College	**Business and Economics**
Department	**Management**
Course Title	**Mathematics for Management**
Instructor	**Azmeraw Y.**

UNIT 1

LINEAR EQUATIONS AND THEIR INTERPRETATIVE APPLICATION

Introduction

Mathematics, old and newly created, coupled with innovative applications of the rapidly evolving electronic computer and directed toward management problems, resulted in a new field of study called quantitative methods, which has become part of the curriculum of colleges of business. The importance of quantitative approaches to management problems is now widely accepted and a course in mathematics, with management applications is included in the core of subjects studied by almost all management students. This manual develops mathematics in the applied context required for an understanding of the quantitative approach to management problems.

1.1. Linear Equations, Functions and Graphs

***Equation* is a** mathematical statement which indicates two algebraic expressions are equal

Example: Y = 2x + 3

Algebraic expressions: is a mathematical statement which indicating that numerical quantities are linked by mathematical operations.

Example: X + 2

Linear equations: - are equations with a variables & a constant with degree one.

- Are equations whose terms (the parts) are separated by +, -, = signs
- Whose terms are a constant, or a constant times one variable to the first power
- Linear equations are equations whose graph in a coordinate plane is a straight line with a constant slope.

Linear equations having two variables x and y has the following general form

$$Y = mx + b$$

Where Y= Dependent Variable

M = slope

X = Independent Variable

b = Constant (Y – intercept)

linear equations have the following basic characteristics.

1. Linear equations have variables with only power one or first degree.
2. The product of two variables should not avail in the linear equation.
3. The coefficient of linear equation variables all should not be zero
4. The graphs of a linear equation s are a straight line

Example: 2x – 3y = 7

- The degree (the power) of the variables is 1
- The constant or the fixed value is 7
- The terms of the equation are 2x and 3y separated by – sign

However, 2x + 3xy = 7 isn't a linear equation, because 3xy is a constant times the product of 2 variables.

No x^2 terms, x/y terms, and xy terms are allowed in linear equations.

Dependent and Independent Variables

The relationship between x and y expressed in y = mx +b is called functional relationship because for each value of X there is one and only one corresponding value of Y. Therefore, Y is the dependent variable since its value depends upon what value we assign to x but X is the independent variable.

Slope(m): Slope is the rate of change in the dependent variable as a result of a unit change in the independent variable. Hence a slope of a straight line joining the two points (x_1, y_1) and (x_2,y_2) is given as follows;

$$\text{Slope (m)} = \frac{\Delta Y}{\Delta X} = \frac{rise(fall)}{run} = \frac{Y_2 - Y_1}{X_2 - X_1} \text{ if } X_1 \neq X_2$$

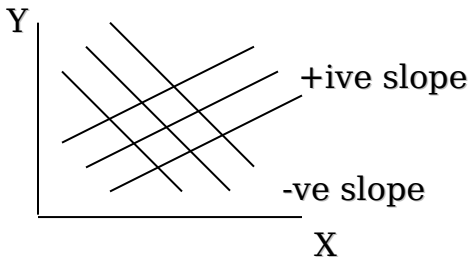
Eg.Find the slope for the following line segment joining the f following points(3,10) and (4,12)

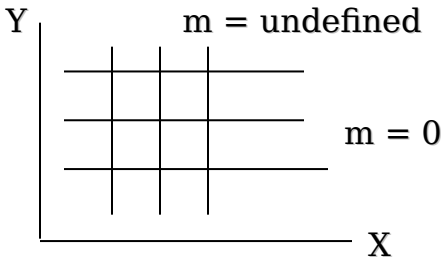
$X_1 = 3$ $Y_1 = 10$

$X_2 = 4$ $Y_2 = 12$

$$m = \frac{Y2 - Y1}{X2 - X2} = \frac{12 - 10}{4 - 3} = 2$$

Slope measures the steepness of a line. The larger the slope the steeper (steeper) the line is, both in value and in absolute value.





- A line that is parallel to the X-axis is the gentlest of all lines i.e. m = 0
- A line that is parallel to the Y-axis is the steepest of all lines i.e. m = undefined or infinite.

The slope of a line is defined as the change-taking place along the vertical axis relative to the corresponding change taking place along the horizontal axis, or the change in the value of Y relative to a one-unit change in the value of X.

Developing Equation of a Line

There are three ways of developing the equation of a line. These are:

1. The slope-intercept form
2. The slope-point form
3. Two-point form

1. The slope-intercept form

This way of developing the equation of a line involves the use of the slope & the intercept to formulate the equation.

Example find the equation of a line

Given Slope = 10

Y-intercept = 20, then

Y = mx + b

Y = 10X + 20

2. The slope - point form

In a straight line we designate points as (X_1, Y_1) and (X_2, Y_2) and then we let (X, Y) represents any other point on the same straight line. Thus we have (X1, Y1), (X2, Y2) and (X, Y). The slope of the line must be the same for any pair of its points. i.e,

$$M = \frac{Y - Y1}{X - X1} = \frac{Y - Y2}{X - X2} = \frac{Y2 - Y1}{X2 - X1}$$

$$M = \frac{Y - Y1}{X - X1} \qquad M = \frac{Y - Y2}{X - X2}$$

$y - y_1 = m(x-x_1)$ $\qquad$ $y - y_2 = m(x-x_2)$

$y = m(x-x_1) + y_1$ $\qquad$ $y = m(x-x_2) + y_2$

So $Y = m(X-X_1) + Y_1$ or $Y = m(X-X_2) + Y_2$ is the formula to develop the equation of a line.

Given, slope = 4 and

Point = (1, 2)

The equation of a non-vertical line, L, of slope, m, that passes through the point (X_1, Y_1) is: defined by the formula $Y - Y_1 = m(X - X_1) = Y = m(X-X1) + Y1$

Example $\Rightarrow$ $Y = 4(X - 1) + 2$

$Y = 4X - 4 + 2$

$Y = 4X - 2$

3. Two - point form

In this method, first we will compute the slope, then use this value of m together with either point in the point-slope form $Y = m(X - X_1) + Y_1$ to generate the equation of a line.

Example: find the equation of a line that passes through points (100, 200) and (150,275)

$M = \frac{275-200}{150-100} = 1.5$

Equation: $Y = m(X-X_1) + Y_1$ or $Y = m(X-X_2) + Y_2$

$= 1.5(X-100) + 200$ $\qquad$ $= 1.5(x-150) + 275$

$= 1.5X - 150 + 200$ $\qquad$ $= 1.5x - 225 + 275$

$= 1.5x + 50$ $\qquad$ $= 1.5x + 50$

1.1.1. Special formats

a) Horizontal & vertical lines

If the Y values are the same and the slope is zero, the line is a horizontal line. The equation for such a line is given as Y= Constant number

Example: Given the points (3, 6) & (8, 6) the line through them is horizontal because both Y-coordinates are the same i.e. 6

The equation of the line becomes Y = 6, which is different from the form Y = mx + b

If the X values are the same and the slope is undefined, the line is a vertical line. The equation for such a line is given as X = Constant number

Example (5, 2) & (5, 12) the line through them is vertical, & its equations is X = 5 i.e. X is equal to a constant.

b) Parallel & perpendicular lines (// and ⊥)

X	0	1	2	3	4	5
Y	0	2	4	6	8	10

Two lines are parallel if the two lines have the same slope, & two lines are perpendicular if the product of their slope is –1 or the slope of one is the negative reciprocal of the slope of the other.

Example: L_1: Y = 2X – 10 &

L_2: Y = 2X + 14 are parallel because their slope are equal i.e. 2

L_1: Y = 3/2X + 10 &

L_2: Y = -2/3X + 100 are perpendicular to each other because the multiplication result of the two slope are –1 i.e. $\frac{3}{2} x \frac{-2}{3} = -1$

c) Lines through the origin

Any equation in the variables X & Y that has no constant term other than zero will have a graph that passes through the origin. Or, a line which passes through the origin has both X and Y intercepts are zero.

Y

Example Y= 2x Y = 2x

X

1.2. Application of linear equations

1.2.1. linear Cost - Output Relationships: – Variable Cost, Fixed Cost , Total Cost , Average Cost , Marginal Cost , Total Revenue and Total Profit

Some application of linear equations and straight lines to problems in business and economics such as costs, revenues, and profit of Manufacturing and Merchandizing firms shall be discussed.

Manufacturing Firms

Manufacturing firms are those businesses which engaged in the production of goods and services from the resource available. To produce goods and services, those firms incur costs in the form of fixed and variable costs

In this section we shall consider a Manufacturer who produces Q units of outputs at a unit variable cost(VC), fixed cost(FC) and sells the product at a unit selling price (P).

A. **Linear Cost Function/Equation**

Accountants and Economists often define total cost in terms of two components

1. Total Variable Cost (TVC)
2. Total Fixed Cost (TFC)

1. **Variable Costs**: they are costs that can vary with the level of output (they are costs that depends on the level of production). But variable cost per unit is constant.

   Example: Costs of raw materials, costs of labor etc.

   $$TVC = VC.Q$$

2. **Fixed Costs:** they are costs that cannot vary with the levels of production(output)

   Example: Rent, interest on a loan etc.

   Therefore, Total Cost = Total Variable Cost + Total Fixed Cost

$$TC = TVC + TFC$$

$$TC = VC.\ Q + FC$$

$$TC = VC.\ Q +$$

Where TC = Total Cost of production.

VC = Variable cost per unit.

Q = Units produced/level of production.

FC = Fixed cost of production.

Example: A firm which produces a single product is interested in determining the function that expresses annual total cost as a function of units produced (Q). The fixed expenditure each year is Birr 50,000. The raw material costs for each unit produced are Birr 5.50 and labor costs per unit are Birr 1.5 in the assembly department, birr 0.75 in the finishing department and Birr 1.25 in the packaging and shipping department.

Required:

A. Develop a total cost function for the firm
B. Find total cost, if the volume of production is 1000 units
C. Determine Average cost and Marginal cost
D. Draw the graph.

Solution:

B. **Linear Revenue Function/Equation**

Revenue is the amount of money received from selling products or rendering services

TR =

Example: A firm sells a single product for $65 per unit

Required:

A. Construct the revenue function of the firm in terms of quantity sold.
B. Plot the graph
C. What is the total revenue of the firm for selling 200 units of output?

Solution

C. **Linear Profit Function/Equation**

Profit is the difference between total revenue and total costs.

Profit (π) = TR – TC

π =PQ – (VC. Q + FC)

π = PQ -VC. Q – FC

π = Q (P – VC) – FC

π = Q (P – VC) -

Example: a Manufacturing firm producing a single product has a fixed cost of $60,000, variable cost of $4 per unit and selling price of per unit is $7 .

Required:

A. Construct revenue, cost and profit function of the firm.
B. Determine the profit if 40,000 units are produced and sold.
C. Draw the graph.

Solution.

Given

FC = Birr 50,000

Raw material costs per unit = Birr 5.50

Labor costs per unit: assembly department = Birr 1.50

Finishing department = Birr 0.75

Packaging & shipping department = Birr1.25

VC = Birr 9 i.e, 5.50 +1.50 + 0.75 + 1.25

A. TC function =?

Solution:

TC = VC.Q + FC

TC = 9Q +50,000

B. If Q = 1000 units TC =?

Solution

TC = 9Q + 50,000

TC = 9(1000) + 50,000

TC = 9000 +50,000

TC = Birr 59,000

C. AC and MC =?

Solution $AC = \frac{TC}{Q}$ $MC = \frac{dTC}{dQ}$

$AC = \frac{9Q + 50000}{Q}$ $MC = \frac{d(9Q + 50000)}{dQ}$

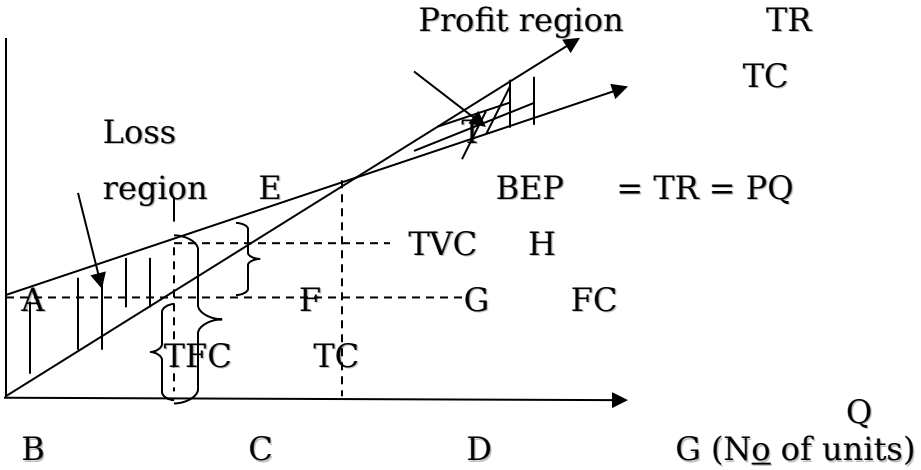
$AC = 9 + \frac{50000}{Q}$ $MC = Birr9$

1.2.2. Break- Even Analysis

Break-even point is the point at which there is no loss or profit to the company. It can be expressed as either in terms of production quantity or revenue level depending on how the company states its cost equation.

Break- even analysis is concerned with determining the output level that would result in zero profit.

