

UNIT-7

The set of complex numbers

$$\sqrt{4} = 2$$

$$\sqrt{-4} = \square$$

$\mathbb{R}$: *real numbers*

**consist of all
positive and
negative**

integers, all

rational numbers

**Rational numbers
are of the form p/q ,
where p , q are
integers.**

Irrational numbers

$\pi, e, \sqrt{2}, \sqrt[3]{4}, \dots$

Complex number notation

Notation

n

Algebra form

$$z = a + bi$$

Trigonometric

$$z = r(\cos\theta + i\sin\theta)$$

Complex numbers

are defined as
numbers of the form :

$$a + b\sqrt{-1}$$

or

$$a + ib$$

$\sqrt{-1}$ is represented by i

$a + bi$ are real numbers.

A complex number
consists of 2 parts.

Real part and
imaginary part.

Notes:

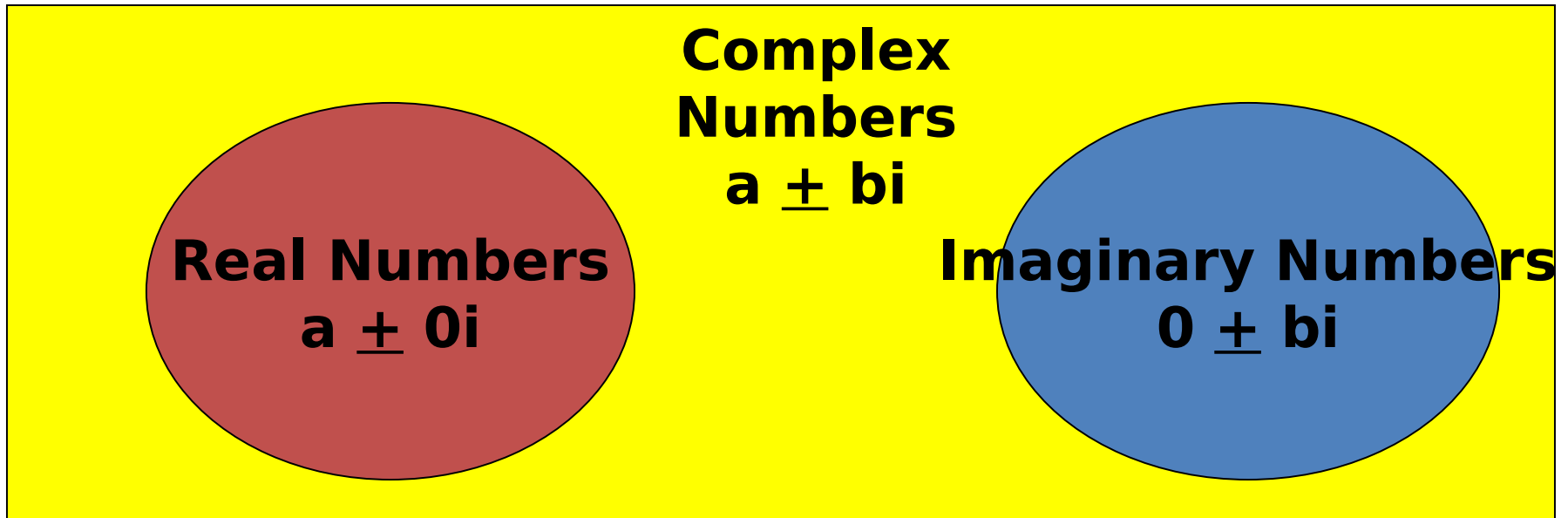
$$a + ib$$

- 1. When $b=0$, the complex number is Real**
- 2. When $a=0$, the complex number is imaginary**
- 3. The complex number is zero iff $a=b=0$**

VENN DIAGRAM

Representation

- All numbers belong to the Complex number field, **C**. The Real numbers, **R**, and the imaginary numbers, **i**, are **subsets** of **C** as illustrated below.



