

AASTU
DEPARTMENT OF MATHEMATICS
Mathematics for Natural Sciences (Math 1011)

Worksheets

Chapter 1

Part I: Propositional Logic.

- (1) Which of the following sentences are propositions? Determine the truth value whenever possible?
 - (a) $5 < 9$ and there are infinitely many prime numbers.
 - (b) Nairobi is the capital of Kenya.
 - (c) Thank you for your consideration.
 - (d) Answer this question.
 - (e) $x^2 = y^2$ provided that $x = y$.
 - (f) How many courses are you taking this semester?
 - (g) $x + 5 = 11$.
 - (h) The first human kind lived in Ethiopia.
- (2) What is the negation of each of the following propositions?
 - (a) Today is Friday.
 - (b) There is no pollution in Europe.
 - (c) $6 + 4 = 10$.
 - (d) The summer in Dire Dawa is hot and sunny.
- (3) Let p and q be propositions:
 p : It is below freezing.
 q : It is snowing.
Write the following propositions using p , q and logical connectives.
 - (a) It is below freezing and snowing.
 - (b) It is below freezing but not snowing.
 - (c) It is not below freezing and it is not snowing.
 - (d) It is either snowing or below freezing (or both).
 - (e) If it is below freezing, it is also snowing.
 - (f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing.
 - (g) That it is below freezing is necessary and sufficient for it to be snowing.
- (4) Given the following propositions
 p : You have the flu.
 r : You pass the course.
 q : You miss the final examination.
Express each of the following propositions as an ordinary English sentence.
 - a) $\neg q \Leftrightarrow r$
 - b) $\neg(p \wedge q)$
 - c) $(p \Rightarrow \neg r) \vee (q \Rightarrow \neg r)$.
- (5) Determine the truth value of p if
 - (a) $(q \Rightarrow \neg p) \vee r$ is False
 - (b) $\neg q$ and $p \Rightarrow q$ are True
 - (c) $(p \vee \neg q) \Rightarrow (\neg r \vee p)$ is False.
- (6) If $\neg[\neg r \Rightarrow \neg(p \wedge q)]$ is true, then find the truth value of $[(p \Leftrightarrow r) \vee q] \Leftrightarrow (\neg p \wedge r)$.
- (7) Prove that

- (a) $(p \wedge q) \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$
 (b) $(q \vee r) \Rightarrow p \equiv (q \Rightarrow p) \wedge (r \Rightarrow p)$.
- (8) For the following propositions, indicate whether it is a tautology, a contradiction, or neither. Use a truth table to decide.
 (a) $[p \wedge (p \Rightarrow q)] \Rightarrow q$
 (b) $(p \Rightarrow q) \wedge \neg(\neg q \Rightarrow \neg p)$.
 (c) $(\neg q \Rightarrow \neg p) \Rightarrow ((\neg q \Rightarrow p) \Rightarrow q)$.
- (9) Determine whether the following statements are valid or invalid.
 (a) If I do not wake up, then I cannot go to work.
 If I cannot go to work, then I will not get paid.
 Therefore, if I do not wake up, then I will not get paid.
 (b) If I study, then I will not fail Math 1011.
 If I do not play cards too often, then I will study.
 I failed Math 1011.
 Therefore, I played cards too often.
 (c) Today is either Thursday or Friday.
 On Thursdays I have to go to a lecture.
 Today is not Friday.
 Therefore, I have to go to a lecture today.
- (10) Let $P(x) : x$ is an integer greater than 5. $Q(x) : x$ is a natural number.
 Determine the truth value of the following propositions.
 (a) $P(3) \wedge Q(-2)$
 (b) $P(2) \Rightarrow Q(2)$
 (c) $[P(3) \Rightarrow Q(4)] \vee Q(9)$.
- (11) Let $P(x) : x$ is a prime number $Q(x) : x$ is an even number.
 $R(x) : x$ is an odd number. $S(x) : x$ is an integer.
 Write a sentence, which corresponds to each of the following:
 (a) $R(x) \vee Q(x) \Rightarrow P(x)$
 (b) $P(x) \Leftrightarrow Q(x) \vee R(x)$.
- (12) If $U = \mathbb{R}$, then find the truth value of $(\forall x)(\exists y)(\sqrt{x^2} = 16 \Rightarrow y + x = 10)$.
- (13) Let $P(x) : x$ is a composite number. $Q(x) : x$ is a prime number.
 Find the truth values of (i) $(\exists x)[P(x) \Leftrightarrow Q(x)]$ (ii) $(\forall x)[P(x) \Leftrightarrow Q(x)]$.
- (14) Find the truth values of the following where $U = \mathbb{R}$
 (a) $(\exists x)(\forall y)(x^2 < y^2)$
 (b) $(\forall x)[x \neq 0 \Rightarrow (\exists y)(xy = 4)]$.