Manufacturing companies usually state their cost equation in terms of quantity (because they produce and sell) where as retail (merchandizing) business state their cost equation in terms of revenue (because they purchase and sell)



Interpretation of the graph:

1. The vertical distance between AB, FC, GD is the same because Fixed Cost is the same at any levels of output.
2. There is no revenue without sales (because Total Revenue function passes through the origin), but there is cost without production (because of Fixed Cost) & the TC function starts from A & doesn't pass through the origin
3. Up to point T, Total Cost is greater than Total Revenue ⇒ results in loss. While at point T, (Total Revenue = Total Cost) i.e. Breakeven. (0 profit), & above point T, TR > TC ⇒ +ve profit.
4. TFC remains constant regardless of the number of units produced. Given that there is no any difference in scale of production.
5. As production increases, Total Variable Cost increases at the same rate and Marginal cost is equal with Unit Variable Cost (MC = VC) only in linear equations.
6. As production increases TC increases by the rate equal to the AVC = MC (average cost equal to marginal cost)

7. AVC is the same throughout any level of production, however Average Fixed Cost (AFC) decreases when Quantity increases & ultimately ATC decreases when Q increases because of the effect of the decrease in AFC.
8. As Quantity increases TR increases at a rate of P. and average revenue remains constant.

$$AR = \frac{TR}{Q} = \frac{P.Q}{Q} = P \Rightarrow AR = P \text{ in linear functions}$$

Case 1: Manufacturing Companies

Consider a Company with equation

Total cost = Total Variable cost + Total Fixed cost

TC = TVC + TFC

Total Revenue = Price x Quantity

TR = PQ

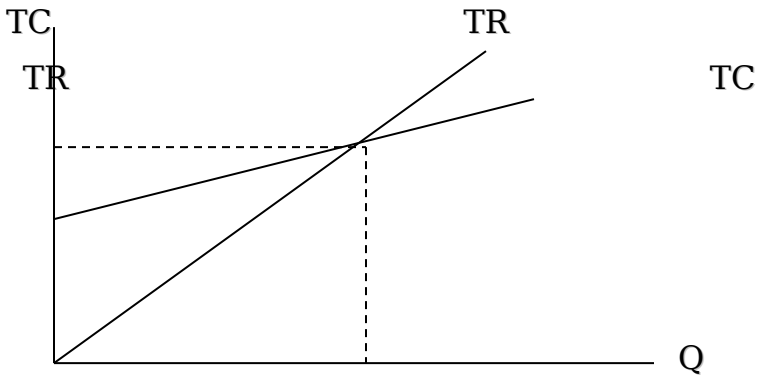
At Break-even point, TR = TC i.e TR – TC = 0

PQ = VC + FC where Qe = Breakeven Quantity

PQ – VC.Q = FC FC = Fixed cost

Q (P – VC) = FC P = unit selling price

$$Qe = \frac{FC}{P - VC}$$ VC = unit variable cost



$$Qe = \frac{FC}{P - VC}$$

Example 1 A manufacturing Co. has a Total Fixed Cost of Br. 10,000 & a Unit Variable Cost of Br. 5. if the co. can sell what it produces at a price of Br. 10,

a) Write the Revenue, cost & Profit functions
b) Find the breakeven point in terms of quantity and sales volume

c) Show diagrammatically the Total Revenue, Total Cost, Total Profit, Fixed Cost and Variable Costs.

d) Interpret the results

e) Find profit or loss at 40% of the firm's capacity and 80% of its capacity.

Given

FC = Br 10,000
VC = Br 5
P = Br 10

;Answer

a) TC = VC + FC TR = PQ Profit (Π) = TR – TC = 10Q – (5Q + 10,000)

TC = 5Q + 10,000 TR = 10Q = 5Q – 10,000

b) At break -even point TR = TC

$10Q = 5Q + 10{,}000$

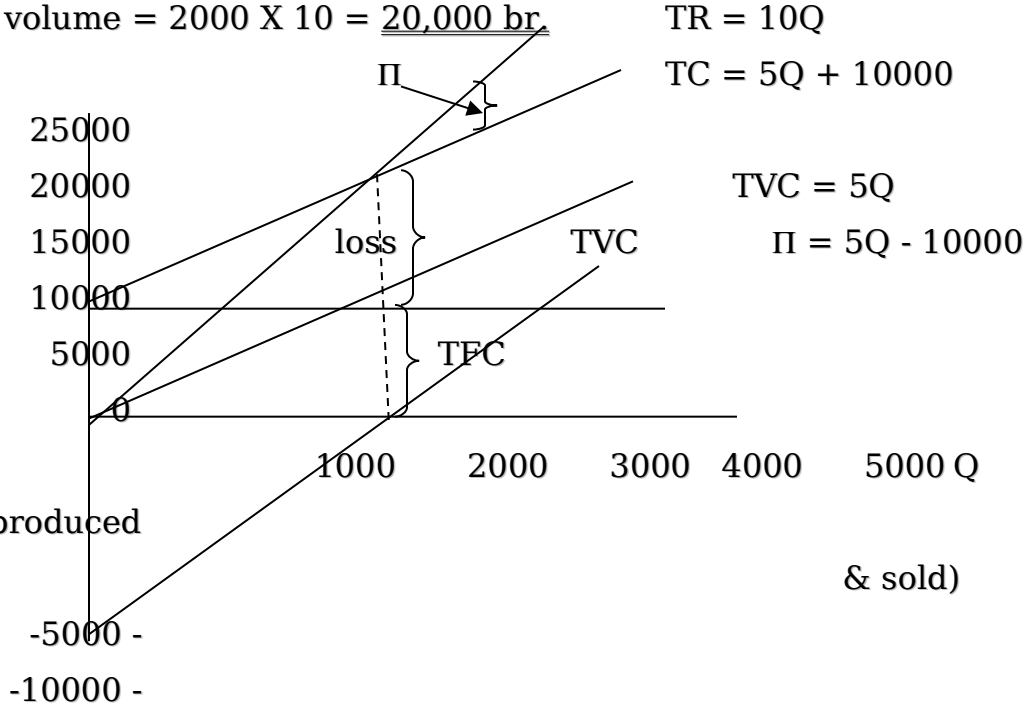
$5Q - 10{,}000 = 0$

$Qe = \frac{10{,}000}{5}$

Qe = 2000 units

i.e. Breakeven Quantity is 2000 units

Sales volume = 2000 X 10 = 20,000 br.



d. Interpretation:

When a co. produces & sells 2000 units of output, there will not be any loss or gain (no profit, no loss)

e. see your exercise book solution for question e

Finding the quantity level which involves profit or loss

BEP = $\frac{FC+0}{P-V}$, any Q is related to the cost , profit

Π = TR – Tc

Π = PQ – (VC.Q + FC)

Π = (P.Q – VC.Q) – FC

Π = Q (P – VC) – FC

$\frac{\pm\Pi+FC}{P-VC} = for\ any\ qty\ level\ Q = \frac{FC\pm\Pi}{P-VC}$

Where: BEP = breakeven point

Π = Profit

TR = Total revenue

TC = Total cost

Q = Quantity

VC = Unit variable cost

Example 1: For the above manufacturing co. if it wants to make a profit of 25000 br. What should be the quantity level?

TR = 10Q

TC = 5Q + 10,000

Π = 25,000

Q = ?

Q = $\frac{FC\pm\Pi}{P-V}$ when there isΠ, the quantity produced & Sold have to be greater than the Break-even quantity

= $\frac{10{,}000+25000}{10-5}$

= 7000 units

If it expects a loss of br. 5000 what will be the quantity level.

Q = $\frac{FC\pm VC}{P-VC} = \frac{10{,}000-5000}{10-5}$ * when there is loss, the qty produced & sold should be less than the BEQ

Case 2 Merchandising /Retail Business

The total cost of merchandizing/retail business can be expressed as follows

$Y = mx + b$

Where Y= Total Cost

m = Variable Cost per Dollar/Birr

b = Fixed cost

x =Sales Revenue (Volume)

At break-even point, Total Cost = Total revenue = break –even revenue

$Y = X = X_e$ where X_e is break even sales revenue

Y = mx + b

xe = mxe +b

$xe - mxe = b$

$xe(1- m) = b$

$$\frac{xe(1- m)}{1- m} = \frac{b}{1- m}$$

$$xe = \frac{b}{1- m}$$

PC = 1 – Margin

Where PC = Purchasing Cost per Dollar/Birr

m = Variable cost per Dollar/Birr

Other VC = other variable costs or additional costs

Example1: Suppose a retail business sale its commodities at a margin of 25% on all items purchased & sold. Moreover, the company uses 5% commission as selling expense & Br. 12,000 as a Fixed Cost.

Required

A. Find the Break- even revenue for the retail business after developing the equation

B. Find profit/loss if the sales are 70, 000Br

Given

b = Br 12,000

Margin = 25% = 0.25

Other VC = 5% = 0.05

A. Solution

Y = mx +b

m = PC + other VC

PC = 1- Margin

= 1 - 0.25

= 0.75

m = PC + other VC

= 0.75 + 0.05

= 0.80

So Y = mx + b

Y = 0.8x + 12,000

Xe =?

Solution $xe = \frac{b}{1- m}$

$$xe = \frac{12,000}{1- 0.8}$$

$$xe = \frac{12,000}{0.2}$$

$$xe = 60,000 Br$$

B. Π = ?

Solution

$\Pi = x - y$

$Y = 0.8x + 12{,}000$

$Y = 0.8(70{,}000) + 12{,}000$

$Y = 56{,}000 + 12{,}000$

$Y = 68{,}000$ Birr

So $\Pi = x - y$

$\Pi = 70{,}000 - 68{,}000$

$\underline{\Pi = 2{,}000 \text{ Birr}}$

The Break-even revenue (BER = $\frac{FC \pm \Pi}{1- m}$) method is useful, because we can use a single formula for different goods so far as the company uses the same amount of profit margin for all goods.

Eg2: it is estimated that the sales in the coming week will be Birr 6000 and fixed cost will be Birr 1000 and variable cost is Birr 3,600.

Required:

A. Develop the total cost function/equation
B. Find the break- even revenue of the firm
C. Find the profit of the firm.

Solution

See your Exercise book.

Market Equilibrium Analysis (Reading Assignment)

UNIT 2: MATRIX ALGEBRA AND ITS APPLICATION

Unit Objectives

After reading this unit, you will be able to:

- Explain a matrix
- Define the different types of matrices
- Perform matrix operations
- Find inverse of a square matrix
- Explain how to solve a systems of linear equations
- Understand the concept of Markov chain Analysis

✪ Introduction

Brevity in mathematical statements is achieved through the use of symbols. The price paid for brevity, of course, is the effort spent in learning the meaning of the symbol.

In this unit we shall learn the symbols for matrices, and apply them in the statement and solution of input-output problems and other problem involving linear systems

2.1. Matrix Algebra Concepts

Algebra is a part of mathematics, which deals with operations (+, -, x,÷).

A matrix is a rectangular array of real numbers arranged in m rows & n columns. It is symbolized by a bold face capital letter enclosed by a bracket or parentheses.

eg. $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} - - - - - - a_{1n} \\ a_{21} & a_{22} - - - - - - a_{2n} \\ a_{m1} & a_{m2} - - - - - - a_{mn} \end{bmatrix}$ in which a_{jj} are real numbers

Each number appearing in the array is said to be an element or component of the matrix. Element of a matrix are designated using a lower-case form of the same letter used to symbolize the matrix itself. These letters are subscripted as a_{ij}, to give the row & column location of

the element with in the array. The first subscript always refers to the raw location of the element; the second subscript always refers to its column location. Thus, component a_{ij} is the component located at the intersection of the i^{th} raw and j^{th} column.

The number of rows (m) & the number of columns (n) of the array give its order or its dimension, m x n (reads "m" by "n")

Eg. The following are examples of matrices

$$A = \begin{pmatrix} 1 & 7 \\ 5 & 3 \\ 4 & 2 \end{pmatrix}$$

this is 3 x 2 matrix

element $a_{12} = 7$

$a_{21} = 5$

$a_{32} = 2$

$$X = \begin{pmatrix} 1 & 5 & 9 & 15 \\ 2 & 6 & 10 & 20 \\ 3 & 7 & 11 & 30 \\ 4 & 8 & 12 & 45 \end{pmatrix}$$

This is a 4 x 4 matrix

Element $X_{44} = 45$

$X_{34} = 30$

$X_{42} = 8$

$X_{32} = 7$

2.2. Dimensions &Types of Matrices

There are deferent types of matrices. These are

1. **Vector matrix** – it is a matrix, which consists of just one row or just one column. It is an m x 1 or 1 x n matrix.

   i. Row vector is a 1 x n matrix i.e. a matrix with 1 row

   eg. $W = \begin{pmatrix} -1 & 0 & 6 \end{pmatrix}$ 1 x 3

   ii. Column vector: is an m x 1 matrix i.e. a matrix with one column only

   eg. $Z = \begin{pmatrix} 0 \\ 20 \\ 5 \end{pmatrix}$ 3 x 1

2. **Square matrix**: - a matrix that has the same number of rows & columns. It is also called n-th order matrix

eg. 2 x 2, 3 x 3, n x n $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2 x 2

3. **Null or zero Matrix:** -it is a matrix that has zero for every entry.

It's generally denoted by Om x n eg. $Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. **Identity (unit) matrix:** -it is a square matrix in which all of the primary/main diagonal entries are ones & all of the off diagonal entries are zeros. Its denoted by I.

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2 x 2 $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 4 x 4

N.B. Each identity matrix is a square matrix

* Primary diagonal represents: $a_{11}, a_{22}, a_{33}, a_{44}$---------$a_{nn}$ entries element

A x I = A & I x A = A that is, the product of any given matrix & the identity matrix is the given matrix itself. Thus, the identity matrix behaves in a matrix multiplication like number 1 in an ordinary arithmetic.

