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Mathematical Economics

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Chapter One: Derivative in Use

In this unit we shall revise what you have studied in Quantitative method for economists I and II such as basic concepts from calculus which incorporates differential calculus, integral calculus, differential and difference equations. In addition to these, we will revise matrix algebra which includes matrix representation, determinants, inverse of a matrix, solutions of simultaneous linear equation systems, Leontief input - output model and linear programming. Therefore, you should read your module of the above two courses to understand this unit properly.

1.1 Basic Concepts from Calculus

1.1.1 Differential Calculus - Differentiation

Given a function y = f(x), where x and y are independent and dependent variables respectively, if Δx is an increment in the value of x and Δy is an increment in the value of y, then $\Delta y / \Delta x$ is referred to as the average rate of change or the rate of change of y with respect to the given change in x. This limiting value is also known as derivative which is the instantaneous rate of change of y resulted from a very small change in x represented by dy/dx.

This means

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}$$

Thus, differentiation is the process of finding the rate of change of the dependent variable (y) with respect to a given change in the independent variable (x). In other words, it is a process of determining the slope of the function y= f(x) at any point of x in the domain of the function.

$$\frac{dy}{dx} = f'(x)$$

Rules OF Differentiation

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Given the function y= f(x) and other functions such as g(x) and h(x) which are differentiable, the following are the rules of differentiation.

1.Constant function Rule

Given a function y = f(x) = k, where k is constant

$\frac{dy}{dx} = \frac{dk}{dx} = 0$, That is the derivative of the constant function is zero.

Example If $f(x) = 4 = 4x^0$, $\frac{dy}{dx} = 0\ (4x^{0-1}) = 0$

1.2 Linear function Rule

Given a function y = ax + b, where a and b are constants

$$\frac{dy}{dx} = a$$

Example If $f(x) = 3x + 5$, then $f'(x) = 3$

If f(x) = 5x, then $f'(x) = 5$

2.Power function rule

Given a function $f(x) = x^n$ where $x \neq 0$ and n is any real number

$f'(x) = n\, x^{n-1}$

Example If $f(x) = 2x^5$, $f'(x) = 5\ (2x^{5-1}) = 10x^4$

If $f(x) = 1/x^3 = x^{-3}$, $f'(x) = -3\ (x^{-3-1}) = -3x^{-4}$

$$f'(x) = -3/x^4$$

If $f(x) = \sqrt{x} = x^{½}$, $f'(x) = ½\, x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

3.Sum and Difference Rule

When $y = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$

Example If $g(x) = 2x+3$ and $h(x) = 5x^2 + 2x$, then

$f'(x) = g'(x) \pm h'(x)$

$= 2 \pm (10x + 2)$

$g'(x) + h'(x) = 10x + 4$

$g'(x) - h'(x) = 2 - (10x+2)$

$= -10x$

4.Product rule

This rule enables us to differentiate two functions which are multiplied together.

If $f'(x) = g'(x)\, h(x) + h'(x)\, g(x)$

a) Example

1. If $f(x) = (7.5 + 0.2x^2)\ (4 + 8x^{-1})$, What is $f'(x) = \frac{dy}{dx}$?

This function can be multiplied out and differentiated without using the product rule. But, let us first use the product rule and then compare the answers obtained by the two methods.

Let $g(x) = 7.5 + 0.2x^2$ and $h(x) = 4 + 8x^{-1}$

Therefore, $g'(x) = 0.4x$ and $h'(x) = -1\ (8x^{-2}) = -8x^{-2}$

Using the product rule

$f'(x) = 0.4x\ (4 + 8x^{-1}) + -8x^{-2}\ (7.5 + 0.2x^2)$

$= 1.6x + 3.2 - 60x^{-2} - 1.6$

$f'(x) = -60x^{-2} + 1.6x + 1.6$ ---------- (1)

If we multiply the original function, we get

$f(x) = (7.5 + 0.2x^2)(4+8x^{-1})$

$= 30+ 60x^{-1} + 0.8x^2 + 1.6x$

Therefore

$f'(x) = -60x^{-2} + 1.6x + 1.6$ ---------- (2)

These two derivatives, i.e. (1) and (2) are the same. But in some cases, it is not possible to multiply out the different components of a function and then we must use the product rule to differentiate.

Example

2.A firm faces a non - linear demand function $P = (650 - 0.25q)^{1.5}$

Determine the marginal revenue (MR) function of this firm. In this case first we should determine the total revenue function and then differentiate it with respect to the level of output using the product rule.

$TR = p \times q = q(650 - 0.25q)^{1.5}$

To differentiate TR with respect to q using the product rule,

Let $u = (650 - 0.25q)^{1.5}$ and $v=q$

Then $\frac{du}{dq} = 1.5(650 - 0.25q)^{0.5}$

$\frac{dv}{dq} = 1$

Therefore, $\frac{dTR}{dq} = 1(650 - 0.25q)^{1.5} + (q)(-0.375)(650 - 0.25q)^{0.5}$

$MR = (650 - 0.25q)^{0.5}(650 - 0.625q)$

5.Quotient rule

Dear colleague! This rule allows as differentiating two functions where one function is divided by the other function.

If $f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{(h(x))^2}$

Example

1.Determine $f'(x)$ if $f(x) = \frac{4x^2}{8+0.2x}$

Let $g(x) = 4x^2$ and $h(x) = 8+0.2x$, $g'(x) = 8x$ and $h'(x) = 0.2$

According to the quotient rule,

$f'(x) = \frac{8x(8+0.2x) - 0.2(4x^2)}{(8+0.2x)^2}$

$= \frac{64x + 1.6x^2 - 0.8x^2}{(8+0.2x)^2}$

$f'(x) = \frac{64x + 0.8x^2}{(8+0.2x)^2}$

2. If a monopoly faces a non- linear demand function

$$p = \frac{252}{(4+q)^{0.5}}$$

Derive the marginal revenue function (the rate of change of TR with respect to output).
As we know, TR = p x q,

$$TR = \frac{252q}{(4+q)^{0.5}}$$

If $u = 252q$ and $v = (4+q)^{0.5}$, then

$\frac{du}{dq} = 252$ And $\frac{dv}{dq} = 0.5\,(4+q)^{-0.5}$

Now we can find the marginal revenue function using the quotient rule as follows.

$$MR = \frac{dTR}{dq} = \frac{252(4+q)^{0.5} - 252q(0.5)(4+q)^{-0.5}}{[(4+q)^{0.5}]^2}$$

$$= \frac{252(4+q)^{0.5} - 252q\,(0.5)\,(4+q)^{-0.5}}{(4+q)}$$

$$= \frac{(4+q)252 - 126q}{(4+q)^{1.5}}$$

$$= \frac{1{,}008 + 252q - 126q}{(4+q)^{1.5}}$$

$$MR = \frac{1{,}008 + 126q}{(4+q)^{1.5}}$$

7. Chain Rule

The chain rule enables us to differentiate' functions within functions ', for instance, given a function $y = f(z)$ and $z = g(x)$, then $y = f[g(x)]$.

To differentiate y with respect to x, we use chain rule which states that

$$\frac{dy}{dx} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)$$

Example

1. Given a non- linear demand function, $p = (150 - 0.2q)^{0.5}$, find its slope.

Let $z = 150 - 0.2q$ as a result $p = z^{0.5}$

$$\frac{dz}{dq} = \frac{dp}{dz}\,\frac{dz}{dq} = (0.5z^{-0.5})(-0.2)$$

$$= -0.1(150-0.2q)^{-0.5}$$

$$= \frac{-0.1}{(150 - 0.2q)^{0.5}}$$

2. If the present value of a one birr due in 8 years time is given by the formula

$$PV = \frac{1}{(1+r)^8}$$

Where r is the given interest rate, find the rate of change of PV with respect to r.
Let (1+r) = z, then PV= $1/z^8 = z^{-8}$

This means $\frac{dPV}{dz} = -8z^{-9} = \frac{-8}{z^9}$ and $\frac{dz}{dr} = 1$

Thus, using the chain rule

$$\frac{dPV}{dr} = \frac{dPV}{dz}(\frac{dz}{dr}) = -8(1+r)^{-9}(1) = -8(1+r)^{-9}$$
$$\frac{dPV}{dr} = \frac{-8}{(1+r)^9}$$

As you know in your microeconomics course the marginal revenge productivity theory of demand for labor states that profit is maximized when $MRP_L = MC_L$.

Where MRP_L = marginal revenue product of labor
MC_L = Marginal cost of labor.

$$MRP_L = \frac{dTR}{dL} \quad \text{and} \quad MC_L = \frac{dTC_L}{dL}$$

Where TR is total revenue and TC_L is total cost of labor.

3. A firm is a monopoly seller of good q and faces the demand schedule P = 200 - 2q, where p is price in birr and the short run production function is given as $q = 4L^{0.5}$ Determine the marginal revenue productivity of labor (MRP_L) function.

$$MRP_L = \frac{dTR}{dq}, \frac{dq}{dL}$$

Thus, first we need TR, given P = 200 - 2q

$TR = (200 - 2q)\, q = 200q - 2q^2$

Therefore,

$\frac{dTR}{dq} = 200 - 4q$, $\frac{dq}{dL} = 2L^{-0.5}$

As a result, $MRP_L = (200 - 4q)(2L^{-0.5})$
$= [200-4(4L^{0.5})]\, 2L^{-0.5}$
$= (200- 16\, L^{0.5})\, 2\, L^{-0.5}$

$$MRP_L = \frac{400 - 32L^{0.5}}{L^{0.5}}$$

6.Implicit function Rule

Functions of the form y = f(x) express the dependent variable y explicitly in terms of the independent variable x and are called explicit functions. Where as functions of the form f (x, y) =0 do not express y in terms of x. These functions are referred to as implicit functions. If the implicit function f(x, y) = 0 exists, and the first order partial derivative of the function with respect to y is different from zero ($f_y \neq 0$), then the total differential

$$f_x dx + f_y dy = d0$$

$$f_x dx = -f_y dy$$

Rearranging these terms,

$$\frac{dy}{dx} = \frac{-f_x}{f_y}, \text{ where } \quad f_x = \frac{\partial(f(x,y)}{\partial x}$$

$$f_y = \frac{\partial f(x,y)}{\partial y}$$

Example

1. Given the function f (x, y) = $x^2 + y^2 - 9 = 0$, find $\frac{dy}{dx}$.

To do this we should partially differentiate the function with respect to x and with respect to y, and then determine $\frac{dy}{dx}$.

$$f_x = 2x \quad \text{and} \quad f_y = 2y$$

$$\text{Thus, } \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

2. Given the implicit function f (x, y) = $x^3 - 2x^2 y + 3xy^2 - 22 = 0$, find $\frac{dy}{dx}$.

$$f_x = 3x^2 - 4xy + 3y^2 \quad \text{and} \quad f_y = -2x^2 + 6xy$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{-(3x^2 - 4xy + 3y^2)}{-2xy^2 + 6xy}$$

9. Inverse function rule

Given the function y= f(x), it has an inverse if and only if each value of y corresponds with one and only one value of x. According to the inverse function rule, the derivative of the inverse function is the reciprocal of the derivative of the original function. The derivative of the given original function is $\frac{dy}{dx}$. Thus, the derivative of the inverse function x = f^{-1} (y) which is $\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$

Example If Q = $P^3 + 2P^2 + 7p$, find the derivative of P with respect to Q

$\frac{dQ}{dp}$ = 3 P^2 + 4P + 7, Thus, $\frac{dp}{dQ} = \frac{1}{3p2+4P+7}$

10. Logarithm and Exponential Function Rules

a. Logarithmic function rule

If we have two numbers 8 and 64 that can be related with each other by the equation $8^2 = 64$ the exponent 2 can be expressed as the logarithm of 64 to the base 8. This means.

