Lecture Note with ESSLCE Questions

For

Mathematics for Natural Science (Remedial Program)

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Text book (Grade 9-12), different web sources are used while the lecture note is prepared.

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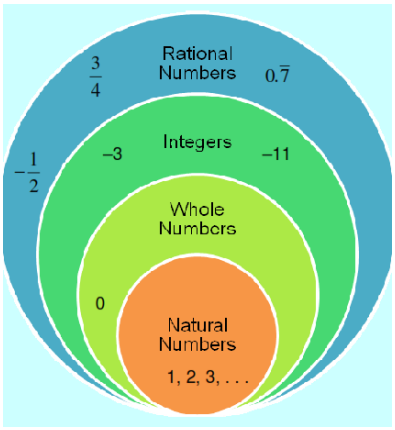
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Chapter 1
Solution of Equation



1.1. Revision on Number System

$$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

- If $c = ab$, c is called a **multiple** of a and b.
- If $c = ab$, a and b are called **factors** or divisor of c.
- A natural number that has exactly distinct factors, namely 1 and itself is called **prime number**..
- A natural number that has more than two factors is called a **composite number**.

Note: 1 is neither prime nor composite.

Note: $i)\ a^n a^m = a^{n+m}$ $ii)\ a^{-n} = \frac{1}{a^n}$ $iii)\ \frac{a^n}{a^m} = a^{n-m}$

$iv)\ a^n b^n = (ab)^n$ $v)\ a^{1/n} = \sqrt[n]{a}, n > 1$ $vi)\ (a^n)^m = a^{nm}$

$vii)\ \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $vii)\ \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Note: $i)\ \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$ $ii)\ \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ $iii)\ \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

1.2. Equations Involving Exponents and Radicals

Classwork: Express each of the following in power form: i) 125 ii) 1296 iii) 625

Note: i) If $a > 0$, $a^x = a^y \Leftrightarrow x = y$.

ii) $a^x = b^x, (x \neq 0) \Leftrightarrow a = b$.

Example: Solve each of the following

i) $2^x = 32$ ii) $x^3 = 27$ iii) $\left(\frac{1}{16}\right)^{3x-1} = 32$

Solution: i) $2^x = 32 \Longrightarrow 2^x = 2^5 \Longrightarrow x = 5.$

ii) $x^3 = 27 \Longrightarrow x^3 = 3^3 \Longrightarrow x = 3.$

iii) $\left(\frac{1}{16}\right)^{3x-1} = 32 \Longrightarrow (2^{-4})^{3x-1} = 2^5 \Longrightarrow 2^{-12x+4} = 2^5 \Longrightarrow -12x + 4 = 5 \Longrightarrow x = -\frac{1}{12}$

Home Work: Solve for x

i) $\sqrt[3]{81^{2x-1}} = 3^x$ ii) $9^{2x+2}\left(\frac{1}{81}\right)^{x+2} = 243^{-3x-2}$

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1.3. Systems of Linear Equations in Two Variables

- A ***system of equations*** is two (or more) equations with the *same variables*.
- An equation is ***linear*** if the degree of the equation is one and no product of the variables.

Example: i) $\begin{cases} x+y=5 \\ 2x-y=3 \end{cases}$ ii) $\begin{cases} x-3y=8 \\ 2x=4 \end{cases}$ are example of system of linear equation.

Example: The following are not system of linear equation:

i) $\begin{cases} x+y^2=5 \\ 2x-y=3 \end{cases}$ ii) $\begin{cases} x-y=3 \\ \frac{1}{x}+\frac{1}{y}=5 \end{cases}$ iii) $\begin{cases} x+xy=3 \\ x-y=5 \end{cases}$

General form of system of two linear equations is given by

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are constants.

1.3.1. Types of Solutions

Class work: Solve the equation $-2(2x-4)=-2(-3x+6)$.

The ***solution*** to a system of equations is the ordered pair(s) that ***satisfies*** **(solves)** ***all*** **of the equations**.

There are three types of solutions to any system of equations:

1) Unique solution: ***exactly one*** ordered pair (x, y) will solve all equations.

Example: $\begin{cases} x+2y=3 \\ 2x-y=1 \end{cases}$ have unique solution $(1, 1)$.

2) No solution: no ordered pair (x, y) will ever solve all equations.

Example: $\begin{cases} x+2y=3 \\ x+2y=6 \end{cases}$

3) Infinitely many solutions: an **infinite number** of ordered pairs will solve all equations.

Example: $\begin{cases} x+2y=3 \\ 2x+4y=6 \end{cases}$ have solutions $(3, 0), (1, 1), (5, -1), \cdots$.

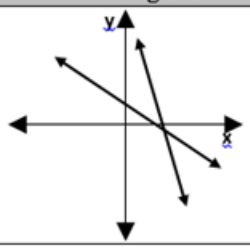
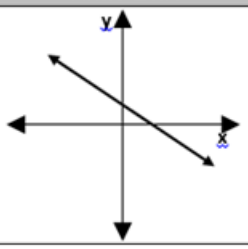
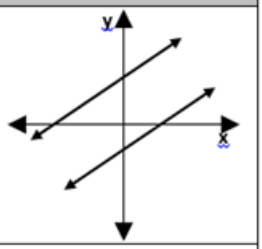
<u>There are three solving methods for system of equations:</u>

1) Graphing
2) Substitution
3) Elimination

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Method 1: Graphing

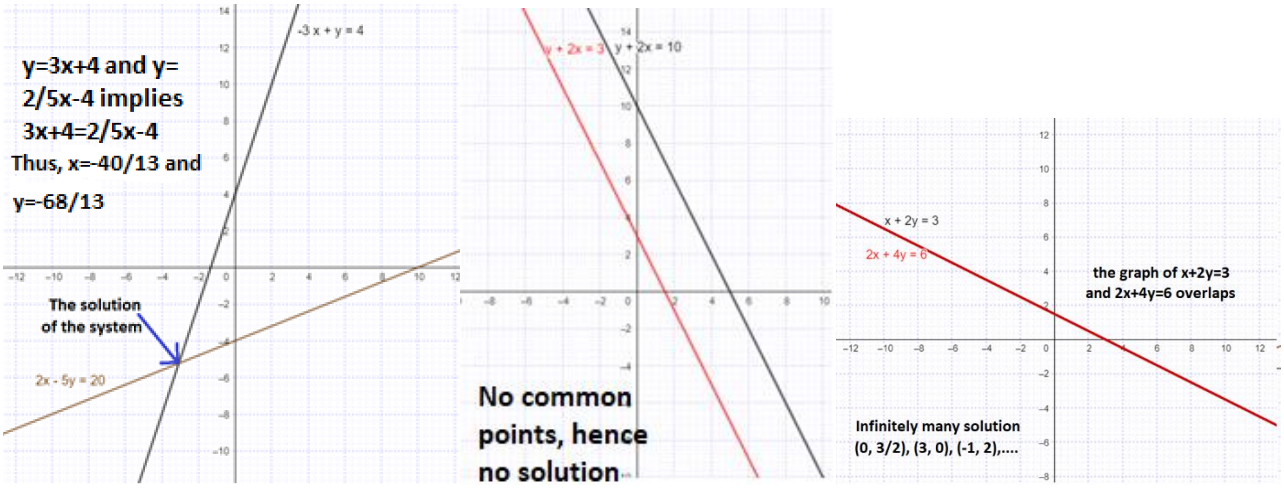
The method involves to find the intersection of lines.

	Intersecting Lines	Same Line	Parallel Lines
Graph of a System			
Number of Solutions	Exactly One Solution	Infinitely Many Solutions	No Solutions
Terminology	Consistent and Independent	Consistent and Dependent	Inconsistent

Note: The point where the two lines intersect is the only solution. An inconsistent system has no solution.

Example: Solve the following graphically: i) $\begin{matrix}2x-5y=20\\-3x+y=4\end{matrix}$ ii) $\begin{cases}x+2y=3\\2x+4y=6\end{cases}$ iii) $\begin{cases}y+2x=3\\y+2x=10\end{cases}$

Solution:



Method 2: Substitution - Substitute one equation into the other

Example: Solve $\begin{matrix}2x-5y=20\\-3x+y=4\end{matrix}$ by substitution method.

Solution: From the second equation $y = 4 + 3x$, then replacing it in the value of y in the first equation gives,

$$2x - 5(4 + 3x) = 20 \Longrightarrow 2x - 20 - 15x = 20 \Longrightarrow -13x = 40 \Longrightarrow x = -\frac{40}{13}$$

Then $y = 4 + 3x = 4 - \frac{120}{13} = -\frac{68}{13}$.

Method 3: Elimination - Eliminate one variable from both equations

Example: Solve $\begin{matrix}3x-7y=-14\\5x+2y=45\end{matrix}$ by elimination.

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Solution: $\begin{matrix}-5\\3\end{matrix}\begin{cases}3x-7y=-14\\5x+2y=45\end{cases}\Rightarrow\begin{cases}-15x+35y=70\\15x+6y=135\end{cases}\Rightarrow 41y=205\Rightarrow y=\frac{205}{41}$

Note: Consider the general system of two linear equations

$$\begin{cases}a_1x+b_1y=c_1\\a_2x+b_2y=c_2\end{cases}$$

Case I: If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ the system has infinite solutions. The system is called dependent. The graphs overlap.

Case II: If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ the system has no solutions. The system is called inconsistent. The graphs are parallel (Inconsistent).

Case III: If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ the system has unique solutions. Thus the system is called independent. The graphs intersects.

1.3.2. System of Equation Word Problems

1. Read the problem carefully and identify any important information
2. Define Variables (x and y)
3. Write a system (2 equations) with variables
4. Solution Method: Based on your equations, decide on the best method(Graph, Substitution, Elimination)
5. Answer the question.

Examples:

1. The length of a rectangle is equal to triple the width. Which system of equations can be used to find the dimensions of the rectangle if the perimeter is 86 centimeters?

$$\begin{cases}l=3w\\2(l+w)=86\end{cases}$$

2. Fatuma's Shop sells orange for 10 birr and banana for 5birr. During night, she sold 8 more sundaes than banana splits and made 185 birr.

$$\begin{cases}10s+5b=185\\s=b+8\end{cases}$$

3. A collection of 105 coins consists of 1.00 cents and 5.00 cents. If the total value is 205.00 cents, find the number of coins of each denomination in the collection.

$$\begin{cases}x+y=105\\x+5y=205\end{cases}$$

4. A merchant paid 48,000birr for some dresses and shoes. He paid 400birr for each dress and 1,000birr for each pair of shoes. He sold the dresses at a profit of 20% and the shoes at a profit of 50%. If his total profit was 18,000 birr. How many dresses and shoes did he buy?

$$\begin{cases}400d+1000s=48000\\400*20\%d+1000*50\%s=18000\end{cases}\Rightarrow\begin{cases}10d+25s=12000\\80d+500s=18000\end{cases}$$

Homework

1. Find b and c given that the graph of $y=x^2+bx+c$ passes through $(3,14)$ and $(-4,7)$.

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2. Solve $\begin{cases} \frac{2}{x}+\frac{3}{y}=-2 \\ \frac{4}{x}-\frac{5}{y}=1 \end{cases}$

1.4. System of linear inequality

- There are five inequalities: $<, \leq, >, \geq, \neq$.
- A **system of linear inequalities** is a collection of two or more linear inequalities to be solved simultaneously.
- A **solution** of system of two linear inequality is the set of order pair that satisfies both inequalities.
- Best way to solve a system of linear inequalities is to use the **graphical method.**
- Treat the inequality as a linear equation and graph the line as either a **solid line (≤** or **≥)** or a **dashed line (<** or **>)** depending on the inequality sign. The solution set will be the **overlapped region** of all the inequalities.
- A solution region of as system of linear inequalities is said to be **bounded** if it can be enclosed by a rectangle, otherwise it is **unbounded**.

Intervals

- Interval is a set of real number that lies between two numbers.

Closed interval	$[a,b]$	$\{x \in R: a \leq x \leq b\}$	
Open interval	(a,b)	$\{x \in R: a < x < b\}$	
Half-open interval	$[a,b)$	$\{x \in R: a \leq x < b\}$	
Half-open interval	$(a,b]$	$\{x \in R: a < x \leq b\}$	

Example: The solutions for the inequality $2x-6<-4x$ is $\{x \in \mathbb{R}: x<1\}=(-\infty,1)$.

Example: The solutions for the inequality $2x-6\geq-4x$ is $\{x \in \mathbb{R}: x\geq 1\}=[1,\infty)$.

Example: The solutions for the inequality $-8\leq x-6\leq-4$ is $\{x \in \mathbb{R}: -2\leq x\leq 2\}=[-2,2]$.

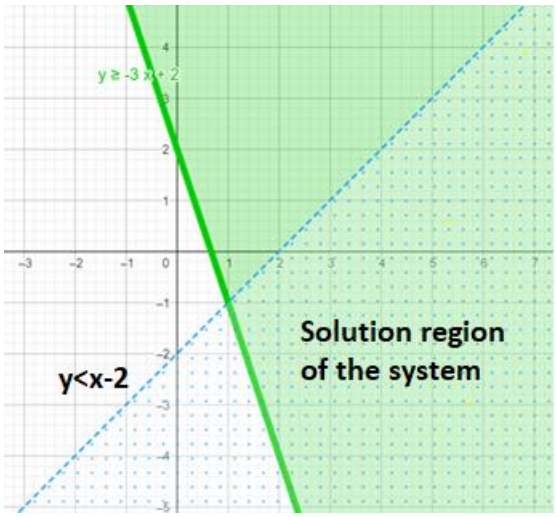
1.4.1. Solution of system of equations

Example: The order pair $(1,2),(0,1),(-2,-2),\cdots$ are a solutions to $2x-3y<2$.

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Example: Find the solution of the following system of inequalities graphically:

$$y \geq -3x + 2$$
$$y < x - 2$$



The solution region is the intersection of the two regions.

The solution set of $\begin{matrix} y \geq -3x + 2 \\ y < x - 2 \end{matrix}$ is shown by the cross-shaded region in the diagram. From the inequality we observe

$-3x + 2 \leq y < x - 2$ and $-3x + 2 = x - 2 \Longrightarrow x = 1$

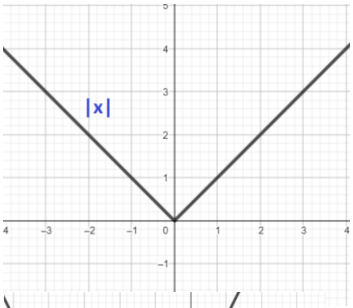
Hence, the solution set of the system is expressed as

$$\{(x, y): -3x + 2 \leq y < x - 2 \text{ and } 1 < x < \infty\}.$$

1.5. Equations Involving Absolute Value

Absolute value of x denoted by $|x|$ is the distance of x from the origin.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

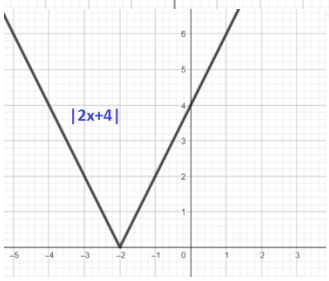


Note: For any real number x, $|x| = |-x|$.

Example: $|-3 - 7| =$

Note: i) If $a > 0$, then $|x| = a$ has two solution $x = -a$ or $x = a$.

ii) If $a < 0$, then $|x| = a$ have no solution.



Note:

$$|2x + 4| = \begin{cases} 2x + 4 \text{ if } x \geq -2 \\ -(2x + 4) \text{ if } x < -2 \end{cases}$$

Example: Solve the following

i) $|3 - x| = 4$ ii) $|2x - 5| = 6$ iii) $|x + 3| = -4$

Solution: i) $|3 - x| = 4 \Longrightarrow 3 - x = 4$ or $3 - x = -4$. i.e., $x = -1$ or $x = 7$.

ii) $|2x - 5| = 6 \Longrightarrow 2x - 5 = 6$ or $2x - 5 = -6$. i.e., $x = \frac{11}{2}$ or $x = -\frac{1}{2}$.

iii) $|x + 3| = -4$, this equation has no solution.

Note: If $|x| = |y|$ then $x = y$ or $x = -y$.

Example: Solve

i) $|4x - 3| = |x + 6|$ ii) $|2x + 2| = |2x - 1|$

Solution: i) $|4x - 3| = |x + 6| \Longrightarrow 4x - 3 = x + 6$ or $4x - 3 = -(x + 6)$

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$\Rightarrow 3x = 9$ or $3x = -3$ i.e., $x = 3$ or $x = 1$. Hence S.S. $\{1, 3\}$.

ii) $|2x + 2| = |2x - 1| \Rightarrow 2x + 2 = 2x - 1$ or $2x + 2 = -(2x - 1)$

$\Rightarrow 2 = -1$ or $4x = -1$ i.e., $x = -\frac{1}{4}$ is the solution.

1.5.1. Properties of absolute value

i) $x \leq |x|$

ii) $|xy| = |x||y|$

iii) $\sqrt{x^2} = |x|$

iv) $|x + y| \leq |x| + |y|$ (Triangular inequality)

1.6. Solving absolute value inequality

Note: i) For $a > 0$, $|x| \leq a$ has solution $-a \leq x \leq a$.

ii) For $a > 0$, $|x| \geq a$ has solution $x \leq -a$ or $x \geq a$.

Example: Solve i) $|2x + 1| < 7$ ii) $\left|\frac{2}{5} - 3x\right| \geq 1$

Solution: i) $|2x + 1| < 7 \Rightarrow -7 < 2x + 1 < 7 \Rightarrow -8 < 2x < 6 \Rightarrow -4 < x < 3$

The solution set is $\{x: -4 < x < 3\} = (-4, 3)$.

ii) $\left|\frac{2}{5} - 3x\right| \geq 1 \Rightarrow \frac{2}{5} - 3x \geq 1$ or $\frac{2}{5} - 3x \leq -1 \Rightarrow -3x \geq \frac{3}{5} \Rightarrow x \leq -\frac{1}{5}$ or $-3x \leq -\frac{7}{5} \Rightarrow x \geq \frac{7}{15}$.
Hence the solution set is $\left\{x: x \geq \frac{7}{15} \text{or } x \leq -\frac{1}{5}\right\} = \left[-\frac{1}{5}, \frac{7}{15}\right]$.

Homework: Solve i) $|5x + 15| = 0$ Ans: $x = -3$

ii) $|2 - 7x| - 1 > 4$ Ans: $\left(-\infty, -\frac{3}{7}\right) \cup (1, \infty)$

1.7. Quadratic Equations

- Linear equation $ax + b = 0$ where $a \neq 0$.
- Standard Form of Quadratic Equations: $ax^2 + bx + c = 0$ where $a \neq 0$.

When solving quadratic equations, we can use three methods:

1) Method of factorization
2) Method of Completing the square
3) Quadratic Formula

1) Method of factorization

Note: i) $a^2 - b^2 = (a - b)(a + b)$ ii) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

Class Work: Find $550^2 - 250^2$.
Solution: $550^2 - 250^2 = (550 - 250)(550 + 250) = (300)(800) = 240{,}000$

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Example: Factorize i) $9x^2 - 64$ ii) $5x^2 - 16$

Solution: i) $9x^2 - 64 = (3x)^2 - 8^2 = (3x - 8)(3x + 8)$

ii) $5x^2 - 16 = \left(\sqrt{5}x\right)^2 - 4^2 = \left(\sqrt{5}x - 4\right)\left(\sqrt{5}x + 4\right)$

Zero-product rule: If $ab = 0$, then either $a = 0$ or/and $b = 0$.

Example: Solve 1) $x^2 + 4x = 5$ 2) $m(m - 3) = 4$ 3) $4y^2 + 3y - 12 = 6y^2 - 7y - 60$

4) $5x^2 + 14x + 9 = 0$ 5) $2x^2 - 6x + 7 = 3$ 6) $\sqrt{x + 5} = x + 3$

Solution:

1) $x^2 + 4x = 5 \Rightarrow x^2 + 4x - 5 = 0 \Rightarrow (x - 1)(x + 5) = 0 \Rightarrow x = 1$ or $x = -5$.

2) $m(m - 3) = 4 \Rightarrow m^2 - 3m - 4 = 0 \Rightarrow (m + 1)(m - 4) = 0 \Rightarrow m = -1$ or $m = 4$.

3) $4y^2 + 3y - 12 = 6y^2 - 7y - 60 \Rightarrow y^2 - 5y - 24 = 0$

$\Rightarrow (y - 8)(y + 3) = 0 \Rightarrow y = 8$ or $y = -3$.

4) $5x^2 + 14x + 9 = 0 \Rightarrow 5x^2 + 5x + 9x + 9 = 0 \Rightarrow 5x(x + 1) + 9(x + 1) = 0$

$\Rightarrow (5x + 9)(x + 1) = 0 \Rightarrow x = -9/5$ or $x = -1$.

5) $2x^2 - 6x + 7 = 3 \Rightarrow 2x^2 - 6x + 4 = 0 \Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x - 1)(x - 2) = 0 \Rightarrow x = 1$ or $x = 2$.

2) Method of Completing the square

Note: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

To complete square $ax^2 + bx + c = 0$, add $\left(\frac{1}{2}b\right)^2$ on both sides.

Example: Express in completing square i) $x^2 + 7x + 1$ ii) $x^2 - 5x + 6$.

Solution: i) $x^2 + 7x + 1 = x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 1 = \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + 1 = \left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$

ii) $x^2 - 5x + 6 = x^2 - 5x + \left(\frac{-5}{2}\right)^2 - \left(\frac{-5}{2}\right)^2 + 6 = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$.

Example: Solve $x^2 - 4x + 1 = 0$ by completing the square.

Solution: $x^2 - 4x + 1 = x^2 - 4x + 4 - 4 + 1 = (x - 2)^2 - 3$

$x^2 - 4x + 1 = 0 \Rightarrow (x - 2)^2 - 3 = 0 \Rightarrow (x - 2)^2 = 3$

$\Rightarrow x - 2 = \pm\sqrt{3} \Rightarrow x = 2 - \sqrt{3}$ or $x = 2 + \sqrt{3}$.

Example: Solve $x^2 - 6x + 10 = 0$

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Solution: $x^2 - 6x + 10 = x^2 - 6x + \left(\frac{-6}{2}\right)^2 - \left(\frac{-6}{2}\right)^2 + 10$

$= x^2 - 6x + 9 - 9 + 10 = (x-3)^2 + 1 = 0$

$\Rightarrow (x-3)^2 = -1 \Rightarrow x - 3 = \pm\sqrt{-1}$ which is undefined.

Hence the equation has **no real solution**.

3) Method of Quadratic Formula

The formula for solving quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

☞$b^2 - 4ac$ is called **discriminant**.

- If $b^2 - 4ac < 0$, the equation does not have any real solution.
- If $b^2 - 4ac = 0$, then the only solution is $x = \frac{-b}{2a}$.
- If $b^2 - 4ac > 0$, the equation have two solutions.

Note: If $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ are two solutions of the quadratic equation $ax^2 + bx + c = 0$, then $r_1 r_2 = \frac{c}{a}$ and $r_1 + r_2 = -\frac{b}{a}$.

Example: Solve i) $4x^2 = 2x + 1$. ii) $2x^2 + \frac{3}{2}x + \frac{1}{4} = 0$ iii) $2x^2 + 2x + 5 = 0$

Solution: i) $4x^2 = 2x + 1 \Rightarrow 4x^2 - 2x - 1 = 0 \Rightarrow x = \frac{2\pm\sqrt{4+16}}{8} = \frac{2\pm\sqrt{20}}{8} = \frac{2\pm2\sqrt{5}}{8} = \frac{1\pm\sqrt{5}}{4}$

ii) $2x^2 + \frac{3}{2}x + \frac{1}{4} = 0 \Rightarrow x = \frac{-\frac{3}{2}\pm\sqrt{\frac{9}{4}-2}}{4} = \frac{-\frac{3}{2}\pm\sqrt{\frac{1}{4}}}{4} = \frac{-3\pm8}{8}$

iii) $2x^2 + 2x + 5 = 0$, Since $b^2 - 4ac = 4 - 40 = -36 < 0$, there is no real solution.

1.8. Word problem

Two different square have a total area of $274\ cm^2$ and the sum of their perimeters is $88\ cm$. Find the lengths of the sides of the squares.

Answer: $x^2 + y^2 = 274$ and $4x + 4y = 88$

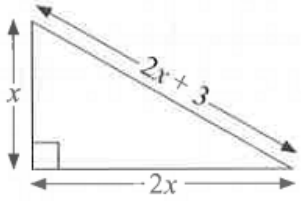
Homework

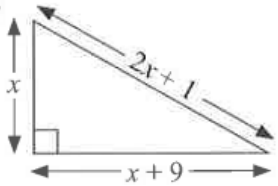
1. Solve i) $6x - 1 = \frac{4}{4x+1}$ ii) $9x^2 + 25 = 30x$

2. For what value(s) of k does the equation $x^2 + (5-k)x + 9 = 0$ have
   i) one real root? ii) Two distinct real roots? iii) no real root?

3. Find the value of x for each of the triangle shown below. (Hint use Pythagoras' Theorem)

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1.9. Quadratic Inequalities

For a constant a, b and c, $ax^2 + bx + c \leq 0, ax^2 + bx + c \geq 0, ax^2 + bx + c < 0, ax^2 + bx + c > 0$ are called a quadratic inequality.

- The solution set of $ax^2 + bx + c > 0$ is the set of all real numbers if $a > 0$, and empty set if $a < 0$.
- The solution set of $ax^2 + bx + c < 0$ is the set of all real numbers if $a < 0$, and empty set if $a > 0$.

1.9.1. Solving Quadratic Inequalities Using Product Property

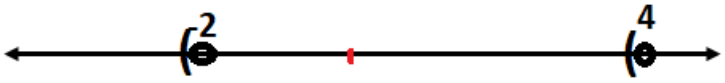
Note: i) $ab > 0$ iff $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

ii) $ab < 0$ iff $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

Example: Solve i) $(x + 2)(x - 4) > 0$ ii) $x^2 - x - 2 \leq 0$

Solution: i) $(x + 2)(x - 4) > 0 \Rightarrow x + 2 > 0$ and $x - 4 > 0$ or $x + 2 < 0$ and $x - 4 < 0$

Case I: $x + 2 > 0$ and $x - 4 > 0$ implies $x > -2$ and $x > 4$. The intersection is $x > -2$.



Hence, the solution set for the first case is

$$S_1 = \{x: x > -2 \text{ and } x > 4\} = (-2, \infty) \cap (4, \infty) = (4, \infty).$$

Case II: $x + 2 < 0$ and $x - 4 < 0$ implies $x < -2$ and $x < 4$. The intersection is $x < -2$.

Hence, the solution set for the second case is

$$S_2 = \{x: x < -2 \text{ and } x < 4\} = (-\infty, -2) \cap (-\infty, 4) = (-\infty, -2).$$

Therefore, the solution set of $(x + 2)(x - 4) > 0$ is

$$S_1 \cup S_2 = (4, \infty) \cup (-\infty, -2)$$

ii) $x^2 - x - 2 \leq 0 \Rightarrow (x + 1)(x - 2) \leq 0$

Case I: $x + 1 \leq 0$ and $x - 2 \geq 0$

$$S_1 = \{x: -1 \leq x \leq 2\} = [-1, 2]$$

Case II: $x + 1 \geq 0$ and $x - 2 \leq 0$

$$S_2 = \emptyset$$

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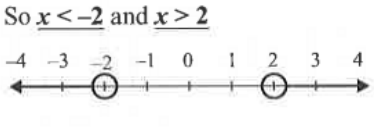
Therefore, the solution set for $x^2 - x - 2 \leq 0$ is $S_1 \cup S_2 = [-1, 2]$.

Example: Solve the inequality $x^2 > 4$.

Solution: Given that $x^2 - 4 > 0$. Which is a parabola $f(x) = x^2 - 4$.

Factorizing $x^2 - 4$ we have $(x - 2)(x + 2) > 0$

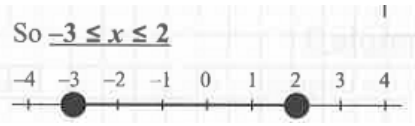
The quadratic equation is > 0 when the graph is above the x-axis. Hence $x < -2$ and $x > 2$



Example: Solve $x \leq 6 - x^2$

Rearranging the inequality $x^2 + x - 6 \leq 0$.

Factorizing gives, $(x + 3)(x - 2) \leq 0$



1.9.2. Solving Quadratic Inequalities Using the Sign Chart Method:

Example: Solve $x^2 + x - 12 > 0$.

Solution: $x^2 + x - 12 = (x - 3)(x + 4) > 0$

		-4		3	
$x - 3$	$- - -$		$- - -$		$+ + +$
$x + 4$	$- - -$		$+ + +$		$+ + +$
$(x - 3)(x + 4)$	$+ + +$		$- - -$		$+ + +$
	> 0		< 0		> 0

Hence the solution set is $\{x \in \mathbb{R}: x < -4 \text{ or } x > 3\} = (-\infty, -4) \cup (3, \infty)$.

Homework: Solve the following quadratic inequalities

i) $4x^2 - x - 8 < 3x^2 - 4x + 2$ ii) $x - 1 \leq x^2 + 2$ iii) $9x^2 - \frac{1}{64} < 0$

1.9.3. Solving quadratic inequalities graphically

Quadratic inequality in one variables:

$ax^2 + bx + c < 0, ax^2 + bx + c > 0$, $ax^2 + bx + c \geq 0, ax^2 + bx + c \leq 0$

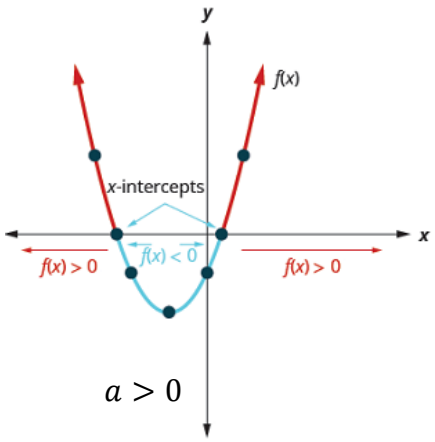
Solving $ax^2 + bx + c < 0$ we mean we are finding intervals when $f(x) < 0$. We want to know when the parabola is below the x-axis.

Solving $ax^2 + bx + c > 0$ we mean we are finding intervals when $f(x) > 0$. We want to know when the parabola is above the y-axis.

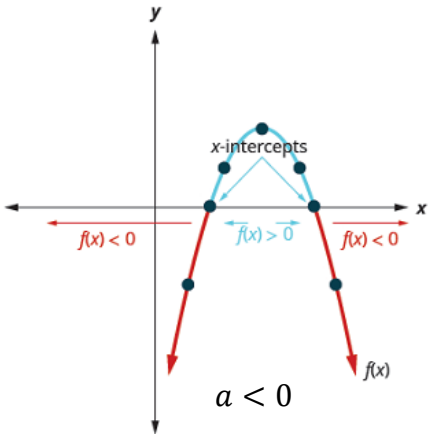
To solve $ax^2 + bx + c < 0$, graph $f(x) = ax^2 + bx + c$, which is parabola.

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- If $a > 0$ then the graph is upward parabola
- If $a < 0$ then the graph is downward parabola
- Axis of symmetry is the line $x = -\frac{b}{2a}$.
- The vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- To find x-intercepts solve for x from $ax^2 + bx + c = 0$.
- To find y-intercept find $f(0) \Longrightarrow f(0) = c$. Thus, $(0, c)$ is y-intercept.



$a > 0$

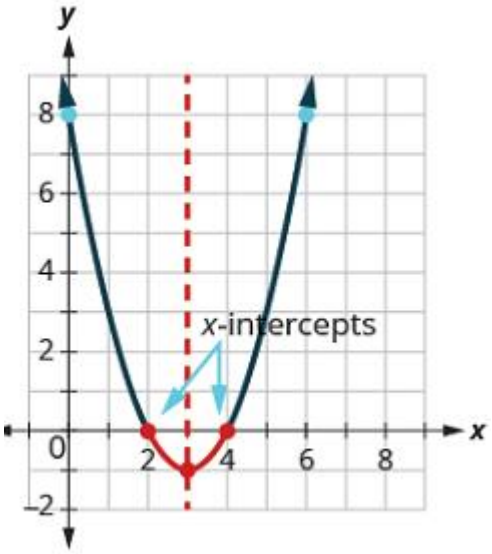


$a < 0$

Example: Solve $x^2 - 6x + 8 < 0$ graphically.

Solution: Since $a = 1 > 0$, the parabola opens upward.

- Axis of symmetry: $x = -\frac{b}{2a} = 3$
- The vertex is $(3, -1)$.
- x-intercepts: $x^2 - 6x + 8 = 0$ implies $x = 2$ or $x = 4$. The x-intercepts are $(2, 0)$ and $(4, 0)$.
- y −intercept: Since $f(0) = 8$, the y-intercept is $(0, 8)$.



The inequality asks for the values of x which make the function less than 0, i.e., values of x which make the parabola below the x-axis.

Thus, the solution interval is $(2, 4)$.

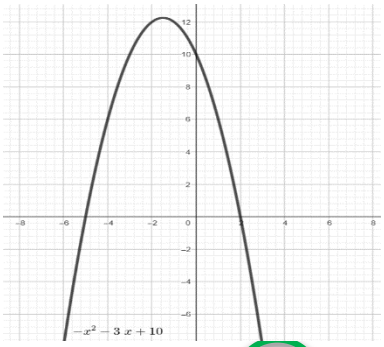
Example: Solve $x^2 - 6x + 8 \geq 0$ graphically.

The inequality asks for the values of x which make the function greater than or equal to 0, i.e., values of x which make the parabola above the x-axis.

Thus, the solution interval is $(-\infty, 2] \cup [4, \infty)$.

Example: Solve $-x^2 - 3x + 10 \leq 0$

Solution: $(-x + 2)(x + 5) \leq 0$

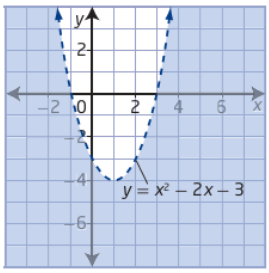


Quadratic inequality in two variables

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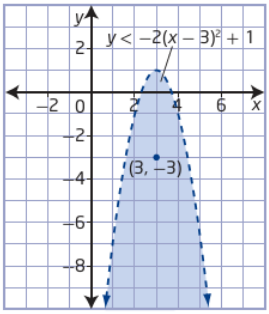
Quadratic inequality in two variables are of type: $y < ax^2 + bx + c$,

$y > ax^2 + bx + c, y \geq ax^2 + bx + c,$ and $y \leq ax^2 + bx + c$



Example: Solve $y < x^2 - 2x - 3$.

Solution: Graph the related parabola $y = x^2 - 2x - 3$. Since the inequality symbol is $<$ the curve is drawn as a dashed line. To determine which region is the solution region, choose a test point from either above or below the parabola and identify which will satisfy the inequality.



Example: Solve $y < -2(x-3)^2 + 1$ and determine if the point $(2, -4)$ is a solution to the inequality.

1.10. Rational expressions: Simplification and Solution

If both the numerator and denominator are polynomials, the fractional expression is called a rational expression.

Rational expression: $\frac{P(x)}{Q(x)}, Q(x) \neq 0$ where $P(x)$ and $Q(x)$ are polynomials.

Note: Non-zero constant is a polynomial of degree 0.

Example: The following are rational expressions: i) $\frac{x^3-2}{5x^4-3x+7}$ ii) $\frac{1}{x^2-4}$

Example: Find the domain of i) $\frac{x^2-1}{x-1}$ ii) $x + 1$ iii) $\frac{x}{x^2+3x-10}$

Solution: i) Domain of $\frac{x^2-1}{x-1}$ is $\{x \in \mathbb{R}: x \neq 1\}$

ii) Domain of $x + 1$ is $\mathbb{R}$.

iii) Domain of $\frac{x}{x^2+3x-10}$ is $\{x \in \mathbb{R}: x \neq -5 \text{ and } x \neq 2\}$.

Note: A rational expression has been reduced to lowest terms if all common factors from the numerator and denominator have been canceled.

To reduce rational expression to lowest terms (simplify) a rational expression:

i) Find the domain.

ii) Factorize the numerator and denominator completely.

iii) Cancel like terms in the numerator and denominator.

Example: Reduce the rational expression $\frac{x^2+x-2}{x^2-x-6}$ to lowest terms.

Solution: $\frac{x^2+x-2}{x^2-x-6} = \frac{(x+2)(x-1)}{(x+2)(x-3)} = \frac{x-1}{x-3}, x \neq -2, 3.$

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1.10.1. Operations with Rational Expressions

Note: i) $-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$ ii) $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ iii) $\frac{a/b}{c/d}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\frac{d}{c}=\frac{ad}{bc}$

iv) $\frac{a}{b/c}=\frac{a}{1}\frac{c}{b}=\frac{ac}{b}$ v) $\frac{a/b}{c}=\frac{a}{b}\frac{1}{c}=\frac{a}{bc}$

Classwork: Operate

i) $\frac{x^2+3x}{x^2+2x-15}-\frac{2x+12}{x^2+2x-15}$ ii) $\frac{3}{x^2-x}-\frac{2}{x^2+x-2}$ iii) $\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}$

1.10.2. Partial Fraction

Example: Show that $\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)=\frac{1}{x^2-1}$

Proper fractions: The degree of the numerator is less than the degree of the denominator.

Example: $\frac{x}{x^2-5},\frac{x^2+5}{x^4+x-1}$

Improper fractions: The degree of the numerator is greater than or equal of the denominator.

Example: $\frac{x^2}{x^2-5},\frac{x^4+5}{x^2+x+2}$

Factor in the denominator	Corresponding term in the partial fraction
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^n$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Note: For improper fraction apply long division before finding the partial fraction.

Example: Find the partial fraction of

i) $\frac{x}{(x+1)(x+5)}$ ii) $\frac{7x+6}{x^2+x-6}$ iii) $\frac{5x+1}{x^2(x^2+4)}$ iv) $\frac{(x+3)^2}{(x^2+1)(x-3)}$ v) $\frac{x-18}{x(x-3)^2}$

Solution: i) $\frac{x}{(x+1)(x+5)}=\frac{A}{x+1}+\frac{B}{x+5}\Rightarrow\frac{x}{(x+1)(x+5)}=\frac{A(x+5)+B(x+1)}{(x+1)(x+5)}\Rightarrow(A+B)x+(5A+B)=x\Rightarrow$

$A+B=1$ and $5A+B=0$ solving we obtain $A=-\frac{1}{4}$ and $B=\frac{5}{4}$. Hence

$$\frac{x}{(x+1)(x+5)}=\frac{-1}{4(x+1)}+\frac{5}{4(x+5)}$$

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ii) $\frac{7x+6}{x^2+x-6}=\frac{7x+6}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}=\frac{(A+B)x-2A+3B}{(x+3)(x-2)}$

$\Rightarrow (A+B)x-2A+3B=7x+6\Rightarrow \begin{cases} A+B=7 \\ -2A+3B=6 \end{cases}\Rightarrow A=3, B=4$. Thus,

$$\frac{7x+6}{x^2+x-6}=\frac{3}{x+3}+\frac{4}{x-2}$$

iii) $\frac{5x+1}{x^2(x^2+4)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}=\frac{Ax(x^2+4)+B(x^2+4)+x^2(Cx+D)}{x^2(x^2+4)}$

$$\Rightarrow 5x+1=4Ax+(A+C)x^3+(B+D)x^2+4B\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ 4A=5 \\ 4B=1 \end{cases}$$

$\Rightarrow A=\frac{5}{4}, B=\frac{1}{4}, C=\frac{-5}{4}, D=\frac{-1}{4}$

Then, $\frac{5x+1}{x^2(x^2+4)}=\frac{5}{4x}+\frac{1}{4x^2}+\frac{-\frac{5}{4}x-\frac{1}{4}}{x^2+4}$

iv) $\frac{(x+3)^2}{(x^2+1)(x-3)}=\frac{-13x-9}{5(x^2+1)}+\frac{18}{5(x-3)}$

v) $\frac{x-18}{x(x-3)^2}=-\frac{2}{x}+\frac{2}{x-3}-\frac{5}{(x-3)^2}$

Homework: Express $\frac{4x^3+10x+4}{x(2x+1)}$ in partial fractions. **Answer**: $\frac{4x^3+10x+4}{x(2x+1)}=2x-1+\frac{4}{x}+\frac{3}{2x+1}$

1.10.2. Solving Rational Equations

Example: Solve i) $\frac{x+1}{3}=5-\frac{x+2}{7}$ ii) $\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}$ iii) $\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5x}$

Solution: i) $\frac{x+1}{3}=5-\frac{x+2}{7}\Rightarrow \frac{x+1}{3}=\frac{33-x}{7}\Rightarrow 7x+7=-3x+99\Rightarrow 10x=92\Rightarrow x=\frac{46}{5}$

ii) $\frac{3x}{5}-\frac{x-3}{2}=\frac{x+2}{3}\Rightarrow 18x-15(x-3)=10(x+2)\Rightarrow -7x=-25\Rightarrow x=\frac{25}{7}$

iii) $\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5x}\Rightarrow 9x=12\Rightarrow x=\frac{12}{9}$

Homework: Solve $\frac{x-1}{x-2}+\frac{x}{x-3}=\frac{1}{x^2-5x+6}$. (**Answer**: $x=\frac{3}{2}\pm\frac{\sqrt{5}}{2}$)

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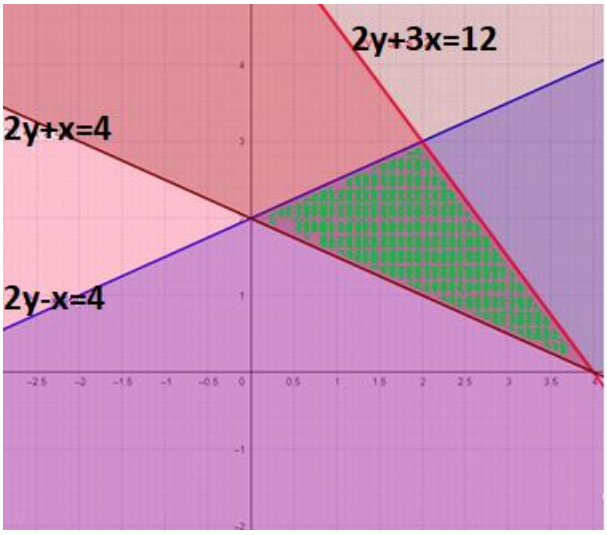
ESSLCE Question

1. The simplified form of $\sqrt[3]{-27} \times \sqrt{0.02} \div \sqrt{18}$ is equal to (Social Sci ESSLCE, 2014)

A. $-\frac{1}{10}$ B. -10 C. 10 D. $\frac{1}{10}$

Solution: $\sqrt[3]{-27} \times \sqrt{0.02} \div \sqrt{18} = \frac{(-27)^{\frac{1}{3}}(0.02)^{\frac{1}{2}}}{(18)^{\frac{1}{2}}} = \frac{(-3^3)^{\frac{1}{3}}(2\times10^{-2})^{\frac{1}{2}}}{(2\times9)^{\frac{1}{2}}} = \frac{-3(2)^{\frac{1}{2}}(10^{-2})^{\frac{1}{2}}}{3(2)^{\frac{1}{2}}} = -10^{-1} = -\frac{1}{10}$

2. Consider the regions R shown in the figure below. (Social Sci ESSLCE, 2014)



For which of the following systems of inequality does the region R on the above figure represent? (Social Sci ESSLCE, 2014)

A. $\begin{cases} 2y - x \leq 4 \\ 2y + 3x \leq 12 \\ 2y + x \geq 4 \\ x \geq 0, y \geq 0 \end{cases}$

B. $\begin{cases} 2y - x \geq 4 \\ 2y + 3x \geq 12 \\ 2y + x \leq 4 \\ x \geq 0, y \geq 0 \end{cases}$

C. $\begin{cases} 2y - x \leq 4 \\ 2y + 3x \geq 12 \\ 2y + x \leq 4 \\ x \geq 0, y \geq 0 \end{cases}$

D. $\begin{cases} 2y - x \leq 4 \\ 2y + 3x \leq 12 \\ 2y + x \leq 4 \\ x \geq 0, y \geq 0 \end{cases}$

Solution: From the graph by taking test points we observe that

$2y + x \geq 4$, $2y + 3x \leq 12$ and $2y - x \leq 4$

3. Which one of the following is equal to $81^{\frac{1}{8}}$? (SS ESSLCE, 2014)

A. $\sqrt{3}$ C. $\sqrt[8]{9}$

B. $\sqrt{81}$ D. $\sqrt[4]{9}$

Solution: $81^{\frac{1}{8}} = (3^4)^{1/8} = 3^{1/2} = \sqrt{3}$

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4. Which one ot the following is the solution set of the equation $-2x^2 + 10x = 6$? (SS ESSLCE, 2014)

A. $\{0, 5\}$ C. $\{-\sqrt{13}, \sqrt{13}\}$

B. $\left\{-\frac{5}{2}, \frac{5}{2}\right\}$ D. $\left\{\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right\}$

Solution: Given $-2x^2 + 10x - 6 = 0$. Then, by quadratic formula

$$x = \frac{-10 \pm \sqrt{100 - 48}}{-4} = \frac{-10 \pm \sqrt{52}}{-4} = \frac{-10 \pm \sqrt{4 \times 13}}{-4} = \frac{-5 \pm \sqrt{13}}{-2} = \frac{5 \mp \sqrt{13}}{2}$$

5. What is the solution set of the equation $\left(\frac{64}{729}\right)^x - \left(\frac{9}{4}\right)^{2x-1} = 0$?

A. $\left\{\frac{1}{5}\right\}$ C. $\left\{\frac{5}{2}\right\}$

B. $\left\{\frac{2}{5}\right\}$ D. $\{5\}$

Solution: $\left(\frac{64}{729}\right)^x - \left(\frac{9}{4}\right)^{2x-1} = 0 \implies \left(\frac{8}{27}\right)^{2x} = \left(\frac{9}{4}\right)^{2x-1} \implies \left(\frac{2}{3}\right)^{6x} = \left(\frac{3}{2}\right)^{4x-2}$

$\implies \left(\frac{2}{3}\right)^{6x} = \left(\frac{2}{3}\right)^{2-4x} \implies 6x = 2 - 4x \implies 10x = 2 \implies x = \frac{1}{5}$

6. What is the solution set of the quadratic inequality $-2x^2 + 5x - 3 < 0$? (SS ESSLCE, 2014)

A. $(-\infty, 1) \cup \left(\frac{3}{2}, \infty\right)$ C. $(-\infty, \infty)$

B. $\left(1, \frac{3}{2}\right)$ D. $(-\infty, -1) \cup (1, \infty)$

Solution: $-2x^2 + 5x - 3 < 0 \implies -2x^2 + 2x + 3x - 3 < 0 \implies -2x(x-1) + 3(x-1) = (-2x+3)(x-1) < 0$

Thus, $ab < 0 \implies a < 0$ and $b > 0$ or $a > 0$ and $b < 0$. So, $-2x + 3 < 0$ and $x - 1 > 0$. That is $x > \frac{3}{2}$ and $x > 1$. Hence, $S_1 = \left(\frac{3}{2}, \infty\right) \cap (1, \infty) = \left(\frac{3}{2}, \infty\right)$. Also, from the case $-2x + 3 > 0$ and $x - 1 < 0$, we obtain $x < \frac{3}{2}$ and $x < 1$. Hence $S_1 = \left(-\infty, \frac{3}{2}\right) \cap (-\infty, 1) = (-\infty, 1)$

Therefore, the solution set is $\left(\frac{3}{2}, \infty\right) \cup (-\infty, 1)$.

7. What is the solution set of the system of linear equation $\begin{cases} -2x + z = 1 \\ x - y + 2z = 0 \\ y + z = -2 \end{cases}$? (Social Sci ESSLCE, 2014)

A. { } C. $\{(x, y, z): x, y, z \in \mathbb{R}\}$

B. $\left\{-\frac{5}{7}, -\frac{11}{7}, -\frac{3}{7}\right\}$ D. $\{(-2 - 3t, -2 - t, t), t \in \mathbb{R}\}$

8. What is the solution set of the rational equation $\frac{3}{x^2+x} + 3 = \frac{2x-1}{x+1}$? (SS ESSLCE, 2014)

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A. $\{-3,-1\}$ C. $\{-3\}$

B. $\{-1,0\}$ D. $\emptyset$

Solution: $\frac{3}{x^2+x}+3=\frac{2x-1}{x+1}\Rightarrow\frac{3+3x^2+3x}{x^2+x}=\frac{2x-1}{x+1}\Rightarrow\frac{3x^2+3x+3}{x(x+1)}=\frac{2x-1}{x+1}$

$$\Rightarrow\frac{3x^2+3x+3}{x}=2x-1,\text{for } x\neq 0$$

$\Rightarrow 3x^2+3x+3=2x^2-x\Rightarrow x^2+4x+3=0\Rightarrow(x+1)(x+3)=0\Rightarrow x=-1 \text{ or } x=-3$

Hence the solution set is $\{-3,-1\}$.

9. for $x\neq -1$ and $x\neq 1$, the simplified form of $\frac{x^2+x}{2-2x}\div\frac{x^2+2x+1}{x^2-1}-x$ is: (Natural S, December 2014)

A. $\frac{1}{2(x+1)}$ C. $\frac{3}{2}x$

B. $-\frac{1-x}{2}$ D. $-\frac{3}{2}x$

Solution: $\frac{x^2+x}{2-2x}\div\frac{x^2+2x+1}{x^2-1}-x\Rightarrow\frac{x^2+x}{2-2x}\times\frac{x^2-1}{x^2+2x+1}-x\Rightarrow\frac{x(x+1)}{2(1-x)}\times\frac{(x-1)(x+1)}{(x+1)^2}-x$

$$\Rightarrow\frac{x(x+1)^2(x-1)}{2(1-x)(x+1)^2}-x\Rightarrow-\frac{x}{2}-x\Rightarrow\frac{-x-2x}{2}\Rightarrow\frac{-3x}{2}$$

10. If the sum and the product of the roots of the quadratic equation $x^2+(k+1)x+mk=0$ are -3 and 7 respectively, what is the value of m? (Natural S, December 2014)

A. 2 C. $-\frac{7}{2}$

B. $\frac{7}{2}$ D. -2

Solution: If r_1 and r_2 are any two roots of the quadratic equation, then

$r_1r_2=\frac{c}{a}\Rightarrow\frac{mk}{1}=7$ and $r_1+r_2=-\frac{b}{a}\Rightarrow-\frac{k+1}{1}=-3$

$\Rightarrow mk=7$ and $k+1=3$

$k=2$ and $m=\frac{7}{k}=\frac{7}{2}$.

11. The simplified form of $\sqrt[3]{-27}\times\sqrt{0.02}\div\sqrt{18}$

A. -10 B. $-\frac{1}{10}$ C. $\frac{1}{10}$ D. 10

Solution: $\sqrt[3]{-27}\times\sqrt{0.02}\div\sqrt{18}\Rightarrow(-27)^{\frac{1}{3}}\times(2\times10^{-2})^{\frac{1}{2}}\div18^{\frac{1}{2}}\Rightarrow(-3)^{3\frac{1}{3}}\times2^{\frac{1}{2}}(10^{-2})^{\frac{1}{2}}\div(9\times2)^{\frac{1}{2}}$

$$\Rightarrow-3\times\sqrt{2}\frac{1}{10}\div3\sqrt{2}\Rightarrow-3\times\sqrt{2}\frac{1}{10}\times\frac{1}{3\sqrt{2}}\Rightarrow-\frac{1}{10}$$

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Chapter 2

Relations and Functions

2.1. Revision on Set theory

2.1.1. Definition and Representation of Set

Definition: Set is a well-defined collection of objects.

- Set can be described by three methods:

i) Partial Listing method

Example: The set of non-negative integer $A = \{0, 1, 2, \cdots\}$

ii) Complete listing method

Example: The set of even numbers between 1 and 10 is $B = \{2, 4, 6, 8, 10\}$

iii) Set-builder method

Example: The set of even number between 1 and 10 $D = \{x \in \mathbb{Z} | x \text{ is an even and } 1 \leq x \leq 10\}$

Definition: Let A and B be two set, then set A is a **subset** of B or A is contained in B if every element of A is an element of B.

Notation: If A is a subset of B we denote it by $A \subseteq B$. If A is not a subset of B it is denoted by $A \subsetneq B$.

2.1.2. Ordered pair

An **ordered pair** is a pair formed by two elements that are separated by a comma and written inside the parentheses.

For example, (x, y) represents an ordered pair, where 'x' is called the **first element/component** and 'y' is called the **second element/component** of the ordered pair.

Example: If $A = \{1, 2, a, b, -3, -5\}$, then $(a, 1), (-3, 2), (-5, b)$ are some examples of ordered pairs

Classwork: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. List all ordered pairs (x, y) from set A which satisfy the following sentences,

i) x is a multiple of y ii) x is the square of y

Answer: i) $(4, 2), (6, 2), (6, 3), (8, 2), (8, 4), (8, 1), (9, 3), (9, 1)$

ii) $(1, 1), (4, 2), (9, 3)$

Note: If $(x, y) = (a, b)$ then $x = a$ and $y = b$.

Example: Identify the quadrants in which the following ordered pairs lie without plotting them.

(a) $(2, -3)$ (b) $(-2, -3)$.

Solution:

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(a) In $(2,-3)$, x-coordinate is positive and y-coordinate is negative. So it lies in Quadrant IV.
(b) In $(-2,-3)$, both x and y coordinates are negative. So it lies in Quadrant III.

Example 2: Find the values of x and y if $(2x - 3, 3y + 2) = (5, 8)$.

Solution: By the equality property of ordered pairs,

$$\begin{cases} 2x - 3 = 5 \\ 3y + 2 = 8 \end{cases}$$

Then, $x = 4$ and $y = 2$.

2.1.3. Cartesian Product of Sets

Let A and B be two non-empty sets. Then the set of all ordered pairs (x, y), where $x \in A$, $y \in B$ is called Cartesian product of A and B and Cartesian product of A and B is denoted by $A \times B$ (read A cross B). thus, $A \times B = \{(x, y) | \, x \in A, y \in B\}$.

Note: For any non-set A and B, $A \times B = B \times A$ if and only if $A = B$.

For example, if $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then the Cartesian product is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$$

$$B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2), (a, 3), (b, 3), (c, 3)\}$$

2.2. Definition and examples of relation

Any subset of the Cartesian product is called a **relation**.

Definition: For any sets A and B, any subset of $A \times B$ is called a relation from A to B.

For example, if $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ then $\{(1, a), (1, b), (3, c)\}$ is a relation from A into B.

- If an ordered pair $(x, y) \in R$, then it is denoted by xRy (x "is R related to" y).
- Let R be a relation from a set A into a set B, then $R \subseteq A \times B$.

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**.

Example: 1) Let $A = \{(a, b) | \, a, b \in Z^+ \text{and } a + 2b = 7\}$, then $A = \{(1, 3), (3, 2), (5, 1)\}$.

2) Let A= {1, 2, 3, 4, 5, 6} and $S = \{(a, b) | \, a, b \in A \text{ and } a \text{ divides } b\}$. Then

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

Example: Let $A = \{2, 3, 4, 5\}$ and $B = \{4, 5, 6, 8\}$, a relation R from A to B is defined as

$$R = \{(a, b): a \text{ is a factor of } b, a \in A, b \in B\}$$

Then, $R = \{(2, 4), (2, 6), (2, 8), (3, 6), (4, 4), (4, 8), (5, 5)\}$. Notice that here $R \subseteq A \times B$.

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2.2.1. Domain and range of a relation

Domain: The set of all **first elements** of the ordered pairs in a relation R from a set A to a set B, is called the domain of R and is denoted by $Dom(R)$.
Range: The set of all **second elements** of the ordered pairs in a relation R from a set to a set B, is called the range of R and is denoted by $Range(R)$..

Mathematically,

$$\boldsymbol{Dom(R) = \{a: (a,b) \in R \text{ for some } b \in B\}}$$

$$\boldsymbol{Range(R) = \{b| (a,b) \in R \text{ for some } a \in A\}}$$

Example: 1) Let $A = \{1,2,3,4\}, B = \{a,b,c,d,e\}$ and $R = \{(1,a),(2,c),(3,a),(2,a)\}$. Then

$Dom(R) = \{1,2,3\}$ and $Range(R) = \{a,c\}$.

(2) Let $A = \{2,3,4\}, B = \{2,3,4,5,6,7,8\}$ and $R = \{(a,b) \in A \times B | a \text{ divides } b\}$. Then

$R = \{(2,2),(2,4),(2,6),(2,8),(3,3),(3,6),(4,4),(4,8)\}$

$Dom(R) = \{2,3,4\}$ and $Range(R) = \{2,3,4,6,8\}$

Theorem: Let $A, B \neq \emptyset$ be sets with $n(A) = m$ and $n(B) = n$. Then the number of relations from A to B is 2^{mn}.

Example: Let $A = \{1,2,3\}$ and $B = \{a,b\}$. then $n(A) = 3$ and $n(B) = 2$ and $n(A \times B) = n(A) \cdot n(B) = 3(2) = 6$. Therefore there are exactly $64 (= 2^6)$ relations from A to B.

2.2.2. Representing a relations

A relation can be represented in the following ways:

Ordered Pairs

$(1,2)$

$(-2,3)$

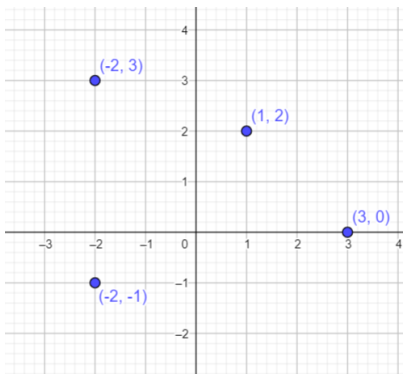
$(-2,-1)$

$(3,0)$

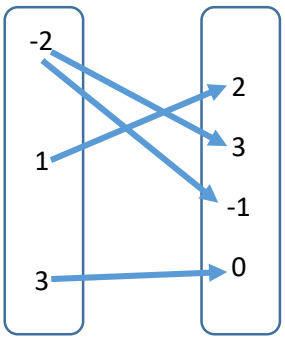
Table

x	y
1	2
-2	3
-2	-1
3	0

Graph



Mapping Diagram



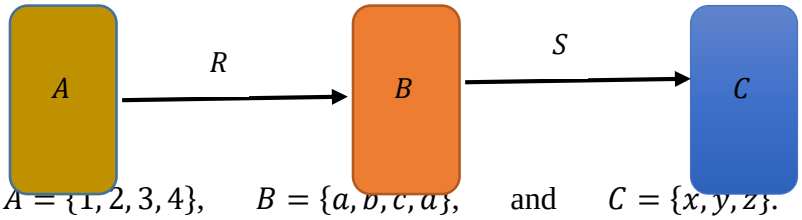
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Composition of relation

Definition: Let A, B and C be sets, R a relation from A to B and S a relation from B to C. Define

$$R \circ S = \{(a, c) \in A \times C | \text{ there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

Then $R \circ S$ is a relation from A to C.



Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$. Let $R = \{(1, c), (2, d), (2, a), (3, d)\}$ and $S = \{(a, y), (b, x), (b, y), (a, z)\}$. Then R is a relation form A to B and S is a relation form B to C.

$Dom(R) = \{1, 2, 3\}$ and $Range(R) = \{a, c, d\}$

$Dom(S) = \{a, b\}$ and $Range(S) = \{x, y, z\}$

$R \circ S$ a relation from A into C.

Since $(2, a) \in R$ and $(a, y) \in S$, we have $(2, y) \in R \circ S$

Since $(2, a) \in R$ and $(a, z) \in S$, we have $(2, z) \in R \circ S$

Hence $S \circ R = \{(2, y), (2, z)\}$.

☞ However $R \circ S$ is not defined.