5. **Scalar matrix:** - is a square matrix where elements on the primary diagonal are the same. "An identity matrix is a scalar matrix but a scalar matrix may not be an identity matrix". e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

6. **Diagonal Matrix:** it is a square matrix whose all elements except the main diagonal element are zero. e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C =$

0 1 2x2 0 2 2x2

0 2

2.3. Matrix Operations and Techniques (Addition, Subtraction, Multiplication)

i. Matrix Addition or Subtraction

Matrices are added or subtracted by adding or subtracting the corresponding entries… Two matrices of the same dimension are said to be conformable for addition. Adding corresponding elements from the two matrices & entering the result in the same raw-column position of a new matrix perform the addition.

If A & B are two matrices, each of site m x n, then the sum of A & B is the m x n matrix C whose elements are:

$C_{ij} = a_{ij} + b_{ij}$ for i = 1, 2 --------m $C_{11} = a_{11} + b_{11}$

j = 1, 2 ---------n $C_{22} = a_{22} + b_{22}$

$C_{12} = a_{12} + b_{12}$ etc

eg. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{3} + \begin{bmatrix} 7 & 9 \\ 8 & -10 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 10 & -6 \end{bmatrix}$

eg. $\begin{bmatrix} -2 & 7 \\ 4 & 6 \end{bmatrix}_{2} + \begin{bmatrix} 8 & 7 \\ 9 & 6 \end{bmatrix}$ These two matrices aren't conformable for addition because they aren't of the same dimension.

Laws of matrix addition

The operation of adding two matrices that are conformable for addition has these two basic properties.

1. A + B = B + A → The commutative law of matrix addition

2. (A + B) + C = A + (B + C) → the associative law of matrix addition.

3. Additive Inverse

A + 0 = 0 +A

ii. Matrix Multiplication

A. Scalar Multiplication

A matrix can be multiplied by a constant by multiplying each component in the matrix by a constant. The result is a new matrix of the same dimension as the Original matrix. If K is any real number & A is an M x n matrix, then the product KA is defined to be the matrix whose components are given by K times the corresponding component of A; i.e.

$$KA = [K_{aij}] \quad (m \times n)$$

eg. If $A = \begin{bmatrix} 6 & 5 & 7 \end{bmatrix}$, and K = 2 then KA = 2A = $\begin{bmatrix} (2 \times 6) & (2 \times 5) & (2 \times 7) \end{bmatrix}$

$$KA = \begin{bmatrix} 12 & 10 & 14 \end{bmatrix}$$

Laws of Scalar Multiplication

The operation of multiplying a matrix by a constant (a scalar) has the following basic properties. If X & Y are real numbers & A & B are m x n matrices, conformable for addition, then

1. XA = AX
2. (X + Y) A = XA + YA
3. X (A + B) = XA + XB
4. X (YA) = XY (A)

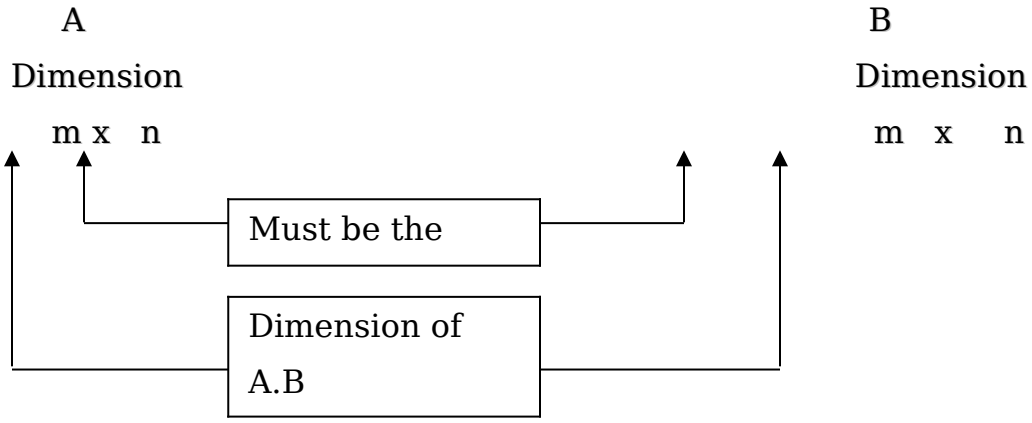
B. Matrix by matrix multiplication

If A & B are two matrices, the product AB is defined if and only if the number of Columns in the first matrix (A) is equals to the number of rows in the second matrix (B)

If this requirement is met., A is said to be conformable to B for multiplication. The matrix resulting from the multiplication has dimension equivalent to the number of rows in A & the number columns in B

If A is a matrix of dimension m x n (which has m rows and n columns) & B is a matrix of dimension m x n (which has m rows and n columns and if n in A and m in B aren't the same product A.B is not defined. That is, multiplication of matrices is possible only if the number of columns of the first matrix equals to the number of rows of the second.

If A has dimension m x n & if B has dimension m x n, then the product A.B has dimension m x n



eg. $A = \begin{pmatrix} 2 & 3 & 4 \\ 6 & 9 & 7 \end{pmatrix}_{2 \times 3}$ $B = \begin{pmatrix} -1 & 7 \\ 0 & 8 \\ 5 & 1 \end{pmatrix}_{3 \times 2}$

$$AB = \begin{pmatrix} 2x(-1)+3x0+4x5 & 2x7+3x8+4x1 \\ 6x(-1)+9x0+7x5 & 6x7+9x8+7x1 \end{pmatrix}_{2x2}$$

$$\therefore AB = \begin{pmatrix} 18 & 42 \\ 29 & 121 \end{pmatrix}$$

Find BA =

$$B = \begin{pmatrix} -1 & 7 \\ 0 & 8 \\ 5 & 1 \end{pmatrix}_{3 \times 2} \quad A = \begin{pmatrix} 2 & 3 & 4 \\ 6 & 9 & 7 \end{pmatrix}_{2 \times 3}$$

B (3 x 2) A (2 x 3) result 3 x 3 matrix

Conformable

$$BA = \begin{bmatrix} (-1 \times 2) + (7 \times 6) & (-1 \times 3) + (7 \times 9) & (-1 \times 4) + (7 \times 7) \\ (0 \times 2) + (8 \times 6) & (0 \times 3) + (8 + 9) & (0 \times 4) + (8 \times 7) \\ (5 \times 2) + (1 \times 6) & (5 \times 3) + (1 \times 9) & (5 \times 4) + (1 \times 7) \end{bmatrix}$$

3 x 3

$$BA = \begin{bmatrix} 40 & 60 & 45 \\ 48 & 72 & 56 \\ 16 & 24 & 27 \end{bmatrix}$$ 3 x3

Special Properties of Matrix Multiplication

1. The associative & distributive laws of ordinary algebra apply to matrix multiplication.

Given three matrices A, B & C which are conformable for multiplication,

i. A (BC) = (AB) C → Associative law

ii. A (B+C) = AB + AC → Distributive property

iii. (A + B) C = AC + BC → Distributive property

2. On the other hand, the commutative law of multiplication doesn't apply to matrix multiplication. For any two real numbers X & Y, the product XY is always identical to the product YX. But for two matrices A & B, it is not generally true that AB equals BA.

3. In many instances for two matrices, A & B, the product AB may be defined while the product BA may not be defined or vice versa.

In some special cases, AB does equal BA. In such special cases A & B are said to be Commutative $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2x2}$ $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}_{2x2}$ $AB = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}_{2 \times 2}$

$$BA = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

4. Another un usual property of matrix multiplication is that the product of two matrices can be zero even though neither of the two matrices themselves is zero: we can't conclude from the result AB = 0 that at least one of the matrices A or B is a zero matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 7 & -10 & 4 \\ 8 & 3 & 2 \end{pmatrix} \quad AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

5 Also we can't, in matrix algebra, necessarily conclude from the result AB = AC that B= C even if A ≠ 0. Thus the cancellation law doesn't hold, in general, in matrix multiplication

eg. $A = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$AB = AC = \begin{pmatrix} 10 & 14 \\ -20 & -28 \end{pmatrix} \quad \text{but } B \neq C$$

2.4. The Multiplicative Inverse of a Matrix

If A is a square matrix of order n, then a square matrix of its inverse (A^{-1}) of the same order n is said to be the inverse of A, if and only if $A \times A^{-1} = I = A^{-1} \times A$

Two square matrices are inverse of each other, if their product is the identity matrix.

$AA^{-1} = A^{-1} A = I$

Not all matrices have an inverse. In order for a matrix to have an inverse, the matrix must, first of all, be a square matrix.

Still not all square matrices have inverse. If a matrix has an inverse, it is said to be **invertible** or **Non- singular.** A matrix that doesn't have an inverse is said to be singular. An invertible matrix will have only one inverse; that is, if a matrix does have an inverse, that inverse will be unique.

Note: i. Inverse of a matrix is defined only for square matrices

ii. If B is an inverse of A, then A is also an inverse of B

iii. Inverse of a matrix is unique

iv. If matrix A has an inverse, A is said to be invertible & not all square matrices are invertible.

Finding the inverse of a matrix

Let's begin by considering a tabular format where the square matrix A is augmented with an identity matrix of the same order as [A / I] i.e. the two matrices separated by a vertical line

Now if the inverse matrix A^{-1} were known, we could multiply the matrices on each side of the vertical line by A^{-1} as

$$[AA^{-1} / A^{-1} I]$$

Then because $AA^{-1} = I$ & $A^{-1}I = A^{-1}$, we would have $[I / A^{-1}]$. We don't follow this procedure, because the inverse is not known at this juncture, we are trying to determine the inverse. We instead employ a set of permissible row operations on the augmented matrix [A / I] to transform A on the left of the vertical line in to an identity matrix (I). As the identity matrix is formed on the left of the vertical line, the inverse of A is formed on the right side. The allowable manipulations are called Elementary raw operations Elementary Row Operations: are operations permitted on the rows of a matrix.

In a matrix Algebra there are three types of row operations

Type 1: Any pair of rows in a matrix may be interchanged / Exchange operations

Type 2: a row can be multiplied by any non-zero real number / multiple operations

Type 3: a multiple of any row can be added to any other row. / Add or subtract a multiple operation

In short the operation can be expressed as

1. Interchanging rows
2. The multiplication of any row by a non-zero number.

3. The addition / subtraction of (a multiple of) one row to /from another row

eg.1. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 6 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ ⇒ interchanging rows

2. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 6 & 4 \\ -2 & 6 & 7 \end{bmatrix}$ Multiplying the first rows by 2

3. $A = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 6 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 & 2 \\ 6 & 12 & 11 \end{bmatrix}$ multiplying the 1st row by 2 & add to the 2nd row. This case there is no charge to the first row.

Basic Procedures to Find the Inverse of a Square Matrix

1. To set ones first in a column & next zero (within a given column)
2. To set zeros first in a matrix & next one.

<u>Ones First Method</u>

Find the inverse of the following matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

augment A with the same dimension ⇒ identity matrix first

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

Interchange rows (row 1 with row 2)

1	1	0	1
3	2	1	0

Multiply R_1 by –3 & add to R_2

($-3R_1 + R_2$) i.e. No change to R_1

1	1	0	1
0	-1	1	-3

Multiply R_2 by –1 = ($-R_2$)

1	1	0	1
0	1	-1	3

Multiply R_2 by –1 & add to R_1

1	0	1	-2
0	1	-1	3

Ones first: try to set ones first in a column and then zeros of the same column. Goes from left to right

Therefore, inverse of A is

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

2.5. Matrix Applications

1. Solving Systems of Linear Equations

I. n by n systems

Systems of linear equations can be solved using different methods. Some are:

i. Matrix method
   1. Inverse method
   2. Gaussian method

ii. Estimation method

1. Inverse method:

Steps 1: Change the system of linear equation into matrix form. The result will be 3 different matrices constructed using coefficient

of the variables, unknown values and right- hand side (constant) values

Step 2: Find the inverse of the coefficient matrix

Step 3: Multiply the inverse of coefficient matrix with the vector of constant, and the resulting values are the values of the unknown matrix.

eg. $2X + 3Y = 4$ Given this system of linear equation applying

$X + 2Y = 2$ inverse method we can find the u unknown values.

Step 1: Change it into matrix form

- Using coefficient construct one matrix i.e. coefficient matrix

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$ = Coefficient matrix

- Using the unknown variables construct unknown matrix & it is a column vector (a matrix which has one column)

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$ = vector of unknown

-Using the constant values again construct vector of constant

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$ = vector of constant

Step 2: Find inverse of the coefficient matrix

Now we are familiar how to find an inverse for any square matrix. Assuming once first method find the inverse for matrix $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

Its inverse become $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Step 3: Multiply the coefficient inverse with the vector of constant

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore, the resulting matrix that is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is the value for the unknown variables i.e. $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Then X = 2 and Y = 0 that is unique solution

* The logic is this given three matrices, coefficient matrix, unknown matrix and vector of constant in the following order.

AX = B

Given this we can apply different Operations, say multiply both sides Of the expression by A^{-1}

A = coefficient matrix

X = vector of unknown

B = vector of constant

$\underline{A^{-1}AX} = A^{-1}B$

$\underline{IX} = A^{-1}B$

$X = A^{-1}B$ this implies that multiplying inverse of the coefficient matrix will gives us the value of the unknown matrix

Limitations of inverse method

- ✓ It is only used whenever the coefficient matrix is square matrix
- ✓ In addition to apply the method the coefficient matrix needs to have an inverse
- ✓ It doesn't differentiate between no solution and infinite solution cases.

