Example.

① Let f be subset of $\mathbb{Q} \times \mathbb{Z}$ defined by $f = \{(\frac{p}{q}, p) : p, q \in \mathbb{Z}, q \neq 0\}$. Is f a function?

② Let f be subset of $\mathbb{Z} \times \mathbb{Z}$ defined by $f = \{(mn, m+n) : m, n \in \mathbb{Z}\}$.

Is f a function?

1. $\left(\frac{1}{2}, 1\right), \left(\frac{2}{4}, 2\right) \in f$.

2. $(0 \times 1, 0+1)$, $(0 * -1, 0+(-1))$.

$(0, 1), (0, -1) \in f$.

3. $(2+(-2), -4)$, $(1+(-1), -1)$.

$\Rightarrow (0, -4), (0, -1) \in f$.

Example:

Let $f = \{(2,4), (5,6), (8,-1), (10,-3)\}$.

$g = \{(2,5), (7,1), (8,4), (10,13), (11,5)\}$.

Find:

(a) $f+g = \{(2,9), (8,3), (10,10)\}$ //

(b) $f-g = \{(2,-1), (8,-5), (10,-16)\}$.

(c) $fg = \{(2,20), (8,-4), (10,-39)\}$.

(d) $\frac{f}{g} = \{(2, \frac{4}{5}), (8,-1/4), (10,-3/13)\}$ //

(c) $f \circ g = \{(2,6), (11,6)\}$.

(d) $g \circ f =$ _____.

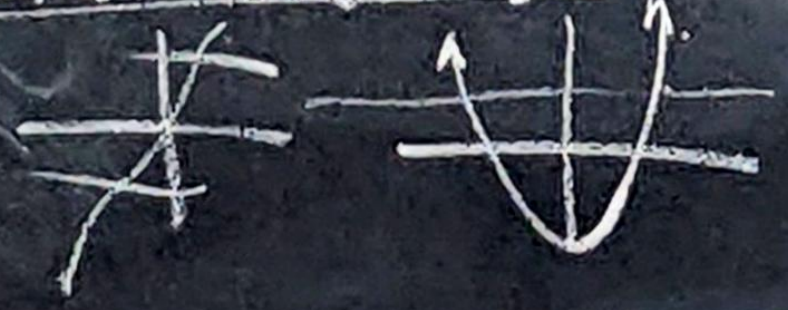
* Equality of two function.

$f = g$. 1. $\text{Dom}(f) = \text{Dom}(g)$.
2. functional values of the 2 functions must be equal.

* Types of Functions.

(1) 1-1 function / Injective.

- $\forall x_1, x_2 \in$ Domain
  If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
- Horizontal line Test.



② Onto function / Surjective.

$f: A \to B$: f is onto if Range$(f) = B$.

$\forall y \in B. \exists x \in A \ni f(x) = y$.

③ 1-1 Correspondency / Bijective.

Example.

Which one of the following is 1-1 Correspondency.

A. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

B. $f: \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$.

C. $f: [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$.

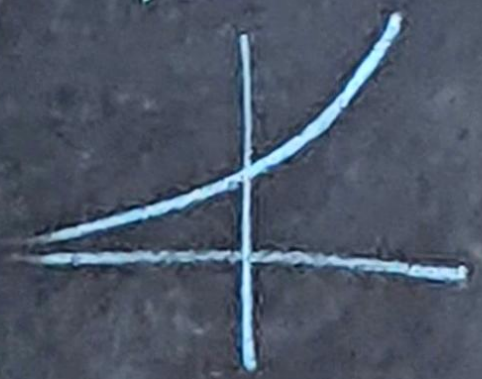
D. $f: [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$.

① A. $f: \mathbb{R}\setminus\{1\} \to \mathbb{R}$ defined by $f(x) = \frac{2}{1-x}$.

B. $f: \mathbb{R} \to [0, \infty)$ defined by $f(x) = 3^x$.

C. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \ln x$.

D. $f: [-1, 1] \to [0, 2]$ defined by $f(x) = x^3 + 1$.



$\forall\ x_1, x_2 \in [-1, 1]$

Let $f(x_1) = f(x_2)$

$x_1^3 + 1 = x_2^3 + 1$

$x_1^3 = x_2^3$.

$x_1 = x_2$

$x^5 = x^5$

Exception

$x_1^2 = x_2^2$

$\sqrt{x_1^2} = \sqrt{x_2^2}$.