Example: Suppose $A = \{a, b, c\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f: A \to B$ be the function $f = \{(a, 0), (b, 1), (c, 0)\}$, let $g: B \to C$ be the function $g = \{(0, 3), (1, 1)\}$. Then

$$g \circ f = \{(a, 3), (b, 1), (c, 3)\}.$$

2.2.3. Inverse of Relation

Definition: Let A and B be non-empty sets and R a relation from A to B then the inverse of R denoted by R^{-1} is defined as the set $\{(b, a) \in B \times A | (a, b) \in R\}$. Thus

$$R^{-1} = \{(b, a) \in B \times A | (a, b) \in R\} = \{(a, b) : (b, a) \in R\}.$$

Note: i) If R is a relation from A to B, then R^{-1} is a relation form B to A

ii) If $(x, y) \in R$, then $(y, x) \in R^{-1}$.

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iii) $R \circ R^{-1}$ is a relation on B and $R^{-1} \circ R$ is a relation on A.

iv) $(R^{-1})^{-1} = R$

v) Let A, B and C be non-empty sets and R a relation form A to B and S a relation form B to C, then

$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Homework: Let $A = \{x: 1 \leq x < 10\}$ and $B = \{2, 4, 6, 8\}$. If R is a relation from A to B given by $R = \{(x, y): x + y = 12\}$, then find

i) R and R^{-1} ii) domain (R) and domain (R^{-1})

iii) range (R) and range (R^{-1}) iv) $R \circ R^{-1}$

Answer: i) $R = \{(8, 4), (6, 6), (4, 8)\}$ $R^{-1} = \{(4, 8), (6, 6), (8, 4)\}$

ii) domain $(R) = \{4, 6, 8\} \subseteq A$ and domain $(R^{-1}) = \{4, 6, 8\} \subseteq B$

iii) range $(R) = \{4, 6, 8\} \subseteq B$ and range $(R^{-1}) = \{4, 6, 8\} \subseteq A$

iv) $R \circ R^{-1} = \{(4, 4), (6, 6), (8, 8)\}$

Note: i) domain (R^{-1}) = Range (R)

ii) $Range\ (R^{-1})$ = Domain(R)

Example: Let $R = \{(x, y): y \geq x - 4 \text{ and } y < -3x - 6\}$. Find R^{-1}.

Solution: $R^{-1} = \{(y, x): y \geq x - 4 \text{ and } y < -3x - 6\}$

$= \{(x, y): x \geq y - 4 \text{ and } x < -3y - 6\}$

$= \{(x, y): x + 4 \geq y \text{ and } 3y < x - 6\}$

Thus, $R^{-1} = \left\{(x, y): y \leq x + 4 \text{ and } y < \frac{1}{3}x - 2\right\}$.

2.2.4. Graphs of relations involving inequalities

Example: Sketch the graph of the relation and determine their domain and range.

i) $R_1 = \{(x, y): y \leq 2x - 1, x, y \in \mathbb{R}\}$

ii) $R_2 = \{(x, y): y > 2x - 1 \text{ and } y > -2x, x, y \in \mathbb{R}\}$

ii) $R_3 = \{(x, y): y \leq 2x - 1 \text{ and } y > 2x, x, y \in \mathbb{R}\}$

iii) $R_4 = \{(x, y): y \leq x^2 - 2x - 1, x, y \in \mathbb{R}\}$

Solution:

i) To graph the relations R_1 consider the graph of $y = 2x - 1$, and to determine the region by taking test points from each quadrant.

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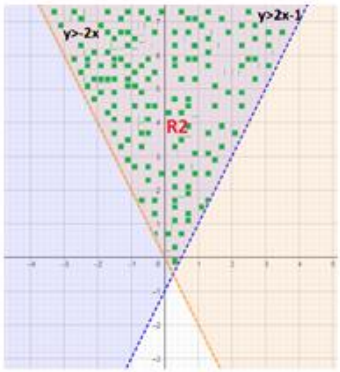
i)



Domain of $R_1 = \{x: x \in \mathbb{R}\}$

Range of $R_1 = \{y: y \in \mathbb{R}\}$

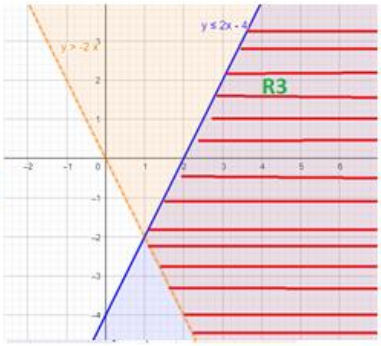
ii)



Domain of $R_2 = \{x: x \in \mathbb{R}\}$

Range of $R_2 = \left\{y: \ y > \frac{1}{4}\right\}$

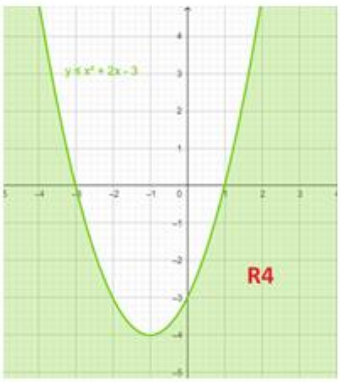
iii)



Domain of $R_3 = \{x: x \geq 1\}$

Range of $R_3 = \{y: y \in \mathbb{R}\}$

iv)



Domain of $R_4 = \{x: x \geq -3 \text{ or } x \geq 1\}$

Range of $R_4 = \{y: y \in \mathbb{R}\}$

Domain of $R_1 = \{x: x \in \mathbb{R}\}$ and Range of $R_1 = \{y: y \in \mathbb{R}\}$

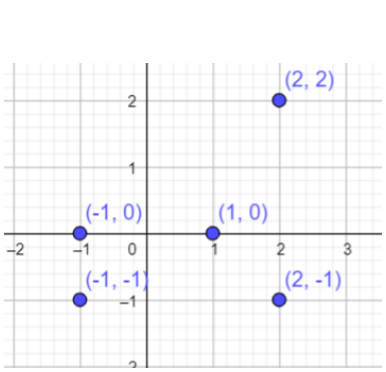
Domain of $R_2 = \{x: x \in \mathbb{R}\}$ and Range of $R_2 = \left\{y: \ y > \frac{1}{4}\right\}$

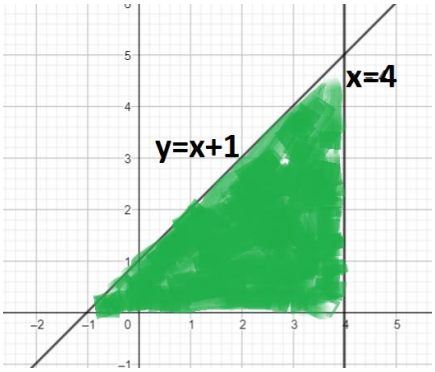
Domain of $R_3 = \{x: x \geq 1\}$ and Range of $R_3 = \{y: y \in \mathbb{R}\}$

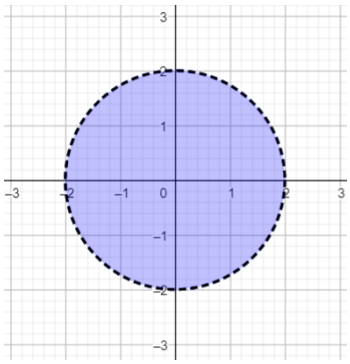
Domain of $R_4 = \{x: x \geq -3 \text{ or } x \geq 1\}$ and Range of $R_4 = \{y: y \in \mathbb{R}\}$

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Homework: Write the relation and find the domain and range of the following.



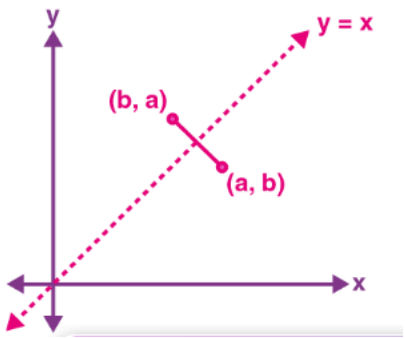


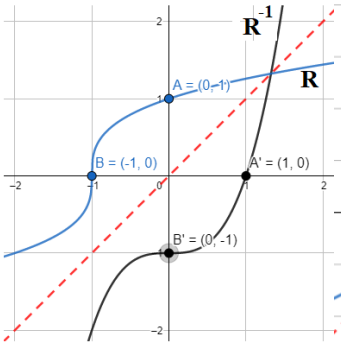


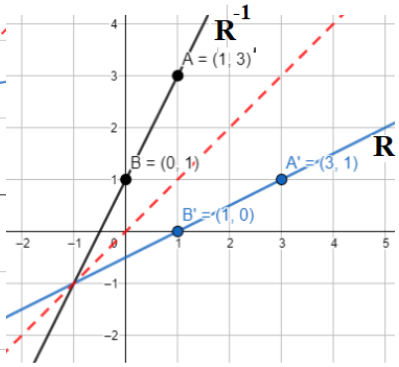
2.2.5. Graphs of inverse Relations

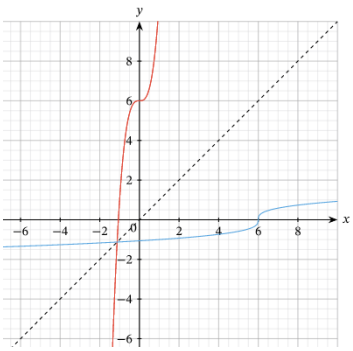
If the graphs of both functions are symmetric with respect to the line $y = x$, then we say that the two functions are inverses of each other. That is, their graphs are mirror images over the line of reflection $y = x$.

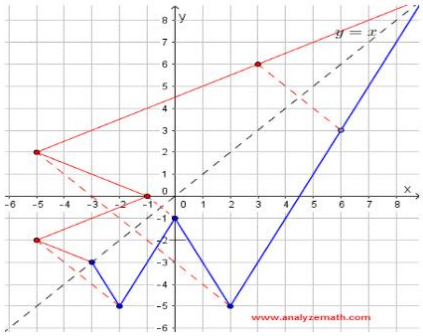
Note: A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half.

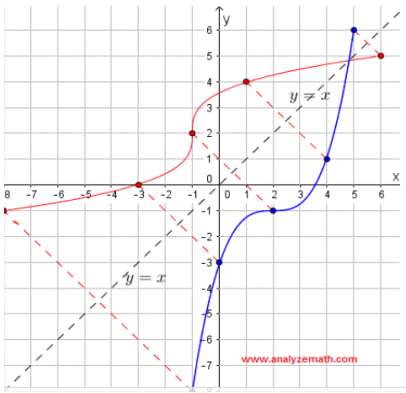












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2.3. Definition and examples of function

2.3.1. Definition of Function

The word "**function**" is derived from a Latin word meaning **operation**. For example, when we multiply a given real number x by 2, we are performing an operation on the number x to get another number $2x$.

A function f is a relation with a special property: each x is related by f to one and only one y. That is, two ordered pairs (x, y) and (x, z) in f imply that y = z.

Definition: A relation f from a set A to a set B is said to be **function** if every element of set A has one and only one image in set B. In other words, a function f is a relation such that no two pairs in the relation has the same first element.

If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x.

Note: A relation $R \subseteq A \times B$ is called a function from A into B if the following hold:

1. For each $a \in A$, there exists $b \in B$ such that $(a, b) \in R$.
2. If $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

The ***independent variable*** is the variable that represents the first element of an ordered pair and the ***dependent variable*** is the variable that represents the second element of an ordered pair.

- ☞ Function is also called a map or a mapping.
- ☞ We often use the notation $y = f(x)$ to mean that the dependent variable y is a function of the independent variable x.

Examples: (1) Let $R = \{(x, 2x) | x \in \mathbb{Z}\}$. Then R is a function from $\mathbb{Z}$ into $\mathbb{Z}$.

(2) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.

i) Let $R_1 = \{(1, a), (2, a), (3, b), (4, b)\}$. Then R_1 is a function from A into B.

ii) Let $R_2 = \{(1, a), (2, b), (3, c), (3, a), (4, a)\}$. Then R is not a function from A into B, since we have two ordered pairs $(3, c)$ and $(3, a)$ in R whose first components are equal and the second components are different.

iii) If $R_3 = \{(1, a), (2, b), (4, c)\}$, then R_3 is not a function of A into B, since $Dom(S) \neq A$.

Example: Determine whether or not each set is a relation. Is it a function?

i) $R = \{y: -1 < y < 1\}$

Answer: Not a relation and hence *not a function*. B/C, it is not collection of ordered pairs.

ii) $R = \{(x, y): x + y = 5\}$

Answer: A relation and a function.

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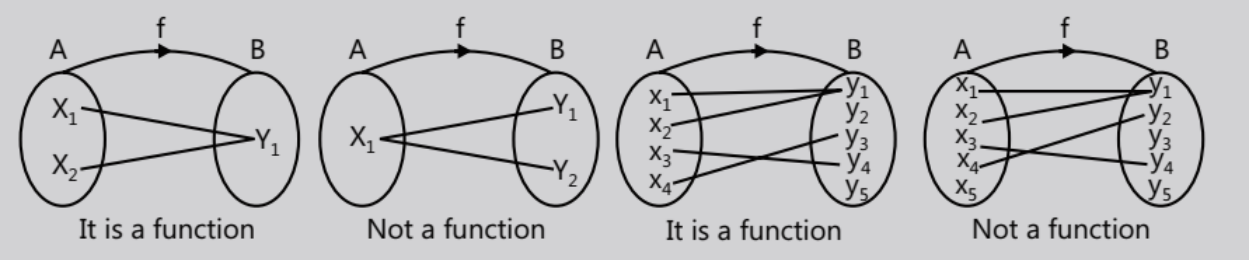
iii) $R = \{(x, y): y = x^2\}$

Answer: A relation but not a function. B/C consider $(1, 1)$ and $(-1, 1)$

iv) $R = \{(x, y): x + y \leq 5\}$

Answer: A relation but *not a function*. B/C $(0, 1) \in R$ and $(0, 2) \in R$ have the same first element with different second element.

Note: Every function is a relation, but every relation may not necessarily be a function.



Note: A relation $R \subseteq A \times B$ is a function if $Dom(R) = A$.

Notation:

1. If R is a function from A into B and $a \in A$, then the unique element b in B such that $(a, b) \in R$ is denoted by $R(a)$.

2. Usually functions will be denoted by lower case letters $f, g, h, \ldots$.

3. If f is a function from A into B, then we denote this by $f: A \to B$.

4. If $f: A \to B$ is a function and $a \in A$, then there exists a unique element b in A such that $(a, b) \in f$. This unique element is denoted by $f(a)$. We write $f(a) = b$ to say that $(a, b) \in f$ or afb. Some authors also write $(a)f = b$ or simply $af = b$ to say that $(a, b) \in f$. However in this chapter we prefer to use $f(a) = b$.

Homework: List all four different functions $f: \{a, b\} \to \{0, 1\}$.

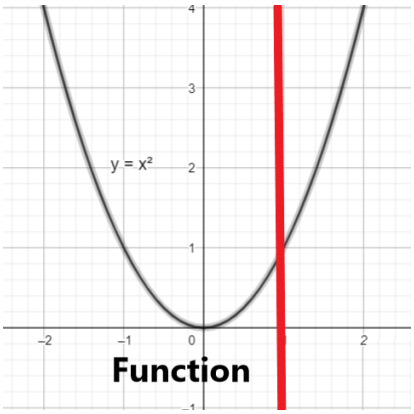
Answer: $f_1 = \{(a, 1), (b, 1)\}$, $f_2 = \{(a, 0), (b, 0)\}$, $f_3 = \{(a, 1), (b, 0)\}$, $f_2 = \{(a, 0), (b, 1)\}$

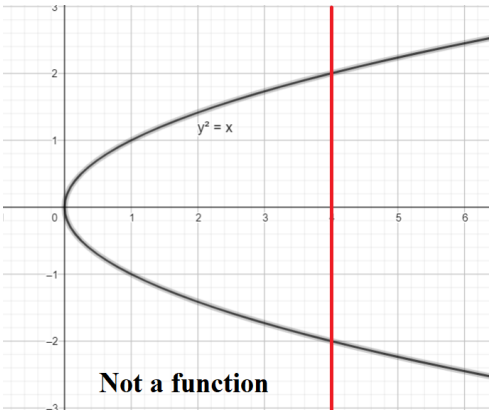
Vertical Line Test

You can use the graph of a relation to determine whether it is a function by applying the vertical line test.

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A relation is a *function* if and only if *no vertical line intersects* the graph of the relation at more than one point.





2.3.2. Domain and Range Of Function

Definition: Let $f: A \to B$ be a function. Then A is called the domain of f and is denoted by $Dom(f)$. B is called the co-domain of f and is denoted by $codom(f)$.

The set of allowable values of x is called the natural domain of the function. The range of the function f is the set of all possible output values.

The range of f is also called the image of f or the image of A under f and is denoted by $Im(f)$. That is,

$$Im(f) = \{f(a) | a \in A\}$$

Note: $Dom(f) \subseteq A$.

If $f(a) = b$, then b is called the ***image*** of a under f and a is called a ***pre-image*** of b.

Note: For any $a \in A$, the image of a under f is unique. But, for $b \in B$, there may be several pre-images of b.

Type	Square Roots	Logarithms	Rationals	Bounded Region

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	$\sqrt{f(x)}$	$\log f(x)$	$\frac{1}{f(x)}$	
Restriction	$f(x) \geq 0$	$f(x) > 0$	Denominator$\neq 0$ $f(x) \neq 0$	
Example	$y = \sqrt{-3x+4} - 1$ $-3x + 4 \geq 0$ $x \leq 4/3$ $Dom = \{x: x \leq 4/3\}$ **or** $(-\infty, \mathbf{4/3}]$	$y = \log(4 + 5x)$ $4 + 5x > 0$ $x > -4/5$ $Dom = \{x: x > -4/3\}$ **or** $(\mathbf{-4/5}, \infty)$	y= $\frac{1}{2x^2+11x-6}$ $2x^2 + 11x - 6 \neq 0$ $(2x - 1)(x + 6) \neq 0$ $\mathbf{x \neq 1/2}$, $\mathbf{x \neq -6}$ $\boldsymbol{Dom} = \mathbb{R} \setminus \{\mathbf{-6, 1/2}\}$	$(x-3)^2+(y+2)^2=16$ Center is (3,-2) with radius of 4 $3 + 4 = 7, 3 - 4 = -1$ $Dom = \{x: -1 \leq x \leq 7\}$ **or** $[\mathbf{-1, 7}]$

Examples: Define a function $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. That is, $f = \{(x, x^2) | x \in \mathbb{R}\}$.

Here x^2 is the image of any $x \in \mathbb{R}$. Note that x^2 is always non-negative for any $x \in \mathbb{R}$ and hence a negative real number has no pre-image under f. For example, there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

Here both the ***domain*** and ***co-domain*** of the function are $\mathbb{R}$ and the ***image*** of f (or range of f) is equal to the set of non-negative real numbers.

Example: Let f be the function defined by

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$$

Find out the domain of f.

The expression of the right-hand side has meaning for all real numbers except when $x = 6$ or $x = 2$. Therefore, the domain of f is the set at all real number other than 6 and 2, that is,

$$Dom(f) = \mathbb{R} - \{2, 6\}$$

Example: Find the range of $f(x) = \frac{x^2}{1+x^2}$.

The inverse of $f(x) = \frac{x^2}{1+x^2}$ is $y = \pm\sqrt{\frac{x}{1-x}}$

provided $\frac{x}{1-x} \geq 0$, that is $0 \leq x < 1$. Thus the range of f is $[0, 1)$.

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Example: Let $f(x) = \frac{\sqrt{x^2-6}}{x+5}$. Find, i) $f(3)$ ii) $f(-1)$ iii) $f(h-5)$ iv) $f(-5)$

Table: Domain and range of some standard functions

Functions	**Domain**	**Range**
Polynomial functions	$\mathbb{R}$	$\mathbb{R}$
Identity function	$\mathbb{R}$	$\mathbb{R}$
Constant function	$\mathbb{R}$	$\mathbb{R}$
Modulus function $\|x\|$	$\mathbb{R}$	$\mathbb{R}^+ \cup \{x\}$
Signum function $sgn(x)$	$\mathbb{R}$	$\{-1, 0, 1\}$
Greatest integer	$\mathbb{R}$	$\mathbb{Z}$
Exponential function a^x	$\mathbb{R}$	$\mathbb{R}^+$
Logarithmic function $\log_a x$	$\mathbb{R}^+$	$\mathbb{R}$
$\boldsymbol{sin\, x}$	$\mathbb{R}$	$[-1, 1]$
$\boldsymbol{cos\, x}$	$\mathbb{R}$	$[-1, 1]$
$\boldsymbol{tan\, x}$	$\mathbb{R} - (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$	$\mathbb{R}$
$\boldsymbol{sec\, x}$	$\mathbb{R} - (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$	$(-\infty, -1] \cup [1, \infty)$
$\boldsymbol{csc\, x}$	$\mathbb{R} - k\pi, k \in \mathbb{Z}$	$(-\infty, -1] \cup [1, \infty)$
$\boldsymbol{cot\, x}$	$\mathbb{R} - k\pi, k \in \mathbb{Z}$	$\mathbb{R}$
$\boldsymbol{sin^{-1}x}$	$(-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\boldsymbol{cos^{-1}x}$	$[-1, 1]$	$[0, \pi]$
$\boldsymbol{tan^{-1}x}$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\boldsymbol{cot^{-1}x}$	$\mathbb{R}$	$(0, \pi)$
$\boldsymbol{sec^{-1}x}$	$\mathbb{R}/(-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\boldsymbol{csc^{-1}x}$	$\mathbb{R}/(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

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2.3.3. Equality of functions

Two functions $f: A \to B$ and $g: A \to C$ are said to be **equal** if their

i) $Dom(f) = Dom(g)$ and

ii) $f(x) = g(x)$ for all elements $x \in Dom(f)$.

Note: f and g can have different codomains but can still be equal.

2.4. Real-Valued Function

Definition: If $f: A \to B$ is a function and $a \in A$ then the image $f(a)$ is also called a value of f at a. If the value of f at each $a \in A$ is a real number, then f is called a ***real-valued function*** on A ; that is, any function from a set A into a subset of the real number system is called a real-valued function on A.

If $f: A \to B$ is a function and $B \subseteq C$, then f can be treated as a function from A into C as well. Therefore, a real-valued function on A is just a function from A into $\mathbb{R}$.

Let f and g be real valued functions on a set A. Then we define the real-valued functions $f + g$, $-f$, $f - g$ and $f \times g$ on A as follows:

1. $(f + g)(x) = f(x) + g(x)$
2. $(-f)(x) = -f(x)$
3. $(f - g)(x) = f(x) - g(x)$
4. $(f \times g)(x) = f(x)f(x)$
5. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0$

Example: Let $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{1}{x^2-1}$, then

$$f(x) + g(x) = \frac{x}{x-1} + \frac{1}{x^2-1} = \frac{x(x^2-1)-(x-1)}{x^2-1} = \frac{x^3-2x+1}{x^2-1} = \frac{(x-1)(x^2+x-1)}{(x-1)(x+1)} = \frac{x^2+x-1}{x+1}, x \neq 1$$

$$f(x)g(x) = \left(\frac{x}{x-1}\right)\left(\frac{1}{x^2-1}\right) = \frac{x}{(x-1)(x^2-1)} = \frac{x}{x^3-x^2-x+1}$$

$$\frac{f(x)}{g(x)} = \left(\frac{x}{x-1}\right) / \left(\frac{1}{x^2-1}\right) = \frac{x}{(x-1)}\frac{(x^2-1)}{1} = \frac{x^3-x}{x-1}$$

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2.4.1. The Domain of the Sum, Difference, Quotient or Product Function

i) $Dom\{f(x) \pm g(x)\} = Dom\{f(x)\} \cap Dom\{g(x)\}$

ii) $Dom\{f(x)g(x)\} = Dom\{f(x)\} \cap Dom\{g(x)\}$

iii) $Dom\left\{\frac{f(x)}{g(x)}\right\} = Dom\{f(x)\} \cap Dom\{g(x)\} - \{g(x) = 0\}$

iv) $Dom\{\sqrt{f(x)}\} = Dom\{f(x)\} \cap \{x: f(x) \geq 0\}$

Example: Let $f(x) = \sqrt{x}$ and $g(x) = |x - 4|$, then $Dom\{f(x)\} = \{x: x \geq 0\} = [0, \infty)$ and $Dom\{g(x)\} = \{x: x \in \mathbb{R}\} = (-\infty, \infty)$. Then,

i). $f(x) + g(x) = \sqrt{x} + |x - 4|$ and $Dom\{f(x) + g(x)\} = Dom\{f(x)\} \cap Dom\{g(x)\} = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$.

ii). $f(x) - g(x) = \sqrt{x} - |x - 4|$ and $Dom\{f(x) - g(x)\} = Dom\{f(x)\} \cap Dom\{g(x)\} = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$.

iii). $f(x)g(x) = \sqrt{x}|x - 4|$ and $Dom\{f(x)g(x)\} = Dom\{f(x)\} \cap Dom\{g(x)\} = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$.

iv). $\frac{f(x)}{g(x)} = \frac{\sqrt{x}}{|x-4|}$ and $Dom\left\{\frac{f(x)}{g(x)}\right\} = Dom\{f(x)\} \cap Dom\{g(x)\} - \{g(x) = 0\} = [0, \infty) \cap (-\infty, \infty) - \{4\} = [0, \infty) - \{4\} = [0, 4) \cup (4, \infty)$.

2.5. Composition of functions

The composition of functions is **the process of combining two or more functions into a single function**.

Definition: Let $f: A \to B$ and $g: B \to C$ be functions. Then the composition of f with g is defined as the function $g \circ f: A \to C$ given by

$$(g \circ f)(a) = g\big(f(a)\big) \text{ for all } a \in A.$$

Note that $g \circ f$ is defined only when the range of f is contained in the domain of g.

If $f: A \to B$ is a function and $g: D \to C$ is another function such that $Range(f) \subseteq D = Dom(g)$, then $g \circ f$ can be defined as a function from A into C.

Example: Let $A = \{1, 2, 3, 4\}, B = \{a, b, c\}$ and $C = \{x, y, z\}$. Let $f: A \to B$and $g: B \to C$ be functions defined by $f = \{(1, a), (2, c), (3, b), (4, a)\}$ and $g = \{(a, y), (b, z), (c, x)\}$. Find $g \circ f$.

Solution: We have $f: A \to B$ and $g: B \to C$ are functions. Then $g \circ f: A \to C$ is given by

$$g \circ f = \{(1, y), (2, x), (3, z), (4, y)\}.$$

Theorem: Let $f: A \to B$ and $g: B \to C$ be functions. Then $Dom(g \circ f) = Dom(f)$ and $codom(g \circ f) = codom(g)$.

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Example: Let $f:\mathbb{Z}\to\mathbb{Z}$ and $g:\mathbb{Z}\to\mathbb{Z}$ be defined by $f(x)=x^2$ and $g(x)=x+2$ for all$x\in\mathbb{Z}$. Show that $g\circ f\neq f\circ g$.

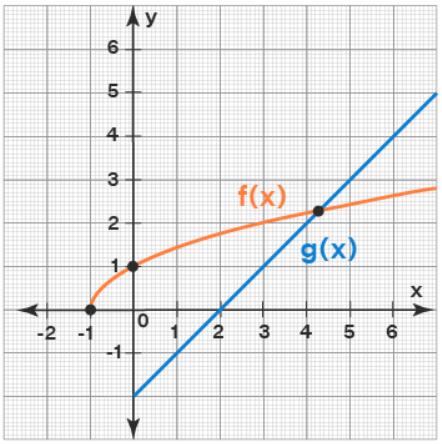
Solution: We have $(g\circ f)(x)=g\big(f(x)\big)=g(x^2)=x^2+2$

$$(f\circ g)(x)=f\big(g(x)\big)=f(x+2)=(x+2)^2$$

Therefore $g\circ f\neq f\circ g$.

Theorem: Let $:A\to B$, $g:B\to C$ and $h:C\to D$ be functions. Then $h\circ(g\circ f)=(h\circ g)\circ f$.

Note: $f\circ g\neq fg$.



Example: Find $f\big(g(5)\big)$ from the following graph.

Solution: From the graph we see, $g(5)=3$ and $f(3)=2$.

Hence, $f\big(g(5)\big)=f(2)=3$.

Example: If $f(x)=x^2$ and $g(x)=5x-1$, then find i) $(f\circ g)(2)$ ii) $(g\circ f)(2)$

Solution: i) $(f\circ g)(2)=f\big(g(2)\big)=f(9)=81$.

ii) $(g\circ f)(2)=g\big(f(2)\big)=g(4)=19$.

Classwork: Let $f(x)=2x$ and $g(x)=2^x$, then find $(f\circ g)(x)$. Answer: $(f\circ g)(x)=2^{x+1}$.

Homework: Let $f(x)=x^2$ and $g(x)=x+2$, then for what value of x does $(f\circ g)(x)=(g\circ f)(x)$?

Answer: $(f\circ g)(x)=(g\circ f)(x)\Rightarrow x^2+4x+4=x^2+2\Rightarrow x=-1/2$.

Homework: Find the domain of $(g\circ f)(x)$ if i) $f(x)=x+5$ and $g(x)=\frac{1}{x-5}$. ii)) $f(x)=|x|$ and $g(x)=\frac{1}{x-1}$.

Answer: i) Since, $(g\circ f)(x)=\frac{1}{x}$, $Dom\{(g\circ f)(x)\}=\{x,x\neq 0\}=(-\infty,0)\cup(0,\infty)=\mathbb{R}\setminus\{0\}$.

ii) $(g\circ f)(x)=\frac{1}{|x|-1}$. Then, $Dom\{(g\circ f)(x)\}=\mathbb{R}\setminus\{-1,1\}=(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

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Homework: If $f(x)=\frac{1}{x-1}$ and $g(x)=\frac{1}{x}$, then find the domain of $(f\circ g)(x)$?

Answer: $Dom\{(f\circ g)(x)\}=\{x, x\neq 0,1\}=\mathbb{R}\setminus\{0\}=(-\infty,0)\cup(0,1)\cup(1,\infty)$.

Homework: If $f(x)=x^2-3x$ and $g(x)=x^2+x-2$, then find the function h such that $g\circ h=f$?

Answer: $h(x)=x-2$.

Composed of function With Itself

Let $f(x)=5x-2$, then

$$(f\circ f)(x)=f\big(f(x)\big)=f(5x-2)=5(5x-2)-2=25x=12.$$

Even Functions and Odd Functions

Even Functions

Definition: Let X be a symmetric set and $f: X\to\mathbb{R}$ a function. Then f is said to be an ***even function*** if $f(-x)=f(x)$ for all $x\in X$.

Note: If the function f is defined in ordered pair, f is even function if $(x,y)\in f$, then $(-x,y)\in f$.

Graphically: Even function is symmetrical around the **y-axis** (Reflect across y-axis)

Example: 1) If $f:\mathbb{R}\to\mathbb{R}$ is the function defined by $f(x)=x^2$ for all $\in\mathbb{R}$, then f is an even function, since, for any $x\in\mathbb{R}$

$$f(-x)=(-x)^2=x^2=f(x)$$

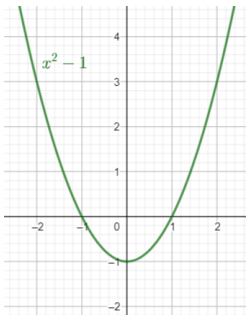
2) If $f:\mathbb{R}\to\mathbb{R}$ is the function defined by $f(x)=|x|$ for all $\in\mathbb{R}$, then f is an even function, since, for any $x\in\mathbb{R}$

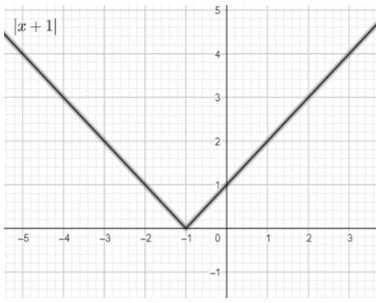
$$f(-x)=|-x|=|x|=f(x).$$

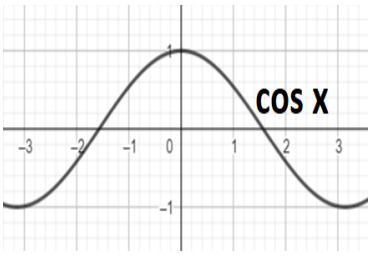
3) Any constant function $f:\mathbb{R}\to\mathbb{R}$ is even, that is, for any $c\in\mathbb{R}$, the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=c$ for all $x\in\mathbb{R}$, is even.

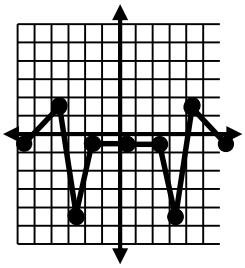
The following functions are even function:

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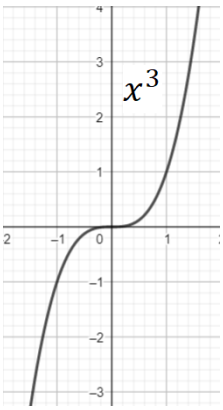


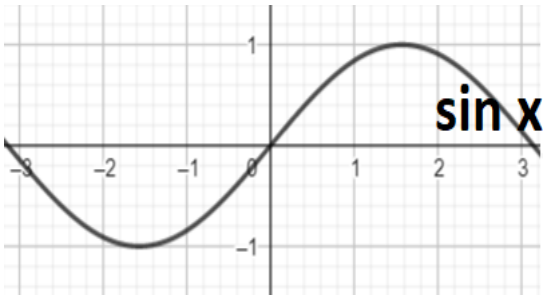


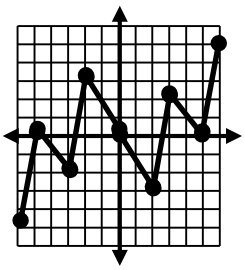
Odd Functions

Definition: Let X be a set and $f: X \to \mathbb{R}$ is a function. Then f is said to be an ***odd function*** if $f(-x) = -f(x)$ for all $x \in X$.
Note: If the function f is defined in ordered pair, f is odd function if $(x, y) \in f$, then $(-x, -y) \in f$.
Graphically**:** Even function is symmetrical around the origin (Reflect across x and y axis)
Examples: (1) The identity function $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3$ for all $x \in \mathbb{R}$, is an odd function, since $f(-x) = (-x)^3 = -x^3 = -f(x)$ for all $x \in \mathbb{R}$.

The following functions are odd function:







Remark: Unlike in integers, a function can be neither even nor odd. For example, consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + x + 1$ for all $x \in \mathbb{R}$. Then f (-1) = 1 and f (1) = 3 and hence

$f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$.

Therefore f is neither even nor odd. Next, note that a function f is both even and odd if and only if $f(x) = 0$ for all x.

Theorem: Let X be a symmetric set and f and g functions of X into $\mathbb{R}$. Then, the product fg is an even function if both f and g are even or both f and g are odd.

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Theorem: For any real-valued functions f and g defined on a symmetric set X, the product fg is an odd function if one of f and g is odd and the other is even.

Theorem: Any function can be expressed as a sum of an even function and an odd function.

Properties of even & odd function:

(a) The graph of an even function is always symmetric about y-axis.

(b) The graph of an odd function is always symmetric about origin.

(c) Product of two even or odd function is an even function.

(e) Product of an even or odd function is an odd function.

(f) Sum of even and odd function is neither even nor odd function.

(d) Sum & difference of two even (odd) function is an even (odd) function.

(g) Zero function, i.e. $f(x) = 0$, is the only function which is both even and odd.

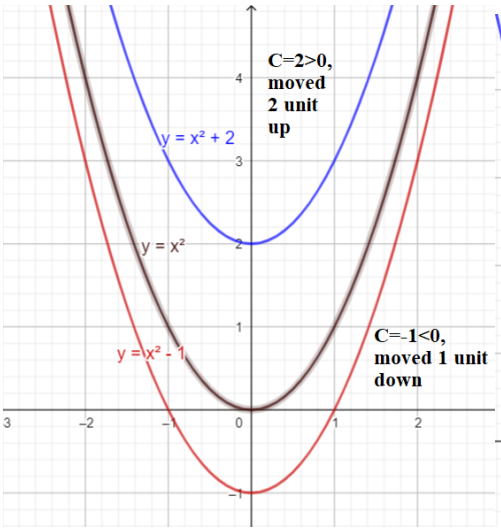
(i)A given function can be expressed as sum of even and odd function.

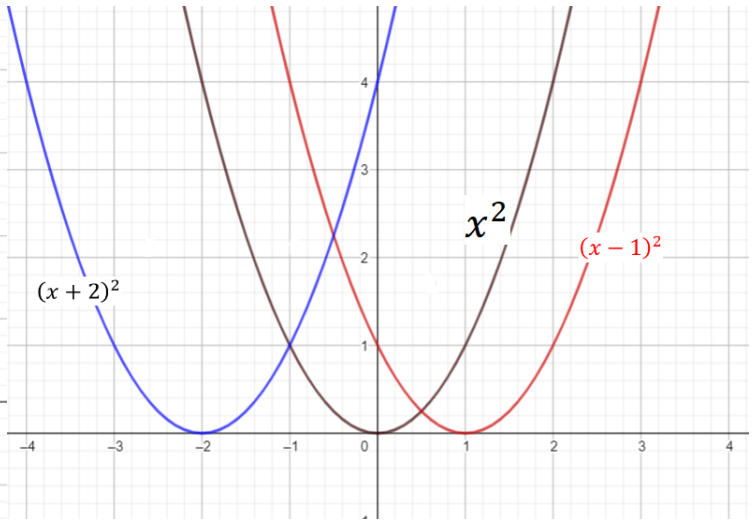
2.6. Graphs of Functions

2.6.1. Transformations of functions

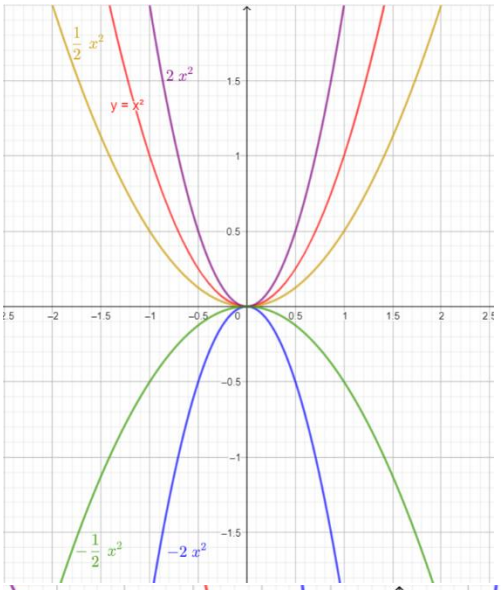
Let $f(x)$ be a function, then

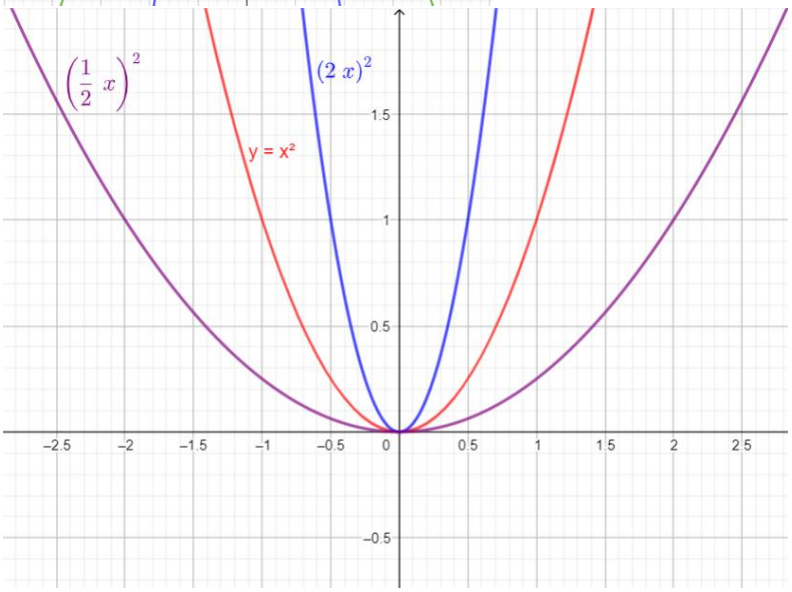
$f(x) + C$	$f(x + C)$	$Cf(x)$	$f(Cx)$
If $C > 0$ moves it up C units	If $C > 0$ moves it left C units	If $C > 1$ stretches it in the y-direction	If $C > 1$ compress it in the x-direction
If $C < 0$ moves it down	If $C < 0$ moves it right	If $0 < C < 1$ compress it	If $0 < C < 1$ stretches it





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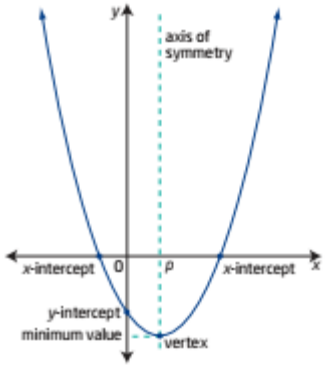
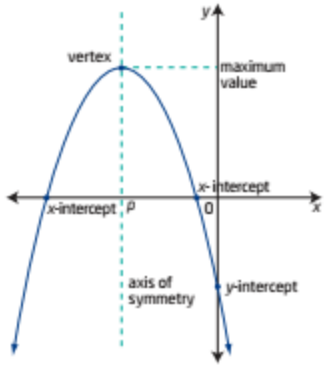
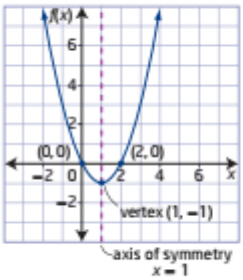
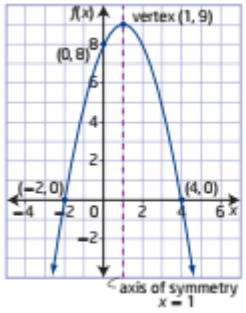




2.6.2. Graph of quadratic functions: $f(x) = ax^2 + bx + c$

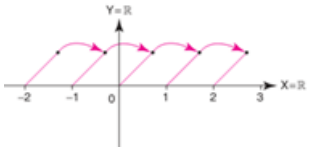
If $a > 0$	If $a < 0$
Opens upward	Opens downward
Has a y-intercept at $(0, c)$	
Vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$	Vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
Has a minimum value $f\left(-\frac{b}{2a}\right)$	Has a maximum value $f\left(-\frac{b}{2a}\right)$
Domain: $\mathbb{R}$	Domain: $\mathbb{R}$

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Range: $\left[f\left(-\frac{b}{2a}\right), \infty\right)$	Range: $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$
	
b) $f(x) = x^2 - 2x$  • opens upward • vertex: (1, −1) • minimum value of y of −1 whe $x = 1$ • axis of symmetry: $x = 1$ • y-intercept occurs at (0, 0) and value of 0 • x-intercepts occur at (0, 0) and (and have values of 0 and 2 • domain: all real numbers, or $\{x \mid x \in R\}$ • range: all real numbers greater or equal to −1, or $\{y \mid y \geq -1,$	c) $f(x) = -x^2 + 2x + 8$  • opens downward • vertex: (1, 9) • maximum value of y of 9 when • axis of symmetry: $x = 1$ • y-intercept occurs at (0, 8) and value of 8 • x-intercepts occur at (−2, 0) and and have values of −2 and 4 • domain: all real numbers, or $\{x \mid x \in R\}$ • range: all real numbers less tha equal to 9, or $\{y \mid y \leq 9, y \in R\}$

2.7. Power function, Absolute value function, Signum function and greatest integer function

Identity function: A function $f: A \to A$ is called an identity function if $f(x) = x$ for all $x \in A$ and is denoted by I_A.



$$Domain\ of\ f = \mathbb{R}$$

$$Range\ of\ f = \mathbb{R}$$

Periodic function: Let A be a subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ a function. A positive real number p is called a period of f if $f(x + p) = f(x)$ whenever x and $x + p$ belong to A. A function with a period is called ***periodic function***. Among the periods of f, the least one (if it exists) is called the least period.

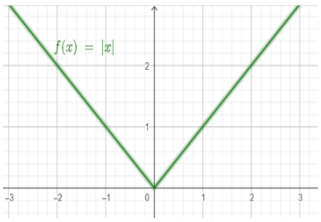
Step function (greatest integer function): Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x]$ for all $x \in \mathbb{R}$ where $[x]$ is the largest integer less than or equal to x. This function f is called step function.



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The Modulus function: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = |x| = \begin{cases} -x, if\ x < 0 \\ x, if\ x \geq 0 \end{cases}$$



for all $x \in \mathbb{R}$ is called the **modulus** function.

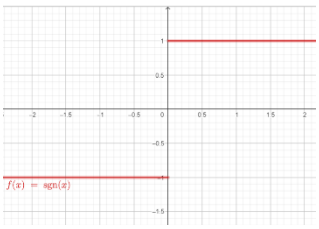
Domain of $f = \mathbb{R}$ and Range of $f = \mathbb{R}^+ \cup \{0\}$.

Homework: Solve $|x + 3| = 2x + 1$.

Answer: $x = 2$. (Solving analytically $x = -\frac{4}{3}$ is also solution, but verifying it does not keep the equality.)

Signum function: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

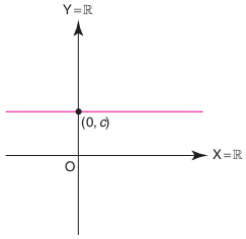
$$f(x) = \begin{cases} -1 & if \quad x < 0 \\ 0 & if \quad x = 0 \\ 1 & if \quad x > 0 \end{cases}$$



is called Signum function and is written as $sign(x)$.

Domain of $f = \mathbb{R}$ and Range of $f = \{-1, 0, 1\}$.

Constant function: The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = C$ for all $x \in \mathbb{R}$ where C is a constant in $\mathbb{R}$, is a constant function.



Domain of $f = \mathbb{R}$

Range of $f = \{C\}$

2.8. Exponential functions and their graphs

2.8.1. Exponential function

Exponential function is a function of the form $y = a^x$, where a is a constant and x is a variable.

Properties of Exponents

Property	Definition $(n, m \in \mathbb{Z})$	Example
Product	$a^m a^n = a^{m+n}$	$2^3 2^4 = 2^{3+4} = 2^7$
Power of a power	$(a^m)^n = a^{mn}$	$(x^2)^3 = x^{3(2)} = x^6$

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Power of a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3} = \frac{27}{125}$
Power of a product	$(ab)^n = a^n b^n$	$(2x)^3 = 2^3 x^3 = 8x^3$
quotient	$\frac{a^m}{a^n} = a^{m-n},\ b \neq 0$	$\frac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$

Definition: For any real number $a > 0$ and any integer $n > 1$

i) $a^{\frac{1}{n}} = \sqrt[n]{a}$

ii) $a^{m/n} = \sqrt[n]{a^m}$

Note: i) For $a < 0$ and n is odd, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

ii) For $a, b \neq 0$ and $n > 0$, i) $a^{-n} = \frac{1}{a^n}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

iii) For any non-zero number , $a^0 = 1$.

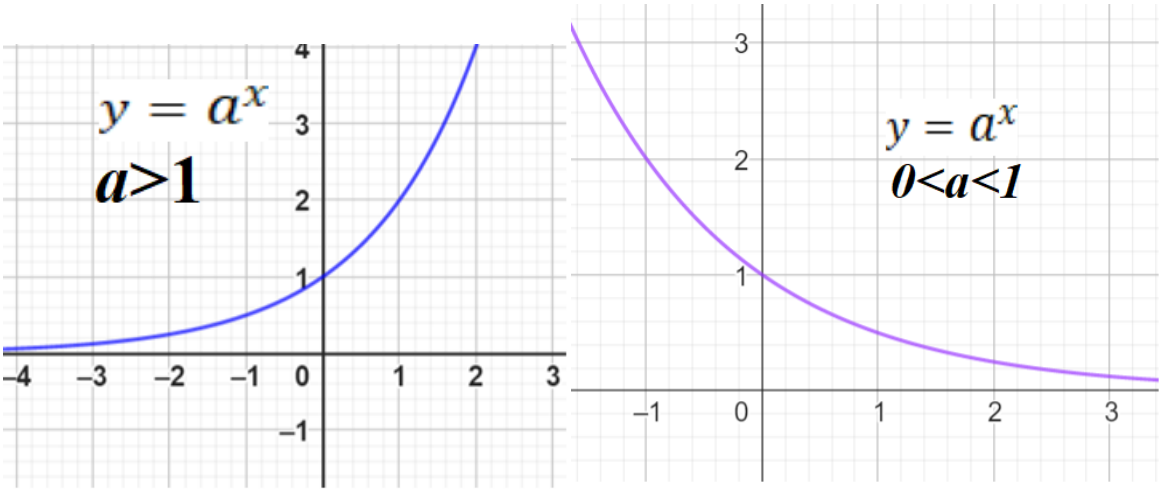
Example: Evaluate $\frac{3^7 9^4}{\sqrt{27^6}}$

Solution: $\frac{3^7 9^4}{\sqrt{27^6}} = \frac{3^7 3^{2(2)}}{(27^6)^{\frac{1}{2}}} = \frac{3^{7+4}}{27^{6\left(\frac{1}{2}\right)}} = \frac{3^{11}}{27^3} = \frac{3^{11}}{3^9} = 3^{11-9} = 3^2 = 9$

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2.8.2. Graph of Exponential function

The graph of an exponential function , such as $y = a^x$, is increasing for $a > 1$, decreasing for $0 < a < 1$, and neither increasing nor decreasing for $a = 1$.



$y = a^x, a > 1$

- The function is defined for all values of x. Therefore, the domain is $\{x|x \in \mathbb{R}\}$.
- The function has only positive values for y.Therefore, the range is $\{y|y > 0, y \in \mathbb{R}\}$.
- The graph never intersects the x-axis, so there is no x-intercept.
- The graph crosses the y-axis at $y = 1$, so the y-intercept is $y = 1$.
- The values of y increase as the values of x increase. Therefore, the function is increasing over its domain.
- Since the graph approaches the line $y = 0$ as the values of x get very small, $y = 0$ is the horizontal asymptote.
- When $x \to -\infty$ then $f(x) \to 0$ and when $x \to \infty$ then $f(x) \to \infty$.

$y = a^x, 0 < a < 1$

- The function is defined for all values of x. Therefore, the domain is $\{x|x \in \mathbb{R}\}$.
- The function has only positive values for y.Therefore, the range is $\{y|y > 0, y \in \mathbb{R}\}$.
- The graph never intersects the x-axis, so there is no x-intercept.
- The graph crosses the y-axis at $y = 1$, so the y-intercept is $y = 1$.
- The values of y decrease as the values of x increase. Therefore, the function is decreasing over its domain.
- Since the graph approaches the line $y = 0$ as the values of x get very small, $y = 0$ is the horizontal asymptote.
- When $x \to -\infty$ then $f(x) \to \infty$ and when $x \to \infty$ then $f(x) \to 0$.

☞ **Why is the definition of an exponential function restricted to positive values of a?**

Classwork: Solve i) $16^{2m-3} = 32^{m,+3}$ ii) $\left(\frac{1}{8}\right)^{2x+1} = 32^{x-3}$

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2.5.4. Logarithmic functions, graphs of logarithmic functions and equations involving logarithms

For the exponential function $y = a^x$, the inverse is $x = a^y$. This inverse is called a **logarithmic function**. It is written as $y = \log_a x$, where a is a positive number other than 1.

Example: Solve $\log_{10} 0.001$

Solution: Let $\log_{10} 0.001 = x$. Then in exponential form

$10^x = 0.001 \Longrightarrow 10^x = 10^{-3} \Longrightarrow x = -3$. Therefore, $\log_{10} 0.001 = -3$.

Example: Determine the value of x for i) $\log_{64} x = \frac{2}{3}$ ii) $\log_x 64 = 2$

Solution: i) Expressing in exponential form $\log_{64} x = \frac{2}{3} \Longrightarrow 64^{\frac{2}{3}} = x \Longrightarrow \left(64^{\frac{1}{3}}\right)^2 = x \Longrightarrow 4^2 = x \Longrightarrow x = 16$

ii) Expressing in exponential form $\log_x 64 = 2 \Longrightarrow x^2 = 64 \Longrightarrow x^2 = 2^6 \Longrightarrow x = 6$

2.8.3. Laws of logarithm

If a, b, c, x and y are positive numbers and $a \neq 1, b \neq 1$ then

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b x^k = k \log_b x$ for any real number k
4. $\log_b b = 1$, $\log_b 1 = 0$
5. $\log_b a = \frac{\log_c a}{\log_c b}$
6. $b^{\log_b x} = x$

- $\log_e x = \ln x$ is called the natural logarithm of x
- The function $f(x) = b^x, b > 0$ and $b \neq 1$ defines an exponential function. The function $f(x) = e^x$ is called the natural exponential function.
- Logarithms to base 10 are called common logarithms.

Property of equality for logarithmic equations

If b, x and y are positive numbers with $b \neq 1$, then

$\log_b x = \log_b y$ if and only if $x = y$.

Example: If $\log_3 6 = \log_3 x$, then $x = 6$.

Homework: Find x if:

i) $\log_{49} 1/7 = x$ ii) $\log_{1/3} 9 = x$

iii) $\log_2(x^2 + 3x + 2) = 1$ iv) $\log_4(x + 1) + \log_4(x + 3) = 1$

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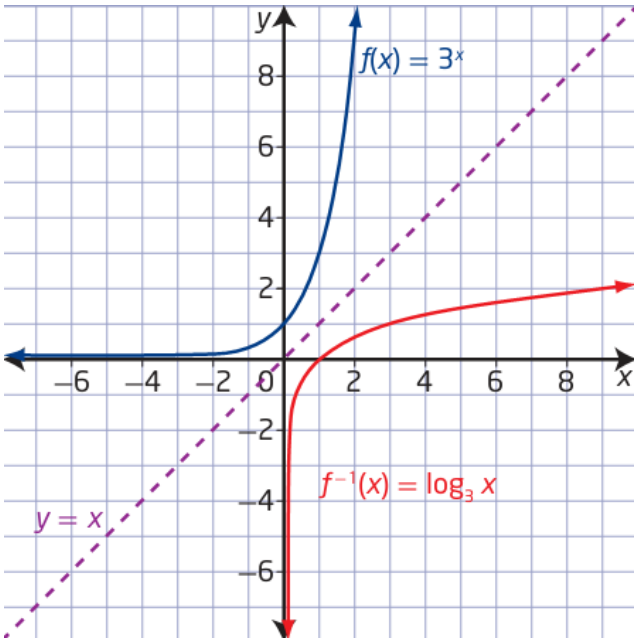
2.8.3. Graph of Logarithmic function

Example: Graph the inverse of an exponential function $y = 3^x$.

Solution: The inverse of $y = 3^x$ is $y = \log_3 x$. Then set up tables of values for both the exponential function and its inverse.

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



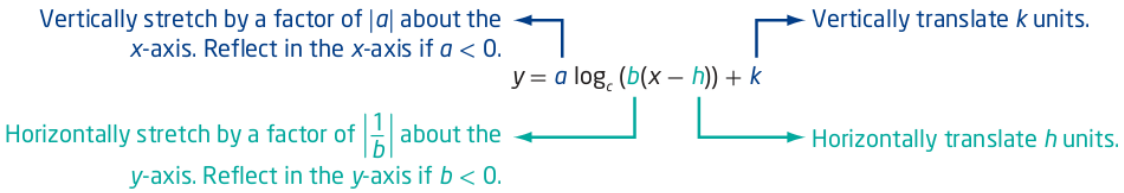
The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.

For the logarithmic function $y = \log_a x, a > 0, a \neq 1$,

- the domain is $\{x | x > 0, x \in \mathbb{R}\}$
- the range is $\{y | y \in \mathbb{R}\}$
- the x- intercept is 1
- the vertical asymptote is $x = 0$, or the y-axis.

2.8.4. Transformation of logarithmic functions graph

The effect of the parameters $a, b, h,$ and k in $y = a \log_c b(x - h) + k$ on the graph of the logarithmic function $y = \log_c x$



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2.9. Classification of functions

2.9.1. Injection/One-One function

Definition: A function $f: A \to B$ is called an **injection** or "**one-one function**" if $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$ for any $a_1, a_2 \in A$.

Or, If $a_1 \neq a_2$ in A then $f(a_1) \neq f(a_2)$ for any $a_1, a_2 \in A$.

Example: Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 2$ for all $x \in \mathbb{Z}$.

Then f is an injection, since, for any $x, y \in \mathbb{Z}$

$$f(x) = f(y) \Longrightarrow x + 2 = y + 2 \Longrightarrow x = y$$

Example: Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{Z}$.

Then f is not an injection, since two distinct elements have the same image; for example, $1 \neq -1$ but $f(1) = 1^2 = (-1)^2 = f(-1)$.

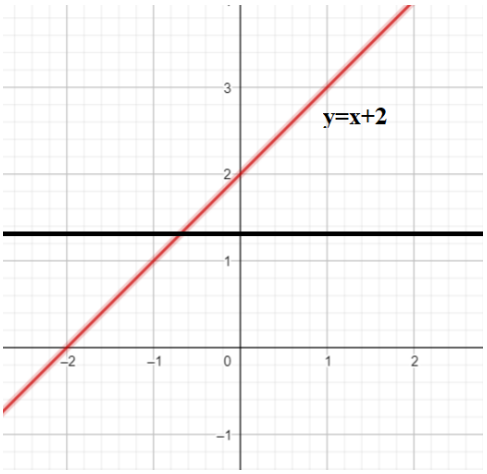
Example: Determine if the function $F = \{(-1, 1), (0, 2), (2, 1)\}$ is one-to-one.

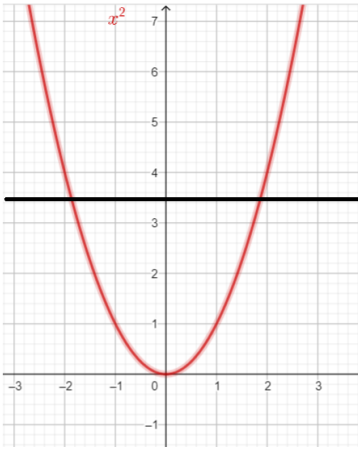
Solution: Here $F(-1) = 1$ and $F(2) = 1$, but since $-1 \neq 2$, we have established that F is not one-to-one.

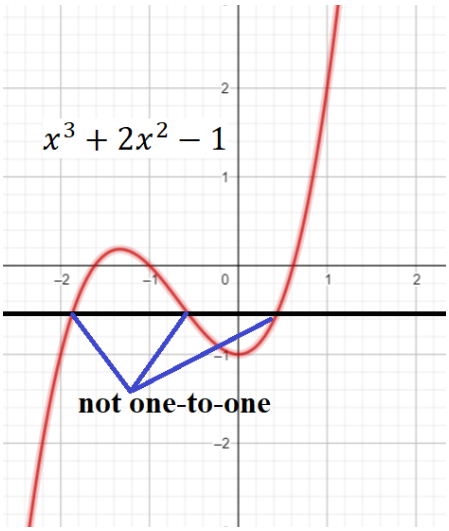
Horizontal Line test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the horizontal line test. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

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Theorem: $f: A \to B$ and $g: B \to C$ be functions. Then the following hold.

1. If f and g are injections, then so is $g \circ f$

2. If $f \circ g$ is an injection, then f is an injection.

2.9.2. Surjection/Onto function

Definition: A function $f: A \to B$ is called a **surjection** or "**onto function**" if the range of f is equal to the co-domain B; that is, for each $b \in B$, $b = f(a)$ for some $a \in A$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$. Then for any element y in the co-domain $\mathbb{R}$, we have $\frac{y-1}{2}$ is in the domain $\mathbb{R}$ and

$$f\left(\frac{y-1}{2}\right) = \frac{2(y-1)}{2} + 1 = y.$$

Therefore f is a surjection. Note that f is an injection also, since

$$f(x) = f(y) \Longrightarrow 2x + 1 = 2y + 1 \Longrightarrow x = y.$$

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Example: Let $\mathbb{N}$ be the set of all non-negative integers. Define $f:\mathbb{Z}\to\mathbb{N}$ by $f(x)=|x|$ for all $x\in\mathbb{Z}$. Then f is a surjection, since $f(x)=x$ for all $x\in\mathbb{N}$ and $\mathbb{N}\subseteq\mathbb{Z}$.