2. Gaussian method

It is developed by a mathematician Karl F. Gauss (1777-1855). It helps to solve systems of linear equations with different solution approaches i.e. unique solution, No solution and infinite solution cases.

"n" by "n" systems

Step: 1. Change the system of linear equation into a matrix form

2. Augment the coefficient matrix with the vector of constant.

3. Change the coefficient matrix into identity form by applying elementary ro w operation and apply the same on the vector of constant.

4. The resulting values of the vector of constant will be the solution or the value of the unknown

Example: $\begin{cases} 2X + 3Y = 4 \\ X + 2Y = 2 \end{cases}$

Step 1: Change it into matrix form

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Coefficient Matrix | unknown matrix | vector of constant

Step 2: Augmentation

$$\left[\begin{array}{cc|cc} 2 & 3 & 4 \\ 1 & 2 & 2 \end{array}\right]$$

Step: 3. Change the coefficient matrix into identity form by applying elementary row operation (use ones first method)

$$\left[\begin{array}{cc|cc} \bigcirc\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{array}\right]$$

Change first the primary diagonal entry from the first row into positive one. Possible operation is exchange row one with row two.

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 \\ \bigcirc\ 2 & 3 & 4 \end{array}\right]$$

Next change the remaining numbers in the first column into zero, this case number 2

Now multiply the 1st row by –2 & add the result to row –2

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 \\ 0 & -1 & 0 \end{array}\right]$$

Then proceed to column 2 and change the primary diagonal entry i.e. –1 into 1

Multiply the 2nd row by –1 ($-1R_2$)

$$\begin{array}{cc|cc} 1 & & 2 & 2 \\ 0 & & 1 & 0 \end{array}$$

Now change the remaining number with in the same column (column –2) into zero i.e. number 2

Multiply 2nd row by –2 and add the result to the 1st row

$$\begin{array}{cc|cc} 1 & 0 & 2 \\ 0 & 1 & 0 \end{array}$$

Therefore X = 2 and Y = 0

Example 2. $\begin{cases} X + Y = 2 \\ 2X + 2Y = 4 \end{cases}$

Step-1

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \begin{bmatrix} \\ \end{bmatrix} \begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} \\ \end{bmatrix} = \begin{matrix} 2 \\ 4 \end{matrix}$$

Step-2

$$\begin{array}{cc|cc} 1 & & 1 & 2 \\ 2 & & 2 & 4 \end{array}$$

Multiply Row-1 by –2 & add the result to raw-1 ($-2R_1 + R_2$)

$$\begin{array}{cc|cc} 1 & & 1 & 2 \\ 0 & & 0 & 0 \end{array}$$

The next step is changing the primary diagonal entry in the 2nd row to 1. But there is no possible operation that can enable you to change it in to number 1

Therefore, the implication is that you can't go further but we can observe something from the result. And it is implying an infinite solution case

Example 3: $\begin{cases} X + Y = 5 \\ X + Y = 9 \end{cases}$

Step 1: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix} \begin{bmatrix} \\ \end{bmatrix} \begin{matrix} X \\ Y \end{matrix} \begin{bmatrix} \\ \end{bmatrix} = \begin{matrix} 5 \\ 9 \end{matrix}$

Step 2:
$$\begin{array}{cc|cc} 1 & & 1 & 5 \\ \textcircled{1} & & 1 & 9 \end{array}$$

Change the encircled number above in to zero

Multiply the first row by –1 & add the result to the 2nd row.

$$\begin{array}{cc|cc} 1 & & 1 & 5 \\ 0 & & 0 & 4 \end{array}$$

$0 = 4$ **no solution**

There is no possible operation that we can apply in order to change the primary diagonal entry in the 2nd column without affecting the first column structure. Therefore, stop there, but here we can observe something i.e. it is no solution case.

Therefore, Gaussian method makes a distinction between No solution & infinite solution. Unlike the inverse method

Summarizing our results for solving an "n" by "n" system, we start with the matrix.

(A/B), & attempt to transform it into the matrix (I/C) one of the three things will result.

1. an "n" by "n" matrix with the unique solution.

eg.
$$\begin{array}{cc|cc} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}$$

2. A row that is all zeros except in the constant column, indicating that there are no solutions,

eg.
$$\begin{array}{cc|cc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 7 \end{array}$$

3. A matrix in a form different from (1) & (2), indicating that there are an unlimited number of solutions. Note that for an n by n system, this case occurs when there is a row with all zeros, including the constant column.

Eg.
$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

II. M by n linear systems

The m x n linear systems are those systems where the number of rows (m) and number of columns (n) are unequal or it is the case where the number of equations (m) & the number of variables (n) are unequal. And it may appear as m > n or m < n.

Linear equation where m > n

To solve an m by n system of equations with m > n, we start with the matrix (A/B) and attempt to transform it into the matrix (I/C).

One of the three things will result:

1. An m by n identifying matrix above m – n bottom rows that are all zeros, giving the unique solution:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} 3X_1 + 2X_2 + X_3 = 23 \\ X_1 + 3X_2 + 2X_3 = 26 \\ 2X_1 + X_2 + 2X_3 = 10 \\ 4X_1 + 5X_2 + 3X_3 = 49 \end{array} \quad \left\{3, -5, 4\right\}$$

2. A row that is m – n bottom raw is all zeros except in the constant column, indicating that there are no solutions

eg.
$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} 2X_1 + X_2 = 30 \\ X_1 + 2X_2 = 24 \\ 4X_1 + 5X_2 = 72 \end{array}$$

3. A matrix in a form different from (1) & (2), indicating that there are an unlimited number of solutions

eg.
$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -4 \end{array}\right] \quad 3X_1 + 2X_2 + X_3 = 6$$

0	1	3	8	$6X_1 + 4X_2 + 3X_3 = 12$
0	0	0	0	$9X_1 + 6X_2 + 3X_3 = 18$
0	0	0	0	$15X_1 + 10X_2 + 5X_3 = 30$

Linear Equations where m < n

Our attempts to transform (A/B) into (I/C) in the case where m < n will result in:

1. A raw which is all zeros except in the constant columns, indicating that there are no solutions, or
2. A matrix in a form different from number one above indicating that there are an unlimited number of solutions.

"Every system of linear equations has No solution, exactly one Solution or infinitely many solutions."

Example: Solve the following systems of linear equations

1) $\begin{cases} 4X_1 + 6X_2 - 3X_3 = 12 \\ 6X_1 + 9X_2 - 9/2X_3 = 20 \end{cases}$

No solution

$$\begin{cases} X_1 + 3X_2 + X_3 = 6 \\ -X_1 + X_2 + X_3 = 2 \end{cases}$$

Unlimited solution

$$\begin{cases} 2X_1 + 3X_2 + 4X_3 + X_4 = 37 \\ X_1 + 2X_2 + 3X_3 + 2X_4 = 24 \\ 3X_1 + X_2 + X_3 + 3X_3 = 33 \end{cases}$$

Unlimited solution

2. Markov Chains: Concepts, Model and Solutions

This model is a forecasting model. It is probabilistic (stochastic) model. A Russian Mathematician called Andrew Markov around 1907 develops this model.

Markov chains are models, which are useful in studying the evolution of certain systems over repeated trials. These repeated trials are often

successive time periods where the state (outcome condition) of the systems in any particular time period can't be determined with certainty. Therefore, a set of transition probabilities is used to describe the manner in which the system makes transition from one period to the next. Hence we can predict the probabilities of the system being in a particular state at a given time period. We can also talk about the long run or equilibrium or steady state.

The necessary assumptions of the chain:

1. The system condition (outcome) or state in any given period depends on its state in the Preceding period & on the transition probabilities
2. The transition probabilities are constant overtime
3. Change in the system will occur once & only once each period
   eg. If it's a week, it's only once in a week
4. The transition period occurs with regularities
   if we start with days, we use the day until we reach our end.

Information flow in the analysis

The Markov model is based on two sets of input data

- The set of transition probabilities
- The existing or initial or current conditions or states

The Markov process, therefore, describes the movement of a system from a certain state in the current state/time period to one of n possible states in the next stage. The system makes in an uncertain environment, all that is known is the probability associated with any possible move or transition. This probability is known as transition probability, symbolized by P_{ij}. It is the likelihood that the system which is currently in state i will move to state j in the next period. From these inputs the model makes two predictions usually expressed as vectors.

1. The probabilities of the system being in any state at any given future time period
2. The long run (equilibrium) or steady state probabilities.

The set of transition probabilities are necessary for both prediction (time period n, & steady state), but the initial state is needed for only the first prediction.

Example

Currently it's known that 80% of customers shop at store 1 & 20% shop at store 2. In reviewing a past data suppose we find that out of all customer who shopped at store 1 in a given week 90% remain loyal for the next week (store one again), 10% switch to store 2. On the other hand, out of all customers who shopped at store 2, in a given week 80% remains loyal for the next week (store 2 again), 20% switch to store 1. What will be the proportion of customers shopping at store 1 & 2 in each of the next two weeks and in the long run?

Solution

Let S_1 be the proportion of store 1

S_2 be the proportion of store 2

- Initial state/current state probability matrix for store one and two will be:

$$V_0 = (0.8 \quad 0.2)$$

The transition probability matrix can be developed after summarizing the problem in the following form

		Next period (week) shopping	
This Period (Week) shopping		**Store 1(S_1)**	**Store 2(S_2)**
	Store 1(S_1)	**0.9**	**0.1**
	Store 2(S_2)	**0.2**	**0.8**

$$P = \begin{array}{c} \\ S_1 \\ S_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

Transition probability matrix is a square matrix such that each entry indicates the probability of the system moving from a given state to another state.

- The sum of rows in the transition matrix should be 1
- We have to be consistent in writing the elements

Markov Chain Formula

a) V n = [V (n – 1)] x P where P = Transition matrix

Vn = Vector for period n

V (n – 1) = Vector for period n – 1

$V_0 = (0.8\ 0.2)$ ⟶ current share

$V_1 = V(_{1-1}) \times P$ ⟶ the proportion of customers after one week

$= V_0 \times P$

$$(0.8\ 0.2)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (V_1\ V2)$$

$V_1 = (0.8 \times 0.9) + (0.2 \times 0.2)$ $V_2 = (0.8 \times 0.1) + (0.2 \times 0.8)$

$= 0.72 + 0.04$ $0.08 + 0.16$

$= \underline{0.76}$ $\underline{0.24}$

$V_1 = (0.76 \quad 0.24)$ the proportion of customers after one week for store 1 & 2 respectively

$V_2 = V(2 - 1)$ ~~x P~~ ⟶ the proportion of customers for each store after two weeks

$= V_1 \times P$

$$(0.76\ 0.24)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (V_1\ V_2)$$

$V_1 = (0.76 \times 0.9) + (0.24 \times 0.2)$ $V_2 = 0.76\ (0.1) + (0.24 \times 0.8)$

$(0.732 \quad 0.268)$

b) Long run market share

Assumption

In the long run the share of the systems is assumed to be constant.

Let - the share of store 1 in the long run be V_1

- the share of store 2 in the long run be V_2

$$\underset{n-1}{(V_1\ V_2)} \times \underset{p}{\begin{pmatrix} 0.9 & 0.1 \end{pmatrix}} = \underset{n}{(V_1\ V_2)}$$

0.2 0.8

$0.9V_1 + 0.2V_2 = V_1$

$0.1V_1 + 0.8V_2 = V_2$

$0.9V_1 + 0.2V_2 = V_1$

$-V_1 + 0.9V1 + 0.2V_2 = 0$

$-0.1V_1 + 0.2V_2 = 0$ *

$0.1V_1 + 0.8V2 - V_2 = 0$

$0.1V_1 - 0.2V_2 = 0$ *

$-0.1V_1 + 0.2V_2 = 0$ *

$0.1V_1 - 0.2V_2 = 0$ *

$-0.1V_1 + 0.2V_2 = 0$

$-0.1V_1 + 0.2(1-V_1) = 0$

$-0.1V_1 + 0.2 - 0.2V_1 = 0$

$-0.3V_1 = -0.2$

$-0.3V1/-0.3 = -0.2/-0.3$

$\underline{V_1 = 2/3 = 67\%}$

$V_2 = 1 - V_1$

$= 1 - 2/3$

$\underline{V_2 = 1/3 = 33\%}$

In the long run 67% of the customers will shop in store 1 & 33% in shop 2.

UNIT 3: INTRODUCTION TO LINEAR PROGRAMMING

✪ Introduction

This unit begins our coverage of linear programming, which is one of the most popular tools of management science. A linear programming (LP) model is mathematical representation of linear programming problems. Graphic or simplex approaches can be used to find optimal solutions to certain linear programming problems in which the solution must satisfy a given set of requirements, or constraints.

The purpose of this unit is to provide you with an introduction to linear programming models. Emphasis is placed on familiarization with terminology, model formulation, and find solutions for linear programming models using graphic Method and simplex method. The unit is designed to help you to develop two very important skills:

1. The ability to formulate an LP model in mathematical terms.
2. The ability to solve linear programming model

3.1. Concepts of Linear Programming

Linear programming- is an optimization method which shows how to allocate scarce resources in the best possible way subject to more than one limiting condition expressed in the form of inequalities and /or equations.

- It enables users to find optional solution to certain problems in which the solution must satisfy a given set of requirements or constraints.
- Optimization in linear programming implies maximization (max) Profit, revenue, sales, market share or minimization (min) Cost, time, distance, or a certain objective function.

