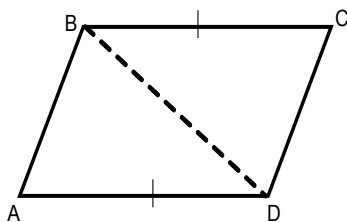


Fig. 5.23

Proof:

Use SAS to show that $\triangle ABD \equiv \triangle CDB$

Then, it follows that $\angle ABD \equiv \angle CDB$.

This in turn implies that $\overline{AB} \parallel \overline{DC}$.

Hence ABCD is a parallelogram.

Theorem 5.9

The diagonals of a parallelogram bisect each other.

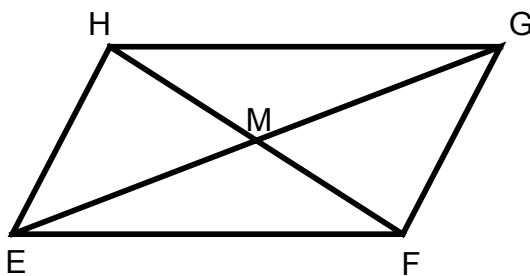


Fig. 5. 24

Proof:

Show that $\triangle HMG \equiv \triangle FME$ and $\triangle HME \equiv \triangle FMG$ (ASA).

The theorem follows from these two congruent triangles that, if two triangles are congruent, the corresponding parts are congruent.

Theorem 5.10

The diagonals of a rhombus are perpendicular bisectors of one another.

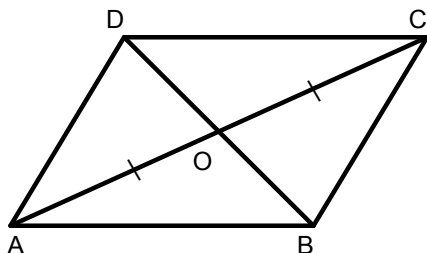


Fig. 5.25

Proof:

It is already proved that the diagonals of a parallelogram bisect each other. It suffices to show that, the diagonals are perpendicular.

1. $\overline{AB} \equiv \overline{BC} \equiv \overline{CD} \equiv \overline{DA}$ Sides of a rhombus
2. $\triangle AOD \equiv \triangle COD$ SSS
3. $\angle DOA \equiv \angle DOC = 90^\circ$
4. Hence, the diagonals are perpendicular.

Theorem 5.11

The line segment joining the midpoint of two sides of a triangle is parallel to the third side and half of it.

Proof:

In Fig. 5.26, points E and F are respectively the midpoints of $\overline{AB}$ and $\overline{AC}$. We want to show that

- a) $\overline{EF} \parallel \overline{BC}$
- b) $EF = \frac{1}{2} BC$

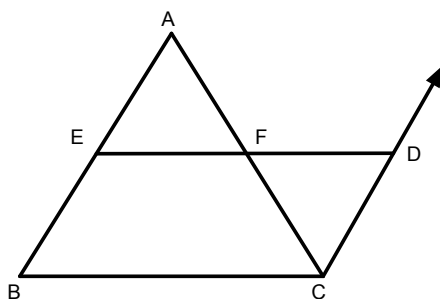


Fig. 5.26

1. Extend $\overline{EF}$ to D such that $\overline{EF} \equiv \overline{DF}$
2. Now $\triangle AEF \equiv \triangle CDF$ SAS
3. Therefore, $\angle EAF \equiv \angle DCF$ Parts of congruent triangles
4. $\overline{AE} \equiv \overline{DC}$ Parts of congruent triangles
5. $\overline{AE} \equiv \overline{BE}$ Given
6. Quadrilateral BCDE is a parallelogram.
7. Hence, the conclusion of the theorem follows

Example 5.7

In $\triangle ABC$ of Fig. 5.27, D, E, and F are respectively, the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E, and F.

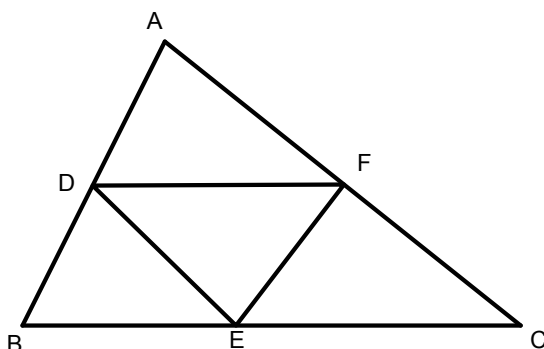


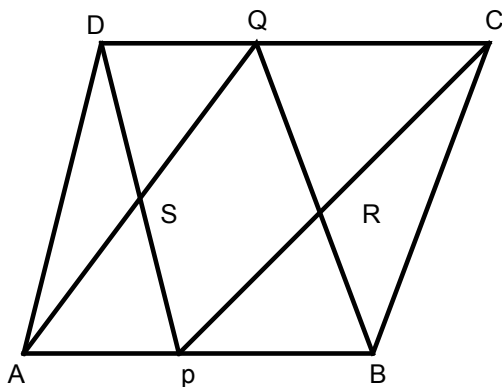
Fig. 5.27

Proof:

1. As D and E are midpoints of sides $\overline{AB}$ and $\overline{BC}$ of $\triangle ABC$,
 $\overline{DE} \parallel \overline{AC}$.
2. Similarly, $\overline{DF} \parallel \overline{BC}$ and $\overline{EF} \parallel \overline{AB}$.
3. Therefore, ADEF, BDFE, and DFCE are all parallelograms.
4. Now $\overline{DE}$ is a diagonal of BDFE, therefore $\triangle BDE \cong \triangle FED$.
5. Similarly $\triangle DAF \cong \triangle FED$ and $\triangle EFC \cong \triangle FED$.
6. Therefore, all the four triangles are congruent.

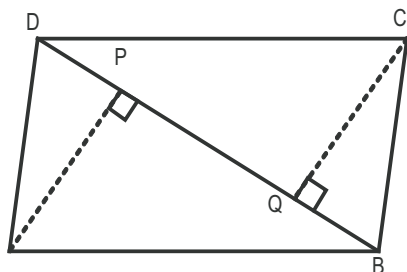
Exercise 5.5

1. In the figure, ABCD is a parallelogram in which P and Q are mid-points of opposite sides $\overline{AB}$ and $\overline{CD}$. If $\overline{AQ}$ intersects $\overline{DP}$ at S and $\overline{BQ}$ intersects $\overline{CP}$ at R, show that:
 - a) APCQ is a parallelogram
 - b) DPBQ is a parallelogram
 - c) PSQR is a parallelogram

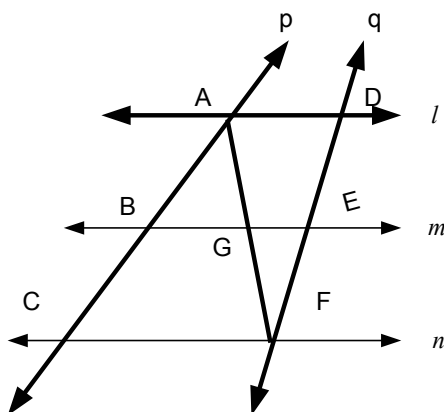


2. ABCD is a rectangle in which diagonal $\overline{AC}$ bisects $\angle A$ as well as $\angle C$. Show that:
 - a) ABCD is a square.
 - b) Diagonal $\overline{BD}$ bisects $\angle B$ and $\angle D$

3. ABCD is a parallelogram and $\overline{AP}$ and $\overline{CQ}$ are perpendicular from vertices A and C on diagonal $\overline{BD}$. Show that
- $\triangle APB \cong \triangle CQD$
 - $\overline{AP} \cong \overline{CQ}$



4. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a square.
5. l , m , and n are three parallel lines intercepted by transversals p and q . If $\overline{AB} \cong \overline{BC}$ show that $\overline{DE} \cong \overline{EF}$.



6. Prove that the figure formed by joining the midpoints of the sides of a rhombus is a rectangle.
7. Prove that the figure formed by joining the midpoints of the sides of a rectangle is rhombus.

Activity 5.9

- Construct a triangle with all sides of different length.
- Measure the length of the sides.
- What do we call such a triangle?
- Draw any triangle ABC as in Fig. 5.28. Find $AB + BC$, $AB + AC$, and $AC + BC$. What do you observe?

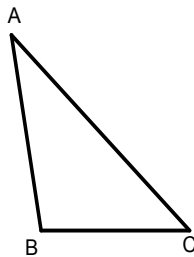


Fig. 5.28

Theorem 5.12

The sum of any two sides of a triangle is greater than the third side.

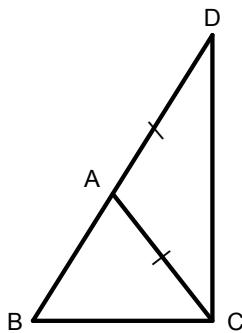


Fig. 5.29

Proof:

We want to show that, in any triangle ABC, $AB + AC > BC$.

Observe the side $\overline{BA}$ has been produced to a point D such that $\overline{AD} \equiv \overline{AC}$.

Therefore, $m(\angle BCD) > m(\angle BDC)$. This implies that $BD > BC$. That is, $AB + AD > BC$. But, $AD = AC$. Hence, $AB + AC > BC$.

Exercise 5.6

1. Which of the following sets of numbers could be the lengths of the sides of a triangle?
 - a) $\{3, 8, 12\}$
 - b) $\{4, 7, 9\}$
 - c) $\{6, 15, 7\}$
 - d) $\{3, 5, 1\}$
2. Suppose that you are to draw a triangle having two sides of length 3cm and 7cm. The third side must have less than _____ and greater than _____
3. Prove that the sum of the length of the diagonals of a quadrilateral is less than the perimeter of the quadrilateral.

5.2 Perimeter and Area

5.2.1 Perimeter of Polygons

Activity 5.10

6. What is the perimeter of a polygon?
7. Find the perimeter of the following:
 - a) A square with each side of length 3 cm.
 - b) A rectangle with sides 4 and 5 units.
 - c) A triangle with sides 3, 4, and 5 units.
 - d) A rectangle with sides a and b units.

The perimeter of a polygon is the sum of the lengths of its sides.

Example 5.8

In figure 5.30 ABCD is a rectangle with $AB = 10$ units and $AD = 15$ units, and $\triangle EFG$ is an equilateral triangle with sides 12 units. Find the perimeter of the shaded region.

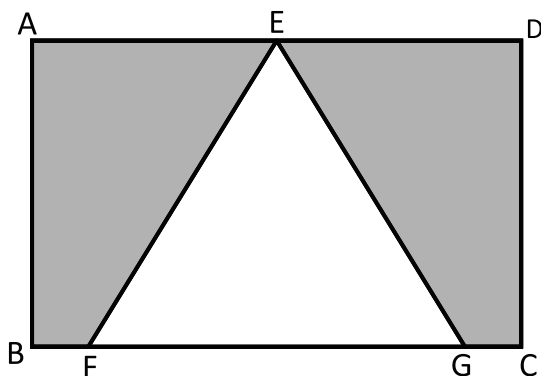


Fig. 5.30

Solution:

The perimeter P of the shaded region is:

$$P = AB + BF + FE + EG + GC + CD + DA.$$

Moreover, $BF + FG + GC = BC$

$$BF + 12 + GC = 15$$

$$BF + GC = 3.$$

Hence, $P = 10 + 3 + 12 + 12 + 10 + 15 = 62$ units.