$\text{Log}^{64}_8 = 2$. In general, if $y = a^x$, then $\log^y_a = x$ where a >0 and a $\neq$ 1

In logarithmic application two numbers are mainly selected as abase. These numbers are 10 and e. If the base is 10, then the logarithmic is referred to as ***common logarithm***. But if we use e as abase, then the logarithm is known as ***natural logarithm***.

Natural logarithmic function rule

Given the function $f(x) = \ell n^{9(x)} = \log_e{}^{9(x)}$ where $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$, $f'(x) = \frac{1}{g(x)}[g'(x)]$

Example If $y = \ell n^{6x+7}$, then $\frac{dy}{dx} = \frac{6}{6x+7}$

Logarithmic function of base a

Given the function $f(x) = \log_a^{g(x)}$, when f(x) and g (x) are differentiable, and g(x) $\neq$ 0,

$$\frac{dy}{dx} = f'(x) = \frac{1}{g(x)}\ [g'(x)]\frac{1}{\ell n^a}$$

Example

1. If $f(x) = x^3 \log_6{}^x$, then $f'(x) = 3x^2\ \ell og_6{}^x + x^3 \frac{1}{x}\frac{1}{\ln^6} = x^2(3\ \ell og_6{}^x + \frac{1}{\ln^6})$

$$f'(x) = 3x^2\ \ell og_6{}^x + \frac{x^2}{\ln^6}$$

2. If $f(x) =, \text{Log}_3^{(3x^2+2x)}$ then $f'(x) = \frac{6x+2}{(3x^2+2x)\ln^3}$

b) Exponential Function Rule

The function $y = a^x$ where a is constant and a > 0, a $\neq$ 1 is known as exponential function to base a. If the base is e, the function is referred to as natural exponential function.

$y = e^x$, where e= 2.71828

In economics, natural exponential functions are particularly useful for analyzing the growth rates of different variables.

Natural exponential function rule

If $f(x) = e^{g(x)}$ where f (x) and g(x) are differentiable, then $f'(x) = e^{g(x)}$ g' (x).

Given the function $y = e^x$, then $\frac{dy}{dx} = e^x$.

Example

1. If $f(x) = 5e^{1-x^2}$, then $f'(x) = -10x\, e^{1-x^2}$

2. If $f(x) = \frac{e^{5x}}{e^{5x}+1}$, then $f'(x) = \frac{5e^{5x}(e^{5x}+1) - 5e^{5x}(e^{5x}-1)}{(e^{5x}+1)^2}$

$$f'(x) = \frac{10e^{5x}}{(e^{5x}+1)^2}$$

Exponential Function Rule for Base a

Given the function $f(x) = a^{g(x)}$ Where g(x) and f(x) are differentiable, a>0, $a \neq 1$,

$$f'(x) = a^{g(x)} \ell n\, a$$

Example.1. $f(x) = a^{3x}$, then $f'(x) = a^{3x}\, 3\ell n^a$

$$f'(x) = 3a^{3x}\ \ell n^a$$

1.1.3 Differentials

Given the function $y = f(x)$ if we know the rate at which x changes, we can find the change in y, $\Delta y = (\frac{\Delta y}{\Delta x})\Delta x$. As $\Delta x \to 0, \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$

Denoting dy as small change in y and dx as small change in x,

$$dy = (\frac{dy}{dx})dx, where \frac{dy}{dx} = f'(x)$$

$\boxed{dy = f'(x)dx}$ It is the differential.

Example

Given $y = (5x^3 + 2x^2 + x)$ $dy = (15x^2 + 4x + 1)\ dx$

Suppose x changes from 2 to 2.01 (i.e. $dx = 0.01$), by what value does the dependent variable y changes?

$$dy = (15\, x^2 + 4x + 1)\ (0.01)$$
$$= [15\ (2^2) + 4\ (2) + 1]\ (0.01)$$
$$= (60 + 8 + 1)\ (0.01)$$
$$dy = (69)0.01 = 0.69$$

From the original function, the actual change in y is

$$\Delta y = f\ (2.01) - f\ (2)$$
$$f\ (2) = 5(2^3) + 2(2^2) + 2$$

$$= 5(8) + 2(4) + 2$$

$$f(2) = 40+8+2 = 50$$

$$f(2.01) = 5(2.01)^3 + 2(2.01)^2 + 2.01$$

$$= 40.606020+8.0802+2.01$$

$$=50.696220$$

$$\Delta y = f(2.01) - f(20 = 50.696220 - 50 = 0.696220$$

$$\Delta y - dy = 0.696220 - 0.69$$

$$= 0.006220$$

The value 0.006220 is the ***error term that is the error of calculation***.

As x changes from 2 to 3, that is $\Delta x = 1$, what is the change in y?

$$dy = [15(2)^2 + 4(2) + 1]\,1$$

$$= 60+8+1 = 69$$

$$\Delta y = f(3) - f(2)$$

But we have determined that

$$f(2) = 50$$

$$f(3) = 5(3^3) + 2(3^2) + 3$$

$$= 5(27) + 2(9) + 3$$

$$f(3) = 135 + 18 + 3 = 156$$

Thus, Δ y = 156 - 50 = 116. The error term is 116-69 = 47

Note.

By now we have realized that the smaller the change in x the smaller error term and the greater the change in x the larger will be the error term.

Total differentials

The concept of differential can be extended to a function of two or more independent variables. Given the utility function U = U (x_1, x_2)

Supposing U as continuous and differentiable

$Ux_1 = \frac{\partial u}{\partial x_1}$ which is marginal utility of x_1.

$Ux_2 = \frac{\partial u}{\partial x_2}$ which is marginal utility of x_2.

The change in total utility that results from small change in x_1 is given by $\frac{\partial u}{\partial x_1} dx_1$ and from small change in x_2 is $\frac{\partial u}{\partial x_2} dx_2$

Then the total change in utility resulted from small change in x and small change in y is represented by

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

Example Find the total differential, given

1. $z = 3x^2 + xy - 2y^3$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = (6x + y)\ dx + (x - 6y^{2})\ dy$$

2. $U = 2x_1^2 + 6x_1x_2 + 4x_2^2$

$$dU = (4x_1 + 6x_2)dx_1 + (6x_1 + 8x_2)dx_2$$

By now you have completed the first section of this unit. Therefore, try to do the following self - test questions in order to examine how you have understood this section.

1.2 Elasticity: Definition and as Logarithmic Derivative

Elasticity is a measure of a proportionate change in the dependent variable which results from a proportionate change in the independent variable. Thus elasticity of demand is a measure of the percentage change is quantity demanded due to a percentage change in the factors which determine demand.

If there is change in quantity demanded from Q to $(Q+ \Delta Q)$ resulting from a given change in price from P to $(P + \Delta P)$, the price elasticity of demand is

$$E_p = \frac{\text{Proportionate change in quantity demanded}}{\text{Proportionate change in price}}$$

$$= \frac{\Delta Q / Q}{\Delta p / p}$$

$$= \frac{\Delta Q}{Q} x \frac{p}{\Delta p} = \frac{\Delta Q}{\Delta p}\left(\frac{p}{Q}\right)$$

It gives us the average price elasticity of demand over the price range of P to $(P+ \Delta p)$. It is possible to use the method of derivative to get the price elasticity of demand at a particular point P, i.e., as Δp approaches to zero.

$$E_p = \lim_{\Delta p \to 0} \frac{p}{Q}\left(\frac{\Delta Q}{\Delta p}\right) = \frac{P}{Q}\lim_{\Delta p \to 0}\frac{\Delta Q}{\Delta P}$$

$$= \frac{P}{Q}\left(\frac{dQ}{dp}\right)$$

Thus price elasticity of demand $\boxed{E_p = \frac{dQ}{dp}\left(\frac{p}{Q}\right)}$

Example

1.Find the price elasticity of demand if the demand function for a product is $Q = P^2 - 6P + 36$, where Q is quantity and P is price of a product. Using the above formula

$$E_p = \frac{dQ}{dp}\left(\frac{p}{Q}\right), \text{ but } \frac{dQ}{dp} = 2p - 6$$

Therefore, $E_p = (2p - 6)\ \frac{p}{P^2 - 6P + 36}$

$$E_p = \frac{2p^2 - 6p}{p^2 - 6p + 36}$$

$$E_p = \frac{2p^2 - 6p}{p^2 - 6p + 36}$$

What is the elasticity of demand at P = 4?

$$E_p = \frac{2(4^2) - 6(4)}{4^2 - 6(4) + 36}$$

$$= \frac{32 - 24}{16 - 24 + 36} = \frac{8}{28} < 1$$

As $E_p < 1$, then demand is said to be price inelastic at P = 4.

2. Find the price elasticity of demand in terms of quantity sold.

$$P = \frac{10}{(Q+1)^2} = 10\ (Q+1)^{-2}$$

$$E_p = \frac{dQ}{dp}\left(\frac{P}{Q}\right), \text{ Therefore, } \frac{1}{Ep} = \frac{dp}{dQ}\left(\frac{Q}{P}\right)$$

Differentiating the demand function with respect to Q gives us

$$\frac{dp}{dQ} = -20\ (Q+1)^{-3} = \frac{-20}{(Q+1)^3}$$

$$\text{Thus, } \frac{1}{E_p} = \left(\frac{-20}{(Q+1)^3}\right)\left(\frac{Q}{10/(Q+1)^2}\right)$$

$$\frac{1}{E_p} = [\frac{-20}{(Q+1)^3}][\frac{(Q+1)^2(Q)}{10}]$$

$$\frac{1}{E_p} = \frac{-2Q}{(Q+1)}$$

This means $\boxed{E_p = \frac{Q+1}{-2Q}}$

3. If the demand function is given as Q= $\sqrt{10-p^2}$, find the price elasticity of demand at P = 2.

$$E_p = \frac{dQ}{dp}\frac{p}{Q} \qquad \frac{dQ}{dp} = \tfrac{1}{2}(10 - p^2)^{-\frac{1}{2}}(-2p)$$

$$= \frac{-2p}{2(10-P^2)^{\frac{1}{2}}}$$

$$\frac{dQ}{dp} = \frac{-p}{\sqrt{10-p^2}}$$

Thus, $E_p = (\frac{-p}{\sqrt{10-p^2}})(\frac{p}{\sqrt{10-p^2}})$

$E_p = \frac{-p^2}{10-p^2}$. Thus, at P = 2, $E_p = \frac{-2^2}{10-2^2} = \frac{-4}{6} = \frac{-2}{3}$

This means, as the price of the product increases by one percent then the quantity demanded for the product decreases by $\frac{2}{3}$ percent at p = 2.