The goal in linear programming is to find the best solution given the constraints imposed by the problem, hence the term constrained optimization.

3.2. Formulation of Linear Programming Model

LP models are mathematical representation of LP problems. Some models have a specialized format where as others have a more generalized format. Despite this, LPMs have certain characteristics in common; knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories: **Components** and **assumptions.** The components relate to the structure of a model, where as the assumptions describe the conditions under which the model is valid.

Components

1. Objective function
2. Decision variables
3. Constraints
4. Parameters and Right. Hand Side Values

} Model Structure

Assumptions

1. Linearity
2. Divisibility
3. Certainty
4. Non-negativity

} Model Validity

1. Components of LP Model

1. **The Objective function:** is the mathematical/ quantitative expression of the objective of the company/ model. The objective in problem solving is the criterion by which all decisions are evaluated. In LPMs a single quantifiable objective must be specified by the decision maker. For example, the objective might relate to profits, or costs or market share, best to only one of these.

Decision Variables

Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time

2. **The Decision Variables:** represent unknown quantities to be resolved for. These decision variables may represent such things as the:
   - Number of units of different products to be sold
   - The number of dollars to invest in various projects
   - The number of ads to place with different media

   Since the decision maker has freedom of choice among actions, these decision variables are controllable variables.
3. **The constraints:** are restrictions which define or limit the attainability (achievability) or feasibility of a proposed course of action. They limit the degree to which the objective can be pursued.

   A typical restriction embodies scarce resources (such as labor supply, RMs, production capacity, machine time, storage space), legal or contractual requirements (leg. Product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.
4. **Parameters-** are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as to the numerical value of each constraint.

The components are the building blocks of an LP model. We can better understand their meaning by examining a simple LP model as follows.

Example:

Maximize: $4X_1 + 7X_2 + 5X_3$ (profit)… objective function subject to

$2X_1 + 3X_2 + 6X_3 \leq 300$ labor hrs
$5X_1 + 4X_3 \leq 200$ raw mater
$3X_1 + 5X_2 + 2X_3 \leq 360$
} System Constrai

$X_1 = 30$
$X_2 \geq 4$
} Individual Constrai

X_1 – qty of product 1
X_2 qty of product 2
X_3 qty of product 3

$X_1, X_2, X_3 \geq 0$ → Non negativity constructs

System constraints- involve more than one decision variables

Individual constraint- involve only one decision variable.

None-negativity constrains- specify that no variable will be allowed to take on a negative value. The non negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

2. Assumption of LP models

1. **Linearity:** The linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. Taking the above example, producing one more unit of products add Br 4 to the total profit. This is true over the entire range of possible values of x_1. The same is true for each of the constraints.
2. **Divisibility:** The divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. For example: 3.5 TV sets/ hr would be acceptable →7TV sets/ 2hr.
3. **Certainty:** The parameters are known and constant. The certainty requirement indicates that the constraint equations do not change, with respect to model parameters (i.e. the numerical values); it is assumed that these values are known and constant. Eg. In the above example each unit of product 1 requires 2 labor hours is known and remain constant, and also the 300 labor available is deemed to be known and constant.
4. **Non-negativity**: The non-negativity constraint is that negative values of decision variables are unrealistic and, therefore, will not be considered in any potential solutions, only positive values and zero will be allowed.

Steps for formulating Linear Programming Models

Just as it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lead to poor decisions.

Formulating linear programming models involves the following steps:

1. **Define the problem/ problems definition**
2. **Identity the decision variables or represent unknown quantities.**
3. **Determine the objective function:** Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is maximization or a minimization problem and the coefficients of each decision variable.
4. **Identify the constraints**
   - system constraints- more than one variable
   - individual constraints- one variable
   - Non-negativity constraints.

3.2.1. The Maximization Problem

Example: A firm that assembles computers equipment is about to start production of two new microcomputers (Type I & Type II). Each type of microcomputer will require assembly time, inspection time, and storage space. The amount of each of these resources that can be devoted to the production of microcomputers is limited. The manager of the firm would like to determine the quantity of each microcomputer to produce per week in order to maximize the profit generated by sales of these microcomputers.

Additional information

In order to develop a suitable model of the problem, the manager has met with the design and manufacturing personnel.

As a result of these meetings, the manager has obtained the following information:

	Micro computers	
	Type 1	Type 2
Assembly time per unit	4hrs	10hrs
Inspection time per unit	2hrs	1hr
Storage space per unit	3 cubic feet	3 cubic ft
Profit per unit	$ 60	$ 50

Weekly Maximum Available Resources for the firm

Resource	**Amount Available**
assembly time	100 hrs
Inspection time	22hrs
Storage space	39 cubic feet

Required: formulate this problem as a linear programming model.

Solution

Step 1: Define the problem

- To determine the number of type I &II microcomputers to be produced (and sold) per week so as to maximize the weekly profit given the restrictions.

Step 2: Identify the decision Variable

- Let X_1 and X_2 be the weekly quantities of type 1 and type 2 microcomputers respectively.

Step 3: Develop the objective function

$Z \max = 60X_1 + 50X_2$

Step 4: identify the Constraint

- System constraints: $4X_1 + 10X_2 \leq 100$hrs Assembly time
  $2X_1 + X_2 \leq 22$hrs Inspection time
  $3X_1 + 3X_2 \leq 39$ cub Feet Storage space
- Non-negativity constraint ….$X_1, X_2 \geq 0$

In summary, the mathematical model for the microcomputer problem is:

$$Z \max = 60X_1 + 50X_2$$

Subject to constraints

$$4X_1 + 10X_2 \leq 100$$

$$2X_1 + X_2 \leq 22$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

3.2.2. Minimization Problems

Example: the agricultural research institute suggested to a farmer to spread out at least 4800 kg of a special phosphate fertilizer and not less than 7200 kg of a special nitrogen fertilizer to raise productivity of crops in his fields. There are two sources for obtaining these mixtures A and B. Both of these are available in bags weighing 100 kg each and they cost $40 and $24 respectively. Mixture A contains phosphate and nitrogen equivalent of 20 kg and 80 kg respectively, while mixture B contains these ingredients equivalent of 50 kg each.

Write this as a linear programming problem and determine how many bags of each type the farmer should buy in order to obtain the required fertilizer at minimum cost.

Required: Formulate the linear programming model for the problem.

1. Define the problem

Determining the number of bags of mixtures A and B that must be purchased, so as to satisfy the requirements set and minimize total cost, given the limiting factors.

2. Identify the decision variables

Let x_1 = the number of bags of mixture A that must be purchased

x_2 = the amount of bags of mixture B that must be purchased

z = the minimum cost

3. Determine the objective function

$$z_{\min} = 40x_1 + 24x_2$$

4. Identify the constraints.

$20x_1 + 50x_2 \geq 4800$ *Phosphate requirme*

$80x_1 + 50x_2 \geq 7200$ *Nitrogen requiremtn*

$x_1, x_2 \geq 0$ *nonnegativity*

Summary
Zmin = 40x1 + 24x2
Subject to constraints
20x1 + 50X2≥ 4,800
80x1 + 50x2 ≥ 7,200
X1, x2 ≥0

3.3 Solving Linear Programming Problems

There are two approaches to solve linear programming problems.

1. The graphic solution method
2. The algebraic solution/ simplex algorithm

3.3.1. The Graphic Solution Method

It is relatively straight forward method for determining the optimal solution to certain linear programming problems.

This method can be used only to solve LP problems that involve two decision variables. However, most linear programming applications involve situations that have more than two decision variables, so the graphic approach is not used to solve these problems.

Steps

1. Graph all the constraints that involve in the model using one coordinate plane and shade the solution region for each constraint
2. Identify the feasible region. That is identifying the common region, which is all area that contains all the points that satisfy the entire set of constraints.
3. Determine the coordinates of all corner points of the feasible region.
4. Calculate the value of the objective function at each corner points of the feasible region by using the coordinates.
   i. If the objective function is maximization, the point that gives the largest value of the objective function is the optimal/best solution.
   ii. If the objective function is minimization, the point that gives the least value of the objective function is the optimal/best solution.

Example: Solving the micro-computer problem with graphic approach.

Z max: $60X_1 + 50X_2$
Subject to constraints
$4X_1 + 10X_2 \leq 100$
$2X_1 + X_2 \leq 22$

B

$3X_1 + 3X_2 \leq 39$

$X_1, X_2 \geq 0$

Solution

$4X_1 + 10X_2 = 100$

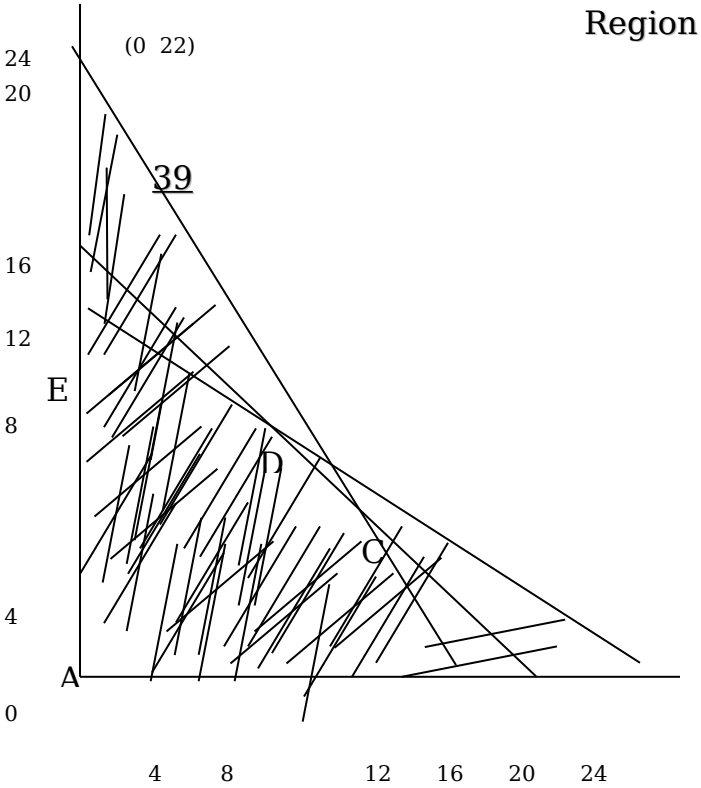
X_1	0	25
X_2	10	0

$2X_1 + X_2 = 22$

X_1	0	11
X_2	22	0

$3X_1 + 3X_2 = 39$

X_1	0	13
X_2	13	0



Region ABCDE is called feasible region

point C = ? $2x_1 + x_2 = 22$ x-3

$3x_1 + 3x_2 =$ 39

$-6x_1 + 3x_2 = -66$

$3x_1 + 3x_2 = 39$

$-3x_1 = -27$

$x_1 = -27/-3 = 9$

$2(9) + x2 = 22$

$x_2 = 22 - 18 = 4$

Point D = $3x_1 + 3x2 = 39$ x - 4

$4x1 + 10x2 = 100$x3

$= -12x1 + 12x2 = -156$

$12x1 + 30x2 = 300$

$18x2 = 144$

$x2 = 8$

$3x1 + 3(8)$

$= 39$

$3x1 = 39 - 24$

$x1 = 15/3 = 5$

To identity the maximum (minimum) value we use the corner point approach or the extreme point approach. The corner point/ extreme point approach has one theorem. It states that:

For problems that have optional solutions, a solution will occur at an extreme, or corner point. Thus if a problem has a single optional solution, it will occur at a corner point. If it has multiple optional solutions, at least one will occur at a corner point consequently, in searching for an optional solution to a problem, we need any consider the extreme points because one of those must be optional. Further, determining the value of the objective function at each corner point, we could identify the optional solution by selecting the corner point that has the best value (i.e. maximum or minimum, depending on the optimization case) of the objective function. Extreme points represent interactions of constraints.

Determine the values of the decision variables at each corner point. Sometimes, this can be done by impaction (observation) and sometimes by simultaneous equation.

Substitute the value of the decision variables at each corner point into the objective function to obtain its value at each corner point.

After all corner points have been so evaluated, select the one with the highest or lowest value depending on the optimization case.

Corner Points	Coordinates X_1	X_2	How determined?	Value of the obi function $Z = 60X_1 + 50X_2$
A	0	0	observation	$0
B	11	0	observation	$660
C	9	4	Simultaneous equation	$740
D	5	8	Simultaneous equation	$700
E	0	10	Observation	$500

Basic Solution

$X_1 = 9$ $X_2 = 4$ Zmax = $740

After we have got the optimal solution, we have to substitute the value of the decision variables into the constraints and check whether all the resources available are used or not. If there is any unused resource we can use it for any other purpose. The amount of unused resource is known as slack- the amount of a scarce resource that is unused by a given solution. The slack can range from zero, for a case in which all of a particular resource is used; to the original amount of the resource that was available (i.e. none of it is used.)

Computing the amount of slack

	Amount used	Originally available	unused Amount of slack (Available-used)
Constraint	X1 = 9X2 = 4		
Assembly	4(9) + 10(4) = 76	100	100 – 76 = 24 hrs
Impective	2(9) 9+ 1(4) = 22	22	22 – 22 = 0 hr
Storage	3(9) + 3(4) = 39	39	39 – 39 = 0 hr

Constraints that have no slack are sometimes referred to as binding constraints since they limit or bind the solution. In the above cases, inspection time and storage space are binding constraints, while assembly time has slack.