Part II: Set Theory.

- (1) Write the following sets in complete listing or partial listing method.
 (a) $A = \{x | x \text{ is an integer and } 3 < x \leq 10\} \cap \{x | x \text{ is even integer}\}$.
 (b) $B = \{x \in \mathbb{N} | (x - 1)(x - 3) \neq 0 \Leftrightarrow x + 1 = x\}$.
 (c) $C = \{x \in \mathbb{N} | x \geq 3 \Leftrightarrow x < 0\}$.
 (d) $D = \{x | x \in \mathbb{N} \wedge (x - 1)(x - 3) \neq 0 \Leftrightarrow x + 1 = x\}$.
- (2) If $n(A \setminus B) = 18$, $n(A \cup B) = 70$ and $n(A \cap B) = 25$, then find $n(B)$.
- (3) For all subsets A, B and C of some Universal set U , prove or disprove the following statements.

- (a) $(A \subseteq B \wedge A \subseteq C) \Rightarrow A \subseteq B \cap C$
 (b) $A \subseteq B \Leftrightarrow A \cap B = A$
 (c) $A^c \setminus B^c = B \setminus A$
 (d) $A \subseteq B \Rightarrow A \cup (B \setminus A) = B$.
- (4) Let $U = \{x \in \mathbb{Z} \mid -12 \leq x \leq 6\}$.
 $A = \{x \in \mathbb{Z} \mid x = 2n \wedge x = 3m, \text{ for } -4 \leq n \leq 8 \text{ and } -2 \leq m \leq 3\}$.
 $B = \{0, 1, 3\}$ $C = \{x \in \mathbb{Z} \mid x = \frac{n}{3}, n \in A\}$.
 Then find
 (a) $A \setminus [B \cup C]^c$
 (b) $(A \setminus B) \setminus (A \cup C)^c$
 (c) $(A \cap B) \setminus C$.
- (5) (a) For every natural number n , define $A_n = \{x \in \mathbb{Z} \mid \frac{-24}{n} \leq 2x + 1 \leq \frac{24}{n}\}$.
 Then find
 (i) $\cup_{n=1}^6 A_n$
 (ii) $[(\cup_{n=1}^4 A_n) \setminus (\cap_{n=1}^3 A_n)]^c$
 (iii) $\cap A_n, n \in \mathbb{N}$.
- (b) For every natural number n , define
 $A_n = \{x \in \mathbb{N} \mid x = n \Rightarrow x < n\}$.
 Then find
 (i) $\cup_{n=1}^{10} A_n$
 (ii) $\cup A_n, n \in \mathbb{N}$.