Conjugate complex number

Conjugate complex

The complex numbers

$$a + ib$$

and $a - ib$ are called

conjugate numbers.

$$z = a + ib$$

$$\bar{z} = a - ib$$

$\bar{z}$ is conjugate of z .

e.g. 1

ve the quadratic equation

$$x^2 + x + 1 = 0$$

Soln:

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm \sqrt{3}i}{2} \end{aligned}$$

e.g. 2

factorise

$$(x + y)^2 + z^2$$

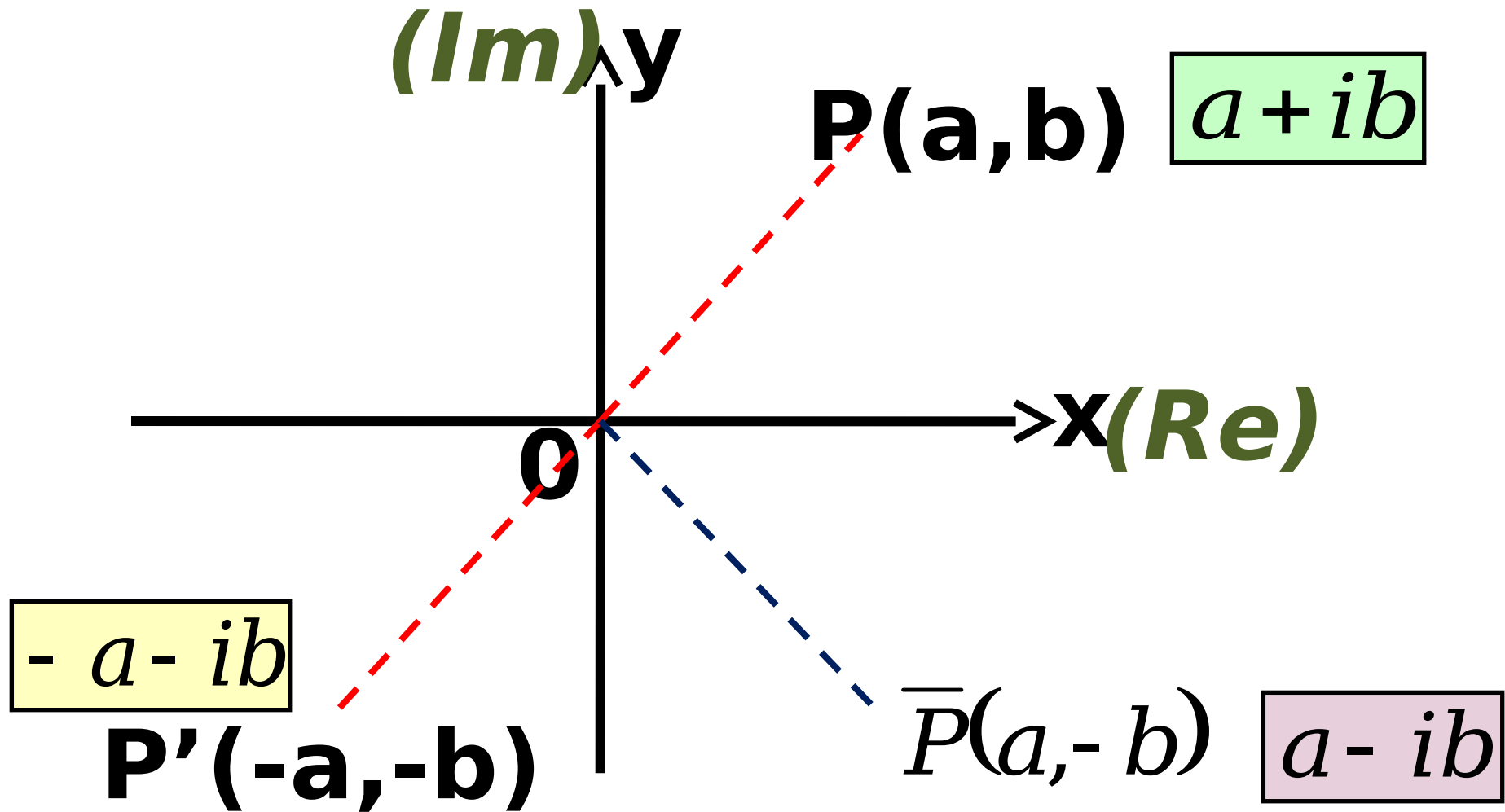
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Soln:

$$(x + y)^2 + z^2 = (x + y)^2 - (iz)^2$$

$$= (x + y + iz)(x + y - iz)$$

Representation of complex number in an Argand diagram

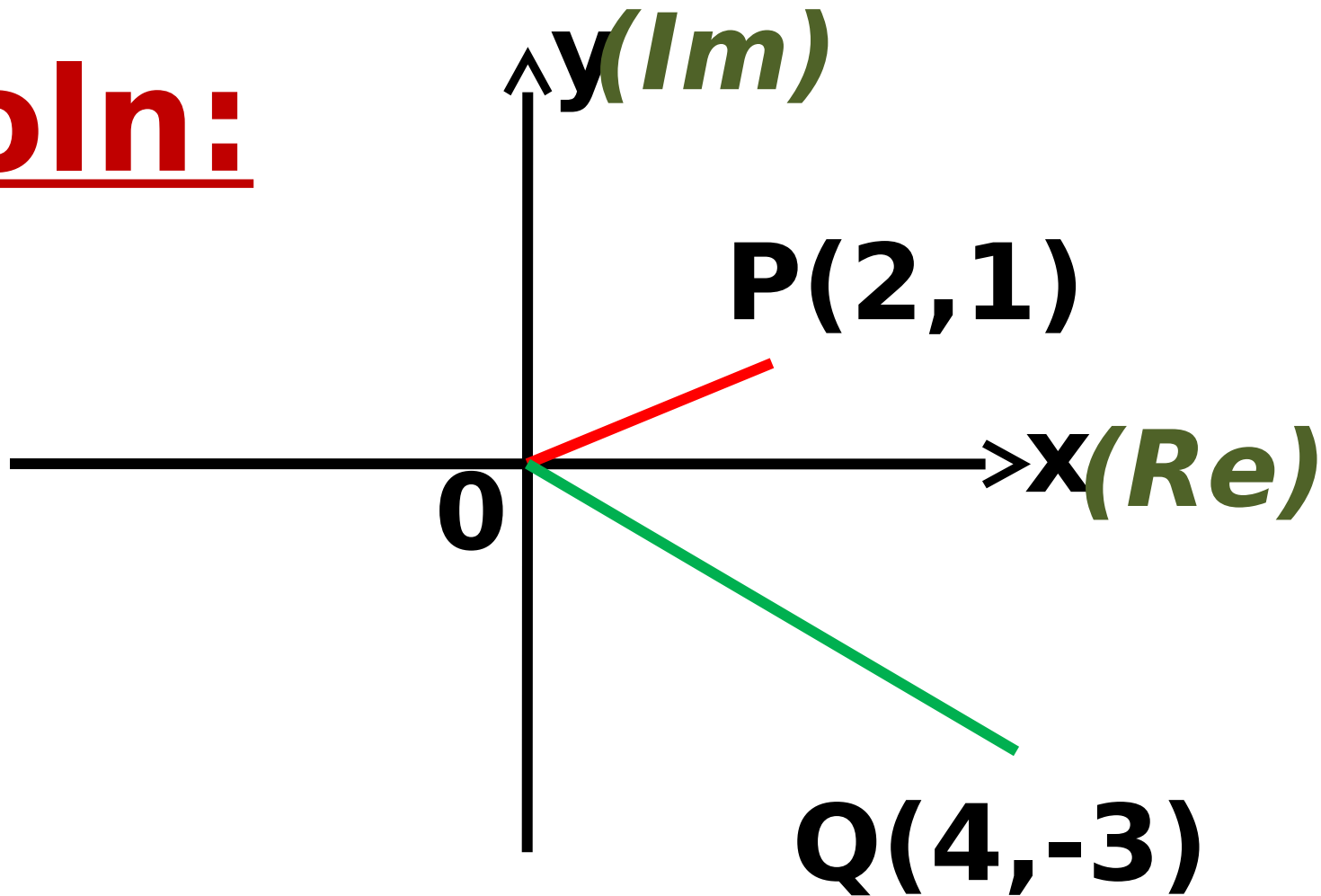


Argand diagram

e.g. 3

If P , Q represent the complex numbers $2+i$, $4-3i$ in the Argand diagram, what complex number is represented by the mid-point of PQ ?

Soln:



Mid-point of PQ is (3,-1)

$\therefore 3-i$ is the complex number.

$$i^2 = -1$$

$$i^6 = -1$$

$$i^{10} = -1$$

$$i^3 = -i$$

$$i^7 = -i$$

$$i^{11} = -i$$

$$i^4 = 1$$

$$i^8 = 1$$

$$i^{12} = 1$$

$$i^5 = i$$

$$i^9 = i$$

$$i^{13} = i$$

Example

∴ Simplify the following complex number

1. $i^{15} + 4i^{24} - 5i - 10$

2. $5i^{40} + i^{27} + 6i - 11$

Class activity

1. Solve each of the following quadratic function

a. $x^2 + 2x + 4$

b. $x^2 - 3x + 5$

2. Simplify each of the following complex number

a. $i^{102} - 3i^{49} + 2i - 5$

b. $4i^{97} + 5i^{55} + 5i - 10$

Exercise

1. Solve each of the following

a. $x^2 + 4$

b. $x^2 + 2x + 10$

c. $x^2 + 100$

2. Simplify each of the following and write as $Z = x + yi$

a. $i^{11} - 5i^{33} + i - 5$

b. $i^{4n+3} + (i^{2n+2})^2$

c. $i^{10} - 3(i^4 + 2i^{13}) - 5i - 3$

Addition of complex numbers

If $z_1 = a + ib ; z_2 = c + id$

then

$$z_1 + z_2 = (a + c) + i(b + d)$$

Subtraction n of complex numbers