$|x_1| = |x_2|$

② A. $f: \mathbb{R}\setminus\{1\} \to \mathbb{R}$ defined by $f(x) = \frac{2}{1-x}$.

B. $f: \mathbb{R} \to [0, \infty)$ defined by $f(x) = 3^x$.

C. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \ln x$.

D. $f: [-1, 1] \to [0, 2]$ defined by $f(x) = x^3 + 1$.

Example.

Determine whether the following pairs of functions are equal or not.

② $f(x) = x^2$; $g(x) = |x|^2$

1. $\text{Dom}(f) = \text{Dom}(g)$.

2. ___

$\therefore f = g$

⑥ $f(x) = \frac{x^2 - 25}{x - 5}$; $g(x) = x + 5$.

$\text{Dom}(f) \neq \text{Dom}(g)$.

$\therefore f \neq g$.

⑦ $f(x) = \frac{x^4 - 1}{x^2 + 1}$; $g(x) = x^2 - 1$.

$\therefore f = g$.

⑧ $f(x) = x^2$; $g(x) = x|x|$.

Example.

Given $f: \mathbb{R} \setminus \{2/3\} \to \mathbb{R}$ defined by $f(x) = \frac{x-5}{2-3x}$. Show that f is 1-1, but not onto.

Proof: 1-1: $\forall x_1, x_2 \in \mathbb{R} \setminus \{2/3\}$

Let $f(x_1) = f(x_2)$.

$$\frac{x_1 - 5}{2 - 3x_1} = \frac{x_2 - 5}{2 - 3x_2}$$

$$(x_1 - 5)(2 - 3x_2) = (x_2 - 5)(2 - 3x_1)$$

$$2x_1 - 3x_1x_2 - 10 + 15x_2 = 2x_2 - 3x_1x_2 - 10 + 15x_1.$$

$2x_1 + 15x_2 = 2x_2 + 15x_1$.

$2(x_1 - x_2) = 15(x_1 - x_2)$.

$-13(x_1 - x_2) = 0$

$x_1 - x_2 = 0$.

$x_1 = x_2$. $\therefore f$ is 1-1 function.

Onto.

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \mid (2/3) \ni f(x) = y$.

$\frac{x-5}{2-3x} = \frac{y}{1} \Rightarrow x - 5 = y(2-3x)$.

$x - 5 = 2y - 3xy$.

$x + 3xy = 2y + 5$

$x(1+3y) = 2y + 5$

$x = \frac{2y+5}{3y+1}$.

$\Rightarrow x$ is not defined for all $y \in \mathbb{R}$.

$\therefore f$ is Not onto.

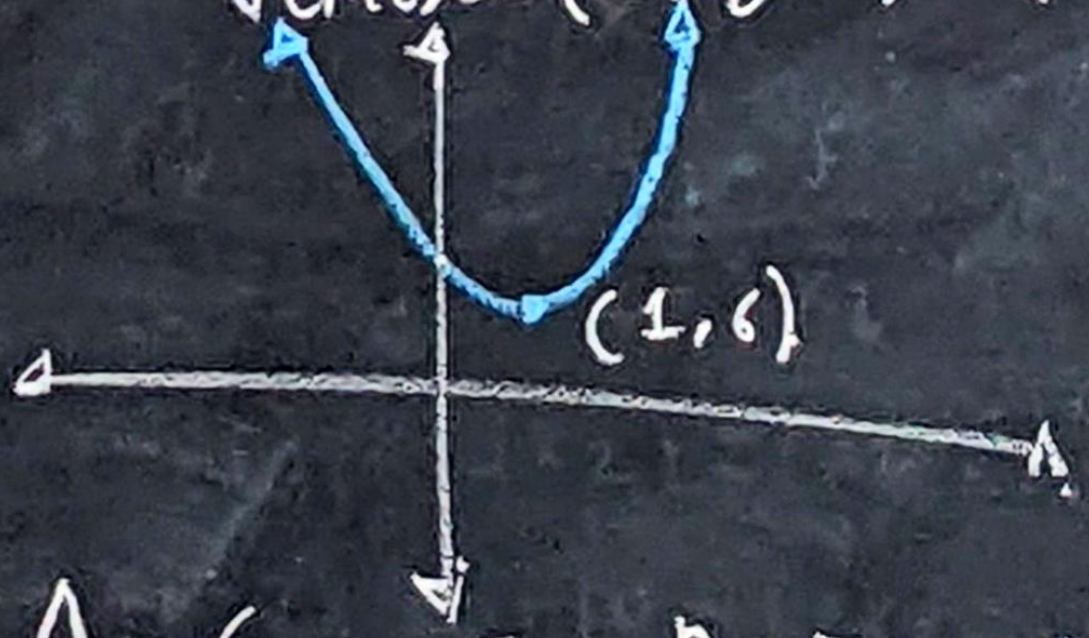
Example.