Chapter 2

Part I: The real number system

- (1) Two incandescent lamps are turned on at the same time. One blinks every 4 seconds and the other blinks every 6 seconds. In 60 seconds, how many times will they blink at the same time?
- (2) Write the following sets as an interval or union of intervals. In each case $x \in \mathbb{R}$.
 - (a) $A = \{x | x \neq 5 \wedge x \neq 8\}$.
 - (b) $B = \{x | x^2(x^3 - 1) \neq 0\}$.
 - (c) $C = \{x | x^3 - 2x \neq 0\}$.
 - (d) $D = \{x | (x - 2)(x + 3)(x + 4) = 0 \Rightarrow x \geq 2\}$.
- (3) Simplify each of the following sets and give the final result as an interval or union of intervals.
 - (a) $([-3, 2] \cap (0, 5))^c$.
 - (b) $((-3, 5.6] \cap (-5, 2.4\bar{3})) \cup (0.\bar{3}, 11)$.
 - (c) $(2, 4]^c \cap ((-\infty, 5)) \cup [0, \infty))$.
 - (d) $(-0.\bar{3}, 0.0\bar{3}) \cup ((-\infty, \frac{1}{30}) \cap [5, \infty))$.
- (4) Find the supremum and infimum of each of the following sets of real numbers:
 - (a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q , and r take on all positive integer values.
 - (b) $S = \{3x^2 - 10x + 3 < 0\}$.
- (5) Find the smallest and the largest elements of the given set. Also the lub and glb whenever they exist.
 - (a) $A = \{x | x = 5 - \frac{1}{n}; n \in \mathbb{N}\}$.
 - (b) $B = \{x | x \in (-1, 1) \cup \{\frac{1}{n} | n \in \mathbb{N}\}\}$.
 - (c) $C = \{x | x = \frac{n}{n+1}; n \in \mathbb{N}\}$.
 - (d) $D = \{x | x = \frac{n+1}{n}; n \in \mathbb{N}\}$.
 - (e) $E = \{x | x = \frac{n^3}{2n^2+1}; n \in \mathbb{N}\}$.
- (6) Using the Principle of Mathematical Induction, prove that
 - (a) $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n-1)}{2}$ for $n \geq 1$.
 - (b) $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ for $n \geq 1$.
 - (c) $2^{2n+1} + 7$ is a multiple of 3 for $n \geq 1$.
 - (d) $n^2 - n$ is even for $n \geq 1$.
 - (e) $6^n - 1$ is divisible by 5 for $n \geq 1$.
 - (f) $n! > 3^n$ for $n \geq 7$.
- (7) Consider the sequence of real numbers defined by the relations

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for } n \geq 1.$$

Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \geq 1$.

- (8) Prove that $\sqrt{2} + \sqrt{3}$ is irrational number.
- (9) If $\frac{a}{b} < \frac{c}{d}$ with $b > 0$, $d > 0$, prove that $\frac{a+c}{b+d}$ lies between the two fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Part II: The Complex number system.

