Mathematics grade 12

Unit 1

Sequence and series

1.1 sequence

- A sequence is a function whose domain is the collection of all integers greater than or equal to a given integer m (usually 0 or 1_).
- A sequence is denoted by a_n
- The functional values a1 , a2 a3 ….. called terms of **sequence.**
- a_n is called general term
- there are 2 types of sequences finite and infinite depending on their last term
- finite sequence sequence with last term
- infinite sequence that doesn't have last term

Example

Find the first 3 term of the sequence with general term a_n= 2n-1

Solution a1=2(1)-1=1

A2=2(2)-1=3

A3= (3)-1=5

Fibonacci's sequence

HISTORICAL NOTE

Leonardo Fibonacci (circa 1170, 1240)



Italian mathematician Leonardo Fibonacci made advances in number theory and algebra. Fibonacci, also called Leonardo of Pisa, produced numbers that have many interesting properties such as the birth rates of rabbits and the spiral growth of leaves on some trees.

He is especially known for his work on series of numbers, including the Fibonacci series. Each number in a Fibonacci series is equal to the sum of the two numbers that came before it. Fibonacci sequence arose when he was trying to solve a problem of the following kind concerning the breeding of rabbits.

"Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at the age of two months. If we start with one new born pair, how many pairs of rabbits will we have in the n^{th} month?"

Fibonacci sequence is defined as: $F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$

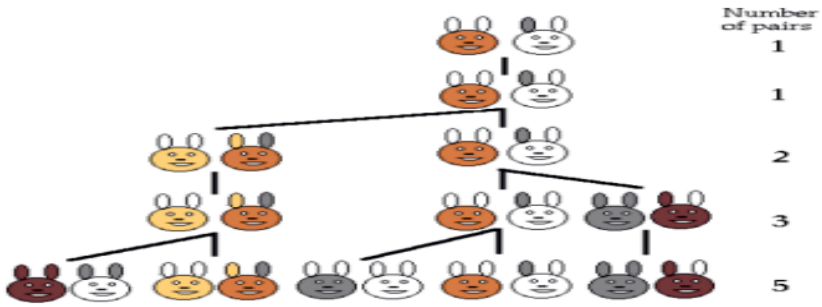


Figure 1.3

Mulatu sequence

HISTORICAL NOTE

Mulatu Lemma



Professor Mulatu Lemma is an Ethiopian Mathematician. He pursued an education through a university in Ethiopia, earning a Bachelor of Science in 1977 from Addis Ababa University. He continued his studies at the aforementioned university and obtained a Master of Science in applied mathematics in 1982. Following these accomplishments, he was enrolled at Kent State University, where he pursued a Master of Arts in pure mathematics in 1993. Prof. Mulatu completed his academic journey with a Doctor of Philosophy from Kent State University in 1994. In 2011, he introduced the Mulatu's Number (named after him) to the mathematical community and to the world.

Professor **mulatu** introduced a sequence of the form

$$Mn\begin{cases} 4 \ if \ n=0 \\ 1 \ if \ n=1 \\ Mn-1+Mn-2 \ if \ n\geq 2 \end{cases}$$

Recursive sequence

A sequence that relates to the general term an of sequence where one or more of the terms that come before it is said to be defined **recursive** .

Domain of recursive sequence can be whole number.

1.2 arithmetic sequence

Arithmetic sequence or progression is sequence in which each term except the first term is obtained by adding fixed number to the **preceding number .**

The fixed number called common **difference(d) .**

Example

1 Find the first term,6^{th} term and common difference for the following arthimetic sequence

A) 2,4,6,8,10

1^{st} term is 2

Common difference = 4-2=6-4=10-8=2

6^{th} term is 10+2=12

example2 given 1^{st} term 10 and common difference 2 find the 5^{th} term

Solution 2^{nd} term =10+2=12 3^{rd} term= 12+2=14 4^{th} term= 14=2=16 5^{th} term=16+2=18

General term of arithmetic sequence = $A1 + (n-1)d$

A1 = first term

D = common difference

n = n^{th} term

Example

Find the general term of the sequence A_n, when the first term is 5, and the common difference is 4.