However, f is not an injection since $f(-1)=|-1|=1=f(1)$ and $-1\neq 1$.

(3) Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2+1$ for all $x\in\mathbb{R}$. Then f is neither an injection nor a surjection. It is not an injection, since $f(-1)=2=f(1)$ but $-1\neq 1$. f is not a surjection, since we cannot find an element x in such that $x^2+1=0$; that is $f(x)=0$.

(4) Define $f:\mathbb{Z}\to\mathbb{R}$ by $f(x)=x+2$ for all $x\in\mathbb{Z}$. Then f is an injection and is not a surjection. It is not an injection, since $f(-1)=2=f(1)$ and $-1\neq 1$. f is not a surjection, since we cannot find an integer x such that $f(x)=1/2$. Note that $f(x)=x+2$ is always an integer for any integer x.

Theorem: Let $f:A\to B$ and $g:B\to C$ be functions. Then the following hold.

1. If f and g are surjections, then so is $g\circ f$
2. If $f\circ g$ is a surjections, then f is a surjections.

It is a convention that, when $f:A\to B$ is a surjection, we often denote this by saying "f is a function of A onto B" or f is a surjection of A onto B.

2.9.3. Bijection

Definition: A function $f:A\to B$ is said to be a **bijection** or a one-one and onto function or a one-to-one function if f is both injective and surjective.

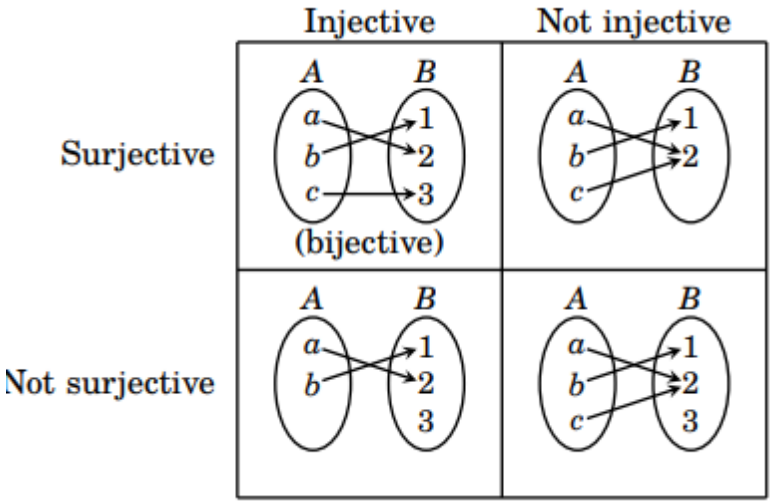
Let E be the set of all even integers and $\mathbb{Z}$ the set of all integers. Define $f:E\to\mathbb{Z}$ by

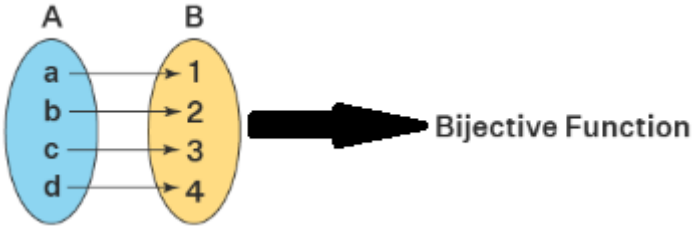
$$f(x)=\begin{cases}2y & if\ x=4y\\ y & \text{if } x=2y \text{ and } y \text{ is odd}\end{cases}.$$

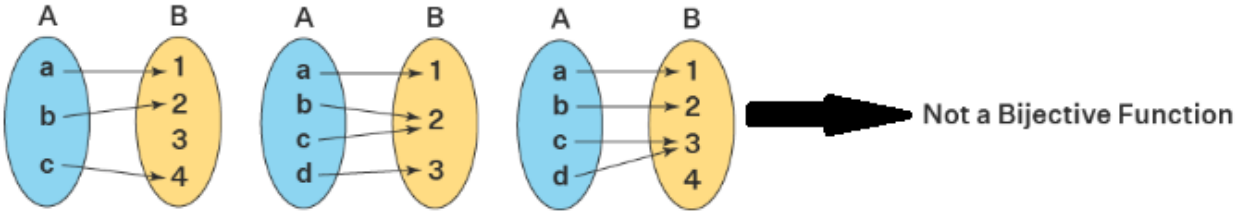
Then f is a bijection. One can verify that $f(0)=0, f(2)=1, f(4)=2, f(6)=3, f(8)=4, f(-2)=-1, f(-4)=-2, .. f(-n)=-f(n)$.

Theorem: If $f:A\to B$ and $g:B\to C$ are bijections, then $g\circ f:A\to C$ is a bijection.

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From the above examples of bijective function, we can observe that every element of set B has been related to a distinct element of set A. The non-bijective functions have some element in set B which do not have a pre-image in set A, or some of the elements in set B is the image for more than one element in set A.

The bijective functions need to satisfy the following four conditions.

- ✓ Every element of set A must be paired with an element of set B.
- ✓ The element of set A must not be paired with more than one element of set B.
- ✓ Each element of set B must be paired with an element of set A.
- ✓ The element of set B must not be paired with more than one element of set A.

Note: A function $f: A \to B$ is injective when, for each $x_1, x_2 \in A$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Note: A function $f: A \to B$ is surjective when, if for each $y \in B$ there exists some x in X such that $f(x) = y$.

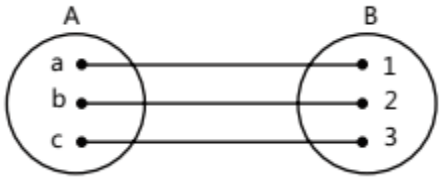
Note: A function $f: A \to B$ is bijective if f is both one-one and onto.
Example: $f(x) = e^{-x}$ is one–one function (Injective mapping)

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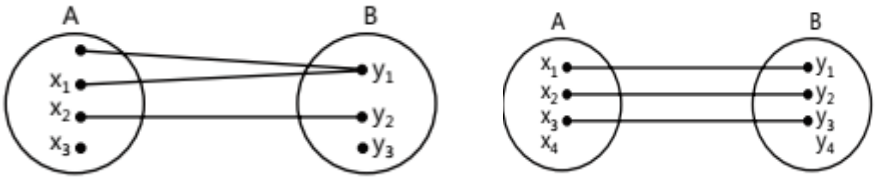
Solution: Since, $e^{-x} = e^{-y} \Longrightarrow -x = -y \Longrightarrow x = y$.
Example: Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$, $f: A \to B$ defined as $f(x) = x^2$ is surjective.

Example: Let $A = \{-1,\ 2, 3\}$ and $B = \{1, 4, 9\}$, $f: A \to B$ defined as $f(x) = x^2$ is bijective.

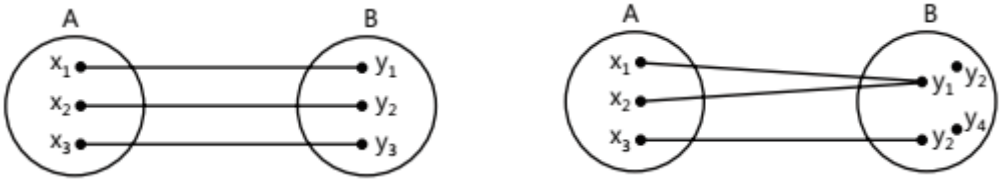
Diagrammatically an injective mapping is shown as follows:



Diagrammatically surjective mapping is shown as follows:



Diagrammatically into function is shown as follows:



2.10. Inverse function

If f is a one-to-one function with domain A and range B, we can define an inverse function f^{-1} (with domain B) by the rule

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

This is a sound definition of a function, precisely because each value of y in the domain of f^{-1} has exactly one x in A associated to it by the rule $y = f(x)$.

Note that the domain of f^{-1} equals the range off and the range of f^{-1} equals the domain of f.

Note: If f^{-1} is the inverse of function f, then the following holds from the definition if $f^{-1}(y) = x$ then $f(x) = y$.

Note: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Example: If $f = \{(a, 2), (b, 3), (c, 3)\}$ is a function from $A = \{a, b, c\}$ into $B = \{2, 3\}$, find f^{-1}.

Solution: $f^{-1} = \{(2, a), (3, b), (3, c)\}$, f^{-1} is not a function. Notice also, f is not injective.

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Note: f^{-1} exists if f is injective and surjective.

2.10.1. Steps for finding the Inverse of a One-to-one Function

1. Write $y = f(x)$.
2. Interchange x and y.
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$.

Example: Let $f(x) = \frac{2x+1}{x-3}$, find a formula for $f^{-1}(x)$.

Answer: $f^{-1}(x) = \frac{3x+1}{x-2}$.

$$(f \circ f^{-1})(x) = f\left(f^{-1}(x)\right) = f\left(\frac{3x+1}{x-2}\right) = \frac{2\left(\frac{3x+1}{x-2}\right)+1}{\frac{3x+1}{x-2}-3} = \frac{7x}{7} = x.$$

Example: Find inverse of $f(x) = 4x^2 + 12x + 15$.

Solution: Solving for y from $x = 4y^2 + 12y + 15$ we obtain, $(2y+3)^2 + 6 = x$

$\Longrightarrow 2y + 3 = \sqrt{x-6}$. Thus, $f^{-1}(x) = \frac{\sqrt{x-6}-3}{2}$.

Example: Find the inverse of the function $\frac{1-2x}{5}$ one-to-one functions. Check your answers analytically using function composition.

Solution: Let $f(x) = \frac{1-2x}{5}$. Then $y = \frac{1-2x}{5}$. Then switch x and y. $x = \frac{1-2y}{5}$. Solving for y, $y = \frac{1-5x}{2}$. Then we have

$$(f \circ f^{-1})(x) = f\left(f^{-1}(x)\right) = f\left(\frac{1-5x}{2}\right) = \frac{1-2\left(\frac{1-5x}{2}\right)}{5} = \frac{1-(1-5x)}{5} = \frac{5x}{5} = x.$$

Similarly,

$$(f^{-1} \circ f)(x) = f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{1-2x}{5}\right) = \frac{1-5\left(\frac{1-2x}{5}\right)}{2} = \frac{1-(1-2x)}{2} = \frac{2x}{2} = x$$

Therefore, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ for all x in the domain of f.

Homework: Find the inverse of $y = 2^{x+3}$ ii) $y = \ln(x-2) + 5$

2.10.2. Graph of inverse function

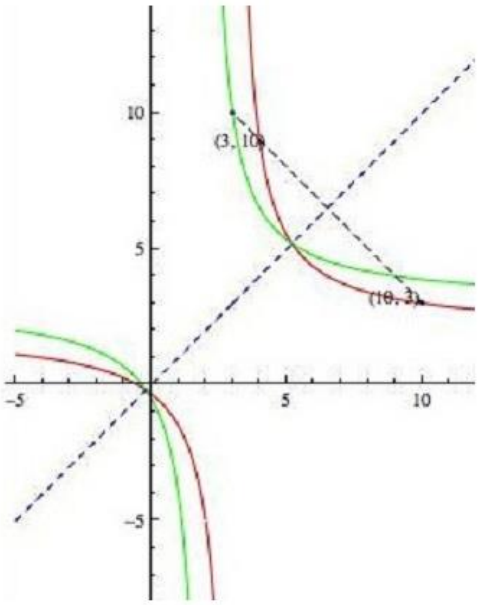
Since the equation $y = f^{-1}(x)$ is the same as the equation $x = f(y)$, the graphs of both equations are identical. To graph the equation $x = f(y)$, we note that this equation results from switching the roles of x and y in the equation $y = f(x)$. This transformation of the equation results in a

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transformation of the graph amounting to reflection in the line $y = x$. Thus the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$ and vice versa.

Note: The reflection of the point (x_1, y_1) in the line $y = x$ is (y_1, x_1). Therefore if the point (x_1, y_1) is on the graph of $y = f^{-1}(x)$, we must have (y_1, x_1) on the graph of $y = f(x)$.

The graphs of $f(x) = \frac{2x+1}{x-3}$ and $f^{-1}(x) = \frac{3x+1}{x-2}$ are shown below.



Example: If $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$, then $f^{-1}(-2) \cup f^{-1}(17)$

(A) $\{\pm 4\}$ (B) $\{\pm 1, \pm 4\}$ (C) $\{4\}$ (D) $\{1, 4\}$

Solution: Here $f^{-1}(-2) = \{x \in \mathbb{R} | f(x) = -2\} = \{x \in \mathbb{R} | x^2 + 1 = -2\} = \emptyset$.

$$f^{-1}(17) = \{x \in \mathbb{R} | f(x) = 17\} = \{x \in \mathbb{R} | x^2 + 1 = 17\} = \{x \in \mathbb{R} | x^2 = 16\} = \{\pm 4\}$$

2.11. Polynomial functions

Definition: The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$ is a polynomial function of degree n where n is a nonnegative integer. The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial function. The number a_n is called **the leading coefficient** and a_0 is called the **constant coefficient.**

Classwork: If $f(x) = 2x^3 - 5x^2 - 4x + 7$: *Leading coefficient*___, *constant*___, and *degree*_____.

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2.11.1. Graph of polynomial

Two-Step Process for Determining the End Behavior of a Polynomial Function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Step 1: Determine the sign of the leading coefficient a_n:

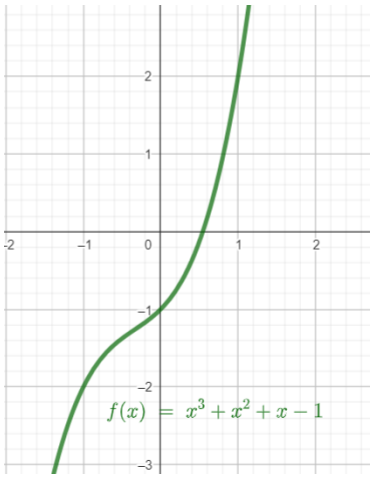
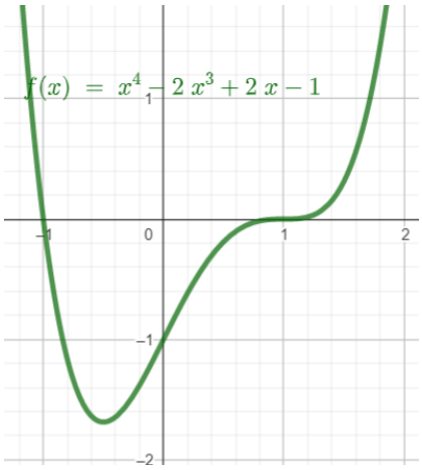
- If $a_n > 0$ right-hand end of graph finishes in an upward direction.
- If $a_n < 0$ right-hand end of graph finishes in an downward direction.

Step 2: Determine the degree

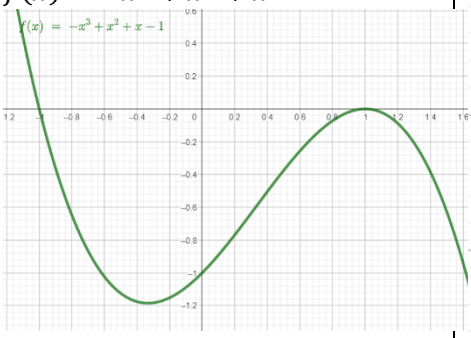
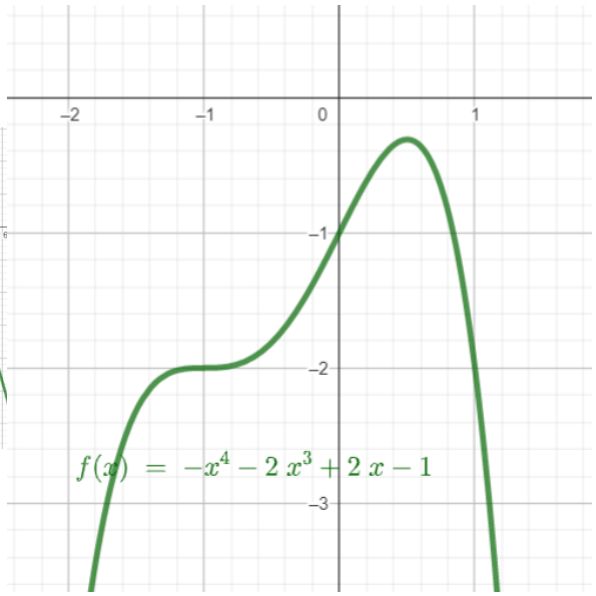
If the degree *n* is odd, the graph has opposite left-hand and right-hand end behavior, that is, the graph "starts" and "finishes" in **opposite** directions.

- Odd degree polynomials have opposite left-hand and right-hand end behavior.
- Even degree polynomials have the same left-hand and right hand end behavior.
- If the degree *n* is even, the graph has the same left-hand and right-hand end behavior, that is, the graph "starts" and "finishes" in the **same** direction.

2.11.2. End Behavior of Polynomial Functions

	Odd degree	Even Degree
Positive leading coefficients	$f(x) = x^3 + x^2 + x - 1$  As $x \to \infty, f(x) \to \infty$ As $x \to -\infty, f(x) \to -\infty$	$f(x) = x^4 - 2x^3 + 2x - 1$  As $x \to \infty, f(x) \to \infty$ As $x \to -\infty, f(x) \to \infty$

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Negative leading coefficients	$f(x) = -x^3 + x^2 + x - 1$  As $x \to \infty, f(x) \to -\infty$ As $x \to -\infty, f(x) \to \infty$	$f(x) = x^4 - 2x^3 + 2x - 1$  As $x \to \infty, f(x) \to -\infty$ As $x \to -\infty, f(x) \to -\infty$

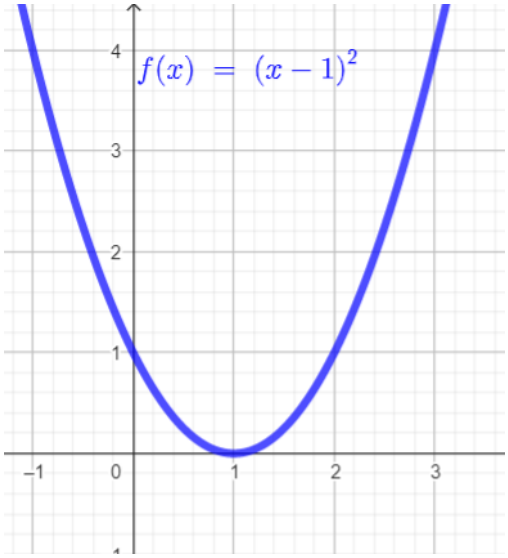
Note: i) The graph of a polynomial function of degree n has at most $n-1$ turning points and intersects the x-axis at most n times.

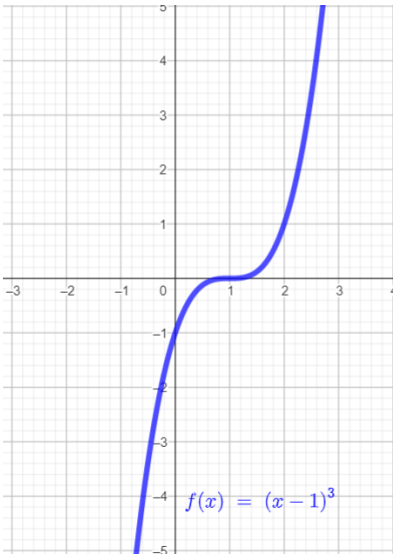
ii) The graph of every polynomial function has no sharp corners, it is a smooth and continuous curve.

Example: For the function $f(x) = (x-1)^2$, $x = 1$ is called a zero of multiplicity **two** since the factor $(x-1)$ occurs two times in the factorization of f.

Example: For $g(x) = (x-1)^3$, $x = 1$ is a zero of multiplicity 3.

The graph of a polynomial function will touch the x-axis at a real zero of even multiplicity and will cross the x-axis at a real zero of odd multiplicity.





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The Shape of the Graph of a Polynomial Function Near a Zero of multiplicity *k*.

Suppose c is a real zero of a polynomial function f of multiplicity k, that is, $(x-c)^k$ is a factor of f. Then the shape of the graph of f near c is as follows:

- If $k > 1$ is even, then the graph touches the *x*-axis at *c*.
- If $k > 1$ is odd, then the graph crosses the *x*-axis at *c*.

Classwork:

i. $f(x) = -5x^5 - 6x^3 + 3x - 7$
Degree _____
Leading Coefficient _____
As $x \to +\infty$ *then* $f(x) \to$ _____
As $x \to -\infty$ *then* $f(x) \to$ _____
Left/right-hand end behavior? _____

ii. $f(x) = 6x^8 - 5x^3 - 3x + 7\ 4$
Degree _____
Leading Coefficient _____
As $x \to +\infty$ *then* $f(x) \to$ _____
As $x \to -\infty$ *then* $f(x) \to$ _____
Left/right-hand end behavior? _____

2.12. Rational Function

***Definition*:** A **rational function** is a function of the form $f(x) = \frac{g(x)}{h(x)}$ where g and h are polynomial functions such that $h(x) \neq 0$. If $h(x) = C$ where C is a real number, then we will consider the function $f(x) = \frac{g(x)}{h(x)} = \frac{g(x)}{C}$ to be a polynomial.

2.12.1. Graphing Rational Functions

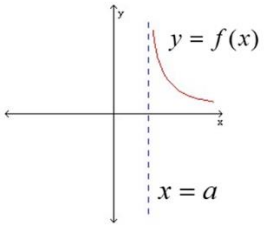
Vertical asymptote

***Definition*:** The vertical line $x = a$ is a **vertical asymptote** of a function $y = f(x)$ if *at least* one of the following occurs:

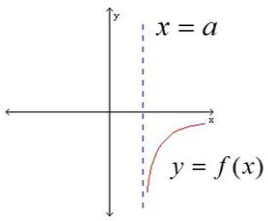
i) $f(x) \to \infty$ as $x \to a^+$ **ii)** $f(x) \to -\infty$ as $x \to a^+$

iii) $f(x) \to \infty$ as $x \to a^-$ **iv)** $f(x) \to -\infty$ as $x \to a^-$

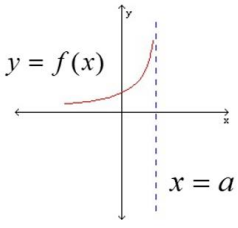
The figures below illustrate each of these cases.



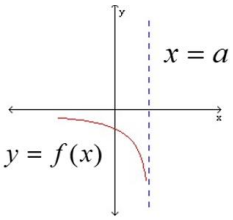
$f(x) \to \infty$ as $x \to a^+$



$f(x) \to -\infty$ as $x \to a^+$



$f(x) \to \infty$ as $x \to a^-$



$f(x) \to -\infty$ as $x \to a^-$

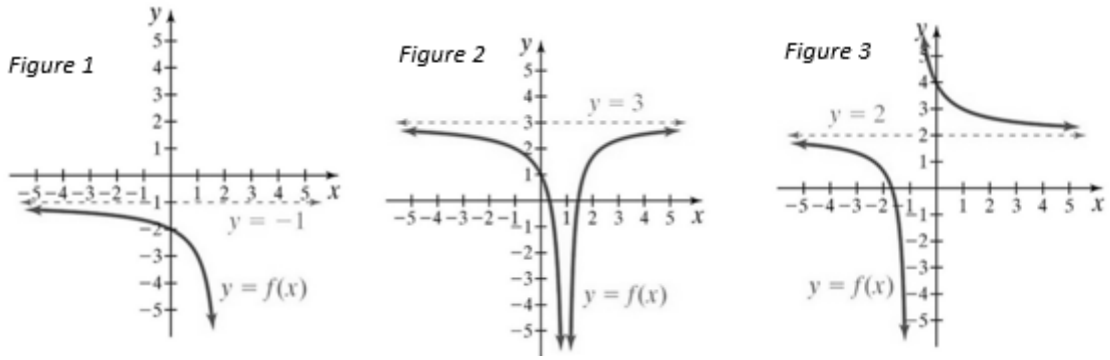
A rational function of the form $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ have no common factors will have a vertical asymptote at $x = a$ if $h(a) = 0$.

Note: It is essential to cancel any common factors before locating the vertical asymptotes.

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Horizontal Asymptotes

***Definition*:** A horizontal line $y = L$ is a **horizontal asymptote** of a function f if the values of $f(x)$ approach some fixed number L as the values of x approach ∞ or $-\infty$.



- ✓ In Figure 1 above on the left, the line $y = -1$ is a horizontal asymptote because the values of $f(x)$ approach -1 as x approaches $-\infty$.
- ✓ In Figure 2 above in the middle, the line $y = 3$ is a horizontal asymptote because the values of $f(x)$ approach 3 as x approaches $\pm\infty$.
- ✓ In Figure 3 above on the right, the line $y = 2$ is a horizontal asymptote because the values of $f(x)$ approach 2 as x approaches $\pm\infty$.

Finding Horizontal Asymptotes of a Rational Function

Let $f(x) = \frac{g(x)}{h(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$, $a_n \neq 0, b_m \neq 0$ where f is written in lowest terms, n is the degree of g, and m is the degree of h.

- If $m > n$, then $y = 0$ is the horizontal asymptote.
- If $m = n$, then the horizontal asymptote is $y = \frac{a_n}{b_m}$, the ratio of the leading coefficients.
- If $m < n$, then there are no horizontal asymptotes.

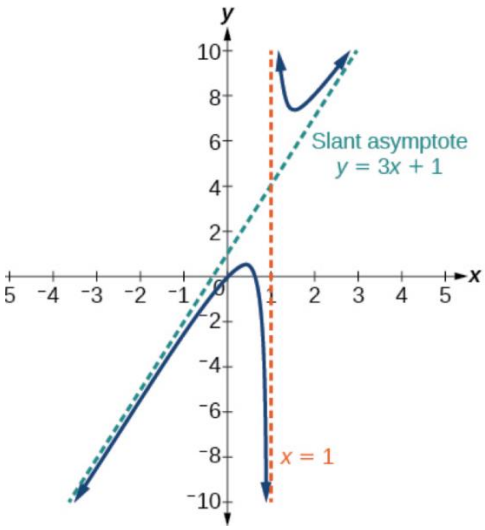
If degree(nume)<degree(denom) then the horizontal asymptote is $y = 0$.	If degree(nume)=degree(denom) then $\frac{a_n}{b_n}$ is the horizontal asymptote where a_n and b_n are the lead coefficients of the numerator and denominator.	If degree(nume)>degree(denominator), there is no horizontal asymptote.
$f(x) = \frac{x}{x^2-1}$ HA is $y = 0$. $f(x) = \frac{5x^2-4}{x^5-1}$ HA is $y = 0$.	$f(x) = \frac{x+3}{2x-1}$ HA is $y = \frac{1}{2}$. $f(x) = \frac{9x^3+x+3}{3x^3-1}$ HA is $y = 3$.	$f(x) = \frac{x^3+3}{2x-1}$ has no HA.

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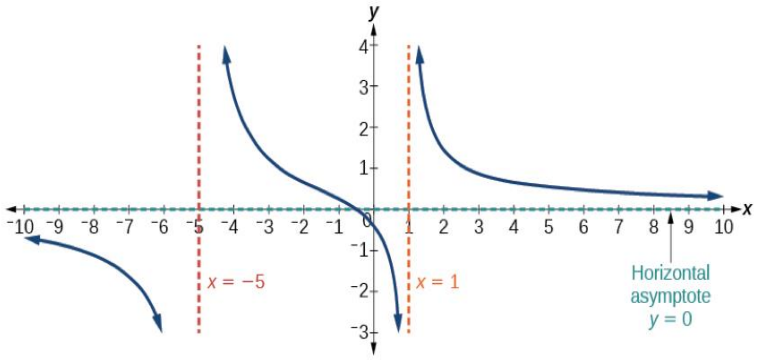
Oblique asymptote:

Example: Graph $f(x) = \frac{3x^2-2x+1}{x-1}$.

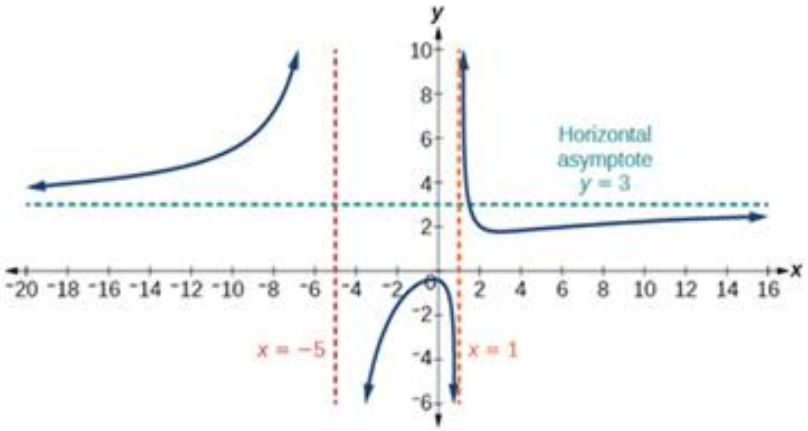
Solution: Since $\frac{3x^2-2x+1}{x-1} = 3x + 1 + \frac{2}{x-1}$, the line $y = 3x + 1$ is a oblique/slant asymptote.



Example: Graph $f(x) = \frac{4x+2}{x^2+4x-5}$.



Example: Graph $f(x) = \frac{3x^2+2}{x^2+4x-5}$



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Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote. It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction.

Examples: For the functions listed, identify the horizontal or slant asymptote.

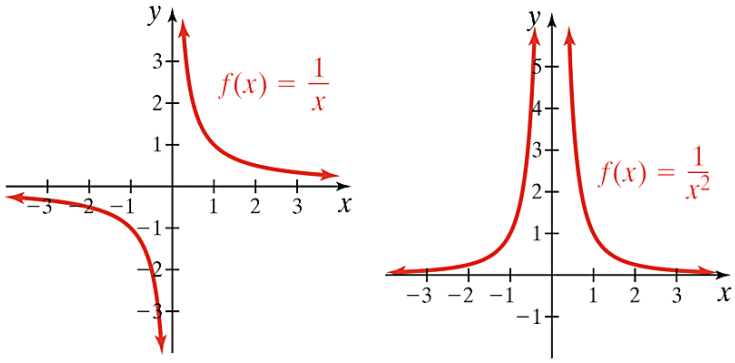
i) $g(x) = \frac{6x^3-10x}{2x^3+5x^2}$ HA: $y = \frac{6}{2} = 3$

ii) $h(x) = \frac{x^2-4x+1}{x+2}$ Oblique or Slant : $y = x - 6$

iii) $k(x) = \frac{x^2+4x}{x^3-8}$ HA: $y = 0$

Example: The graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$

- ✓ Domain: $(-\infty, 0) \cup (0, \infty)$
- ✓ Range of $f(x) = \frac{1}{x}$: $(-\infty, 0) \cup (0, \infty)$
- ✓ Range of $f(x) = \frac{1}{x^2}$: $(0, \infty)$
- ✓ No intercepts
- ✓ Vertical Asymptote: $x = 0$
- ✓ Horizontal Asymptote: $y = 0$
- ✓ $f(x) = \frac{1}{x}$ is an odd function. Its graph is symmetric about the origin and $f(-x) = -f(x)$. $f(x) = \frac{1}{x^2}$ is an even function. Its graph is symmetric about the *y*-axis and $f(x) = f(-x)$.



2.12.2. Sketching Rational Functions Having Removable Discontinuities

A rational function $f(x) = \frac{g(x)}{h(x)}$ may sometimes have a "hole" in its graph. In calculus, these "holes" are called **removable discontinuities.** Removable discontinuities occur when $g(x)$ and $h(x)$ share a common factor.

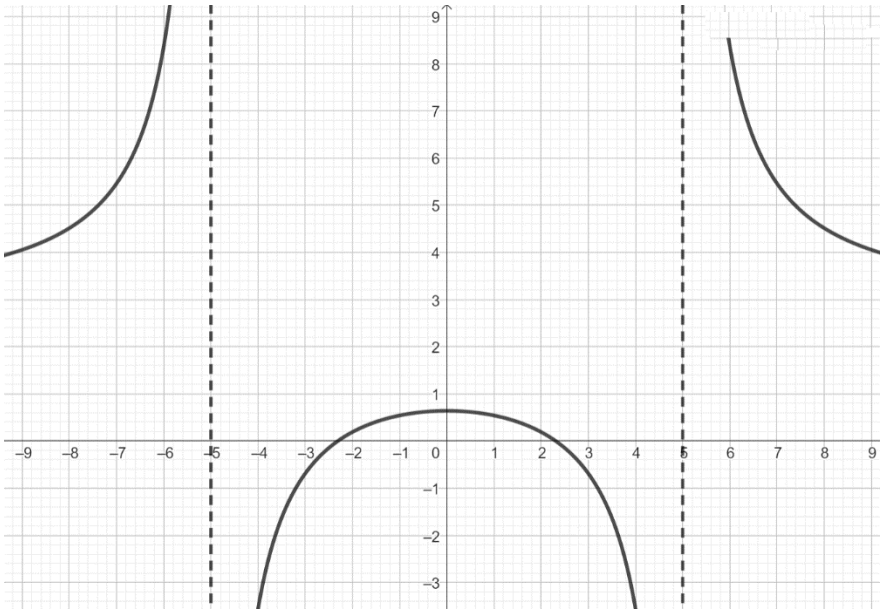
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Classwork: Graph $f(x) = \frac{3x^2-16}{x^2-25}$

x-intercept(s): _____ *y*-intercept: _____

Vert. asymptotes:_____ horiz. asymptotes: _____

Oblique asymptote: _____ Domain: _____ Hole: _____

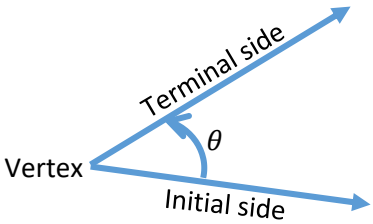


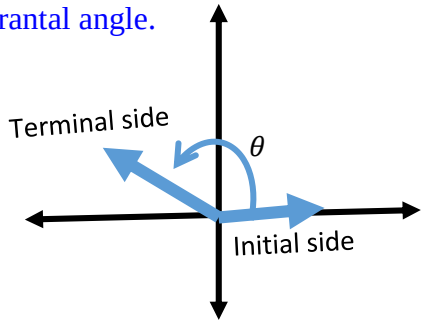
2.13. Trigonometric functions

As derived from the Greek language, the word trigonometry means "measurement of triangles." Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying.

2.13.1. Angles and Degree Measure

An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. An angle with its vertex at the origin and its initial side along the positive *x*-axis is said to be in **standard position.** If the terminal side of an angle lies **"on"** the axes (such as 0º, 90º, 180º, 270º, 360º), it is called a quadrantal angle.





If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

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The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. A complete circular rotation contains 360 degrees.
Angles may be specified in either degrees or ***radians***. You need to be competent in both. This is easy.

Note: There are **360** degrees in a complete circle or **2π radians in a circle**, or about 6.28 radians because π is approximately 3.14.

Thus we have the critical conversion formula: **$360° = 2\pi$ radians or $180° = \pi$ radians.**

Relationship between Degrees and Radians:

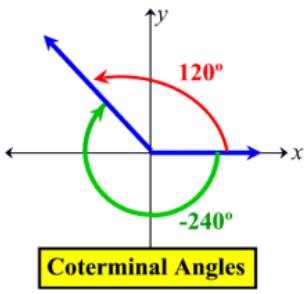
$$Radians = degree \times \frac{\pi}{180°} \qquad Degree = radinas \times \frac{180°}{\pi}$$

Classwork: Change the degree measures to radian measures and the radian measures to degree measures

a. $145°$ b. $-250°$ c. $\frac{7\pi}{15}$ d. -3.5π

The angle is measured by the amount of rotation from the initial side to the terminal side. If measured in a **counterclockwise direction** the measurement is **positive**. If measured in a **clockwise** direction the measurement is **negative.** (A negative associated with an angle's measure refers to its "direction" of measurement, clockwise.)

Coterminal Angles: Two angles are said to be coterminal if they have the same initial and terminal sides. Finding coterminal angles is as simple as adding or subtracting $360°$ or 2π to each angle, depending on whether the given angle is in degrees or radians



Note: 1.The angle $\theta \pm n \times 360º$ or $\theta \pm 2n\pi, n \in \mathbb{Z}$ are coterminal angles with θ. The value of $\boldsymbol{n}$ is a positive integer if the rotations are counterclockwise and a negative integer if the rotations are clockwise.

2. Every angle has infinitely many coterminal angles.

Type of Angle	Description
Acute Angle	is less than 90°
Right Angle	is 90° exactly
Obtuse Angle	is greater than 90° but less than 180°
Straight Angle	is 180° exactly
Reflex Angle	is greater than 180°
Full Rotation	is 360° exactly

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Example: The following are coterminal angle with $20°$.

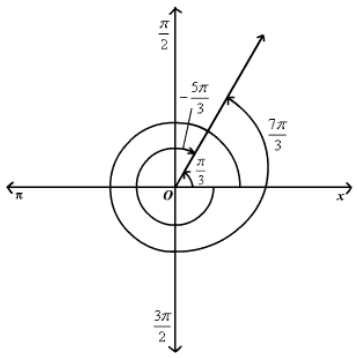
i) $(20° + 360°) = 380°, (20° + 2 \times 360°) = 740°, (20° - 360°) = -340°, (20° - 2 \times 360°) = -700°, \cdots$ are coterminal angle with $20°$.

ii) $\left(\frac{\pi}{3} + 2\pi\right) = \frac{7\pi}{3}, \left(\frac{\pi}{3} + 4\pi\right) = \frac{13\pi}{3}, \left(\frac{\pi}{3} - 2\pi\right) = -\frac{5\pi}{3}, \left(\frac{\pi}{3} - 4\pi\right) = -\frac{11\pi}{3}, \cdots$ are coterminal angle with $\frac{\pi}{3}$.

Example: Find the coterminal angle for $-1297°$.

Solution: Using division $-\frac{1297}{360} = -3.602777778$, thus $-1297° = -3(360°) - 217°$. Thus, the angle is $-217°$. But the coterminal angle needs to be positive: $360° - 217° = 143°$.

The coterminal angle is $\theta = 143°$ which lies in the second quadrant.



Example: Find the coterminal angle with $\frac{\pi}{3}$.

- $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}, \frac{\pi}{3} + 4\pi = \frac{12\pi}{3}, \cdots$ are coterminal angles with $\frac{\pi}{3}$ in positive directions.
- $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}, \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}, \cdots$ are coterminal angles with $\frac{\pi}{3}$ in negative directions.

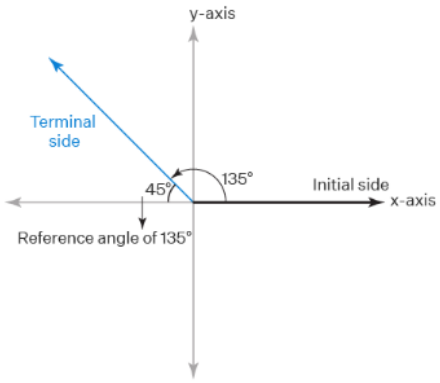
2.13.1. Reference angle

Reference Angle: The reference angle is the smallest possible acute angle between the terminal arm/terminal side and the x-axis. The reference angle is always positive and measures between 0° and 90°.

Note: i) A reference angle is always positive and is always less than 90º.

ii) If the starting angle is greater than 360º, start by subtracting 360º from the starting angle (as many times as needed) to arrive at a new "starting" angle that lies between 0º and 360º. (If working in radians, subtract 2π until the starting angle is between 0 and 2π radians.)

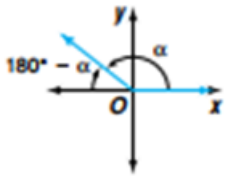
iii) Angles in quadrant I are their own reference angles.



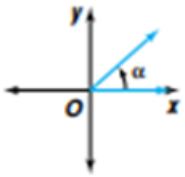
Example: The reference angle of 135° is drawn below:

Here, 45° is the reference angle of 135°.

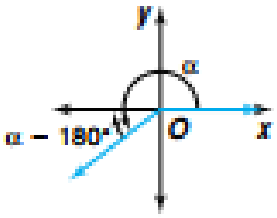
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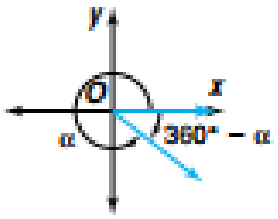
Reference angle=$180° - \theta = \pi - \theta$



Reference angle=θ



Reference angle=$\theta - 180° = \theta - \pi$



Reference angle=$360° - \theta = 2\pi - \theta$

Quadrant	Angle, θ	In degree	In Radian
I	$0 < \theta < 90$ $0 < \theta < \pi/2$	θ	θ
II	$90 < \theta < 180$ $\pi/2 < \theta < \pi$	$180° - \theta$	$\pi - \theta$
III	$180 < \theta < 270$ $\pi/2 < \theta < 3\pi/2$	$\theta - 180°$	$\theta - \pi$
IV	$270 < \theta < 360$ $3\pi/2 < \theta < 2\pi$	$360° - \theta$	$2\pi - \theta$

Steps to Find Reference Angles

Step 1: Find the coterminal angle of the given angle that lies between 0° and 360°.

Step 2: Find the reference angle of the given angle by identifying the quadrant of it.

Example: Determine the reference angle of 130°.

Answer: The reference angle becomes $180° - 130° = 50°$.

Example: Determine the reference angle of $-240°$.

Answer: The negative angle $\mathbf{-240°}$ is equal with the coterminal angle $\mathbf{-240°} + 360° = \mathbf{120°}$, which is in second quadrant. Thus, the reference angle becomes $180° - 120° = 60°$

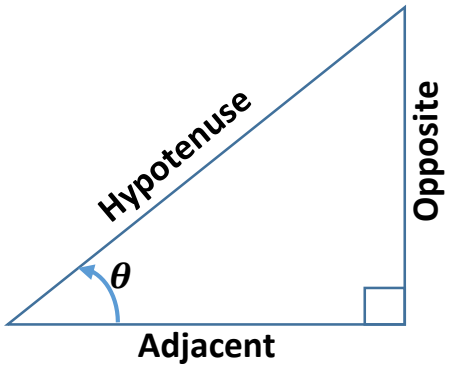
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Example: Determine the reference angle of -23°.

Answer: The Coterminal angle is $-23° + 360° = 337°$. And hence the reference angle of -23 degree is $360° - 337° = 23°$.

Quadrantal angle: An angle in standard position whose terminal arm lies on one of the axes. Examples are 0°, 90°, 180°, 270°, and 360°.

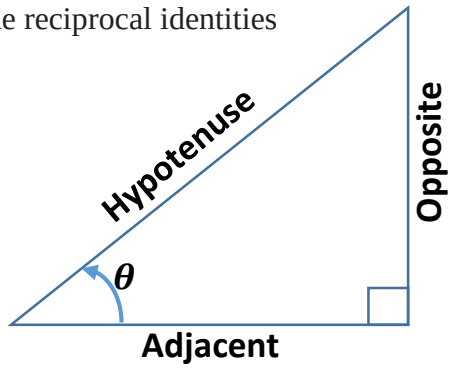
2.13.2. Sine, Cosine and Tangent



- **SOH**: $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$
- **CAH**: $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
- **TOA**: $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$

2.13.3. The functions $y = \sec x$, $y = cosec\ x$ and $y = \cot x$

In addition to the trigonometric ratios sine, cosine, and tangent, there are three other trigonometric ratios called ***cosecant, secant, and cotangent***, which are called the reciprocal identities



$$\textbf{\textit{cosecant}} \Rightarrow \csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{Opposite}$$

$$\textbf{\textit{secant}} \Rightarrow \sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent}$$

$$\textbf{\textit{cotangent}} \Rightarrow \cot\theta = \frac{1}{\tan\theta} = \frac{adjacent}{Opposite}$$

Note: 1. Sine, Cosine and tangent functions are the basic trigonometric functions.

2. Cosecant is the reciprocal of sine, secant is the reciprocal cosine and cotangent is the reciprocal of tangent.

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2.13.4. Common angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	1	0	$\nexists$	0	$\nexists$
$30°$ or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	1	$\sqrt{3}$
$45°$ or $\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$	1
$60°$ or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$90°$ or $\frac{\pi}{2}$	1	0	$\nexists$	1	$\nexists$	0

Note that from above table, $sin\,30° = cos\,60°$ and $sin\,60° = cos\,30°$. That is if θ is an acute angle, $\sin\theta = \cos(90° - \theta)$. Here we say the sine and cosine are **co-functions**. In general,

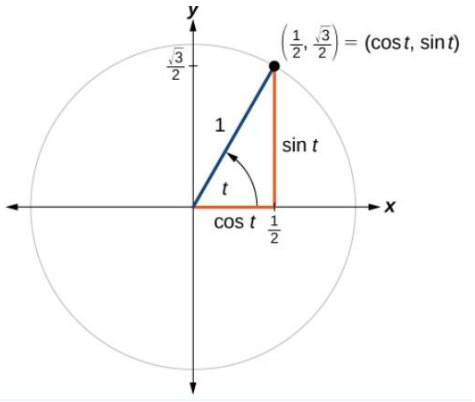
$sin\,\theta = cos(90° - \theta) = cos\left(\frac{\pi}{2} - \theta\right)$ $\qquad$ $cos\,\theta = sin(90° - \theta)$

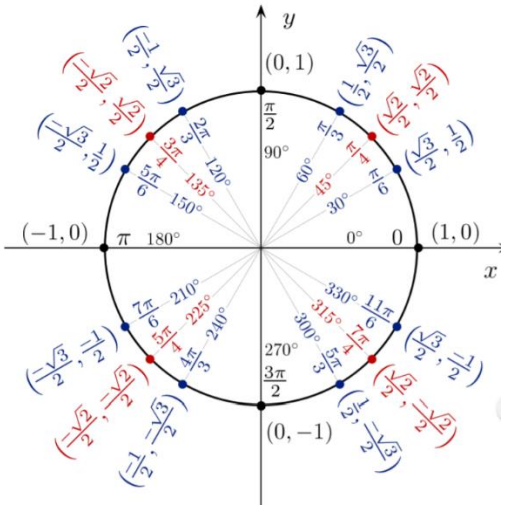
$tan\,\theta = cot(90° - \theta) = cot\left(\frac{\pi}{2} - \theta\right)$ $\qquad$ $cot\,\theta = tan(90° - \theta)$

$sec\,\theta = csc(90° - \theta) = csc\left(\frac{\pi}{2} - \theta\right)$ $\qquad$ $csc\,\theta = sec(90° - \theta)$

Note: Sine and cosine, Secant and Cosecant and tangent and cotangent are co-functions.

Note: If the sum of the measures of two acute angles is 90°, then the two angles is said to be **complementary angles.**



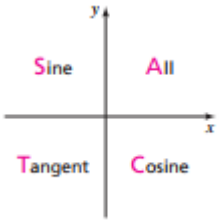


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Signs of the trigonometric functions in various quadrants:

$\frac{\pi}{2}, -\frac{3\pi}{2}$

$\pi, -\pi$	$\sin\theta, \csc\theta > 0$ $\cos\theta, \sec\theta, \tan\theta, \cot\theta < 0$ $\frac{\pi}{2} \le \theta \le \pi$ II	I $\sin\theta, \csc\theta > 0$ $\cos\theta, \sec\theta > 0$ $\tan\theta, \cot\theta > 0$ $0 \le \theta \le \frac{\pi}{2}$	$0, -2\pi$
	$\tan\theta, \cot\theta > 0$ $\sin\theta, \csc\theta, \cos\theta, \sec\theta <$ ($\pi \le \theta \le \frac{3\pi}{2}$ III	IV $\sin\theta, \csc\theta, \tan\theta, \cot\theta <$ 0 $\cos\theta, \sec\theta > 0$ $\frac{3\pi}{2} \le \theta \le 2\pi$	



$\frac{3\pi}{2}, -\frac{\pi}{2}$

Example: Determine $\cos 135°$.
Solution: Since $135°$ is quadrant II, cosine is $-ve$ and the reference angle is $180° - 135° = 45°$.

Thus $\cos 135° = -\cos 45° = -\frac{\sqrt{2}}{2}$.

Classwork: Find the exact value of the following.

i) $\cos 120° = -\cos(180° - 120°) = -\cos 60° = -\frac{1}{2}$ ($120°$ is in the II quadrant and cosine is -ve)

ii) $\sin 210° = -\sin(210° - 180°) = -\sin 30° = -\frac{1}{2}$ ($210°$ is in the III quadrant and sine is -ve)

iii) $\sin(\frac{13\pi}{6}) = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

iv) $\cot(-\frac{9\pi}{2}) = \cot(-4\pi - \frac{\pi}{2}) = \cot(-\frac{\pi}{2}) = \frac{\sin(-\frac{\pi}{2})}{\cos(-\frac{\pi}{2})} = -\frac{1}{0} = \nexists$

v) $\cos(-\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

vi) $\tan(495°) = \tan(360° + 135°) = \tan 135° = -\tan 45° = -1$ ($135°$ is in quadrant II, where sine is positive.)

vii) $\sec(-\frac{31\pi}{6})$

viii) $\tan\left(\frac{-\pi}{3}\right)$ Ans: There reference angle for $\frac{-\pi}{3}$ is $\frac{\pi}{3}$. Since the terminal point of $\frac{-\pi}{3}$ is in quadrant IV, $\tan\left(\frac{-\pi}{3}\right)$ is negative. Thus,

$$\tan\left(\frac{-\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}.$$

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Note: If θ is obtuse angel ($\frac{\pi}{2} \leq \theta \leq \pi$), then

- ✓ $sin\,\theta = sin(\pi - \theta)$
- ✓ $cos\,\theta = -cos(\pi - \theta)$

Even and Odd trigonometric functions

Cosine and secant are *even*	**Sine, cosecant, tangent and cotangent are *odd* functions**
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$ $\quad \csc(-x) = -\csc x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$ $\quad \cot(-x) = -\cot x$

2.13.5. Trigonometric Identities

☞ **Pythagorean Identities:**

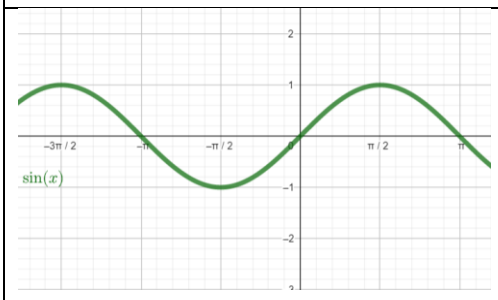
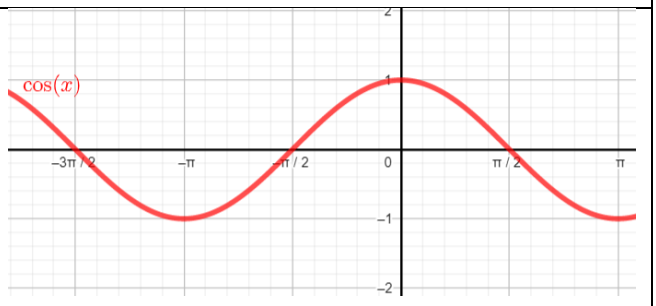
$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta = \tan^2\theta + 1$

$\cot^2\theta + 1 = \csc^2\theta$

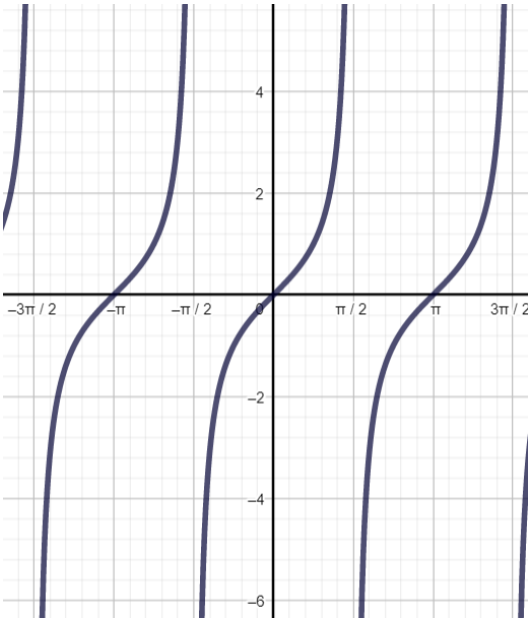
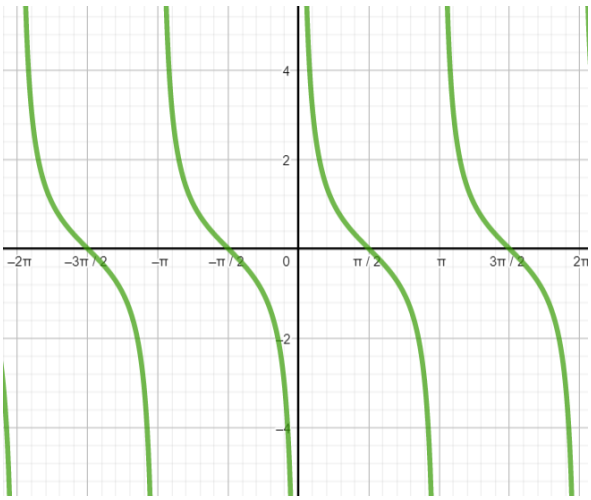
2.13.6. Graphs of some trigonometric functions

Graph of Sine Function and Cosine Function

Sine Function: $y = \sin x$	Cosine Function: $y = \sin x$
Period: 2π (Length Of One Cycle) Domain: $\{x \mid x \in \mathbb{R}\}$ Range: $\{y \mid -1 \leq y \leq 1\}$	Period: 2π Domain: $\{x \mid x \in \mathbb{R}\}$ Range: $\{y \mid -1 \leq y \leq 1\}$
	

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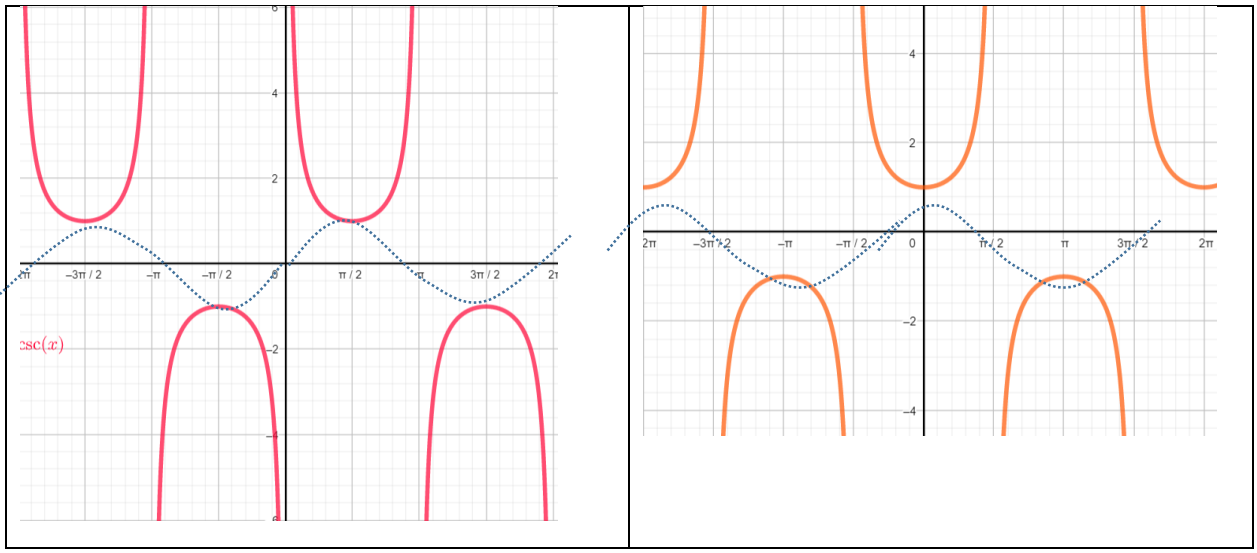
Graph of Tangent Function and Cotangent Function

Tangent Function: $y = \tan x$	Cotangent Function: $y = \cot x$
Period: π (Length Of One Cycle) Domain: $\left\{x \mid x \neq \frac{2(n+1)\pi}{2}, n \in \mathbb{Z}\right\}$ Range: $\{y \mid y \in \mathbb{R}\} = \mathbb{R}$ 	Period: π Domain: $\{x \mid x \neq n\pi,\ n \in \mathbb{Z}\}$ Range: $\{y \mid y \in \mathbb{R}\} = \mathbb{R}$ 

Graph of Cosecant Function and Secant Function

Cosecant Function: $y = \csc x$	Secant Function: $y = \sec x$
Period: 2π (Length Of One Cycle) Domain: $\{x \mid x \neq n\pi, n \in \mathbb{Z}\}$ Range: $(-\infty, -1] \cup [1, \infty)$	Period: 2π Domain: $\left\{x \mid x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\}$ Range: $(-\infty, -1] \cup [1, \infty)$

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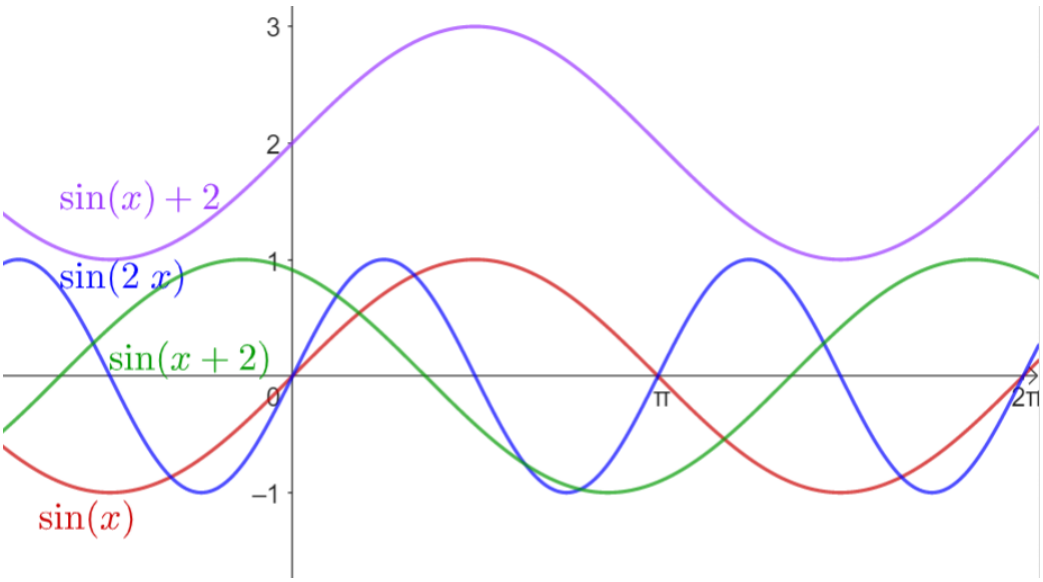
Transformation of Graph of $\boldsymbol{sin\ x}$ and $\boldsymbol{cos\ x}$

Given an equation in the form $f(x) = A\sin(Bx)$ or $f(x) = A\cos(Bx)$, here, A is the vertical stretch factor, and its absolute value $|A|$ is the amplitude and defined by

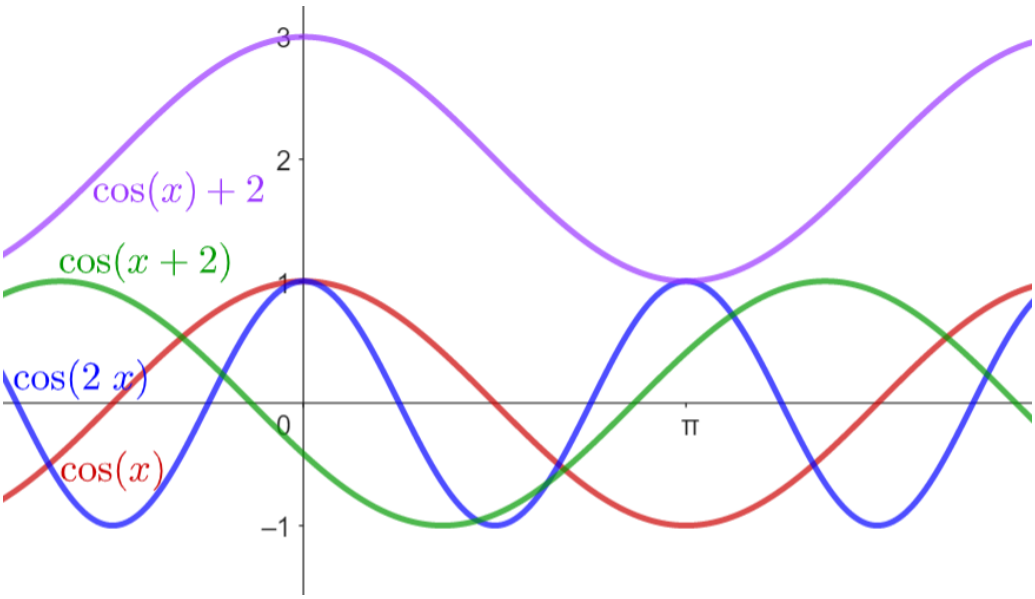
$$A = \frac{1}{2}|maximum - minimum|$$

Note: If $|A| < 1$, the function is compressed.