- (1) Change the following complex numbers to polar form
 - (a) $-2 - 4i$.
 - (b) $1 - \sqrt{2}i$.
 - (c) -27 .
 - (d) $5i$.
- (2) Find the modulus and argument of
 - (a) $(1 - \sqrt{2}i)^2$.
 - (b) $(1 + i)^{11}$.
 - (c) $(\frac{1}{2} - \frac{\sqrt{3}i}{2})^{15}$.
- (3) Find $z \in \mathbb{C}$ such that
 - (a) $\operatorname{Re}(z(1 + i)) + z\bar{z} = 0$;
 - (b) $\operatorname{Re}(z^2) + i\operatorname{Im}(\bar{z}(1 + 2i)) = -3$;
 - (c) $\operatorname{Im}(2 - i)z = 1$.
- (4) Express the following complex numbers in the form $a + bi$.
 - (a) $(1 + i)^3$.
 - (c) $i^5 + i^{16}$.
 - (b) $\frac{2+3i}{3-4i}$.
 - (d) $\frac{1}{2}(1 + i)(1 + i^{-8})$.
- (5) Compute the absolute value and the conjugate of
 - (a) $z = (1 + i)^6$.
 - (b) $w = i^{17}$.
- (6) In each case, determine all real x and y which satisfy the given relation.
 - (a) $x + iy = |x - iy|$.
 - (b) $x + iy = (x - iy)^2$.
 - (c) $\sum_{k=0}^{100} i^k = x + iy$.
- (7) Solve the following equations.
 - (a) $z^2 = 1 + i$.
 - (b) $z^2 - 4z + 6 = 0$.
 - (c) $z^4 + 4z^2 + 16 = 0$.
 - (d) $(\bar{z} - i)^3 = 1$.
 - (e) $z^5 = 32$.
- (8) Compute
 - (a) the square roots of $z = -1 - i$.
 - (b) the three cube roots of $z = -8$.
 - (c) the four fourth roots of $z = -1 + \sqrt{3}i$.
- (9) Write $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$. (Hint: use $(\cos \theta + i \sin \theta)^4$)
- (10) Find the locus of points $z = x + iy$ that satisfy
 - (a) $|z + 1| = |z - i|$.
 - (b) $3|z| = |z + 3i|$.
 - (c) $|z - 1| = 2$.
 - (d) $z^2 \cdot \bar{z} = z$.
- (11) If $\alpha = \cos \theta + i \sin \theta$ for $0 < \theta < 2\pi$, show that $\frac{1+\alpha}{1-\alpha} = i \cot \frac{\theta}{2}$.
- (12) If $z \neq -1$ and $\operatorname{Im}(z) \neq 0$, show that $\frac{z}{(1+z)^2}$ is real iff $z \cdot \bar{z} = 1$.
- (13) If for $z = x + iy$, $f(z) = e^x(\cos y + i \sin y)$, then show that
 - (a) $f(z_1 + z_2) = f(z_1)f(z_2)$.
 - (b) $\overline{f(z)} = f(\bar{z})$.
- (14) Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.