Elasticity as Logarithmic Derivative

In a function **y =** f (t), when the dependent variable y is a function of time t, then the instantaneous rate of growth is determined as

$$\text{Rate of growth} = \frac{dy/dt}{y} = \frac{marginalfunction}{totalfunction}$$

This ratio exactly represents the derivative of $\ell n\ f$ (t). This means, the instantaneous rate of growth of the function is the derivative of natural logarithm of the function with respect to time. That is

$$\text{Rate of growth} = \frac{d \ln f(t)}{dt}$$

By now, given the function y = f(x), let us observe the result if we carry out differentiation of $\ln y$ with respect to $(\ln x)$. To start it, suppose u = $\ln^y$ and v = $\ln^x$. Then the chain of relationship relating u to y and v to x is that

u = ℓn^y , y = f(x), v = $\ell n^x, x = e^v$

Finding the derivative of $\ln y$ with respect to $\ln x$ gives us

$$\frac{d(\ell n^y)}{d(\ell n^x)} = \frac{du}{dv} = (\frac{du}{dy})(\frac{dy}{dx})(\frac{dx}{dv})$$

Using the method of substitution

$$\frac{d(\ell n^y)}{d(\ell n^x)} = [\frac{d(\ell n^y)}{dy}][\frac{dy}{dx}][\frac{d(e^v)}{dv}]$$

$$= (\frac{1}{y})(\frac{dy}{dx})e^v$$

$$= (\frac{1}{y})(\frac{dy}{dx})x \quad \text{as } e^v = x$$

$$\boxed{\frac{d(\ell n^y)}{d(\ell n^x)} = \frac{dy}{dx}(\frac{x}{y})}$$

As you remember this expression is the point elasticity of the function. Hence, one can generalize that for a function y = f(x), the point elasticity of y with respect to x is

$$E_{xy} = \frac{d(\ell n^y)}{d(\ell n^x)}$$

Example

1. Find the point elasticity of demand given Q = K/ P^n where K and n are positive constants

$$Q = K\, p^{-n}$$

$$\ln^Q = \ln^K - n\ln^P$$

$$Ep = \frac{d(\ln^Q)}{d(\ln p)} = -n$$

$$|E_p| = \frac{d\ln Q}{d\ln P} = n$$

2. Find the point elasticity of demand provided that x = a p^b where a and b are positive constants, and x and p are quantity demanded and price of a product respectively.

$$\ln^x = \ln^a + b\ln^P$$

Thus, $E_p = \frac{d(\ln x)}{d(\ln p)} = b$

Relationship between Marginal Revenue, Average Revenue and Elasticity of Demand

Given the total revenue function TR= PQ where P is the price at which the product is sold and Q is the quantity sold. Then marginal revenue that is the rate of change of total revenue with respect output is

$$\text{MR} = \frac{d(PxQ)}{dQ}$$

Using the product rule of differentiation

$$MR = p\frac{dQ}{dQ} + Q\frac{dp}{dQ}$$

$$\text{MR} = \text{P} + Q\frac{dp}{dQ}$$

We know form the above analysis that $\text{E}_\text{p} = \frac{dQ}{dp}\frac{p}{Q}$ which implies that $\frac{1}{Ep} = \frac{dp}{dQ}(\frac{Q}{p})$ multiplying both sides by p gives us $\frac{p}{Ep} = (\frac{dp}{dQ})(\frac{Q}{p})p$

$$\frac{p}{Ep} = Q\frac{dp}{dQ}$$

Substituting it in the above marginal revenue function , we get

$$MR = P + \frac{P}{Ep}$$

However, we also know that p is equal to average revenue (AR), thus

$$MR = AR + \frac{AR}{Ep}$$

$$\text{MR} = \text{AR}\ (1+ \frac{1}{Ep}) = \text{AR}\ (1-\frac{1}{|E_p|})$$

$$\boxed{MR = AR(1-\frac{1}{|E_p|})}$$

From this equation we realize that

When demand is unitary elastic, i.e. $|E_p| = 1$, MR = 0. This means, total revenue is maximized.

When demand in elastic, i.e., $|E_p| > 1$, MR > 0 which implies that total revenue is increasing. However, as demand is inelastic, i.e., $|E_p| < 1$, MR < 0 which implies that total revenue in decreasing.

1.3 Higher Order Derivatives

We know that the first order derivative $f'(x)$ of a function y = f (x) is a function of x. As a result we can determine the rate of change of $f'(x)$ with respect to x if $f'(x)$ is differentiable. The result of this differentiation is referred to as ***second order derivative*** of the function y = $f(x)$. It is represented by $f''(x)$ where the double prime shows that the primitive function has been differentiated with respect to x twice. The expression (x) next to the double prime indicates that the second order derivative in a function of x.

Alternatively this second order derivative can be represented by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(dy \big/ dx\right)$$

The above expression indicates that the second derivative is a function of x. Thus, it is possible to differentiate this function with respect to x in order to get the ***third order derivative*** of the function $f'''(x)$ or $f^3(x)$ which in turn can be a source of fourth order derivative and so on provided that the differentiable condition is satisfied.

Symbolically, these successive higher order derivatives are

f^3 (x), f^4(x), f^5(x) ----------, f^n (x).

Or

$$\frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \quad \frac{d^5y}{dx^5}, \text{ -----}, \frac{d^n(x)}{dx^n}$$

Example

1. Find the first through the fourth derivative of the following function.

$$f(x) = 5x^3 + 2x^2 + 3x + 4$$

These derivatives are $f'(x) = 15x^{2+} 4x + 3$

$f''(x) = 30x + 4$

$f'''(x) = 30$

$f^4(x) = 0$

We have seen that each successive derivative gives us a simpler expression than it's precedence until we get the fourth derivative. However, it is not always true. The fourth order derivative is equal to zero does not imply that it does not exist.

2. Find the first four derivatives of the function given as

$$f(x) = \frac{2x}{1-x}, \quad x \neq 1$$

$$f'(x) = \frac{2(1-x)-(-1)(2x)}{(1-x)^2}$$, using the quotient rule

$$f'(x) = \frac{2(1-x)-(-1)(2x)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$f''(x) = 2(-2)\,(1\text{-}x)^{-3}\,(-1)$

$f''(x) = -4(-1)\,(1\text{-}x)^{-3}$ using the chain rule

$= 4\,(1\text{-}x)^{-3}$

$f^3(x) = (-3)(4)(1-x)^{-4}(-1)$

$f^3(x) = 12\,(1\text{-}x)^{-4}$

$f^4(x) = (-4)(12)(1-x)^{-5}(-1)$

$f^4(x) = 48(1-x)^{-5}$

1.3.1 Convexity and Concavity of a Function

Given the function y = f(x), it is clear that the derivative of the function $f'(x)$ indicates the rate of change of the primitive function with respect to x. Similarly, $f''(x)$ shows the rate of change of $f'(x)$ with respect to x. Observing the sign of the first derivative of the original function at any point of x, one can determine whether a function is increasing or decreasing at that point. However, the sign of the second derivative indicates whether the function is concave or convex at that particular point. In the original function is y = f(x),

- $f'(a) > 0$ implies that the function is increasing at x = a and
- $f'(a) < 0$ implies that the function is decreasing at x = a
- $f'(a) = 0$, implies that the function is at its optimum point at x = a

Similarly

- If $f''(a) > 0$, then the function is convex at x = a
- If $f''(a) < 0$, then the function is concave at x = a
- If $f''(a) = 0$, x = a is the point at which the curvature of the function is changed

In general, a positive first derivative coupled with a positive second derivative at x = a shows that the function increases at an increasing rate at x = a. A positive first derivative coupled with a negative second derivative at x = a, indicates that the original function is increasing at a decreasing rate at this point. A negative first derivative with a positive second derivative shows that the function is decreasing at an increasing rate where as a negative first derivative with negative second derivative at x= a, implies that the function decreases at a decreasing rate at x = a.

Example

If $f(x) = x^3 - 12x^2 + 36x + 8$,

a) Is it increasing or decreasing at x = 3?
b) Is it convex or concave at x = 3?

Solution

First we should find the first and second derivative of this function at x = 3 to answer these questions.

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24$

Therefore, $f'(3) = 3(9) - 24(3) + 36$

$f'(3) = 27 - 72+36 = 63- 72 = -9 < 0$

$f''(x) = 6(3) - 24 = 18 -24 = -6 < 0$

This implies that the function is decreasing at decreasing rate at x= 3 and it is concave at this point.

1.3.2 Linear Approximation

As we know there are complicated functions in economics. In order to avoid this complexity, we sometimes try to find a simpler function which approximates the original function. Given the function y = f(x), the equation of a straight line which passes through point (x_1, y_1) and having a slope b is represented as

$$b = \frac{y - y_1}{x - x_1}$$

$$b(x - x_1) = (y - y_1)$$

$$y = b(x - x_1) + y_1$$

This means, the equation of the tangent line which passes through the graph of the function y = $f(x)$ at x = a provided that f(x) is continuous and smooth at x = a, is expressed as

$$y - f(a) = f'(a)(x - a)$$

$$\boxed{y = f'(a)(x - a) + f(a)}$$

When we approximate the graph of $f(x)$ by its tangent line at x = a. The resulting approximation is referred to as ***linear approximation.***

Example

1. Find the linear approximation of the function $f(x) = \sqrt[4]{x}$, about x = 1.

First we should determine the value of $f(x)$ and $f'(x)$ at x = 1 to answer this question. Thus,

$$f'(x) = \frac{1}{4}x^{(\frac{1}{4}-1)} = \frac{1}{4}x^{(-\frac{3}{4})}$$

$$f'(x) = \frac{1}{4}(1)^{-\frac{3}{4}} = \frac{1}{4}$$

$$f(1) = \sqrt[4]{1} = 1$$

Therefore, the linear approximation of the function is

$$f(x) \approx \frac{1}{4}(x-1) + 1$$

If x = 1.02, $f(x) \approx \frac{1}{4}(1.02 - 1) + 1$

$$\approx \frac{1}{4}(0.02) + 1$$

$$f(x) \approx \frac{1}{200} + 1 = 1.005$$

However, the actual value is f (1.02) = $\sqrt[4]{1.02}$ =1.00496 which is very close to the approximate value of the function, i.e., 1.005.

2. Determine the linear approximation of the function

$$f(x) = (1 + \frac{3}{2}x + \frac{1}{2}x^{2})^{\frac{1}{2}} \quad \text{at } x = 0$$

$$f'(x) = \frac{1}{2}(1 + \frac{3}{2}x + \frac{1}{2}x^2)^{-½}(\frac{3}{2} + x)$$

This means $f'(0) = \frac{1}{2}(1 + \frac{3}{2}(0) + \frac{1}{2}(0)]^{-½}(\frac{3}{2} + 0)$

$$f'(0) = \frac{1}{2}(1)^{-\frac{1}{2}}(\frac{3}{2}) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$$

And $f(0) = [1 + \frac{3}{2}(0) + \frac{1}{2}(0)]^{\frac{1}{2}}$

$$f(0) = (1)^{\frac{1}{2}} = 1$$

Therefore, at x = 0 the linear approximation of the function is

$$f(x) \approx f'(0)(x - 0) + f(0)$$

$$f(x) \approx \frac{3}{4}(x - 0) + 1$$

$$f(x) \approx \frac{3}{4}x + 1$$

2. Identify an approximate value of $(1.001)^{50}$

Solution

In order to approximate it, initially we should convert the equation in to $f(x) = x^{50}$, when x is close to 1.Then we should evaluate the original function and the first derivative of this function at x=1.

$$f'(x) = 50x^{49}$$

$$f'(1) = 50(1)^{49} = 50$$

And $f(1) = (1)^{50} = 1$

Thus, $f(x)) \approx f(1) + f'(1)\ (x-1)$

$\approx 1 + 50\ (x-1) = 1+50x-50 = 50x - 49$

$$f(1.001) \approx (1.001)^{50} = 1 + 50(1.001 - 1)$$

$$f(1.001) \approx 1 + 50(0.001) = 1 + 0.05 = 1.05$$

However, the actual value is $(1.001)^{50} = 1.0512$ which is very close to the approximate value i.e., 1.05.

1.3. 3 Polynomial Approximation

Approximation using linear functions is not as such accurate. Therefore, it is necessary to use quadrate approximation or higher order approximation so as to minimize this problem.

a) Quadratic Approximation

Given a function y = f(x), how can it be approximated by the second degree polynomial of the form $P(x) \approx A + B(x-a) + C(x-a)^2$ as x is close to a?

In this function we have seen three unknown coefficients such as A, B and C. Therefore, we ought to put the following three conditions in order to determine these three unknowns.