Knowledge of unused capacity can be useful for planning. A manager may be able to use the remaining assembly time for other products, or, perhaps to schedule equipment maintenance, safety seminars, training sermons or other activities

Interpretation: The Company is advised to produce 9 units of type 1 microcomputer and 4 units of type 2 microcomputers per week to maximize its early profit to $ 740, and in doing so the company would be left with unused resource of 24 assembly hrs which can be used for other purposes.

Special Cases in graphic method

1. No Feasible Solution
2. Unbounded problems
3. Redundant Constraints
4. Multiple optimal solution

3.3.2 Simplex Method

Simplex Method is a general-purpose approach for solving linear programming problems regardless of the numbers of the decision variables. It is an iterative technique that begins with a feasible solution which is not optimal but serves as a starting point.

The solution is improved until the no further improvement is possible (until the optimal solution has been identified).

Slack: Slack is the amount of resources that is unused by the given solution. This amount can range from zero (cases in which all a particular resource is used) to the original amount of that particular resource. Slack potentially exists in a $\leq$ constraints

Surplus: Surplus is defined as the amount by which the left hand side of the constraint exceeds the right hand side of the constraints. Surplus potentially exists in a $\geq$ constraints

Both slack and surplus variables are accounted in the objective function by using coefficient of zero for each variable

Steps in simplex Method

The following steps are involving in solving LPPs using Simplex Method

Step 1: Convert a linear programming model into a standard form by introducing slacks and surplus variables and the inequalities should be changed into equalities.

Step 2: Develop the initial tableau.

Step 3: Determine the entering variable (i.e, non -basic variable becomes basic variable)

Decision Rule: The variable on the column with the largest positive valaue in the Cj-Zj row is the entering variable and the column is the pivot column

Step 4: Determine the leaving variable (i.e, basic variable becomes non- basic variable)

Decision Rule: Divided the right- hand side value by the corresponding positive elements in the pivot column, the variable in the row with the least value is the leaving variable.

The process is continued until:

i. All values of Cj – Zj ≤ 0 if the objective function is maximization
ii. All values of Cj – Zj >0 if the objective function is minimization

Example solve the following linear programming model using simplex method

$$Z \max = 60X_1 + 50X_2$$

Subject to constraints

$$4X_1 + 10X_2 \leq 100$$
$$2X_1 + X_2 \leq 22$$
$$3X_1 + 3X_2 \leq 39$$
$$X_1, X_2 \geq 0$$

UNIT 4
MATHEMATICS OF FINANCE

4.1 INTRODUCTION

The basic concept of mathematics of finance is that money has time value which is described either as present value or future. **Present value** is the value of money today; **future value** is the value of money at some point in the future. The different between money now and the same money in the future is called interest. Interest have a wide spread influence over decisions made by businesses and every of us in our personal lives. Therefore, the basic objective of this unit is to discuss interest rates

and their effects on the value of money. Specifically, it covers simple interest, compound interest, annuity and mortgage problems.

4.2 INTERESTS

Interest is the price paid for the use of a sum of money over a period of time. It is a fee paid for the use of another's money, just rent is paid for the use of another's house. A savings institution (Banks) pays interest to depositors on the money in the savings account since the institutions have use of those funds while they are on deposit. On the other hand, a borrower pays interest to a lending agent (bank or individual) for use of the agent's fund over the term of the loan.

Interest is usually computed as percentage of the principal over a given period of time. This is called **interest rate**. Interest rate specifies the rate at which interest accumulates per year through out the term of the loan. The original sum of money that is lent or invested/ borrowed is called the **principal.**

Interests are of two types: simple interest and compound interest.

4.2.1 Simple Interest

If interest is paid on the initial amount of money invested or borrowed only and not on subsequently accrued interest, it is called **simple interest**. The sum of the original amount (principal) and the total interest is the **future amount or maturity value or in short amount**. Simple interest generally used only on short-term loans or investments –of duration less than one year. Simple interest is given by the following formula.

$I = prt$ ..... ---------- (1)

Where: I = Simple interest

P = principal amount

r = Annual simple interest rate

t = time in years, for which the interest is paid

If any three of the four variables are given, you can solve for the fourth (unknown variable) and their relationship is as follows:

Amount (A) = P + I

= P + Prt. Factor out the common term P

= P (1 + rt) ..... (2)

P = I/rt or P = $\frac{A}{1+rt}$ ..... (3)

r = I/pt.....(4)

t = I/pr .....(5)

Example 1 Ato Kassahun wanted to buy TV which costs Br. 10, 000. He was short of cash and went to Commercial Bank of Ethiopia (CBE) and borrowed the required sum of money for 9 months at an annual interest rate of 6%. Find the total simple interest and the maturity value of the loan. A = P + I

Solution: p = Br. 10,000

t = 9 months = 9/12 = ¾ year

r = 6% per year = 0.06

I =? A =?

Interest (I) = Prt

= P (1 + rt)

= 10, 000 (1 + 0.06 x ¾)

= 10, 000 x 1.045

= Br. 10, 450

= 10, 000 x 0.06 x ¾

= Br. 450

The total amount which will have to be repaid to CBE at the end of the 9th month is Br. 10, 450 (the original borrowed amount plus Br. 450 Interest).

4.2.2 Compound Interest

If the interest, which is due, is added to the principal at the end of each interest period (such as a month, quarter, and year), then this interest as well as the principal will earn interest during the next period. In such a case, the interest is said to be **compounded**. The result of compounding interest is that starting with the second compounding period; the account earns interest on interest in addition to earning interest on principal during the next payment period. Interest paid on interest reinvested is called **compound interest**.

The sum of the original principal and all the interest earned is the **compound amount**. The difference between the compound amount and the original principal is the compound interest.

The compound interest method is generally used in long-term borrowing unlike that of the simple interest used only for short-term borrowings. The time interval between successive conversions of interest into principal is called the **interest period,** or **conversion period**, or **Compounding period**, and may be any convenient length of time. The interest rate is usually quoted as an annual rate and must be converted to appropriate rate per conversion period for computational purposes. Hence, the rate per compound period (i) is found by dividing the annual nominal rate (r) by the number of compounding periods per year (m): i = r/m

Example if r = 12%, i is calculated as follows:

Conversion period (m)	Rate per compound period (i)
1. Annually (once a year)	i = r/1 = 0.12/1 = 0.12
2. Semi annually (every 6 months)	i = r/2 = 0.12/2 = 0.06
3. Quarterly (every 3 months)	i = r/4 = 0.12/4 = 0.03
4. Monthly	i = r/12 = 0.12/12 = 0.01

Example 1 Assume that Br. 10, 000 is deposited in an account that pays interest of 12% per year, compounded quarterly. What are the compound amount and compound interest at the end of one year?

Solution P = Br. 10, 000

r = 12%

t = 1 year

m = No. of conversion periods = 4 times per quarter. This means interest will be computed at the end of each three month period and added in to the principal.

i = r/m 12%/4 = 3%

In general, if **p** is the principal earning interest compounded **m** times a year at an annual rate of **r**, then (by repeated use of the simple interest formula, using i = r/m, the rate per period, the amount **A** at the end of each period is:

(1) A = p (1 + i).....compound amount at the end of first period.

(2) Compound amount = Compound amount + Interest earned during
After two periods after one period the 2nd period

$A = p(1 + i) + [P(1 + i)](i)$

$= P(1 + i)^2$

(3) Compound amount = Compound amount + Interest earned

After three periods after two period during the 3rd period

$A = P(1 + i)^2 + [P(1 + i)^2](i)$

Factor out p and $(1 + i)^2$ from the terms on the right side of the equation and it gives you:

$A = P(1 + i)^2(1 + i)$

$= P(1 + i)^3$

(4) Compound amount

After n^{th} period $= P(1 + i)^n$

The compound amount formulas developed so far are summarized below:

1. Compound amount after one period = $p(1 + i)^1$
2. Compound amount after two periods = $p(1 + i)^2$
3. Compound amount after three periods = $p(1 + i)^3$
4. Compound amount after n^{th} periods = $p(1 + i)^n$

$A = P(1 + i)^n$ ..... Compound amount formula.

Where: A = amount (future value) at the end of n periods.

P = Principal (present value)

i = r/m = Rate per compounding period.

n = mt = total number of conversion periods

t = total number of years

m = number of compounding/ conversion periods per Year

r = annual nominal rate of interest

Now let us solve the above problem.

$A = 10,000(1.03)^1$ = Br. 10, 300.....1st quarter

$A = [10,000(1.03)](1.03) = 10,000(1.03)^2 = 10,609$ .....2nd quarter.

$A = [10,000(1.03)^2](1.03) = 10,000(1.03)^3 = 10,927.27$ .....3rd quarter.

$A = [(10,000)(1.03)^3](1.03) = 10,000(1.03)^4 = 11,255.088$ .....4th quarter.

In general, the compound amount can be found by multiplying the principal by $(1 + i)^n$. So for the above problem the amount at the end of the year, using the general formula, is equal to:

$A = P(1 + i)^n$ $n = mt = 4 \times 1 = 4$

$= 10,000(1.03)^4$ $i = r/m = 12\%/4 = 3\%$

= Br. 11, 255.088

Compound Interest = Compound amount – original principal

= 11, 255.088 – 10, 000

= Br. 1, 255.088

We may evaluate "A" in several different ways. Among the possible alternatives are:

1. Use a hand-held calculator with a Y^x function key. This is the procedure most often used.
2. Using Logarithms

Restate the equation by finding for log A (or In A) and then finding the antilog, using either a hand-held calculator with logarithmic functions or a table of logarithms. Let us illustrate this alternative.

$A = 10,000 (1.03)^4$

$\log A = \log 10,000 + \log (1.03)^4$

$= \log 10,000 + 4 \log 1.03$

$= \log 10^4 + 4 \log 1.03$

$= 4 + 4(0.01284)$

$= 4 + 0.05135$

$\log A = 4.05135$

$A = \text{Antilog } 4.05135$

$= \text{Br. } \underline{11255.117}$

Basic rules of logarithm

1. $a^x = b$ Eg. $2x = 5$

   $\log a^x = \log b$ $\log 2^x = \log 5$

   $x \log a = \text{lob } b$ $x \log 2 = \log 5$

   $x = \log b / \log a$ $x = \log 5 / \log 2$

   $x = \underline{2.322}$

2. $ab^x + c = d$

   $ab^x = d - c$

   $b^x = d - c/a$

   $\log b^x = \log (\frac{d - c}{a})$ let: $\frac{d-c}{a} = K$

   $X \log b = \log K$ divide both side of the equality by log b.

   $X = \frac{\log k}{\log b}$

   Eg. $4 (3^x) + 10 = 18$ find the value of X

   $4 (3^x) = 18 - 10$

   $4 (3^x) = 8$ (Divide by 4)

   $3^x = 2$

   $\log 3^x = \log 2$

   $X \log 3 = \log 2$

   $X = \frac{\log 2}{\log 3}$

   $= \underline{0.63093}$

3. $X^3 = a$ Eg. $X^3 = 1,000$

   $\log X^3 = \log a$ $\log X^3 = \log 1,000$

   $3 \log x = \log a$ $3 \log X = \log 10^3$

$\log x = \frac{\log a}{3}$; Let $\frac{\log a}{3} = K$ $\log X = 3/3$

X = Antilog K $\log X = 1$

X = Antilog 1

X = 10

4. $a = b(c + x)^d$ find the value of X

$(c + x)^d = a/b$

$\log (c + X)^d = \log a - \log b$

$d \log c + X = \log a = \log b$

$\log c + X = \frac{\log a - \log b}{d}$ Let: $\frac{\log a - \log b}{d} = K$

$\log c + x = K$

$c + x = \text{Antilog } K$

$X = \text{Antilog } K - C$

Eg. $1,000 = 250 (5 + X)^5$

$(5 + X)^5 = \frac{1,000}{250} = 4$

$\log (5 + x)^5 = \log 4$

$5 \log 5 + x = \log 4$

$\log 5 + X = \frac{\log 4}{5}$

$\log 5 + X = 0.120412$

$X = \text{antilog } 0.120412 - 5$

$= 1.319508 - 5$

$= -3.6805$

3. The third way of finding the compound amount is using specially prepared tables which provide values of $(1 + i)^n$ for selected values of i and n.

Therefore, to calculate the value of "A" or other variables in the compound interest formula, you can use any of these three approaches which ever convenient to you.

Example 2. Find the compound amount and compound interest after 10 years if Br. 15, 000 were invested at 8% interest;

a) If compounded annually
b) If compounded semiannually
c) If Compounded quarterly
d) If compound monthly
e) If compounded weekly

The present value

Frequently it is necessary to determine the principal P which must be invested now at a given rate of interest per conversion period in order that the compound amount A be accumulated at the end of n conversion periods. Under these conditions, p is called the present value of A. This process is called discounting and the principal is now a discounted value of future income A. If:

$A = P(1 + i)^n$ then dividing both sides by $(1 + i)^n$ leads to

$P = \frac{A}{(1+i)^n} = A(1 + i)^{-n}$

$P = A(1 + i)^{-n}$ ....* Present value of compound amount.

Example 4. How much should you invest now at 8% compounded semiannually to have, Br. 10, 000 toward your brother's college education in 10 years?

Solution A = Br. 10, 000 $P = A(1 + i)^{-n}$

t = 10 years p =? $= 10,000 (1.04)^{-20}$

m = 2 $= 10,000 (0.456387)$

n = mt = 20 = Br. 4563.87

r = 8%

i = r/m = 4% = 0.04

4.3 EFFECTIVE RATE

An effective rate is the simple interest rates that would produce the same return in one year had the same principal been invested at simple interest without compounding. In other words, the effective rate r converted m times a year is the simple interest rate that would produce an equivalent amount of interest in one year. It is denoted by r_e.