If $z_1 = a + ib ; z_2 = c + id$

then

$$z_1 - z_2 = (a - c) + i(b - d)$$

Multiplicati on of complex numbers

If $z_1 = a + ib ; z_2 = c + id$

then

$$\begin{aligned} z_1 \bullet z_2 &= (a + ib)(c + id) \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Division of complex numbers

If $z_1 = a + ib ; z_2 = c + id$

then

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(a + ib)}{(c + id)} \\ &= \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(ac + bd) + i(bc - ad)}{(c^2 + d^2)}\end{aligned}$$

e.g. 6

Express

$$\frac{2-i}{3+i} \text{ in the form } a+ib$$

Soln:

$$\begin{aligned} \frac{2-i}{3+i} &= \frac{2-i}{3+i} \frac{3-i}{3-i} = \frac{6-1-5i}{9+1} \\ &= \frac{5-5i}{10} = \frac{1}{2}(1-i) \end{aligned}$$

Multiplicative inverse

For every $Z \neq 0$ in complex number there is multiplicative

invers of $\frac{1}{Z}$ such that

$$Z \cdot \frac{1}{Z} = 1 = \frac{1}{Z} \cdot Z$$

Example 1. $2 + 2i$ 2. $3 - 4i$ 3. $1 - \sqrt{3}i$

Equality of complex numbers

The complex numbers
 $a+ib$ and $c+id$ are
said to be equal if,
and only if,
 $a=c$ and $b=d$.

e.g. 4

Find the values of x
and y if $(x+2y)+i(x-y)=1+4i$.

Soln:

$$x+2y=1; \quad x-y=4$$

$$2y+y=1-4; \quad 3y=-3, \quad y=-1$$

$$x-(-1)=4, \quad x=4-1=3$$

Example. Solve for x and y for each of the following .