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the phase shift and D is the vertical shift.



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Classwork: Compare the graph of $f(x) = \cos x$ with the graph of $\cos 2x$, $\cos(x+2)$, $\cos x + 2$

2.13.7. Inverse of trigonometric functions

- Define notation for $arcsin\, x$, $sin^{-1}\, x$ and inverse sine functions of x.

Note: The notation of $sin^{-1}\, x$ is not exponential notation. Thus, $sin^{-1}\, x \neq \frac{1}{\sin x}$.

Note: $(\sin x)^{-1} = \frac{1}{\sin x}$, $\sin(x^{-1}) = \sin\left(\frac{1}{x}\right)$, $\sin^{-1} x \neq \frac{1}{\sin x}$

- ✓ $\sin^{-1} x = y$ iff $x = \sin y$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- ✓ $\cos^{-1} x = y$ iff $x = \cos y$ for $0 \leq y \leq \pi$.
- ✓ $tan^{-1} x = y$ iff $x = \tan y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- ✓ $csc^{-1} x = y$ iff $x = \csc y$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0, |x| \geq 1$
- ✓ $sex^{-1} x = y$ iff $x = \text{sex}\, y$ for $0 \leq y \leq \pi, y \neq \frac{\pi}{2}, |x| \geq 1$
- ✓ $cot^{-1} x = y$ iff $x = \cot y$ for $0 < y < \pi, -\infty < x < \infty$

Example: Evaluate i) $\cos^{-1}\left(\frac{1}{2}\right)$ ii) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ iii) $\cos^{-1}\left(\frac{1}{2}\right)$ iv)$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution: i) We are looking for an angle in the interval $[0, \pi]$ with a cosine value of $\frac{1}{2}$. Let θ be the angle. Then, $\cos^{-1}\left(\frac{1}{2}\right) = \theta \Longrightarrow \cos\theta = \frac{1}{2}$.

$\cos\frac{\pi}{3} = \frac{1}{2}, \cos -\frac{\pi}{3} = \frac{1}{2}, \cos\frac{5\pi}{3} = \frac{1}{2}, cos -\frac{5\pi}{3} = \frac{1}{2}$. The possible value of θ are

The angle that satisfies this is:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

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ii) We are looking for an angle in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ with sine value of $-\frac{\sqrt{3}}{2}$. Let θ be the angle. Then, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta \Rightarrow \sin\theta=-\frac{\sqrt{3}}{2}$.

iii) Remember that $\cos\frac{\pi}{3}=\frac{1}{2}, \cos-\frac{\pi}{3}=\frac{1}{2}, \cos\frac{5\pi}{3}=\frac{1}{2}, cos-\frac{5\pi}{3}=\frac{1}{2}$. The possible value of θ are $\frac{\pi}{3},-\frac{\pi}{3},\frac{5\pi}{3},-\frac{5\pi}{3},\cdots$. But range of $\cos\theta$ is

- Relationship between functions and their inverses:

$$f\left(f^{-1}(x)\right)=x \text{ and } f^{-1}\left(f(x)\right)=x$$

✓ $\sin(\sin^{-1}x)=x$ if $-1\le x\le 1$	$\sin^{-1}(\sin x)=x$ if $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$
✓ $\cos(\cos^{-1}x)=x$ if $-1\le x\le 1$	$\cos^{-1}(\cos x)=x$ if $0\le x\le\pi$
✓ $\tan(\tan^{-1}x)=x$ for all real x	$\tan^{-1}(\tan x)=x$ if $-\frac{\pi}{2}<x<\frac{\pi}{2}$

- Odd and even functions to inverse trigonometric functions:

✓ $sin^{-1}x$ is odd,	$\sin^{-1}(-x)=-\sin^{-1}x$
✓ $cos^{-1}x$ is neither	$\cos^{-1}(-x)=\pi-\cos^{-1}x$
✓ $tan^{-1}x$ is odd	$\tan^{-1}(-x)=-\tan^{-1}x$

Note: $\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$

Note: Domain of f=Range of f^{-1} and Range of f =Range of f^{-1}

Example: Compute i) $sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$ ii) $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

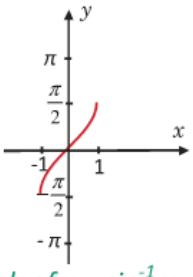
Solution: i) Let $\theta=\cos^{-1}\left(\frac{3}{5}\right)$. Then $\cos\theta=\frac{3}{5}$. Using Pythagoras's theorem, adjacent=3, hypotenuse=5 and then opposite=4. Hence $\sin\theta=\frac{opp}{hyp}=\frac{4}{5}$.

ii) Let $\theta=\sin^{-1}\left(\frac{3}{5}\right)$. Then $\sin\theta=\frac{3}{5}$. Using Pythagoras's theorem, adjacent=3, hypotenuse=5 and then opposite=4. Hence, $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)\right)=\tan\theta=\frac{hyp}{opp}=\frac{5}{3}$.

Homework: Simplify $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$. Answer: $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)=\frac{\sqrt{9-x^2}}{3}$.

2.13.8. Graph of Inverse Trigonometric Functions

$\sin^{-1}x$

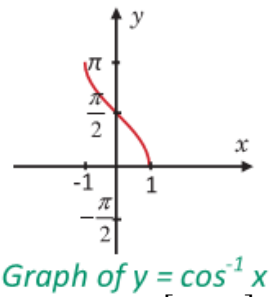


Graph of $y=sin^{-1}x$

Domain=$[-1,1]$

Range=$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$cos^{-1}x$

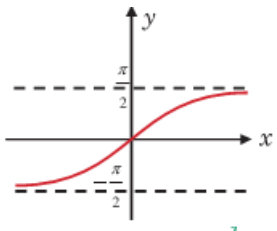


Graph of $y=cos^{-1}x$

Domain= $[-1,1]$

Range= $[0,\pi]$

$tan^{-1}x$



Graph of $y=tan^{-1}x$

Domain= $\mathbb{R}$

Range= $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

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2.13.9. Applications of trigonometric functions

Example: Show that

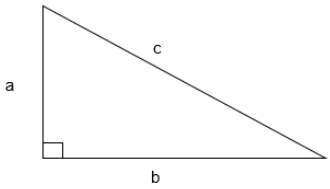
i) $(\tan y + \cot y)\sin y \cos y = 1$ ii) $\frac{1+sec\,x}{(tan\,x+sin\,x)} = csc\,x$

Solution: i) $(\tan y + \cot y)\sin y \cos y = \left(\frac{\sin y}{\cos y} + \frac{\cos y}{\sin y}\right)\sin y \cos y$

$$= \left(\frac{\sin^2 y + \cos^2 y}{\cos y \sin y}\right)\sin y \cos y = \sin^2 y + \cos^2 y = 1.$$

Pythagoras's Theorem: is used to find the length of the sides of a right-angle triangle.

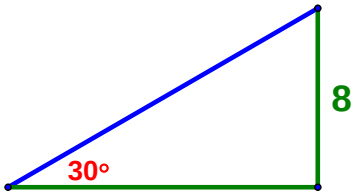
The theorem states that for a right-angle triangle with side lengths a, b and c:



$$a^2 + b^2 = c^2$$

a and b are the two shorter sides, and c is always the longest side, which is the one opposite the right angle.

Example: Complete the following triangle



Solution: let's get the hypotenuse (say H) first. By definition

$$sin(30) = 8/H$$

We know $sin(30) = \frac{1}{2}$ so $\frac{1}{2} = 8/H$ or $H = 8/(1/2) = 16$

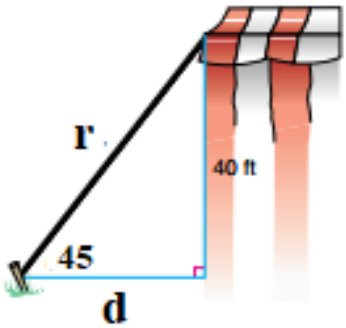
To get the horizontal side, which we call A, we know that

$$cos(30) = \frac{A}{H} = \frac{A}{16}$$

$$\frac{A}{16} = \frac{\sqrt{3}}{2}$$

and $A = (16)\frac{\sqrt{3}}{2} = 8\sqrt{3}.$

Example: If the angle that the rope makes with the level ground is 45°, how long is the rope? What is the distance between the bottom of the tent and the stake?



Solution: Let r be the length of the rope and d be the distance between the bottom of the tent and the stake. Then,

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{40}{r} \Rightarrow r = 40\sin 45° = 40\left(\frac{\sqrt{2}}{2}\right) = 20\sqrt{2} \approx 28.28$$

Hence the rope is about 28.28 ft long.

Again, using the tangent function**:**

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$$tan\,45° = \frac{opposite}{adjacent} = \frac{40}{d} \Longrightarrow d = 40\,tan\,45° = 40(1) = 40.$$

The distance between the bottom of the tent and the stake is about 40 feet.



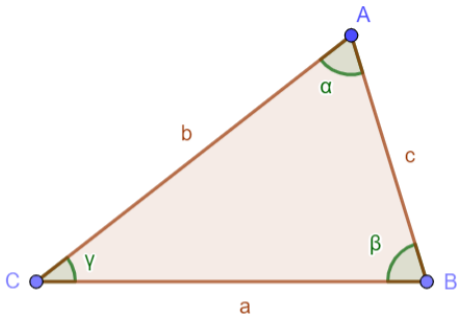
Homework: When rounding a curve, the acute angle θ that a runner's body makes with the vertical is called the angle of incline. It is described by the equation $tan\,\theta = \frac{v^2}{gr}$, where $\boldsymbol{v}$ is the velocity of the runner, $\boldsymbol{g}$ is the acceleration due to gravity, and $\boldsymbol{r}$ is the radius of the track. The acceleration due to gravity is a constant 9.8 meters per second squared. Suppose the radius of the track is 15.5 meters.

a. What is the runner's velocity if the angle of incline is 15°? $(tan\,15° = 0.26795)$

b Should a runner increase or decrease her velocity to increase his or her angle of incline?

2.13.10. Non-Right Triangles: Laws of Sines and Cosines

The sine law is a relationship between the sides and angles in any triangle. Let $\triangle ABC$ be any triangle, where a, b, and c represent the measures of the sides opposite $\angle A, \angle B$, and $\angle C$, respectively. Then,



$$\frac{\boldsymbol{a}}{\boldsymbol{sin\,\alpha}} = \frac{\boldsymbol{b}}{\boldsymbol{sin\,\beta}} = \frac{\boldsymbol{c}}{\boldsymbol{sin\,\gamma}}$$

or

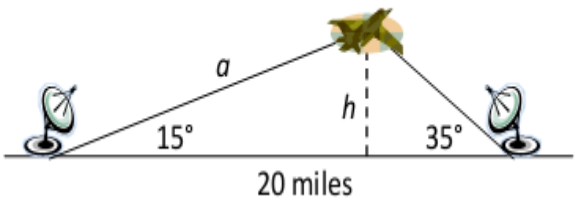
$$\frac{\boldsymbol{sin\,\alpha}}{\boldsymbol{a}} = \frac{\boldsymbol{sin\,\beta}}{\boldsymbol{b}} = \frac{\boldsymbol{sin\,\gamma}}{\boldsymbol{c}}$$

The law of sines is generally used to find the unknown angle or side of a triangle. This law can be used if certain combinations of measurement of a triangle are given.

1. ASA Criteria: Given two angles and included side, to find the unknown side.
2. AAS Criteria: Given two angles and a non-included side, to find the unknown side.

Example: Find the elevation of the UFO from the beginning of the section.

Solution: To find the elevation of the UFO, we first find the distance from one station to the UFO, such as the side in the picture, then use right triangle relationships to find the height of the UFO, h.



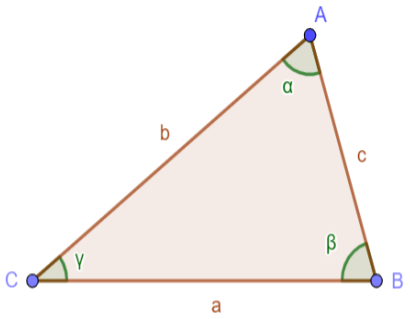
$\frac{sin\,130}{20} = \frac{sin\,35}{a} \Longrightarrow a = 14.975$

$sin\,15° = \frac{h}{14.975} \Longrightarrow h = 3.876\,miles$

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2.13.11. Law of Cosines

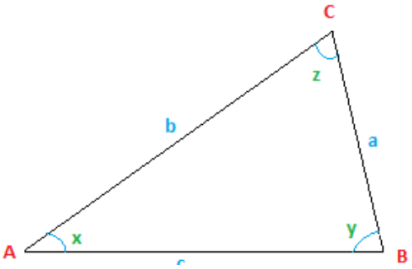
$$a^2 = b^2 + c^2 - 2bc\ cos\ (\alpha)$$



The Law of Cosines is sometimes called the Generalized Pythagorean Theorem, since it extends the Pythagorean Theorem to non-right triangles.

The Law of Cosines is useful for finding:

- the third side of a triangle when we know **two sides and the angle between** them (like the example above)
- the angles of a triangle when we know **all three sides**



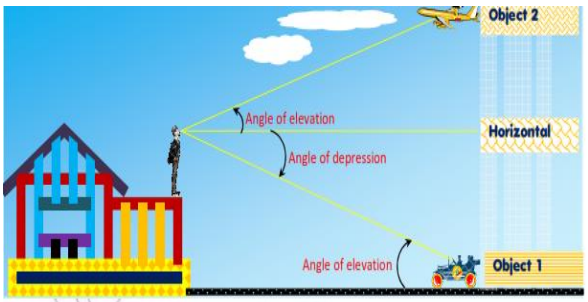
Example: A triangle ABC has sides $a = 10cm, b = 7cm$ and $c = 5cm$. Now, find its angle 'x'.

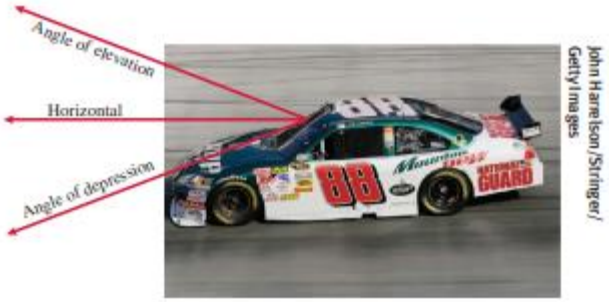
$$\cos x = \frac{b^2+c^2-a^2}{2bc} = \frac{7^2+5^2-10^2}{2 \times 7 \times 5} = -0.37$$

$$x = \cos^{-1} -0.37° = 68.3°$$

Angles of elevation and angles of depression

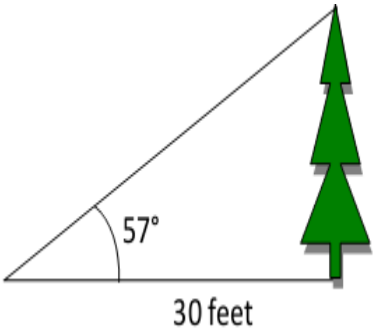
- An angle of elevation is the angle between a horizontal line and the line of sight from an observer to an object at a higher level.
- An angle of depression is the angle between a horizontal line and the line of sight from the observer to an object at a lower level.
- The angle of elevation and the angle of depression are equal in measure because they are alternate interior angles.





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Example: To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 57 degrees. Find the height of the tree.



Solution: We can introduce a variable, h, to represent the height of the tree. The two sides of the triangle that are most important to us are the side opposite the angle, the height of the tree we are looking for, and the adjacent side, the side we are told is 30 feet long.

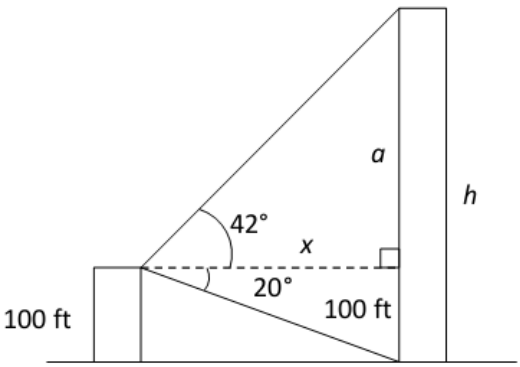
The trigonometric function which relates the side opposite of the angle and the side adjacent to the angle is the tangent.

$$\tan(57) = \frac{opposite}{adjacent} = \frac{h}{30}$$

$$h = 30\tan 57 = 46.2\ ft$$

Example: A person standing on the roof of a 100 foot building is looking towards a skyscraper a few blocks away, wondering how tall it is. She measures the angle of declination from the roof of the building to the base of the skyscraper to be 20 degrees and the angle of inclination to the top of the skyscraper to be 42 degrees.

Solution: To approach this problem, it would be good to start with a picture. Although we are interested in the height, h, of the skyscraper, it can be helpful to also label other unknown quantities in the picture –



in this case the horizontal distance x between the buildings and , the height of the skyscraper above the person.

To start solving this problem, notice we have two right triangles. In the top triangle, we know one angle is 42 degrees, but we don't know any of the sides of the triangle, so we don't yet know enough to work with this triangle.

In the lower right triangle, we know one angle is 20 degrees, and we know the vertical height measurement of 100 ft. Since we know these two pieces of information, we can solve for the unknown distance x.

$$\tan(20°) = \frac{100}{x} \Longrightarrow x = \frac{100}{\tan(20°)} = 274.75$$

$$tan(42°) = \frac{a}{x} = \frac{atan\,20°}{100} \Longrightarrow a = \frac{100\,tan(42°)}{tan(20°)} =$$

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ESSLCE Exam

1. If $f(x) = x^2 + px + 1$ where $f(0)$ and $f(1)$ have opposite signs so that f has a root in $(0, 1)$, then which one of the following number is a possible value p? (NOVEMBER 2022, NATURAL)

   A. -2 B. -3 C. 3 D. 2

2. Let $R = \{(x, y) | \, y \geq x^2 + 1 \text{ and } y \leq 5\}$ be a relation. Then which one of the following defines the inverse of R? (NOVEMBER 2022, NATURAL)

   A. $\{(x, y) | \, x \geq y^2 + 1 \text{ and } x \geq 5\}$ B. $\{(x, y) | \, x \geq y^2 + 1 \text{ and } x \leq 5\}$

   C. $\{(x, y) | \, x \geq y^2 - 1 \text{ and } x \leq 5\}$ D. $\{(x, y) | \, x \leq y^2 + 1 \text{ and } x \geq 5\}$

3. The Radian measure of an angle of $120°$ is equal to: (NOVEMBER 2022, NATURAL)

   A. $\frac{\pi}{2}$ B. $\frac{\pi}{8}$ C. $\frac{2\pi}{3}$ D. $\frac{\pi}{4}$

4. Which one of the following is the simplified form of $\frac{4x^4 - 64}{2 - x}$ for $x \neq 2$. (NOVEMBER 2022, NATURAL)

   A. $-4(x + 2)(x^2 + 4)$ B. $-4(x - 2)(x^2 + 4)$

   C. $4(x + 2)(x^2 + 4)$ D. $4(x - 2)(x^2 - 4)$

5. Which one of the following is an onto function from $\mathbb{R}$ on to $[0, \infty)$? (NOVEMBER 2022, NATURAL)

   A. $f(x) = |x| + 2$ B. $f(x) = \sqrt{x^2}$ C. $f(x) = 2^x$ D. $f(x) = x^2 + 1$

6. Which one of the following trigonometric value is correct? (NOVEMBER 2022, NATURAL)

   A. $tan(-120°) = -1$ B. $tan(-60°) = \frac{\sqrt{3}}{2}$

   C. $\sin(-120°) = 0.5$ D. $\cos(-120°) = -0.5$

7. Which of the following is true about the trigonometric values of the given pairs of angles? (NOVEMBER 2022, NATURAL)

   A. $sin(75°) = \cos 105°$ B. $tan(75°) = \tan 105°$

   C. $\sin(120°) = \sin 60°$ D. $\cos(120°) = \cos 60°$

8. The solution set of the equation given by $\log(x^2 - 3) = 2\log(x - 1)$ is (NOVEMBER 2022, NATURAL)

   A. $\{4\}$ B. $\emptyset$ C. $\left\{\frac{1}{2}\right\}$ D. $\{2\}$

9. Which one of the following is **true**? (NOVEMBER 2022, NATURAL)

   A. The range of arctangent function is $[-\pi, \pi]$

   B. The range of arcsine function is $(-\infty, \infty)$

   C. The domain of arccosine function is $[-1, 1]$

   D. The domain of arcsine function is $[0, \pi]$

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10. If r_1 and r_2 are the roots of the equation $3x^2 + 9x + 16 = 0$, which one of the following is true? (NOVEMBER 2022, NATURAL)

A. $r_1 + r_2 = 3$ B. $r_1^2 + r_2^2 = -\frac{5}{3}$ C. A. $r_1 r_2 = -\frac{16}{3}$ D. $\frac{1}{r_1} + \frac{1}{r_2} = \frac{9}{16}$

11. Which one of the following is NOT true about the graph of $f(x) = 5 + \frac{|x|}{x+1}$? (NOVEMBER 2022, NATURAL)

A. $y = 6$ is its horizontal asymptote of f.

B. $y = 5 + x$ is its oblique asymptote of f.

C. $x = -1$ is its vertical asymptote of f.

D. $y = 4$ is its horizontal asymptote.

Solved Problems

1. If $R = \{(x, y): x, y \in \mathbb{Z}, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers $\mathbb{Z}$, then the domain of R^{-1} is:

(A) $\{-2, -1, 1, 2\}$ (B) $\{0, 1\}$ (C) $\{-2, -1, 0, 1, 2\}$ (D) $\{-1, 0, 1\}$

Solution: Since $R = \{(x, y): x, y \in \mathbb{Z}, x^2 + 3y^2 \leq 8\}$. Therefore

$R = (1, 1), (2, 1), (1, -1), (0, 1), (1, 0)$. Which implies, $domain(R^{-1}) = \{-1, 0, 1\}$.

2. Let f and g be two functions given by $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$ and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)\}$ then $Domain\ of\ \{f + g\}$ is _____

Solution: Since Domain of f = Df = {2, 5, 8, 10} and Domain of g = Dg = {2, 7, 8, 10, 11}, therefore the domain of $f + g = \{x | x \in D_f \cap D_g\} = \{2, 8, 10\}$.

3. The domain of the function f defined by $f(x) = \frac{1}{\sqrt{x - |x|}}$ is

Solution: Given that $f(x) = \frac{1}{\sqrt{x - |x|}}$, where $x - |x| = \begin{cases} x - x = 0\ if\ x \geq 0 \\ x - (-x) = 2x\ if\ x < 0' \end{cases}$

Thus $f(x) = \frac{1}{\sqrt{x - |x|}}$ is not defined for any $x \in \mathbb{R}$. Hence f is not defined for any $x \in \mathbb{R}$.

4. Find the domain and range of the function f given by $f(x) = \frac{|x-4|}{x-4}$ and $\sqrt{16 - x^2}$.

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Solution: $f(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{x-4}{x-4} = 1, x \geq 1 \\ \frac{-(x-4)}{x-4} = -1, x < 1 \end{cases}$

Thus range of $\frac{|x-4|}{x-4}$ is $[-1, 1]$.

Again the domain of f where $f(x) = \sqrt{16 - x^2}$ is given by $[-4, 4]$.

For the range, let $y = \sqrt{16 - x^2}$. Then $y^2 = 16 - x^2$, or $x^2 = 16 - y^2$

Since $x \in [-4, 4]$, thus $range\ of\ f = [0, 4]$.

5. If $f(x) = \begin{cases} x + 2, x \leq -1 \\ x^2, -1 \leq x \leq 1 \\ 2 - x, x \geq 1 \end{cases}$. Then value of $f(-1.75) + f(0.5) + f(1.5)$ is

A. 0 B. 2 C. 1 D. -1

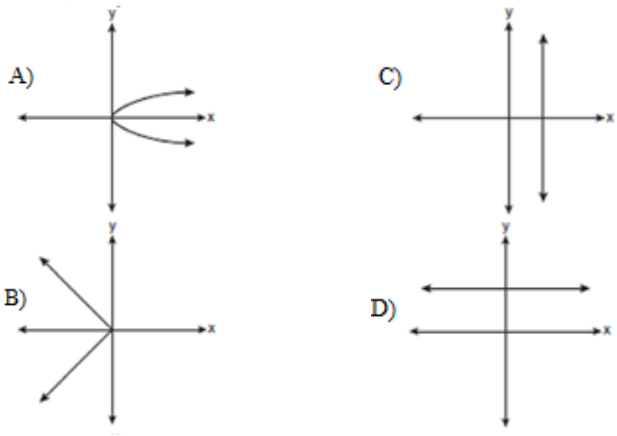
Answer: C

6. The domain of the function f defined by $f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}}$ is equal to

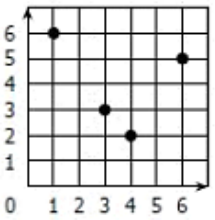
(A) $(-\infty, -1) \cup (1, 4)$ (B) $(-\infty, -1] \cup (1, 4]$

(C) $(-\infty, -1) \cup [1, 4]$ (D) $(-\infty, -1) \cup [1, 4)$

7. Which graph represents a function?



8. A relation is graphed as shown. What is the domain of this relation?



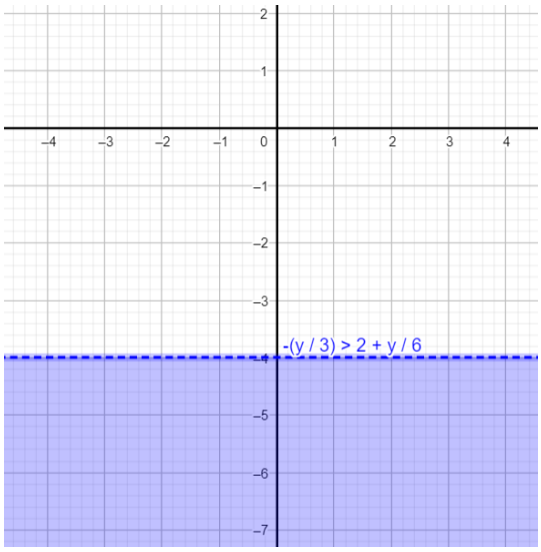
(A) $\{1, 3, 4, 6\}$ (B) $\{2, 3, 5, 6\}$ (C) $\{1, 2, 3, 4, 5, 6\}$

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9. In order to calculate the volume of a prism, we use the formula $V = \frac{1}{3}Bh$. What is the dependent variable?

   (A) h (B) $1/3$ (C) V (D) B

10. Graph the following relation in a Cartesian plane: $S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} | -\frac{y}{3} > 2 + \frac{y}{6}\right\}$ and determine its domain and range.



$Domain = \mathbb{R}$

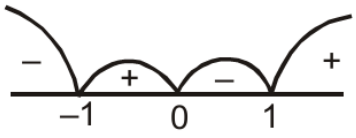
$Range = \{y \in \mathbb{R}: y < -4\} = [-4, \infty]$

11. Domain of definition of the function $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is

   (a) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ (b) $(a, 2)$

   (c) $(-1, 0) \cup (a, 2)$ (d) $(1, 2) \cup (2, \infty)$.

   Solution: $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$



   $4 - x^2 \neq 0; x^3 - x > 0; x \neq \pm\sqrt{4}$ and $-1 < x < 0$ or $1 < x < \infty$.

   $\therefore D = (-1, 0) \cup (1, \infty) - \{\sqrt{4}\}$

   $D = (-1, 0) \cup (1, 2) \cup (2, \infty)$

12. If $1 - p$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its root are

   (a) -1, 2 (b) -1, 1 (c) 0, -1 (d) 0, 1

   Solution: Let the second root be α. Then $\alpha + (1 - p) = -p \Rightarrow \alpha = -1$. Also $\alpha(1 - p) = 1 - p$

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$$\Rightarrow (\alpha - 1)(1 - p) = 0 \Rightarrow p = 1[\because \alpha = -1]$$

Therefore roots are $\alpha = -1$ and $p - 1 = 0$

13. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is

(a) 4 (b) 12 (c) 3 (d) $49/4$

Solution: 4 is a root of $x^2 + px + 12 = 0$ implies $16 + 4p + 12 = 0 \Rightarrow p = -7$.

Now, the equation $x^2 + px + q = 0$ has equal roots

$$\therefore\ p^2 - 4q = 0 \Rightarrow q = \frac{p^2}{4} = \frac{49}{4}$$

14. Which of the following is the solution set of $\log\left(1 - \frac{3}{x}\right) + 2\log x = 1$? (ESSLCE, December 2014)

A. $\{-2, 5\}$ B. $\{-5, -2\}$ C. $\{-2\}$ D. $\{5\}$

15. Which of the following is equal to $cos(-570°)$? (ESSLCE, December 2014)

A. $\frac{1}{2}$ B. $-\frac{1}{2}$ C. $-\frac{\sqrt{3}}{2}$ D. $\frac{\sqrt{3}}{2}$

16. If θ is a fourth quadrant angle, which of the following is NOT a correct relationship? (ESSLCE, December 2014)

A. $\sin(180° - \theta) = \sin\theta$ B. $\sin(\theta - 180°) = -\sin\theta$

C. $\sin(\theta - 180°) = \cos\theta$ D. $\tan(180° - \theta) = -\tan\theta$

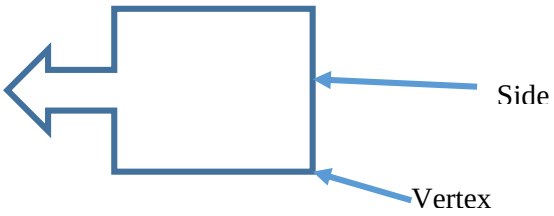
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Chapter 3

Geometry and Measurement

3.1. Polygon

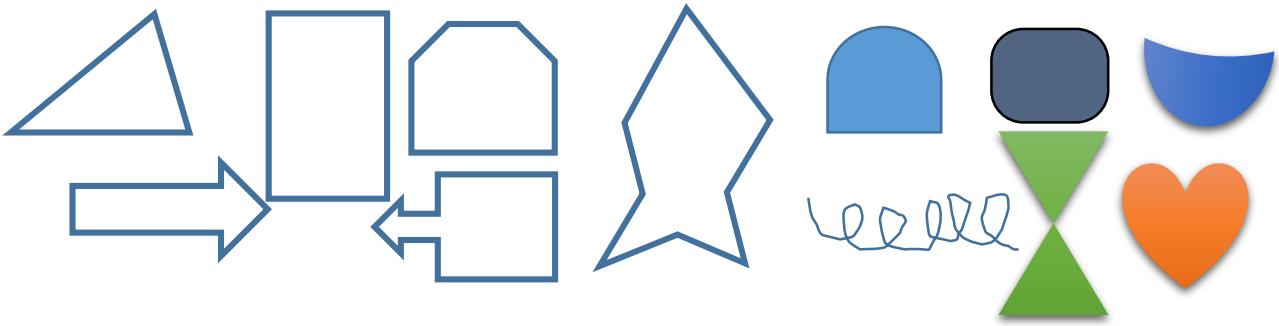
A polygon is a simple closed curve line segments, no two of which in succession are collinear. The line segments are called the *sides* of the polygon and the end points of the *vertices*.



The polygon have 11 sides and 11 vertices.

Example: The following are example of polygons and Non-polygons.

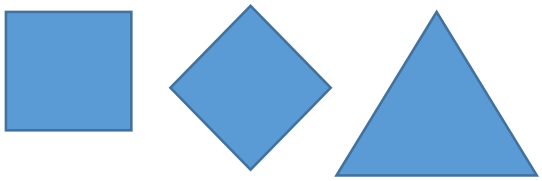
Polygons **Non-polygons**



3.1.1. Regular Polygon

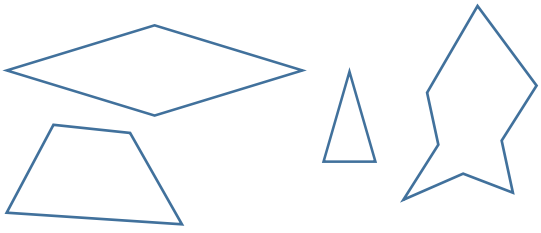
A **regular polygon** is a convex polygon in which the lengths of *all of its sides are equal* and the *measures of all of its angles are equal*. If any one of the conditions is not met, it becomes an **irregular polygon**.

Regular Polygon



Both sides and angles at each vertices are equal.

Irregular Polygon



Irregular quadrilateral Irregular octagon

Angles or sides may be equal, but not both.

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3.1.2. Regular Polygons vs. Irregular Polygons

Let's now compare and contrast regular and irregular polygons.

1. All sides of a regular polygon are congruent.
   Irregular polygons generally don't have equal sides.
   Rhombus is an example of an irregular polygon; it has equal sides, but unequal angles.
2. Regular polygons have equal interior/exterior angles.
   In irregular polygons, the measure of all interior/ exterior angles is not equal.
   A **rectangle** is an example of an irregular polygon where all the angles are equal but all sides are not of equal length.
3. Regular polygons have equal sides and equal angles. Conversely, irregular polygons can either have equal sides or equal angles or none.

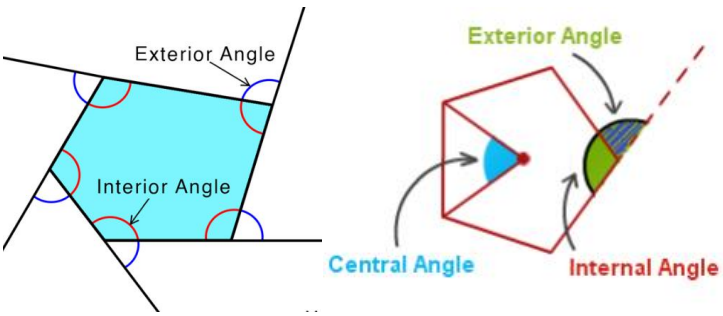
3.1.3. Interior and exterior angles of a polygon

- Collinear points are the points that lying on a single line.

Interior angle: Is an angle in the interior of a polygon at a vertex. Interior angles are angles between a polygon's two adjacent sides.

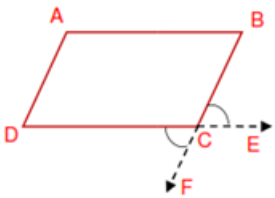
Exterior angle: If the side of a polygon is extended, the angle formed outside the polygon is the exterior angle. An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called exterior angle.

Central angle:



The interior angle-sum formula for an **irregular polygon** is the same as the sum of interior angle in a regular polygon.

Example: In the polygon ABCD $\angle DCB$ is an interior angle, $\angle BCE$ and $\angle DCF$ are exterior angles of the polygon at the vertex C.



Note: There are two possible exterior angles at any vertex, which are equal.

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Example: Find the measure of the exterior angle of a decagon when its corresponding interior angle is 144°.

Solution: Exterior angle$= 180° -$Adjacent interior angle

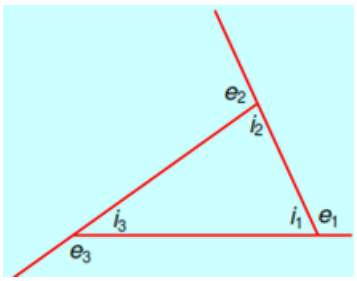
$= 180° - 144°$

$= 36°$

3.1.4. The sum of the measures of the interior angles of a Polygon

Consider a polygon of 3 sides, i.e., triangle, let i_1, i_2, i_3 be the measures of the interior angles of the given triangle, and let e_1, e_2 and e_3 be the measures of the exterior angles. Then,

$$i_1 + e_1 = 180°$$
$$i_2 + e_2 = 180°$$
$$i_3 + e_3 = 180°$$
$$i_1 + i_2 + i_3 = 180°$$

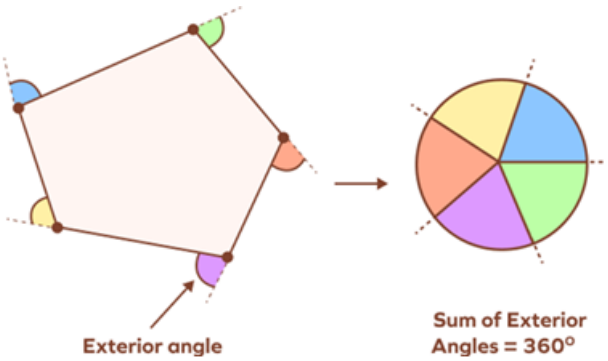


Then, from the above three equation

$$(i_1 + e_1) + (i_2 + e_2) + (i_3 + e_3) = 180° + 180° + 180°$$
$$(i_1 + i_2 + i_3) + (e_1 + e_2 + e_3) = 3(180°)$$
$$180° + e_1 + e_2 + e_3 = 3(180°)$$
$$e_1 + e_2 + e_3 = 360°$$

Thus, the sum of the measures of the exterior angles of a triangle, taking one angle at each vertex is $360°$.

1. The sum of the measures of the three interior angles of any triangle is $180°$.
2. If the number of sides of a polygon is n, then the sum of the measures of all its interior angles is equal to $(n-2)180°$.
3. **For any polygon, regular or non-regular, the sum of its exterior angles is 360 degrees.**



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Note that the measure of an interior angle of an n −sided regular polygon is $\frac{S}{n}$, where $S = (n-2)\times 180°$ is the sum of the measures of all of its interior angles. Hence, we have the following

The measure of each interior angle of a regular n-sided polygon is $\frac{(n-2)180°}{n}$.

- Measure of each interiro angle$= \frac{(n-2)180°}{n}$
- Measure of each exteriror angle$= \frac{360°}{n}$
- Measure of each central angle$= \frac{360°}{n}$

Example: Find the measure of each interior angle, exterior angle and each central angle of a regular polygon with 3 sides.

Solution: Since the sum of interior angles of a triangle is 180°, each interior angle is

$$\frac{180}{3} = 60°$$

The measure of each central angle of an equilateral triangle is

$$\frac{360°}{3} = 120°$$

Recall that a 3-sided regular polygon is an equilateral triangle.

Homework: If each interior angle is equal to 144°, then how many sides does a regular polygon have? **Answer**: The polygon has 10 sides.

Example: Find the measure of the unknown interior angle in an irregular hexagon with angles 130°, 90°, 140°, 150°, and 90°.

Solution: As we know,

Sum of interior angles$= (n-2)\times 180°$, here n=6

$= (6-2)\times 180°$

$= 720°$

Let the unknown angle be $x°$

Now,

$130° + 90° + 140° + 150° + 90° + x = 720°$

$x = 720° - 600°$

$x = 120°$

Example: How many exterior angles are there in an irregular octagon?

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Solution: An irregular octagon has eight sides and eight interior angles. Therefore, it has eight exterior angles because the number of interior angles of a polygon is equal to its exterior angles.

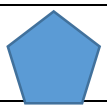
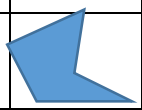
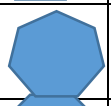
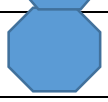
Example: If the sum of the interior angles of an irregular polygon with "n" sides is 1800 degrees, what is n?

Solution: Given the sum of the interior angles$= 1800°$

$$(n-2) \times 180° = 1800°$$

$$n - 2 = 10$$

So, $n = 12$. Therefore, the polygon has 12 sides.

Number of sides	Name of polygon	Regular	Irregular	Number of interior angles	Sum of interior angles	Sum of exterior angles	Each interior angle of regular polygon
3	Triangle			3	180°	360°	60°
4	Quadrilateral			4	360°	360°	90°
5	Pentagon			5	540°	360°	108°
6	Hexagon			6	720°	360°	120°
7	Heptagon			7	900°	360°	128.57°
8	Octagon			8	1080°	360°	135°
9	Nonagon			9	1260°	360°	140°
10	Decagon			10	1440°	360°	144°

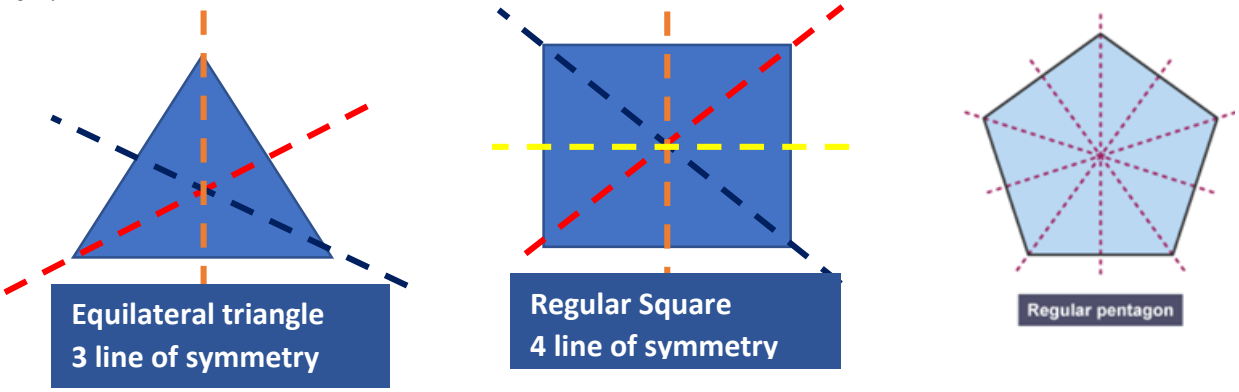
***Note**: The number of vertices, angles* and *sides* of a polygon are the same.

Note: In a regular polygon, all the interior angles are of the same measure.

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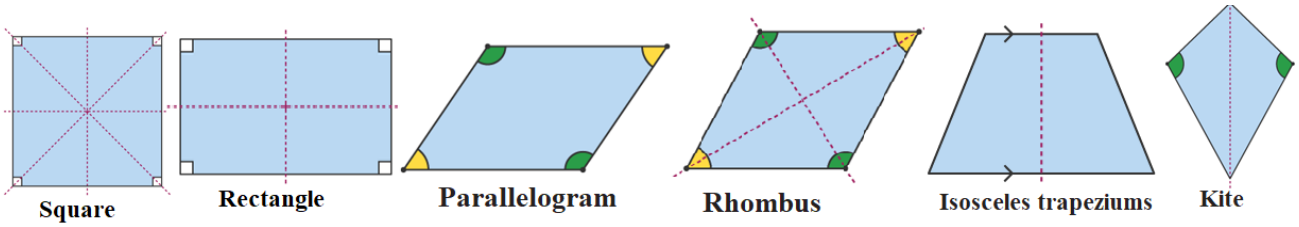
3.1.5. Properties of Regular Polygons

A figure has a line of symmetry, if it can be folded so that one half of the figure coincides with the other half.



A regular n-sided polygon always has n lines of symmetry.

Note: There are six special quadrilaterals (Square, Rectangle, Parallelogram, Rhombus, Trapezium and Kite) with different symmetrical properties.



A *square* has four lines of symmetry.

A *rectangle* has two lines of symmetry.

A *parallelogram* has no lines of symmetry.

A *rhombus* has two lines of symmetry.

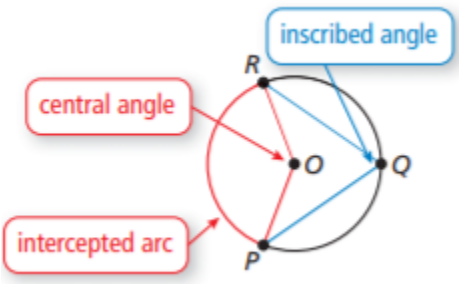
Isosceles *trapeziums* has one line of symmetry.

A *kite* has one line of symmetry.

3.1.6. Inscribed and subscribed Polygon

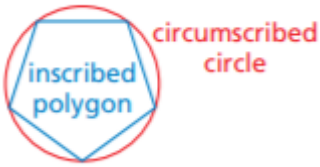
An inscribed angle is an angle whose vertex is on a circle and whose sides contain chords of the circle. An arc that lies between two lines, rays, or segments is called an intercepted arc. A polygon is an inscribed polygon when all its vertices lie on a circle.

A polygon is said to be inscribed in a circle if all of its vertices lie on the circle.

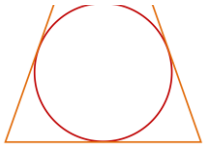


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A polygon is an inscribed polygon when all its vertices lie on a circle. The circle that contains the vertices is a circumscribed circle.

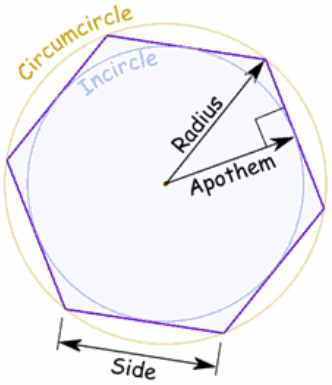


For any regular n −sided polygon: A circle can always be inscribed in or circumscribed about any given regular polygon.



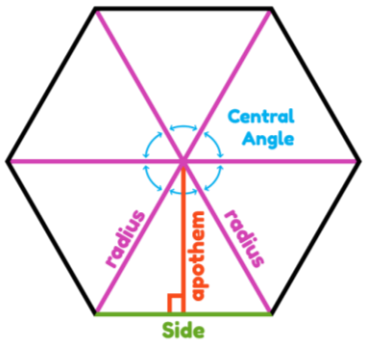
3.1.7. Apothem and Radius of regular polygon

Apothem: The apothem (sometimes abbreviated as apo) of a regular polygon is a line segment from the center to the midpoint of one of its sides. In other words, a **straight line** from a **midpoint of the regular polygon perpendicular to any of its sides** is known as **apothem**.



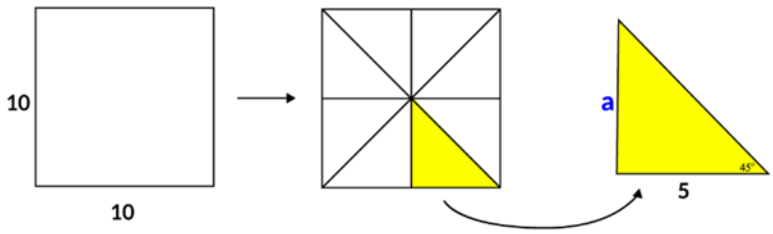
An apothem is a perpendicular segment from the center of a regular polygon to one of the sides.

Radius: The radius of a regular polygon is the distance from the center to any vertex. It will be the same for any vertex. The radius is also the radius of the polygon's circumcircle, which is the circle that passes through every vertex. The radius of the polygon is the radius of the circumcircle.



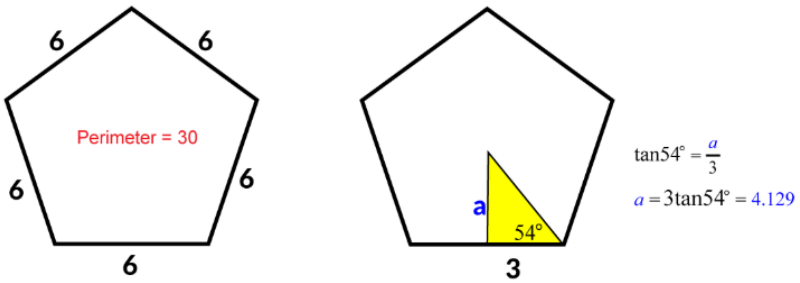
Note: All apothems and radii are congruent.

Example: Find the measure of the apothem of a square



$$tan\ 45° = \frac{a}{5} \Longrightarrow a = 5tan\ 45° = 5.$$

Example: Find the area of a hexagon



$$A = \frac{1}{2}aP = \frac{1}{2}(4.129)(30) = 61.935$$

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Theorem: Formulae for the length of side s, apothem a, perimeter P and area A of a ***regular polygon*** with n sides and radius r are

$$s = 2rsin\frac{180°}{n} \qquad P = 2nr\sin\frac{180°}{n}$$

$$a = r\cos\frac{180°}{n} \qquad A = \frac{1}{2}aP$$

Proof: $\sin\left(\frac{1}{2}\frac{(n-2)180°}{n}\right) = \frac{a}{r} \Rightarrow \sin\left(90°\frac{(n-2)}{n}\right) = \frac{a}{r}$

$$\Rightarrow \sin\left(90° - \frac{180°}{n}\right) = \frac{a}{r}$$

$$\Rightarrow \cos\frac{180°}{n} = \frac{a}{r}$$

Example:

a) Find the length of the side of an equilateral triangle if its radius is $\sqrt{12}\ cm$.

b) Find the area of a regular hexagon whose radius is $5\ cm$.

c) Find the apothem of a square whose radius is $\sqrt{8}\ cm$.

Solution: a) Since $r = \sqrt{12}$ and $n = 3$ is give,

Length of the side, $s = 2r\sin\frac{180°}{n} = 2\sqrt{12}\sin\frac{180°}{3} = \sqrt{36} = 6$.

Therefore, the length of the side of the equilateral triangle is $6\ cm$.

b) Here a is the apothem and P is the perimeter of regular hexagon $n = 6$

$$a = r\cos\frac{180°}{n} = 5\cos\frac{180°}{6} = \frac{5\sqrt{3}}{2}$$

$$P = 2nr\sin\frac{180°}{n} = 2\times 6\times 5\sin\frac{180°}{6} = 2\times 6\times 5\times\frac{1}{2} = 30$$

$$A = \frac{1}{2}aP = \frac{1}{2}\left(\frac{5\sqrt{3}}{2}\right)(30) = \frac{75\sqrt{3}}{2}$$

c) $r = \sqrt{8}\ cm,\ n = 4$

$$a = r\cos\frac{180°}{n} = \sqrt{8}\ \cos\frac{180°}{4} = \sqrt{8}\ \cos 45° = \sqrt{8}\ \frac{\sqrt{2}}{2} = 2cm$$

3.2. Different Types of Quadrilaterals

A quadrilateral is a polygon which has the following properties

- 4 vertices and 4 sides enclosing 4 angles and number of diagonals are two
- The sum of all interior angles and sum of all exterior angles of a quadrilateral is 360°

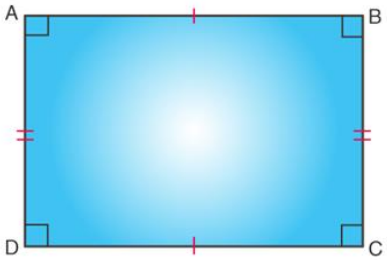
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A quadrilateral, in general, has sides of different lengths and angles of different measures. However, squares, rectangles, etc. are special types of quadrilaterals with some of their sides and angles being equal.

There are six basic types of quadrilaterals. They are:

1. Trapezium 2. Parallelogram 3. Rectangle
4. Rhombus 5.Square 6. Kite

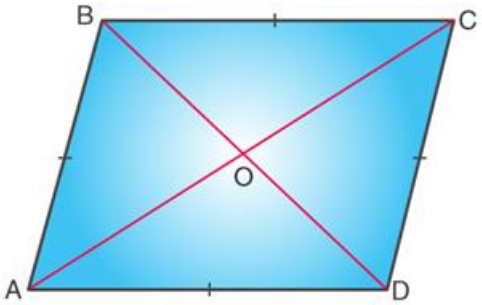
3.2.1. Rectangle



It is a quadrilateral with all the 4 angles of equal measure, that is, each of them is 90°. Both the pairs of opposite sides are parallel and equal in length.

- The opposite sides are parallel and equal
- Each interior angle is a right angle
- The diagonals bisect each other and have the same length
- Area = Length × Breadth
- Perimeter = 2(Length + Breadth)

3.2.2. Rhombus



It is a quadrilateral with all four sides having equal lengths. The Opposite sides of a rhombus are parallel and opposite angles are equal.

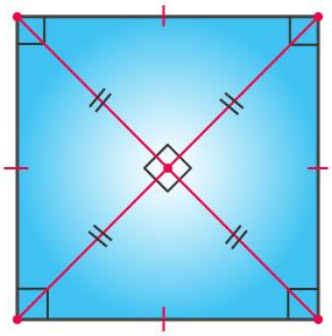
- All sides of the rhombus are congruent, and the opposite sides are parallel
- Opposite angles of a rhombus are congruent
- Diagonals bisect each other at right angles
- Diagonals bisect the interior angles of a rhombus
- The sum of two adjacent angles is equal to 180 degrees
- Area $= \frac{1}{2}(d_1 d_2) = \frac{1}{2}$ (Product of the length of diagonals)
- Perimeter = 4(side length)

Question: Is a rhombus a square?

No, rhombus is not a square but a square is a rhombus. In rhombus all the interior angles are not equal (only the opposite angles are equal) even though they have equal sides. However, the interior angles of a rhombus add up to 360 degrees.

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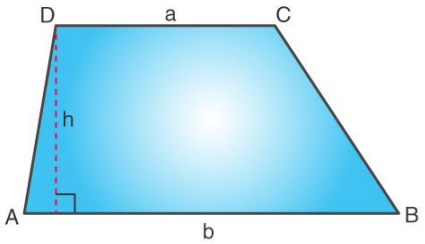
3.2.3. Square



It is a quadrilateral in which all the sides and angles are equal. Every angle is a right angle (i.e. 90° each). The pairs of opposite sides are parallel to each other.

- All four sides of the square are equal, and the opposite sides of the square are parallel to each other
- The measure of each interior angle is 90 degrees
- The diagonals of the square are equal to each other and bisect each other at right angles.
- Area = (side)²
- Perimeter = 4(side)

3.2.4. Trapezium

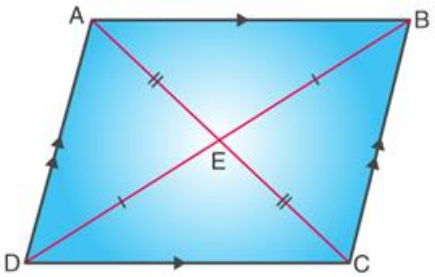


It is a quadrilateral with one pair of opposite parallel sides. In the trapezium, $ABCD$, side $\overline{AB}$ is parallel to side $\overline{CD}$. Diagonals intersect each other.

The length of the mid-segment is equal to half the sum of the parallel bases

- Area of trapezium$= \frac{1}{2}(a+b)h = \frac{1}{2}$(Sum of two parallel sides) × Height
- Perimeter = Sum of all the sides

3.2.5. Parallelogram



It is a quadrilateral with two pairs of parallel sides. The opposite sides are parallel and equal in length. The opposite angles are equal in measure. In the parallelogram, $ABCD$, side $\overline{AB}$ is parallel to side $\overline{CD}$ and side $\overline{AD}$ is parallel to side $\overline{BC}$.

Also, the two diagonals formed to intersect each other at the midpoints. As in the figure given below, E is the point where both the diagonals meet. So

Length $\overline{AE} = \overline{EC}$, & Length $\overline{BE} = \overline{ED}$.

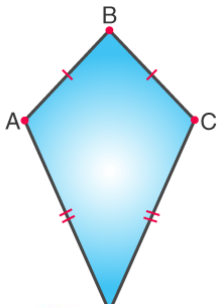
- The opposite sides are parallel and equal
- The opposite angles are equal

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- The sum of two consecutive angles is 180 degrees
- If any one of the angles is a right angle, then all the other angles will be right angles
- Diagonals bisect each other
- Each diagonal bisects the parallelogram into two congruent triangles
- Area = Base × Height
- Perimeter = Sum of all the side

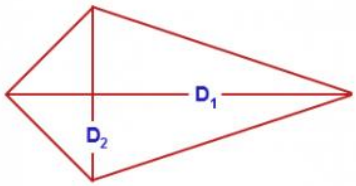
3.2.6. Kite

It is a quadrilateral that has 2 pairs of equal-length sides and these sides are adjacent to each other.



Some points about quadrilaterals to be kept in mind are:

- Square, rectangle, and rhombus are types of parallelograms.
- A square is a rectangle as well as a rhombus.
- The rectangle and rhombus are not a square.
- A parallelogram is a trapezium.
- A trapezium is not a parallelogram.
- Kite is not a parallelogram.



- Area $= \frac{1}{2}(D_1 + D_2) = \frac{1}{2}$(Product of the length of diagonals)
- Perimeter = Sum of all the sides

Example: If the perimeter of a square is 72 cm, then find its area.

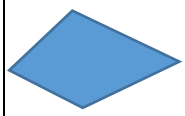
Solution: Let a be the side of a square. Then Perimeter of a square $P = 4a$

$$4a = 72\ cm \Longrightarrow a = \frac{72}{4} = 18$$

Thus, side of the square $= 18\ cm$

Area of the square $= a^2 = (18)^2 = 324\ cm^2$

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Quadrilateral	Image	Sides	Angles	Diagonals	Symmetry
Rectangle		Two pairs of equal sides Opposing sides are parallel	All equal angles (90°) Two adjacent angels are supplementary	Bisect each other with the same length	Two lines of symmetry
Square		Four equal sides Opposing sides are parallel	All equal angles (90°) Two adjacent angels are supplementary	Bisect each other with the same length at meet at 90°	Four lines of symmetry
Parallelogram		Two pairs of equal sides Opposing sides are parallel	Two opposing pairs of equal angles Two adjacent angels are supplementary	Bisect each other	No line of symmetry
Rhombus		Four equal sides Opposing sides are parallel	Two opposing pairs of equal angles Two adjacent angels are supplementary	Bisect each other with the same length at meet at 90°	Two lines of symmetry
Kite		Two pairs of equal sides	One opposing pair of equal angles Two adjacent angels are supplementary	Meet at 90°	One line of symmetry
Trapezium		One pair of parallel sides	No equal angles	No special properties	No lines of symmetry
Isosceles trapezium		One pair of equal sides, one pair of parallel sides	Two pairs of equal angles	Bisect each other with the same length	One line of symmetry

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Example: The area of a trapezium is 180 cm², and its height is 9 cm. If one of the parallel sides is longer than the other by 6 cm, find the two parallel sides.

Solution: Let x be the length of the shorter parallel side. So, the length of the longer side $= (x + 6)\ cm$

Height of a trapezium (distance between two parallel sides) $h = 9\ cm$

As we know,

Area of trapezium $= \frac{1}{2} \times$ (sum of parallel sides) $\times$ Height

Thus, $180 = \frac{1}{2}(x + x + 6) \times 9 \Longrightarrow 2x + 6 = 40 \Longrightarrow x = 17cm$

Now, $x + 6 = 17 + 6 = 23\ cm$

Therefore, the length of the two parallel sides will be 17 cm and 23 cm.

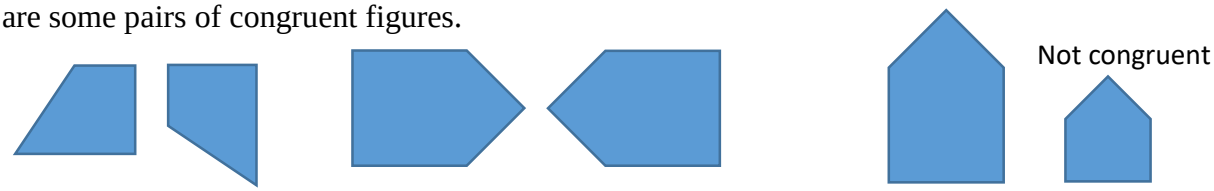
3.3. Congruency and Similarity

Two plane figures are congruent if they are exact copies of each other.