If principal p is invested at an annual rate r, compounded m times a year, then in one year,

$A = P (1 + r/m)^m$

What simple interest rate will produce the same amount A in one year? We call this simple interest rate the effective rate. To find r_e we proceed as follows:

(Amount at simple interest after 1 year) = (Amount at compound interest after 1 year)

$P (1 + r_e) = p (1 + r/m)^m$ .....Divide both sides by p

$1 + r_e = (1 + r/m)^m$ .....isolate r_e on the left side and gives you:

$r_e = (1 + r/m)^m - 1$ .....* effective interest rate formula.

Where: r = nominal annual rate of interest

m = no. of conversion periods per year.

$r_e = e^r - 1$ .....** effective interest rate in continuous compounding.

Example 5. An investor has an opportunity to invest in two investment alternatives A and B which pays 15% compounded monthly, and 15.2% compounded semi-annually respectively. Which investment is better investment, assuming all else equal?

Solution

Nominal rate with different compounding periods cannot be compared directly. We must find the effective rate of each nominal rate and then compare the effective rates to determine which investment will yield the larger return.

Effective rate for investment A

r = 15%

m = 12

i = 1.25% = 0.0125

$r_{eA} = (1 + i)^m - 1$

$= (1.0125)^{12} - 1$

= 16.076%

Effective rate for Investment B

r = 15.2%

m = 2

i = 7.6% = 0.076

$r_{eB} = (1.076)^2 - 1$

= 15.778%

Since the effective rate for investment A (16.076%) is greater than the effective rate for investment B (15.778%), A is the preferred investment alternative.

Example 6. What is the effective rate corresponding to a nominal rate of 16% compounded quarterly?

$r_e = (1 + 0.16/4)^4 - 1 = (1.04)^4 - 1 = 1.169859 - 1 = \underline{16.99\%}$

4.4 ANNUITIES

An annuity is any sequence of **equal periodic payments**. The payments may be made weekly, monthly, quarterly, annually, semiannually or for any fixed period of time. The time between successive payments is called **payment period** for the annuity. If payments are made at the **end of each payment period**, the annuity is called an **ordinary annuity**. If payment is made at **the beginning of the payment period**, it is called **annuity due**. In this course we will discuss only ordinary annuities. The **amount, or future value**, of an annuity is the sum of all payments plus the interest earned during the term of the annuity.

The term of an annuity refers to the time from the beginning of the first payment period to the end of the last payment period.

4.4.1 Ordinary Annuity

An ordinary annuity is **a series of equal periodic payments** in which each payment is made at the end of the period. In an ordinary annuity the first payment is not considered in interest calculation for the first period because it is paid at the end of the first period for which interest is calculated. Similarly, the last payment does not qualify for interest at all since the value of the annuity is computed immediately after the last payment is received.

Future value (Amount) of an ordinary annuity.

Example1. What is the amount of an annuity if the size of each payment is Br. 100 payable at the end of each quarter for one year at an interest rate of 4% compounded quarterly?

Solution

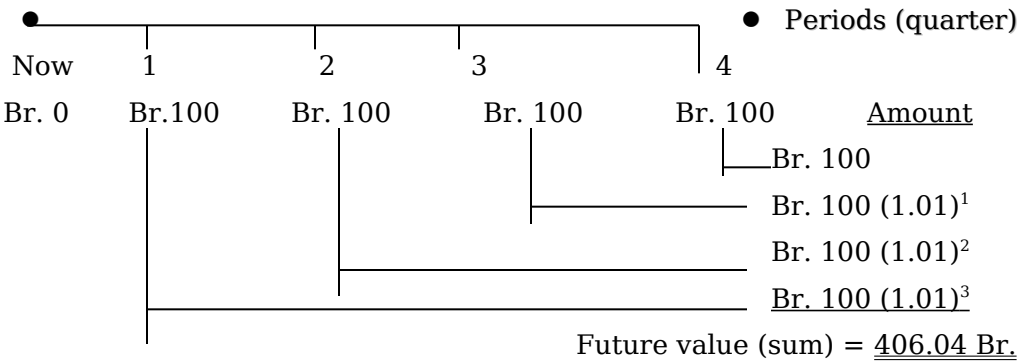
Periodic payment (R) = Br. 100

Payment interval (Conversion period) = quarter

Nominal (annual rate) = r = 4%

Interest per conversion period (i) = r/m = 4%/4 = 1%

Future value of an annuity =?



Future value (sum) = $\underline{406.04 \text{ Br.}}$

Compound interest = Amount – R (n)

= 406.04 – 100 (4)

= $\underline{\text{Br. } 6.04}$

$$A = \left[\frac{R(1+i)^n - 1}{i}\right]$$

..... Amount of an ordinary annuity

Where: A = Amount (future value) of an ordinary annuity at the end of its term

R = Amount of periodic payment

i = interest rate per payment period

n = (mt) total no. of payment periods

Solve the above problem (example 1) using the annuity formula:

$$A = \left[\frac{100(1.04)^4 - 1}{0.04}\right]$$

= Br. 406.04

4.4.2 Present value of an ordinary annuity

The present value of an ordinary annuity is the sum of the present values of all the payments, each discounted to the beginning of the term of the annuity. It represents the amount that must be invested now to purchase the payments due in the future.

The present value of an annuity can be computed in two ways:

- Discounting all periodic payments to the present (beginning of the term individually) or
- Discounting the future value (amount) of an annuity to the beginning of the term

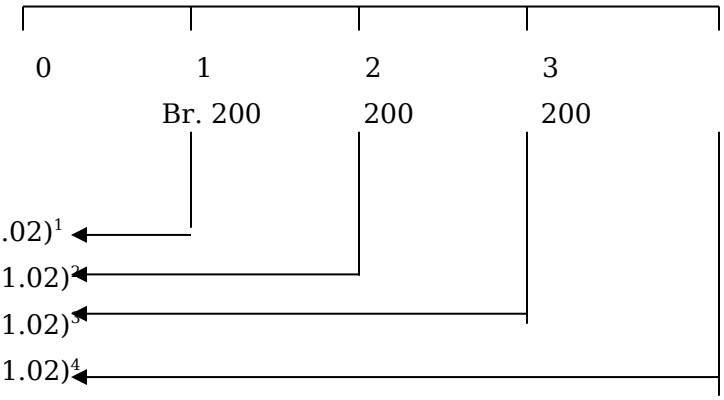
Example 6. What is the present value of an annuity if the size of each payment is Br. 200 payable at the end of each quarter for one year and the interest rate is 8% compounded quarterly?

Solution R = Br. 200 m = 4

r = 8%, n = 4

i = 2% p = ?

Using the first approach (discounting each payment individually), the present value will be:



761.56 Br = Present value.

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$ .....* Present value of an ordinary annuity.

Using the formula, the present value of the above example is computed as:

R = Br. 200

r = 8%, i = 2%

m = 4

t = 4

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right] = 200\left[\frac{1-(1.02)^{-4}}{0.02}\right]$$

$$= 200\ (3.80773) = \underline{\text{Br.}}$$

4.4.2 SINKING FUND

A sinking fund is a fund in which equal periodic payments are made in order to accumulate a definite amount of money up on a specific date. Sinking funds are generally established in order to satisfy some financial obligations or to reach some financial goal.

If the payments are to be made in the form of an ordinary annuity, then the required periodic payment into the sinking fund can be determined by reference to the formula for the amount of an ordinary annuity, i.e.

$$A = R\left[\frac{(1+i)^n - 1}{i}\right] \quad \text{then,} \quad R = \frac{A}{\frac{(1+i)^n - 1}{i}}$$

$$R = A\left[\frac{i}{(1+i)^n - 1}\right]$$

Example 4. How much will have to be deposited in a fund at the end of each year at 8% compounded annually, to pay off a debt of Br. 50, 000 in five years?

Solution:

A = Br. 50, 000

t = 5 years

m = 1, n = 5 (5 x 1)

$$R = A\left[\frac{i}{(1+i)^n - 1}\right] \quad R = 50,000$$

$$\left[\frac{(0.08}{(1.08)^5 - 1}\right]$$

r = 8% i = r = 8%

R =?

The total amount of deposit over the 5 year period is equal to 5 x 8, 522.80 = Br.42, 614

4.4.3 Amortization

Amortization means retiring a debt in a given length of time by equal periodic payments that include compound interest. After the last payment, the obligation ceases to exist it is dead and it is side to have been amortized by the periodic payments. Prominent examples of amortization are loans taken to buy a car or a home amortized over periods such as 5, 10, 20 or 30 years.

In amortization the interest is to determine the periodic payment, **R**, so as to amortize (retire) a debt at the end of the last payment. Solving the present value of ordinary annuity formula for R in terms of the other variable, we obtain the following amortization formula:

$$R = P\left[\frac{i}{\quad}\right]$$ ..... Amortization formula

Where: R = Periodic payment

P = Present value of a loan

i = Rate per period

n = Number of payment periods

Example: 1

1. Ato Elias borrowed Br. 15, 000 from Commercial Band of Ethiopia and agree to repay the loan in 10 equal installments including all interests due. The banks interest charges are 6% compounded Quarterly. How much should each annual payment be in order to retire the debt including the interest in 10 years?

Solution

Pv = Br. 15, 000

t = 10 ye n = 10 x 4 = 40

m = 4

r = 6% i = 6%/4 = 1.5%

R =?

$$R = 15,000\left[\frac{0.015}{1-(1.015)^{-40}}\right]$$

$= 15,000\ (0.033427)$

$= \text{Br. } \underline{501.4}$

Interest = [501.4 x 40] – 15, 000

= 20056 – 15,000

4.4.4 Mortgage Payments

In a typical home purchase transactior n cash and borrows the remaining needed, usually itions. The buyer amortizes the indebtedness by periodic payments over a period of time. Typically **payments** are **monthly** and the time period is long such as 30 years, 25 years and 20 years. Mortgage payment and amortization are similar. The only differences are:

- the time period in which the debt/ loan is amortized /repaid/
- The amount borrowed.

In mortgage payments m is equal to 12 because the loan is repaid from monthly salary or Income, but in amortization money takes other values. Similarly stated mortgage payments are of amortization in nature involving the repayment of loan monthly over an extended period of time.

Therefore, in mortgage payments we are interested in the determination of monthly payments.

Taking:

A = total debt

R = monthly mortgage payments

r = stated nominal rate per annum

n = 12 x number of years (period of the loan)

R can be determined as follows:

$$R = A\left[\frac{r/12}{1-(1+r/12)^{-n}}\right] \quad \text{or} \quad R = A\left[\frac{i}{1-(1+i)^{-n}}\right]$$

$$A = R\left[\frac{1-(1+r/12)^{n}}{r/12}\right]$$

Example: 1 Ato Assefa purchased a house for Br. 115, 000. He made a 20% down payment with the balance amortized by a 30 year mortgage at an annual interest of 12% compounded monthly so as to amortize/ retire the debt at the end of the 30th year.

Required:

1. Find the periodic payment
2. Find the interest charged.

Find the interest charged.

Solution:

Selling price = Br. 115, 000 r = 12%, i = 0.01

Less: Down payment = 23, 000 (20% x 115,000) m = 12, n = 360

Mortgage (A) = Br. 92, 000 t = 30 years

R =?

$$R = A\left[\frac{r/12}{1-(1+r/12)^{-n}}\right] = 92,000\left[\frac{0.01}{1-(1.01)^{-360}}\right]$$

$$= 92,000\ (0.010286125)$$

$$= \text{Br. } 946.32$$

Ato Assefa should pay Br. 946.32 every month to retire the debt within 30 years or 360 monthly payments.

Interest = Actual payment – mortgage (loan)

= (946.32 x 360) – 92, 000

= Br. 340, 675.20 – 92, 000

= Br. 248, 675.20

Throughout the 30 years period the loan earns an interest of Br, 284,675.20

UNIT 5

ELEMENTS AND APPLICATIONS OF CALCULUS

Unit Objectives

After reading this unit students must be able to:

- Explain the concepts of limits and continuity
- Provide an understanding of average rate of change
- Provide an understanding of derivative
- Illustrate a wide variety of applications of optimization procedures
- Understand skills in problem formulation
- Reinforce skills of interpretation of mathematical skills

Introduction

This unit examines the calculus and its application to business, economics and other areas of problem solving. The major areas of study within the calculus are differential calculus and integral calculus. Differential calculus focuses on rates of change in analyzing a situation.

Integral calculus involves summation of a special type. Graphically the concepts of area in two dimensions or volume in three dimensions are important in integral calculus. The goal in this chapter is to provide an appreciation for what the calculus is and where it can be applied. Though it would take several semester of intensive study to understand most of the finer points of the calculus, your coverage will enable you to understand the tools for conducting analysis at an elementary level.

Calculus is a mathematical tool used to solve problems in business, Economics and other areas.

- Differential calculus and
- Integral calculus

5.1. Differential Calculus

Differential calculus focuses on rates of change in analyzing a situation. It broadens the c

- Limits
- Continuity
- Derivatives

Application areas:

- Optimization problems i.e minimizing cost and /or maximizing profit, revenue---

1) Integral calculus involves summation of a special type, total change -------

   It is the inverse of Differential calculus or vice versa like that of log and antilog.

5.1.1. Rules of Differentiation:

The rules of differentiation have been developed using the limit approach. The mathematics involved in providing these rules can be finally complicated. For our purposes it will suffice the rules without proof.

The rules of differentiation apply to functions, which have specific structural characteristics. A rule will state that if a function has specific characteristics, then the derivative of the function will have resulting form.