At x = a, it is assumed that

- $f(a) = P(a)$
- $f'(a) = P'(a)$
- $f''(a) = P''(a)$

Given the polynomial function $P(x) = A + B(x-a) + C(x-a)^2$

$$P'(x) = B + 2C\ (x - a)$$

$$P''(x) = 2C$$

When $x = a$, $P(a) = A + B\ (a-a) + C\ (a-a)^2$

$$= A = f(a)$$

$$P'(a) = B + 2C\ (a-a)$$

$$P'(a) = B = f'(a)$$

$$P''(a) = 2C = f''(a),\ C = \frac{1}{2} f''(a)$$

Substituting these values in the given quadratic function gives us the ***quadratic approximation*** of the function which is represented by

$f(x) \approx f(a) + f'(a)\ (x-a) + \frac{1}{2} f''(a)\ (x-a)^2$ as x is close to a.

Example

1. Determine the ***quadratic approximation*** to a function f(x) = $\sqrt[4]{x}$ about x= 1. In this case, first we should find the value of $f(x)$, $f'(x)$ and $f''(x)$ at x = 1 . Thus,

$$f(x) = x^{1/4},\ f(1) = 1^{1/4} = 1$$

$$f'(x) = \frac{1}{4}x^{-3/4},\ f'(1) = \frac{1}{4}(1)^{-3/4} = \frac{1}{4}$$

$$f''(x) = \frac{1}{4}\left(\frac{-3}{4}\right)x^{-7/4},\ f''(1) = \frac{-3}{16}$$

Thus,

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

$$f(x) \approx 1 + \frac{1}{4}(x-1) + \frac{1}{2}\left(\frac{-3}{16}\right)(x-1)^2$$

$$f(x) \approx 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$$

Compare the actual value and the approximate value of this function when $x = 1.02$.

2. Find the quadrate approximation to a function $f(x) = (5x+3)^{-2}$ about x = 0.

Solution

$$f(0) = \frac{1}{9}$$

$$f'(x) = -2(5x+3)^{-3}(5) = -10(5x+3)^{-3}$$

$$f'(0) = -10[5(0)+3]^{-3} = -10(3)^{-3} = \frac{-10}{27}$$

$$f''(x) = 30(5x+3)^{-4}(5) = 150(5x+3)^{-4}$$

$$f''(0) = 150(5(0)+3)^{-4} = 150(3)^{-4} = \frac{150}{81}$$

Therefore, the quadratic approximation of the function is

$$f(x) \approx f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2$$

$$f(x) \approx \frac{1}{9} + \left(-\frac{10}{27}\right)(x-0) + \frac{1}{2}\left(\frac{150}{81}\right)(x-0)^2$$

$$f(x) \approx \frac{1}{9} - \frac{10}{27}x + \frac{75}{81}x^2$$

B. Higher Order Approximation

As it is indicated above quadratic approximation is more accurate than linear approximation of a function. However quadratic approximation is not as such accurate. Therefore, functions with higher order derivatives can be better approximated near one point by using ***polynomial function of higher degree***.

Given a function y = $f(x)$, how can we approximate it around x = a by an n^{th} degree polynomial of the form

$$P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + ---+ A_n x-a)^n$$

As we have seen this function has (n+1) unknown coefficients. Therefore, we should impose (n+1) conditions on the polynomial to determine the value of the unknown coefficients. These are, at x = a, it is assumed that

$$\begin{aligned} f(a) &= P(a) \\ f'(a) &= P'(a) \\ f''(a) &= P''(a) \\ &\vdots \\ f^n(a) &= P^n(a) \end{aligned}$$

Given the above polynomial function,

$$P'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + 4A_4(x-a)^3 + -----+nA_n(x-a)^{n-1}$$
$$P''(x) = 2A_2 + 6A_3(x-a) + 12A_4(x-a)^2 + 20A_5(x-a)^3 + ---+ n(n-1)A_n(x-a)^{n-2}$$
$$P^3(x) = 6A_3 + 24A_4(x-a) + 60A_5(x-a)^2 + ---+ n(n-1)(n-2)A_n(x-a)^{n-3}$$

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$$P^n(x) = n(n-1)(n-2)(n-3) - (3)(2)(1)A_n$$

Therefore, when x is close to a,

$$P(a) = 0!A_0 = f(a) \Rightarrow A_0 = \frac{f(a)}{0!}$$

$$P'(a) = 1!A_1 = f'(a) \Rightarrow A_1 = \frac{f'(a)}{1!}$$

$$P''(a) = 2!A_2 = f''(a) \Rightarrow A_2 = \frac{f''(a)}{2!}$$

$$P^3(a) = 3!A_3 = f^3(a) \Rightarrow A_3 = \frac{f^3(a)}{3!}$$

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$$P^n(a) = n!A_n = f^n(a) \Rightarrow A_n = \frac{f^n(a)}{n!}$$

Substituting these values of the unknown coefficients in the polynomial function gives us the polynomial approximation of the function as x is close to a, which is represented by

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \text{---} + \frac{f^n(a)}{n!}(x-a)^n$$

Find the ***fourth order approximation*** of the function

$f(x) = \sqrt{1+x}$, about x = 0

In this case $f(x) \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^3(0)}{3!}(x-0)^3 + \frac{f^4(0)}{4!}(x-0)^4$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \quad f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = 1/2$$

$$f''(x) = \frac{-1}{4}(1+x)^{-3/2}, \quad f''(0) = -1/4$$

$$f^3(x) = \frac{3}{8}(1+x)^{-5/2}, \qquad f^3(0) = \frac{3}{8}$$

$$f^4(x) = \frac{-15}{16}(1+x)^{-7/2}, \quad f^4(0) = \frac{-15}{16}$$

Therefore,

$$\sqrt{1+x} \approx 1 + \frac{1}{2}(x-0) + (\frac{-1}{4})(\frac{1}{2})(x-0)^2 + \frac{3}{8}(\frac{1}{3!})(x-0)^3 + (-\frac{15}{16})(\frac{1}{4!})(x-0)^4$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{384}x^4$$

1.3.4 Estimation of Functions (Maclaurin and Taylor Series)

A. Maclaurin Series (Expansion of a function around x = 0)

It is an expansion of a function y = f(x) that is differentiable, around x = 0. In other words, it represents the polynomial approximation of the function around x= 0. Therefore, given the function

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + ... + A_nx^n$$

Repeatedly differentiating this function gives us

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + 6A_6x^5 + ... + nA_nx^{n-1}$$

$$f''(x) = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + 30A_6x^4 + ... + n(n-1)A_nx^{n-2}$$

$$f^3(x) = 6A_3 + 24A_4x + 60A_5x^2 + 120A_6x^3 + ... + n(n-1)(n-2)A_nx^{n-3}$$

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$$f^n(x) = (n)(n-1)(n-2)(n-3).....(3)(2)(1)A_n$$

(Where n is Positive integer)

We have understood that the number of constant terms is decreased by one until the n^{th} derivative is reached. We can determine the value of each rate of change at various values of x. In this case, let us evaluate these derivatives at x = 0. When we determine the value of the derivatives at x = 0, then all terms containing x will be eliminated.

$f'(0) = A_1$, $f''(0) = 2A_2$, $f^3(0) = 3(2)\,A_3$, $f^4(0) = (4)\,(3)\,(2)\,A_4$ ---------, $f^n(0) = n\,(n-1)\,(n-2)\,(n-3)\,...\,(3)\,(2)\,(1)\,A_n$

Using the symbol (n!) which can be read as n – factorial, where n! = n (n-1) (n-2) (n-3)---- (3) (2) (1).

$$A_1 = \frac{f'(0)}{1!},\ A_2 = \frac{f''(0)}{2!},\ A_3 = \frac{f^3(0)}{3!},\ \text{----------}\ A_n = \frac{f^n(0)}{n!}$$

Now the primitive function f(x) can be expressed as a new polynomial function by substituting these terms in it as follows

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + ... + \frac{f^n(0)}{n!}x^n$$

In this case, the unknown coefficients of the primitive function are replaced by the derivatives evaluated at x = 0. Zero factorial represents one (i.e., 0! =1). This power series representation is known as ***Maclaurin series*** of the primitive function $f(x)$ around x = 0.

Example

1. Determine the Maclaurin series of the function

$$f(x) = 5x^3 + 2x^2 + 3x + 1$$

This function has the following derivatives

$$f'(x) = 15x^2 + 4x + 3$$
$$f''(x) = 30x + 4$$
$$f^3(x) = 30$$

So that
$$f(0) = 1$$
$$f'(0) = 15(0)^2 + 4(0) + 3 = 3$$
$$f''(0) = 30(0) + 4 = 4$$
$$f^3(0) = 30$$

Therefore, the Maclaurin series of this function is

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^3(0)}{3!}(x-0)^3$$
$$f(x) = 1 + 3(x-0) + \frac{4}{2}(x-0)^2 + \frac{30}{6}(x-0)^3$$
$$f(x) = 1 + 3x + 2x^2 + 5x^3$$

This result shows that the Maclaurin series really represents the given function f (x)**.**

2. Find the Maclaurin series of the function

$$f(x) = 3x^2 + 2x + 4$$

The derivatives from this function are

$$f'(x) = 6x + 2$$
$$f''(x) = 6$$

So that $f(0) = 4,\ f'(0) = 2,\ f''(0) = 6$

Thus, the Maclaurin series of the function is

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$$
$$f(x) = 4 + 2(x-0) + \frac{6}{2}(x-0)^2 = 4 + 2x + 3x^2$$

B. Taylor's Series

It deals about the expansion of the primitive function f (x) provided that the function is continuous and smooth, around any point $x = x_0$. Thus given the function y = f(x), we will explain it using a specific quadratic function for the sack of simplicity.

Let us consider the expansion of the function

$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + ... + A_nx^n$ at some point x = x_0

First let us write this function in terms of the power of (x-x_0)

$$f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + A_3(x-x_0)^3 + A_4(x-x_0)^4 + ... + A_n(x-x_0)^n$$

This function has the following successive derivatives

$$f'(x) = A_1 + 2A_2(x-x_0) + 3A_3(x-x_0)^2 + 4A_4(x-x_0)^3 + 5A_5(x-x_0)^4 + ... + nA_n(x-x_0)^{n-1}$$
$$f''(x) = 2(1)A_2 + 3(2)A_3(x-x_0) + 4(3)A_4(x-x_0)^2 + .5(4)A_5(x-x_0)^3 + ... + n(n-1)A_n(x-x_0)^{n-2}$$
$$f^3(x) = 3(2)(1)A_3 + 4(3)(2)A_4(x-x_0) + 5(4)(3)A_5(x-x_0)^2 + ... + n(n-1)(n-2)A_n(x-x_0)^{n-3}$$

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$$f^n(x) = (n)(n-1)(n-2)(n-3)..... .....(3)(2)(1)A_n$$

So that

$$f(x_0) = 0!A_0 \Rightarrow A_0 = \frac{f(x_0)}{0!}$$
$$f'(x_0) = 1!A_1 \Rightarrow A_1 = \frac{f'(x_0)}{1!}$$
$$f''(x_0) = 2!A_2 \Rightarrow A_2 = \frac{f''(x_0)}{2!}$$

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$$f^n(x_0) = n!A_n \Rightarrow A_n = \frac{f^n(x_0)}{n!}$$

Therefore

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + ... + \frac{f^n(x_0)}{n!}(x-x_0)^n$$

This power series representation is referred to as ***Taylor's series***.