3.3.1. Congruent Polygons

Two polygons that have the exact same shape and the exact same size, meaning that all corresponding sides have the same length, and all corresponding angles have the same measure are called congruent polygon.

Here are some pairs of congruent figures.



Note: Congruent polygons are polygons with the same shape and size, although they can be rotations, translations, or mirror images of each other.

In order to prove that two polygons are congruent, we need to show that

i) all corresponding sides are congruent, which means they have the same length;

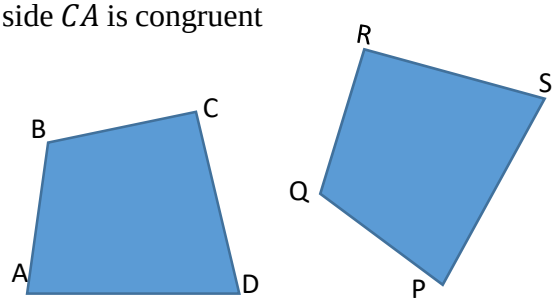
ii) all corresponding interior angles are congruent, which means they have the same measure.

Notation: The symbol ≅ means that the two objects are congruent.

For polygons ABC and XYZ,the notation $ABC \cong XYZ$ implies that the interior angle at vertex A is congruent to the one at vertex X,the interior angle at vertex B is congruent to the one at

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vertex Y,and the interior angle at vertex $C$is congruent to the one at vertex Z. Furthermore, side AB is congruent to side XY,side BC is congruent to side YZ,and side CA is congruent to side ZX.



These two parallelograms $ABCD \cong PQRS$ if

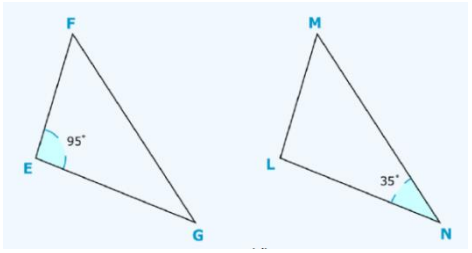
- $\angle A \cong \angle P, \angle B \cong \angle Q, \angle C \cong \angle R, \angle D \cong \angle S$ and
- $\overline{AB} \cong \overline{PQ}, \overline{BC} \cong \overline{QR}, \overline{CD} \cong \overline{RS}$ and $\overline{DA} \cong \overline{SP}$.

Example: Triangle ABC is congruent with triangle PQR. If angle $B = 54°$, what is the measure of angle Q?

Solution: Since angle B is congruent to angle Q, $\angle Q = 54°$.

Example: What is the measure of $\angle M$?

Solution: First, identify the angle that corresponds with angle ***L***.



$$\angle L = \angle E = 95°$$

Next, understand that the sum of three interior angles is 180o and write an equation.

$$95 + 35 + \boldsymbol{x} = 180°$$

Then, solve for the missing angle measure.

$$130 + \boldsymbol{x} = 180°$$

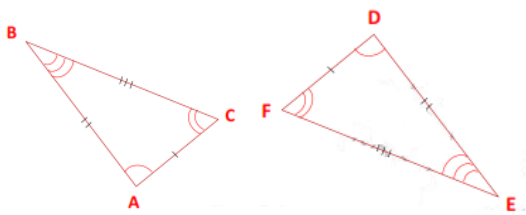
The answer is that the measure of angle M is $50°$.

3.3.2. Congruency of Triangles

Congruence of triangles: Two triangles are said to be congruent if *all three corresponding sides are equal and all the three corresponding angles are equal in measure*.

If two triangles, ΔABC and ΔDEF are congruent like those given below, then we denote this as

$$\Delta ABC \cong \Delta DEF.$$



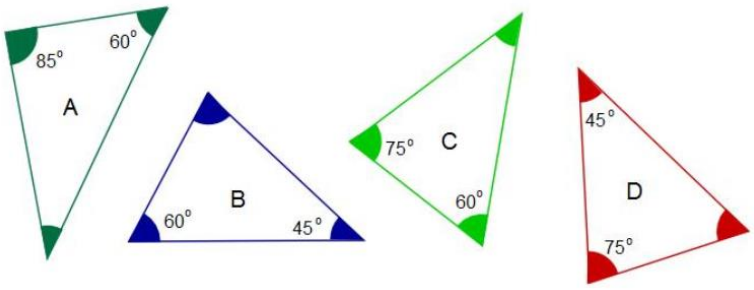
The notation "≅" means "is congruent to".

Congruent angles: $\angle A \cong \angle D; \angle B \cong \angle E; \angle C \cong \angle F$

Congruent sides: $\overline{AB} \cong \overline{DE}; \overline{BC} \cong \overline{EF}; \overline{AC} \cong \overline{DF}$

Classwork: Which triangle is not congruent to the other three?

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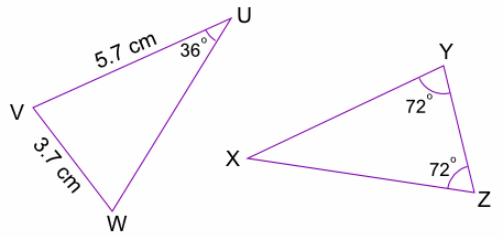
Solution: Triangles B, C and D all have exactly the same angles: 45°, 60° and 75°. (Remember the angles of a triangle always add to 180°)

- Triangle A has some different angles 35°, 60° and 85°.
- Triangle A, therefore, is not congruent to the other three.

Example: Given that $\Delta ABC \cong \Delta DEF$. Find the value of x, if $\angle B = x° + 30°$ and $\angle E = 3x°$.

Solution: Since $\Delta ABC \cong \Delta DEF, \angle B \cong \angle E$.that is $x° + 30° = 3x°$. Hence $x = 15°$.

Example: The two triangles are congruent. What is the perimeter of triangle XYZ?



The triangle has been flipped and rotated, but the size and shape is exactly the same. Since triangle UVW has a 36° angle and triangle XYZ has two 72° angles, the two triangles must be isosceles.

$$36° + 72° + 72° = 180°$$

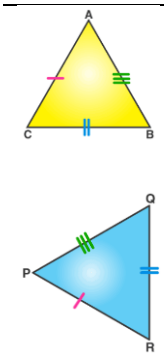
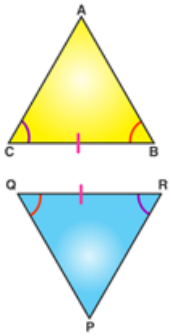
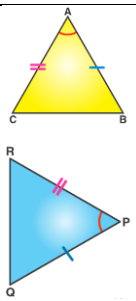
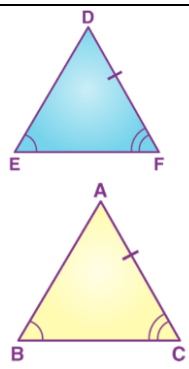
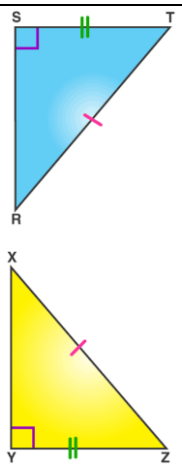
So

$$XY = 5.7\ cm, XZ = 5.7\ cm \text{ and } YZ = 3.7\ cm$$

Thus the perimeter of triangle $\triangle XYZ = 5.7cm + 5.7cm + 3.7cm = 15.1\ cm$.

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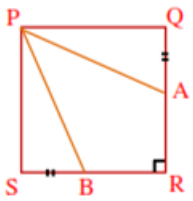
Congruent triangles

Three sides are congruent. SSS(Side-Side-Side)	Two angles and the included side are congruent. ASA (Angle-Side- Angle)	Two sides and the included angles are congruent. SAS(Side-Angle-Side)	Two angles and the included side are congruent. **AAS(Angle-Angle-Side)**	A right angle, hypotenuse and a side are congruent. RHS
				
$AB = PQ, BC = QR$ and $AC = PR$. Hence $\Delta ABC \cong \Delta PQR$	$\angle B = \angle Q, \angle C = \angle R$ and sides between $\angle B$ and $\angle C, \angle Q$ and $\angle R$ are equal to each other i.e. $BC = QR$. Hence, $\Delta ABC \cong \Delta PQR$.	$AB = PQ$, $AC = PR$ and $\angle A = \angle P$. Hence, $\Delta ABC \cong \Delta PQR$	$\angle B = \angle E, \angle C = \angle F$ and $AC = DF$. Hence, $\Delta ABC \cong \Delta DEF$	$XZ = RT$ and side $YZ = ST$. Hence $\Delta XYZ \cong \Delta RST$

In short, if $\Delta ABC \cong \Delta PQR$, then

- $AB = PQ, BC = QR$ and $\overline{AC} = \overline{PR}$ (SSS)
- $\angle B = \angle Q, \overline{BC} = \overline{QR}$ and $\angle C = \angle R$ (ASA)
- $AB = PQ, \angle A = \angle P$ and $\overline{AC} = \overline{PR}$ (SAS)
- $\angle B = \angle Q, \angle C = \angle R$ and $\overline{AC} = \overline{PR}$ (AAS)

Example: If $PQRS$ is a square. A and B are points on $\overline{QR}$ and $\overline{SR}$, such that $\overline{QA} \cong \overline{SB}$. Prove that $\angle PAQ \cong \angle PBS$.

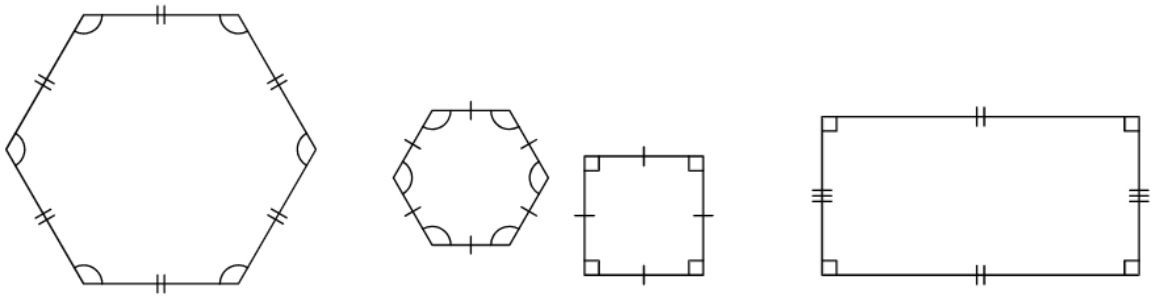


Solution: Since $PQRS$ is a square, $\overline{PQ} \cong \overline{PS}$ and it is given $\overline{QA} \cong \overline{SB}$.Since $\angle Q = \angle S = 90°$, hence $\angle Q \cong \angle S$. Therefore by SAS $\triangle PQA \cong \triangle PSB$. Hence $\angle PAQ \cong \angle PBS$.

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3.3.3. Similar Polygons

Two polygons are **similar** if their corresponding *angles are congruent* and their *corresponding sides are in proportion*. Two polygons that have the same shape, but their sizes may vary.

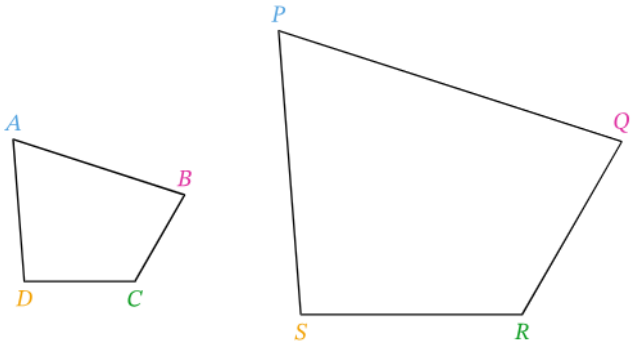


Regular hexagons are similar

The square and rectangle are not similar

- We can use the similarity statement to identify corresponding sides and angles, and we must ensure that the letter ordering is correct when writing a similarity relationship between polygons.

Let's consider the two quadrilaterals $ABCD$ and $PQRS$ below



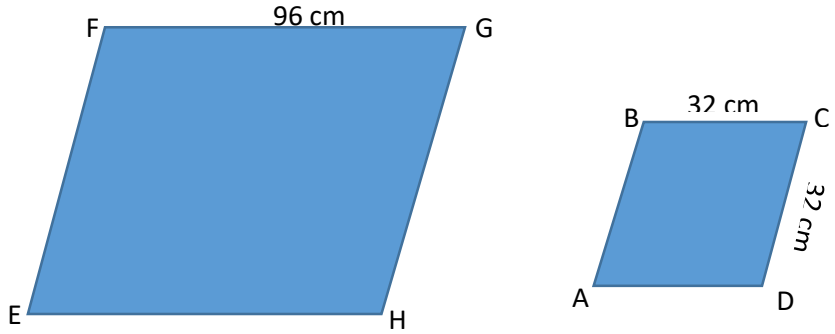
If we are given that $ABCD \sim PQRS$ ($ABCD$ is similar to $PQRS$), we have $m\angle A = m\angle P, m\angle B = m\angle Q, m\angle C = m\angle R, m\angle D = m\angle S$.

We can also observe the corresponding sides are in the same proportion, and we can write

$$\frac{\overline{PQ}}{\overline{AB}} = \frac{\overline{QR}}{\overline{BC}} = \frac{\overline{RS}}{\overline{CD}} = \frac{\overline{SP}}{\overline{DA}}.$$

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Example: Given that $\boldsymbol{ABCD \sim EFGH}$, determine the length of $\boldsymbol{\overline{GH}}$.



Solution: We are given that the two polygons $ABCD$ and $EFGH$ are similar. This means that their corresponding angles are congruent, and their corresponding sides are in proportion. We can use the proportionality of the sides to help us find the unknown side length, $\boldsymbol{\overline{GH}}$. We can write that

$$\frac{FG}{BC} = \frac{GH}{CD}$$

Substituting the given lengths, we have

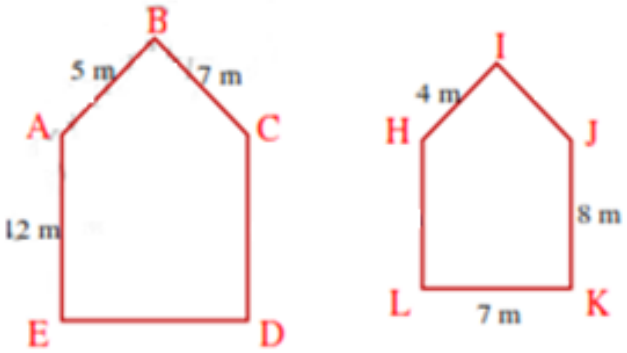
$$\frac{96}{32} = \frac{GH}{35}$$

Hence, $\overline{GH} = 105$.

Note: All regular polygons are similar to other regular polygons with the same number of sides.

Example: If $ABCD \sim HIJKL$, then find the lengths of

i) $\overline{IJ}$ ii) $\overline{CD}$ iii) $\overline{HL}$



Solution: Since $ABCD \sim HIJKL$

$$\frac{AB}{HI} = \frac{BC}{IJ} = \frac{CD}{JK} = \frac{DE}{KL} = \frac{AE}{HL}$$

i) Thus, to find the length of $\overline{IJ}$, we use

$$\frac{AB}{HI} = \frac{BC}{IJ}$$

$$\frac{5}{4} = \frac{7}{x} \Longrightarrow x = \frac{28}{5} = 5.6$$

Therefore, the length of side $\overline{IJ} = 5.6$.

ii) In the same way, $\frac{AB}{HI} = \frac{CD}{JK}$

$$\frac{5}{4} = \frac{CD}{8}$$

Therefore, the length of side $\overline{CD} = 10$.

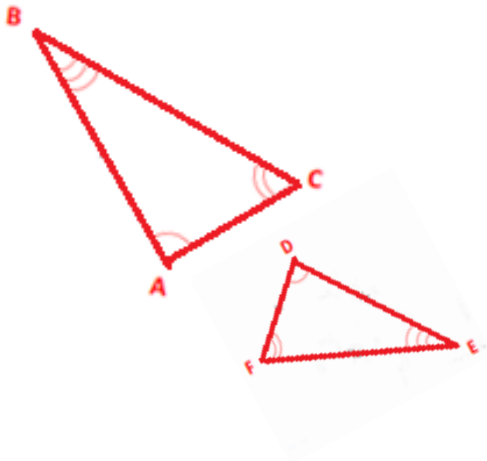
ii) In the same way, $\frac{AB}{HI} = \frac{AE}{HL}$

$$\frac{5}{4} = \frac{12}{HL}$$

Therefore, the length of side $\overline{HL} = 9.6$

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3.3.4. Similarity of Triangles



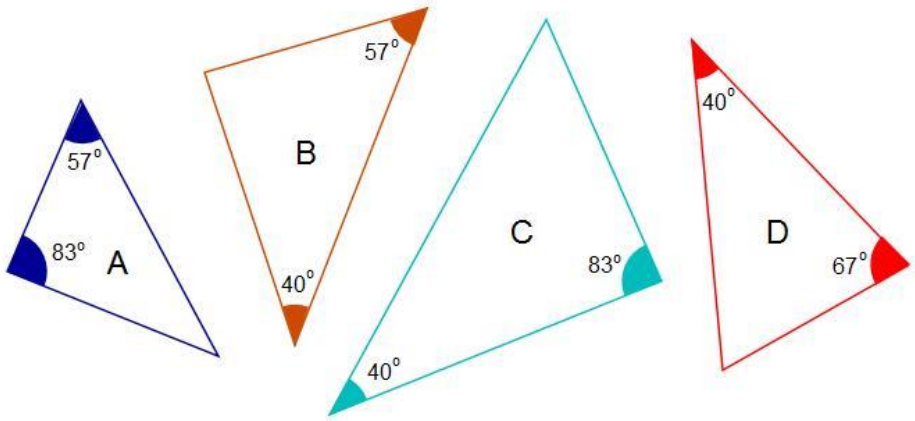
If two triangles, ΔABC and ΔDEF are similar, then we denote this as

$$\Delta ABC \sim \Delta DEF.$$

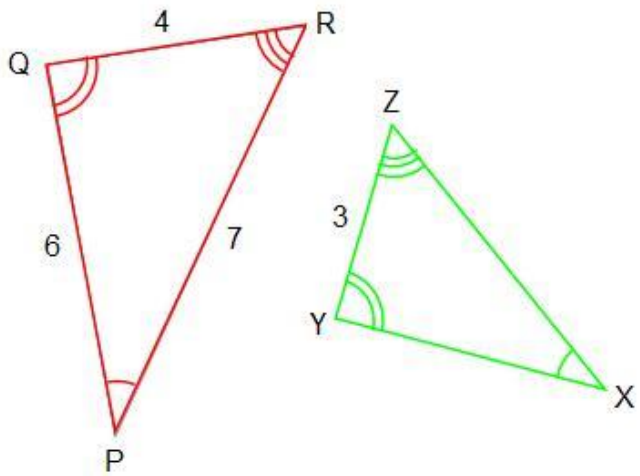
The notation "~" means "is similar to". Thus if $\Delta ABC \sim \Delta DEF$

1. Corresponding angles are congruent:
$$\angle A \cong \angle D; \angle B \cong \angle E; \angle C \cong \angle F$$
2. Corresponding sides are proportional (have equal ratio):
$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}$$

Classwork: Which triangle is NOT similar to the other three?



Example: Triangles PQR and XYZ are similar. What is the length of $\overline{XZ}$?



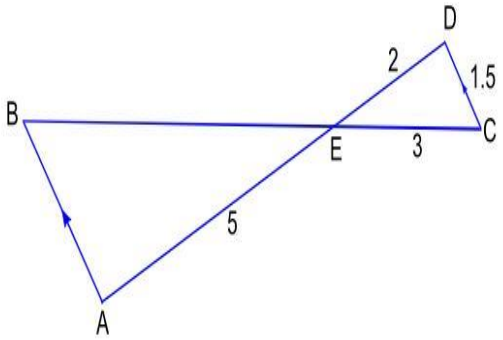
The lengths QR and YZ both face the angle marked with one arc. So we can match 4 with 3, and so the ratio of sides in triangle ΔPQR to sides in triangle ΔXYZ is 4: 3

The lengths $\overline{XZ}$ and $\overline{PR}$ both face the angle marked with two arcs.

Therefore the length of XZ = 3/4 × the length of $PR = \frac{3}{4} \times 7 = \frac{21}{4} = 5.25$

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Example: If $\overline{AB}$ is parallel to $\overline{CD}$, then what is the length of $\overline{BE}$?

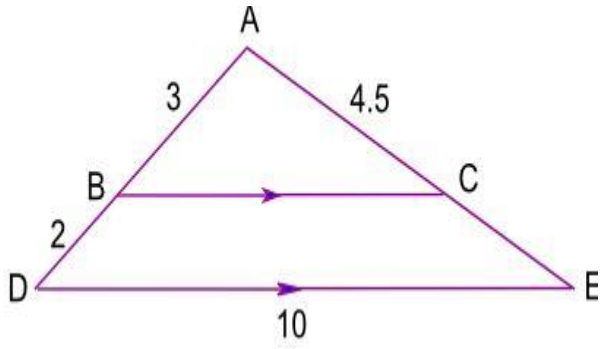


Solution: Since triangles ABE and DCE are similar.

$$\frac{BE}{CE} = \frac{AE}{DE}$$

$$\Rightarrow \frac{BE}{3} = \frac{5}{2} \Rightarrow BE = 3 \times \frac{5}{2} = \frac{15}{2} = 7.5.$$

Example: BC is parallel to DE. What is the length of $\overline{BC}$ and $\overline{CE}$?



So triangles ABC and ADE are similar. And we can work out the length by comparing sides we know.

So $\frac{BC}{DE} = \frac{AB}{AD}$

$$\Rightarrow \frac{BC}{10} = \frac{3}{3+2} \Rightarrow \frac{BC}{10} = \frac{3}{5} \Rightarrow \overline{BC} = 10 \times \left(\frac{3}{5}\right) = 6$$

Similarly $\frac{AE}{AC} = \frac{AD}{AB}$

$$\Rightarrow \frac{AE}{4.5} = \frac{3+2}{3} \Rightarrow \frac{AE}{4.5} = \frac{5}{3} \Rightarrow AE = 4.5 \times \frac{5}{3} = 7.5$$

Therefore $\overline{CE} = AE - AC = 7.5 - 4.5 = 3$.

3.4. Theorems on Similarity of Triangles

SSS Similarity	**SAS Similarity**	**AA Similarity**
The three sides of one triangle are proportional to the three corresponding sides of another triangle	Two pairs of corresponding sides of the two triangles are proportional and the included angles b/n these sides are congruent.	Two angles of one triangle are congruent to two corresponding angles of another triangle
$\Delta ABC \sim \Delta DEF$ if $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}$	$\Delta ABC \sim \Delta DEF$ if $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}}$ and $\angle A \cong \angle D$	$\Delta ABC \sim \Delta DEF$ if $\angle A \cong \angle D$ and $\angle C \cong \angle F$

In short, if $\Delta ABC \sim \Delta DEF$, then

- $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}}$ (SSS)
- $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{DF}}$ and $\angle A \cong \angle D$ (SAS)
- $\angle A \cong \angle D$ and $\angle C \cong \angle F$ (AA)

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Theorem: If the lengths of the corresponding sides of two similar triangles is k, then

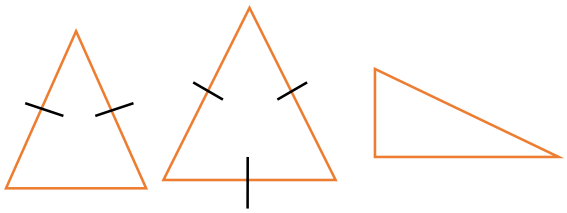
i) the ratio of their perimeters is k

ii) the ratio of their areas is k^2.

That is, if $\Delta ABC \sim \Delta DEF$, then

- $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}} = k$, and hence $\frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta DEF} = k$.
- $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}} = k$ and hence $\frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF} == \frac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \frac{b}{b'}\frac{h}{h'} = k \times k = k^2$.

Properties and types of triangles.

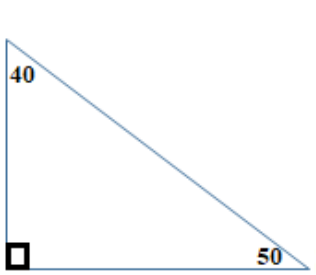


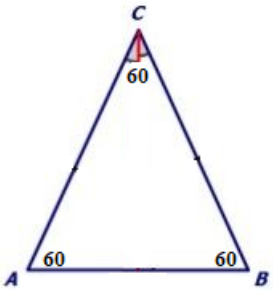
Classify Triangles by Sides

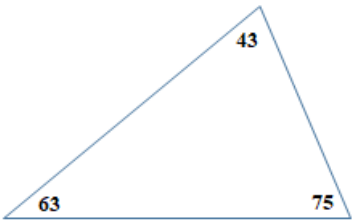
- **Isosceles Triangle:** Two congruent sides
- **Equilateral Triangle:** All congruent sides
- **Scalene Triangle:** No congruent sides

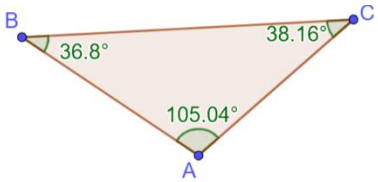
Classify Triangles by Angles

- Acute Triangle: 3 acute angles
- Right Triangle: 1 right angle
- Equiangular Triangle: All congruent angles
- Obtuse Triangle: 1 obtuse angle









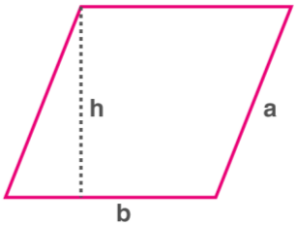
3.5. Areas of Triangle and parallelogram

The **area** represents the amount of **planar surface** being covered by **a closed geometric figure**.

3.5.1. Area of a Parallelogram

Area of a parallelogram$= b \times h$

where 'b' is the **base** and 'h' is the corresponding **altitude** (Height).

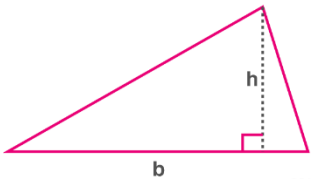


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3.5.2. Area of a Triangle

Area of a triangle $= \frac{1}{2}bh$

where "b" is the **base** and "h" is the corresponding **altitude**.



3.5.3. Application

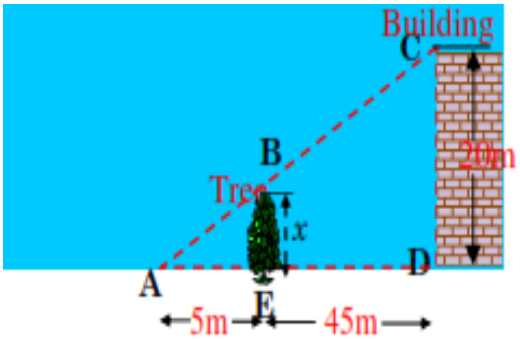
When Ali planted a tree 5m away from point A, the tree just blocked the view of a building 50m away. If the building was 20m tall, how tall was the tree?

Solution: Label the figure as shown. Let x be the height of the tree. We have

$$\frac{BE}{CD} = \frac{AE}{AD}$$

$$\frac{x}{20} = \frac{5}{50}$$

$$x = 20 \times \frac{5}{50} = 2m$$



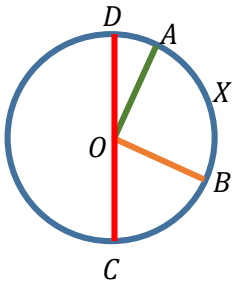
The height of the tree was $2m$.

3.6. Circles

A circle is defined as the set of points in a given plane, each of which is at fixed point of the plane. The fixed point is called the ***center*** of the circle, and the distance is the radius of the circle.
A line segment through the center of a circle with end points on the circle is called a diameter.

Chord: A chord is a straight line segment that has endpoints on the circumference of the circle.

- A circle is 360° all the way around.



Central angles are angles formed by any two radii in a circle whose vertex is at the center. In Figure below $\angle$**AOB** is a central angle. In this case, we say that $\angle AOB$ is subtended by the arc AXB (or by the chord AB). Here, we may also say that the angle $\angle AOB$ intercepts the arc $\widehat{AXB}$.

Arc: A part of a circle between any two points on the circle, say between A and B, is called an arc and is denoted by $\widehat{AB}$.

- $\widehat{AB}$ indicates the arc AB, and m$\widehat{AB}$ indicates the degree measure of arc AB.

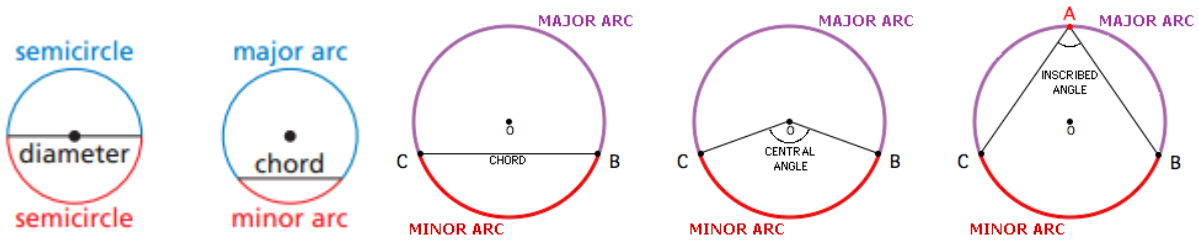
Semicircle: an arc whose endpoints are the endpoints of a diameter. An arc whose measure equals 180 degrees is called a semicircle

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Minor arc: an arc that is less than a semicircle. An arc whose measure is less than 180 degrees is called a minor arc.

Major arc: an arc that is more than a semicircle. An arc whose measure is greater than 180 degrees is called a major arc.

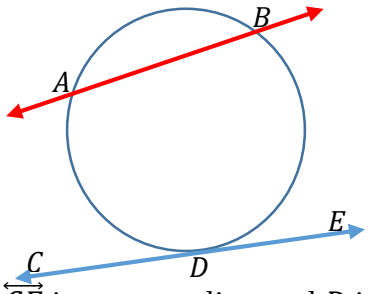
In the above figure, $\overline{OA}, \overline{OB}$ are radius, $\overline{DC}$ is a diameter, $\widehat{DBC}$ is semicircle, $\widehat{BC}$ is a minor arc and $\widehat{BDC}$ is major arc of the circle.



- **Degree measure of a semicircle:** This is 180°. Its unit length is half of the circumference of the circle.
- **Degree measure of a minor arc:** The same as the measure of its corresponding central angle. Its unit length is a portion of the circumference. Its length is always less than half of the circumference.
- **Degree measure of a major arc:** This is 360° minus the degree measure of the minor arc that has the same endpoints as the major arc. Its unit length is a portion of the circumference and is always more than half of the circumference.

3.5.1. Secant and Tangent line

- **Secant:** Any line that contains a chord is a secant.
- **Tangent:** Any line in the same plane as a circle and intersecting the circle at exactly one point is a tangent.
- **Point of tangency:** The point where a tangent line intersects a circle is the point of tangency.

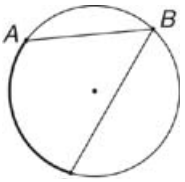


- $\overleftrightarrow{AB}$ is a secant line, $\overleftrightarrow{CE}$ is a tangent line, and D is the point of tangency for $\overleftrightarrow{CE}$.

- **Inscribed angle:** Observe that the vertex of an inscribed angle $\angle ACB$ is on the arc ACB. This arc, ACB, can be a semicircle, a major arc or a minor arc. Here also, the inscribed angle $\angle ACB$ is said to be subtended by the arc $\widehat{AXB}$ (or by the chord AB).
- **Intercepted arc:** Corresponding to an angle, this is the portion of the circle that lies in the interior of the angle together with the endpoints of the arc.

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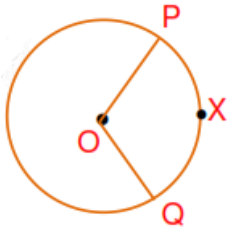
For example, in Figure beside, $\angle ABC$ is an inscribed angle and $\widehat{AC}$ is its intercepted arc. Here also, the inscribed angle $\angle ABC$ is said to be subtended by the arc $\widehat{AC}$. Observe that the vertex of an inscribed angle $\angle ACB$ is on the arc ACB. This arc, ACB, can be a semicircle, a major arc or a minor arc.



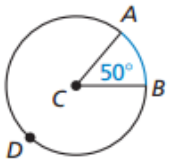
3.5.2. Angle formed by chord

Note: The measure of a central angle equals the angle measure of the arc it intercepts.

$$m\angle(POQ) = m(\widehat{PXQ})$$



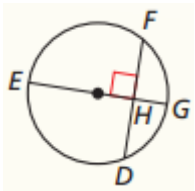
Example: If $m\angle ACB = 50°$, find $m\widehat{ADB}$.



Solution: $m\widehat{ADB} = 360° - m\widehat{AB} = 360° - 50° = 310°$.

Theorem (Perpendicular Chord Bisector Theorem): If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and its arc.

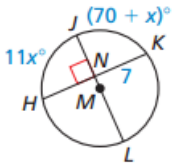
If $\overline{EG}$ is a diameter and $\overline{EG} \perp \overline{DF}$, then $HD \cong HF$ and $\widehat{GD} \cong \widehat{GF}$.



Example: Find i) $\overline{HK}$ ii) $m\widehat{HK}$

Solution: i) Diameter $\overline{JL}$ is perpendicular to $\overline{HK}$. So, by the perpendicular chord bisection theorem, $\overline{JL}$bisects $\overline{HK}$, and $HN = NK$.

So, $HK = 2(NK) = 2(7) = 14$.



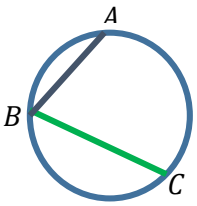
ii) Diameter $\overline{JL}$ is perpendicular to $\overline{HK}$. So, by perpendicular chord bisector theorem, $\overline{JL}$ bisects $\widehat{HK}$, and $m\widehat{HJ} = m\widehat{JK}$.

$$m\widehat{HJ} = m\widehat{JK}$$
$$11x° = (70 + x)°$$
$$10x = 70$$
$$x = 7$$

So, $m\widehat{HJ} = m\widehat{JK} = (70 + x)^{\circ} = 77°$ and $m\widehat{HK} = 2(m\widehat{HJ}) = 2(77)° = 154°$

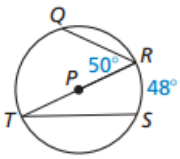
Theorem (Measure of an Inscribed Angle): The measure of an inscribed angle is one-half the measure of its intercepted arc.

$$m\angle(ABC) = \frac{1}{2}m(\widehat{AC})$$



Corollary: Angles inscribed in the same arc of a circle (*i.e., subtended by the same arc*) are equal.

Example: Find i) $m\angle T$ ii) $m(\widehat{QR})$



Solution: i) $m\angle T = \frac{1}{2}m(\widehat{RS}) = \frac{1}{2}(48°) = 24°$

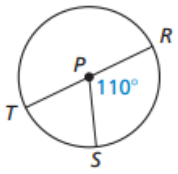
ii) $m(\widehat{TQ}) = 2m\angle R = 2 \times 50° = 110°$.

Since $\widehat{TQR}$ is a semicircle, $m(\widehat{QR}) = 180° - m(\widehat{TQ}) = 180° - 100° = 80°$

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Example: If P is the center and $\overline{RT}$ is a diameter, find the measure

i) $\widehat{RTS}$ ii) $\widehat{RST}$



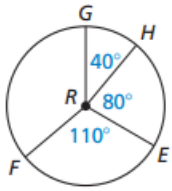
Solution: i) Since $\widehat{RS}$ is minor arc, so $m(\widehat{RS}) = m\angle RPS = 110°$.

Hence, $m(\widehat{RTS}) = 360° - m(\widehat{RS}) = 360° - 110° = 250°$.

ii) Since $\overline{RT}$ is a diameter, so $\widehat{RST}$ is a semicircle, and $m(\widehat{RST}) = 180°$

Example: Find the measure of each arc:

i) $\widehat{GE}$ ii) $\widehat{GEF}$ iii) $\widehat{GF}$

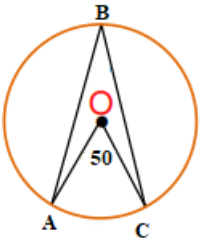


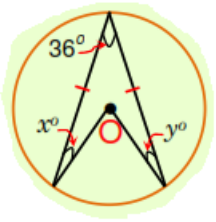
Solution: i) $m(\widehat{GE}) = m(\widehat{GH}) + m(\widehat{HE}) = 40° + 80° = 120°$

ii) $m(\widehat{GEF}) = m(\widehat{GE}) + m(\widehat{EF}) = 120° + 110° = 230°$

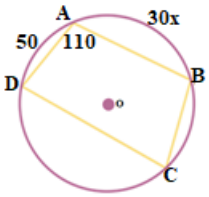
iii) $m(\widehat{GF}) = 360° - m(\widehat{GEF}) = 360° - 230° = 130°$

Homework: Find i) $m(\angle ABC)$ **ii)** the measure of the angles x and y.





Example: Solve for x.



Solution: The full circle is 360°. If we can find the measure of arc DB, we can set up an equation to solve for $\boldsymbol{x}$, because $m\widehat{DB} + m\widehat{BA} + m\widehat{AD} = 360°$, and we know that $m\widehat{BA} = 30x$ and $m\widehat{AD} = 50$.

From the diagram we can see that $m\angle DAB = 110°$. The intercepted arc that belongs to this angle is arc $\widehat{DB}$. The intercepted arc has a measure of twice the inscribed angle

$$m\widehat{DB} = 2\angle DAB = 110° = 2(110°) = 220°$$

Now we can use the equation we wrote earlier to solve for x.

$$m\widehat{DB} + m\widehat{BA} + m\widehat{AD} = 360°$$

$$220° + 30x + 50° = 360°$$

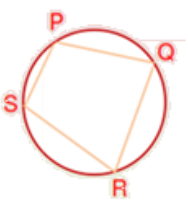
$$270° + 30x = 360°$$

$$30x = 90°$$

$$x = 3°$$

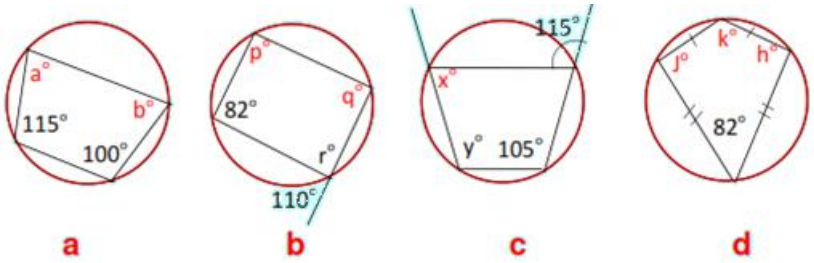
Corollary: Points P, Q, R and S all lie on a circle. They are called concyclic points. Joining the points P, Q, R and S produces a cyclic quadrilateral. The opposite angles of a cyclic quadrilateral are supplementary, i.e.,

$m(\angle P) + m(\angle R) = 180°$ and $m(\angle S) + m(\angle Q) = 180°$



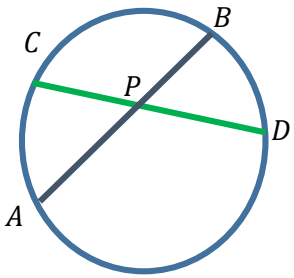
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Homework: Calculate the lettered angles in each of the following:



Theorem (Angles inside the Circle Theorem)

The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

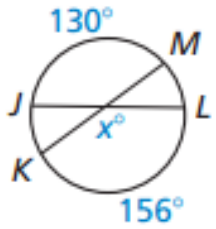


$$m(\angle BPD) = \frac{1}{2}\left(m(\widehat{AC}) + m(\widehat{BD})\right)$$

Theorem: If two chords intersect inside a circle, then the product of the segments of one chord equals the product of the segments of the other chord. Then

$$(\overline{CP})(\overline{PD}) = (\overline{AP})(\overline{PB}).$$

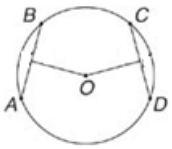
Example: Find the value of x



Solution: Since the chords $\overline{JL}$ and $\overline{KM}$ intersect inside the circle,

$$x = \frac{1}{2}\left(m(\widehat{JM}) + m(\widehat{LK})\right) = \frac{1}{2}(130° + 156°) = 143°$$

Theorem: In a circle, two chords are equal in measure if and only if they are equidistant from the center.

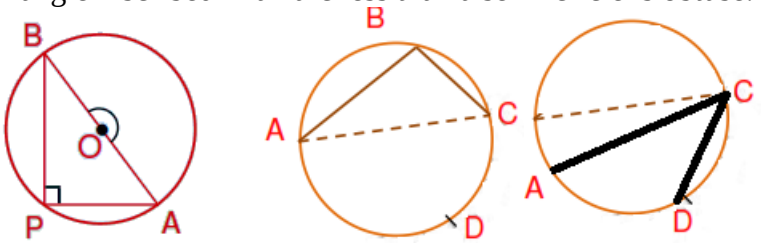


AB = CD if and only if OX = OY.

Corollary: i) The angle inscribed in a semi-circle is a right angle.

ii) An angle inscribed in an arc greater than a semi-circle is acute.

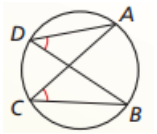
iii) An angle inscribed in an arc less than a semi-circle is obtuse.



$m(\angle ABC) = 90°$ $m(\angle ABC) > 90°$ $m(\angle ACD) < 90°$

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Theorem (**Inscribed angles of a circle**): If two inscribed angles of a circle intercept the same arc, then the angles are congruent.

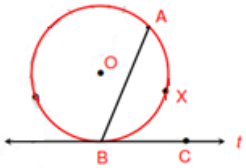


$$\angle ADB \cong \angle ACB$$

3.7. Angles and Arcs Determined by Lines Intersecting Outside a Circle

3.7.1. Angle formed by tangent

Theorem: An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

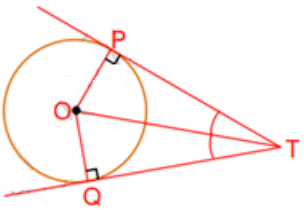


$$m(\angle ABC) = \frac{1}{2}m(\widehat{AXB})$$

Theorem: If two tangent segments are drawn to a circle from an external point, then,

i the tangents are equal in length, and

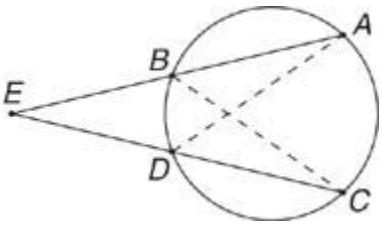
ii the line segment joining the center to the external point bisects the angle between the tangents.



If TP is a tangent to a circle at P whose centre is O and TQ is another tangent to this circle at Q, then,

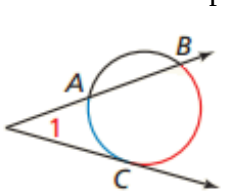
i $TP = TQ$ **ii** $m(\angle OTP) = m(\angle OTQ)$

Theorem: If two secant segments intersect outside a circle, then the product of the secant segment with its external portion equals the product of the other secant segment with its external portion.

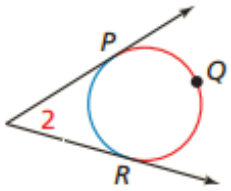


Then, $(EB)(EA) = (ED)(EC)$.

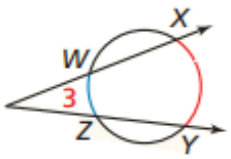
Theorem (Angles outside the Circle Theorem): If a tangent and a secant, two tangents, or two secants intersect outside a circle, then the measure of the angle formed is one-half the difference of the measures of the intercepted arcs.



$m\angle 1 = \frac{1}{2}(m\widehat{BC} - m\widehat{AC})$



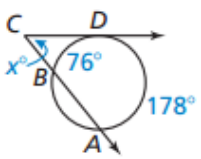
$m\angle 2 = \frac{1}{2}(m\widehat{PQR} - m\widehat{PR})$



$m\angle 3 = \frac{1}{2}(m\widehat{XY} - m\widehat{WZ})$

Theorem: If two tangent segments intersect outside a circle, then the tangent segments have equal measures.

Example: Find the value of x.

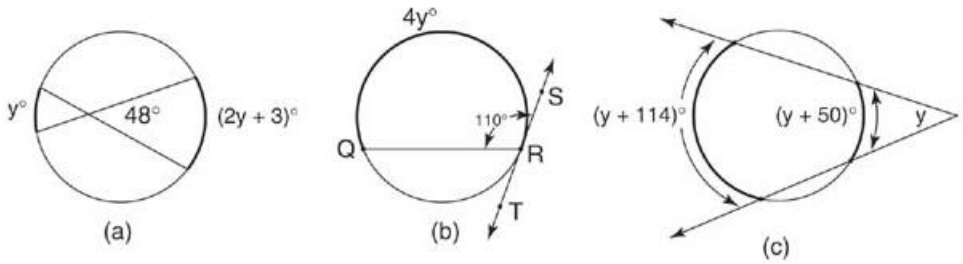


Solution: The tangent CD and the secant CB intersect outside the circle, hence

$$m\angle BCD = \frac{1}{2}(m\widehat{AD} - m\widehat{BD}) = \frac{1}{2}(178° - 76°) = 51$$

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Example: Find the value of y

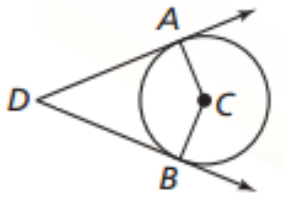


Solution: (a) $48° = \frac{1}{2}\big(y + (2y + 3)\big) \Longrightarrow 96° = 3y + 3 \Longrightarrow y = 31°$.

(b) $110° = \frac{1}{2}(4y) \Longrightarrow y = 55°$

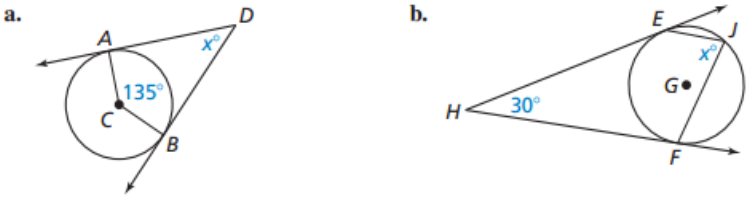
(c) $y = \frac{1}{2}\big((y + 114) - (y + 50)\big) \Longrightarrow 2y = y + 114 - y - 50 \Longrightarrow 2y = 64° \Longrightarrow y = 32°$.

Theorem: (Circumscribed Angle Theorem)
The measure of a circumscribed angle is equal to 180° minus the measure of the central angle that intercepts the same arc.



$$m\angle ADB = 180° - m\angle ACB$$

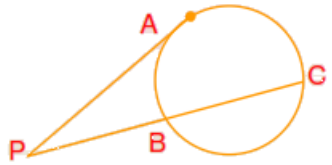
Example: Find the value of x.



Solution: a) $x = m\angle ADB = 180° - m\angle ACB = 180° - 135° = 45°$

b) $x = m\angle EJF = \frac{1}{2}m\widehat{EF} = \frac{1}{2}m\angle EGF = \frac{1}{2}(180° - m < EHF) = \frac{1}{2}(180° - 30°) = 75°$

Theorem: If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by

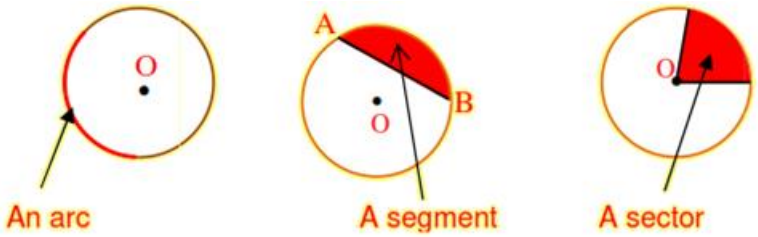


$$(PA)^2 = (PA)(PC)$$

3.7.2. Arc lengths, perimeters and areas of segments and sectors

- Circumference of a circle= $2\pi r$ or πd.
- Area of a circle= πr^2
- Part of the circumference of a circle is called an arc.
- A segment of a circle is a region bounded by a chord and an arc.
- A sector of a circle is bounded by two radii and an arc.

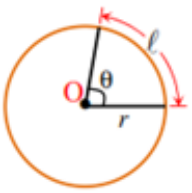
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Arc length

- The length l of an arc of a circle of radius r that subtends an angle of θ at the center is given by

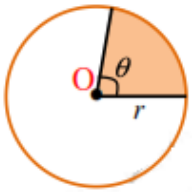
$$l = \frac{\theta}{360°} 2\pi r = \frac{\pi r \theta}{180°}$$



The area and perimeter of a sector

- The area A of a sector of radius r and central angle θ is given by

$$A = \frac{\theta}{360°} \times \pi r^2 = \frac{\pi r^2 \theta}{360°}$$

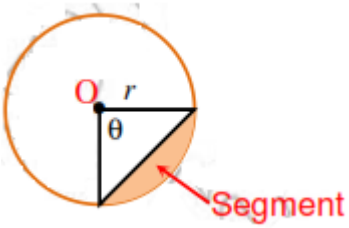


- The perimeter P of the sector is the sum of the radii and the arc that bound it.

$$P = 2r + \frac{\pi r \theta}{180°}$$

The area and perimeter of a segment

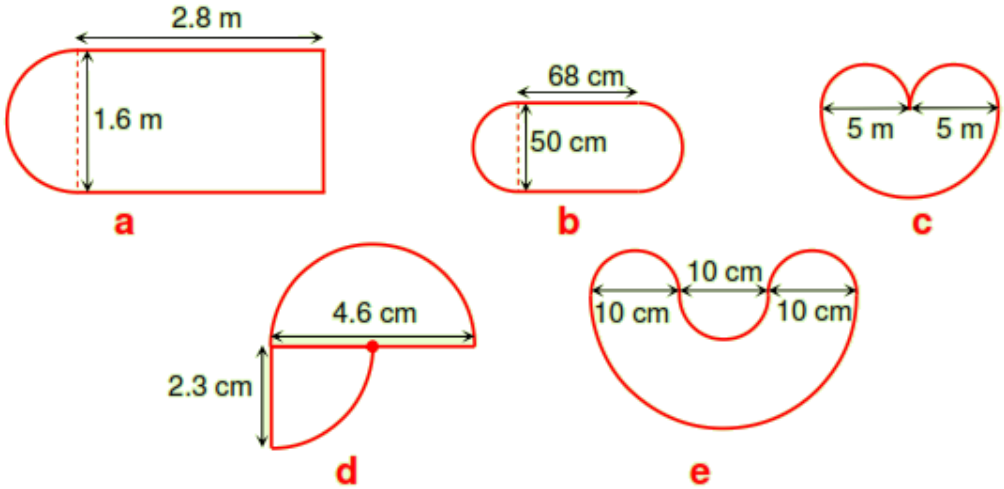
The area A and perimeter P of a segment of a circle of radius r, cut off by a chord subtending an angle θ at the center of a circle are given by



$$A = \frac{\pi r^2 \theta}{360°} - \frac{1}{2} r^2 \sin\theta \ (sector\ area\ -\ triangle\ area)$$

$$P = 2r \sin\frac{\theta}{2} + \frac{\pi r \theta}{180°} \ (chord\ length\ +\ arc\ length)$$

Homework: Calculate the perimeter and area of each of the following figures. All curves are semicircles or quadrants.



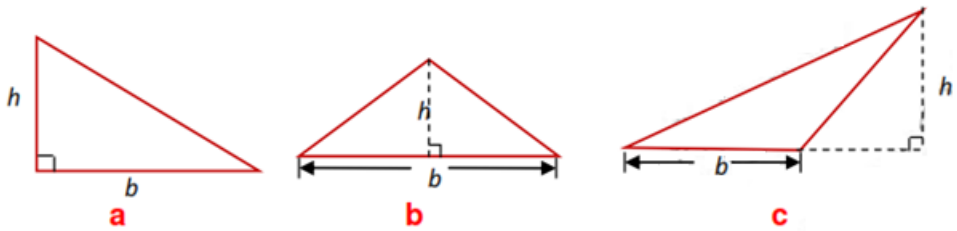
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3.8. Areas of Triangles and Parallelograms

3.8.1. Area of Triangle

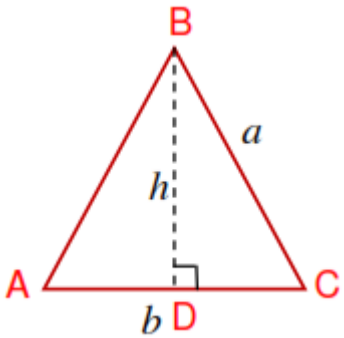
The area A of a right angle triangle with perpendicular sides of length a and b is given by

$$A = \frac{1}{2}bh$$



The area A of any triangle with sides a and b units long and angle C ($\angle C$) included between these sides is

$$A = \frac{1}{2}ab\ sin\angle C$$



Heron's formula, which is often used to find the area of a triangle when its three sides are given.

Theorem: Heron's formula

The area A of a triangle with sides a, b and c units long and semi- perimeter $s = \frac{1}{2}(a+b+c)$ is given by

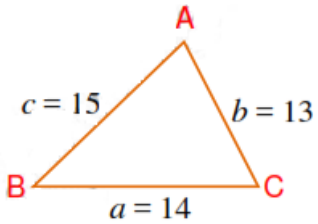
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example: Given ΔABC. If $AB = 15$ units, $BC = 14$ units and $AC = 13$ units, find

a) the area of ΔABC.

b) the length of the altitude from the vertex A.

c) the measure of ∠B.



Solution: Since $a = 14 \Longrightarrow s - a = 7$

Since $b = 13 \Longrightarrow s - b = 8$

Since $c = 15 \Longrightarrow s - c = 6$

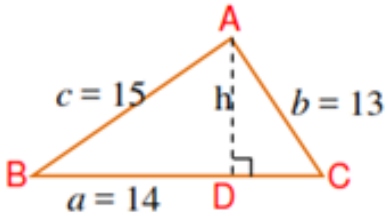
$a + b + c = 42 \Longrightarrow (s-a) + (s-b) + (s-c) = 21$

Therefore, $s = \frac{a+b+c}{2} = \frac{42}{2} = 21$

Thus, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(7)(8)(6)} = 84$

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b. Let the altitude from the vertex A to the corresponding vase BC be h, meeting BC at D as shown. Then area of $\Delta ABC = \frac{1}{2}BC \times h$



$$84 = \frac{1}{2}14h = 7h$$

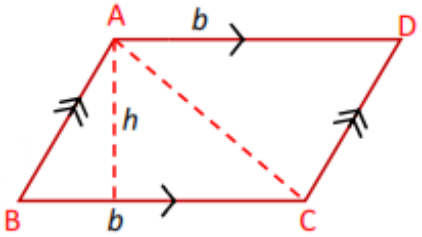
$$h = \frac{84}{7} = 12 \text{ units.}$$

Therefore, the altitude of ΔABC from the vertex A is 12 units long.

c. In the right angle triangle ABD we see that

$$\sin(\angle B) = \frac{AD}{AB} = \frac{h}{c} = \frac{12}{15} = 0.8$$

Then, $\angle B = \sin^{-1} 0.8 = 53°$.



3.8.2. Area of parallelograms

Theorem: The area A of a parallelogram with base b and perpendicular height h is

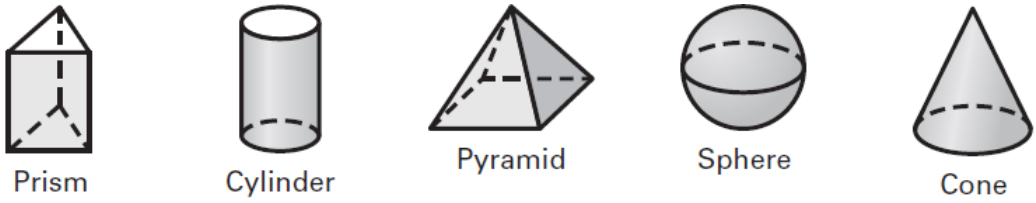
$$A = bh$$

3.9. Surface area and volume of solid figures (Prism, Cylinder, Cone and Sphere)

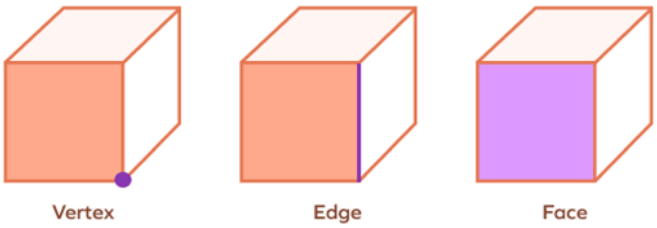
Solid shapes are three-dimensional (3D) geometric shapes that occupy some space and have length, breadth, and height. **Solid** shapes are classified into various

Example of solid figures include cone, cube, cylinder, pyramid, rectangular prism, and sphere.

Example of non-solid figures include Circle, triangle, Square (these are some 2 dimensional shapes and hence not solids).



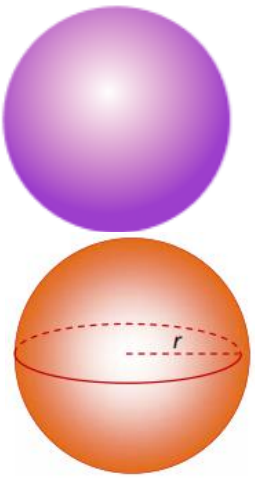
- **Faces**: A face is a single flat surface of a solid shape, and there can be more than one face of a shape.
- **Vertices**: A vertex is a point where two or more lines meet, forming a corner. It is also the point of intersection of edges.
- **Edges**: An edge is a line segment at the boundary of a solid shape that joins one vertex to another. The edges serve as the junction of two faces.



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- **Volume**: The volume helps in calculating how much matter can be stored or how much a a solid shape can hold inside it.
- **Surface area: The** amount of external space that covers a three-dimensional shape is called it's surface area. The surface area of a solid shape is categorized into three types, namely ***lateral surface area, curved surface area, total surface area***.
- **Lateral Surface Area**: The lateral surface area is known as the area of all the faces except the bottom and top faces or bases.
- **Curved Surface Area:** The curved surface area is known as the area of all the curved regions of the solid.
- **Total Surface Area:** The total surface area is the area of all the faces (including top and bottom faces) of the solid object.

3.9.1. Sphere



A sphere is round in shape, like the moon or a ball. Every point on a sphere is located such that it is at an equal distance from a central point on the sphere.

A **sphere** does not have any vertices or edges. So, it has only one surface.

The following are formulas related to the sphere shape:

Total Surface Area (A) and the volume (V) of a sphere of radius r

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

A sphere is a solid figure with a round shape. It has a curved surface, defined in three-dimensional space. Every point on the surface of a cube is equidistant from the center.

Properties of a Sphere

1. A sphere has no edges or vertices (corners).
2. It has one surface, i.e., curved.
3. It is shaped like a ball and is perfectly symmetrical.
4. All points on the surface of the cube are the same distance (r) from the center.

Example: Find the volume and surface area of a spherical football with a diameter of $10\ m$.

Solution: Since $d = 2r, r = \frac{10}{2} = 5m$.

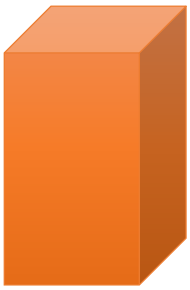
Surface area: $A = 4\pi r^2 = 4\pi 5^2 = 100\pi m^2$

Volume: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi 5^3 = \frac{500}{3}\pi m^3$.