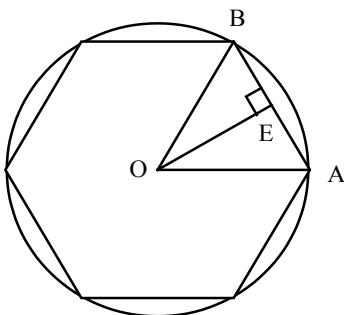
Activity 5.11

1. Draw regular polygons of the following sides and find the perimeter of each polygon.
 - a) An equilateral triangle with each side of length 2 cm.
 - b) A square with each side of length 2 cm.
 - c) A regular pentagon with each side of length 2 cm.
2. What is the perimeter of:
 - a) an equilateral triangle, where each side is s units?
 - b) a square with side s units?
 - c) a regular pentagon with side s units?
3. Find a formula for the perimeter of a regular polygon of n sides, where each side is s units.

The perimeter P of a regular polygon of n sides each of length s units is: $P = ns$ units.

Activity 5.12

1. a) Draw a circle of radius 2 cm. In the circle inscribe a square.
b) Find the sides of the square.
c) Find the perimeter of the square.
2. In Fig. 5.31 a regular hexagon is inscribed in a circle of radius r units.

**Fig. 5.31**

To find the perimeter of the hexagon we have to find the lengths of the sides.

- a) $OB = r$ units. Why?
- b) The point E is the midpoint of $\overline{AB}$. Why?
- c) $m(\angle AOB) = \frac{360^\circ}{6} = 60^\circ$. Why?
- d) $\frac{1}{2}EB = r \times \sin(30^\circ)$. Why?
- e) $AB = 2r \times \sin(30^\circ)$. Why?
- f) The perimeter P of the hexagon inscribed in the circle with radius r units is equal to:

$$P = 6 \times 2r \sin 30^\circ$$

$$= 12r \times \frac{1}{2} = 6r \text{ units. Why?}$$

- Repeat task 2 for a regular pentagon inscribed in a circle with radius r units; for regular octagon and a regular decagon. (i.e. Use the discussions in the above a) to f) to find the perimeter).
- Try to suggest a conjecture about the formula for the perimeter of regular n sided polygon inscribed in a circle with radius r units.

The perimeter P of a regular n sided polygon inscribed in a circle

with radius r units is: $P = 2nr \sin\left(\frac{180^\circ}{n}\right)$

Example 5.9

Find the perimeter of a regular decagon inscribed in a circle of radius 5 cm.

Solution:

In a decagon we have 10 sides and r is equal to 5 cm. Then the perimeter P of the inscribed decagon is given by:

$$P = 2nr \sin\left(\frac{180^\circ}{n}\right) = 2(10)5 \sin\left(\frac{180^\circ}{10}\right) = 100 \sin 18^\circ = 31 \text{ cm.}$$

Activity 5.13

- In the figure 5.32 a regular hexagon is circumscribed about a circle of radius r units.

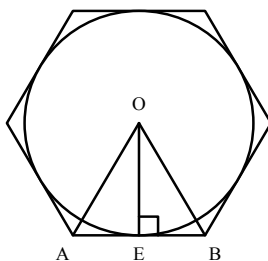


Fig. 5.32

To find the perimeter of the hexagon we have to find the lengths of the sides.

- a) $OE = r$ units. Why?
- b) The point E is the midpoint of $\overline{AB}$. Why?
- c) $m(\angle AOB) = \frac{360^\circ}{6} = 60^\circ$. Why?
- d) $\frac{1}{2}AE = r \times \tan(30^\circ)$. Why?
- e) $AB = 2r \times \tan(30^\circ)$. Why?
- f) The perimeter P of the hexagon circumscribed about the circle with radius r units is equal to:

$$P = 6 \times 2r \tan 30^\circ. \text{ Why?}$$

$$= 12r \times \frac{1}{\sqrt{3}}$$

$$= 12 \times \frac{\sqrt{3}}{3} r$$

$$= 4\sqrt{3} r \text{ Why?}$$

2. Repeat task 1 for a square circumscribed about a circle with radius r units; for regular pentagon and a regular octagon. i.e. use the discussions in a) to f) above to find the perimeter.
3. Try to suggest a conjecture about the formula for the perimeter of regular n-sided polygon circumscribed about a circle with radius r units.

The perimeter P of a regular n-sided polygon circumscribed about a circle with radius r units is:

$$P = 2nr \tan \left(\frac{180^\circ}{n} \right)$$

Example 5.10

Find the perimeter of a regular nonagon circumscribed about a circle of radius 6 cm.

Solution

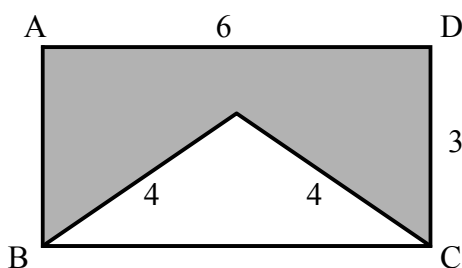
In a nonagon we have 9 sides and the radius r is 6 cm. Then the perimeter P is equal to:

$$P = 2nr \tan\left(\frac{180^\circ}{n}\right) = 2(9)6 \tan\left(\frac{180^\circ}{9}\right) = 108 \tan 20^\circ$$

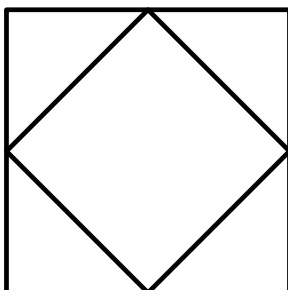
$$= 38.8 \text{ cm.}$$

Exercise 5.7

- Find the perimeter of the shaded region inside the rectangle ABCD.



- Given a square with a side of length 8 cm, a quadrilateral is formed by joining the midpoints of the square as shown in the figure below.
 - Find the lengths of the sides of the quadrilateral.
 - Find the perimeter of the quadrilateral.



3. A 36 cm piece of wire is bent to form an equilateral triangle. Find the length of the side of the equilateral triangle.
4. Find the perimeter of the following regular polygons inscribed in a circle.
 - a) Equilateral triangle inscribed in a circle of radius 3 cm.
 - b) Heptagon inscribed in a circle of radius 4 cm.
 - c) Octagon inscribed in a circle of radius 2 cm.
5. Find the perimeter of the following regular polygons circumscribed about a circle.
 - a) A square circumscribed about a circle of radius 3 cm.
 - b) Pentagon circumscribed about a circle of radius 4 cm.
 - c) Decagon circumscribed about a circle of radius 2 cm.

5.2.2 Area of Polygons

Activity 5.14

1. Find the area of the following.
 - a) A square with side 4 cm.
 - b) A rectangle with sides 3 and 4 units, respectively.
 - c) A rectangle with sides a and b units, respectively.
2. Consider the rectangle ABCD with $AB = a$ units and $BC = b$ units.

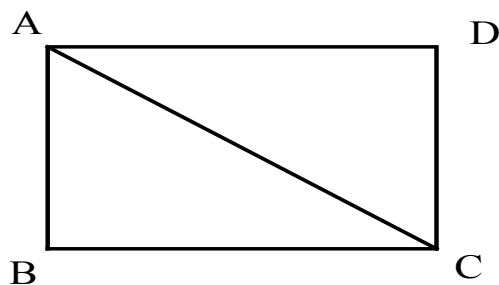


Fig. 5.33

- a) The area A of rectangle $ABCD$ is $A = ab$ sq. units. Why?
- b) $a(\square ABCD) = a(\triangle ABC) + a(\triangle ADC)$. Why?
- c) $\triangle ABC \equiv \triangle ADC$. Why?
- d) $a(\triangle ABC) = a(\triangle ADC)$. Why?
- e) $a(\triangle ABC) = \frac{1}{2}ab$. Why?

3. Consider $\triangle ABC$ (figure 5.34), with AB is b units, CD is h units.

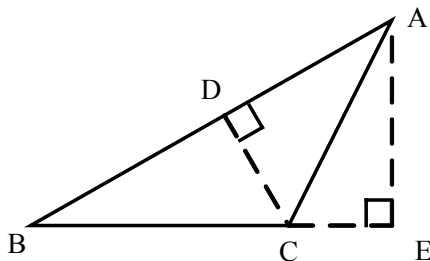


Fig. 5.34

- a) $a(\triangle BCD) = \frac{1}{2}(BD)(CD)$ sq. units. Why?
- b) $a(\triangle ADC) = \frac{1}{2}(CD)(AD)$ sq. units. Why?
- c) $a(\triangle ABC) = a(\triangle ACD) + a(\triangle BCD)$. Why?
- d) $a(\triangle ABC) = \frac{1}{2}(CD)(BD)$ sq. units
 $+ \frac{1}{2}(CD)(AD)$ sq. units. Why?
- e) $a(\triangle ABC) = \frac{1}{2}(CD)(AD + DB)$ sq. units
 $= \frac{1}{2}(CD)(AB)$ sq. units. Why?
 $= \frac{1}{2}bh$ sq. units. Why?

4. Consider $\triangle ABC$ (figure 5.34), where BC is c units, AE is k units.

Show that $a(\triangle ABC) = \frac{1}{2}ck$ sq. units.

The area A of a right triangle with legs a and b is given by:

$$A = \frac{1}{2}ab.$$

The area A of $\triangle ABC$ with $AC = b$ and the altitude $\overline{BD}$ to the

side $\overline{AC}$ is h is given by: $A = \frac{1}{2}bh$.

Example 5.11

Find the area of the quadrilateral $ABCD$ as shown in Fig. 5.35.

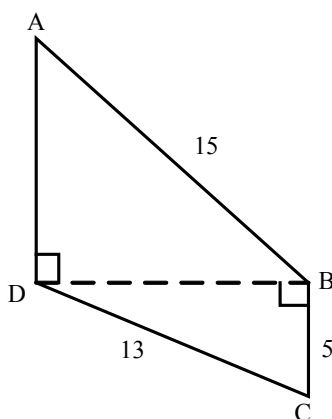


Fig. 5.35

Solution:

Note: $\triangle ABD$ and $\triangle CBD$ are right triangles.

Area of the quadrilateral $ABCD = a(\triangle ABD) + a(\triangle CBD)$

In order to find the area of the two triangles we have to find the legs of each right triangle.