If we consider 3 as a point of expansion, then

$$f(x) = f(3) + f'(3)(x-3) + \frac{f^2(3)}{2!}(x-3)^2 + ... + \frac{f^n(3)}{n!}(x-3)^n$$

Example

Find the Taylor's series of the function $f(x) = 3x^2 + 2x + 5$, around $x = x_0$

The successive derivatives of this function are

$$f'(x) = 6x + 2$$
$$f''(x) = 6$$
$$f(x_0) = 3x_0^2 + 2x_0 + 5$$

So that $f'(x_0) = 6x_0 + 2$

$$f''(x_0) = 6$$

Therefore,

$$f(x) = (3x_0^2 + 2x_0 + 5) + (6x_0 + 2)(x - x_0) + \frac{6}{2}(x - x_0)^2$$
$$f(x) = (3x_0^2 + 2x_0 + 5) + (6x_0x - 6x_0^2 + 2x - 2x_0) + 3(x^2 - 2xx_0 + x_0^2)$$
$$f(x) = 3x_0^2 + 2x_0 + 5 + 6x_0x - 6x_0^2 + 2x - 2x_0 + 3x^2 - 6x_0x + 3x_0^2$$
$$f(x) = 6x_0^2 + 5 - 6x_0^2 + 2x + 3x^2$$
$$f(x) = 5 + 2x + 3x^2 = 3x^2 + 2x + 5$$

This realizes that the ***Taylor's series*** correctly represents the given function.

1.4 Intermediate Value Theorem and Newton Method

A. Intermediate Value Theorem

Suppose that there is a function f(x) that is continuous for all values of x in the close interval [a, b], and assume that the values of the function at x = a and at x = b are not equal to each other and each value has opposite sign, i.e., f (a) and f (b) have different signs. The intermediate value theorem states that there is at least one value of x say c in the interval [a, b] such that $f(c) = 0$.

Example

1. Prove that the equation $x^6 + 3x^2 - 2x - 1 = 0$ has at least one solution in between 0 and 1, i.e. [0, 1]

You ought to follow the steps below to solve this Problem.

- First put the equation in a polynomial form of $f(x) = x^6 + 3x^2 - 2x - 1$
- Second check whether the function is continuous in between 0 and 1 or not.
- Third determine the value of the function at x = 0 and x = 1.

As we know the function f(x) is referred to as continuous in the interval (a, b) iff

i) f (x) is continuous at x = a [i.e., lim $^{f(x)}$ = f(a)]

$x \to a^+$

ii) f(x) is continuous at x = b [i.e. $\lim^{f(x)}$ = f (b)]

$x \to b^-$

iii) f(x is continuous at x = c[i.e., a < c < b]

As a given function in our example is a ***polynomial function***, it is continuous in between 0 and 1.

$f(0) = 0^6 + 3(0)^2 - 2(0) - 1 = -1$

$f(1) = 1^6 + 3(1)^2 - 2(1) - 1 = 1+3-2-1 = 4-3 = 1$

We have seen that $f(0)$ and $f(1)$ have different signs. Therefore, according to the intermediate value theorem there is at least one number that is c ε [0, 1] so that $f(c) = 0$. In other words, the above equation has at least one solution in between 0 and 1.

2. Prove that the equation $\sqrt{2x^2 + 3x} = 2$ has at least one solution in between 0 and 1, i.e. [0, 1].

First make it in to ***polynomial function*** by squaring both sides of the equation as

$(\sqrt{2x^2 + 3x})^2 = 2^2$

$2x^2 + 3x = 4$

$2x^2 + 3x - 4 = 0$, then $f(x) = 2x^2 + 3x - 4$

As the function is polynomial it is continuous in between 0 and 1.

$f(0) = 2(0)^2 + 3(0) - 4 = -4$, $f(1) = 2+3 - 4 = 1$

The value of this function has different signs at x= 0 and x = 1 which implies that there is at least one solution for the equation in between 0 and 1.

3. Show that the equation $\sqrt{x^2 + 1} = 3x$ has at least one solution in between 0 and 1.

Solution

Initially change the equation in to polynomial function by squaring both sides as

$(\sqrt{x^2 + 1})^2 = (3x)^2$

$x^2 + 1 = 9x^2$

$-8x^2 + 1 = 0$

Therefore, $f(x) = -8x^2 + 1$

As the function is a ***polynomial function,*** it is continuous in between 0 and 1.

$f(0) = 1$ and $f(1) = -8(1)^2 + 1 = -7$

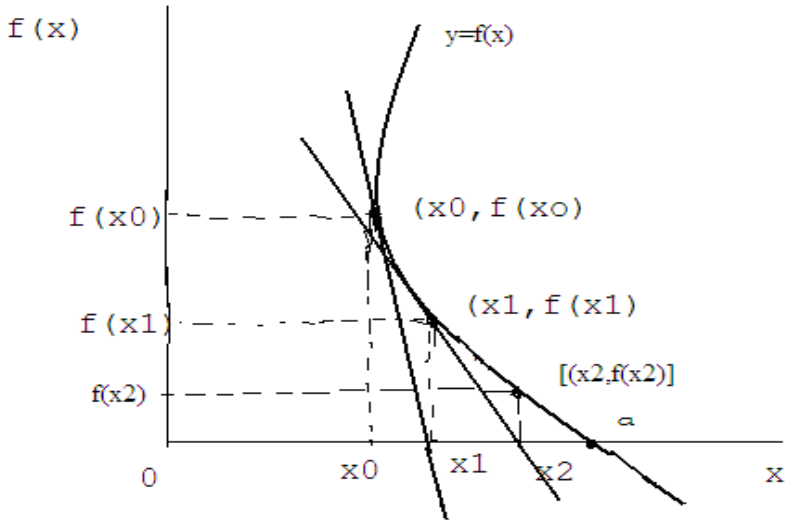
We have observed that $f(0)$ and $f(1)$ have different signs. As a result, the equation has at least one solution in between 0 and 1.

B. Newton's Method

The ***intermediate value theorem*** does not provide information concerning the location at which the solution resides. Rather it indicates the range in which the solution can be found. ***Newton's method***, however, leads to a better approximate solution of the equation under consideration.

Given the function y = f(x) and assume f(x) = 0 at x = *a*, estimate *a*.

To carry out this activity, let us begin with an initial estimate x_0 of *a*. Constructing a tangent line which passes through $[x_0, f(x_0)]$ we get x_0 near to *a* as shown in the following figure.



Undertaking the same procedure we get the estimate x_1 by drawing a tangent line at $[x_1, f(x_1)]$. Continuing this process enables us to estimate the value of a. However, we can also easily find the formula for x_n derived by the ***Newton's method***.

What is the equation of the tangent line which passes through $[x_0, f(x_0)]$ and with slope m? The slope of this line can be calculated as

$$\text{m} = \frac{y - y_0}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\frac{f(x) - f(x_0)}{(x - x_0)} = m \quad \text{Using cross multiplication}$$

$f(x) - f(x_0) = \text{m}(x - x_0)$, m represents the slope of the function at $x = x_0$

$$f(x) - f(x_0) = f'(x_0)(x - x_0)$$

When $x = x_1$, $f(x) = 0$ (for the tangent line passing through $[x_0, f(x_0)]$

Therefore, $0 - f(x_0) = f'(x_0)(x_1 - x_0)$

Rearranging this equation gives as

$$\frac{-f(x_0)}{f'(x_0)} = (x_1 - x_0)$$

This implies that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

In general the points generated by the ***Newton method*** are obtained by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

1. Find the approximate values of the equation $f(x) = x^6 + 3x^2 - 2x - 1$ in the interval [0, 1] using the ***Newton's method once***.

Solution

In this case n = 0, and $x_0 = 1$

We know that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x_0) = f(1) = 1+3 -2- 1 = 4-3 =1$

$f'(x_0) = 6x^5 + 6x - 2 \Rightarrow f'(x_0) = f'(1) = 6+6 -2 = 10$

Therefore, $x_1 = 1 - \frac{1}{10} = 0.9$

If we check for $f(x_1) = f(0.9)$

$= (0.9)^6 + 3(0.9)^2 - 2(0.9) - 1$

$= 0.53441 + 2.43 - 1.8 - 1$

$f(0.9) = 0.161441$

Applying Newton's method twice

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.9 - \frac{f(0.9)}{f'(0.9)}$$

$$= 0.9 - \frac{0.161441}{6.94294}$$

$x_2 = 0.877$

Thus, $f(0.877) = (0.877)^6 + 3(0.877)^2 - 2(0.877) - 1$

$= 0.0084$ which is more approximate to zero.

2. Determine the approximate value of the equation $\sqrt{x^2+1} = 3x$ in the interval [0, 1] using Newton's method once.

Solution

Squaring both sides of the equation gives us $f(x) = -8x^2 + 1$. In this case also n = 0, $x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = (1)^2(-8) + 1 = -7$

$f'(x_0) = f'(1) = -16(1) = -16$

Therefore

$$x_1 = 1 - \frac{(-7)}{(-16)}$$

$= 1 - 0.4375$

$x_1 = 0.5625 \approx 0.56$

If we check the original function at this value of x, we get $f(0.56) = -8(0.56)^2 + 1$

$= -2.5088 + 1$

$f(0.56) = -1.5088$

Applying the Newton method twice gives us

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.56 - \frac{f(0.56)}{f'(0.56)}$$

$$x_2 = 0.56 - \frac{(-1.5088)}{(-8.96)} = 0.392$$

Then $f(0.392) = -8(0.392)^2 + 1 = -0.2293$ which is more approximate to the solution than the earlier value. This shows that we can determine the most approximate value of the solution by applying this procedure continuously.

Exercise

Do the following questions based on the information in this section.

1.Given the function $f(x) = 2x^4 + 3x^3 + 2x^2 + 3x$, find the first five derivatives of the function with respect to x. ----------

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2. If the demand function of a monopolist is given by P = 60 - 4Q where P is price in Birr and Q is quantity. Determine whether the total revenue function of the firm is concave or convex at Q = 7. ----------
---

3. If the total cost function of a product is given by
C (Q) = Q^3 - 315 Q^2 + 27,000 Q + 20,000, where Q is quantity produced. Determine whether the average cost function (AC) is convex or concave when the firm produces two units of output. What happens to the AC at this point? ----------
---
---

4. Find the linear approximation of the function f(x) = $3x^3 + 2x^2$ + x, about x = 1 ----------
---

5. Determine the quadratic approximation of the function given in question number four ----------
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---

6. Given the function f(x) = $2x^2$ + 3x + 5, find its Maclaurin series. ----------
---

7. Find the Taylor's series of the function given in question 6 above around $x = x_0$ ----------
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8. Given the equation $x^5 + 2x^4 + 3x^2 + x - 3 = 0$

- Show that the equation has at least one solution in [0, 1]
- Determine the approximate value of the equation in the interval [0, 1] using the Newton's method once.

2.3 Multivariate Calculus

What are multivariate functions? ----------
Some functions involve one dependent and more than one independent variable. These functions are said to be ***multivariate functions***. For example, the production function Q = f (K, L) has one dependent and two independent variables, labor L and capital K.

2.3.1 Partial Derivatives

What is partial differentiation? ----------

Partial differentiation is a technique of deriving the rate of change of the function with respect to change in one of the independent variable when all other variables in the function are held constant. Thus, if the production function Q = f (K, L) is differentiated with respect to L, with K being constant, we obtain the rate of change of total product with respect to labor, i.e., ***Marginal product of labor*** (MP_L).

Partial differentiation follows the normal rules of differentiation apart from the fact that all variables other than the one the function is being differentiated with respect to are assumed to be constant.