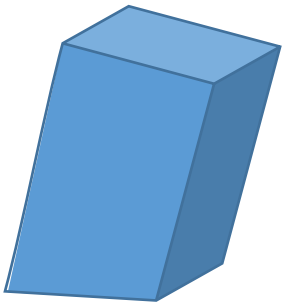
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3.9.2. Prisms

- A prism is a solid figure formed by two congruent polygonal regions in parallel planes, along with three or more parallelograms, joining the two polygons. The polygons in parallel planes are called bases.
- A prism is named by its base. Thus, a prism is called triangular, rectangular, pentagonal, etc., if its base is a triangle, a rectangle, a pentagon, etc., respectively.
- In a prism, the lateral edges are equal and parallel, and the lateral faces are parallelograms.
- A right prism is a prism in which the base is perpendicular to a lateral edge.
- Otherwise it is an oblique prism.



Right Prism



Oblique Prism

- In a right prism
  - ✓ All the lateral edges are perpendicular to both bases.
  - ✓ The lateral faces are rectangles.
  - ✓ The altitude is equal to the length of each lateral edge.
- A regular prism is a right prism whose base is a regular polygon.
- The area A of a regular n −sided polygon with radius r is

$$A = \frac{1}{2} nr^2 \sin\frac{360°}{n}$$

Surface area and volume of prisms

- The lateral surface area (A_L) of a prism is the sum of the areas of its lateral faces.

$$A_L = Ph, P \text{ is perimeter of the base}$$

- The total surface area (A_T) of a prism is the sum of the lateral areas and the area of the bases (A_B).

$$A_T = 2A_B + A_L$$

- The volume (V) of any prism is equal to the product of its base area (A_B) and its altitude (h).

$$V = A_B h$$

Example: Fin the total surface area and volume of hexagonal prism whose base edge is 6cm and height 8cm.

Solution: Lateral surface area (A_L) of a hexagonal prism:

$$A_L = Ph = 6 \times 6 \times 8 = 288\ cm^2$$

Area of hexagon with edge 6cm ($r = 6\ cm$) is,

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$$A_B = \frac{1}{2}nr^2 \sin\frac{360°}{n} = \frac{1}{2}6(6^2)\sin 60° = 54\sqrt{3}\ cm^2$$

Total surface area (A_T) of a hexagonal prism:

$$A_T = A_L + 2A_B = 288\ cm^2 + 54\sqrt{3}\ cm^2$$

Volume (V) of a hexagonal prism:

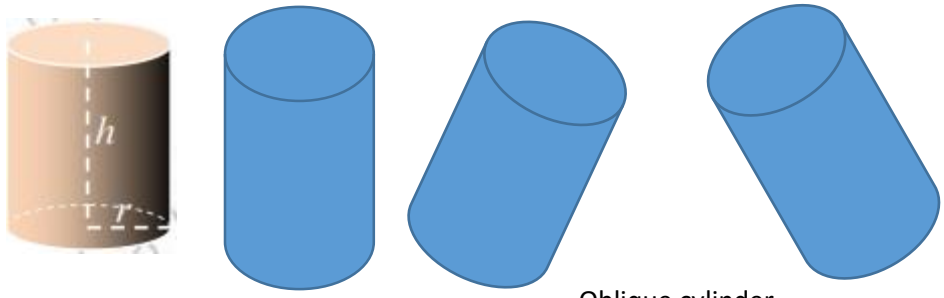
$$V = A_B h = \left(54\sqrt{3}\ cm^2\right)(8\ cm) = 432\sqrt{3}\ cm^2$$

3.9.3. Cylinder

Cylinders are the same as prisms except the bases are circles.

A cylinder is a solid shape that holds two parallel bases that are circular in shape and are joined by a curved surface (like a tube), at a fixed distance.

Right circular cylinder: The line segment joining the centers of the bases is perpendicular to the bases.



Right circular cylinder Oblique cylinder

A cylinder is a solid shape with a curved surface joining its top and bottom circular bases. Think of it as a can of toffees.

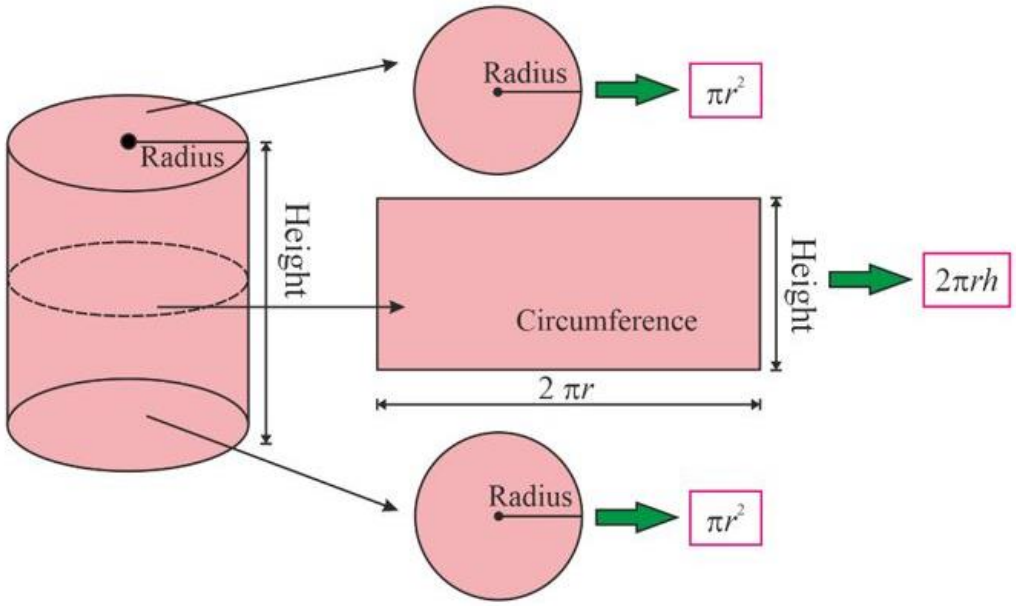
The following are formulas related to the cylinder shape:

Surface Area: $A_L = hC = 2\pi rh$, where r is the radius of the base and h is the height and C is the circumference of its base.

Total Surface Area: $A_T = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$, where r is the radius of the base and h is the height.

Volume: $V = \pi r^2 h$, where r is the radius of the base and h is the height.

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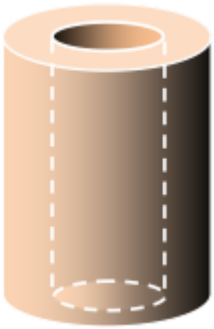


Properties of a Cylinder

1. A cylinder has two flat surfaces, i.e., base and top.
2. It has one curved surface.
3. The bases of the cylinder are always congruent and parallel.
4. It has two identical ends that are either circular or oval in shape.

Example: A circular hole of radius 2 units is drilled through the center of a right circular cylinder whose base has radius 3 units and whose altitude is 4 units. Find the total surface area of the resulting figure.

Solution: Let *R* be the radius of the bigger cylinder and *r* be the radius of the smaller cylinder then



i Area of the resulting base $= 2(\pi R^2 - \pi r^2)$

$$= 2(\pi \times 3^2 - \pi \times 2^2)\ unit^2$$
$$= 10\pi\ unit^2$$

ii Lateral surface area of the resulting figure

= lateral surface area of the bigger cylinder + lateral surface area of inner (smaller) cylinder

$$= (2\pi Rh + 2\pi rh)unit^2$$
$$= [2\pi(3)4 + 2\pi(2)4]unit^2$$
$$= 40\pi\ unit^2$$

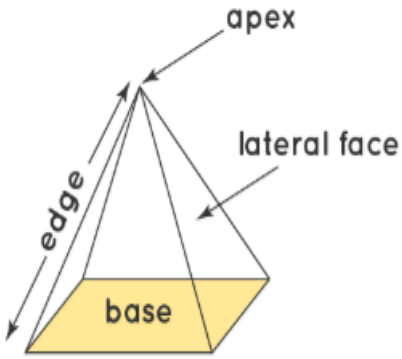
Therefore, total surface area of the resulting figure$= (10\pi + 40\pi) = 50\pi\ unit^2$.

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3.9.4. Pyramids

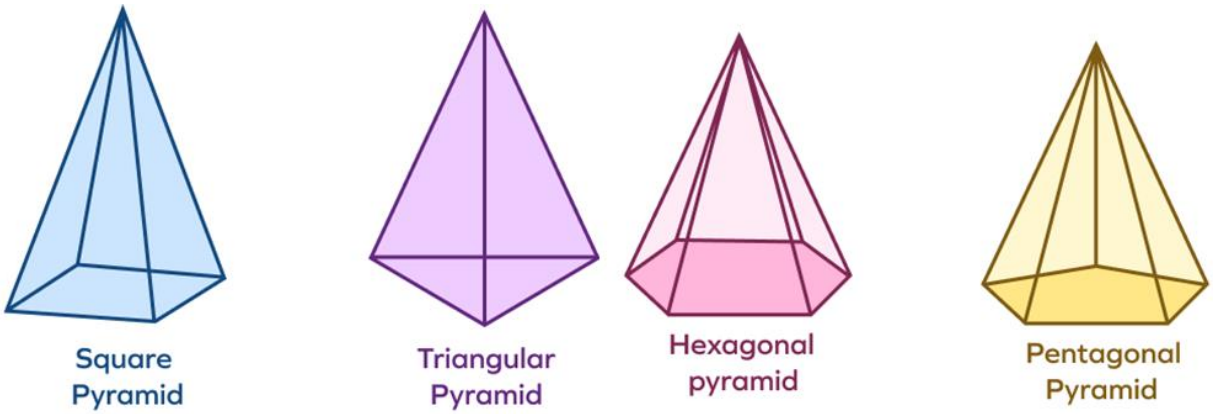
A **pyramid** is a solid that has the following characteristics.

- It has one **base,** which is a polygon.
- The vertices of the base are each joined to a point, not in the plane of the base. This point is called the **vertex** of the pyramid.
- The triangular sides, all of which meet at the vertex, are its **lateral faces**. The faces of the pyramid except the base are called lateral faces.
- The segments where the lateral faces intersect are **lateral edges**.
- The perpendicular segment from the vertex to the plane of the base is the **altitude** of the pyramid.
- **Apex** is the vertex that is opposite to the base and gives the shape to the pyramid.



A pyramid is a polyhedron with a polygon base and all its lateral faces are triangular in shape.

Pyramids are typically categorized by the shape of their bases.



A pyramid with:

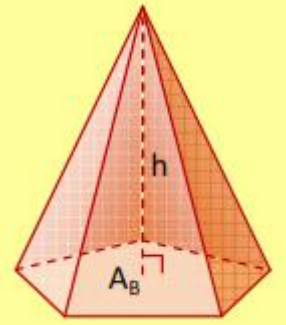
1. A triangular base is known as a tetrahedron.
2. A quadrilateral base is known as a square pyramid.
3. A pentagon base is known as a pentagonal pyramid.
4. A regular hexagon base is known as a hexagonal pyramid.

Properties of a Pyramid

1. A pyramid is a polyhedron with a polygon base and a vertex with straight lines.
2. It has one curved surface and one flat surface.
3. A cone is a rotated triangle.

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Surface Area of a Pyramid



A_B = base area, P = perimeter, A = altitude, and l = slant height

Lateral surface area of a regular pyramid:

$$A_L = \frac{1}{2}Pl$$

Total Surface Area:

$$A_T = A_B + A_L = \frac{1}{2}Pl + A_B$$

Volume of a pyramid

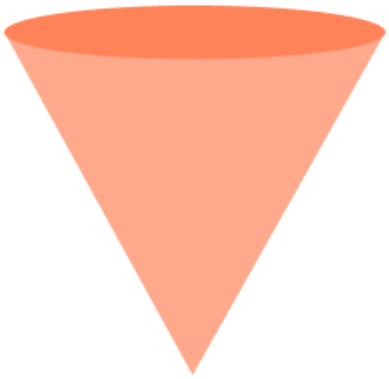
$V = \frac{1}{3}A_B h$ cubic units

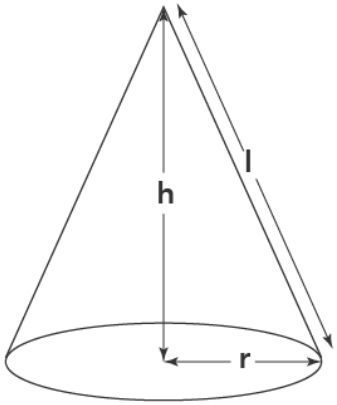
3.7.4. Cone

A cone is a distinctive three-dimensional geometric figure with a flat and curved surface pointed toward the top. A cone has 3 dimensions—its radius, height, and slant height.

A cone is a solid shape that has a flat surface and a curved surface, pointing towards the top. It is formed by a set of line segments connected from the circular base to a common point, which is known as the apex or vertex.

A birthday cap and a funnel are some examples of the cone shape.





The following are formulas related to the cone shape:

Surface Area: $A_L = \pi rl$, where r is the radius of the base and l is the slant height. Using pytagorus theorem, $l = \sqrt{h^2 + r^2}$.

Total Surface Area: $A_T = \pi r(l + r)$, where r is the radius of the base and l is the slant height.

Volume: $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

Properties of a Cone

1. It has a circular or oval base with an apex or a vertex.
2. It has one curved surface and one flat surface.

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3. A cone is a rotated triangle.

Example: A cone and a cylinder have the same volume of 180 cm³, and the same height, $h = 15\ cm$. Which of these two solids has the larger surface area?
Solution: Given $V_{cy} = 180\ cm^3 = \pi r^2 h$, height of the cylinder$h = 15\ cm$

Hence, radius of the cylinder $r = \sqrt{\frac{180}{\pi h}} = 1.95\ cm.$

Thus the surface area of a cylinder$= 2\pi r^2 + 2\pi rh = 2\left(\frac{22}{7}\right) \times 1.95 + 2\left(\frac{22}{7}\right) \times 1.95 \times 15 = 207.76\ cm^2$.

The volume of the cone $V_{cone} = 180\ cm^3 = \frac{1}{3}\pi r^2 h.$

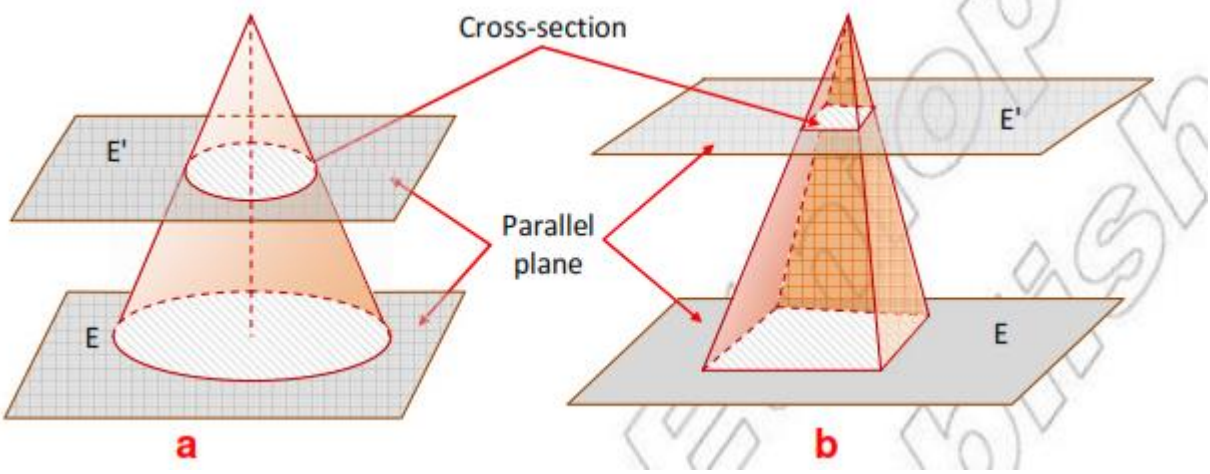
Hence, radius of the cylinder $r = \sqrt{\frac{3\times180}{\pi h}} = 3.38\ cm.$

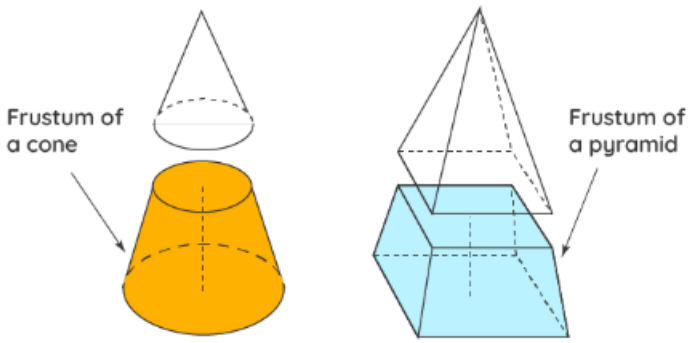
To get the slant height, s, use the pythagoras' theorem:$s = \sqrt{3.38^2 + 15^2} = 15.38\ cm$

Then, The surface area of a cone $= \pi r^2 + \pi rs = \frac{22}{7} \times 3.38^2 + \frac{22}{7} \times 3.38 \times 15.38\ cm = 199.28\ cm^2$.

3.10. Frustum of pyramids and cones

Definition: If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal crosssection of the pyramid (or the cone).





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Frustum of a pyramid

Definition: A frustum of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.
The base of the pyramid and the cross-section made by the plane parallel to it are called the bases of the frustum. The other faces are called lateral faces. The total surface of a frustum is the sum of the lateral surface and the bases.

The altitude of a frustum of a pyramid is the perpendicular distance between the bases.

Note:
i. The lateral faces of a frustum of a pyramid are trapeziums.
ii. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
iii. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
Iv. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces.

$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

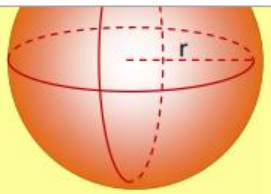


Figure 7.63

Frustum of a pyramid

$A_L = \frac{1}{2}\ell\,(P + P')$

$A_T = \frac{1}{2}\ell\,(P + P') + A_b + A'_b$

$V = \frac{1}{3}\,h'\left(A_b + A'_b + \sqrt{A_b A'_b}\right)$

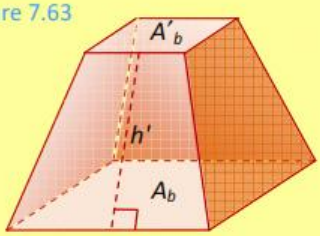


Figure 7.64

Frustum of a cone

$A_L = \frac{1}{2}\,\ell(2\pi r + 2\pi r') = \ell\,\pi(r + r')$

$A_T = \frac{1}{2}\ell(2\pi r + 2\pi r') + \pi r^2 + \pi(r')^2 = \ell\pi(r + r') + \pi(r^2 + r'^2)$

$V = \frac{1}{3}h'\,\pi(r^2 + (r')^2 + rr')$

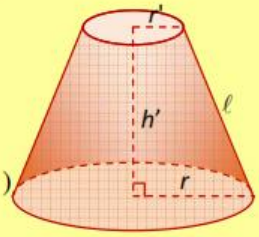
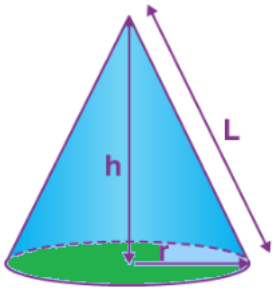
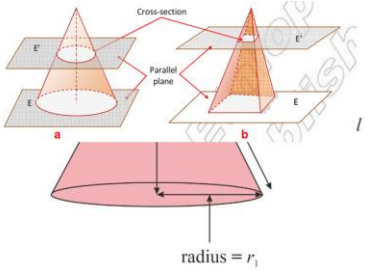


Figure 7.65

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Name	Figure			**Volume Formula**	**Surface Area Formula**	**Lateral Surface Area Formula**
Sphere		does not have any vertices or edges		$\frac{4}{3}\pi r^3$	$4\pi r^2$	$4\pi r^2$
Cone		1 flat face	1 corner	$\frac{1}{3}\pi r^2 h$	$\pi r(r+h)$	πrl
Pyramid				$\frac{1}{3}A_B h$	$B+\frac{1}{2}Ps$	
Cylinder		has 2 flat faces 1 curved face	has no corners	$\pi r^2 h$	$2\pi r(r+h)$	$2\pi rh$

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Frustum of a cone				$\frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$	$\pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$ where $l = \sqrt{h^2 + (r_1 - r_2$	
Rectangular prism				lwh	$2(lw + lh + wh)$	
Triangular prism				Bh	$2B + Ph$	

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ESSLCE Exam

1. In ΔABC,if $AB = 10\ units$, $BC = 14\ units$ and $AC = 8\ units$, then the area of the triangle is (NOVEMBER 2022, NATURAL)

A. $16\sqrt{6}$ square unit B. $16\sqrt{2}$ square unit

C. $10\sqrt{6}$ square unit D. $16\sqrt{3}$ square unit

2. Which of the following is the measure of an exterior angle of a 10 sided regular polygons? (NOVEMBER 2022, NATURAL)

A. 180° B. 36° C. 72° D. 144°

3. What is the surface area of a sphere with radius 6 cm? (NOVEMBER 2022, NATURAL)

A. $36\pi\ cm^2$ B. $24\pi\ cm^2$ C. $144\pi\ cm^2$ D. $48\pi\ cm^2$

4. If the volume of a right circular cone is $64\pi\ cm^3$ and the diameter of its base is 8 cm, what is the height of the cone? (NOVEMBER 2022, NATURAL)

A. 12 cm B. 4 cm C. 3 cm D. 8 cm

5. If the measure of each interior angle of a regular polygon is 157.5°, what is the number of sides of the polygon?

A. 14 B. 15 C. 16 D. 18

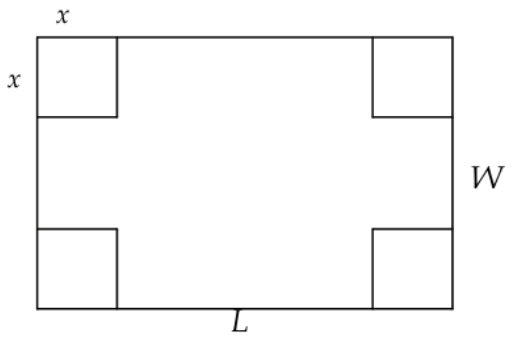
6. A hemispherical container with negligible thickness has a circular edge of perimeter 4π units. What is the outer surface area of this container?

A. $4\pi\ sq.unit$ B. $8\pi\ sq.unit$ C. $8\sqrt{2}\pi\ sq.unit$ D. $16\pi\ sq.unit$

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Worksheet

1. The figure shows a rectangular box without a top formed by cutting out squares x units on each side from a rectangular piece of length L and width W. Select the formula for calculating the surface area SA of the box.



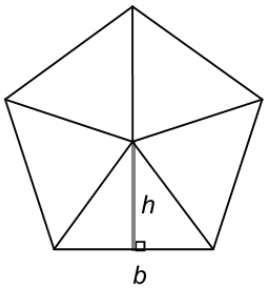
   A. $SA = LW - 4x^2$

   B. $SA = LW - x^2$

   C. $SA = (L - 2x)W - 2x) - 4x^2$

   D. $SA = (L - 2x)(W - 2x)$

2. Study the figure showing a regular pentagon. Then select the formula for computing the total area A of the pentagon.



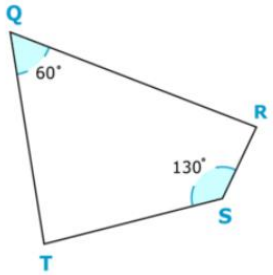
   A. $A = \frac{5}{2}hb$ B. $A = 5(h + b)$

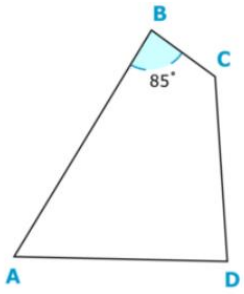
   C. $A = 5h + b$ D. $A = 5hb$

3. Calculate the area A of a regular hexagon with a side equal to 8.

   A. $96\sqrt{3}$ B. $72\sqrt{3}$ C. $48\sqrt{3}$ D. $36\sqrt{3}$

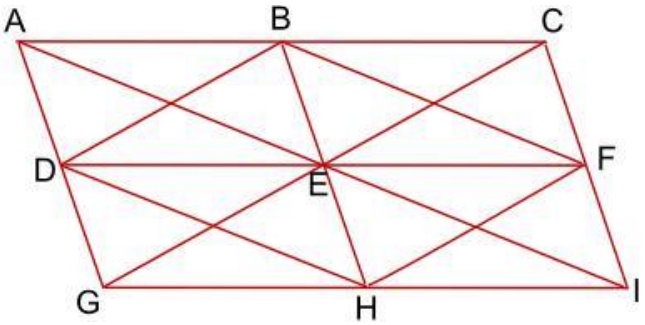
4. Use the relationships between congruent figures to find the measure of $\angle T$. Show your work.





5. Triangles ABC and DEF are congruent. Does this mean that their angle measures are the same? Why?

6. How many triangles congruent to triangle ABE (including itself) are there in the following diagram?



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Chapter 4

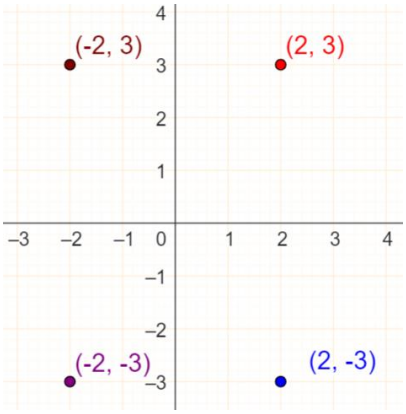
Coordinate Geometry

4.1. Introduction

Coordinate geometry is that branch of geometry in which two real numbers, called coordinates, are used to indicate the position of a point in a plane.

In a two-dimensional figure a point in plane has two coordinates. Generally, the first coordinate is read on the x −axis and the second coordinate on the y −axis. The horizontal distance of the point from the y −axis is called the x −coordinate or the *abscissa* and the vertical distance of the point from the x − axis is called the y −coordinate or the *ordinate*.

Locate the points $(2,3), (-2,3), (2,-3)$ and $(-2,-3)$.

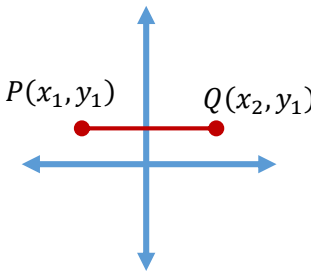


4.1. Distance between two points

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on the *xy*-coordinate plane. We can find the distance between the two points P and Q by considering three cases.

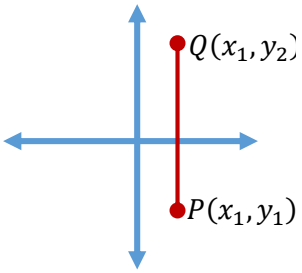
Case i: When P and Q are on a line parallel to the *x*-axis (that is, PQ is a horizontal segment) as in Figure. Since the two points P and Q have the same *y*-coordinate (ordinate), the distance between P and Q is

$$PQ = |x_2 - x_1|$$



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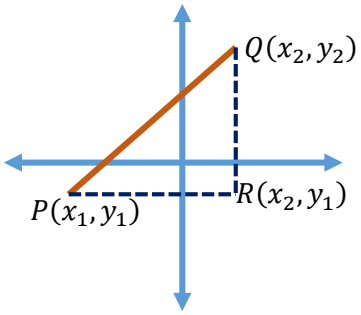
Case ii: When P and Q are on a line parallel to the y-axis (that is, PQ is a vertical segment) as in Figure.



Since the two points have the same x-coordinate (abscissa), the distance

$$PQ = |y_2 - y_1|$$

Case iii: When PQ is neither vertical nor horizontal (the general case).



To find the distance between the points P and Q, draw a line passing through P parallel to the x-axis and draw a line passing through Q parallel to the y-axis.

The horizontal line and the vertical line intersect at $R(x_2, y_1)$.

Using case i and case ii, we have

$$PR = |x_2 - x_1| \text{ and } RQ = |y_2 - y_1|.$$

Since ΔPRQ is a right angled triangle at R, you can use Pythagoras' Theorem to find the distance between points P and Q as follows:

$$PQ^2 = PR^2 + RQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

In general, **the distance d between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by**

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

This is called the *distance formula*.

Example: Find the distance between the points $P(5, -4)$ and $Q(-1, 10)$.

Solution: Let $P(5, -4)$ be denoted by $P(x_1, y_1)$ and $Q(-1, 10)$ be denoted by $Q(x_2, y_2)$.

Therefore the required distance is, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$

$$\sqrt{(-1 - 5)^2 + \big(10 - (-4)\big)^2} = \sqrt{36 + 196} = \sqrt{232} = 2\sqrt{58}$$

Solved Problems

1. Determine if the points $(1, 5), (2, 3)$ and $(-2, -11)$ are collinear.

Solution: Let $A = (1, 5)$, $B = (2, 3)$ and $C = (-2, -11)$.

Therefore, $AB = \sqrt{(1 - 2)^2 + (5 - 3)^2} = \sqrt{5}$

$$BC = \sqrt{\big(2 - (-2)\big)^2 + \big(3 - (-11)\big)^2} = \sqrt{4^2 + 14^2} = \sqrt{212}$$

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$$CA = \sqrt{\left(1-(-2)\right)^2 + \left(5-(-11)\right)^2} = \sqrt{265}$$

Since $AB + BC \neq CA$. Therefore, the points $(1,5), (2,3)$ and $(-2,-11)$ are not collinear.

2. Check whether $(5,-2), (6,4)$ and $(7,-2)$ are representing the vertices A, B, and C of the triangle respectively.

Solution: Let the points $(5,-2), (6,4)$ and $(7,-2)$ are representing the vertices A, B, and C of the given triangle respectively.

$$AB = \sqrt{(5-6)^2 + (-2-4)^2} = \sqrt{1+36} = \sqrt{37}$$

$$BC = \sqrt{(6-7)^2 + \left(4-(-2)\right)^2} = \sqrt{1+36} = \sqrt{37}$$

$$CA = \sqrt{(5-7)^2 + \left(-2-(-2)\right)^2} = \sqrt{4+0} = 2$$

Therefore, $AB = BC$. As two sides are equal in lenth, therefore, ABC is an isosceles triangle.

3. Name the type of quadrillateral formed, if any, by the following points, and give reasons for your answer: $(-1,-2), (1,0), (-1,2), (-3,0)$.

Solution: Let the points $(-1,-2), (1,0), (-1,2), (-3,0)$ be represetning the vertices A, B, C, and D of the given quadrillateral reapectively. Thus,

$$AB = \sqrt{(-1-1)^2 + (-2-0)^2} = \sqrt{8} = 2\sqrt{2}$$

$$BC = \sqrt{\left(1-(-1)\right)^2 + (0-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$CD = \sqrt{\left(-1-(-3)\right)^2 + (2-0)^2} = \sqrt{8} = 2\sqrt{2}$$

$$AD = \sqrt{\left(-1-(-3)\right)^2 + (-2-0)^2} = \sqrt{8} = 2\sqrt{2}$$

Diagonal $AC = \sqrt{\left(-1-(-1)\right)^2 + (-2-2)^2} = \sqrt{16} = 4$

Diagonal $BD = \sqrt{\left(1-(-3)\right)^2 + (0-0)^2} = \sqrt{16} = 4$

It can be observed that all sides of this quadrillateral are of the same length and also, the diagonals are of the same length. Thefrfore, the given points are the vertices of a square.

4. Find the point on the x −axis which is equidistant from $(2,-5)$ and $(-2,9)$.

Solution: We have to find a point on x −axis. Therefore, its y −coordinate will be 0. Let the point on x −axis be $(x,0)$.

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Distance between $(x, 0)$ and $(2, -5)$ is $\sqrt{(x-2)^2 + \big(0-(-5)\big)^2} = \sqrt{(x-2)^2 + (5)^2}$

Distance between $(x, 0)$ and $(-2, 9)$ is $\sqrt{\big(x-(-2)\big)^2 + \big(0-(-9)\big)^2} = \sqrt{(x+2)^2 + (9)^2}$

By the given condition, these distances are equal in measure.

$$\sqrt{(x-2)^2 + (5)^2} = \sqrt{(x+2)^2 + (9)^2}$$

$$(x-2)^2 + (5)^2 = (x+2)^2 + (9)^2$$

$$x^2 - 4x + 29 = x^2 + 4x + 85$$

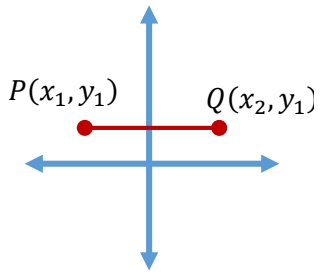
$$8x = -56$$

$$x = -7$$

Therefore, the point is $(-7, 0)$.

4.2. Division of a line segment

Note: A line segment passing through two points *A* and *B* is horizontal if the two points have the same *y*-coordinate. i.e., a line segment whose end-points are $P(x_1, y)$ and $Q(x_2, y)$ is a horizontal line segment as shown in Figure.



The coordinates of a point $R(x, y)$ dividing a line in the ratio of $m:n$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. The coordinates of the point R using the following formula:

$$x = \frac{nx_1 + mx_2}{n+m} \text{ and } y = \frac{ny_1 + my_2}{n+m}.$$

This is called the ***section formula***.

Example: Find the coordinates of the point which divides the join of $(-1, 7)$ and $(4, -3)$ in the ratio $2:3$.

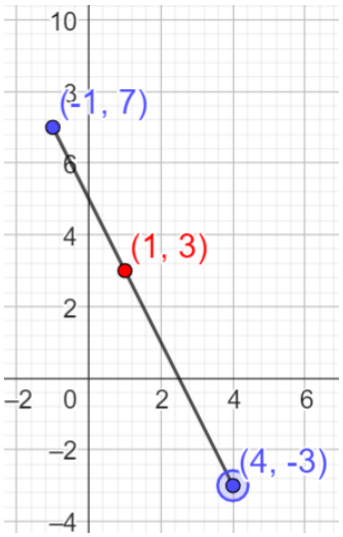
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Solution: Let $P(x, y)$ be the required point. Using the section formula, we obtain

$$x = \frac{2 \times 4 + 3 \times (-1)}{2 + 3} = \frac{8 - 3}{5} = \frac{5}{5} = 1$$

$$y = \frac{2 \times (-3) + 3 \times 7}{2 + 3} = \frac{-6 + 21}{5} = \frac{15}{5} = 3.$$

Therefore, the point $(1, 3)$.

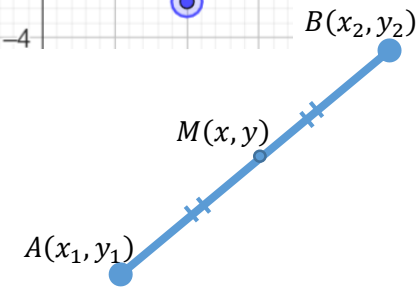


Midpoint of two given points

A point that divides a line segment into two equal parts is the mid-point of the segment.

Midpoint of line segment AB which join $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

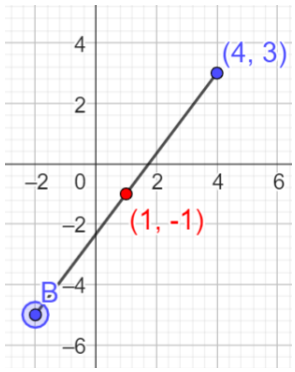


Example: Find the midpoint of $(-4, 6)$ and $(-2, 10)$

Solution:

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + (-2)}{2}, \frac{6 + 10}{2}\right) = (-3, 8)$$

Example: A line segment has one end-point at $A(4, 3)$. If its mid-point is at $M(1, -1)$, where is the other end-point?



Solution: Let $B(x_2, y_2)$ be the end point. Then the midpoint of AB is

$$\left(\frac{4 + x_2}{2}, \frac{3 + y_2}{2}\right) = (1, -1)$$

Thus, $\frac{4+x_2}{2} = 1$ and $\frac{3+y_2}{2} = -1$. Thus, $x_2 = -2$ and $y_2 = -5$.

Hence, the end-point is $(-2, -5)$.

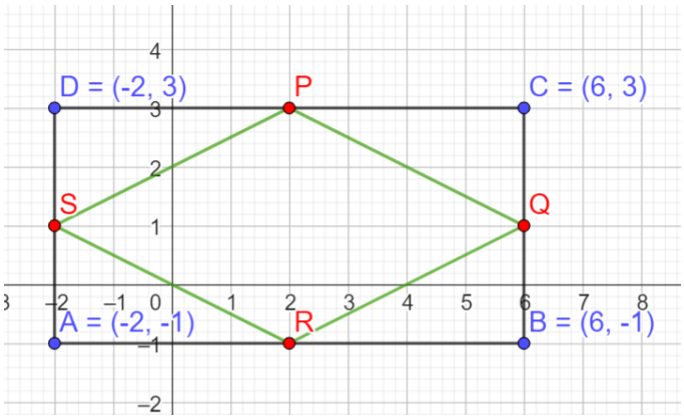
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Example: Let $A(-2,-1), B(6,-1), C(6,3)$ and $D(-2,3)$ be vertices of a rectangle. Suppose P, Q, R and S are mid-points of the sides of the rectangle.

i. What is the area of rectangle $ABCD$?

ii. What is the area of quadrilateral $PQRS$?

iii. Give the ratio of the areas in i and ii.



Solution: For the rectangle $ABCD$

$$d_{AB} = \sqrt{(6+2)^2 + (-1+1)^2} = 8$$
$$d_{CD} = |6-(-2)| = 8$$
$$d_{AD} = \sqrt{(-2+2)^2 + (-1-3)^2} = 4$$
$$d_{BC} = |-1-3| = |-4| = 4$$

Therefore, the area of rectangle $ABCD = d_{AB}d_{BC} = 8(4) = 32$.

Since P, Q, R and S are mid-points of the sides of the rectangle

$$P = \left(\frac{6+(-2)}{2}, \frac{3+3}{2}\right) = (2,3) \qquad Q = \left(\frac{6+6}{2}, \frac{3+(-1)}{2}\right) = (6,1)$$
$$R = \left(\frac{6+(-2)}{2}, \frac{-1+(-1)}{2}\right) = (2,-1) \qquad S = \left(\frac{-2+(-2)}{2}, \frac{-1+3}{2}\right) = (-2,1)$$

For the quadrilateral $PQRS$

$$d_{PQ} = \sqrt{(6-2)^2 + (1-3)^2} = \sqrt{16+4} = 2\sqrt{5}$$
$$d_{QR} = \sqrt{(2-6)^2 + (-1-1)^2} = \sqrt{16+4} = 2\sqrt{5}$$
$$d_{RS} = \sqrt{(-2-2)^2 + \big(1-(-1)\big)^2} = \sqrt{16+4} = 2\sqrt{5}$$
$$d_{SP} = \sqrt{\big(2-(-2)\big)^2 + (3-1)^2} = \sqrt{16+4} = 2\sqrt{5}$$

Therefore, the area of quadrilateral $PQRS = d_{PQ}d_{RS} = 2\sqrt{5}(2\sqrt{5}) = 20$.

Ratio of the areas:

$$k = \frac{\text{area of rectangle}}{\text{area of quadrilateral}} = \frac{32}{20} = \frac{8}{5}$$

Ratio: $8:5$

Homework: 1) The points R and S have coordinates $(-5,1)$ and $(7,-3)$ respectively. Point P lies on the line segment RS such that $RP = 3PS$. Find the coordinates of point P.

Answer: $P(4,-2)$

2) Find the value of k if the points $A(-3,3k), B(0,k)$ and $C(3,-2)$ are collinear.

Answer: $k = 2$

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Example: Determine the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points $A(2, -2)$ and $B(3, 7)$

Solution: Let the given line divide the line segment joining the points $A(2, -2)$ and $B(3, 7)$ in a ratio $k: 1$.

Coordinates of the point of division= $\left(\frac{3k+2}{k+1}, \frac{7k-2}{k+1}\right)$

This point also lies on $2x + y - 4 = 0$. Thus,

$$2\left(\frac{3k+2}{k+1}\right) + \frac{7k-2}{k+1} - 4 = 0$$

$$9k - 2 = 0$$

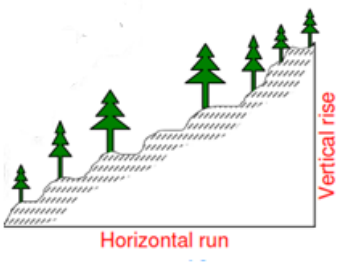
$$k = \frac{2}{9}$$

Therefore, the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points $A(2, -2)$ and $B(3, 7)$ is $2: 9$.

4.3. Equation of a straight line

Slope (Gradient) of a line

The number that describes the steepness of a hill is called the gradient (slope) of the hill. We measure the gradient of a hill by the ratio of the vertical rise to the horizontal run.

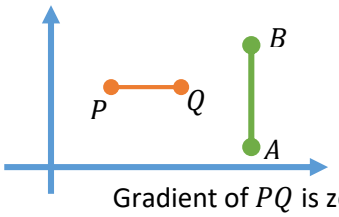


Definition: If (x_1, y_1) and (x_2, y_2) are points on a line with $x_1 \neq x_2$, then the ***gradient*** of the line, denoted by m, is given by

$$m = \frac{change\ in\ y - coordinates}{change\ in\ x - coordinates} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note: 1. If $x_1 = x_2$, that is for a vertical line ***the gradient is not defined***.

2. If the segment is vertical, the horizontal run is zero and the gradient of the segment is not defined. This is shown by segment AB.



Gradient of PQ is zero.
Gradient of AB is not defined.

Example: Given three points $P(-1, -5)$, $Q(1, -2)$ and $R(5, 4)$, find the gradient of PQ and QR. What do you conclude from your result?

$$m_{PQ} = \frac{-2 - (-5)}{1 - (-1)} = \frac{-2 + 5}{1 + 1} = \frac{3}{2}$$

$$m_{QR} = \frac{4 - (-2)}{5 - 1} = \frac{4 + 2}{4} = \frac{6}{4} = \frac{3}{2}$$

Hence, $m_{PQ} = m_{QR}$. Hence the line formed by the point PQ and QR are parallel.

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Example: A line passes through the point (5, 7) and has gradient $\frac{2}{3}$. Find the x-coordinate of a point on the line when $y = 13$.

Solution: Let $(x, y) = (x, 13)$ be a second point. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 7}{x - 5}$$

Hence, $\frac{6}{x-5} = \frac{2}{3}$. Solving for x

$$2(x - 5) = 6(3)$$

$$x = 14.$$

Slope of a Line in Terms of Angle of Inclination

The angle measured from the positive x-axis to a line, in anticlockwise direction, is called the inclination of the line or the ***angle of inclination of the line***. This angle is always less than 180°. The slope of a line may be expressed in terms of the coordinates of two points (x_1, y_1) and (x_2, y_2) on the line as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = tan\,\theta$$

Example: Find the slope of a line, if its inclination is 135°.

Solution: $m = \tan\theta = \tan 135° = -\tan 45° = -1$.

Example: Find the angle of inclination of the line containing the points $(2, 4)$ and $(0, 3)$.

Solution: $tan\,\theta = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-4}{0-2} = \frac{1}{2}$,

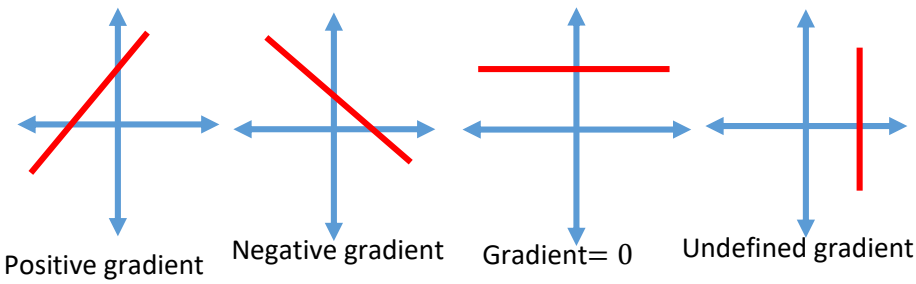
Then the angle of inclination $\theta = \tan^{-1}\left(\frac{1}{2}\right) =$

Note: Let m be the slope of a non-vertical line.

i. If $m > 0$, then the line rises from left to right.

ii. If $m < 0$, then the line falls from left to right.

iii. If $m = 0$, then the line is horizontal.



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Equations of the Straight Line

Slope-intercept form: The equation of the line with slope m and $y-$intercept b is given by $y = mx + b$.

Two-intercept form: Let a straight line intersect the coordinate axes making intercepts of a and b on $x-$axis and $y-$axis respectively, then the required equation of the line is;

$$y = \frac{a}{b}x + b$$

Slope-point form: The equation of a straight line having a slope m and passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Two-point form: If a straight line is passing through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the equation of the straight line is,

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

The most general equation of a straight line is

$$Ax + By + C = 0.$$

If $A = 0$ then the line is horizontal, and if $B = 0$ then the line is vertical. Slope, $m = -\frac{A}{B}$ and y-intercept $b = -\frac{C}{B}$.

Note: 1) The graph of every first degree (linear) equation $Ax + By + C = 0, A, B \neq 0$ is a straight line and every straight line is a graph of a first degree equation.

2) The graph of the equation $x = c$ is the vertical line through $P(c, 0)$ and has no slope.

3) The graph of the equation $y = d$ is the horizontal line through $P(0, c)$ and has slope 0.

Example: Find the slope of the line whose equation is $2x + 3y - 7 = 0$

Solution: The equation $2x + 3y - 7 = 0$ can be written as $3y = -2x + 7$. Therefore $y = -\frac{2}{3}x + \frac{7}{3}$.

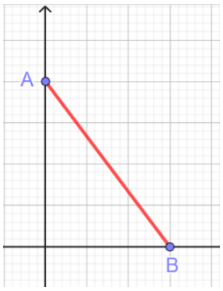
Hence the required slope is $-\frac{2}{3}$.

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Homework: Find the slope of the line passing through the point $(-2, 1)$ and the origin.

Example: Given the points $P(5, 7)$, $Q(8, 0)$ and $R(-4, 8)$, find the equation of a straight line that passes through P and the midpoint of QR.

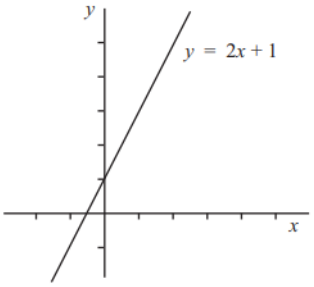
Answer: $y = x + 2$.



Intercepts

The x −intercept of a line is the point at which it crosses the x −axis.
The y −intercept of a line is the point at which it crosses the y −axis.
In the diagram the y −intercept is at A and the x −intercept at B.

Example: Consider the straight line with equation $y = 2x + 1$. This equation is in a slightly different form from those we have seen earlier. To draw a sketch of the line, we must calculate some values.



x	y
0	1
1	3
2	5

Notice that when $x = 0$ the value of y is 1. So this line cuts the y −axis at $y = 1$.

Homework: 1) Find the gradient and y −intercept for the straight lines

i) $2y - 10x = 8$ ii) $-2x + y + 3 = 0$

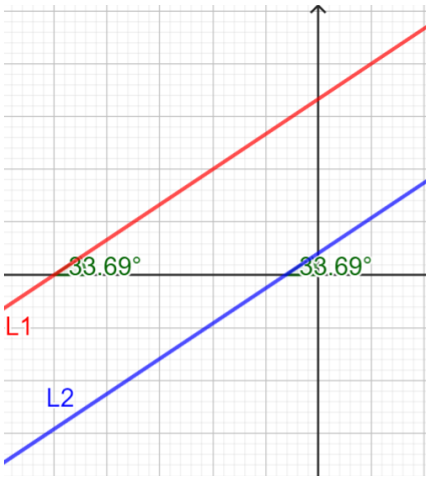
2) Find the equation of the lines with gradient ¼ passing through $(0, 5)$.

Parallel and Perpendicular Lines

Parallel lines

If two lines l_1 and l_2 are parallel then corresponding angles are equal. Conversely, if corresponding angles are equal then the lines are parallel. i.e., if $l_1 \parallel l_2$, then $\tan\theta_1 = \tan\theta_2$.

Theorem: Two lines are parallel if they have the same gradient and conversely, two lines with the same gradient are parallel.



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Example: Find the equation of the line that is parallel to the line $y = -2x + 6$ and passing through the point $A(1, 10)$.

Solution: The gradient of the line $y = -2x + 6$ is –2.

Therefore the line through the point $A(1, 10)$ parallel to $y = -2x + 6$ has equation:

$$y - y_1 = m(x - x_1)$$
$$y - 10 = -2(x - 1)$$
$$y = -2x + 12$$

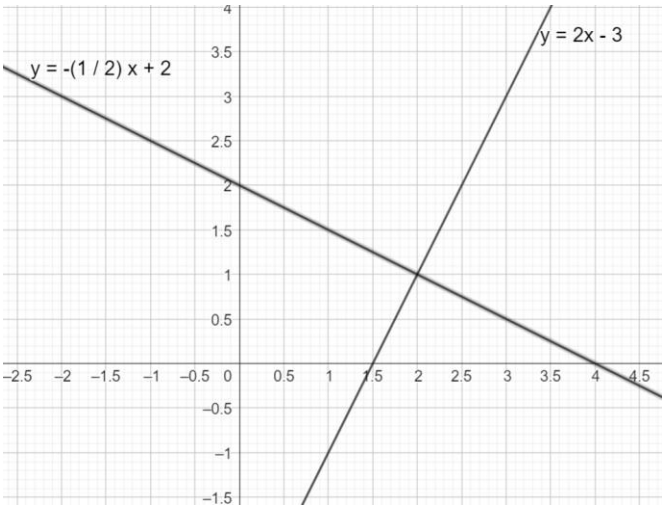
Perpendicular lines

If two non -vertical lines are perpendicular then the product of their gradients is –1. Conversely if the product of the gradients of two lines is –1 then they are perpendicular.

Theorem: Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 m_2 = -1$.

Example: The lines $l_1: y = 2x - 3$ and $l_2: y = -\frac{1}{2}x + 2$ are perpendicular.

Solution: Slope of l_1 is $m_1 = 2$, and slope of l_2 is $m_2 = -\frac{1}{2}$. Moreover, $m_1 m_2 = 2\left(-\frac{1}{2}\right) = -1$.



Homework: Find the equation of the line through $(1, -3)$ which is perpendicular to the line $x - 3y + 4 = 0$.

Point of intersection of two lines

Solve two simultaneous equations to get point of intersection.

Example: Find the point of the straight lines $y = -2x + 1$ and $y = \frac{1}{2}x + 6$.

Solution: $\begin{cases} y + 2x = 1 \\ 2y - x = 12 \end{cases}$

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Solving the system of equation we obtain $(-2, 5)$ as point of intersection.

Homework: A straight line $x + ky = 10$ is perpendicular to the straight line $y - 4x = 28$. Find

a) The value of k. Answer: $k = 4$

b) The point of intersection of the two lines. Answer: $(-6, 4)$.

Angle between two intersecting lines

The angle between tow intersecting lines l_1 and l_2 is defined to be the angle β measured counter-clockwise from l_1 to l_2. If m_1 is the slope of l_1 and m_2 is the slope of l_2, then the tangent of the angle between two lines l_1 and l_2 measured counter-clockwise from l_1 to l_2 is given by

$$tan\,\beta = \frac{m_2 - m_1}{1 + m_1 m_2}, m_1 m_2 \neq -1$$

Example: Given points $P(2, 3), Q(-4, 1), C(2, 4)$ and $D(6, 5)$, find the tangent of the angle between the line that passes through P and Q and the line that passes through C and D when measured from the line that passes through P and Q to the line that passes through C and D counter-clockwise.

Solution: Let m_1 be the slope of the line through P and Q and m_2 be the slope of the line through C and D. Then

$$m_1 = \frac{1-3}{-4-2} = \frac{1}{3} \text{ and } m_2 = \frac{5-4}{6-2} = \frac{1}{4}$$

Thus, the tangent of the angle β

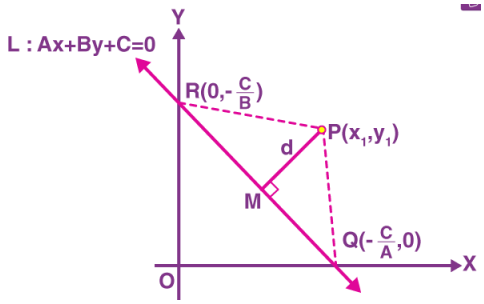
$$tan\,\beta = \frac{1/4 - 1/3}{1 + (1/3)(1/4)} = -\frac{1}{13}$$

4.4. Distance between a *point* and a *line* and between two lines.

4.4.1. Distance between a Point and a Line

Definition: Suppose a line $l: Ax + By + C = 0$ and a point $P(x_1, y_1)$ are given. If P does not lie on l, then we define the distance d from P to l as

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$



Note: If P is on l, the distance is taken to be ***zero***.

Example: Find the distance of the point $(-3, 5)$ from the line $4x - 3y - 26 = 0$.

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Solution: Given equation of a line is: $4x - 3y - 26 = 0$.

Point: $(-3, 5)$

Comparing these with the standard forms: $A = 4, B = -3, C = -26$

$$x_1 = -3, y_1 = 5$$

We know that the perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Substituting the values,

$$d = \frac{|4(-3) + (-3)(5) + (-26)|}{\sqrt{16 + 9}} = \frac{53}{5}$$

Hence, the required distance is $\frac{53}{5}$ units.

4.4.2. Distance between Two Parallel Lines

The shortest distance between the two parallel lines can be determined using the length of the perpendicular segment between the lines. It does not matter which perpendicular line you are choosing, as long as two points are on the line.
Thus, we can now easily calculate the distance between two parallel lines and the distance between a point and a line.

If we consider the general form of the equation of straight line, and the lines are given by:

$$L1: Ax + By + C_1 = 0$$

$$L2: Ax + By + C_2 = 0$$

Then, the distance between them is given by:

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Example: Find the distance between two lines $3x + 4y = 9$ and $6x + 8y = 15$.
Solution: Given equations of lines are:

$$3x + 4y = 9$$
$$6x + 8y = 15 \text{ or } 3x + 4y = \frac{15}{2}$$

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Let us check whether the given lines are parallel or not.

$$y = -\frac{3}{4}x + \frac{9}{4}$$

And

$$y = -\frac{3}{4}x + \frac{15}{2}$$

Thus, the slope of the given lines is equal so they are parallel to each other. Now, by comparing with the standard form of parallel lines equations, we get:

$A = 3, B = 4, C_1 = -9, C_2 = -\frac{15}{2}$

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} = \frac{\left|-9 - \left(-\frac{15}{2}\right)\right|}{\sqrt{3^2 + 4^2}} = \frac{3}{10}$$

Therefore, the distance between the given lines is $\frac{3}{10}$ units.

Example: The distance between two parallel lines $5x - 12y + 2 = 0$ and $5x - 12y + k = 0$ is given by $\frac{5}{13}$ units. Find the value of k.

Solution: The given lines can be rewritten $y = \frac{5}{12}x + \frac{2}{12}$ and $y = \frac{5}{12}x + \frac{k}{12}$.

Here, the slope of the lines are $m_1 = \frac{5}{12}$ and $m_2 = \frac{5}{12}$.

Thus, the slope of the given lines is equal so they are parallel to each other. Now, by comparing with the standard form of parallel lines equations, we get:

$$A = 5, B = -12, C_1 = 2, C_2 = k$$

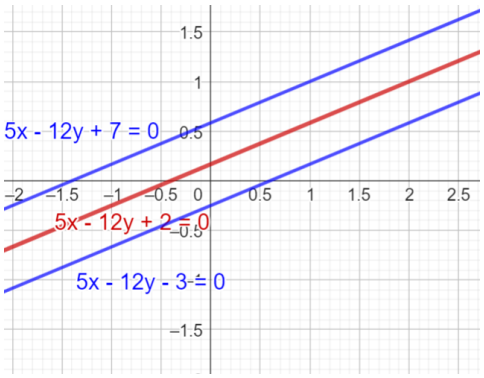
$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

$$\frac{5}{3} = \frac{|2 - k|}{\sqrt{5^2 + (-12)^2}}$$

$$\frac{5}{3} = \frac{|2 - k|}{13}$$

$$|2 - k| = 5$$

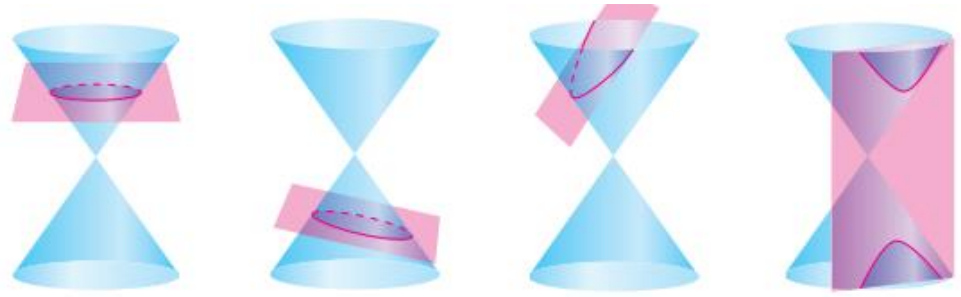
$$k = -3, 7.$$



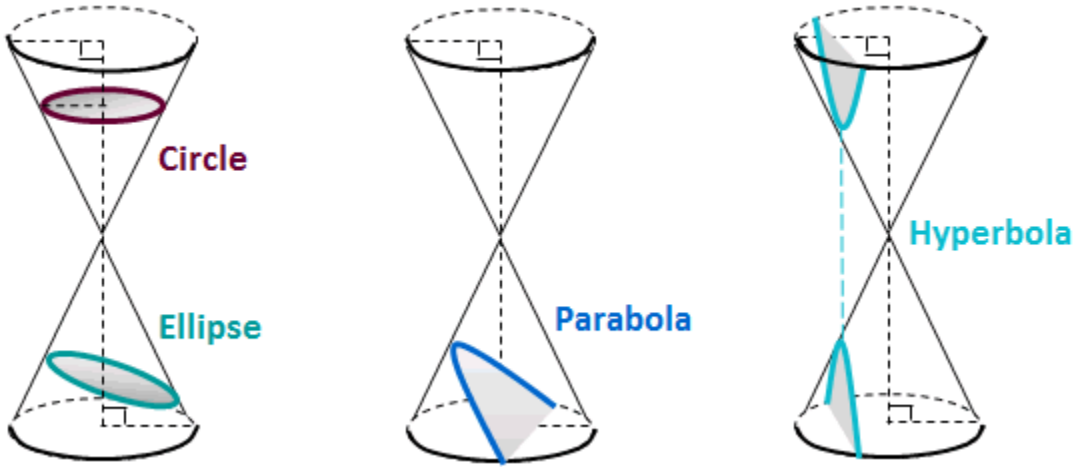
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4.5. Conic Sections

A conic section (or simply conic) is the intersection of a plane and a double-napped cone.



Figures: Basic Conics



- If a horizontal plane intersects through one of the cones, the section formed is a circle.
- If a slanted plane intersects through one of the cones, then the section formed is either an ellipse of a parabola.
- If a vertical plane intersects through the pair of cones, then the section formed is a hyperbola.

4.5.1. Circles

Definition: A circle is the locus of a point that moves in a plane with a fixed distance from a fixed point. The fixed distance is called the radius of the circle and the fixed point is called the center of the circle.

For any point $P(x, y)$ on a circle with center $C(h, k)$ and radius r we have

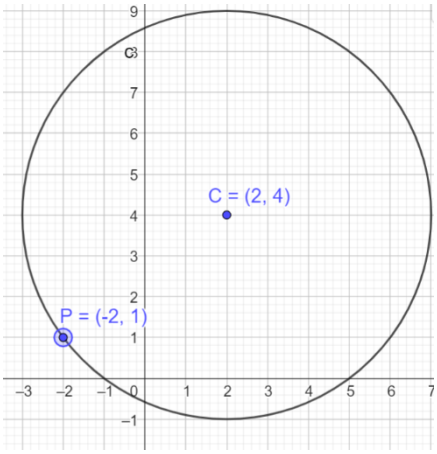
$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Thus, the standard form of the equation of a circle with center $C(h, k)$ and radius r

$$(x-h)^2 + (y-k)^2 = r^2$$

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Example: Write the standard from of the equation of the circle with center at $C(2,4)$ and that passes through the point $P(-2,1)$.