Using Pythagoras' Theorem we have the following:

$$DB^2 + 5^2 = 13^2$$

$$DB^2 = 169 - 25 = 144. \text{ Hence, } DB = 12$$

$$AD^2 + 12^2 = 15^2$$

$$AD^2 = 225 - 144 = 121. \text{ Hence, } AD = 11.$$

Therefore,

$$\text{area of the quadrilateral } ABCD = a(\triangle ABD) + a(\triangle CBD)$$

$$= \frac{1}{2}(11)(12) + \frac{1}{2}(5)(12)$$

$$= 96$$

Activity 5.15

1. Consider an equilateral triangle with side a units, as shown in the figure below.

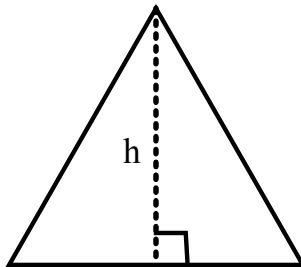


Fig. 5.36

- a) Express h in terms of a .
- b) Find the area A of the equilateral triangle
- c) Find the perimeter P of the equilateral triangle.
- d) Show that the area A of the equilateral triangle is equal to:

$$A = \sqrt{s(s-a)(s-a)(s-a)}, \text{ where } s = \frac{1}{2}P$$

2. In Fig. 5.37 below, $\triangle ABC$ is a right triangle with $AB = b$, $AC = c$ and $CB = a$.

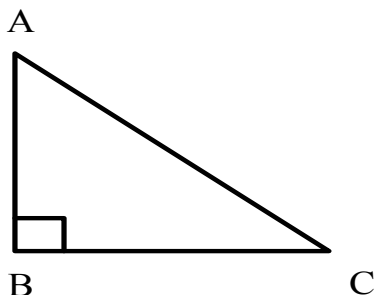


Fig. 5.37

- Find the area A of $\triangle ABC$.
- Express c in terms of a and b using Pythagoras' Theorem.
- Find the perimeter P of $\triangle ABC$ in terms of a and b .
- Show that the area A of the right triangle is equal to:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}P$$

Using the above discussions, we have the following area formula for any triangle, whenever the three sides are given.

The area A of $\triangle ABC$ with $AC = b$, $CB = a$ and $AB = c$ is equal to: $A(\triangle ABC) = \sqrt{s(s-c)(s-a)(s-b)}$, where $\frac{1}{2}(a+b+c) = s$

Example 5.12

Find the area of the quadrilateral ABCD, where $m(\angle BAD) = 90^\circ$.

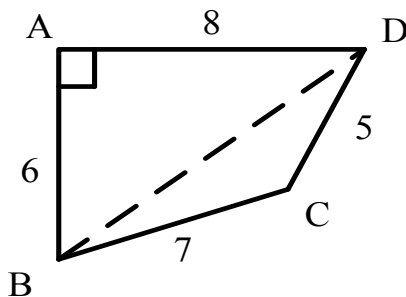


Fig. 5.38

Solution:

$$\text{Area of } ABCD = a(\triangle ABD) + a(\triangle BCD)$$

$$a(\triangle ABD) = \frac{1}{2}(6)(8) = 24$$

To find $a(\triangle BCD)$, we have to find BD.

$$BD^2 = 6^2 + 8^2 = 100, \text{ since } \triangle ABD \text{ is a right triangle.}$$

$$BD = 10.$$

$$\text{Then we have } s = \frac{1}{2}(10 + 7 + 5) = 11.$$

Therefore,

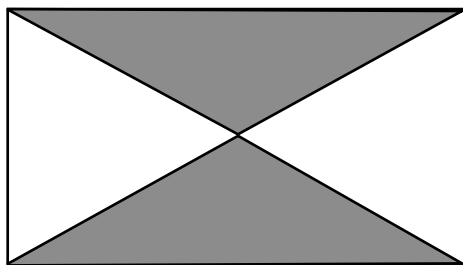
$$\begin{aligned} a(\triangle BCD) &= \sqrt{s(s-c)(s-a)(s-b)} = \sqrt{11(11-10)(11-7)(11-5)} \\ &= 2\sqrt{66}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Area of } ABCD &= a(\triangle ABD) + a(\triangle BCD) \\ &= 24 + 2\sqrt{66} = 2(12 + \sqrt{66}) \text{ sq. units} \end{aligned}$$

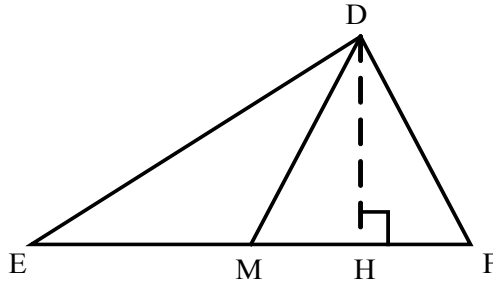
Exercise 5.8

1. In the figure below the quadrilateral is a rectangle with sides 8 and 3 units respectively. Find the area of the shaded region.

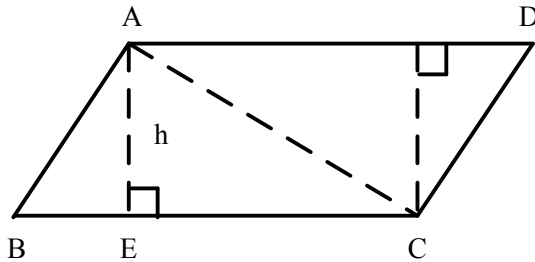


2. A 36 cm piece of wire is bent to make a rectangle with sides x and $2x$ cm respectively.
 - a) Find x .
 - b) Find the area of the rectangle.

3. In figure below $\overline{DH} \perp \overline{EF}$ and M is the midpoint of $\overline{EF}$ ($\overline{DM}$ is the median). Show that the area of $\triangle DME$ is half of the area of $\triangle DEF$.

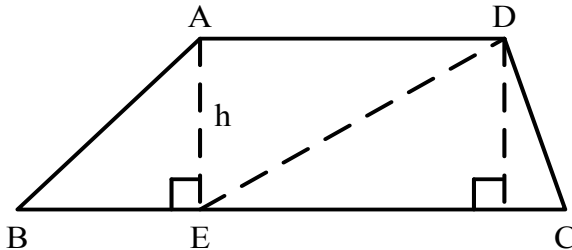


4. In the figure below ABCD is a parallelogram with $AD = b$ units and the altitude to BC is h units. Show that the area A of the parallelogram ABCD is $A = bh$ sq. units.



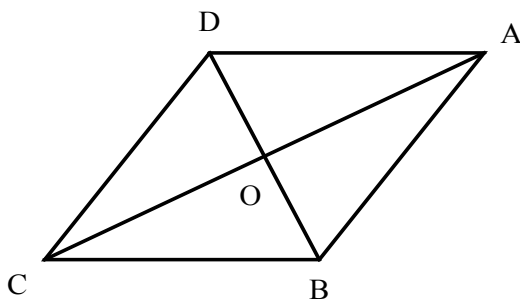
5. In the figure below ABCD is a trapezium ($\overline{AD} \parallel \overline{BC}$), with $AD = b_1$ units, $BC = b_2$ and the altitude to BC is h units. Show that the area A of the trapezium ABCD is:

$$A = \frac{1}{2}(b_1 + b_2)h \text{ sq. units.}$$



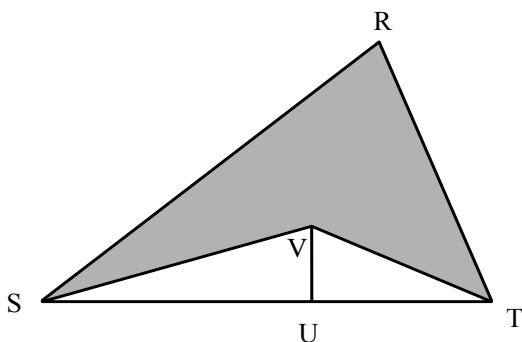
6. Show that the area A of the rhombus $ABCD$ is:

$$A = \frac{1}{2}ab \text{ sq. units, where } AC = a \text{ units and } BD = b \text{ units.}$$



7. Find the area of the shaded region as given in the figure below.

$$RS = 21, RT = 13, ST = 20, UV = 4 \text{ and } \overline{UV} \perp \overline{ST}.$$



Activity 5.16

1. a) Draw a square with side a units.
- b) Draw the two diagonals of the square.
- c) Show that the four triangles formed by the two diagonals are isosceles triangles and they are congruent.
- d) Show that the area A of each isosceles triangle is:

$$A = \frac{1}{4}a^2 \text{ sq. units, where } a \text{ is the side of the square.}$$

2. Draw a regular hexagon with side s units.
 - a) Construct the three diagonals that intersect at point O , where O is the centre of the regular hexagon.
 - b) How many triangles are formed by the construction in (a)?
 - c) Show that the triangles formed by the diagonals are equilateral triangles and they are congruent.
 - d) Show that the area A of each equilateral triangle is:

$$A = \frac{\sqrt{3}}{4}s^2 \text{ sq. units, where } s \text{ is the side of the hexagon.}$$

- e) Show that the area A of the regular hexagon with s units is

$$A = \frac{3\sqrt{3}}{2}s^2 \text{ sq. units.}$$

Example 5.13

Find the area of a regular hexagon whose side is 4 cm.

Solution:

The area A of a regular hexagon with s units is given by:

$$A = \frac{3\sqrt{3}}{2}s^2 \text{ sq. units.}$$

Hence, if $s = 4$, then the area A is equal to:

$$A = \frac{3\sqrt{3}}{2}(4)^2 \text{ sq. cm}$$

$$= 24\sqrt{3} \text{ sq. cm.}$$

Activity 5.17

1. Consider $\triangle ABC$ with $m(\angle ACB) = t$, $AC = b$ and $BC = a$, as shown in Fig. 5.39 below.

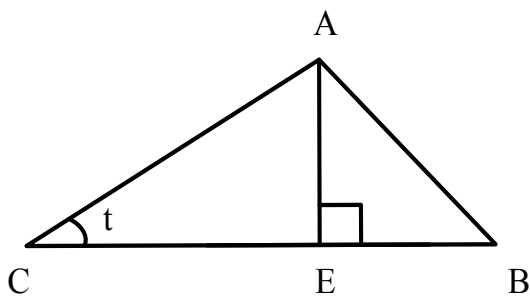


Fig. 5.39

- a) $\triangle AEC$ is a right triangle. Why?
 - b) $\sin t = \frac{AE}{b}$. Why?
 - c) $AE = b \sin t$. Why?
 - d) $a(\triangle ABC) = \frac{1}{2}a(AE)$. Why?
 - e) $a(\triangle ABC) = \frac{1}{2}ab \sin t$. Why?
2. In figure 5.40, a regular pentagon is inscribed in a circle of radius r units.

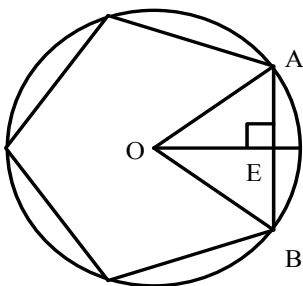


Fig. 5.40

- a) $OA = r$ units. Why?
- b) $\triangle AOB$ is an isosceles triangle. Why?
- c) How many triangles are there in the pentagon which are congruent to $\triangle AOB$?

d) $m(\angle AOB) = \frac{360^\circ}{5} = 72^\circ$. Why?

e) $a(\Delta AOB) = \frac{1}{2}OA \times OB \sin 72^\circ$. Why?

f) $a(\Delta AOB) = \frac{1}{2}r^2 \sin 72^\circ$. Why?

g) Area A of the pentagon ABCDE = $5 a(\Delta AOB)$. Why?

h) The area A of the pentagon = $5 \left(\frac{1}{2} r^2 \sin 72^\circ \right)$. Why?