Solution

The given sequence is arithmetic sequence with $A_1 = 5$ and $d = 4$.

From the theorem 1.1, we have the formula $A_n = A_1 + (n-1)d$. So,

$$A_n = A_1 + (n-1)d = 5 + (n-1)(4) = 5 + 4n - 4 = 4n + 1.$$

EXAMPLE 2

When the third term is 10 and the sixth term is 1,

a. Find the general term of sequence A_n.

b. Find A_8.

Solution

a. Applying arithmetic sequence formula and substituting existing values yields:

$$A_3 = A_1 + (3-1)d, \qquad 10 = A_1 + 2d$$
$$A_6 = A_1 + (6-1)d, \qquad 1 = A_1 + 5d$$
$$\begin{cases} A_1 + 2d = 10 \\ A_1 + 5d = 1 \end{cases}$$

Subtracting these two equations,

$$3d = -9 \Rightarrow d = -3\ ,\ A_1 + 2d = 10,\ A_1 + 2(-3) = 10,\ A_1 = 10 + 6 = 16.$$

Therefore, the general term becomes

$$A_n = A_1 + (n-1)d = 16 + (n-1)(-3) = 16 - 3n + 3 = 19 - 3n.$$

b. $A_n = 19 - 3n \Rightarrow A_8 = 19 - 3(8) = 19 - 24 = -5.$

Arithmetic mean between two numbers

Arithmetic mean means term of arithmetic sequence that lie between two given terms.

Example1

Given

arthimetic sequence 2,x,5 find x

solution

x-2=5-x

2x=7

X=7/2

Example2

The first and sixth terms of an arithmetic sequence are 4 and 29. Find the values of terms 2, 3, 4 and 5.

Solution

Let the four terms be A_2, A_3, A_4, A_5.

So, $4, A_2, A_3, A_4, A_5, 29$ form an arithmetic sequence. Then, $A_n = A_1 + (n-1)d$.

$$A_1 = 4$$
$$A_6 = A_1 + (6-1)d$$
$$29 = 4 + 5d$$
$$d = 5$$

Thus, $A_n = 4 + (n-1)5 = 5n-1$

$$A_2 = 4 + 5 \times 1 = 9$$
$$A_3 = 4 + 5 \times 2 = 14$$
$$A_4 = 4 + 5 \times 3 = 19$$
$$A_5 = 4 + 5 \times 4 = 24.$$

1.2.2 geometric sequence

- **Geometric sequence** is one in which the ratio between consecutive terms is non-zero constant r .
- G_n is geometric sequence if and only r $= \frac{Gn-1}{Gn}$ r $r \in \mathbb{R} \,\&\, r \neq 0$ r is the common ratio.
- $G_n = G1r^{n-1}$ nth term of geometric sequence.

EXAMPLE

FIND G_n when first term is 3 and r=2

$G_n = G1r^{n-1} = 3x2^{n-1}$ … ans

Geometric mean between two numbers

When a, m, and b are terms in a geometric sequence, then m is called the geometric mean between a and b. $(a \neq 0,\ b \neq 0,\ m \neq 0)$. In a geometric sequence, the ratio between consecutive terms is constant.

$$r = \frac{m}{a} = \frac{b}{m}$$
$$m^2 = ab$$
$$m = \pm\sqrt{ab}$$

- **Example1**

When 2,x,5 is geometric sequence find x

$$\frac{x}{2} = \frac{5}{x}$$

$x^2=10$

$x=\pm\sqrt{10}$

TUTORIAL

1.3 the sigma notation and partial sum

Partial sum

Given the sequence a_n

$S_1 = a_1$, S_1 is the first term of the sequence.

$S_2 = a_1 + a_2$, S_2 is the sum of the first two terms of the sequence.

$S_3 = a_1 + a_2 + a_3$, S_3 is the sum of the first three terms of the sequence.