Solution: Let r be the radius of the circle. Then the equation of the circle is

$$(x-2)^2+(y-4)^2=r^2$$

Since the point $P(-2,1)$ pass through the circle,

$$(-2-2)^2+(1-4)^2=r^2 \Rightarrow 25=r^2 \Rightarrow r=5.$$

Therefore, the equation of the circle is

$$(x-2)^2+(y-4)^2=25.$$

Homework: Write the standard form of the equation of the circle with center $(7,1)$ that also contains the point $(-1,-5)$.

Example: Give the center and radius of the circle $x^2+y^2+6x-8y=0$.

Solution: By completing the square method,

$$x^2+y^2+6x-8y=0 \Rightarrow x^2+6x+9+y^2-8y+16=9+16 \Rightarrow (x+3)^2+(y-4)^2=5^2$$

Therefore, the center of the circle is $C(-3,4)$ and the radius of the circle is $r=5$.

Example: Find the center and radius of the circle $3x^2+3y^2=27$

Solution: The given equation can be rewritten as $x^2+y^2=9$. i.e., $(x-0)^2+(y-0)^2=3^2$.

Thus, the center of the circle is at origin $(0,0)$ and radius $r=3$.

Homework: Find the center and radius of the circle $x^2+y^2-12y+11=0$. Answer: $C(0,6)$ and $r=5$.

Intersect of a circle with line

- If the perpendicular distance from the center of a circle to a line is less than the radius of the circle, then the line intersects the circle **at two points**. Such a line is called a **secant line** to the circle.
- If the perpendicular distance from the center of a circle to a line is equal to the radius of the circle, then the line intersects the circle at only one point. Such a line is called a **tangent line** to the circle and the point of intersection is called **the point of tangency**.
- If the perpendicular distance from the center of a circle to a line is greater than the radius of the circle, then **the line does not intersect the circle**.

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A line with equation $Ax + By + C = 0$ intersects a circle with equation $(x-h)^2 + (y-k)^2 = r^2$, if and only if,

$$\frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}} \leq r$$

How to find point of intersection: If a line with equation $Ax + By + C = 0$ intersects a circle with equation $(x-h)^2 + (y-k)^2 = r^2$ is a quadratic equation in x. If $B = 0$, then $x = -\frac{C}{A}$ is a vertical line.

$(y-k)^2 = r^2 - \left(-\frac{C}{A} - h\right)^2 = r^2 - \left(\frac{C+hA}{A}\right)^2$, which is a quadratic in y. Solving this equation you can get points of intersection of the line and the circle.

Example: Find the intersection of the circle with equation $(x-1)^2 + (y+1)^2 = 25$ with the line $4x - 3y - 7 = 0$.

Solution: From $4x - 3y - 7 = 0$, we have $y = \frac{4x-7}{3}$. Replacing it into the equation of the circle we obtain

$$(x-1)^2 + \left(\frac{4x-7}{3} + 1\right)^2 = 25$$

$$25x^2 - 50x - 200 = 0$$

$$x^2 - 2x - 8 = 0$$

$$x + 2x - 4 = 0$$

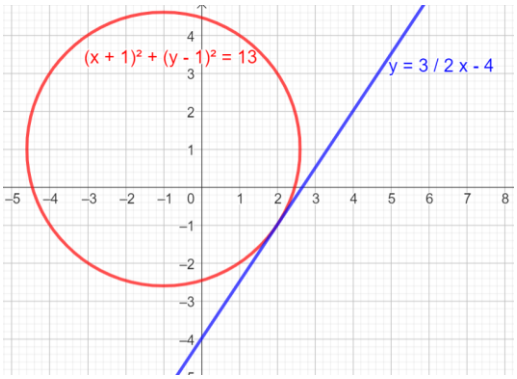
$$x = -2 \text{ or } x = 4.$$

Example: For the circle $(x+1)^2 + (y-1)^2 = 13$, show that $y = \frac{3}{2}x - 4$ is a tangent line.

Solution: The distance from $C(-1, 1)$ to the line $-3x + 2y + 8 = 0$ is

$$d = \frac{|-3(-1) + 2(1) + 8|}{\sqrt{(-3)^2 + 2^2}} = \frac{|13|}{\sqrt{13}} = \sqrt{13} = r$$

Hence, $y = \frac{3}{2}x - 4$ is a tangent line to the circle $(x-1)^2 + (y+1)^2 = 25$.



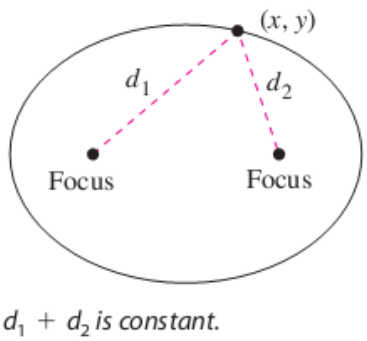
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Note: If a line l is tangent to a circle $(x-h)^2+(y-k)^2=r^2$ at a point $T(x_0,y_0)$, then the equation of l is given by

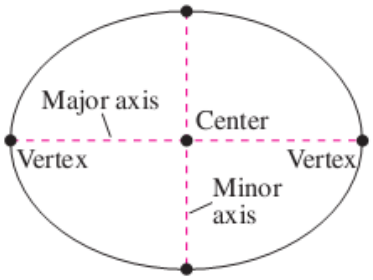
$$\frac{y-y_0}{x-x_0}=-\frac{x_0-h}{y_0-k}.$$

4.5.2. Ellipse

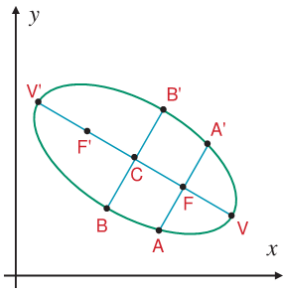
An ellipse is the set of all points (x,y) in a plane, the sum of whose distances from two distinct fixed points (foci) is constant.



$d_1 + d_2$ is constant.



- The line through the foci intersects the ellipse at two points called ***vertices***.
- The chord joining the vertices is the ***major axis***, and its midpoint is the ***center*** of the ellipse.
- The chord perpendicular to the major axis at the center is the ***minor axis*** of the ellipse.
  - F and F' are ***foci***.
  - V, V', B and B' are called ***vertices*** of the ellipse.
  - $\overline{V'V}$ is called the major axis and $\overline{B'B}$ is called the ***minor axis***.
  - C, which is the intersection point of the major and minor axes is called the ***center*** of the ellipse.



Standard Equation of an Ellipse

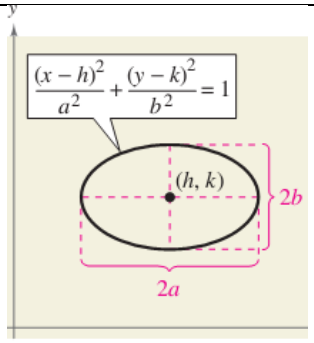
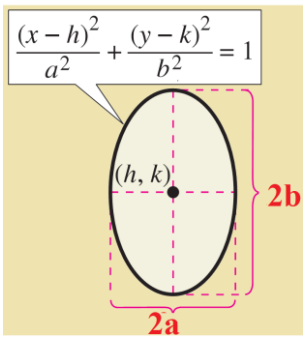
The standard form of the equation of an ellipse, with center (h,k) and major and minor axes of lengths $2a$ and $2b$ respectively, where $0<b<a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad (a>b \text{ major axis is horizontal})$$

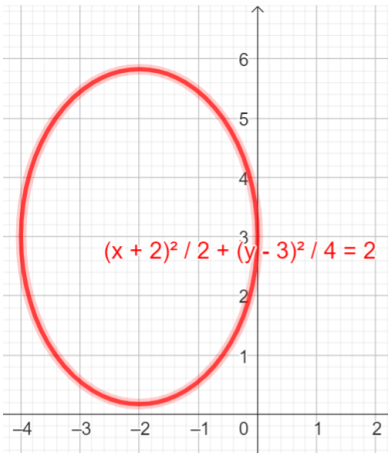
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad (b>a \text{ major axis is vertical})$$

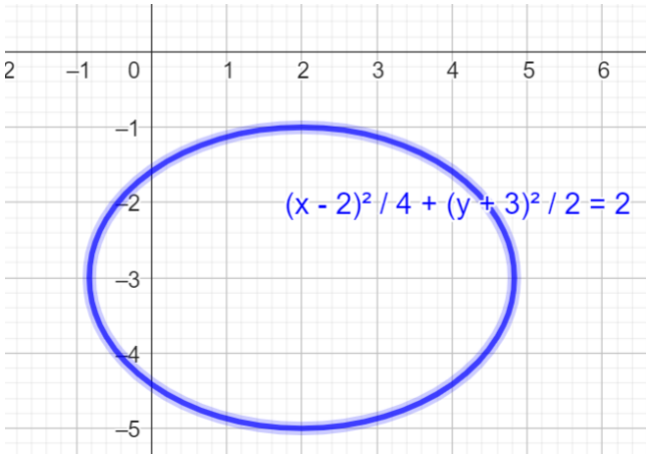
The foci lie on the major axis, c units from the center, with $c^2=a^2-b^2$.

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Standard form of the equation of ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$		
	$a > b$	$b > a$
Major axis	Horizontal 	Vertical 
Center	(h,k)	(h,k)
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Vertex	$(h \pm a, k)$	$(h, k \pm b)$
Focus	$(h \pm c, k)$	$(h, k \pm c)$, $c^2 = a^2 - b^2$
Graph		

Example: Graph of $\frac{(x+2)^2}{2}+\frac{(y-3)^2}{4}=2$ and $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{2}=2$.





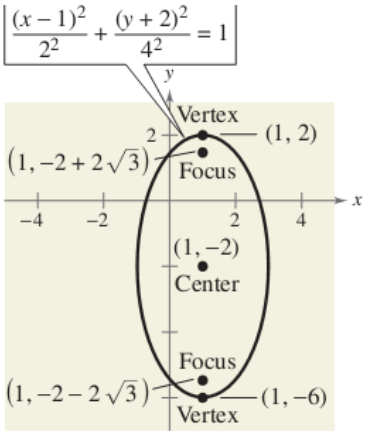
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Example: Find the center, vertices and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution: By completing the square,

$$4x^2 - 8x + y^2 + 4y - 8 = 0 \Rightarrow 4(x-1)^2 + (y+2)^2 = 16$$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

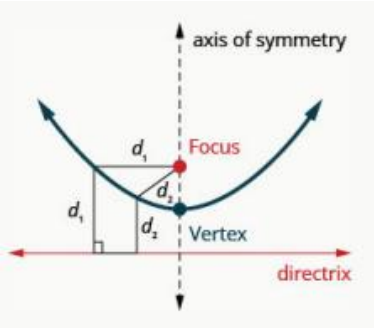


Thus the major axis is vertical with length $2b = 32$

Center: $(1, -2)$ Vertices: $(1, -6)$ Foci: $\left(1, -2 \pm 2\sqrt{3}\right)$

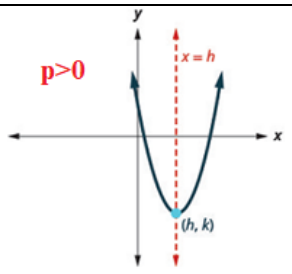
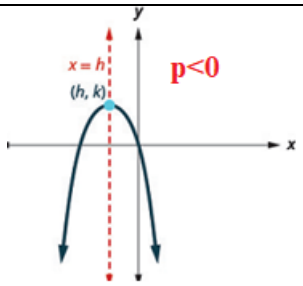
Homework: Find the center, vertices and foci of the ellipse $x^2 + 4y^2 - 6x + 20y - 2 = 0$.

4.5.3. Parabola



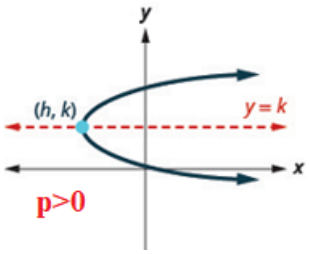
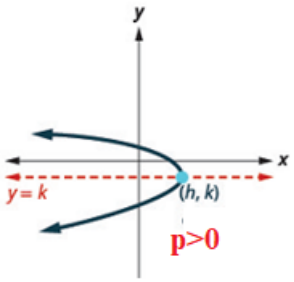
A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point that is not on the line. The fixed point is called the **focus**, and the fixed line is called the ***directrix*** of the parabola.

Vertical parabolas

	Standard form: $(x-h)^2 = 4p(y-k)$	
Orientation	If $p > 0$ opens upward	If $p < 0$ opens down
Axis of symmetry	$x = h$	$x = h$
Vertex	(h, k)	(h, k)
Focus	$(h, k+p)$, where p is the focal length	$(h, k-p)$
Directrix	$y = k - p$	$y = k + p$
Graph		

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Horizontal parabolas

Standard form: $(y-k)^2 = 4p(x-h)$		
Orientation	If $p > 0$ opens right	If $p < 0$ opens left
Axis of symmetry	$y = k$	$y = k$
Vertex	(h, k)	(h, k)
Focus	$(h+p, k)$	$(h-p, k)$
Directrix	$x = h - p$	$x = h + p$
Graph		

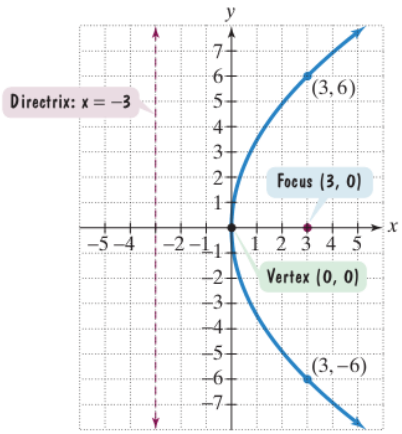
Example: Find the focus and directrix of the parabola given by $y^2 = 12x$. Then graph the parabola.

Solution: The given equation, $y^2 = 12x$ is in the standard form $y^2 = 4px$, so $4p = 12 \Longrightarrow p = 3$. Hence $p = 3$ is the focal length.

Because p is positive, the parabola, with its symmetry, opens to the right. The focus is 3 units to the right of the vertex, $(0, 0)$.

Focus: $(p, 0) = (3, 0)$

Directrix: $x = -p \Rightarrow x = -3$.



Example: Find the vertex, focus, and directrix of the parabola given by $y^2 + 2y + 12x - 23 = 0$.

Solution: We convert the given equation to standard form by completing the square on the variable y.

$$y^2 + 2y + 12x - 23 = 0 \Longrightarrow y^2 + 2y + 1 = -12x + 23 + 1 \Longrightarrow (y+1)^2 = -12(x-2).$$

We see that $h = 2$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (2, -1)$. Because $4p = -12 \Rightarrow p = -3$. Based on the standard form of the equation, the axis of symmetry is horizontal. With a negative value for p and a horizontal axis of symmetry, the parabola opens to the left. Because $p = -3$ the focus is located 3 units to the left of the vertex $(2, -1)$, Likewise, the directrix is located 3 units to the right of the vertex.

Focus: $(h+p, k) = (-1, 1)$

Directrix: $x = h - p \Rightarrow x = 5$.

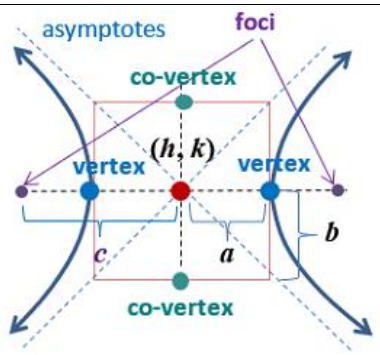
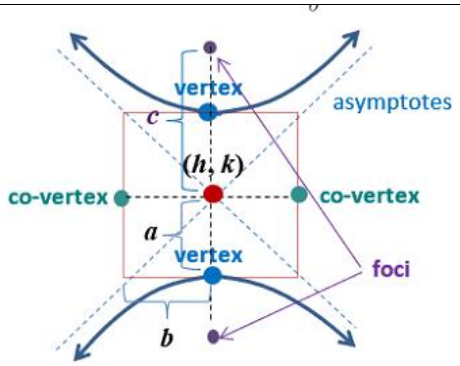
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Homework: Find the vertex, focus, and the directrix and graph each of

i) $y^2 - 2y - x - 5 = 0$ ii) $y = -x^2 - 4x + 4$

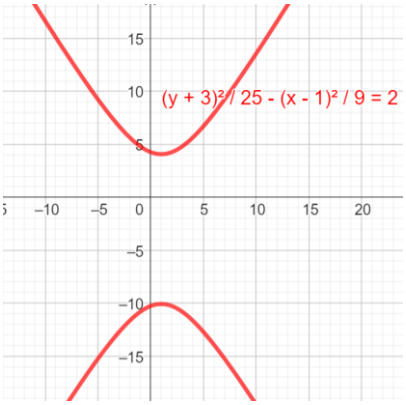
4.5.3. Hyperbolas

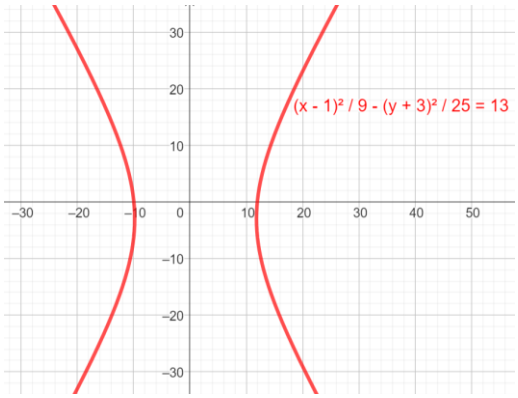
Definition:

Standard form of the equation of hyperbola		
	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Center	(h, k)	(h, k)
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Vertex	$(h \pm a, k)$	$(h, k \pm b)$
Focus	$(h \pm c, k)$	$(h, k \pm c)$, $c^2 = a^2 + b^2$
Major axis	Horizontal	Vertical
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$
Graph		

Example: The graph of hyperbolas i) $\frac{(y+3)^2}{25} - \frac{(x-1)^2}{9} = 2$ ii) $\frac{(x-1)^2}{9} - \frac{(y+3)^2}{25} = 13$.

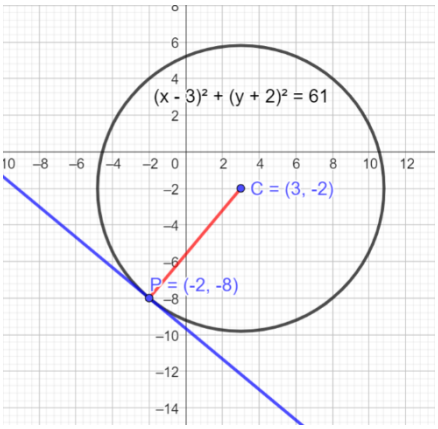
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Solved Problems

1. Write the equation of tangent line to the circle $(x-3)^2+(y+2)^2=61$ at the point $(-2,-8)$.



Solution: Since we only have one point on the tangent line $(-2,-8)$, we'll have to get the slope of the line to get its equation.

We can get the slope of the line that connects the center of the circle $(3,-2)$ and the point on the tangent line $(-2,-8)$, and then take the negative or opposite reciprocal to get the slope of the tangent line.

The slope of the line that contains $(3,-2)$ and $(-2,-8)$ is

$$m_{PC}=\frac{y_2-y_1}{x_2-x_1}=\frac{-8-(-2)}{-2-3}=\frac{-6}{-5}=\frac{6}{5}$$

Then $m_l m_{PC}=-1 \Longrightarrow m_l=-\frac{1}{m_{PC}}=-\frac{5}{6}$.

Thus using slope of the line $m_l=-\frac{5}{6}$ and point on the line $P(-2,-8)$

$$y-y_0=m(x-x_0) \Longrightarrow y=-\frac{5}{6}x-\frac{29}{3}.$$

2. Write the equation of a circle if the points $(-6,8)$ and $(0,18)$ are on the circle.

Solution: Since the center of a circle is midpoint between any two points of the diameter, we can use the Midpoint Theorem

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$$(h,k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0 + (-6)}{2}, \frac{18 + 8}{2}\right) = (-3, 13)$$

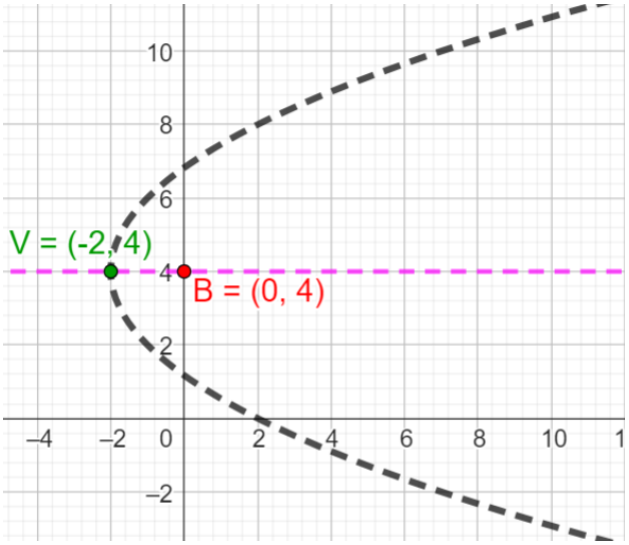
To get the radius of the circle, we can use the Distance Formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to get the distance between the center and one of the points; let's pick $(0, 18)$:

$$\sqrt{\big(0 - (-3)\big)^2 + (18 - 13)^2} = \sqrt{34}$$

Thus, the equation of circle is $(x + 3)^2 + (y - 13)^2 = 34$.

3. Write the equation of a parabola with a vertex of $(-2, 4)$ and focus point $(0, 4)$.

Solution: It's best to first plot the points, so we can see the direction of the parabola. We can see that it's a horizontal parabola.



Since the focal length (length from the vertex to the focus) is 2, and the vertex is $(-2, 4)$, we have

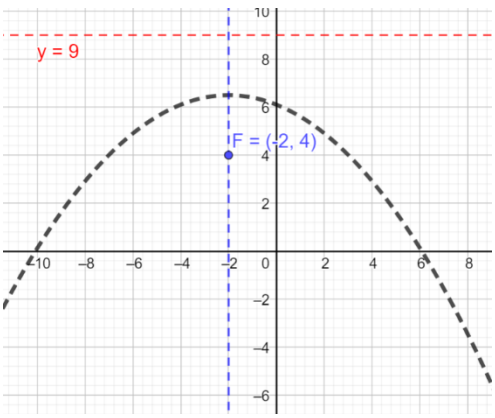
$$(y - k)^2 = 4p(x - h)$$

$$\Rightarrow (y - 4)^2 = (2)4\big(x - (-2)\big)$$

$$\Rightarrow (y - 4)^2 = 8(x + 2)$$

4. Write the equation of a parabola with a focus of $(-2, 4)$ and directrix $y = 9$.

Solution: To know the direction of the parabola, let first plot the points. Since the directrix is horizontal and the focus is below it, we can see that it's a vertical parabola that opens down.



We know that the parabola is in the form of $(x - h)^2 = 4p(y - k)$.

The length from focus to directrix is $|9 - 4| = 5$, the focal length is $2p = 5 \Rightarrow p = \frac{5}{2} = 2.5$. hence, the vertex is $(-2, 4 + p) = (-2, 6.5)$.

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Thus, the equation is:

$$(x-h)^2 = -4p(y-k) \Rightarrow (x+2)^2 = -4\left(\frac{5}{2}\right)(y-6.5)$$

$$\Rightarrow 10(y-6.5) = -(x+2)^2.$$

4. Find the center and vertices of the hyperbola $11x^2 - 25y^2 + 22x + 250y - 889 = 0$.

Answer: Center: $(-1, 5)$, vertices $(-6, 5), (4, 5)$

ESSLCE Questions

1. Consider the lines $l_1: x + y = 0$ and $l_2: 2x - y + 4 = 0$ in the xy-plane. Let β be the angle between l_1 and l_2 measured from l_1 to l_2. What is the value of $\tan\beta$?

   A. 3 B. -3 C. $\frac{1}{3}$ D. $-\frac{1}{3}$

2. An ellipse has center (h, k) and its foci are on the line $x = h$. What is the standard form of equation of the ellipse if the lengths of the semi-major and semi-minor axes are 3 and 2 units, respectively?

   A. $\frac{(x-h)^2}{9} + \frac{(y-k)^2}{4} = 1$ B. $\frac{(y-k)^2}{4} + \frac{(x-h)^2}{4} = 1$ C. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ D. $\frac{y^2}{9} + \frac{x^2}{4} = 1$

3. Consider the following statement: A plane curve is the locus of points in the plane such that the difference between the distances from two fixed points is a constant. Which one of the following is defined by the statement?

   A. Circle B. Parabola C. Hyperbola D. Ellipse

4. What is the equation of the circle with center $(-2, 3)$ and radius 2 units?

   A. $x^2 + y^2 - 4x + 6y + 9 = 0$ B. $x^2 + y^2 - 6x + 4y + 9 = 0$

   C. $x^2 + y^2 + 4x - 6y + 9 = 0$ D. $x^2 + y^2 + 6x - 4y + 9 = 0$

5. Which one of the following is the equation of a line through $(2, 1)$ and perpendicular to the line with equation $4x - 2y = 3$?

   A. $x + 2y - 4 = 0$ B. $2x + y - 5 = 0$ C. $2x - y - 3 = 0$ D. $x - 2y - 6 = 0$

6. A line in the xy −plane has inclination of $150°$ with the positive x-axis. What is the slope of the line?

   A. $-\sqrt{3}$ B. $-\frac{\sqrt{3}}{3}$ C. $\frac{\sqrt{3}}{3}$ D. -1

7. Consider the equation of the sphere $x^2 + y^2 + z^2 - 2x + 2y - 4z + 5 = 0$. What are its center and radius respectively?

   A. $(1, -1, 2)$ and 1 B. $(1, 1, 2)$ and 1 C. $(-1, 1, 2)$ and $\sqrt{5}$ D. $(-1, 1, 2)$ and $\sqrt{5}$

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Worksheet

1. Plot the points with the following coordinates: $P(-5,-4), Q(-4,5), R(2,4), S(1,-5)$.
2. In which quadrant does $(-3,4)$ lie?
3. The points $(4,2)$ and $(-3,-5)$ form with another point (x,y) an equilateral triangle. Find x and y.
4. Find the point of x −axis which is equidistant from the points $P(7,6)$ and $Q(-3,4)$.
5. Find the area of a the triangle whose vertices are $(-2,4), (4,6)$ and $(3,2)$.
6. Find the value of x if the distance between the points $(x,3)$ and $(-4,-1)$ is 5. Answer: $x = -1,-7$.
7. The distance between point $A(-k,1)$ and the origin is the same as the distance between point $B(k,1)$ and point $C(3,5)$. Find the value of k. Answer: $k = 4$.
8. The coordinates of points R and S are $(-4,3)$ and (x,y) respectively. Point $P(-1,2)$ is the midpoint of RS. Find the values of x and y. Answer: $(2,1)$.
9. A line segment has end-points P (-1, 5) and Q (5, 2). Find the coordinates of the points that trisect the segment.
10. Find the equation of the line making intercepts 2 and –3 on the x-axis and y-axis respectively.
11. Find the equation of a line passing through the origin and parallel to the line $3x - 2y + 1 = 0$.
12. Find the length of perpendicular from the point $(4,1)$ to the line $3x - 4y + 12 = 0$
13. Find the equation of a straight line passing through the point of intersection of the lines $x - 2y + 3 = 0, 2x - 3y + 4 = 0$ and parallel to the line joining the points $(1,1)$ and $(0,-1)$.
14. Write the standard form of the equation of the circle with the given center with point on the circle. i) Center $(-5,6)$ with point $(-2,3)$ ii) Center $(4,4)$ with point $(2,2)$
15. Find the center and radius of i) $x^2 + y^2 - 6x - 8y = 0$ ii) $x^2 + y^2 - 14x + 13 = 0$

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Chapter 5

Vectors and Transformation of the Plane

5.1. Introduction to vectors and scalars (Grade 9, Page 291)

In general, there are two types of physical measurements; those involving only magnitude and no direction, called scalars and others involving magnitude and a definite direction, called vectors.

A vector has both magnitude (length) and direction, and both these properties must be given in order to specify it. A quantity with magnitude (length) but no direction is called a scalar.

Example: The velocity of a car is 80 km/h in the direction of north. This is a vector.

Example: Force, velocity, acceleration, and momentum are example of vector.

Note: Speed ≠ velocity (Since speed=magnitude of velocity).

Distance ≠ displacement (Since distance = magnitude of displacement).

Classwork: What are these quantities, vectors or scalars?

a) *A* temperature of $100°C$

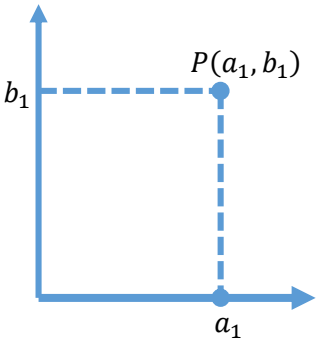
b) An acceleration of $9.8142\ m/s$ towards earth

c) The weight of a 2 kg mass

e) A North easterly wind of 20 knots

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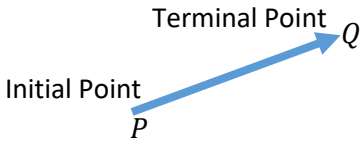
5.2. Representation of vectors



Note: $\mathbb{R}^2 = \{(x, y): x, y \in \mathbb{R}\}$.

Points P in the plane are described by pairs (a_1, b_1) of real numbers, where a_1 and b_1 stand for the x and y coordinates of the point P. This means, when we project P perpendicularly on the x-axis, then we get a_1 and when we project P on the y-axis, then we get b_1, see Figure.

A vector can be represented either algebraically or geometrically.



A vector quantity can be represented diagrammatically by a directed straight line segment $\overrightarrow{PQ}$ from a point P to a point Q in space (with an arrow head) whose length of the segment is equal to the magnitude of the vector, and whose direction represents the direction of the vector. Here, P is called the ***initial point*** and Q is called the ***terminal point***. The direction is indicated by an arrow joining the initial and terminal points of the line regiment.

Analytically, the vector is represented by either $\overrightarrow{PQ}$ and the magnitude which is the length of the line $\overrightarrow{PQ}$ denoted by $|\overrightarrow{PQ}|$ or $\| \overrightarrow{PQ} \|$. Another name for length is **norm**.

Notation: To distinguish them from scalars, vectors will be denoted by lowercase symbols; i.e., the vector v is represented by either of the symbols v, or $\vec{v}$. But it is inconvenient in handwriting. In writing, we normally put an arrow on top of, or sometimes underneath, the letter: $\vec{v}$ or $\underline{v}$.

To distinguish between vector and scalar quantities, various ways are used. These include:

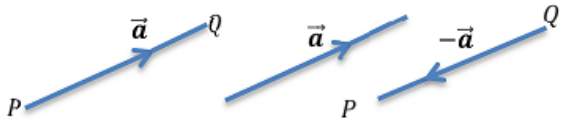
i. bold print $\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w},$

ii. two capital letters with an arrow above them to denote the sense of direction, e.g. $\overrightarrow{PQ}$, where P is the starting point and Q the end point of the vector

iii. a line over the top(or underline) of letters, e.g. $\overline{PQ}$ or $\bar{a}, \underline{a}$

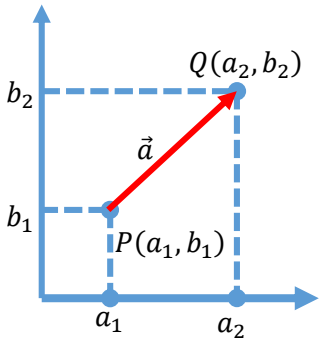
iv. letters with an arrow above, e.g. $\vec{a}, \vec{A}$

Sometimes the notation $\overrightarrow{PQ}$ is used to represent the vector $\vec{a}$. Then the vector $\overrightarrow{QP}$ is represented by $-\vec{a}$.



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5.3. Components of a Vector



The vector $\vec{a}$ from $P(a_1, b_1)$ to $Q(a_2, b_2)$ can be written as

$$\vec{a} = \overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1 \rangle$$

Its tail is P and its tip is Q.

$$x = a_2 - a_1, y = b_2 - b_1$$

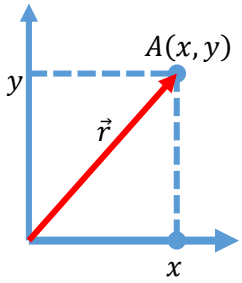
are called the ***components of vector*** $\vec{a}$ with respect to Cartesian coordinate system, and we write simply $\vec{a} = \langle x, y \rangle$. The element x is called the *first component*, y the *second component* of $\vec{a}$.

i.e., $\vec{a} = \overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1 \rangle$

Example: The vector $\vec{a}$ with initial point $P: (4,0)$ and terminal point $Q: (6, -1)$ has the components $x = 6 - 4 = 2, y = -1 - 0 = -1$; Hence $\vec{a} = \langle 2, -1 \rangle$.

Class work: 1. If we choose $(-1,5)$ as the initial point of $\vec{a} = \langle 2, -1 \rangle$, then, find the corresponding terminal point. Answer: $(1,4)$

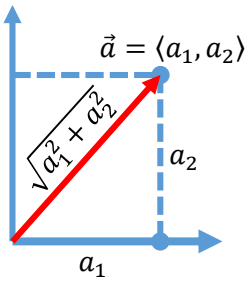
2. If we choose the origin $(0,0)$ as the initial point of $\vec{a}$, find the corresponding terminal point. Answer: $(2, -1)$; its coordinates equal the components of $\vec{a}$. This suggests that we can determine each point in space by a vector called the position vector of the point, as follows.



Sometimes vectors are referred to a fixed point, an origin. Such a vector is called a **Position vector.** In the given Cartesian coordinate system, the position vector $\vec{r}$ of a point $A(x, y)$ is the vector with the origin $(0,0)$ as the initial point and A as the **terminal point.** Thus $\vec{r} = \langle x, y \rangle$.

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We note that we have developed two approaches to vectors: ***geometric***, in which vectors are represented by arrows or directed line segments, and ***analytic***, in which vectors are represented by pairs or triples of numbers called ***components***.



5.4. Magnitude of a vector

The length (magnitude) $\| \vec{a} \|$ of a vector $\vec{a} = \langle a_1, a_2 \rangle$, illustrated in Figure is

$$\| \vec{a} \| = \sqrt{a_1^2 + a_2^2}$$

It is the Euclidean distance between the points $(0,0)$ and (a_1, a_2), or the length of the line segment that joins these two points.

Example: The length of the vector $\vec{a} = \langle 2,3 \rangle$ is $\| \vec{a} \| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

Example: The length of the vector from $Q = (1,2)$ to $P = (5,3)$ is

$$\| \overrightarrow{QP} \| = \| (4,1) \| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

5.5. Types of Vectors

Unit vector: Any vector whose magnitude (length) is equal to one is called a ***unit vector***. A vector $\vec{v}$ whose magnitude is 1 is called a "unit vector" and the symbol $\hat{v}$ denotes a unit vector in the same direction as v.

Zero vector: Any vector whose magnitude is equal to zero is called a ***zero or null vector*** and is denoted by **0**, or $\vec{\mathbf{0}}$. A zero vector is one whose magnitude is zero, but no definite direction associated with it, for example, if P is a point, $\overrightarrow{PP}$ is a zero vector. It has indeterminate direction.

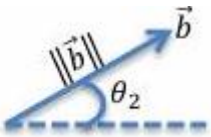
Negative vector: If $\overrightarrow{AB}$ is a vector, then the negative vector of $\overrightarrow{PQ}$ is $\overrightarrow{QP}$ or $\overleftarrow{PQ}$. If the direction of a vector changed, we can get the negative vector i.e $\overrightarrow{QP} = -\overrightarrow{PQ}$

Equal Vectors: Consider two vectors $\vec{a}$ and $\vec{b}$ where the direction of the vector is measured with respect to a fixed axis of a given coordinate system which we'll take to be the $xy-$ plane. Let θ_1 and θ_2 be the angle between the vector and the x-axis, respectively.

Then $\vec{a} = \vec{b} \Leftrightarrow$ 1) $\| \vec{a} \| = \| \vec{b} \|$ (same magnitude)

2) $\theta_1 = \theta_2$ (same direction)





Two vectors are equal if they have the same magnitude, the same direction (i.e. they are parallel). Thus, **we** will not change a vector when we move it to a new position provided we preserve its magnitude and direction. We can move

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vectors anywhere on the plane or equate any two vectors who's tails are at the points P and Q, respectively, and will get that the vectors are treated as if they're the same.

Example: Consider the sketch below.

Each of the directed line segments in the sketch represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we'll use for this vector is, $\vec{v} = \langle -2, 5 \rangle$.

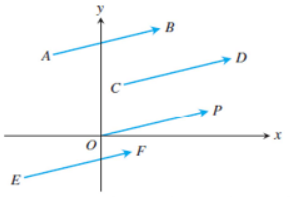


Figure: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$

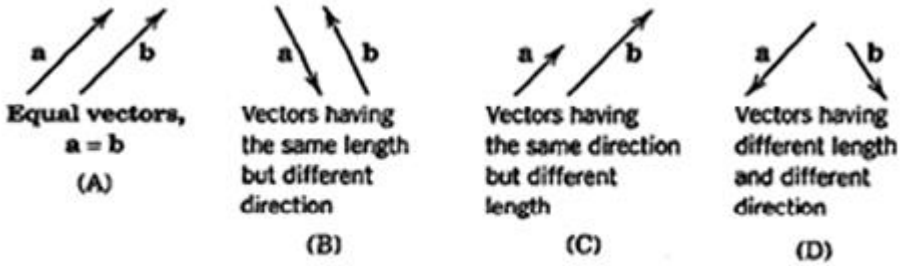
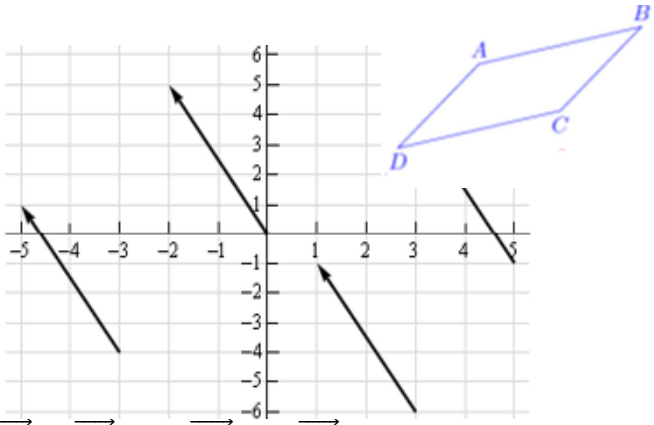


Fig. (A) Equal vectors. (B)–(D) Different vectors

Note: Two vectors $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ are equal, written $\vec{a} = \vec{b}$, iff $a_1 = b_1$, and $a_2 = b_2$.

Class work: Find the value of x and y if the two vectors $\vec{a} = \langle x + 2, 3 \rangle$ and $\vec{b} = \langle y, x \rangle$ are equal.

Home work: Diagram beside shows a parallelogram. Which of the following equations is the correct one?



a) $\overrightarrow{DA} = \overrightarrow{BC}$ b) $\overrightarrow{AD} = -\overrightarrow{CB}$ c) $\overrightarrow{AD} = \overrightarrow{CB}$ d) $\overrightarrow{DA} = -\overrightarrow{CB}$

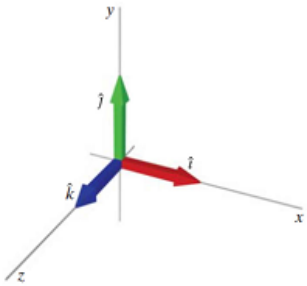
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5.6. Vectors in ijk Notation

Another representation of a vector is

$$\vec{a} = \langle a_1, a_2 \rangle = a_1 i + a_2 j$$

where i, j, k are unit vectors in the positive directions of the axes of a Cartesian coordinate system. In three dimensional space $\mathbb{R}^3$ we have three coordinate axes, often called the xy −axis.



In the plane $\mathbb{R}^2$ it is common to set $i = \langle 1,0 \rangle$ and $j = \langle 0,1 \rangle$.

Example: $\langle 2, -3 \rangle = 2\text{i} - 3\text{j}$.

It is common to refer to i and j as standard basis vectors (unit vector form) of $\mathbb{R}^2$.

Class work: Express the component form $\langle 9,5 \rangle$ in terms of the standard basis vectors.

5.7. Vector Arithmetic

5.7.1. Addition and subtraction of vectors

We shall start by adding (subtracting) two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on. If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ are vectors, their sum (or the resultant, as it is sometimes called), written $\vec{a} + \vec{b}$, is defined as the vector obtained by adding corresponding components of $\vec{a}$ and $\vec{b}$ to give

$$\vec{a} + \vec{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$$

Note: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (triangle law) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} = \overrightarrow{AD}$ (parallelogram law).

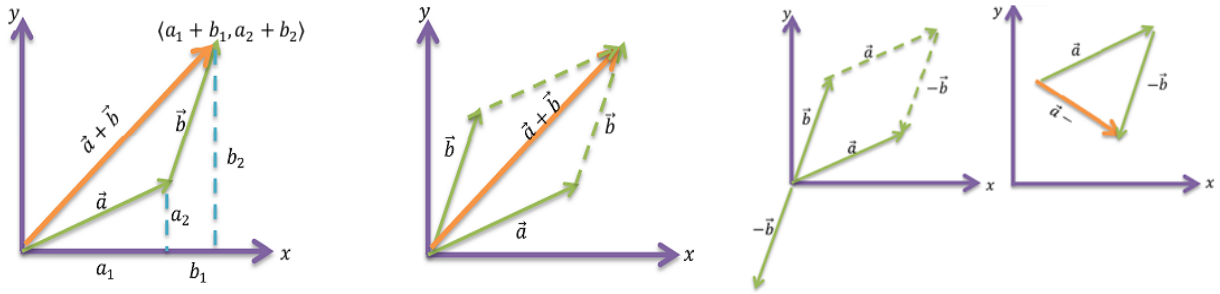
Example: If $\overrightarrow{AC} = \langle 2, -1 \rangle$ and $\overrightarrow{BA} = \langle -1,3 \rangle$ find $\overrightarrow{BC}$.

Computationally, subtraction is very similar. If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ are vectors, their difference, written $\vec{a} - \vec{b}$, is defined as the vector obtained by subtracting corresponding components of $\vec{a}$ and $\vec{b}$ to give

$$\vec{a} - \vec{b} = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$$

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Geometrically interpretation: To determine the resultant of vectors $\vec{a}$ and $\vec{b}$, begin by placing the tail of one vector at the tip (arrow end) of the other. The resultant is then obtained by drawing a line from the tail end of the first vector to the tip of the second.



Graphical addition of vectors

To add two vectors $\vec{a}$ and $\vec{b}$ graphically (see figure)

- Draw the first vector $\vec{a}$,
- Move the second vector $\vec{b}$ (without rotating it) so its tail as located at the tip of the first vector.
- Draw a new vector from the tail of vector $\vec{a}$ to $\vec{b}$.

Graphical subtraction of vectors

To subtract two vectors $\vec{a}$ and $\vec{b}$ graphically (see figure)

- Draw the first vector $\vec{b}$
- Move the second vector $\vec{b}$ (without rotating it)
- Move the second vector $\vec{a}$ (without rotating it) so its tail as located at the tip of the first vector.
- Draw a new vector from the tip of vector $\vec{a}$ to the tip of vector $\vec{b}$

If the end point of first vector and the initial point of the second vector are same, the addition of two vectors can be found as the vector joining the initial point of the first vectors and the end point of the second vector.

Note: $1 \cdot \overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$2 \cdot \overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB} = \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} = \overrightarrow{AD}$

Note: 1. We can add two vectors $\vec{a}$ and $\vec{b}$ by making $\vec{b}$ start where $\vec{a}$ finishes, and completing the triangle. Alternatively, we can make $\vec{a}$ an $\vec{b}$ start at the same place, and take the diagonal of the parallelogram.

2 We can't add or subtract two vectors unless they a have the same number of components.

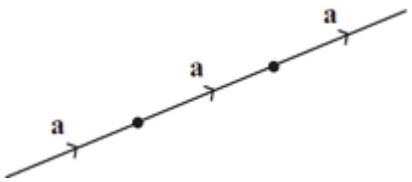
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Properties of vectors addition

1) vector addition is commutative i.e $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

2) vector addition is associative i.e $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Adding a vector to itself: What happens when you add a vector to itself, perhaps several times? We write, for example, $\vec{a} + \vec{a} + \vec{a} = 3\vec{a}$.

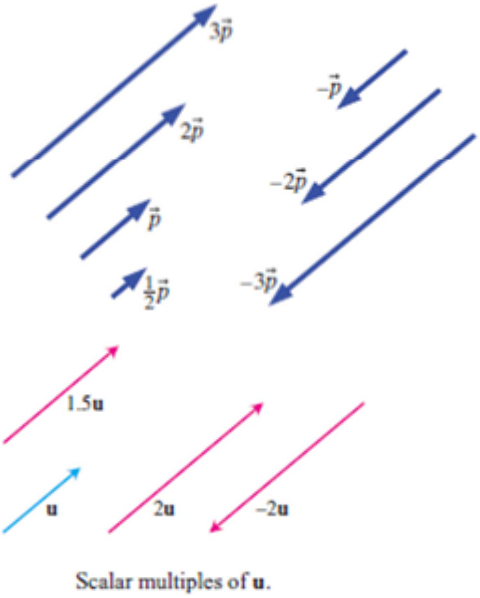


In the same way, we would write $n\vec{a} = \underbrace{\vec{a} + \vec{a} + \cdots + \vec{a}}_{n-times}$

Note: A vector $n\vec{a}$ is in the same direction as the vector $\vec{a}$, but n times as long.

5.7.2. Scalar multiplication

Given the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and any number c the scalar multiplication is, $c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$. so, we multiply all the components by the constant c. A scalar is simply a real number, and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero. We can see that if c is positive all scalar multiplication will do is stretch (if > 1) or shrink (if $c < 1$) the original vector, but it won't change the direction. Properties of vector scalar multiplication: For any scalars k and l, and any **vectors $\vec{a}$ and $\vec{b}$ the following rules hold:**



Scalar multiples of **u**.

- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $(k + l)\vec{a} = k\vec{a} + l\vec{a}$
- $k(l)\vec{a} = (kl)\vec{a}$

Example: Let $\vec{u} = \langle -1,3 \rangle$ and $\vec{v} = \langle 4,7 \rangle$, find a) $2\vec{u} + 3\vec{v}$ b) $\vec{u} - \vec{v}$ c) $\frac{1}{2} \| \vec{u} \|$

Solution: a) $2\vec{u} + 3\vec{v} = \langle -2,6 \rangle + \langle 12,21 \rangle = \langle 10,27 \rangle$

b) $\vec{u} - \vec{v} = \langle -5, -4 \rangle$

c) $\frac{1}{2} \| \vec{u} \| = \left\| \left\langle -\frac{1}{2}, \frac{3}{2} \right\rangle \right\| = \frac{\sqrt{11}}{2}$

Example: $3\vec{a} + 7\vec{a}$ can be simplified to $(3 + 7)\vec{a} = 10\vec{a}$

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Class work: If $\vec{a} = \langle -1,2\rangle$ and $\vec{b} = \langle 2,-2\rangle$ find $\vec{x}$ if $2\vec{b} - 2\vec{x} = -\vec{a}$

Note: 1. for $\alpha \in \mathbb{R}$ if $|\alpha| < 1, \alpha\vec{a}$ contracts $\vec{a}$ and if $|\alpha| > 1, \alpha\vec{a}$ expands $\vec{a}$.

2 If $\alpha > 0, \alpha\vec{a}$ points in the same direction of $\vec{a}$. If $\alpha < 0, \alpha\vec{a}$ points in the opposite direction of $\vec{a}$

Collinear vectors: If $\vec{a}$ and $\vec{b}$ are such that they have the same or opposite directions, they are said to be collinear (parallel) vectors and one is a numerical multiple of the other, i.e $\vec{b} = k\vec{a}$ or $\vec{a} = k\vec{b}$ for some k. Furthermore, if $k > 0$ they are actually in the same direction.

Class work: Determine if the sets of vectors are parallel or not.

(a) $\vec{a} = \langle 2,-4\rangle, \vec{b} = \langle -6,12\rangle$ (b) $\vec{u} = \langle -3,-4\rangle, \vec{v} = \langle -6,-7\rangle$

Example: Show that the points whose position vectors $2i + 3j, 3i + j$ and $6i - 5j$ are collinear.

Solution: Let $\overrightarrow{OA} = 2i + 3j, \overrightarrow{OB} = 3i + j$ and $\overrightarrow{OC} = 6i - 5j$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = i - 2j$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 3i - 6j = 3(i - 2j) = 3\overrightarrow{AB}$$

i.e, $\overrightarrow{BC} = 3\overrightarrow{AB}$

therefore $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors and B is the common point of these two vectors Therefore the given points A, B and C are collinear.

Example: If the vectors $\vec{a} = 2i - 3j$ and $\vec{b} = -6i + mj$ are collinear, find the value of m.

Solution: Given $\vec{a} = 2i - 3j$ and $\vec{b} = -6i + mj$ are collinear, Therefore, $\vec{a} = t\vec{b}$. i.e.

$$2i - 3j = t(-6i + mj) = -6ki + mtj$$

Comparing coefficients of $i \implies 2 = -6t \Rightarrow t = -1/3$

Comparing coefficients of $j, \implies -3 = mt = m = -\frac{3}{2} = 9 = m = 0$

Note: If $\vec{a}$ has magnitude 3, then a unit vector in the direction of $\vec{a}$ is $\frac{1}{3}\vec{a}$. (That means $\frac{\vec{a}}{\|\vec{a}\|}$ is unit vector)

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Example (Finding a Vector's Direction): Find a unit vector $\hat{u}$ in the direction of the vector $\vec{u}$ from $P(1,0)$ to $Q(3,2)$.

Solution: Here the vector is $\vec{u} = \overrightarrow{PQ} = 2i + 2j$. What we're asking for is a new parallel vector (points in the same direction) that happens to be a unit. Here's what we'll do. First let's determine the magnitude of $\vec{u}$. i.e. $\| \vec{u} \| = \sqrt{4+4} = \sqrt{8}$. Now, let's form the following new vector, $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{2}{\sqrt{8}}\mathrm{i} + \frac{2}{\sqrt{8}}\mathrm{j}$

Let check if it is unit vector: $\| \hat{u} \| = \frac{\vec{u}}{\| \vec{u} \|} = \sqrt{\left(\frac{2}{\sqrt{8}}\right)^2 + \left(\frac{2}{\sqrt{8}}\right)^2} = \sqrt{\frac{4}{8} + \frac{4}{8}} = 1$

This vector also points in the same direction as $\vec{u}$ since it is only a scalar multiple of $\vec{u}$(i.e. $\vec{u} = 3\hat{u}$)

Note: Given a vector $\vec{a}, \hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$ will be a unit vector that points in the same direction as $\vec{a}$.

Class work: Find k for the unit vectors 1. $\vec{u} = \left\langle -\frac{1}{2}, k \right\rangle$ 2. $\vec{v} = ki + \frac{2}{3}j$

Class work: Find the unit vector in the direction of 1) $2i$ 2) $-2i - 5j$

Class work: Find a vector $\vec{b}$ if it has opposite direction to $-i - 2j$ and has length 5 units.

Note: Vector $\vec{b}$ of length $k, k > 0$ in the same (opposite) direction as $\vec{a}$ is $\vec{b} = \frac{k}{\|\vec{a}\|}\vec{a}$ $\left(\vec{b} = -\frac{k}{\|\vec{a}\|}\vec{a}\right)$

5.7.3. Scalar product (Dot Product)

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2.$$

Example: Compute the scalar product of $a) \vec{u} = \langle 3, -8 \rangle, \vec{v} = \langle 6,1 \rangle$ $b) \vec{a} = 2i - 3j, \vec{b} = i$

Solution: $a) \vec{u} \cdot \vec{v} = 18 - 8 = 10$ b) $\vec{a} \cdot \vec{b} = 2 + 0 = 2$

Note: A) $i \cdot i = j \cdot j = k \cdot k = 1$

B) $i \cdot j = j \cdot k = k \cdot j = i \cdot k = 0$

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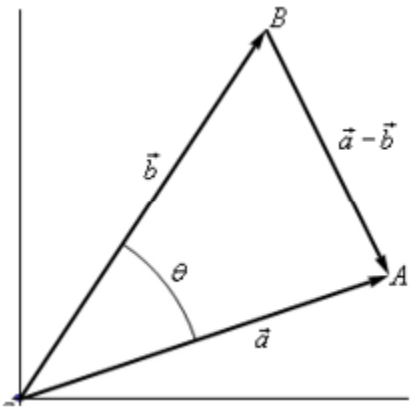
Properties of dot product

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\vec{a} \cdot \vec{a} = \| \vec{a} \|^2$ If $\vec{a} \cdot \vec{a} = 0$ then $\vec{a} = \vec{0}$.

Geometric interpretation to the scalar (dot) product: First suppose that θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$ as shown in the image beside. We can then have the following theorem.

Theorem: $\vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \| \cos\theta$

Proof: The three vectors above form the triangle AOB and note that the length of each side is nothing more than the magnitude of the vector forming that side. The Law of Cosines tells us that,



$$\| \vec{a} - \vec{b} \|^2 = \| \vec{a} \|^2 + \| \vec{b} \|^2 - 2 \| \vec{a} \| \| \vec{b} \| \cos\theta$$

Also using the properties of dot products we can write the left side as,

$$\| \vec{a} - \vec{b} \|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \| \vec{a} \|^2 - 2\vec{a} \cdot \vec{b} + \| \vec{b} \|^2$$

Then, $\| \vec{a} \|^2 - 2\vec{a} \cdot \vec{b} + \| \vec{b} \|^2 = \| \vec{a} \|^2 + \| \vec{b} \|^2 - 2 \| \vec{a} \| \| \vec{b} \| \cos\theta$

$$\Rightarrow -2\vec{a} \cdot \vec{b} = -2 \| \vec{a} \| \| \vec{b} \| \cos\theta \Rightarrow \vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \| \cos\theta$$

With the notation of the dot product, the angle between two vectors $\vec{a}$ and $\vec{b}$ can be written as

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\| \vec{a} \| \| \vec{b} \|}\right)$$

Now, if two vectors are orthogonal then we know that the angle between them is 90 degrees. From (2) this tells us that if two vectors are orthogonal (perpendicular) then $\vec{a} \cdot \vec{b} = 0$,

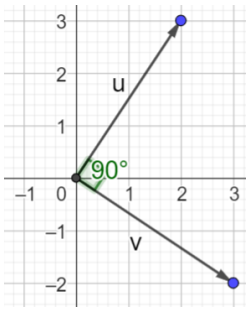
Once again using (2) this would mean that one of the following would have to be true.

- if $\theta = 0$, then $\vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \|$

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- if $\theta = 180$ then $\vec{a} \cdot \vec{b} = \pm \| \vec{a} \| \| \vec{b} \|$

If two vectors are parallel then the angle between them is either 0 degrees (pointing in the same direction) or 180 degrees (pointing in the opposite direction).



Example: $\vec{u} = 2j + 3k$ and $\vec{v} = 3i - 2j$ are orthogonal because

$$\vec{u} \cdot \vec{v} = 6 - 6 = 0.$$

Class work: Find t such that $\vec{a} = -ti + (t+2)j$ and $\vec{b} = \langle 3, -4 \rangle$ are perpendicular.

5.8. Equation of lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^2$. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0)$ and $\vec{v} = \langle a, b \rangle$ is some vector that is parallel to the line. The vector $\vec{v}$ not need to be on the line, but parallel to the line. Let $P = (x, y)$ be any point on the line. Let $\overrightarrow{r_n}$ and $\vec{r}$ be the position vectors for P_0 and P. Let define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0 P}$.

From the graph we can write since the vector $\vec{a}$ and $\vec{v}$ are parallel, there is a scalar t, such that

$$\vec{a} = t\vec{v}$$

Then we get $\vec{r} = \overrightarrow{r_0} + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$

which is called vector form of the equation of a line. Rewriting this equation we get

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

This set of equations is called the **parametric form** of the equation of a line

If we solve each of the equations in the parametric form of the line for t we get

$$\frac{x - x_0}{\mathrm{t}} = \frac{y - y_0}{\mathrm{t}} = \frac{z - z_0}{t}$$

This is called the **symmetric equations of the line**.

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Example: Write down the equation of the line that passes through the point $(2,-1)$ and $(1,4,)$

Solution: To do this we need the parallel vector $\vec{v}$. let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are one the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v} = \langle 1, -5 \rangle$$

To use the vector form we'll need a point on the line. We have got two and so we can use either one.

$$\vec{r} = \overrightarrow{r_0} + t\vec{v} = \langle 2, -1 \rangle + t\langle 1, -5 \rangle = \langle 2 + t, -1 - 5t \rangle$$

which is the vector form.

The parametric equations of the line becomes

$$x = 2 + t$$
$$y = -1 - 5t$$

and here is the symmetric form

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Note: Two lines are said to be parallel if their direction vectors are multiples of each other. They are said to be perpendicular if their direction vectors are perpendicular to each other. More generally, the angle between two intersecting lines is the angle between their directions vectors.

5.9. Transformation of the plane (Reflection, Translation, Rotation)

A transformation changes a figure into another figure. The new figure is called the image.

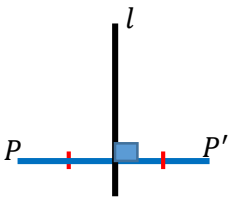
A translation is a transformation in which a figure slides but does not turn. Every point of the figure moves the same distance and in the same direction.

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A rotation, or turn, is a transformation in which a figure is rotated about a point called the center of rotation. The number of degrees a figure rotates is the angle of rotation .In a rotation, the original figure and its image are congruent.

5.9.1. Reflections

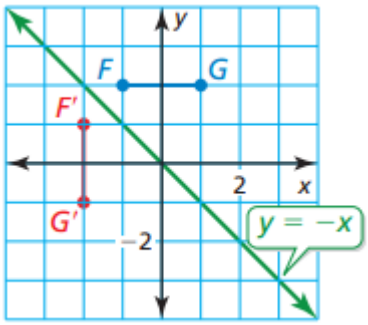
A reflection is a transformation that uses a line like a mirror to reflect a figure. The mirror line is called the ***line of reflection***.



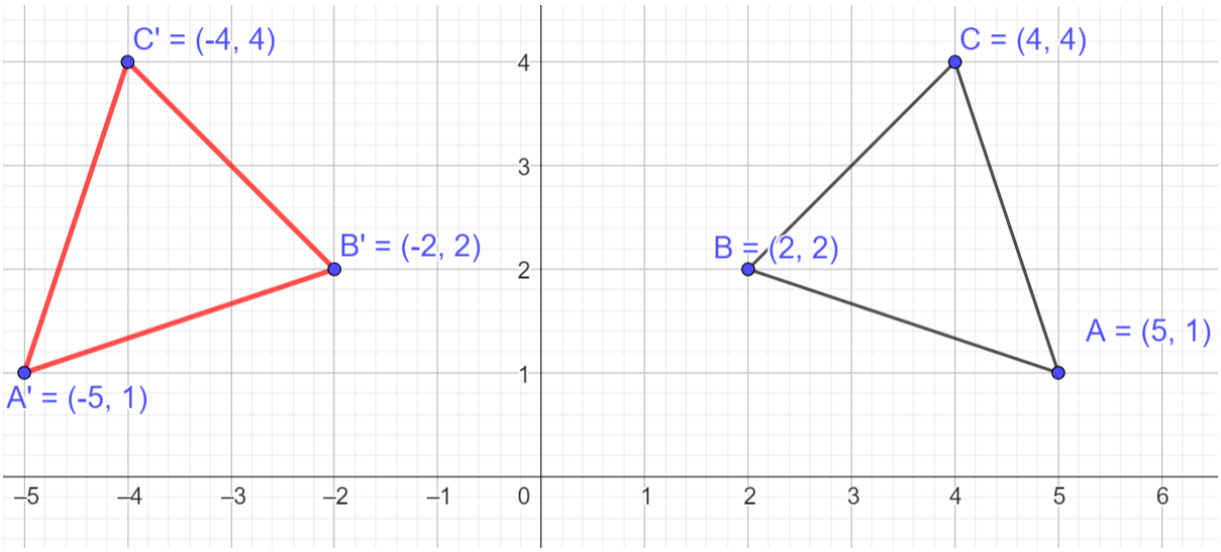
The reflection of point P about a line l is P' such that l is perpendicular bisector of PP'. i.e, $l \perp \overline{PP'}$.