3. Repeat task (2) for a regular hexagon inscribed in a circle with radius r units; for regular octagon and a regular decagon. (i.e use the discussions in Activity 5.17 task (2) in (a to h) above to find the area).

The area A of a regular n -sided polygon inscribed in a circle with radius r units is: $A = \frac{1}{2}nr^2 \sin\left(\frac{360^\circ}{n}\right)$.

Example 5.14

Find the area of a regular hexagon inscribed in a circle of radius 4 cm.

Solution:

To find the area A of the hexagon inscribed in a circle of radius 4cm, we use,

$$A = \frac{1}{2}nr^2 \sin\left(\frac{360^\circ}{n}\right), \text{ where } n = 6 \text{ and } r = 4 \text{ cm.}$$

$$\begin{aligned} \text{Hence, } A &= \frac{1}{2}6(4)^2 \sin\left(\frac{360^\circ}{6}\right) \\ &= 3(16) \sin 60^\circ = 48 \frac{\sqrt{3}}{2} \\ &= 24\sqrt{3} \end{aligned}$$

Activity 5.18

1. In the following figure a regular pentagon is circumscribed about a circle of radius r units.

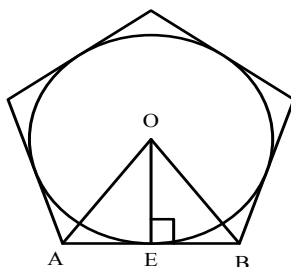


Fig. 5.41

- a) $OE = r$ units. Why?
 - b) $\triangle AOB$ is an isosceles triangle. Why?
 - c) $a(\triangle AOB) = \frac{1}{2}OE \times AB$. Why?
 - d) $m(\angle AOE) = 36^\circ$. Why?
 - e) $AE = r \tan 36^\circ$ and $AB = 2r \tan 36^\circ$. Why?
 - f) $a(\triangle AOB) = r^2 \tan 36^\circ$. Why?
 - g) Area A of the pentagon ABCDE = $5a(\triangle AOB)$. Why?
 - h) The area A of the pentagon = $5r^2 \tan 36^\circ$. Why?
2. Repeat task (1) for a regular hexagon circumscribed about a circle with radius r units; for a square and a regular octagon. (i.e use the discussions in activity 5.18 in (a to h) above to find the area).

The area A of a regular n sided polygon circumscribed about a circle with radius r units is:

$$A = nr^2 \tan \left(\frac{180^\circ}{n} \right).$$

Example 5.15

Find the area of a square circumscribed about a circle of radius 3cm.

Solution:

In this case $n = 4$ and $r = 3$.

$$\text{Hence, } A = 4(3)^2 \tan\left(\frac{180^\circ}{4}\right) = 36 \tan 45^\circ = 36 \text{ sq. cm.}$$

Exercise 5.9

1. Draw the following inscribed regular polygons in a circle and find their areas.
 - a) An equilateral triangle in a circle of radius of 2 cm.
 - b) Heptagon in a circle of radius of 3 cm.
 - c) Octagon in a circle of radius of 4 cm.
 - d) Nonagon in a circle of radius of 4 cm.
2. Draw the following circumscribed regular polygons about a circle and find their areas.
 - a) An equilateral triangle in a circle of radius of 2 cm.
 - b) Heptagon in a circle of radius of 3 cm.
 - c) Octagon in a circle of radius of 4 cm.
 - d) Decagon in a circle of radius of 4 cm.
3. Show that the area A of a regular n sided polygon inscribed in a circle of radius r is equal to:

$$A = nar \sin\left(\frac{180^\circ}{n}\right),$$

where a is the perpendicular distance from the centre to the midpoint of the side of the polygon (known as apothem).

5.2.3 Area and Perimeter of Sectors

Activity 5.19

- What is the circumference of a circle?
 - What is a sector of a circle?
 - Draw a sector with central angle 60° and radius 3 cm; with central angle 135° and radius 2 cm.
- Consider the following circle with centre O, and $\overline{DA}$ a diameter of length 8 cm.

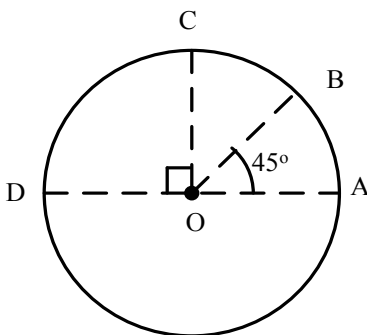


Fig. 5.42

- What is the circumference of the circle?
 - What is the ratio 180° to 360° ?
 - What is the ratio of the length of the semicircle to the circumference of the circle?
 - What is the ratio 90° to 360° ?
 - What is the ratio of the length of $\widehat{ABC}$ to the circumference of the circle?
 - What is the ratio 45° to 360° ?
 - What is the ratio of the length of the minor $\widehat{AB}$ to the circumference of the circle?
- In a circle of radius r , what can we say about the ratio of central angle θ to 360° and the ratio of the length of the arc with central angle θ to the circumference of the circle?

That is, if $\widehat{AB}$ is the arc with central angle θ in a circle with radius r , then $\frac{\theta}{360^\circ} = \frac{m(\widehat{AB})}{2\pi r}$

From the above we have the following:

The arc length L of a sector with central angle θ in a circle of radius r units is given by: $L = \frac{\pi r \theta}{180^\circ}$.

The perimeter P of a sector with central angle θ and radius r units is given by: $P = 2r + L = 2r + \frac{\pi r \theta}{180^\circ}$

Example 5.16

For the sector in a circle with radius of 5 cm formed by 120° central angle:

- Find the arc length.
- Find the perimeter of the sector.

Solution:

- The arc length $L = \frac{\pi r \theta}{180^\circ} = \frac{5\pi(120^\circ)}{180^\circ} = 5\pi \frac{2}{3} = \frac{10\pi}{3}$.
- The perimeter $P = 2r + L = 2(5) + \frac{10\pi}{3} = \frac{10(3 + \pi)}{3}$

Activity 5.20

- Draw the following circles and find their area.
 - A circle with radius 5 cm.
 - A circle with diameter 4 units.

2. Consider the following circle with centre O and radius 3 cm.

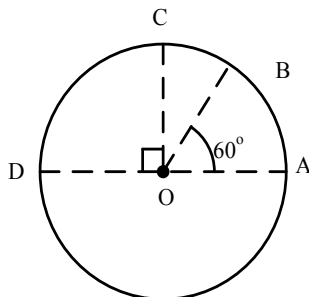


Fig. 5.43

- What is the area of the circle?
 - What is the ratio 180° to 360° ?
 - What is the ratio of the area of the semicircle to the area of the circle?
 - What is the ratio 90° to 360° ?
 - What is the ratio of the area of the sector with central angle 90° to the area of the circle?
 - What is the ratio 45° to 360° ?
 - What is the ratio of the area of the sector with central angle 45° to the area of the circle?
3. In a circle of radius r , what can we say about the ratio of central angle θ to 360° and the ratio of the area of the sector with central angle θ to the area of the circle?

If the area of a sector with central angle θ in a circle of radius r

is A , then $\frac{A}{\pi r^2} = \frac{\theta}{360^\circ}$

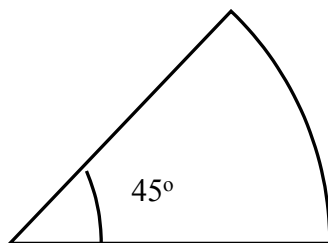
From the above we have the following:

The area A of a sector with central angle θ in a circle of radius r

units is equal to: $A = \frac{\pi r^2 \theta}{360^\circ}$

Example 5.17

Find the area of the following sector.



Radius = 6 units

Fig. 5.44

Solution:

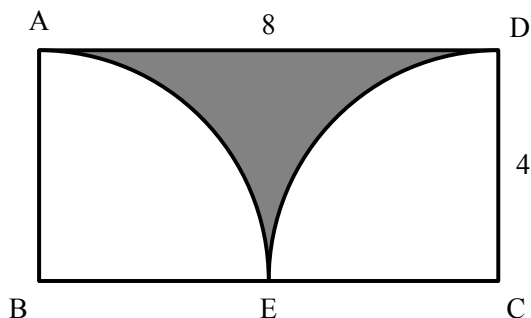
The area A of the sector is given by:

$$\begin{aligned} A &= \frac{\pi r^2 \theta}{360^\circ} = \frac{\pi(6)^2 45^\circ}{360^\circ} = \frac{36\pi}{8} \\ &= 4.5\pi \text{ sq. units} \end{aligned}$$

Exercise 5.10

1. Find the area and perimeter of a sector in a circle with radius of 6 cm and central angle 120° .
2. A sector of a circle has an area of $84\pi \text{ cm}^2$ and a central angle of 70° . Find the radius of the circle.
3. The arc length of a sector with central angle 225° is 10π . Find the radius and the area of the sector.
4. A sector of a circle with radius 6 cm has an area of $15\pi \text{ cm}^2$. Find the arc length.
5. Find the area of a sector formed by central angle of 135° having arc length $9\pi \text{ cm}$.
6. The quadrilateral ABCD is a rectangle with sides 8 cm and 4 cm long. Two circles each with radius 4 cm are drawn, one with

centre B and the other with centre C as shown in the figure below. Find the area and perimeter of the shaded region.



5.2.4 Area of Segments

Activity 5.21

1.
 - a) What is a segment of a circle?
 - b) Draw a sector with radius 3 cm and central angle 45° . Shade the segment of the sector.
 - c) Repeat b) for sectors of radius 3 cm and central angles 90° , 135° , and 60° .
2. Consider the figure below:

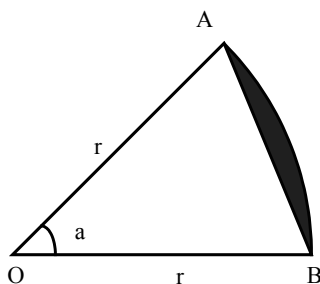


Fig. 5.45

- a) The area of the sector is $\frac{\pi r^2 \theta}{360^\circ}$. Why?
- b) $a(\Delta AOB) = \frac{1}{2} r^2 \sin a$. Why?

c) The area A of the segment is given by:

$$A = \frac{\pi r^2 \theta}{360^\circ} - \frac{1}{2} r^2 \sin a. \text{ Why?}$$

The area A of a segment with central angle θ and radius r units

is equal to: $A = \frac{\pi r^2 \theta}{360^\circ} - \frac{1}{2} r^2 \sin(\theta)$

Example 5.18

Find the area of the segment in Fig. 5.46.