Chapter 3: FUNCTIONS

- (1) Let $A = \{1, 2, 3, 4\}$ and R be a relation on A such that $\{(x, y) | y - x^2 + 2x \leq 0\}$. Describe R by complete listing method and find its domain and range.
- (2) In each case determine whether f is well-defined. Give reason for your answer
- (a) $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = -n, \forall n \in \mathbb{Z}$ (b) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sqrt{x}, \forall x \in \mathbb{R}$
(c) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{1}{x}, \forall x \in \mathbb{R}$ (d) $f : \mathbb{Q} \rightarrow \mathbb{Z}, f(\frac{m}{n}) = mn, \forall m, n \in \mathbb{Z}, n \neq 0$
- (3) Which of the following relations are functions? Which relations have their inverse that are functions?
- (a) $R_1 = \{(1, 2), (2, 3), (3, 1), (4, 3)\}$. (b) $R_2 = \{(x, y) | x, y \in \mathbb{Z} \wedge x - y \text{ is even}\}$.
(c) $R_3 = \{(x, y) | x, y \in \mathbb{R} \wedge x - y = 4\}$.
- (4) Let $A = \{a, b, c, d\}$, $B = \{a, c, d, e\}$. Let $f : A \rightarrow \mathbb{R}$ and $g : B \rightarrow \mathbb{R}$ such that $f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$ and $g = \{(a, 0), (c, 4), (d, 8), (e, 0)\}$. State the domain and find:
- (a) $f + g$ (b) $f - g$ (c) $f \cdot g$ (d) f/g .
- (5) (i) Given $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 1}$, state the domain and find
(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $f \circ g \circ f$
(ii) Given a set A and $f : A \rightarrow A, g : A \rightarrow A$. Is $f \circ g = g \circ f$?
- (6) Which of the following functions are one-to-one, onto and one-to-one correspondence?
- (a) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^n, n \text{ is odd}$ (b) $k : [0, \infty) \rightarrow [1, \infty), k(x) = x^2 + 1$.
(c) $h : E^+ \rightarrow \mathbb{N}, h(n) = \frac{3n+2}{2}$ where $E^+ = \{2n | n \in \mathbb{N}\}$ (d) $g : \mathbb{Z} \rightarrow \mathbb{N}, g(n) = |n|$
- (7) Determine whether each of the following functions has an inverse or not. If an inverse exists, give the domain and the range of the inverse.
- (a) $f(x) = x^5 + 1$ (b) $f(x) = x + |x|$ (c) $f(x) = x + \sin x$ (d) $f(x) = \ln(2 - x)$
- (8) Find all intercepts, asymptotes and sketch the graph of the following rational functions. Find the domain and range of each.
- (a) $f(x) = \frac{x}{x^2 - 4}$ (b) $f(x) = \frac{x^2 + 1}{x^2 - 1}$ (c) $f(x) = \frac{x^2 - x}{x - 3}$ (d) $f(x) = \frac{x + 1}{x^2 - 1}$
- (9) Find the domain of the following functions
- (a) $f(x) = \ln(x^2 - 2x - 15)$ (b) $g(x) = \frac{x}{\ln(5x + 4)}$
- (10) Find the inverse of the following functions
- (a) $y = \ln(3x + 7)$ (b) $y = 2^{x+3} - 1$
(c) $y = (4^x - 1)^{\frac{1}{4}}$ (d) $y = \log_4(3^x + 4)$
- (11) Solve the following equations
- (a) $5^{2x} - 2(5^x) - 15 = 0$ (b) $\ln(n^2 + 12) = \ln(-9n - 2)$
(c) $\log_9(-11x + 2) = \log_9(x^2 + 30)$ (d) $\log_5 6 + \log_5 2x^2 = \log_5 48$
- (12) 15,000 Birr is invested in an account that yields 5% interest per year. After how many years will the account be worth 91,221.04 Birr if the interest is compounded yearly?
- (13) A function f is said to be even if $f(-x) = f(x)$ for all x in the domain of f and f is said to be odd if $f(-x) = -f(x)$ for all x in the domain of f . Classify the following functions as even, odd and neither.
- (a) $f(x) = x \cos x - \tan x$ (b) $f(x) = \cot^2 x - \sin x$
(c) $f(x) = \sec x \csc x$ (d) $f(x) = x \sinh x - \cosh x$
- (14) If α and β are angles of the first quadrant such that $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{1}{7}$, find the quadrant in which $\alpha + \beta, \alpha - \beta, 2\alpha$ and 3β terminate and calculate
- (a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha - \beta)$
(c) $\csc(2\beta)$ (d) $\sec(\frac{\alpha}{2})$
- (15) Find the period and amplitude, and sketch the graph of the following equations
- (a) $y = 3 \sec x$ (b) $y = \sin 4(x - \frac{\pi}{2})$
(c) $y = \cos \frac{1}{3}x$ (d) $y = \tan(x - \frac{\pi}{4})$

- (16) Prove whether the following are identities
 (a) $\sin 3x = 4 \cos^3 x - 3 \cos x$ (b) $\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$
- (17) Solve the following equations in $[0, 2\pi]$
 (a) $\cot^2 \theta \sin \theta - 3 \sin \theta + \cos^2 \theta - 3 = 0$ (b) $2 \cos^4 \theta + 9 \sin^2 \theta - 5 = 0$
 (c) $\sin \theta + 2 \cos \theta = 1$ (d) $2 \cos \theta + \sqrt{4 + \csc^2 \theta} = 0$
- (18) Sketch the graph of the following functions
 (a) $y = 3 \cos 2x$ for $0 \leq x \leq 2\pi$ (b) $y = 3 \sin 2(x + \frac{\pi}{4})$ for $-\frac{\pi}{4} \leq x \leq 2\pi$
- (19) Prove whether the following are identities
 (a) $\frac{2 \tanh \frac{1}{2}x}{1 - \tanh^2 \frac{1}{2}x} = \sinh x$ (b) $\frac{\cosh(\ln x) + \sinh(\ln x)}{\cosh(\ln x) - \sinh(\ln x)} = x^2$