$S_4 = a_1 + a_2 + a_3 + a_4$, S_4 is the sum of the first four terms of the sequence,

and so on.

$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, S_n is the sum of the first n terms of the sequence called the partial sum.

Definition 1.4

Let $\{a_n\}_{n=1}^{n=\infty}$ be a sequence. The sum of the first n terms of the sequence, denoted by S_n is called the partial sum of the sequence. Such summation is denoted as follows.

$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + ... + a_n$, where k is the index of the summation, 1 is the lower limit of summation, n is the upper limit of the summation and $\sum$ is the sigma notation or the summation notation.

Example

Let $a_n = 3n + 1$, find S_6

Solution

$$a_n = 3n + 1$$

$$a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, a_6 = 19$$

$$\begin{aligned} S_6 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 4 + 7 + 10 + 13 + 16 + 19 \\ &= 69 \end{aligned}$$

Sigma notation

- Is method used to write out long sum in short
- It is denoted by Greek letter sigma ∑ .
-

Properties of sigma notation

(1) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, c is a constant

(2) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_i + \sum_{k=1}^{n} b_k$

(3) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(4) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$, $1 \le m \le n$

Sum of arithmetic sequence and geometric sequence

The sum of the first n consecutive natural numbers is $S_n = \frac{n}{2}(n+1)$.

Sum of arithmetic sequence

$$S_n = nA_1 + d\left(\frac{n-1}{2}\right)n = \frac{2nA_1 + n(n-1)d}{2} = \frac{n}{2}[2A_1 + (n-1)d].$$

Sum of geometric sequence

$$S_n = \frac{G_1(1-r^n)}{1-r} \text{ for } r \neq 1$$

Or nG_1 if r=1

1,4 infinite sequence

Suppose we have sequence a1 ,a2 ,a3....

Infinite sum of the form a1 ,+A2+A3+....an+..... is called **infine summation** and using summation we can write as

a1 + a2 + a3 +...+an +....=$\sum_{n=1}^{\infty} a1$

1.4.1 divergence and convergence of infinite sequence

Convergence

A **sequence a_n converges to α : $n \to \infty$, $an \to \alpha$**

Divergence

B sequence a_n diverges to positive infinity : $n \to \infty$, $an \to \infty$.

C sequence a_n diverges to positive infinity : $n \to \infty$, $an \to -\infty$.

D sequence a_n vibrates ; an has no limit , means the value oscilates back and forth

Divergence and convergence of geometric sequence

I. r > 1 when $n \to \infty$ then $Gn \to \infty$ diverge
II. r = 1 when $n \to \infty$ then $Gn \to 1$ converge
III. |r |< 1 when $n \to \infty$ then $Gn \to 0$ converge
IV. r ≤ - 1 when $n \to \infty$ then $Gn \to vibrate$

note

Recall that if $\sum_{n=1}^{\infty} G_1 r^{n-1} = G_1 + G_1 r + G_1 r^2 + \ldots$ is a geometric series with first term G_1 and common ratio r, then $S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r}$.

(i) If $|r| < 1$, as $n \to \infty$, $r^n \to 0$ so that $S_n = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r} \to \frac{G_1}{1-r}, S_\infty = \frac{G_1}{1-r}$.

(ii) If $|r| \geq 1$, then the series diverges or vibrates as follows.

a. If $r = 1$, then $\sum_{k=1}^{n} G_k = nG_1$, when $n \to \infty$, $\sum_{k=1}^{\infty} G_k = \infty$ $(G_1 > 0)$

$\sum_{k=1}^{\infty} G_k = -\infty$ $(G_1 < 0)$

b. If $r \neq 1$, then $S_n = \sum_{k=1}^{\infty} G_k = \frac{G_1(1-r^n)}{1-r}$

Then, If $r \leq -1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k$ vibrates,

If $r > 1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k = \infty$ $(G_1 > 0)$

$\sum_{k=1}^{\infty} G_k = -\infty$ $(G_1 < 0)$

TUTORIAL

Application of sequence questions are written in your text book briefly