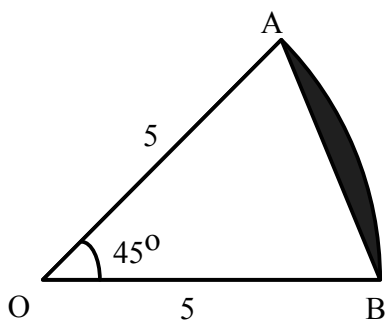


Fig. 5.46

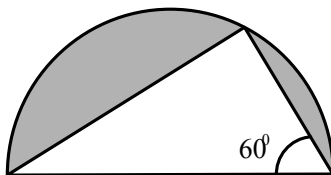
Solution:

The area A of the segment is equal to:

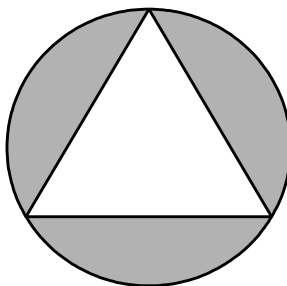
$$\begin{aligned} A &= \frac{\pi(5)^2 45^\circ}{360^\circ} - \frac{1}{2} (5)^2 \sin 45^\circ = \frac{25\pi}{8} - \frac{25}{2} \left(\frac{\sqrt{2}}{2} \right) \\ \frac{25\pi}{8} - \frac{25\sqrt{2}}{4} &= \frac{25}{8} (\pi - 2\sqrt{2}) \end{aligned}$$

Exercise 5.11

1. In a circle of radius of 10 cm find the area of the segment with central angle 30° .
2. Find the area of the segment with radius 4 cm and arc length of the segment π .
3. Find the radius, if the area of a segment with central angle 60° is 12π sq. cm.
4. A rectangle with length 16 cm and width 12 cm is inscribed in a circle. What is the area of the region inside the circle and outside the rectangle?
5. A triangle is inscribed in a semi-circle of radius 6 cm where one side of the triangle is the diameter of the semi-circle, as shown in the figure below. Find the area of the shaded region.



6. In the figure below, an equilateral triangle is inscribed in a circle of radius of 4 cm. Find the area and perimeter of the shaded region.



5.3 Surface Areas and Volumes of Solid Figures

5.3.1 Surface Areas and Volumes of Prisms

Activity 5.22

- Fig. 5.47 shows a cuboid (rectangular prism) of length 4 cm, width 3 cm and height 2 cm.

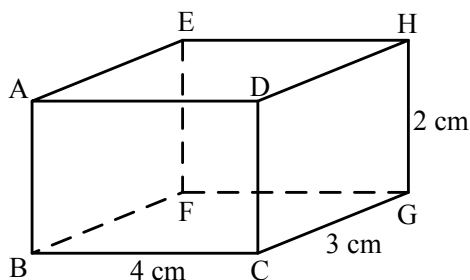


Fig. 5.47

- How many faces does the cuboid have? Name the faces.
 - What are the shapes of the faces of the cuboid?
 - Which of the faces of the cuboid are congruent?
 - What is the area of each face of the cuboid?
 - What is the area of paper needed to exactly cover all faces of the cuboid?
- What is the area of paper needed to exactly cover all faces of the cuboid in Fig. 5.48?

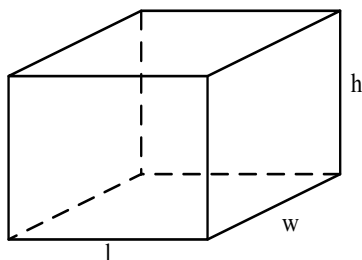


Fig. 5.48

The area of the surface of a solid figure is called the **total surface area** of the solid.

If a rectangular prism has length l , width w and height h , then the lateral surface area (LA), the base area (B) and the total surface area (SA) of the prism are

$$LA = 2lh + 2wh$$

$$B = lw$$

$$SA = LA + 2B = 2lh + 2wh + 2lw$$

Example 5.19

The bases of the right triangular prism in Fig. 5.49 are right-angled triangles. Find the total surface area of the prism.

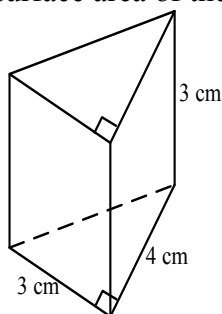


Fig. 5.49

Solution:

Since the bases are right-angled triangles, using the Pythagoras' Theorem the length of the hypotenuse c of the bases is given by

$$c = \sqrt{3^2 + 4^2} = 5 \text{ cm.}$$

Now, the base area B is given by

$$B = \frac{1}{2}ac = \frac{1}{2} \times 3 \times 4 = 6 \text{ cm}^2$$

The lateral surface area LA is the sum of the areas of the three rectangular faces of dimensions: 3 cm by 3 cm, 4 cm by 3 cm, and 5 cm by 3 cm. That is,

$$LA = 3 \times 3 + 4 \times 3 + 5 \times 3 = 36 \text{ cm}^2$$

Hence, the total surface area SA is given by

$$SA = 2B + LA = 2 \times 12 + 36 = 48 \text{ cm}^2.$$

If you look at a can of tomato soup, at a balloon, or at a box of biscuits, you will notice that there are different ways and things that take up space.



Fig. 5.50

In the example of the tomato soup can, the tomato soup inside is taking up space. In the example of the balloon, air or gas is taking up space. When you look at the box of biscuits, you will notice that the biscuits are taking up space. This shows that solids, liquids and gases all take up space.

The **volume** of a solid figure is the amount of space it takes up.

The units used for measuring volume are cubic units depending on the unit of measurement. If the unit of measurement is metre, then the unit of volume is cubic metre which is written as m^3 . If the unit is centimetre, then the unit of volume is cubic centimetre which is written as cm^3 .

One cubic metre is the space taken up by a cube of length 1m, width 1m and height 1m. Similarly, one cubic centimetre is the space taken up by a cube of length 1cm, width 1cm and height 1cm. as shown Fig. 5.51.

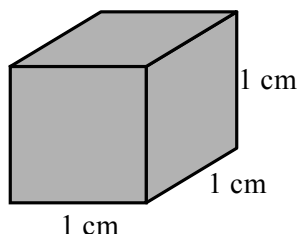


Fig. 5.51: A solid of volume 1 cm^3

Activity 5.23

The solid in Fig. 5.52 is a cuboid made up of **centimetre cubes**: a solid cube of 1cm long, 1cm wide and 1cm high. The cubes were laid in such a way that no space is left among them.

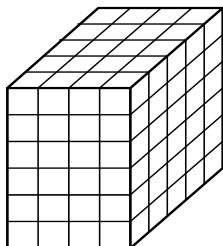


Fig. 5.52

- a) What are the length, width and height of the cuboid?
- b) How many centimetre cubes are on the bottom layer? Why?
- c) What is the area of the bottom base of the cuboid?
- d) How many layers of centimetre cubes are there?
- e) How many centimetre cubes are used to make the cuboid?
- f) What is the volume of each centimetre cube used to make the cuboid?
- g) What is the volume of the cuboid? Why?

Activity 5.24

Suppose you have a certain number of centimetre cubes. You wish to keep your cubes in a box which is in the shape of a cuboid. The box is 6cm long, 5cm wide and 4cm high, as shown in Fig. 5.53.

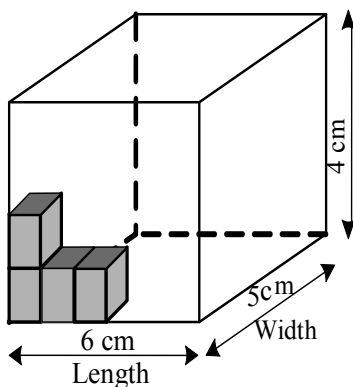


Fig. 5.53

- How many centimetre cubes will fit along the length of the box?
- How many centimetre cubes will fit along the width of the box?
- How many centimetre cubes will fit in the bottom layer of the box? Why?
- What is the area of the bottom base of the box?
- How many layers of centimetre cubes will the box hold?
- How many centimetre cubes will the box hold? Why?
- What is the volume of the box? Why?

Example 5.20

A cuboid has length 9cm, width 7cm and height 5cm. Find the volume of the cuboid.

Solution:

We are given $b = 9$, $w = 7$ and $h = 5$. Then,

$$\begin{aligned} V &= Bh = bwh \\ &= 9 \times 7 \times 5 = 315 \end{aligned}$$

Thus, the volume of the cuboid is 315 cm^3 .

Activity 5.25

The bases of the right triangular prism in Fig. 5.54 are right-angled triangles. Suppose you wish to make an exact model of this prism by fitting together centimetre cubes. The cubes may be sliced to fit each other into the exact shape of the prism as shown in Fig 5.55

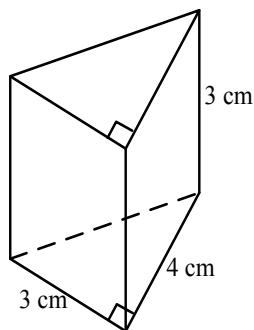


Fig. 5.54

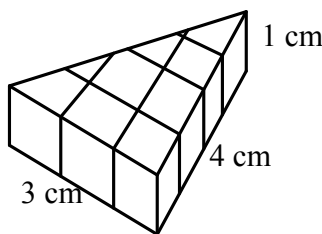


Fig. 5.55

- a) What would be the area of a base of the model prism?
- b) How many centimetre cubes would be fitted on the bottom base of the model prism? Why?
- c) How many layers of centimetre cubes would the model prism contain?
- d) How many centimetre cubes would be used to make the model prism?
- e) What would the volume of the model prism be?
- f) What is the volume of the right triangular prism?

If a right prism has base area B and height h , then the volume V of the prism is given by $V = Bh$

Example 5.21

Find the volume of a right prism of height 10 centimetres when the base is an equilateral triangle with sides of length 12 centimetres.

Solution:

The right triangular prism is shown in Fig 5.58.

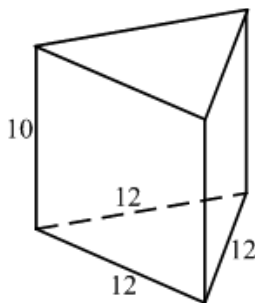


Fig. 5.56

Since the base of the prism is an equilateral triangle with length of a side $s = 12$ and measure each angle 60° , the base area B of the triangle is given by

$$B = \frac{1}{2}s^2 \sin \theta = \frac{1}{2} \times 12^2 \times \sin 60^\circ = 36\sqrt{3}$$

Then,

$$V = Bh = 36\sqrt{3} \times 10 = 360\sqrt{3}$$

Therefore, the volume of the prism is $360\sqrt{3} \text{ cm}^3$.

Volumes of liquids or capacities of liquid containers are usually described using litres (ℓ) and millilitres (ml). Consider the following example.

Example 5.22

A water tank is in the form of a cuboid. It is 120cm long, 80cm wide and 50cm high. Find the capacity of the tank in litres.