Chapter 4: ANALYTIC GEOMETRY

- (1) Find the distance between the point $A(2, -3)$ and the line with equation $L : 3x - 4y + 2 = 0$.
- (2) The point P divides the line segment AB joining the points $A(2, 1)$ and $B(-3, 6)$ in the ratio $2 : 3$. Does the point P lie on the line with equation $x - 5y + 15 = 0$? Justify
- (3) What is the equation of the locus of a point which moves such that 4 times its distance from the x -axis is the square of its distance from the origin?
- (4) Find the equation of the circle if two of its diameters are $x + 4y = 5$ and $7x - y = 6$, and whose radius is $\sqrt{7}$.
- (5) Find the equation of the circle which passes through the points $(1, 2)$, $(2, 2)$ and $(4, -1)$.
- (6) Find the equation of the circle
 - (a) with diameter having end-points $(-8, 5)$ and $(8, -1)$
 - (b) with center $(-4, 1)$ and tangent to the line $3x + 4y = 10$
 - (c) passing through the points $(-1, 1)$, $(3, 2)$ and whose center lies on the line $x - 2y + 2 = 0$
- (7) Find the equation of the parabola
 - (a) with vertex at $(0, 0)$ and focus $(0, -6)$. Sketch the curve
 - (b) with focus at $(5, 0)$ and equation of the directrix $x = -5$
 - (c) with end points of its latus rectum at $(16, 8)$ and $(-16, 8)$ (two possible solutions)
 - (d) symmetric about y -axis, vertex at origin and passes through $(6, -8)$
- (8) Find the vertex, focus, the axis, directrix, latus rectum and length of latus rectum of the following parabola.
 - (a) $y^2 = 12x$ (b) $3(x - 2)^2 + 2y = 0$
- (9) Find the equation of the ellipse
 - (a) with center at $(-2, 1)$, a focus at $(-2, 4)$ and a vertex at $(1, 1)$
 - (b) with foci at $(2, -2)$ and $(4, -2)$
 - (c) with vertices at $(1, -2)$ and $(5, -2)$ (two possibilities)
- (10) Find the foci, the transverse axis, the conjugate axis, the vertices, the latus rectum and the eccentricity of the following ellipse.
 - (a) $25x^2 + 4y^2 = 100$ (b) $(\frac{x}{2} - 1)^2 + (\frac{y}{3} - 1)^2 = 1$
- (11) Find the equation of the hyperbola
 - (a) with vertices at $(9, -3)$ and $(-5, -3)$ and foci at $(\sqrt{53}, -3)$ and $(-\sqrt{53}, -3)$
 - (b) with vertices at $(-5, 0)$ and $(5, 0)$ and conjugate axis of length 12
- (12) Find the foci, the major axis, the minor axis, the vertices, the latus rectum and the eccentricity of the following hyperbola.
 - (a) $4x^2 - 25y^2 = -1$ (b) $(y - 3)^2 - \frac{(x+2)^2}{9} = 1$
- (13) Assuming that the following conic sections are not degenerate, classify the conic sections whose Cartesian equations are
 - (a) $y^2 - 4y + 16 = 4x^2 + 8x$ (b) $2x^2 - 5xy + 3y^2 - x + y = 4$
 - (c) $4y^2 - 4y + 28x + 29 = 0$ (d) $4x^2 + y^2 - 6x + 5 = 0$
- (14) A satellite dish is shaped like a paraboloid. The signals that emanate from a satellite strike the surface of the dish and are reflected to a certain point where the receiver is located. If the dish is 12 feet across its opening, and is 4.5 feet deep at its center, how far from the center of the dish should the receiver be placed?