Let $f: A \to B$ defined by $f(x) = x^2 - 2x + 7$.

Find the values of A and B so that f is 1-1 Correspondency.

Soln: $f(x) = x^2 - 2x + 7$,

Vertex $= (1, f(1)) = (1, 6)$.



a. 1. $A = (-\infty, 1]$, $B = [6, \infty)$.

b. 2. $A = [1, \infty)$, $B = [6, \infty)$

$f = \{(2,5), (3,4), (-4,6), (10,4)\}$.

$f(3) = f(10)$. but $3 \neq 10$.

$f(2) = 1$

$f(10) = 1$

$f(2) = f(10)$

$2 \neq 10$.

$\therefore f$ is Not 1-1.

Inverse Function

* Only 1-1 function is invertible.

Example.

Which one of the following is invertible.

A. $f(x) = x^2 + 7$.

B. $f(x) = x^7 + 5x^5 + 10x^3 + 10$

C. $f(x) = |x|$



D. $f(x) = \operatorname{sgn} x$.

Example.

Find the inverse of the following function.

@ $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Soln

$$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$x = \frac{e^y + e^{-y}}{e^y - e^{-y}} \left(\frac{e^y}{e^y}\right)$$

$$\frac{x = e^{2y} + 1}{1 \quad e^{2y} - 1}$$

$\hookrightarrow x(e^{2y} - 1) = e^{2y} + 1$

$\hookrightarrow x e^{2y} - x = e^{2y} + 1$

$\therefore xe^{2y} - e^{2y} = x+1$

$\therefore e^{2y}(x-1) = x+1$

$\therefore e^{2y} = \frac{x+1}{x-1}$

$\therefore \ln e^{2y} = \ln\left(\frac{x+1}{x-1}\right)$

$\therefore 2y = \ln\left(\frac{x+1}{x-1}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

$\therefore f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

⑥ $f(x) = \ln\left(\frac{x}{x+1} - 2\right), \quad x > -1$

$$y = \ln\left(\frac{x}{x+1} - 2\right)$$

$$x = \ln\left(\frac{y}{y+1} - 2\right)$$

$$e^x = \frac{y}{y+1} - 2$$

$$e^x + 2 = \frac{y}{y+1}$$

$$(e^x + 2)(y+1) = y$$

$$ye^x + 2y + e^x + 2 = y$$

$$ye^x + 2y - y = -2 - e^x$$

$$ye^x + y = -2 - e^x$$

$$\frac{y(e^x+1)}{e^x+1} = \frac{-2-e^x}{e^x+1}$$

$$y = \frac{-2-e^x}{e^x+1}$$

$f(x) = \frac{2-3x}{2x+1}$. find $f^{-1}(0) = 2/3$.

Inverse prop

$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

M1: Let $f^{-1}(0) = y$.

$f(f^{-1}(0)) = f(y)$.

$\frac{0}{1} = \frac{2-3y}{2y+1} \Rightarrow 0 = 2-3y$

$3y = 2 \Rightarrow y = 2/3$.

M2: $y = \frac{2-3x}{2x+1}$

$x = \frac{2-3y}{2y+1}$

$x(2y+1) = 2-3y$.

$2xy + x = 2-3y$.

$2xy + 3y = 2-x$.

$y(2x+3) = 2-x$.

$y = \frac{2-x}{2x+3}$

$f^{-1}(x) = \frac{2-x}{2x+3}$

$f^{-1}(0) = 2/3$ //

Exercise

① Find the inverse of the following function.

(a) $f(x) = \frac{1+e^x}{1-e^x}$

(b) $f(x) = \sqrt[3]{1+e^{-x}}$

② Find the values of A and B so that f is 1-1 correspondency.

(a) $f: A \to B$ defined by $f(x) = |x+3| - 1$.

(b) $f(x): A \to B$ defined by $f(x) = x^2 + 4x + 6$.