Solution:

We are given $b = 120$, $w = 80$ and $h = 50$. Then, the volume V of the tank in cm^3 is given by

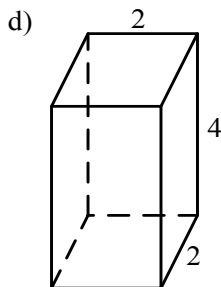
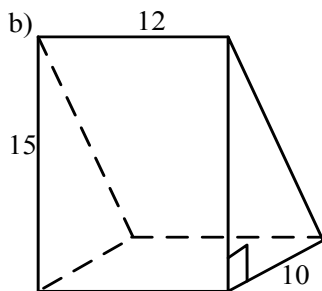
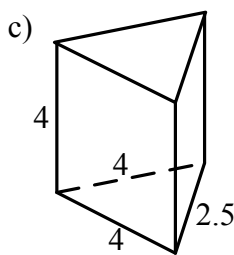
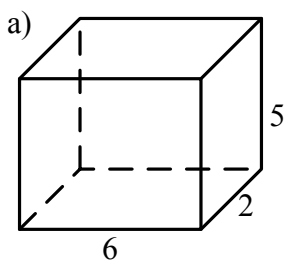
$$\begin{aligned} V &= Bh = bwh \\ &= 120 \times 80 \times 50 \\ &= 480,000 \end{aligned}$$

Since $1\ell = 1000 \text{ cm}^3$, the capacity of the tank in litres is given by

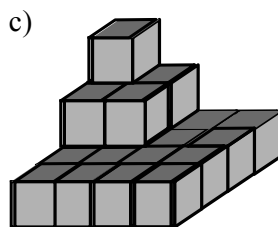
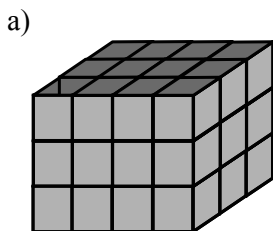
$$\frac{480,000 \text{ cm}^3}{1000 \text{ cm}^3 / \ell} = 480\ell.$$

Exercise 5.12

1. Find the total surface area of each of the following solids. The measures are in centimetres.

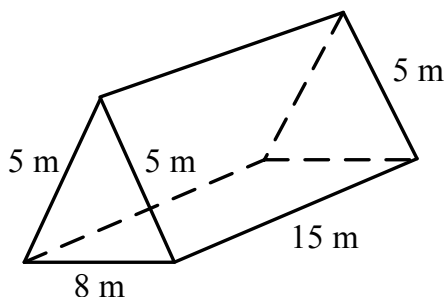


2. Each of the following solids is made up of centimetre cubes. Find the volume of each solid by counting the number of centimetre cubes.

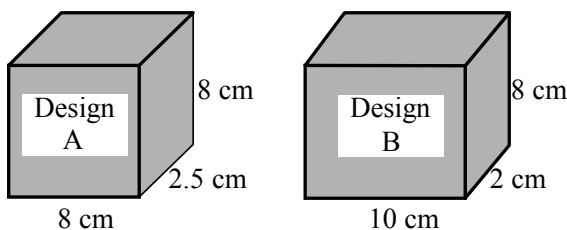


3. Find the volume of each of the solids in problem (1).
4. A cubic metre of water from a sea contains about 1.5 grams of salt. An aquarium is filled with water from the sea. What is the weight of salt in the aquarium if the aquarium is a rectangular prism of length 10m, width 8m and height 6m?
5. A right prism has a lateral edge of 5 cm long and the perimeter of its base is 45 cm. Find the area of the lateral surface of the prism.

6. The altitude of a right pentagonal prism is 10cm and the lengths of the edges of the bases are 3cm, 5cm, 4cm, 7cm and 5cm. Find the area of the lateral surface of the prism.
7. A tent in the shape of an isosceles triangular prism is erected on a flat surface as shown in the following figure. What is the surface area of the entire tent?



8. A manufacturing company is choosing between two box designs of volume 160 cm^3 which are shown below. The company would like to keep the amount of cardboard used for the box to a minimum in order to save costs. Which box design do you think meets the company's requirements?



9. A piece of biscuit has the shape of a cuboid of length 5cm, width 3cm and height 0.5cm. The biscuits are packed into a box in the shape of a cuboid of length 20cm, width 9cm and height 8cm. How many biscuits does each box contain?

5.3.2 Surface Areas and Volumes of Cylinders

Activity 5.26

- Fig. 5.57 shows a right cylinder of height 14 cm and radius 7 cm. Take $\pi = 22/7$ in answering the following tasks.

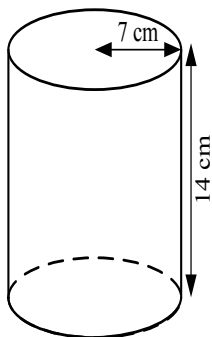


Fig. 5.57

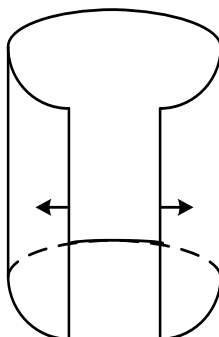


Fig. 5.58

- What is the area of each base of the cylinder?
 - What is the total area of the bases of the cylinder?
 - Supposed the bases of the cylinder are removed and the lateral surface is cut vertically down and unrolled as shown in Fig. 5.58. What is the shape of the resulting surface?
 - What are the dimensions of this unrolled lateral surface?
 - What is the lateral surface area of the cylinder? Why?
 - What is the total surface area of cylinder?
- What is the total surface area of the right cylinder in Fig. 5.59?

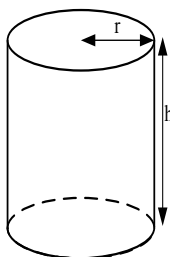


Fig. 5.59

If a right cylinder has radius r and height h , then the lateral surface area (LA), the base area (B) and the total surface area (SA) are given by

$$LA = 2\pi rh$$

$$B = \pi r^2$$

$$SA = LA + 2B = 2\pi rh + 2\pi r^2$$

Example 5.23

Find the total surface area of a right cylinder which has a radius of 14cm and a height of 8cm. Take $\pi = 22/7$.

Solution:

We are given $r = 14$ and $h = 8$. Then, the lateral surface area LA and the base area B of the cylinder are given by

$$LA = 2\pi rh = 2 \times \frac{22}{7} \times 14 \times 8 = 704 \text{ cm}^2$$

$$B = \pi r^2 = \frac{22}{7} \times 14^2 = 616 \text{ cm}^2$$

Therefore, the total surface area SA of the cylinder is given by

$$SA = LA + 2B = 704 + 2 \times 616 = 1936 \text{ cm}^2$$

Activity 5.27

- Fig. 5.60 shows a can in the shape of a right cylinder of radius 3.5cm and height 10cm. Suppose you wish to completely fill up this cylindrical can by fitting together centimetre cubes as shown in Fig. 5.61. The cubes may be sliced to fit each other so that the cylindrical can is filled completely. Take $\pi = 22/7$.

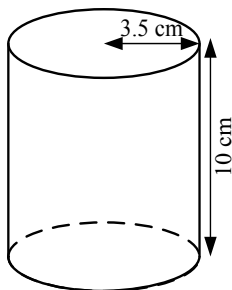


Fig. 5.60

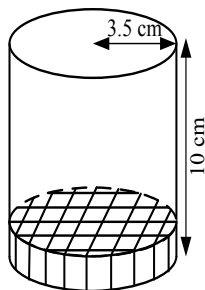


Fig. 5.61

- What is the area of a base of the cylindrical can?
- How many centimetre cubes would be fitted on the bottom base of the cylindrical can? Why?
- How many layers of centimetre cubes would be fitted into the cylindrical can?
- How many centimetre cubes would be needed to completely fill up the cylindrical can?
- What is the volume of the cylindrical can?

2. What is the volume of the right cylinder in Fig. 5.62?

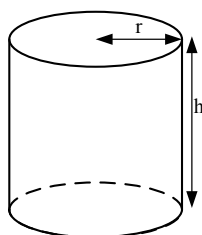


Fig. 5.62

If a right cylinder has base area B and height h , then the volume V of the cylinder is given by $V = Bh = \pi r^2 h$

Example 5.24

Calculate the volume of a right cylinder of base radius 7cm and height 20cm. Take $\pi = 22/7$.

Solution:

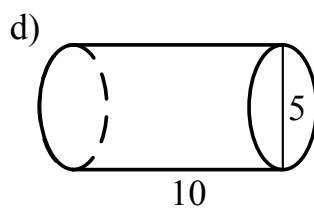
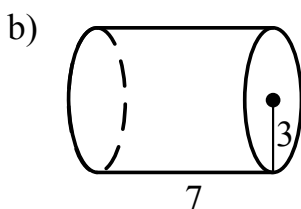
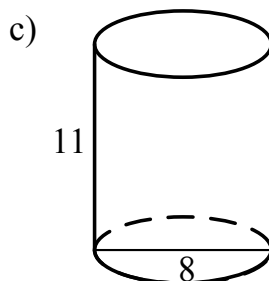
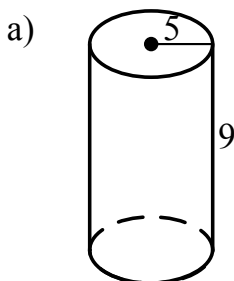
We are given $r = 7$ and $h = 20$. Then,

$$V = \pi r^2 h = \frac{22}{7} \times 7^2 \times 20 = 3080$$

Therefore, the volume of the cylinder is 3080 cm^3 .

Exercise 5.13

1. Find the total surface area of each of the following solids. The measures are in centimetres. Take $\pi = 3.14$.



2. Find the volume of each of the solids in problem (1).
3. A straw is 24cm long and 4mm in diameter. How much liquid can the straw hold? Take $\pi = 3.14$.
4. Cylinder A is eight times as tall as cylinder B, but the diameter of cylinder B is four times the diameter of cylinder A. Which cylinder has larger volume? How many times as large?
5. A tomato paste can has a diameter of 10cm and a height of 15cm. What is the volume of the tomato paste in the can if 0.5cm of space is left at the top of the can to allow for expansion? Take $\pi = 3.14$.
6. An oil can has a diameter of 5cm and a height of 11.5cm. Find the surface area and volume of the can. Take $\pi = 3.14$.
7. A small can of tomato soup is 9cm tall and has a diameter of 6cm. The bases are metal and the rest of the can is cardboard. How much metal and how much cardboard are needed to make a tomato soup can? Take $\pi = 3.14$.

5.3.3 Surface Areas and Volumes of Pyramids

Activity 5.28

1. Fig 5.63 shows a regular square pyramid with slant height 4 cm and sides of the base 6 cm.

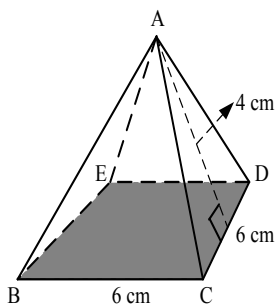


Fig. 5.63

- a) How many faces does the pyramid have? Name the faces.
 - b) What are the shapes of the faces of the pyramid?
 - c) Which of the faces of the pyramid are congruent?
 - d) Draw a net of the pyramid and label its dimensions.
 - e) What is the total area of the net of the pyramid?
 - f) What is the least amount of paper needed to completely cover the faces of the pyramid?
2. What is the least amount of paper needed to completely cover the faces of the regular square pyramid in Fig.5.64?

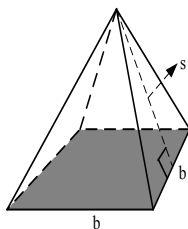


Fig. 5.64

A regular pyramid has a base which is an n -sided regular polygon of side length b . If the slant height of the pyramid is s and the base area is B , then the lateral surface area (LA) and the total surface area (SA) are given by

$$LA = \frac{1}{2}nbs$$

$$SA = LA + B = \frac{1}{2}nbs + B$$

Example 5.25

A regular pyramid has a square base of side length 10cm and height 9cm. What is the total surface area of the pyramid?