⇒ Each function can be graphed and that the derivative is a general expression for the slope of the function.

Notation: $\frac{dy}{dx}$⇒ "This read the derivative of Y with respect to X"

: This notation can be used interchangeably with the notation "**F' (X)** –read **f prime** of **X**" – which represents the derivative of the function **f** at **X**. That is given f(X)

$\frac{dy}{dx} = f'(X)$

⇒ Differentiation by direct application of the limit formula is time consuming and difficult, therefore, functions should be classified into certain groups according to their behavior so that the process of finding a derivative when applied to these groups follows a definite pattern (formula)

1. **Derivative of a constant function**.

   If f(x) = K Eg. F(x) = 10,

$f'(x) = 0$ $f'(x) = 0$ slop $= 0$

If you consider what the function (f(x) = 10) looks like graphically, this result seems reasonable. The function f(x) = 10 is horizontal line intersecting the Y axis at (x, 10). The slope at all points along such function equals 0.

2. Power functions

If $f(x) = X^n$, where n is a real number $f'(X) = nx^{n-1}$

Example

1) If $(X) = X = X^1$ where, n = 1

$f'(X) = nx^{n-1}$

$= 1X^{1-1}$

$= \underline{\underline{1}}$

This implies that for the function f(X) = X, the slope equals 1 at all points. You should recognize that f(X) = X is a linear function with slope 1.

2) f(X) = X5

$f'(X) = \underline{\underline{5x^4}}$

3) $f(X) = \frac{1}{X^3}$

$= X^{-3}$

$f'(X) = -3X^{-4}$

$= \frac{-3}{X^4}$

4) $f(X) = \sqrt[3]{x^2}$

$= X^{2/3}$

$f'(X) = 2/3\ X^{-1/3}$

$= \frac{2x^{\frac{-1}{3}}}{3}$

$= \frac{2}{3X^{\frac{1}{3}}}$

$= \frac{2}{3\sqrt[3]{X}}$

3. A constant times a function

If f (X) = K. g(X), where K is a constant and g is a differentiable function,

$f'(X) = K.g'(X)$

a) $f(X) = 5X^3$

$f'(X) = 15X^2$

b) $f(X) = X^2 - 5X$

f(X) can be expressed as f(X) = g(X) – h(X)

where $g(X) = X^2$

$h(X) = 5x$

$f'(X) = g'(X) - h'(X)$

$= \underline{2x - 5}$

4. **Sum or differences of functions**.

This implies that the derivative of function formed by the sum (difference) of two or more component functions is the sum (difference) of the derivatives of the component functions.

$(X) = [g(X) \pm h(X)]$

$f'(X) = [g'(X) \pm h'(X)]$

If, $f(X) = 6X^4 - 5X^2$ if, $g(X) = 6X^4$ $g'(X) = 24X^3$

Then, $f'(X) = 24X^3 - 10x$ if, $h(X) = 5x^2$ $h'(X) = 10^x$

$g'(X) - h'(X) = 24x^3 - 10^x$

If, $f(X) = 3x^2 + 2x + 1$

$f'(X) = \underline{6x + 2}$

5. **Product of functions**

If $h(X) = f(X) \cdot g(X)$

$h'(X) = f(X) \cdot g'(X) + g(X) \cdot f'(X)$

$h(X) = (2x^2 - 5)(x + 3)$ $f(X) = 2x^2 - 5$

$= (2x^2 - 5) \times 1 + (X + 3)(4x)$ $g(X) = X + 3$

$= \underline{6x^2 + 12x - 5}$ $f'(X) = 4x$

$g'(X) = 1$

$h'(X) = f(X) \cdot g'(X) + g(X) \cdot f'(X)$

$\underline{2x^2 - 5 \times 1 (X + 3)(4x)}$

Eg. 2. $f(X) = (x^2 - 5)(X - X3)$

$g(X) = X^2 - 5$ $g'(X) = 2x$

$h(X) = X - X^3 \qquad h'(X) = 1 - 3X^2$

$f'(X) = g(X) \cdot h'(X) + h(X) \cdot g'(X)$

$= (x^2 - 5)(1 - 3x^2) + (X - X^3)(2x)$

$= \underline{-5x^4 + 18x^2 - 5}$

6. Derivative of the Quotient of function

$h(X) = \frac{f(X)}{g(X)}$

$h'(X) = \frac{g(X).f'(X) - f(X).g'(X)}{[g(X)]^2}$

Eg. 1. $h(X) = \frac{1 - X}{X^2}$

$h'(X) = \frac{(-1)(X^2) - (1 - X)(2X)}{(X^2)^2}$

$= \frac{-x^2 - 2x + 2x^2}{(X^2)^2} = \frac{X^2 - 2x}{X^4}$

Eg. 2. $f(X) = \frac{1}{X^3}$ $\qquad f'(X) = \frac{-3}{X^4}$

3. $F(X) = (3X^2 \not/ 5)\ 1 - X^3$ $\qquad f'(X) = \frac{3x4 - 15x^2 - 6x}{(1 - x^3)^2}$

5.1.2. Higher –Order Derivative (First and Second Derivative for Extreme)

The process of finding a derivative is called differentiation. A set of rules of differentiation exists for finding the derivatives of many common functions.

If a function F has a derivative for each value of X in some specified interval, then the derivative function f' is defined for that interval. If in turn the derivative function itself has a derivative for points in that interval this new derivative function is called the second derivative of the original function f or the first derivative of f'.

5.1.3. Business Applications of Differential Calculus

1. Marginal analysis

Examines Incremental Effects

C(X) = total cost function $\Rightarrow C'(X)$ = marginal Cost

R(X) = total Revenue function $\Rightarrow R'(X)$ = Marginal revenue

P(X) = R(X) – (C(X)) = Profit $\Rightarrow p'(X)$ = marginal Profit

$$p'(X) = R'(X) - C'(X)$$

a. **Marginal Cost**

Marginal Cost is the additional cost incurred as a result of producing and selling one more unit of a product or service. Linear cost functions assume that the variable cost per unit is constant for such functions the marginal cost is the same at any level of output.

- A non-linear cost function is characterized by variable marginal costs.
- For the total cost function C(X), the derivative C'(X) represents
  i) The instantaneous rate of change in TC given a change in the number of units produced.
  ii) A general expression for the slope of the graph of the TC function
  iii) The marginal cost, MC = $C'(X)$

$\therefore$ $C'(X)$ Can be used to approximate the marginal cost associated with producing the next unit.

→ It is the rate of change in total cost per unit change in production at an out put level of X unit. It is also an optimization to the actual cost of making one more unit at any production level X (non-linear functions).

Example: Suppose the total cost C(X) in thousands of dollars for manufacturing X unit is given by the function

$$C(X) = 575 + 25x - \frac{X^2}{4};\ 0 < X \leq 50$$
[0 50]

Required:

1) Find the MC at a production level of X units
2) Find the MC at a production level of 40 unit and interpret the result
3) Find the actual cost of producing the 41st unit and compare this cost with the result found in question number 2

Solution

1) $C'(X) = 25 - x/2$

2) $C'(40) = 25 - 40/2$

$= 5$ Br. $= 5,000$Br. $\Rightarrow$ marginal cost of 41st unit.

3) $C(41) - C(40) = ?$

$C(41) = 575 + 25(41) - (41)2/4$

$= 1179.75$

$C(40) = 575 + 25(40) - (40) - (40)\ 2/4 \quad = 1175$

$C(41) - C(40) = 4750$Br. $\Rightarrow 1179.75 - 1175 = 4.75 \times 1000$

At a production level of 40 units the rate of change of TC relative to production is Br. 5,000. In Other words, the cost of producing 1 more unit at this level of production (40) is approximately 5, 000 Br.

b. Marginal Revenue

Marginal revenue (MR) is the additional revenue derived from selling one more unit of a product or service.

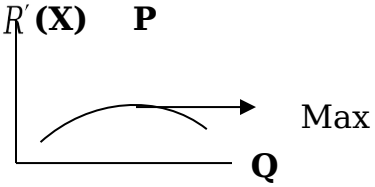
If each unit of a product sells at the same price, the MR is always equal to the price. Eg. R = 10x MR= 10Br.

Marginal revenue for non-linear total revenue function is not constant.

For a total revenue function R (X), the derivative $R'(X)$ represents the instantaneous rate of change in total revenue given a change in the

number of units sold. For the purpose of marginal analysis, the derivative is used to represent the

Marginal revenue or **MR =** R'**(X)** **P**



It represent / approximately the marginal revenue from selling the next unit

Eg. R(X) = 500x – 0.005x^2 X = number of units

R'(X) = 500 – 0.005X

Marginal Profit= Marginal revenue –Marginal cost

P'(X) = R'(X) - C'(X)

- Marginal revenue (Profit) analysis is concerned with the effect on profit if one-additional unit of a product is produced and sold. As long as the additional revenue brought in by the next unit exceeds the cost of producing and selling that unit, there is a net profit from producing and selling that unit and total profit increases.

I. If **MR > MC**, produce the next unit

II. If **MR < MC**, do not produce the next unit.

c. Marginal Profit

Profit maximization criterion

If MR = MC, for the last unit produced and sold, total profit will be maximized.

P(X) = R(X) – C(X) = 0

P'(X) = R'(X) - C'(X) = 0

P'(X) = 0

R' (X) = C'(X)

Example: The market research department of a Company recommends that the Company to manufacture and market a new transistor radio after suitable test. The marketing department also presents the following demand equation.

X=10,000-1000P i.e. P=10-X/1000

Furthermore, the financial department provides the following cost equation:

⇢ C(X) = 7, 000 + 2x

Conduct a marginal analysis for the company.

1) $C'(X) = 2$

2) TR = P.X = $10x - \frac{X^2}{1,000}$

$R'(X) = 10 - \frac{X}{500}$

3) P(X) = R(X) – C(X)

= $(10x - \frac{X^2}{1,000}) – (7, 000 + 2x)$

= $8x - \frac{X^2}{1,000} - 7, 000$

$P'(X) = 8 - \frac{X}{500}$ or

$P'(X) = R'(X) - C'(X)$

= $10 - \frac{X}{500} - 2$

= $8 - \frac{X}{500}$

Average cost, Average revenue, Average profit

→ Average Cost = $\bar{C}(X) = \frac{C(X)}{X}$ ⇒ Cost per unit

Marginal Average cost = $\bar{C}'(X)$ ⇒ the rate of change of average cost

→ Average revenue = $\bar{R}(X) = \frac{R(X)}{X}$ ⇒ Revenue/unit

Marginal Average revenue = $\bar{R}'(X)$ ⇒ R’(X)\X

→ Average Profit = $\bar{P}(X) = \frac{p(X)}{X}$

Marginal Average profit = $\bar{P}'(X)$

Suppose: C(X) = 1, 000 + 25x - $\frac{X^2}{10}$

$$C'(X) = 25 - \frac{X}{5}$$

$$\bar{C}(X) = \frac{1{,}000}{X} + 25 - \frac{X}{10}$$

$$\bar{C}'(X) = \frac{-1{,}000}{X^2} - 1/10$$

$$C'(10) = \frac{1{,}000}{100} - \frac{1}{10}$$

= Br. –10.10 ⇒ Shows that a unit increase in production will decrease the average cost by approximately Br. 10.10 at a production level of 10 units.

If f' set equal to zero

$$\bar{C}'(X) = \frac{1000}{X} + 25 - \frac{-X}{10} = 0$$

X = represents the minimum value of f, If $f'(X) > 0$ i.e.-Cost

$f'(X) < 0$ i.e. –Profit

2. Optimization Problems

Example 1: A company manufactures and sales X units of transistor radios per week. If the weekly cost and demand equations are:

C(X) = 5, 000 + 2x

P = 10 - $\frac{X}{1{,}000}$ [0, 8000]

Finding for each week

a) The production level that leads to maximum revenue and the maximum revenue.
b) The production levels that leads to maximum profit and the maximum profit.
c) The production level that leads to minimum cost and the minimum cost.

Solution

1) $R(X) = 10x - \frac{X^2}{1{,}000} = (10 - \frac{X}{1{,}000}) \cdot (X)$

$R'(X) = 10 - \frac{X}{500}$ $R'(X) = 0$

$= 10 - \frac{X}{500} = 0$

X = <u>5, 000 units</u>

$R''(X) = \frac{-1}{500} = -0.02$, so X = 5, 000 units leads to maximum revenue

TR = XP = 5, 000 x 5 (5, 000 25, 000)

= <u>25, 000 Br</u> R(5, 000) = 10 x 5, 000 - $\frac{(5,000)^2}{1,000}$

= <u>25, 000</u>

2) $P(X) = 10x - \frac{X^2}{1,000} - 5,000 - 2x$

$= 8x - \frac{X^2}{1,000} - 500$ $P''(X) = \frac{1}{500}$

Negative, so it is an optimal solution (4, 000 unit Br. 21, 000)

$P'(X) = 8\ \frac{X}{500}$

$P'(X) = 0$

? Review Question

1. If p is given as p = 110 – 5Q.Then TR= P.Q

   TR = (110 – 5Q) Q

   TR = $110Q – 5Q^2$

   Find MR and AR functions

2. If a total cost function is given as T C = $200 + 5Q + 2Q^2$

   Given the cost function, determine AC (average cost) and MC (Marginal Cost) Functions

AC = <u>200+5Q+2Q²</u>

Q

= <u>200</u> + 5 + 2Q

Q

and MC = <u>∂TC</u> = 5 + 4Q

∂Q

Integral calculus (Reading Assignment)