1. $(x+yi)(1+2i) = 4+7i$

2. $\frac{x+yi}{3-i} = \frac{1}{5} + \frac{2}{5}i$

3. $2x+yi = (3+5i)(2-4i)$

Complex conjugate

Definition : The complex conjugate of a complex number

$Z = x+yi$ denoted by $\bar{Z}$ is given by $\bar{Z} = x-yi$

Properties of complex conjugate

1. If $Z = X+ Yi$, then

a) $\bar{Z} = x - yi$

b) $Z + \bar{Z} = 2x$

c) $Z - \bar{Z} = 2yi$

d) $Z \cdot \bar{Z} = x^2 + y^2$

e.g. 5

If $z = 3 - i$, find the
values of z^2 and $z\bar{z}$

Soln:

(i) (ii)

$$(i) \quad z^2 = (3 - i)(3 - i) = 9 - 1 - 6i = 8 - 6i$$

$$(ii) \quad z\bar{z} = (3 - i)(3 + i) = 9 + 1 = 10$$

Definition :Modulus (the absolute value of complex number)

$Z = x+yi$, denoted by $|z|$ is denoted to be

$$|z| = \sqrt{x^2 + y^2}$$

NOTE :

$$\therefore z \cdot \bar{z} = |z|^2$$

$$\therefore \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\therefore |z| = |\bar{z}|$$

$$\therefore |z^n| = |z|^n$$

$$\therefore |z_1 \cdot z_2| = |z_1| \cdot |z_2| \quad \therefore |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

EXERCISE

1. Solve for x and y for each of the following

a) $(3 + 4i)^2 - 2(x - yi) = x + yi$

b) $(2 - 3i)(x + yi) = 3(x - 2i) + i - 1$

2.

The cube roots of unity

$\tilde{z}$ is a cube root of 1

$$\tilde{z}^3 = 1$$

or

$$\tilde{z}^3 - 1 = 0$$

$$z^3 - 1 = 0$$

$$(z - 1)(z^2 + z + 1) = 0$$

$$\therefore z = 1 ; z^2 + z + 1 = 0$$

$$\therefore z = \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore$ The cube roots of unity are

$$1, \frac{1}{2}(-1 + i\sqrt{3}), \frac{1}{2}(-1 - i\sqrt{3})$$

Notice that the complex roots have the property that one is the square

of the other,

$$\left[\frac{1}{2}(-1 + i\sqrt{3}) \right]^2 = \frac{1}{4}(1 - i2\sqrt{3} - 3) = \frac{1}{2}(-1 - i\sqrt{3})$$

$$\left[\frac{1}{2}(-1 - i\sqrt{3}) \right]^2 = \frac{1}{4}(1 + i2\sqrt{3} - 3) = \frac{1}{2}(-1 + i\sqrt{3})$$

let

$$\omega = \frac{-1 \pm \sqrt{3}i}{2}$$

**So the cube roots of
unity can be
expressed as**

$$1, \omega, \omega^2$$

If we take

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

then

$$\bar{\omega} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

or vice versa.

As $z = \omega$ is a solution

$$z^3 - 1 = 0 \quad \boxed{\therefore \omega^3 = 1}$$

As $z = \omega$ is a solution

$$z^2 + z + 1 = 0$$

$$\boxed{\therefore \omega^2 + \omega + 1 = 0}$$

(3)

$$\omega^2 + \omega = -1$$

(4)

$$\omega^{3n+1} + (\omega^2)^{3n+1} = -1$$

(5)

$$\omega^{3n} + (\omega^2)^{3n} = 2$$

... etc

e.g. 9

solve the equation $(z - 1)^3 = -1$

e.g. 10

If ω is a cube root of unity show that

$$\omega^4 + \omega^2 = -1$$

Soln:

$$\text{We have } \boxed{\omega^2 + \omega + 1 = 0}$$

$$\omega^2(\omega^2 + \omega + 1) = 0$$

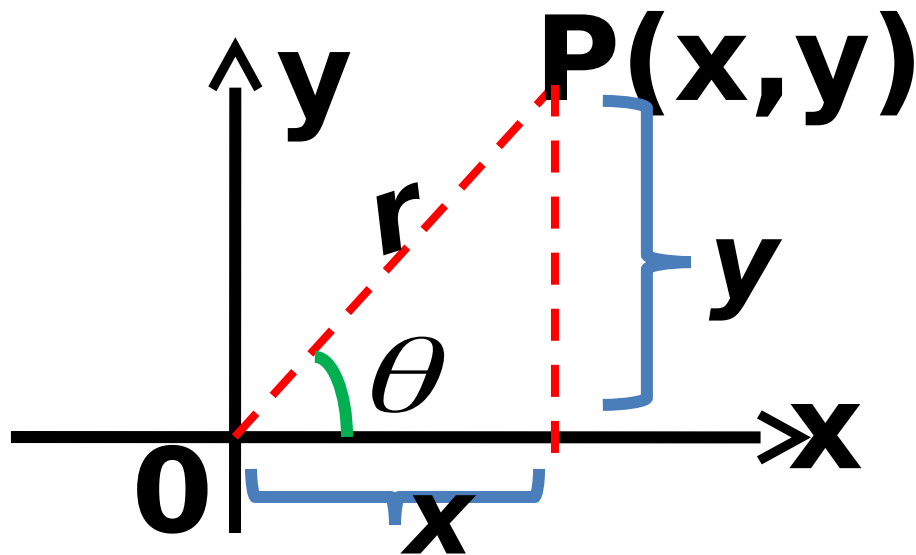
$$\omega^4 + \omega^3 + \omega^2 = 0$$

$$\omega^4 + \omega^2 = -\omega^3$$

$$\therefore \omega^4 + \omega^2 = -1$$

POLAR FORM or The (r, θ) form of a complex number

$$z = x + iy$$



$$-\pi < \theta \leq \pi$$

**Is called the
principal
value**

Argand diagram

r is called the **modulus** of **z**,
θ (in radians) is called the
argument of **z**.

om the Argand diagram,

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$z = x + iy$$

$$= r \cos \theta + i r \sin \theta$$

$$= r(\cos \theta + i \sin \theta)$$

This is called the (r, θ) or
***modulus-argument form of the
complex number***

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + i r \sin \theta \\ &= r (\cos \theta + i \sin \theta) \end{aligned}$$

This is called the (r, θ) or modulus-argument form of the complex number or modulus-amplitude form of the complex number

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg z = \tan^{-1} \left(\frac{y}{x} \right)$$