Solution:

We are given $n = 4$, $b = 10$ and $h = 9$. In a regular pyramid, the altitude, the slant height and a line segment drawn from the centre of the base to the midpoint of a side of the base form a right-angled triangle as shown in Fig. 5.65.

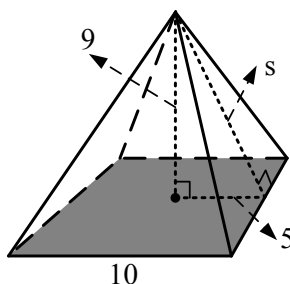


Fig. 5.65

Hence, using the Pythagoras' Theorem, the slant height s of the pyramid is given by:

$$s = \sqrt{\left(\frac{b}{2}\right)^2 + h^2} = \sqrt{\left(\frac{10}{2}\right)^2 + 9^2} = \sqrt{106}$$

Then, the lateral surface area LA is given

$$LA = \frac{1}{2} nbs = \frac{1}{2} \times 4 \times 10 \times \sqrt{106} = 20\sqrt{106}$$

and the base area B is given by

$$B = b^2 = 10^2 = 100$$

Therefore, the total surface area SA of the pyramid is given by

$$SA = LA + B = 20\sqrt{106} + 100 \text{ cm}^2.$$

Activity 5.29

In Fig. 5.66, figure a) shows a cube with vertices A, B, C, D, E, F G and H and side length b. The diagonals of the cube to the vertex C divide the cube into three square pyramids with common apex which is vertex C as shown in figure b). These three square pyramids are congruent and they can be pulled apart, as shown in figures c), d) and e).

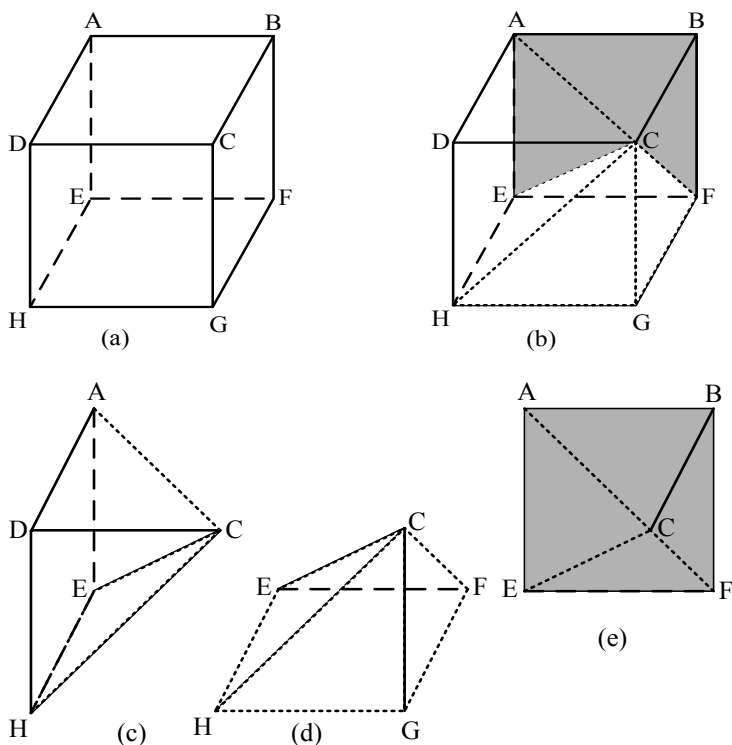


Fig. 5.66

- For the square pyramids in c), d) and e), what are the bases of the pyramids? Are the bases of the pyramids congruent? Why?
- What are the altitudes of the three pyramids? Are they congruent? Why?
- Do all three pyramids have the same volume? Why?
- What is the volume of the cube in figure a)?
- What is the volume of each square pyramid? Why?

If a pyramid has a base area of B and height h , then the volume V of the pyramid is given by: $V = \frac{1}{3}Bh$

Example 5.26

Find the volume of a regular triangular pyramid of height 12cm and base edges of length 8cm.

Solution:

Since the pyramid is a regular triangular pyramid, its base is an equilateral triangle of side length $s = 8$. Then, the base area B of the pyramid is given by:

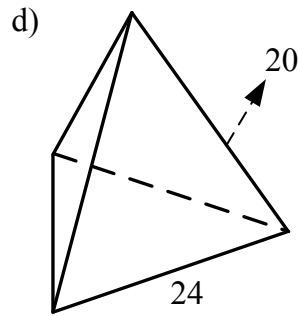
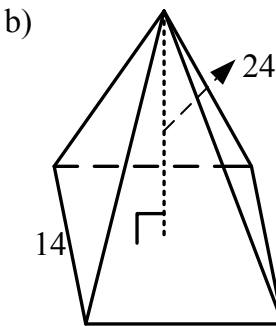
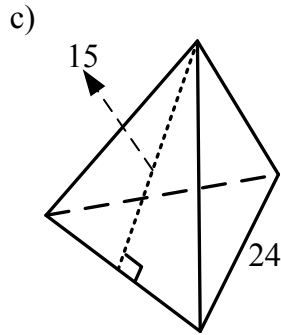
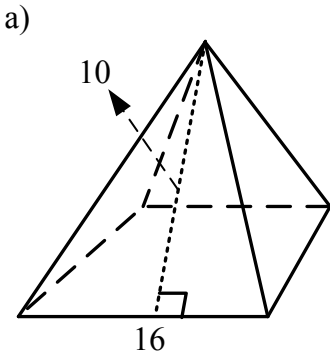
$$B = \frac{1}{2}s^2 \sin \theta = \frac{1}{2} \times 8^2 \times \sin 60^\circ = 16\sqrt{3} \text{ cm}^2$$

Therefore, the volume V of the pyramid is given by:

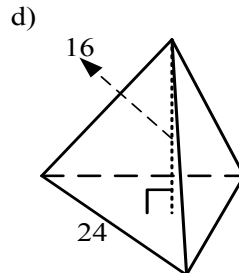
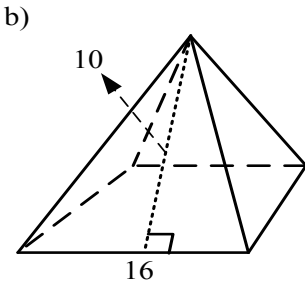
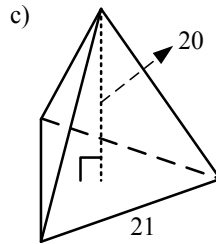
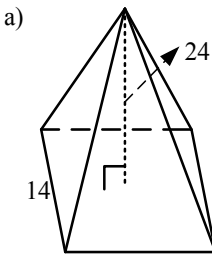
$$V = \frac{1}{3}Bh = \frac{1}{3} \times 16\sqrt{3} \times 12 = 64\sqrt{3} \text{ cm}^3$$

Exercise 5.14

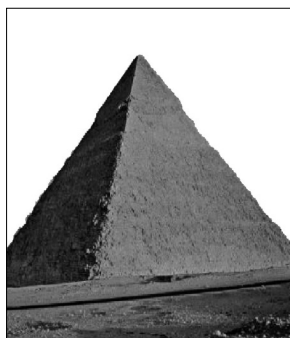
- Find the total surface area of each of the following regular pyramids. The measures are in centimetres.



2. Find the volume of each of the following regular pyramids. The measures are in centimetres.



3. A triangular pyramid has a right-angled triangle base of legs 6cm and 13cm. If the height of the pyramid is 15cm, what is volume of the pyramid?
4. A pyramid has a rectangular base of 18 centimetres by 30cm and a height of 36cm. What is the volume of the pyramid?
5. Find the total surface area of a regular triangular pyramid in which all six edges are 4cm long.
6. The Pyramid of Khafre in Egypt is a regular square pyramid that stands 149 metres tall. The sides of its square base are 216 metres in length. Find the lateral surface area of the pyramid.



5.3.4 Surface Areas and Volumes of Cones

Activity 5.30

1. Fig. 5.67 shows a right cone of height 24cm and radius 7cm. Take $\pi = 22/7$.

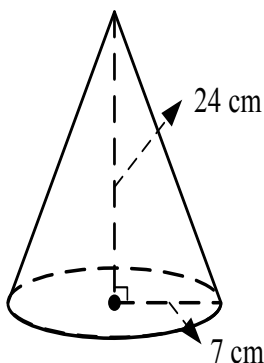


Fig. 5.67

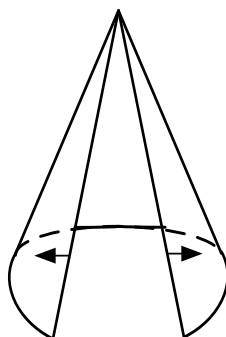


Fig. 5.68

- a) What is the area of the base of the cone?
 - b) What is the slant height of the cone?
 - c) Suppose the base of the cone is removed and the lateral surface is cut vertically down from the vertex to the rim of the base and unrolled as shown in Fig.5.70. What is the shape of the resulting surface?
 - d) What are the dimensions of this surface?
 - e) What is the lateral surface area of the cone? Why?
 - f) What is the total surface area of cone?
2. What is the total surface area of the right cone in Fig. 5.69?

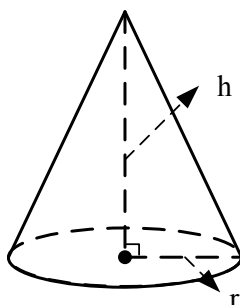


Fig. 5.69

If a right cone has radius r and height h , then the slant height s is given by

$$s = \sqrt{r^2 + h^2}$$

and lateral surface area (LA), the base area (B) and the total surface area (SA) are given by

$$LA = \pi rs$$

$$B = \pi r^2$$

$$SA = LA + B = \pi rs + \pi r^2$$

Example 5.27

Find the total surface area of a right cone which has base radius of 6 cm and a height of 8 cm. Take $\pi = 3.14$.

Solution:

We are given $r = 6$ and $h = 8$. Then, we have

$$s = \sqrt{r^2 + h^2} = \sqrt{6^2 + 8^2} = 10,$$

$$LA = \pi rs = 3.14 \times 6 \times 10 = 188.40 \text{ cm}^2, \text{ and}$$

$$B = \pi r^2 = 3.14 \times 6^2 = 113.04 \text{ cm}^2.$$

Therefore, the total surface area SA of the cone is given by

$$SA = LA + B = 188.40 + 113.04 = 301.44 \text{ cm}^2.$$

Activity 5.31

Fig. 5.70 shows a right cone with radius r and height h while Fig. 5.71 shows the same right cone surrounded by a right cylinder with common base and the same height. As in the case of pyramids, it can be shown that the volume of the cone is one-third of the volume of the surrounding cylinder.

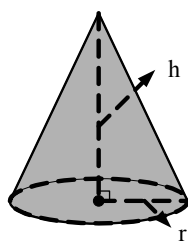


Fig. 5.70

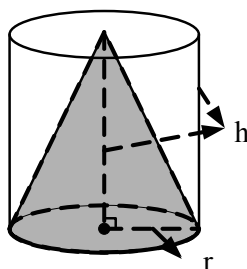


Fig. 5.71

- What is the volume of the cylinder?
- What is the volume of the cone?