Given the function, $y = f(x_1, x_2)$ the usual notation for the partial derivative of the function with respect to x_1 is

$\frac{\partial y}{\partial x_1} = f_{x1}$ and that of with respect to x_2 is $\frac{\partial y}{\partial x_2} = f_{x_2}$

Example

1. If the function is given by $y = 4x_1^2x_2^3 + 6x_1^2x_2 + x_1^4 + 7x_2^3$, find f_{x1} and f_{x_2}.

$f_{x_1} = 8x_1x_2^3 + 12x_1x_2 + 4x_1^3$ (x_2 is assumed to be constant)

$f_{x_2} = 12x_1^2x_2^2 + 6x_1^2 + 21x_2^2$ (x_1 is assumed to be constant)

Economic Applications

Partial differentiation is frequently applied in economic analysis as the economy is a complex system to understand. Economists usually observe the effect of changes in one variable supposing all other variables constant.

A. Elasticity

As we know quantity demanded for a product depends on various factors such as price of the good , (p) consumer's income (y) , price of related goods, and population (n) .

$Q_{dd} = f(p, y, p_c, p_s, n)$ where p_c is price of complement.

p_s is price of substitutes.

And we also know that price elasticity of demand E_p is

$$E_p = \frac{dQ}{dp}(\frac{p}{Q})$$

But with the existence of these determinants in the demand function

$$E_p = \frac{\partial Q}{\partial P}(\frac{P}{Q})$$

Example

Given the demand function as $Q = 35 - 0.4P + 0.15y - 0.25P_c + 0.12P_s + 0.003n$, what is the price elasticity of demand when price is 24?

$E_p = \frac{\partial Q}{\partial P}(\frac{P}{Q})$, you are required to get $\frac{\partial Q}{\partial p}$.

$\frac{\partial Q}{\partial P}$ = - 0.4 (all factors other than the price of the product are constant)

$$E_p = \frac{(-0.4)(24)}{35 - 0.4(24) + 0.15y - 0.25P_c + 0.12P_s + 0.003n}$$

Thus, $$E_p = \frac{(-9.6)}{25.4 + 0.15y - 0.25P_c + 0.12P_s + 0.003n}$$

The exact price elasticity of demand can be calculated if the values of the remaining variables are given.

B. Utility Functions

According to the cardinality approach the consumer who consumes only two products A and B will have the utility function

$$U = U(A, B)$$

We can determine the marginal utility of each good from this function using the technique of ***partial differentiation***.

Example

If the utility function of the consumer is given by $U = 2A^{0.4}B^{0.4}$, where A and B represent the quantity of the two goods consumed; find marginal utility of each good.

$$MU_A = 2(0.4)\, A^{-0.6}\, B^{0.4}$$

$$= \frac{0.8B^{0.4}}{A^{0.6}}$$

$$MU_B = 0.4(2)\, A^{0.4}\, B^{-0.6}$$

$$= \frac{0.8A^{0.4}}{B^{0.6}}$$

As the consumption of good A increases, MU_A decrease and as the consumption of B increases MU_B decreases. As a result the law of diminishing marginal utility holds for both goods.

C) Production Function

When the firm produces a product using two or more variable inputs, then we can determine the rate of change of total product with respect the change in each input using the method of partial differentiation.

Example

If the production function for a product is given by $Q = 20K^{0.5}\, L^{0.5}$ where K is capital and L is labor, Find the marginal Product of each input.

Marginal product of labor $(MP_L) = 0.5\,(20)\, K^{0.5}\, L^{-0.5}$

$$MP_L = \frac{10K^{0.5}}{L^{0.5}}$$

Marginal Product of capital $(MP_{K)} = 0.5\,(20)\, K^{-0.5}\, L^{0.5}$

$$MP_K = \frac{10L^{0.5}}{K^{0.5}}$$

Is the law of diminishing marginal productivity is satisfied for each input? Why? ----------

---

D) Revenue and Cost Functions

It is clear that in some cases firms may produce different products. If they use common production facilities, the costs of each product will be related. Therefore, the marginal cost function of the individual product can be determined by the method of partial differentiation.

Example

A firm produces two products Q_1 and Q_2 and its total cost function is $C(Q_1, Q_2) = 20(Q_1 + Q_2)$, what are the relevant marginal cost functions?

$$C = 20Q_1 + 20Q_2$$

Thus, Marginal cost of $Q_1 = \frac{\partial C}{\partial Q_1}$

$$MC_1 = 20$$

Marginal cost of Q_2 $(MC2) = 20$

In other cases, the firm may produce different products which are ***complementary*** with each other. This means, the price of one good will affect the quantity demanded of the other good. Thus, the marginal revenue of one good is the partial derivative of total revenue with respect to that good assuming the price of the other good constant.

Example

Suppose a firm produces two products A and B which are complements. The relevant demand functions are

$$Q_A = 850 - 12.5P_A - 3.8P_B$$
$$Q_B = 936 - 4.8P_A + 24P_B$$

What are the marginal revenue functions of the two goods? ----------

Marginal revenue is usually expressed in terms of quantity. Thus, we should initially rewrite the demand functions in the form of price in terms of quantity.

For good A,

$$Q_A = 850 - 12.5P_A - 3.8P_B$$
$$12.5P_A = 850 - 3.8P_B - Q_A$$
$$P_A = \frac{850 - 3.8P_B - Q_A}{12.5}$$
$$P_A = 68 - 0.304P_B - 0.08Q_A$$

However, $TR_A = P_A Q_A = 68Q_A - 0.304P_B Q_A - 0.08Q_A^2$

Therefore, the marginal revenue of product A is

$$MR_A = \frac{\partial TR_A}{\partial Q_A}$$
$$MR_A = 68 - 0.304P_B - 0.16Q_A$$

For good B,

$$Q_B = 936 - 4.8P_A - 24P_B$$
$$24P_B = 936 - 4.8P_A - Q_B$$
$$P_B = \frac{936 - 4.8P_A - Q_B}{24}$$
$$P_B = 39 - 0.2P_A - \frac{Q_B}{24}$$

However, $TR_B = P_B Q_B = 39\, Q_B - 0.2\, P_A\, Q_B - \frac{Q_B^2}{24}$

$$MR_B = \frac{\partial TR_B}{\partial Q_B}$$

$$\boxed{MR_B = 39 - 0.2P_A - \frac{1}{12}Q_B}$$

The marginal revenue functions of the above two goods show that when the demand for the two goods are interrelated, the marginal revenue function for one good will depend on the price level of the other good.

E) Keynesian Multiplier

If we incorporate the government sector and foreign trade, the basic Keynesian macroeconomic model will be

$$Y = C + I + G + (X - M)$$

and the functional relationship with consumption is

$$C = c\, Y_d$$

Where c is the marginal Propensity to consume, and imports

$$M = m\, Y_d$$

Where m is marginal propensity to import,

$$Y_d = (1-t)\, Y$$

is the disposable income c, m and t are parameters and Investment I, Government expenditure G and export X are exogenously determined.

Therefore $Y = c\, Y_d + I + G + [X - (m\, Y_d)]$

$$Y = c\,(1-t)\,Y + I + G + X - m\,(1-t)\,Y$$
$$Y - c\,(1-t)\,Y + m\,(1-t)\,Y = I + G + X$$
$$Y\,[1 - c\,(1-t) + m\,(1-t)] = I + G + X$$

$$\boxed{Y = \frac{I + G + X}{1 - c(1-t) + m(1-t)} = \frac{I + G + X}{1 - (c-m)(1-t)}}$$

Dear colleague! If we are interested to determine the government expenditure multiplier, we should partially different income with respect to G, holding Investment L and export X constant as shown below.

$$\frac{\partial Y}{\partial G} = \frac{1}{1-(c-m)(1-t)}$$

However, $\frac{\partial Y}{\partial G} = \frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial I}$

Example

In a Keynesian Macroeconomic model of an economy

$Y = C + I + G + (X-M)$	$I = 820$
$C = 0.75\ Y_d$	$G = 960$
$M = 0.25\ Y_d$	$Y_d = (1-t)\ Y$
$X = 650$	$t = 0.3$

Determine the equilibrium level of national income. Using the export multiplier find out what will happen to the balance of trade if export exogenously increases by 100.

Initially, we should solve the relationship between income (Y) with consumption and Import.

$C = 0.75\ Y_d = 0.75\ (1-t)\ Y = 0.75(1-0.3)\ Y$

$= 0.75\ (0.7)\ Y$

$C = 0.525\ Y$

$M = 0.25 Y_d = 0.25\ (1-0.3)\ Y$

$= 0.25\ (0.7)\ Y$

$M = 0.175\ Y$

Therefore, the equilibrium level of national income is

$Y = 0.525Y + 820 + 960 + 650 - 0.175Y$

$Y - 0.525Y + 0.175Y = 2430$

$0.65Y = 2430$

$Y = \frac{2430}{0.65} = 3{,}738.46$

At this equilibrium level of income, import will be

$M = 0.25\ (0.7)\ (3{,}738.46)$

$M = 654.23$

Thus, the balance of payment will be

$X - M = 650 - 654.23$

$= -4.23$ which is negative trade balance.

However, as export increases by 100, then the equilibrium level of income will be increased by change in export times export multiplier, i.e.,

$$\Delta Y = \Delta X \frac{\partial Y}{\partial X} = 100\ [\frac{1}{1-(c-m)(1-t)}]$$

$$=100[\frac{1}{1-(0.75-0.25)(1-0.3)}]$$

$$=100\ [\frac{1}{1-(0.5)(0.7)}]$$

$$= 100\ [\frac{1}{0.65}]$$

$$\Delta Y = 153.85$$

As a result, equilibrium level of income increases from 3,738.46 to 3,892.31 when export increases from 650 to 750. At this new level of equilibrium income, import will be

M = 0.175 (3,892.31)

M = 681.15

And the new balance of trade will be

X - M = 750 - 681.15

= 68.85

Therefore, the balance of trade changes from 4.23 deficits to 68.85 surpluses when export exogenously increases by 100.

2.3.2 Second Partials and Young's Theorem

As you know, if the first order partial derivatives of the function are continuous and differentiable, differentiating these functions will result in second order partial derivatives. Given the function Z=f(x, y), if it is differentiable, it has two first order partial derivatives such as $Z_x = \frac{\partial Z}{\partial x}$ and $Z_y = \frac{\partial Z}{\partial y}$. In addition to these, this function has four second order partial derivatives (***direct and cross partial derivatives***). These are

$$Z_{xx} = \frac{\partial^2 Z}{\partial x^2}$$

$$Z_{yy} = \frac{\partial^2 Z}{\partial y^2}$$

These two derivatives are referred to as direct second order partial derivatives where a as

$Z_{xy} = \frac{\partial z_y}{\partial x}$ and $Z_{yx} = \frac{\partial Z_x}{\partial y}$ are said to be ***second order cross partial derivatives***.

According to the young's theorem, the mixed (cross) partial derivatives for a given function are always equal if both cross partials exist and they are continuous. This means

$$Z_{xy} = Z_{yx} = \frac{\partial Z_y}{\partial x} = \frac{\partial Z_x}{\partial y}$$

Example

1. Given the demand function for the two commodities Q_1 and Q_2

$Q_1 = P_1^{-1.7} P_2^{0.8}$ And $Q_2 = P_1^{0.5} P_2^{-0.2}$

Find the first and second order partial derivatives of the function and determine whether the two goods are complementary or substitute. Show that whether young's theorem is satisfied or not.

$$\frac{\partial Q_1}{\partial P_1} = -1.7\ P_1^{-2.7}\ P_2^{0.8} \qquad \frac{\partial Q_2}{\partial p_1} = 0.5\ P_1^{-0.5}\ P_2^{-0.2}$$

$$\frac{\partial Q_1}{\partial P_2} = 0.8\ P_1^{-1.7}\ P_2^{-0.2} \qquad \frac{\partial Q_2}{\partial p_2} = -0.2\ P_1^{0.5}\ P_2^{-1.2}$$

From these first order partial derivatives we have observed that an increase in the price of commodity Q_1 results in an increase in the quantity demanded of commodity Q_2. Therefore, Q_1 and Q_2 are ***substitute*** goods.