Notation: The reflection of point P about the line l is denoted by $M(P)$. Thus, $P' = M(P)$.

Example: The reflection of $\overline{FG}$ in the line $y = -x$ is $\overline{F'G'}$.



Example: Below it is shown the reflection of ΔABC in the $y-$axis is also a triangle $\Delta A'B'C'$.



Reflection in the line $y = mx, m = tan\,\theta$, θ is an angle with the positive-axis.

If $b = 0$, then the line of reflection is $l: y = mx$, and the image of $M(x, y) = (x', y')$, where

$$x' = xcos\ 2\theta + ysin\ 2\theta$$

$$y' = xcos\ 2\theta - ysin\ 2\theta$$

- If (a, b) is reflected in the $x-$axis $(\theta = 0)$, then its image is the point $M(a, b) = (a, -b)$.

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- If (a,b) is reflected in the $y-$axis $(\theta=\frac{\pi}{2})$, then its image is the point $M(a,b)=(-a,b)$.
- If (a,b) is reflected in the line $y=x$ $(\theta=\frac{\pi}{4})$, then its image is the point $M(a,b)=(b,a)$.
- If (a,b) is reflected in the line $y=-x$ $(\theta=-\frac{\pi}{4})$, then its image is the point $M(a,b)=(-b,-a)$.
- If (a,b) is reflected in origin, then its image is the point $M(a,b)=(-a,-b)$.

Example: Find the image of the point $P(-1,2)$ when reflected about the line $y=\frac{\sqrt{2}}{2}x$.

Solution: Since $m=tan\,\theta=\frac{\sqrt{2}}{2}\Longrightarrow\theta=45°$. Thus,

$$x'=-1cos\,90+2sin\,90=2$$
$$y'=-1cos\,90-2sin\,90=-2$$

Hence, the reflection of $P(-1,2)$ about $y=\frac{\sqrt{2}}{2}x$ is $(2,-2)$.

Example: Find the image of the point $P(-1,2)$ when reflected about the line $y=2x$.

Solution: Since $m=\tan\theta=2\Longrightarrow\frac{Opp}{Adj}=\frac{2}{1}$. Thus, $\sin\theta=\frac{2}{\sqrt{5}}$ and $\cos\theta=\frac{1}{\sqrt{5}}$

$\sin 2\theta=2\sin\theta\cos\theta=2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)=\frac{4}{5}$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{1}{5}-\frac{4}{5}=-\frac{3}{5}$.

Thus,

$$x'=-1cos\,2\theta+2sin\,2\theta=-1\left(-\frac{3}{5}\right)+2\left(\frac{4}{5}\right)=\frac{11}{5}$$
$$y'=-1cos\,2\theta-2sin\,2\theta=-1\left(-\frac{3}{5}\right)-2\left(\frac{4}{5}\right)=\frac{3}{5}$$

Hence, the reflection of $P(-1,2)$ about $y=2x$ is $\left(\frac{11}{5},\frac{3}{5}\right)$.

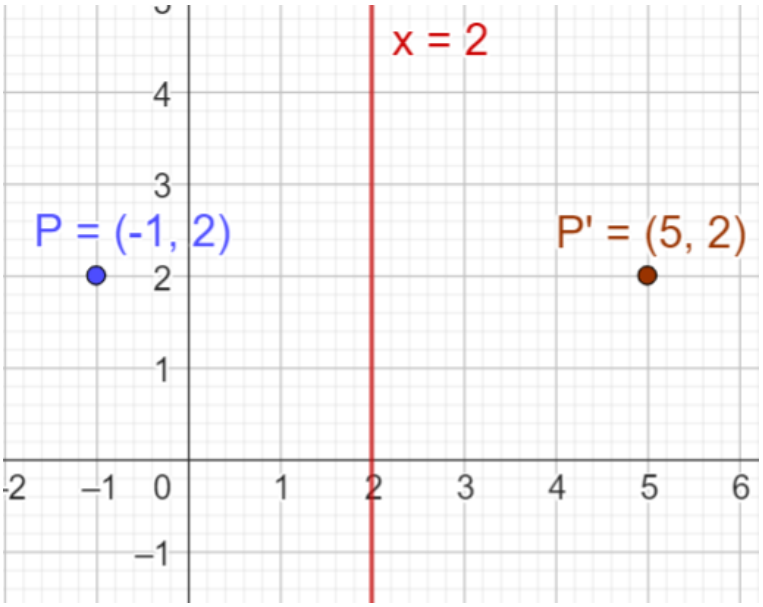
Example: Find the value of a and b if the point $P(a,b)$ is reflected in the $x-$axis to $P'(4,-3)$.

Solution: We know that in the $x-$axis, $M(a,b)=(a,-b)=(4,-3)\Longrightarrow a=4$ and $b=3$.

Example: Find a reflection of a point $P(-1,2)$ in the line $x=2$.

Solution: The reflection is parallel to $y-$axis which is at a distance of 2 units.

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Hence $M(-1,2) = (5,2)$

Reflection of a parallel line

If l' is a line parallel to the line of reflection l, to find the image of l' when reflected about l we follow the following steps:

Step 1: Choose any point P on l'.

Step 2: Find the image of P, $M(P) = P'$.

Step 3: Find the equation of l', which is the line passing through P' with the slope equal to the slope of l.

Reflection in the line $y = mx + b$

To find the image of a point $P(x, y)$ when reflected about a line l, we follow the following four steps:

Step 1: Find the slope of the line ℓ, say m.

Step 2: Find the equation of the line ℓ', which passes through the point P(x, y) and has slope $-\frac{1}{m}$

Step 3: Find the point of intersection A of ℓ and ℓ' which serves as the midpoint of $\overline{PP'}$.

Step 4: Using A as the mid-point of $\overline{PP'}$, find the coordinates of P'.

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Example Find images of the following lines and circles after reflection in the line $y = 2x - 3$.

a $2y + x = 1$ **b** $y = 2x + 1$ **c** $y = 3x + 4$

Solution

a The image of ℓ: $2y + x = 1$ is itself. Explain!

b ℓ: $y = 2x + 1$ is parallel to the reflecting axis.

Hence ℓ' : $y = 2x + b$. We need to determine b.

Let (a, b) be any point on ℓ, say $(0, 1)$, so that its image lies on ℓ'.

By the above reflecting procedure,

$$M((0,1)) = (a', b') \Rightarrow \frac{b'-1}{a'-0} = -\frac{1}{2} \Rightarrow a' = -2b' + 2$$

Also, the midpoint of $(0, 1)$ and (a', b') which is $\left(\frac{a'}{2}, \frac{b'+1}{2}\right)$ lies on the reflecting axis $\Rightarrow \frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}$,

But $a' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2}$

$\Rightarrow b' = -\frac{3}{5} \Rightarrow a' = \frac{16}{5} \Rightarrow \left(\frac{16}{5}, -\frac{3}{5}\right)$ lies on ℓ'

$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b \quad b = -7 \quad \Rightarrow \ell' : y = 2x - 7$

Example Find the image of the line ℓ': $y = -3x - 7$ after a reflection about the line ℓ: $y = -3x + 1$

Solution Pick a point P on ℓ', say P $(1, -10)$.

To find the image of the point P $(1, -10)$ when reflected about the line $y = -3x + 1$, proceed as follows:

Since slope of ℓ is -3, the slope of the perpendicular line is $\frac{1}{3}$. Thus, the equation of the line through $(1, -10)$ with slope $\frac{1}{3}$ is: $\frac{y+10}{x-1} = \frac{1}{3}$

$$\Rightarrow y = \frac{1}{3}x - \frac{31}{3}$$

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Example Find the image of the circle $(x-1)^2+(y+5)^2=1$, when it is reflected about the line $y=2x-1$.

Solution The centre of the circle is $(1,-5)$, the image of the point $(1,-5)$ when reflected about the line $\ell: y=2x-1$ is $\left(\frac{-19}{5},\frac{-13}{5}\right)$

Thus, the equation of the image circle is $\left(x+\frac{19}{5}\right)^2+\left(y+\frac{13}{5}\right)^2=1$

5.9.2. Rotations

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which maps every point P of the plane into the point P' of the plane such that $OP = OP'$ and $m\angle(POP') = \theta$

If $R_\theta(x,y)=(x',y')$, then $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$.

Note:

Let R be a counter-clockwise rotation through an angle θ about the origin. Then

i $\theta=\frac{\pi}{2}\Rightarrow R(x,y)=(-y,x)$

ii $\theta=\pi\Rightarrow R(x,y)=(-x,-y)$

iii $\theta=\frac{3\pi}{2}\Rightarrow R(x,y)=(y,-x)$

iv $\theta=2n\pi$ for $n\in\mathbb{Z}\Rightarrow$ R is the identity transformation.

v Every circle with centre at the centre of rotation is fixed.

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Example Using the formula, find the images of the following points in rotation about the origin through the indicated angle.

a $(4, 0)\ ; 60^{o}$ **b** $(1, 1)\ ; -\frac{\pi}{6}$

Solution

a $x' = x\cos\theta - y\sin\theta;\ x = 4,\ y = 0; \theta = 60^{o}$

$= 4\cos 60^{o} - 0 \times \sin 60^{o} = 2$

$y' = x\sin\theta + y\cos\theta$

$= 4\sin 60^{\circ} + 0 \times \cos 60^{\circ} = 2\sqrt{3}$

$\Rightarrow R_{60^{o}}(4, 0) = \left(2, 2\sqrt{3}\right)$

b $x' = 1 \times \cos\left(-\frac{\pi}{6}\right) - 1 \times \sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

$y' = x\sin\theta + y\cos\theta$

$y' = 1 \times \sin\left(\frac{-\pi}{6}\right) + 1 \times \cos\left(\frac{-\pi}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

$\Rightarrow R_{60^{o}}(1, 1) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2},\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

Note:

One can also obtain the image of a line under a given rotation as follows:

- ✓ Choose two points on the line.
- ✓ Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

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Example Find the equation of the line $\ell: 3x-2y=1$ after it has been Rotated -135^{o} about (–2,3).

Solution According to the note, we choose any two arbitrary points, say (1, 1) and (–1, –2). Together with $(x_o, y_o) = (-2, 3)$ and $\theta = -135^{o}$, we get

$$R(1, 1) = (-2-2.5\sqrt{2},\ 3-0.5\sqrt{2}) \quad \text{and} \quad R(-1,-2) = (-2-3\sqrt{2},\ 3+2\sqrt{2})$$

$$\Rightarrow \text{The slope of } \ell' = \frac{3+2\sqrt{2}-3+0.5\sqrt{2}}{-2-3\sqrt{2}+2+2.5\sqrt{2}} = -5$$

$$\Rightarrow \ell': \frac{y-3-2\sqrt{2}}{x+2+3\sqrt{2}} = -5$$

$$\Rightarrow \ell': y-3-2\sqrt{2} = -5x-10-15\sqrt{2}$$

$$\Rightarrow \ell': y+5x+7+13\sqrt{2} = 0$$

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ESSLCE Questions

1. Let $\overrightarrow{AB}$ be a vector of length 2cm to the west and $\overrightarrow{BC}$ be a vector of length 2 cm to the south. Which one of the following describes the sum of the two vectors? A. $2\sqrt{2}$ cm in the direction East 30° South B. $2\sqrt{2}$ cm in the direction West 30° South C. 4 cm in the direction West 45° South D. 4 cm in the direction West 60° South
2. Which one of the following is a vector quantity?
   A. Size of a plantation farm. B. Velocity of a moving car.
   C. Volume of a normal size bottle. D. Amount of fat in one liter of milk.
3. Let $u = (-1, 4)$ and $v = (1, -3)$ be vectors in the plane. Which one of the following equal to $u - v$? A. $(-2, 1)$ B. $(0, 1)$ C. $(-2, 7)$ D. $(0, 7)$
4. Let $u = (3, -5)$ and $v = (-1, -1)$. What is $u \cdot v$?
   A. -8 B. 2 C. -4 D. 4
5. Let $a = (1, 3, -1)$ and $b = (2, -1, 1)$ be vectors in space. Which one of the following is true?
   A. $a - b = (-1, 4, -2)$ B. $a + b = (3, 4, 0)$
   C. $3a = (3, 9, 3)$ D. $a + 2b = (5, -1, 1)$
6. What is the image of a line $l: y = 2x - 1$ when reflected about the line $y = 2x + 2$?
   A. $y = -2x + 5$ C. $y = 2x - 5$
   B. $y = 2x + 5$ D. $y = -2x - 5$

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Chapter 6

Matrices and Determinant

6.1. Definition of Matrices and types of matrices ….(Grade 11, Page 219)

6.1.1. Definition of Matrices

Consider the stock position of different bulbs of red, blue and green colors of 25, 40, 60 and 100 watts, available in an electrical store and displayed in a tabular form as given below.

$$\begin{matrix} & Red & Blue & Green \\ 25\,W \\ 40\,W \\ 60\,W \\ 100\,W \end{matrix}\begin{bmatrix} 21 & 49 & 37 \\ 80 & 95 & 106 \\ 16 & 30 & 25 \\ 11 & 23 & 31 \end{bmatrix}$$

The table consists of four rows and three columns. The second row informs that there are 80 red, 95 blue and 106 green bulbs of 40 W each available in the store. Such an arrangement of numbers in array of rows and columns is called a matrix.

A rectangular array of ordered elements (numbers, functions or just symbols) is known as a matrix.

The entries in a matrix are usually enclosed in two curved lines or square brackets.

Definition: A set of $m \times n$ numbers or functions, arranged in a rectangular array of m rows and n columns is called $m \times n$ (read "m by n") **matrix** or a matrix of order $m \times n$.

☞ A matrix A of order $m \times n$ is given below

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad \overrightarrow{n\ columns},\ \overleftarrow{m\ rows}$$

Notice here the dimensions of the matrix is $m \times n$, m rows and n coulumns. The row index will always be stated first. We use boldface type to represent a matrix, and we enclose the array itself in square brackets. The horizontal lines are called rows and the vertical lines are called columns.

☞ The individual values in the matrix are called **entries** or **elements**.

☞ We often use capital letters to represent matrices and enclose the array of numbers with brackets or parenthesis; e.g.,$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ or $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. $A = [a_{ij}]$ is a shorthand notation.

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- For the entry a_{ij}, the suffixed i specify the row and the suffices j the column in which the entry appears. Thus a_{ij} is the entry of the matrix located in the i^{th} row and j^{th} column of the matrix. For example a_{21} is the element in row 2, column 1 of the matrix A).
- $m \times n$ matrix contains mn elements.

Example: Suppose $A = \begin{bmatrix} 4 & -7 & 5 \\ 8 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 8 \\ -7 & 1 \\ 5 & 6 \end{bmatrix}$, find $a_{22}, a_{13}, b_{13}$and b_{31}.

Classwork: Form a 3×4 matrix B, such that $b_{ij} = (-1)^{i+j}$.

6.1.2. Algebra of matrices

1. Addition of matrix

Let A and B be two matrices of the same order. Then the addition of A and B, denoted by $A + B$, is the matrix obtained by adding corresponding entries of A and B.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 7 \end{bmatrix}.$$

1+3=4

Thus if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

In general if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m\times n} \text{ then}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}_{m\times n}.$$

Classwork: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ then find $A + B$.

Note: If the two matrices are of different dimensionality then corresponding elements cannot be found, in which case addition is not defined. Matrix addition is defined only when B and A have the same numbers of rows and columns, respectively. When this is the case, the matrices A and B are said to be "conformable in addition."

Note: We can add two matrices if and only if they are of the same order.

Negative of a matrix: If A is any matrix, the negative of A, denoted by $-A$, is the matrix obtained by replacing each entry in A by its negative.

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For example if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$.

Classwork: Find i) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 3 & -1 \\ 2 & 4 \end{bmatrix}$

2. Subtraction of Matrices

Let A and B be two matrices of the same order. Then by $A - B$, we mean $A + (-B)$. i.e., $A - B = A + (-B)$.

Classwork: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ then find $A - B$.

Properties of addition and subtraction of matrices

Matrix addition is commutative: If A and B are two matrices of the same order, then $A + B = B + A$.

Matrix addition is associative: If A, B and C are three matrices of the same order, then $(A + B) + C = A + (B + C)$.

Existence of additive identity: If 0 is the zero matrix of the same order as that of the matrix A, then $A + 0 = A = 0 + A$.

Existence of additive inverse: If A is any matrix then $A + (-A) = 0 = (-A) + A$.

Additive Inverse of Matrix: If two matrices A and B are such that their sum $(A + B)$ is zero matrix, then A and B are called additive inverses of each other.

Example: Since $\begin{bmatrix} 5 & 2 & -1 \\ 1 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -5 & -2 & 1 \\ -1 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

hence $\begin{bmatrix} 5 & 2 & -1 \\ 1 & 4 & 5 \end{bmatrix}$ is the additive inverse of $\begin{bmatrix} -5 & -2 & 1 \\ -1 & -4 & -5 \end{bmatrix}$.

Classwork: Find the additive inverse of $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.

3. Multiplication of a matrix by a scalar

If k is a scalar (a number) and A is a matrix, then the product kA is a matrix, each element of which is k −times the corresponding entry of A. This operation is called scalar multiplication. The result is a matrix of the same size.

$$A = kB \Longrightarrow a_{ij} = kb_{ij} \text{ for } i = 1, \cdots, m \,;\; j = 1, \cdots, n.$$

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Example: If A = $\begin{bmatrix} 3 & 5 & -2 \\ 4 & 2 & 4 \\ 3 & -4 & 3 \end{bmatrix}$ find $3A$?

Solution: 3A = $\begin{bmatrix} 3\times 3 & 3\times 5 & 3\times -2 \\ 3\times 4 & 3\times 2 & 3\times 4 \\ 3\times 3 & 3\times -4 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 9 & 15 & -6 \\ 12 & 6 & 12 \\ 9 & -12 & 9 \end{bmatrix}$

Properties of scalar multiplication

If A and B are two matrices of the same order and if k, k_1 and k_2 are scalar, then

- $k(A+B) = kA + kB$
- $(k_1 + k_2)A = k_1A + k_2A$
- $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

Classwork: Let matrices A and B be of orders 3×2 and 2×2 respectively. Then

a) Is it possible that $A = B$? b) Is $A - B$ defined? c) For a scalar k, is kA defined?

Classwork: Find $3A - 2B$, where $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$.

Homework: 1. Solve for X in the equation, $2X - 3A = 2B$, for matrices $A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$.

2. Find matrix X and Y if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

4. Multiplication of Matrices

The product of an $m\times n$ matrix A and an $n\times p$ matrix B produces an $m\times p$ matrix C, i.e.

$$\underset{m\times n}{A}\ \underset{n\times p}{B} = \underset{m\times p}{C}.$$

Note: Matrix multiplication AB of the matrix A and B is defined if the number of columns in A is equal to the numbers of rows in B.

Example: Consider the following matrices:

$$\text{A} = \begin{bmatrix} 1 & 5 & -2 \\ 3 & 0 & 7 \end{bmatrix}_{2\times 3}, \quad \text{B} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 9 \\ 4 & -3 & 6 \end{bmatrix}_{3\times 3}, \text{C} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 5 & -7 & 6 & -2 \\ -9 & 0 & -1 & 7 \end{bmatrix}_{3\times 4}, \text{D} = \begin{bmatrix} 2 & 1 \\ 5 & 6 \\ 9 & -2 \\ 3 & 7 \end{bmatrix}_{4\times 2}$$

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Notice here the multiplication of A and B, AB is a matrix of order 2×3. Similarly the matrix AC is of order 2×4. BC is a matrix of order 3×4. But, it is not possible to form the multiplication of matric B with A, BA. (Why?)

The product matrix of two matrices is obtained by taking dot products of the rows of the left matrix with the columns of the right matrix.

Below we consider the multiplication of matrix A by matrix B, i.e. $A \times B$.

Dimensions of matrix A	Dimensions of matrix B	Is $A \times B$ possible?	Dimensions of resultant matrix
(2×2)	(2×2)	Yes, number of columns in A equals the number of rows in B	(2×2) => 2 rows and 2 columns
(2×2)	(2×3)	Yes, number of columns in A equals the number of rows in B	(2×3)=> 2 rows and 3 columns
(1×4)	(4×1)	Yes, number of columns in A equals the number of rows in B	(1×1) => a scalar
(2×3)	(3×2)	No, number of columns in A is not equal to the number of rows in B	Not Applicable
(1×4)	(1×4)	No, number of columns in A is not equal to the number of rows in B	Not Applicable

Properties of matrices multiplication.

Let A, B and C be any matrices, and if I is an identity matrix, then

i. $A(B + C) = AB + AC$ $(A + B)C = AC + BC$ distributive law

ii. $A(BC) = (AB)C$ Associative law

iii. $AI = IA = A$ iv. $A0 = 0A = 0$.

Definition: Let $\boldsymbol{A = [a_{ij}]}$ be $\boldsymbol{m \times n}$ and $\boldsymbol{B = [b_{ij}]}$ be $\boldsymbol{n \times p}$ matrices, then

$$\boldsymbol{AB = \sum_{k=1}^{n} a_{ik} b_{kj}.}$$

Note: ***1.Matrix multiplication is not commutative ($AB \neq BA$).***

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Example: Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}$, then $AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}=\begin{bmatrix}8 & 12\\ 18 & 26\end{bmatrix}$

and $BA=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}8 & 12\\ 18 & 26\end{bmatrix}$. Hence $AB=BA$, but not always.

Example: Consider A $=\begin{bmatrix}1 & 2\\ 4 & 1\end{bmatrix}$, B $=\begin{bmatrix}2 & 2\\ 3 & 1\end{bmatrix}$

AB $=\begin{bmatrix}1\times 2+2\times 3 & 1\times 2+2\times 1\\ 4\times 2+1\times 3 & 4\times 2+1\times 1\end{bmatrix}=\begin{bmatrix}8 & 4\\ 11 & 9\end{bmatrix}$

BA $=\begin{bmatrix}2\times 1+2\times 4 & 2\times 2+2\times 1\\ 3\times 1+1\times 4 & 3\times 2+1\times 1\end{bmatrix}=\begin{bmatrix}10 & 6\\ 7 & 7\end{bmatrix}$. Hence $AB\neq BA$.

2. $AB=0$ does not necessary imply $A=0, B=0$.

Example: $A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}, B=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$

$AB=\begin{bmatrix}1\times(-1)+1\times 1 & 1\times 1+1\times(-1)\\ 2\times(-1)+2\times 1 & 2\times 1+2\times(-1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

$BA=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}=\begin{bmatrix}(-1)\times 1+1\times 2 & (-1)\times 1+1\times 2\\ 1\times 1+(-1)\times 2 & 1\times 1+(-1)\times 2\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$.

3. $AB=AC$ does not necessarily imply $B=C$.

Example: $A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}$, $C=\begin{bmatrix}3 & 0\\ 1 & 3\end{bmatrix}$

$AB=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}=\begin{bmatrix}1\times 2+1\times 2 & 1\times 1+1\times 2\\ 2\times 2+2\times 2 & 2\times 1+2\times 2\end{bmatrix}=\begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix}$

$AC=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\cdot\begin{bmatrix}3 & 0\\ 1 & 3\end{bmatrix}=\begin{bmatrix}1\times 3+1\times 1 & 1\times 0+1\times 3\\ 2\times 3+2\times 1 & 2\times 0+2\times 3\end{bmatrix}=\begin{bmatrix}4 & 3\\ 8 & 6\end{bmatrix}$

Classwork: Matrix F is multiplied by matrix B so that the multiplication EB produces a 5×4 matrix. What are the dimensions of matrix F?

Note: A matrix A can be multiplied by itself if and only if it is a square matrix. Then if A is a square matrix, then the product of n As exist and are given the nth power of A (which is of the same order as the order of A):

$$AAA\cdots A\ (n\text{-factors})=A^n.$$

It also follows that $A^pA^q=A^{p+q}$.

Note: If A is square matrix, $A^2=AA, A^3=A^2A=AAA$, ingeneral $A^n=A^{n-1}A=AA\cdots A$ (n factors).

Assignment: If $A=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{3} & \mathbf{-2} & \mathbf{1}\\ \mathbf{4} & \mathbf{2} & \mathbf{1}\end{bmatrix}$ show that $A^3-23A-40I=0$.

Homework: Show that if $A=\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$, then

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(a) $(A+B)^2 \neq A^2+2AB+B^2$

(b) $(A+B)(A-B) \neq A^2-B^2$

Homework: Show that A and B commute, if $A=\begin{bmatrix}2 & -2\sqrt{2}\\ \sqrt{2} & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2\sqrt{2}\\ -\sqrt{2} & 2\end{bmatrix}$.

6. 1. 3. Transpose of a matrix.

Definition: Let A be an $m\times n$ matrix. The transpose of a matrix A, denoted by A' or A^t, is obtained by interchanging the rows and columns of A. Thus the first row of A is the first column of A^t, the second row of A is the second column of A^t and so on.

In general, if $A=(a_{ij})_{m\times n}$ then $A^t=(a_{ji})_{n\times m}$.

i.e., For $A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n}$ the transpose $A^t=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{a22} & \cdots & a_{m2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1n} & a_{2n} & \dots & a_{nm}\end{bmatrix}_{n\times m}$

Example: If $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ then $A^t=\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$.

Properties of matrices transpose:

1. For any matrix A, $(A^t)^t=A$.
2. For any matrix A and scalar k, $(kA)^t=kA^t$.
3. If A and B are of the same order, $(A\pm B)^t=A^t\pm B^t$.
4. If A is $m\times n$ and B is $n\times p$ are matrices then $(AB)^t=B^tA^t$.

Classwork: Verify that $(AB)^t=B^tA^t$ if $A=\begin{bmatrix}-2\\ 4\\ 5\end{bmatrix}$ and $B=[1 \quad 3 \quad -6]$.

6. 1. 4. Special types of matrices

Some frequently used matrices have special names.

i). Row matrix: A matrix having only one row is called a row matrix. Thus the matrix $[a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]_{1\times n}$ is a row matrix of order $1\times n$.

ii). Colum matrix: a matrix having only one column is called a column matrix. Thus the matrix $\begin{bmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{bmatrix}_{m\times 1}$ is a column matrix of order $m\times 1$.

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iii). Zero or Null matrix: A matrix whose entries are all 0 is called a zero or null matrix. It is usually denoted by 0.

iv). Square matrix: An $m \times n$ matrix is said to be a square matrix of order n if $m = n$. i.e., if it has the same number of columns as rows. For example, $\begin{bmatrix} -2 & 4 & -2 \\ 3 & 3 & 3 \\ 4 & -2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are square matrices of order 3 and 2.

In a square matrix of order n, the entries $a_{11}, a_{22}, \cdots, a_{nn}$ which lie on the diagonal are called the main diagonal entries or more simply the main diagonal or leading diagonal.

In the matrix $D = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ 3 & -3 & 1 \end{bmatrix}$ the entries $d_{11} = 1, d_{22} = -2, d_{33} = 1$ constitute the *leading diagonal*.

Note: If the number of rows and number of columns of a matrix are not equal the matrix is called rectangular matrix.

v). Diagonal matrix only has non-zero elements on the main diagonal. These non-zero elements can have any value. The following matrices are diagonal matrices.

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}, \quad \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}, \quad \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

☞ A matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0, \text{for } i \neq j$ and $a_{ij} \neq 0, \text{for at least one } i = j$.

The product of two diagonal matrices is another diagonal matrix, whose main diagonal elements are the products of the corresponding elements of the two given matrices. Clearly, then, diagonal matrix products commute. However, if A is not diagonal, and B is diagonal, the product is not commutative. In BA, the corresponding rows of A are multiplied by the diagonal elements of B, while in AB, the corresponding columns of A are multiplied by the diagonal elements of B.

vi) Scalar matrix: The diagonal elements are not, in general, equal in value. In the cases in which the main diagonal elements are equal, the matrix is called a **scalar matrix**. i.e. A diagonal matrix whose diagonal elements are all equal is called a ***scalar matrix***.

Example: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ is a scalar matrix.

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Note: The sum of the entries of the leading diagonal of a square matrix A of order n is called the trace of A. i.e. trace of $A = \sum_{i=1}^{n} a_{ii}$.

Note: $trace(A + B) = trace(A) + trance(B)$.

vi). Identity Matrix or Unit matrix I: A diagonal matrix whose diagonal elements are all equal to unity is called a unit matrix denoted by I. i.e., A square $(n \times n)$ matrix whose ij elements are zero for $\neq j$, and whose elements ii are unity, is defined as the "unit matrix," I.

☞ A matrix $A = [a_{ij}]$ is identity matrix if $a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$. e.g.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

vii). Triangular matrix: A square matrix whose elements either above (below) the leading diagonal are all zero, is called a **triangular matrix**. If the matrix, A, has all zero elements below the main diagonal, it is known as an "upper triangular" matrix. The transpose of an upper triangular matrix—one with all zero elements above the main diagonal—is called "lower triangular." Then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

are examples of upper and lower triangular matrices.

If A is an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i > j$, we say A is upper triangular.

If A is an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i < j$, we say A is lower triangular.

Note: If A is diagonal, upper triangular, or lower triangular, then $det(A)$ is the product of the diagonal entries of A. i.e.

$$|A| = \prod_{i=0}^{n} a_{ii}$$

Such matrices are very important because (1) their determinant is easily calculated as the product of its main diagonal terms, and (2) its inverse is similarly easy to determine.

viii). Matrix equality: Two matrices A and B are said to be equal, written $A = B$ if and only if they are of the same order and each element of matrix A is equal (identical) to the corresponding element of matrix B; i.e $A = B$ if and only if for each i, j, $a_{ij} = b_{ij}$ and $order(A) = order(B)$.

Activity: Given the matrix equation $\begin{bmatrix} x - y & 6 \\ x + y & 8 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 8 \end{bmatrix}$. Find the value of x and y.

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ix). Orthogonal, symmetric and skew-symmetric matrix

Definition: A matrix which is unchanged under transposition is known as "symmetric." i.e. if $A^t = A$ then A is a ***symmetric*** matrix.

Example: $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix}$, $A^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix}$.

Note: If $A = (a_{ij})_{m\times n}$ is symmetric matrix then $a_{ij} = a_{ji}$ for each i and j.

Definition: If $A^t = -A$ then A is a ***skew-symmetric*** matrix.

Example: The matrix A = $\begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & -4 \\ -7 & 4 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note: If $A = (a_{ij})_{m\times n}$ is skew-symmetric then $a_{ij} = -a_{ji}$ for each i and j. Moreover when $i = j$, then $a_{ii} = -a_{ii}$, i.e., $2a_{ii} = 0$. Hence element of main diagonal of skew-symmetric matrix are all zero.

Definition: A square matrix A is said to be ***orthogonal matrix*** if $AA^t = A^tA = I$.

Example: $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is orthogonal matrix.

$$AA^t = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Properties of symmetric and skew-symmetric matrices

1. For any square matrix A, $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric.
2. If A and B are two symmetric (or skew-symmetric) matrices of the same order, then so is $A + B$.
3. If A is symmetric (or skew-symmetric) matrix, then so is kA.
4. If A and B are two symmetric matrices of the same order, then AB is symmetric iff $AB = BA$.
5. Every matrix $A \in M_n(\mathbb{R})$ can be uniquely written as the sum of a skew-symmetric and symmetric matrix. i.e., $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$.

Classwork: For a matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 4 & 2 \\ 0 & 5 & 1 \end{bmatrix}$ verify that $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric.

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6.2. Determinant and its properties

It is important that students have mastered find the determinant of a 2×2 matrix before progressing to 3×3 and above. The determinants of orders 1 is defined as:

- let $A = [a_{11}]$ be a square matrix of order 1, then determinant of A, denoted by $|A|$ is defined as $|A| = |a_{11}| = a_{11}$
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is written $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

Example: If $A = [10]$, $B = [-15]$ and $C = [0]$ then $|A| = 10$, $|B| = -15$ and $|C| = 0$.

Example: Find det. A if $A = \begin{bmatrix} 1 & 2 \\ 4 & -7 \end{bmatrix}$.

Solution: $|\mathrm{A}| = \begin{vmatrix} 1 & 2 \\ 4 & -7 \end{vmatrix} = 1 \times (-7) - 2 \times 4 = -15.$

Note: The determinants are defined only for square matrices.

Classwork: $\begin{vmatrix} a-b & b \\ b & a+b \end{vmatrix} = a^2 - 2b^2$ (Show!)

6.2.1. Singular and Non-singular Matrices

A square matrix A is called singular if $|A| = 0$ and A is called non-singular if $|A| \neq 0$.

Example: If $A = \begin{bmatrix} 2 & 10 \\ 1 & 5 \end{bmatrix}$, then $|A| = 0$, hence A is singular.

Example: Find k if $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular.

Solution: Since A is singular, $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$$(k-2)(k+2) - 5 = 0 \Longrightarrow k^2 - 4 - 5 = 0 \Longrightarrow k^2 - 9 = 0 \Longrightarrow k = \pm 3.$$

Properties of determinate.

1. $|A| = |A^t|$.
2. $|AB| = |A||B|$.

6.2.2. Minors and Co-factors of element

If we delete the i^{th} rows and j^{th} columns in the determinant of $A_{n\times n}$, we get another determinant of order $(n-1) \times (n-1)$, and we call the new formed determinant minor of the element a_{ij}. It is denoted by M_{ij}.

Consider a determinant $|A|$ of third order

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$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then, the minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, the minor of a_{23} is $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

Cofactors

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij} of the matrix A, where M_{ij} is the minor matrix of order $(n-1) \times (n-1)$ formed by deleting the column and row containing a_{ij}; For example,

$$c_{11} = (-1)^{1+1}|m_{11}| = (1)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{23} = (-1)^{2+3}|m_{23}| = (-1)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$$

Example: Find the cofactors matrix of $= \begin{bmatrix} -2 & 3 \\ 7 & 11 \end{bmatrix}$?

Solution: $C_{11} = (-1)^{1+1}\,|11| = (1).(11) = 11$

$C_{21} = (-1)^{2+1}\,|3|$ = (-1).(3) = -3 , etc…

Example: Find the cofactors matrix of $A = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 2 & 3 \\ 8 & 7 & 3 \end{bmatrix}$

Solution: $c_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 3 \\ 8 & 3 \end{vmatrix} = (-1)[5 \times 3 - 3 \times 8]$ = (-1).(-9) = 9

$c_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 4 \\ 8 & 3 \end{vmatrix} = (-1)[2 \times 3 - 4 \times 8] = (1).(-26) = -26$, etc...

Determinant of a matrix of order greater than 3

If A is a square matrix of order n $(n > 2)$ and if C_{ij} and M_{ij} are the cofactor and minor, respectively, of a_{ij}, then its determinant may be calculated as

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = (-1)^{i+1}a_{i1}C_{i1} + (-1)^{i+1}a_{i2}C_{i2} + \cdots + (-1)^{i+1}a_{in}C_{in}$$

or $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = (-1)^{J+1}a_{1j}M_{1j} + a_{2j}M_{2j} + \cdots + a_{nj}M_{nj}$

Hence the $det\ A$ is the sum of the elements of any row or column multiplied by their corresponding cofactors.

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$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}.\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}.\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}.\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example: Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$, find $|A|$.

Solution: $|A| = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 1 \times -3 - 4 \times -6 + 7 \times -3 = 0$

Or, $|A| = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} = -2 \times -6 + 5 \times -12 + (-8) \times -6 = 0$

Example: Find $det.A$ if $A = \begin{bmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix}$.

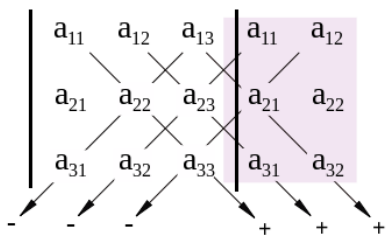
Solution: $|A| = \begin{vmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} - (-5)\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 3(4) + 5(9) + 3(-1) = 54..$

Rules of Sarrus (For third order determinates only)

There is another method and to compute the determinant of a 3×3 matrix. It is named Sarrus' rule or Sarrus' scheme. Consider a 3×3 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then its determinant can be computed by the following scheme:



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

After writing the determinant, repeat the first two columns as below

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Homework: Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$, find $|A|$.

Note: $det(I_n) = 1$, where I_n is an identity matrix of order n.

Adjoint a matrix

Definition: Let $A = (a_{ij})$ be a square matrix of order n and let C_{ij} be the co-factor of (a_{ij}). Then the **adjoin of *A***, denoted by $adj\ A$, is defined as the transpose of the cofactor matrix C_{ij}.

Example: If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, co-factors of A are:

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5, \quad C_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2, \quad C_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{21} = -1, \quad C_{22} = 2, \quad C_{23} = -1$$

$$C_{31} = 3, \quad C_{32} = -2, \quad C_{33} = 3$$

then $C = \begin{bmatrix} 5 & -2 & 1 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$. Hence, $adj\ A = C^t = \begin{bmatrix} 5 & -1 & 3 \\ 2 & 2 & -2 \\ 1 & -1 & 3 \end{bmatrix}$.

6.3. Inverse of a square matrix

Note: The inverse $\boldsymbol{a^{-1}}$ of a scalar (number) $\boldsymbol{a}$ is defined by $aa^{-1} = 1$.

We say that the inverse of a square matrix A is another square matrix B where:

$$AB = I \text{ and } BA = I$$

where I is the identity matrix of the same order.

Such a matrix B, if it exists, is unique and is called the **inverse of *A*** (written as A^{-1}). Consequently we have:

$$AA^{-1} = I = A^{-1}A.$$

Example: Consider these two square matrices: $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$ and $\begin{pmatrix} 7 & 10 \\ -2 & 3 \end{pmatrix}$.

Multiplying the two matrices

$$\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 7 & 10 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly, $\begin{pmatrix} 7 & 10 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence these two matrices are inverses of one another.

Steps to find the inverse of matrix:

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1. Find the determinant $det.A$.
2. Find the cofactors of all elements in A and form a new matrix C of cofactors, where each element is replaced by its cofactor.
3. The inverse of A is now given as:

$$A^{-1} = \frac{C^t}{|A|}$$

Note: The inverse A^{-1} exists if (and only if) $|A| \neq 0$.

To obtain the inverse of a square **matrix of order 2** we:

Step 1: Interchange the elements on the main diagonal,

Step 2: Change the sign of the other two elements,

Step 3: Divide by the determinant corresponding to the original matrix.

Suppose we have a 2×2 matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then the inverse of A is:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Note: 1. A matrix having an inverse is called ***an invertible matrix***.

2. Inverse of a matrix is unique.
3. A square matrix A is invertaible(non-singular) if $|A| \neq 0$ and non-invertible(singular) if $|A| = 0$.
4. $A^{-1} = \frac{1}{|A|} adj\ A$.
5. $|A^{-1}| = \frac{1}{|A|}$.

Example: Consider $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, since $|A| = 0$, the matrix A do not have inverse.

Example: Find the inverse of $\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 2 \\ 3 & -2 & -1 \end{bmatrix}$?

Solution:

$$|\mathrm{A}| = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 1 & 2 \\ 3 & -2 & -1 \end{vmatrix} \begin{matrix} 1 & -1 \\ -3 & 1 \\ 3 & -2 \end{matrix}$$

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$$= 1 \times 1 \times (-1) + (-1) \times 2 \times 3 + 2 \times (-3) \times (-2) - 2 \times 1 \times 3 - 1 \times 2 \times (-2) - (-1) \times (-3) \times (-1)$$

$$= 6$$

Since the determinant is nonzero an inverse exists.

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3 \text{ , } c_{12} = (-1)^{1+2} \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} = 3 \text{ , etc.}$$

Cofactors matrix will be:

$$C = \begin{bmatrix} 3 & 3 & 3 \\ -5 & -7 & -1 \\ -4 & -8 & -2 \end{bmatrix}, \text{then } C^t = \begin{bmatrix} 3 & -5 & -4 \\ 3 & -7 & -8 \\ 3 & -1 & -2 \end{bmatrix}.$$

Finally, the inverse of A is:

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -5 & -4 \\ 3 & -7 & -8 \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-5}{6} & \frac{-2}{3} \\ \frac{1}{2} & \frac{-7}{6} & \frac{-4}{3} \\ \frac{1}{2} & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix}.$$

Check $AA^{-1} = I$ (unit matrix)

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 1 & 2 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-5}{6} & \frac{-2}{3} \\ \frac{1}{2} & \frac{-7}{6} & \frac{-4}{3} \\ \frac{1}{2} & \frac{-1}{6} & \frac{-1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Find the inverse of $R = \begin{bmatrix} -2 & 2 \\ 5 & -4 \end{bmatrix}$ using adjoint method if the inverse exists.

Solution: $|R| = (-2) \times (-4) - 2 \times 5 = -2.$

$c_{11} = (-1)^{1+1}|-4| = -4 \quad c_{12} = (-1)^{1+2}|5| = -5 \quad c_{21} = (-1)^{2+1}|2| = -2 \quad c_{22} = (-1)^{2+2}|-2| = -2$

$$C = \begin{bmatrix} -4 & -5 \\ -2 & -2 \end{bmatrix}, C^t = \begin{bmatrix} -4 & -2 \\ -5 & -2 \end{bmatrix}$$

Hence, $R^{-1} = \frac{1}{|R|} \begin{bmatrix} -4 & -2 \\ -5 & -2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -4 & -2 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2.5 & 1 \end{bmatrix}$

Properties of the inverse of a matrix

1. If A is invertible matrix then $(A^{-1})^{-1} = A$.
2. The transpose of an inverse matrix is equal to the inverse of its transpose. i.e. $(A^{-1})^t = (A^t)^{-1}$.
3. If A and B are two invertible matrices of the same order, then AB is also invertible and moreover, $(AB)^{-1} = B^{-1}A^{-1}$
4. Inverse of a symmetric matrix is also symmetric.

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6.4. Solving system of linear equations with two and three variables

6.4.1. Elementary Row Operations

There are three operations that can be applied to modify the matrix and still preserve the solution to the system of equations.

Type 1: *Interchanging*: Interchanging two rows (say R_i and R_j).

$$R_i \to R_j$$

Type 2: *Replacing*: Replacing one row (say R_i) by th sum of itself and a multiple of another row (say R_j)

$$R_i \to R_i + kR_j$$

Type 3: *Scaling*: Multiply all entries in a row (say R_i) by a non-zero constant(scalar) k

$$R_i \to kR_i$$

Note: The same as for column operation.

If the matrix is associated to a system of linear equations, then these operations do not change the solution set. Therefore, if one's goal is to solve a system of linear equations, then using these row operations could make the problem easier.

Note: Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary row/column operations.

Definition 1. A matrix is in row echelon form if 1. Nonzero rows appear above the zero rows. 2. In any nonzero row, the first nonzero entry is a one (called the leading one). 3. The leading one in a nonzero row appears to the left of the leading one in any lower row.

Definition 2. A matrix is in reduced row echelon form (RREF) if the three conditions in Definition 1 hold and in addition, we have 4. If a column contains a leading one, then all the other entries in that column are zero. Example. $\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form, but not in RREF. Any matrix can be transformed into its RREF by performing a series of operations on the rows of the matrix. The general plan is to first transform the entries in the lower left into zeros. The final step is to transform all the entries above the leading ones into zeros. The allowable operations are called elementary row operations.

N4a 6.4.2. Echelon form

For each row in a matrix, if the row does not consist of only zeros, then the left-most non-zero entry is called the *leading coefficient* (or ***pivot***) of that row. So if two leading coefficients are in the

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same column, then a row operation of type 3 (see **above**) could be used to make one of those coefficients zero. Then by using the row swapping operation, one can always order the rows so that for every non-zero row, the leading coefficient is to the right of the leading coefficient of the row above. If this is the case, then matrix is said to be in **row echelon form**. So the lower left part of the matrix contains only zeros, and all of the zero rows are below the non-zero rows. The word "echelon" is used here because one can roughly think of the rows being ranked by their size, with the largest being at the top and the smallest being at the bottom.

For example, the following matrix is in row echelon form, and its leading coefficients are shown in bold face.

$$\begin{bmatrix} 0 & \mathbf{3} & 1 & -1 \\ 0 & 0 & \mathbf{5} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form because the zero row is at the bottom, and the leading coefficient of the second row (in the third column), is to the right of the leading coefficient of the first row (in the second column).

A matrix is said to be in ***reduced row echelon form*** if furthermore all of the leading coefficients are equal to 1 (which can be achieved by using the elementary row operation of type 2), and in every column containing a leading coefficient, all of the other entries in that column are zero (which can be achieved by using elementary row operations of type 3).

Note: A matrix of numbers can have infinitely many Row-Echelon Forms [that the matrix is row-equivalent to], but it has only one unique RRE Form.

Properties of a Matrix in Row-Echelon Form

1) If there are any “all-0” rows, then they must be at the bottom of the matrix.

A side from these “all-0” rows,

2) Every row must have a “1” (called a “leading 1”) as its leftmost non-0 entry.
3) The “leading 1”s must “flow down and to the right.”

More precisely: The “leading 1” of a row must be in a column to the right of the “leading 1”s of all higher rows.

Properties of a Matrix in Reduced Row-Echelon (RRE) Form

1-3) It is in Row-Echelon form. (See Part F.)

4) Each “leading 1” has all “0”s elsewhere in its column.

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6.4.3. Gaussian Elimination

You may recall from your prior mathematics work that there are three possible conclusions we can make about the solution to a system of equations.

Case 1: There exists one unique solution.

Case 2: There is no solution.

Case 3: There is an infinite number of solutions.

<u>Case 1</u>: There is unique solution: Consider the example…

$$\begin{aligned} x + 3y &= 0 && R_1 \\ x + y + z &= 1 && R_2 \\ 3x - y - z &= 11 && R_3 \end{aligned}$$

To begin, we write the associated **augmented matrix**, which is written in the following form:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$$

To apply the method on a matrix, we use **elementary row operations** to modify the matrix. Our goal is to end up with the **identity matrix**, which is an $n\ x\ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, on the left side of the augmented matrix.

Our solution to the system of equations will be the resulting matrix on the right side of the augmented matrix. This is because the resulting augmented matrix would represent a system of equations in which each variable could be solved for (if a solution exists).

System of equations	Row operation	Augmented matrix
$x + 3y = 0$ $x + y + z = 1$ $3x - y - z = 11$		$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-y + z = 1$ $3x - y - z = 11$	$R_2 \to R_2 - R_1$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-y + z = 1$ $-10y - z = 11$	$R_3 \to R_3 - 3R_1$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-12y = 12$ $-10y - z = 11$	$R_2 \to R_2 + R_3$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -12 & 0 & 12 \\ 0 & -10 & -1 & 11 \end{array}\right]$

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$x + 3y = 0$ $y = -1$ $-10y - z = 11$	$R_2 \to -\frac{1}{12}R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x = 3$ $y = -1$ $-10y - z = 11$	$R_1 \to R_1 - 3R_2$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x = 3$ $y = -1$ $-z = 1$	$R_3 \to R_3 + 10R_2$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$x = 3$ $y = -1$ $z = -1$	$R_3 \to -R_3$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The solution to our system is therefore $\boldsymbol{x = 3, y = -1}$ and $\boldsymbol{z = -1}$.

<u>Case 2</u>: There is no solution.

Consider the system of equations:

$$2x - y + z = 1$$
$$3x + 2y - 4z = 4$$
$$-6x + 3y - 3z = 2$$

Augmented matrix: $\left[\begin{array}{ccc\|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{-6} & \mathbf{3} & \mathbf{-3} & \mathbf{2} \end{array}\right]$

Using row operation $R_3 + 3R_1 \to R_3$, we get $\left[\begin{array}{ccc\|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \end{array}\right]$.

We note that the third row in the augmented matrix is a false statement, so *there is no solution* to this system. WHEN DOES A SYSTEM HAVE NO SOLUTION? If we ever get a row of the form: 0 0 ⋯ 0 | non zero constant.

<u>Case 3</u>: There is an infinite number of solutions.

Consider the system of equations:

$$x - y + 2z = -3$$
$$4x + 4y - 2z = 1$$
$$-2x + 2y - 4z = 6$$

Augmented matrix: $\left[\begin{array}{ccc\|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{4} & \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ \mathbf{-2} & \mathbf{2} & \mathbf{-4} & \mathbf{6} \end{array}\right]$

Using row operations $R_2 - 4R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$, we get $\left[\begin{array}{ccc\|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{8} & \mathbf{-10} & \mathbf{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$.

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This represents a system that leaves us with 2 equations and 3 unknowns. So, we are unable to solve for one variable without expressing it in terms of another. This gives us an infinite number of solutions.

Note: If we get a row of all "0"s, such as: $0 \quad 0 \quad \cdots \quad 0 \quad | \quad 0$, all ordered pairs (x, y) satisfy it. However, a row of all "0"s does not automatically imply that the corresponding system has infinitely many solutions.

1. 1. 8. Solving system of linear equations: $Ax = B$.

Given:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ \ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ \ldots\ldots\ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad\quad \vdots \qquad\quad \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \ \ldots\ldots\ldots + a_{nn}x_n &= b_n \end{aligned}$$

In matrices:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Longrightarrow Ax = B.$$

There are many methods:

1. By inversion of coefficients matrix.

$$Ax = B \ \Longrightarrow x = A^{-1}B$$

Example: Solve the following system of equations:

$$x + 3y = 5$$
$$-x + 2y = 2$$

Solution: $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$|A| = 1 \times 2 - 3 \times (-1) = 5$

$c_{11} = (-1)^{1+1}|2| = 2\,, c_{12} = \ 1\,, c_{21} = -3\,, c_{22} = 1$

Hence $C = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}, C^t = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{C^t}{|A|} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{7}{5} \end{bmatrix}$$

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Example: Solve the following system of equations:

$$8x_1 + 2x_2 + 3x_3 = 30$$
$$x_1 - 9x_2 + 2x_3 = 1$$
$$2x_1 + 3x_2 + 6x_3 = 31$$

Solution: $\begin{bmatrix} 8 & 2 & 3 \\ 1 & -9 & 2 \\ 2 & 3 & 6 \end{bmatrix} . \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 30 \\ 1 \\ 31 \end{bmatrix}$

$c_{11} = (-1)^{1+1} . \begin{vmatrix} -9 & 2 \\ 3 & 6 \end{vmatrix} = -60$, $c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = -2$

$c_{13} = (-1)^4 \begin{vmatrix} 1 & -9 \\ 2 & 3 \end{vmatrix} = 21$, $c_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = -3$, etc.

$$C = \begin{bmatrix} -60 & -2 & 21 \\ -3 & 42 & -20 \\ 31 & -13 & -74 \end{bmatrix}, C^T = \begin{bmatrix} -60 & -3 & 31 \\ -2 & 42 & -13 \\ 21 & -20 & -74 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & 2 & 3 \\ 1 & -9 & 2 \\ 2 & 3 & 6 \end{vmatrix} \begin{matrix} 8 & 2 \\ 1 & -9 \\ 2 & 3 \end{matrix} = (-432) + (8) + (9) - (54) - (48) - (12) = -421$$

(+ + + and - - - mark the diagonal products)

$$A^{-1} = \frac{1}{-421} \begin{bmatrix} -60 & -3 & 31 \\ -2 & 42 & -13 \\ 21 & -20 & -74 \end{bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \frac{60}{421} & \frac{3}{421} & \frac{-31}{421} \\ \frac{2}{421} & \frac{-42}{421} & \frac{13}{421} \\ \frac{-21}{421} & \frac{20}{421} & \frac{74}{421} \end{bmatrix} . \begin{bmatrix} 30 \\ 1 \\ 31 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

2. Cramer's rule.

This method based on replacing the column(s) of variables by the column vector of results, then the value of each variable can be found by dividing the determinate of the new matrix to the determinate of the matrix of coefficients.

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}}{|A|}, x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix}}{|A|}, \dots, x_n = \frac{\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix}}{|A|}$$

Example: Solve the following system of equations:

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Solution: $\begin{bmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

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For $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$x = \frac{\begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}} = \frac{441}{63} = 7 \qquad y = \frac{\begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}} = \frac{-189}{63} = -3 \qquad z =$$

$$\frac{\begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}} = \frac{-252}{63} = -4$$

So the solution set of the system is $x = 7, y = -3, z = -4$.

3. Gaussian's method

The solution using this method has three stages. In the first stage the equations are written in matrix form. In the second stage the matrix equations are replaced by a system of equations having the same solution but which are in triangular form. In the final stage the new system is solved by back-substitution.

Example: Use Gaussian elimination to solve the system of linear equations:

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

Solution: In matrix, $\begin{bmatrix} 1 & 5 \\ -2 & -7 \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$3x_2 = 9 \implies x_2 = 3$

$x_1 + 5x_2 = 7 \implies x_1 = 7 - 5 \times 3 = -8$

Example: Use Gaussian elimination to solve the system of linear equations:

$$2x_1 + x_2 - x_3 + 2x_4 = 5$$
$$4x_1 + 5x_2 - 3x_3 + 6x_4 = 9$$
$$-2x_1 + 5x_2 - 2x_3 + 6x_4 = 4$$
$$4x_1 + 11x_2 - 4x_3 + 8x_4 = 2$$

Solution: In matrix, $\begin{bmatrix} 2 & 1 & -1 & 2 \\ 4 & 5 & -3 & 6 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \\ 2 \end{bmatrix}$

The operation $R_2 \longrightarrow R_2 - 2R_1, R_3 \longrightarrow R_3 + R_1, R_4 \longrightarrow R_4 - 2R_1$ gives

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$$\begin{bmatrix}2 & 1 & -1 & 2\\ 0 & 3 & -1 & 2\\ 0 & 6 & -3 & 8\\ 0 & 9 & -2 & 4\end{bmatrix}.\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}=\begin{bmatrix}5\\ -1\\ 9\\ -8\end{bmatrix}$$

The operation $R_3 \longrightarrow R_3 - 2R_2, R_4 \longrightarrow R_4 - 3R_2$gives

$$\begin{bmatrix}2 & 1 & -1 & 2\\ 0 & 3 & -1 & 2\\ 0 & 0 & -1 & 4\\ 0 & 0 & 1 & -2\end{bmatrix}.\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}=\begin{bmatrix}5\\ -1\\ 11\\ -5\end{bmatrix}$$

The operation $R_4 \longrightarrow R_4 + R_3$gives

$$\begin{bmatrix}2 & 1 & -1 & 2\\ 0 & 3 & -1 & 2\\ 0 & 0 & -1 & 4\\ 0 & 0 & 0 & 2\end{bmatrix}.\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}=\begin{bmatrix}5\\ -1\\ 11\\ 6\end{bmatrix}$$

Then $2x_4 = 6 \Longrightarrow\ x_4 = 3$

$-x_3 + 4x_4 = 11 \Longrightarrow -x_3 + 4(3) = 11 \Longrightarrow\ x_3 = 1$

$3x_2 - x_3 + 2x_4 = -1 \Longrightarrow 3x_2 - 1 + 2(3) = -1 \Longrightarrow x_2 = -2$

$2x_1 + x_2 - x_3 + 2x_4 = 5 \Longrightarrow 2x_1 + (-2) - 1 + 2(3) = 5 \Longrightarrow x_1 = 1$

ESSLCE Questions

1. Let A and B be 33 matrices. Which of the following is NOT true?

   A. If $B = -A, |A| = |B|$.

   B. If r is a non-zero constant, then $|rA| = r^3|A|$.

   C. $|A| = |A^T|$

   D. $|AB| = |A||B|$

2. Let $A = \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$ and $r \in \mathbb{R}$. Which one of the following is equal to rA?

   A. $\begin{pmatrix}a & b & c\\ r^3d & r^3e & r^3f\\ g & h & i\end{pmatrix}$ B. $\begin{pmatrix}ra & rb & rc\\ rd & re & rf\\ rg & rh & ri\end{pmatrix}$

   C. $\begin{pmatrix}ra & rb & rc\\ d & e & f\\ g & h & i\end{pmatrix}$ D. $\begin{pmatrix}ra & b & c\\ rd & e & f\\ rg & h & i\end{pmatrix}$

3. Let $A = \begin{pmatrix}-2 & 0 & 5\\ 3 & -1 & 4\end{pmatrix}$ and $B = \begin{pmatrix}4 & -3\\ 2 & -1\\ 1 & 0\end{pmatrix}$. Which of the following is equal to $A + B^T$?

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A. $\begin{pmatrix} -2 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix}$ B. $\begin{pmatrix} -2 & 2 & 6 \\ 0 & -2 & 4 \end{pmatrix}$ C. $2\begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \end{pmatrix}$ D. $2\begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \end{pmatrix}$

4. Which one of the following matrices is the augmented matrix of the system

$$\begin{aligned} -x + 2y - 4 &= 0 \\ 2x - 3y &= 5 \quad ? \\ -y + 2z &= 0 \end{aligned}$$

A. $\left(\begin{array}{ccc|c} -1 & 2 & -4 & 0 \\ 2 & -3 & 0 & 5 \\ 0 & -1 & 2 & -1 \end{array}\right)$ B. $\left(\begin{array}{ccc|c} -1 & 2 & 0 & -4 \\ 2 & -3 & 0 & 5 \\ 0 & 2 & -1 & 0 \end{array}\right)$

C. $\left(\begin{array}{ccc|c} -1 & 2 & 0 & 4 \\ 2 & -3 & 0 & 5 \\ 0 & -1 & 2 & 0 \end{array}\right)$ D. $\left(\begin{array}{ccc|c} -1 & 2 & -4 & 0 \\ 2 & -3 & 0 & 5 \\ 0 & 2 & -1 & 0 \end{array}\right)$

5. Given the matrix $A = \begin{pmatrix} 3 & -5 & 1 \\ 2 & 0 & 4 \\ -2 & -4 & 5 \end{pmatrix}$, let M_{ij} be the minor of the entry a_{ij} of A and C_{ij} be the cofactor of the entry a_{ij} of A. Which one of the following is correct?

A. $C_{32} = -10$ B. $C_{21} = -21$ C. $M_{31} = 20$ D. $M_{12} = 2$

6. For what value of a constant k is $\det A = -3$, where,

$$A = \begin{pmatrix} 2 & k & 0 \\ 0 & -1 & -3 \\ 1 & k & 5 \end{pmatrix}?$$

A. $-\frac{13}{9}$ B. $-\frac{7}{3}$ C. $\frac{7}{9}$ D. $\frac{7}{3}$

Chapter 7: Limit and Continuity (8 hrs)

7.1. Limit of functions

7.2. Continuity of functions

7.3. Intermediate value theorem

Chapter 8: Derivative and its Application (11 hrs)

8.1. Definition of derivative and its geometric interpretation

8.2. Rules of differentiation

8.3. Derivatives of special functions (exponential, logarithmic and trigonometric)

8.4. Extreme values of functions

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8.5. Minimization and Maximization problems

Chapter 9: Integration and its Application (11 hrs)

9.1. Integration as a reverse process of derivative

9.2. Techniques of integration (Substitution, by partial fractions and by parts)

9.3. Definite integral and Fundamental theorem of calculus

9.4. Area of a region under a curve and between two curves

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