If a right cone has radius r and height h , then the volume V of the cone is given by $V = \frac{1}{3} Bh = \frac{1}{3} \pi r^2 h$

Example 5.28

A right cone of height 12 cm has a circular base of diameter 10 cm. Find the volume of the cone. Take $\pi = 3.14$.

Solution:

We are given diameter $d = 10$ and $h = 12$. Then, the radius r and base area B are:

$$r = \frac{d}{2} = 5, \text{ and}$$

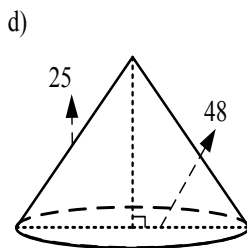
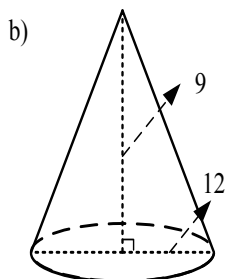
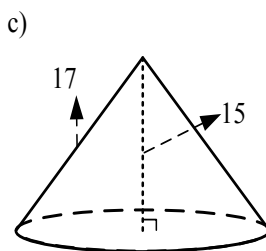
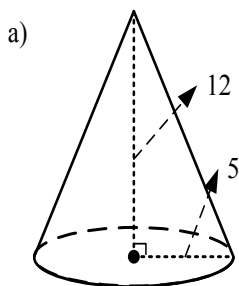
$$B = \pi r^2 = 3.14 \times 5^2 = 78.5 \text{ cm}^2.$$

Therefore, the volume V of the cone is given by

$$V = \frac{1}{3} Bh = \frac{1}{3} \times 78.5 \times 12 = 314 \text{ cm}^3.$$

Exercise 5.15

- Find the total surface area of each of the following right cones. The measures are in centimetres. Take $\pi = 3.14$.



2. Find the volume of each of the right cones in problem (1).
3. A right cone has radius 4cm and a total surface area of $36\pi\text{cm}^2$ square. Find the volume of the cone.
4. A right cone of radius 3cm and a right cylinder of radius 4cm have equal volumes. Find the ratio of the height of the cone to that of the cylinder.
5. A solid metal has the shape of a right cylinder of radius 6cm and height 18cm. The metal is melted down and recast as a solid right cone of radius 9cm. What is the height of the cone?

5.3.5 Surface Areas and Volumes of Spheres

Activity 5.32

The ancient Greeks knew the formulas for surface area and volume of a sphere. Archimedes of Syracuse, a Greek mathematician who lived between 287 and 212 BCE, discovered that:

The surface area of a sphere is the same as the lateral surface area of the smallest right cylinder containing the sphere. (See Fig. 5.72.)

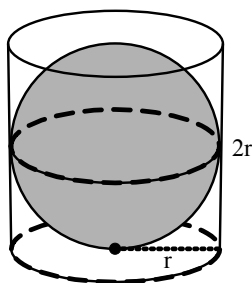


Fig. 5.72

- a) What is the lateral surface area of the right cylinder?
- b) What is the surface area of sphere?

If a sphere has radius r , then the surface area (SA) is given by

$$SA = 4\pi r^2$$

Example 5.29

A ball in the form of a sphere has a diameter of 14 cm long. Find the surface area of the ball. Take $\pi = 22/7$.

Solution:

We are given the diameter $d = 14$. Then, the radius r of the ball is

$$r = \frac{d}{2} = 7.$$

Therefore, the surface area SA of the ball is

$$SA = 4\pi r^2 = 4 \times \frac{22}{7} \times 7^2 = 616 \text{ cm}^2.$$

Activity 5.33

For this activity, you will need a spherical ball and a water container in the shape of a right cylinder whose diameter and height are equal to the diameter of the ball as shown in Fig. 5.73.

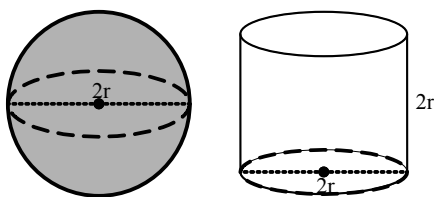


Fig. 5.73

- Fill the cylindrical container with water. What is the volume of water in the cylinder?
- Place the spherical ball in the cylindrical container and press it until the ball touches the bottom of the container. What happens to the water in the container?
- Carefully remove the ball from the container so that the remaining water is retained in the container. Approximately, what is the volume of water left in the container?

- d) What is the volume of water displaced by the spherical ball?
 e) What is the volume of the spherical ball?

If a sphere has radius r , then the volume V of the sphere is given

by: $V = \frac{4}{3}\pi r^3$

Example 5.30

A basin has the shape of a hemisphere of diameter 42 centimetres. How many litres of water will it hold when it is full?

Take $\pi = 22/7$.

Solution:

We are given the diameter $d = 42$. Then, the radius of the hemisphere is:

$$r = \frac{d}{2} = 21$$

Note: A hemisphere is half a sphere, and therefore, its volume is one-half of the volume of the sphere. Hence, the volume V of the hemisphere is given by:

$$V = \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right) = \frac{1}{2} \left(\frac{4}{3} \times \frac{22}{7} \times 21^3 \right) = 6468 \text{ cm}^3.$$

Since $1 \ell = 1000 \text{ cm}^3$, the volume in litres is given by

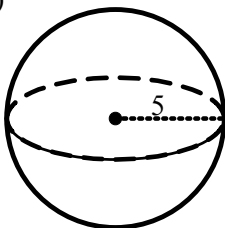
$$\frac{6468}{1000} = 6.468 \text{ lt}$$

Therefore, the capacity of the basin is 6.468 ℓ .

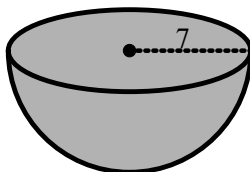
Exercise 5.16

- Find the total surface area of each of the following spherical solids. The measures are in centimetres. Take $\pi = 3.14$.

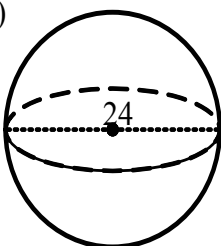
a)



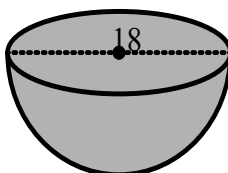
c)



b)



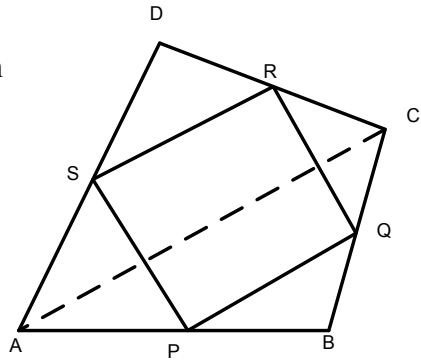
d)



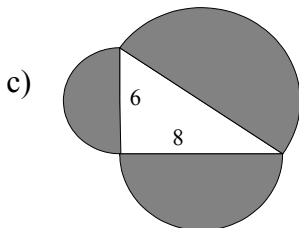
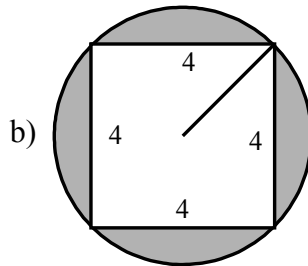
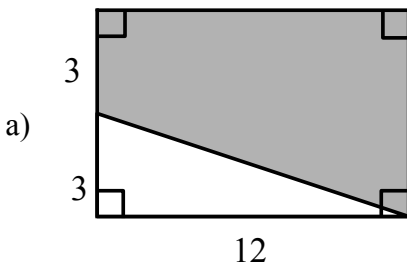
- Find the volume of each of the solids in problem 1.
- An inflated balloon has a spherical shape of diameter 12cm. What is the volume of the balloon?
- Find the volume of a playground ball that has a diameter of 18cm.
- A ball is placed tightly into the smallest possible box that will completely contain the ball. What percentage of the box is filled by the ball?
- A town has three water storage tanks in different shapes: a cylinder, a cone, and a sphere. Each tank has a radius of 20m and a height of 40m. Which tank will hold the most amount of water?

Review Exercise 5

1. Prove that diagonals of a rectangle are equal.
2. ABCD is a quadrilateral in which P, Q, R, and S are the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. $\overline{AC}$ is a diagonal. Show that
 - a) $\overline{SR}$ is parallel to $\overline{AC}$ and $\overline{SR} = \frac{1}{2} \overline{AC}$
 - b) $\overline{PQ} \equiv \overline{SR}$
 - c) PQRS is a parallelogram



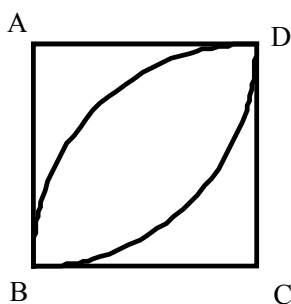
3. Prove that in an isosceles triangle, the bisectors of the base angles are equal.
4. Prove that the angle bisector of each angle of an equilateral triangle is a median of the triangle.
5. Find the perimeter and area of the following shaded regions.



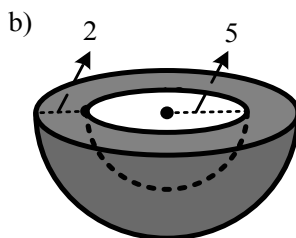
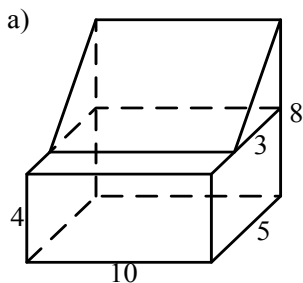
The shaded regions are semicircle.

6. Find the area and the perimeter of a regular dodecagon inscribed in a circle of radius 1cm.
7. Show that the area A of a regular n sided polygon inscribed in a circle of radius r is given by:

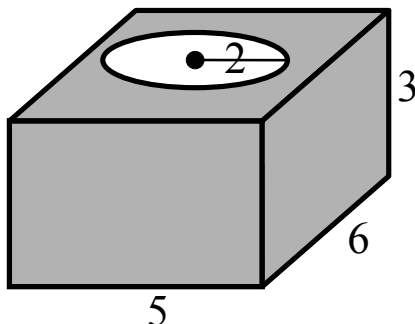
$$A = \frac{1}{2} aP$$
 where P is the perimeter of the polygon and a is the apothem.
8. The quadrilateral ABCD is a square with side 4cm long. Two circles each with radius 4cm are drawn, one with centre A and the other with centre C as shown in the figure below. Find the area of the region inside both sectors.



9. A regular hexagon with side of length 5 cm is inscribed in a circle. What is the area of the region inside the circle and outside the hexagon?
10. Find the total surface area and volume of each of the following solids. The measures are in centimetres. Take $\pi = 3.14$ whenever necessary.



11. A cylinder of radius 2cm and height 3cm is drilled out from a wooden cuboid of length 5cm, width 6cm and height 3cm as shown in the accompanying figure. What is the volume of the remaining solid? Take $\pi = 3.14$.



12. The flask in the accompanying figure has the shape of a sphere with a cylindrical neck. Find the volume and the total surface area of the flask.