The second order direct partial derivatives are

$$\frac{\partial^2 Q_1}{\partial P_1^2} = (-1.7)(-2.7)P_1^{-3.7}\ P_2^{0.8}$$

$$= 4.59\ P_1^{-3.7} P_2^{0.8}$$

$$\frac{\partial^2 Q_2}{\partial P_2^2} = 0.24\ P_1^{0.5}\ P_2^{-2.2}$$

The ***second cross partial derivatives*** are

$$\frac{\partial(\partial Q_1/\partial P_1)}{\partial P_2} = (0.8)\ (-1.7)\ P_1^{-2.7}\ P_2^{-0.2}$$

$$= -1.36 P_1^{-2.7} P_2^{-0.2} \text{ ---------- (a)}$$

$$\frac{\partial(\partial Q_1/\partial P_2)}{\partial P_1} = (-1.7)(0.8)P_1^{-2.7} P_2^{-0.2}$$

$$= -1.36 P_1^{-2.7} P_2^{-0.2} \text{ ---------- (b)}$$

$$\frac{\partial(\partial Q_2/\partial P_2)}{\partial P_1} = (-0.2)\ (0.5)\ P_1^{-0.5} P_2^{-1.2}$$

$$= -0.1\ P_1^{-0.5} P_2^{-1.2} \text{ ----------(c)}$$

$$\frac{\partial(\partial Q_2/\partial p_1)}{\partial P_2} = (-0.2)\ (0.5)\ P_1^{-0.5}\ p_2^{-1.2}$$

$$= -0.1\ P_1^{-0.5} P_2^{-1.2} \text{ ---------- (d)}$$

Equation a is equal to that of b and equation c is equal to equation d. It indicates that the Young's theorem is satisfied.

2. Given the production function $Q= 6K+ 0.3\ K^2L + 1.2L^2$ derive the four second order partial derivatives and interpret their meaning.

The two first order partial derivatives are

$$\frac{\partial Q}{\partial K} = 6+ 0.6\ KL \qquad \text{and} \qquad \frac{\partial Q}{\partial L} = 0.3\ K^2 + 2.4\ L$$

They represent the marginal product function of the two inputs.

The four second order partial derivatives are

i) $\frac{\partial^2 Q}{\partial K^2} = \quad 0.6L$

This represents the rate of change of marginal product of capital function with respect to capital. It indicates that marginal productivity of capital increases as labor increases.

ii) $\frac{\partial^2 Q}{\partial L^2} = 2.4$

It represents the slope of marginal product of labor function which is independent of capital.

iii) $\frac{\partial^2 Q}{\partial L \partial K} = 0.6K$

This tells us that MP_L increases if K is increased, for any given value of L.

iv) $\frac{\partial^2 Q}{\partial L \partial K} = 0.6K$

This tells us that MP_K increases if L is increased, for any given value of L.

2.3.3 Multivariate Chain Rule

As we know many economic models involve composite functions. Thus differentiation of such functions requires the application of chain rule.

Chain rule for functions of one variable

Given y = f(x) and x= g (t), then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Chain rule for multivariate function

Suppose z= *f* (x, y), x = *f* (t) and y = *f* (t), in this case we should use the total derivative.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}(\frac{dx}{dt}) + \frac{\partial z}{\partial y}(\frac{dy}{dt})$$

Given the function y = *f* (x, w) and $x = g(w)$ to get this total derivative, let us initially determine the total differential $dy = f_x dx + f_w dw$. If we divide both sides of this equation by dw, we obtain

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw}$$

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w$$

$f_x \frac{dx}{dw}$ Measures the indirect effect of ***w*** on ***y*** and, f_w shows the direct effect of ***w*** on ***y.***

Example

1. Suppose that the relationship between revenue R, output Q produced and sold each week is given by

$$R = 400Q - Q^2$$

In addition, suppose that Q is a function of t, i.e. time period, Q = f(t)

$$\text{Thus} \quad \frac{dR}{dt} = \frac{dR}{dQ}\frac{dQ}{dt}$$

$$= (400 - 2Q)\frac{dQ}{dt}$$

When Q = 30 and that the Management is considering to increase production by five units per week,

$$\frac{dR}{dt} = [400 - 2(30)]5$$

$$= (400 - 60)5$$

$$\frac{dR}{dt} = (340)5 = 1700$$

Interpretation

The management decision to increase production by five units per week increases total revenue by 1,700 units

2. Given the utility function $U = x^2 + x\,y + y^2$, $x = t^3 + 1$ and $y = t - t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

$$= (2x+y)\,(3t^2) + (y+2y)\,(1-3t^2)$$

2.3.4 Homogenous Functions and Euler's Theorem

A function $y = f(x_1, x_2, ...x_n)$ is referred to as ***homogenous of degree r*** if and only if

$f(tx_1, tx_2, ..., tx_n) = t^r f(x_1, x_x, ..., x_n)$, where t is any parameter.

Multiplying all the independent variables by a factor t will multiply the value of the function by the factor t^{Γ}.

Example

1. If $f(x, y) = 3x^3 + 5x^2y + 2y^3$, then

f (t x, t y) = $3\,(t\,x)^3 + 5\,(t\,x)^2(t\,y) + 2(t\,y)^3$

$$= \quad 3t^3x^3 + 5t^2x^2y + 2t^3y^3$$
$$= \quad 3t^3x^3 + 5t^3x^2y + 2t^3y^3$$
$$= t^3(3x^3 + 5x^2y + 2y^3)$$

Therefore, the function is ***homogenous of degree 3***.

2. Consider the Cobb- Douglas production function

$$Q = bL^{\alpha}K^{1-\alpha}$$
$$f(Lt, kt) = b(Lt)^{\alpha}(kt)^{1-\alpha}$$
$$= b\ t^{\alpha}L^{\alpha}k^{1-\alpha}t^{1-\alpha}$$
$$= b\ t^{\alpha+1-\alpha}L^{\alpha}\ k^{1-\alpha}$$
$$= b\ t\ L^{\alpha}\ k^{1-\alpha}$$
$$= t\ b\ L^{\alpha}\ k^{1-\alpha}$$

Thus, the production function is homogenous of degree 1.

Given the function $f(tx_1, tx_2, \ tx_{3,\ldots}, tx_n) = t^{\Gamma} f(x_1x_2, x_{3,}....x_n)$

If r = 1, a function exhibits constant returns to scale
If r > 1, a function exhibits increasing returns to scale
If r < 1, a function exhibits decreasing rectums to scale

Euler's Theorem 1

If the function y = $f(x_1, x_2, x_3)$ is homogenous of degree r, then according to the Euler's theorem

$$\frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2 + \frac{\partial f}{\partial x_3}x_3 = rf(x_1, x_2, x_3)$$

Example

Find the degree of homogeneity of the function

$f(x, y) = x^4 + x^2y^2$ Using Euler's theorem

$$f_x x + f_y y = r(x^4 + x^2y^2)$$
$$(4x^3 + 2xy^2)x + (2x^2y)y = r(x^4 + x^2y^2)$$
$$= 4x^4 + 2x^2y^2 + 2x^2y^2$$
$$= 4x^4 + 4x^2y^2$$
$$= 4(x^4 + x^2y^2)$$

Thus, the function is ***homogenous*** of degree 4.

Euler's Theorem 2

If a function $f(tx_1, tx_2, \ldots, tx_n) = t^r f(x_1, x_x, \ldots, x_n)$ is homogenous of degree r, then according to this theorem the first partials f_{x_1}, f_{x_2}, f_{x_3} are homogenous of degree $(r-1)$.

Example

1. Given the function $f(x, y) = x^4 + x^2y^2$,

$$f_x = 4x^3 + 2xy^2 \ , \quad f_y = 2x^2y$$

$$f_x(tx,ty) = 4(tx)^3 + 2(tx)(ty)^2$$

$$= 4t^3x^3 + 2t^3xy^2$$

$$= t^3(4x^3 + 2xy^2)$$

Thus, the function f_x is homogenous of degree 3.

$$f_y(tx,ty) = 2(tx)^2(ty)$$

$$= 2t^2x^2ty$$

$$= 2t^3x^2y = t^3(2x^2y)$$

This means, f_y is homogenous of degree 3.

2. Given the ***Cobb - Douglas production function***

$$f\ (\mathrm{L}, \mathrm{K}) = 60\ \mathrm{L}^{\frac{3}{4}}\mathrm{K}^{\frac{1}{4}}$$

Find the degree of homogeneity of f_L and f_k using ***Euler's theorem***.

The first partial derivatives are

$$f_L = 60\left(\frac{3}{4}\right)\mathrm{L}^{-\frac{1}{4}}\ \mathrm{K}^{\frac{1}{4}} \qquad f_{\mathrm{k}} = \frac{1}{4}(60)\ \mathrm{L}^{\frac{3}{4}}\mathrm{K}^{-\frac{3}{4}}$$

$$f_L = \frac{45K^{\frac{1}{4}}}{L^{\frac{1}{4}}} \qquad f_K = \frac{15L^{\frac{3}{4}}}{K^{\frac{3}{4}}}$$

Thus, $f_L(tL,tK) = 45(tL)^{\frac{-1}{4}}(tK)^{\frac{1}{4}}$

$$= 45L^{\frac{-1}{4}}t^{\frac{-1}{4}}K^{\frac{1}{4}}t^{\frac{1}{4}}$$

$$= 45(t)^{(\frac{1}{4}-\frac{1}{4})}L^{\frac{-1}{4}}K^{\frac{1}{4}}$$

$$= t^0(45L^{\frac{-1}{4}}K^{\frac{1}{4}})$$

And $f_K(tL,tK) = 15(tL)^{\frac{3}{4}}(tK)^{\frac{-3}{4}}$

$$= 15L^{\frac{3}{4}}t^{\frac{3}{4}}K^{\frac{-3}{4}}t^{\frac{-3}{4}}$$

$$= 15(t)^{(\frac{3}{4}-\frac{3}{4})}L^{\frac{3}{4}}K^{/\frac{3}{4}}$$

$$= t^0(15L^{\frac{3}{4}}K^{\frac{-3}{4}})$$

What we have seen from this is both of f_L and f_k is homogenous of degree 0. In other words, the original Cobb-Douglas production is homogenous of degree 1.

2.3.5. Total Differential and Implicit Differentiation

Given the function y = f(x), the differential is $dy = f'(x)dx$ .The concept of differential can be extended to functions with two or more independent variables.

Given the utility function, $U = f(x_1, x_2)$, assuming that U is continuous and differentiable

$U_{x_1} = \frac{\partial U}{\partial x_1}$ = Marginal utility of x_1

$U_{x_2} = \frac{\partial U}{\partial x_2}$ = Marginal utility of x_2

Change in U which is resulted from small change in x_1 is $\frac{\partial U}{\partial x_1} dx_1$ and from small change in x_2 is $\frac{\partial U}{\partial x_2} dx_2$.