modulus

argument
(or

$$\theta = \arg z = \tan^{-1} \left(\frac{y}{x} \right)$$

OR

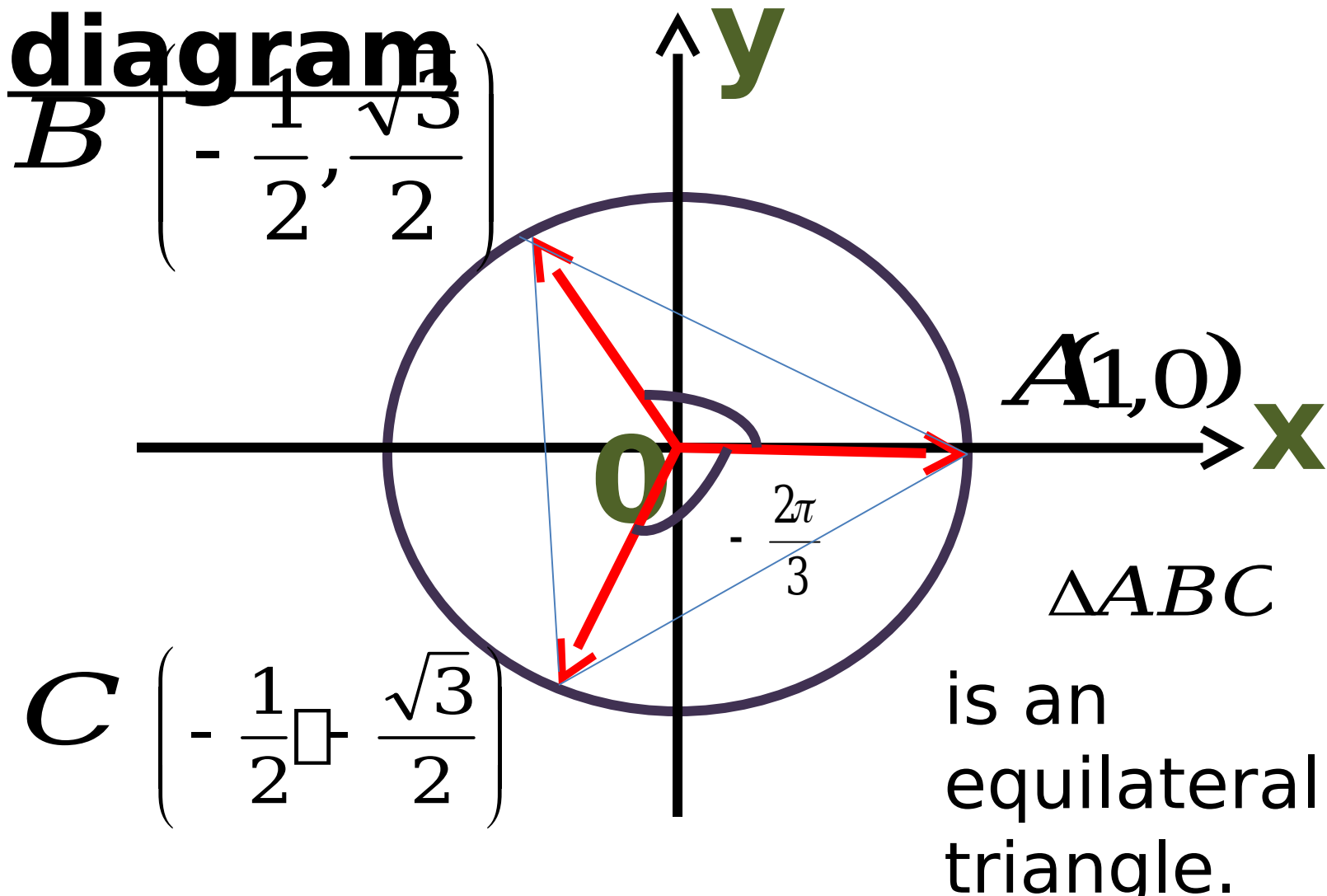
$$\theta = \arg z = \tan^{-1} \left(\frac{y}{x} \right)$$

Some important formulae :

$$|z|^2 = |z|^2 = z \cdot \bar{z}$$

Refer to Example 15 below

Representation of cube roots of unity in Argand diagram



Multiplication and division of two complex numbers (*in*

If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

e.g. 11

If $z = (\cos\theta + i\sin\theta)$

find $\frac{1}{z} = ?$

e.g. 12

If $|4i + \log_3 m| = 5$

, find

the value of m .

e.g. 13

$$\frac{1-z}{1+z} = i, \text{ then } |1-z| = ?$$

e.g. 14

$$\begin{vmatrix} z & i \\ 1 & i \end{vmatrix} = 1 + i, \text{ then } \tilde{z} = ?$$

Miscellaneous examples

e.g. 15

Evaluate

$$(4 - i^5)(6 + 2i^7)$$

$$[ans\ 22 - 14i]$$

e.g. 16

ven

$$z^2 = 8 + 6i$$

,

nd

$$z^3 - 16z - \frac{100}{z}$$

■

e.g. 17

$$|z| = 5, (3 + 4i)z$$

is an imaginary
number $\bar{z} = ?$

e.g. 18

$$\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \text{ then}$$

$$1 + \omega + \omega^2 + \dots + \omega^{13} = \square$$

e.g.

2.3 **If** $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
 $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Show

that

$$|z_1 z_2| = r_1 r_2$$