Thus the total change in utility is

$$dU = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

If the two variables x and y are related by the implicit function $f(x, y) = 0$, the total differential of the function is

$$d0 = f_x dx + f_y dy$$

$$0 = f_x dx + f_y dy$$

If we rearrange this equation, we get

$$0 = f_x dx + f_y dy$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

As you remember in your microeconomic studies, the theory of production states that the slope of the isoquant represents the marginal rate of technical substitution (MRTS) between two inputs. In this case, the use of total differential can help us to show the reality that MRTS between the two inputs is equal to the ratio of the marginal products of the two inputs. If the producer use capital (K) and labor (L) in the production process, $MRTS_{LK} = \frac{dK}{dL}$

Given the production function $Q = f(L, K)$, the total differential is

$$dQ = \frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL$$

Once the isoquant is determined, there is no change in output along the same isoquant. This means change in Q is zero. Therefore,

$$\frac{\partial Q}{\partial K} dK + \frac{\partial Q}{\partial L} dL = 0$$

$$\frac{\partial Q}{\partial K} dK = \frac{-\partial Q}{\partial L} dL$$

$$\frac{dK}{dL} = \frac{-\partial Q / \partial L}{\partial Q / \partial K} = \frac{-MP_L}{MP_K}$$

$$\text{MRTS}_{Lk} = \frac{-MP_L}{MP_k}$$

Exercise

1. Given the demand function for a good as Q= 56.6-0.25 P - 0.03Y+ 0.45P_s + 0.6n
Where Q is quantity demanded per week, P is price per unit, Y is average weekly income, P_s is Price of competing good and n is population in million. Given values are P = 65, Y = 350, P_s = 60 and n = 24. Find the price elasticity of demand. ----------

2. In the Keynesian Macroeconomic model, given the following attributes

Y = C+ I+ G+ x-M	t = 0.2
C = 0.8 Y_d	G = 400
M = 0.2Y_d	I = 300
Y_d= (1-t) Y	X = 288

a. Determine the equilibrium level of income.
b. What increase in G would be necessary to increase Y to 2, 500?
c. Determine the balance of trade at the initial equilibrium level of income

3. Derive all the first order partial derivatives for the production function
Q = 35KL + 1.4LK^2 + 3.2L^2 and interpret their meaning.

6. What do the Euler's theorems state? ----------

7. Given the function $f(x, y, z) = x^3y^2z - xy^5 + y^3z^3$, determine whether the function is homogenous or not.

Chapter Two

Constrained Optimization

In the previous chapter we have studied the optimization of functions without the existence of constraints. In business and economics studies, however, there are many situations in which complete freedom of action is impossible. For example, a firm can maximize output subject to the constraint of a given budget for expenditures on inputs, or it may need to minimize cost subject to a certain minimum output being produced. Such functions which involve constraints are called **constrained functions** and the process of optimization is referred as **constrained optimization**. This chapter explains the ways of solving constrained optimization problems with equality and inequality constraints.

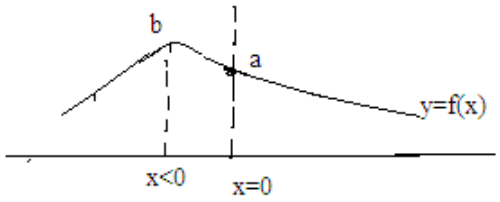
2.1 One Variable Constrained Optimization with Non - Negative Constraint

i) With equality constraint

Maximize y = f(x), subject to $x = \bar{x}$. In this case $y^* = f(\bar{x})$
It simply involves determining the value of the objective function at the given point in the domain.

ii) With non - negativity constraints

Maximize the function $y = f(x)$ subject to $x \geq 0$

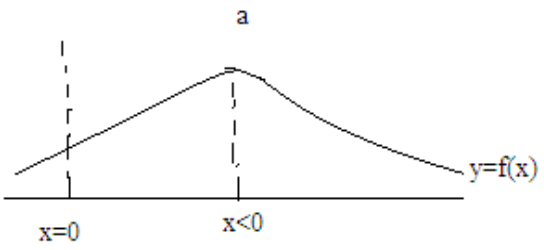


(a)

At $x = 0, f'(x) < 0$

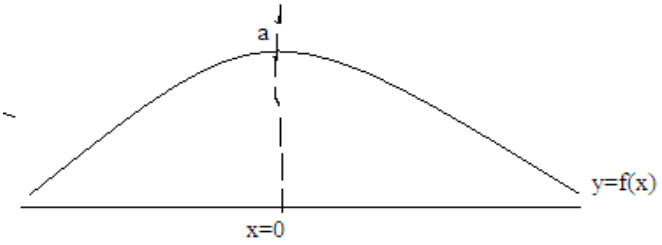
For the above function the unconstrained maximum is attained when x < 0 at point b as shown in the above figure where as the function attains its constrained optimum at point a where $\bar{x} = 0$. $y^* = f(0)$

(b)



In this case the constrained and unconstrained maximum value of the function lie at the same point, i.e., they coincide at point a as shown above. $y^* = f(x)$. At $\bar{x} > 0$, $f'(x) = 0$

(c)



In this case, similar to that of b, the constrained and unconstrained maximum value of the function reside on the same point,

$$y^* = f(0)$$
$$f'(x) = 0$$

Please try to minimize $y = f(x)$, subject to $x \geq 0$ in a similar way.

Given the function $y = f(x)$ subject to the $x \geq 0$

For maximization,

$$f'(x) \leq 0 \quad \begin{cases} \text{If } f'(x) = 0, x \geq 0 \\ \text{If } f'(x) < 0, x = 0 \end{cases}$$

For minimization

$$f'(x) \geq 0 \quad \begin{cases} \text{If } f'(x) = 0, x \geq 0 \\ \text{If } f'(x) > 0, x = 0 \end{cases}$$

Example

Maximize the objective function $y = -3x^2 - 7x + 2$ subject to $x \geq 0$.

First order condition for maximization

$$f'(x) = -6x - 7 = 0$$
$$6x = -7$$

$$x = \frac{-7}{6}$$

Second order condition for maximization

$$f''(x) = -6 < 0$$

Thus, the unconstrained maximum value of the function locates at $x = \frac{-7}{6}$, i.e., x < 0 but the constrained maximum lies at $x = 0$, $f'(0) = -7 < 0$. Thus the constrained maximum is the function is $y = 2$.

2. Minimize $y = x^2 + 2x + 5$, subject to $x \geq 0$

First order condition

$$f'(x) = 2x + 2 = 0$$

$$x = -1 < 0$$

Second order condition

$$f''(x) = 2 > 0$$

Thus the function achieves it's unconstrained minimum at x = -1, i.e., y = 4. However, at $x = 0$, $f'(x) = 2 > 0$. Therefore, the minimum value of the constrained function is $y = f(0) = 5$.

Exercise

Solve the following questions based on the information in the above section.

1. Maximize $y = -x^2 + 5x + 6$, subject to $x = 2$, ----------

2. Maximize $y = -2x^2 + 3x + 4$, subject to $x \geq 0$ ----------

3. Find the maximum value of the function $y = -3x^2 + x + 7$, subject to $x \geq 0$ ----------

4. Minimize the function $y = x^2 + 3x + 4$, subject to $x \geq 0$ ----------

5. Find the minimum value of the function $y = 2x^2 + x + 7$, subject to $x \geq 0$ ----------

2.2 Two Variable Problems with Equality Constraints

This section is emphasized on the two methods of constrained optimization. These are.

- Constrained optimization by substitution
- Lagrange multiplier method

A. Constrained Optimization by Substitution

This method is mainly applicable for problems where the objective function with only two variables is maximized or minimized subject to one constraint.

Consider a firm that wants to maximize output given the production function $Q = f(K, L)$ and suppose P_K and P_L prices of K and L respectively and a fixed budget B. Then, we can determine the amount of K and L that optimize the output Q using the ***method of substitution***.

Example

1. A firm faces the production function $Q= 12K^{0.4} L^{0.4}$ and assume it can purchase K and L at price per unit of 40 birr and 5 Birr respectively and it has a budget of 800 Birr. Determine the amount of K and L which maximizes output.

Solution

The problem is Maximize $Q= 12K^{0.4} L^{0.4}$

Subject to $40K+5L = 800$

According to the theory of production, the optimization condition is written in such away that the ratio of marginal product of every input to its price must be the same. That is

$$\frac{MP_K}{P_K} = \frac{MP_L}{P_L}$$

The marginal products can be obtained using the method of ***partial differentiation*** as follows.

$$MP_K = 4.8K^{-0.6}L^{0.4} \text{ ---------- (1)}$$

$$MP_L = 4.8K^{0.4}L^{-0.6} \text{ ---------- (2)}$$

Substituting these marginal products and the given prices in the constraint function gives us

$$\frac{4.8K^{-0.6}L^{0.4}}{40} = \frac{4.8K^{0.4}L^{-0.6}}{5}$$

$$K^{-0.6}L^{0.4} = 8K^{0.4}L^{-0.6}$$

Multiplying both sides by $K^{0.6}L^{0.6}$ results in

$L = 8K$---------- (3)

Substituting (3) in the budget constraint we get

$40K + 5(8K) =800$

$40K+ 40K = 800$

$80k =800$

$K=10$

Thus, $L= 8(10) =80$

Therefore, this firm should employ 10 units of capital and 80 units of labor in the production process to optimize its output.

2. Suppose the utility function of the consumer is given by $U = 4xy - y^2$ and the budget constraint is $2x+y = 6$. Determine the amount of x and y which will optimize total utility of the consumer.

Solution

Utility is maximized when $\frac{MU_X}{P_x} = \frac{MU_y}{P_y}$

In our example, $MU_X = 4y \;\&\; MU_Y = 4x - 2y$. Therefore, at the point of equilibrium

$$\frac{4y}{2} = \frac{4x - 2y}{1}$$
$$4y = 8x - 4y$$
$$4y + 4y = 8x$$
$$8y = 8x$$
$$x = y \text{ ---------- (4)}$$

Substituting (4) above in the budget constraint gives us

$$2x + x = 6$$
$$3x = 6$$
$$x = 2 = y$$

Therefore, this consumer can optimize his utility when it consumes 2 units of good x and 2 units of good y.

B. Lagrange Multiplier Method

What is ***Lagrange Multiplier*** Method? What are the steps to use this method? How do you interpret the Lagrange multiplier?

The essence of this method is to change a constrained optimization problem in to a form such that the first order condition of the unconstrained optimization problem can still be applicable. This method can be used for most type of constrained optimization problems. Given the function $Z = f(x, y)$ subject to $g(x, y) = P_X x + P_Y y = M$, to determine the amount of x and y which maximize the objective function using the ***Lagrange Multiplier Method***, we should involve the following steps.

Step 1 Rewrite the constraint function in its implicit form as

$$M - xP_x - yP_y = 0$$

Step 2 Multiply the constraint function by the Lagrange multiplier λ

$$\lambda(M - xP_x - yP_y)$$

Step 3 Add the above constraint to the objective function and thereby formulate the ***Lagrange function*** that is a modified form of the objective function which includes the constraints as follows:

$$L(x, y, \lambda) = Z(x, y) + \lambda(M - xP_x - yP_y) \text{ ---------- (5)}$$

Necessary condition, i.e. the first orders condition for maximization is that the first order partial derivatives of the Lagrange function should be equal to zero.

Differentiating L with respect to x, y and λ and equating it with zero gives us.

$$\frac{\partial L}{\partial x} = \frac{\partial Z}{\partial x} - \lambda P_x = 0 \text{ ---------- (6)}$$

$$\frac{\partial L}{\partial y} = \frac{\partial Z}{\partial y} - \lambda P_y = 0 \text{ ---------- (7)}$$

$$\frac{\partial L}{\partial \lambda} = M - xP_x - yP_y = 0 \text{ ---------- (8)}$$

From equation (6) and (7) we get

$$\lambda = \frac{Z_x}{P_x} \text{ and } \lambda = \frac{Z_y}{P_y}$$

This means, $\lambda = \frac{Z_x}{P_x} = \frac{Z_y}{P_y}$ or $\frac{Z_x}{Z_y} = \frac{P_x}{P_y}$

Sufficient condition -To get the second order condition, we should partially differentiate equations (6), (7) and (8). Representing the second direct partial derivatives by Z_{xx} and Z_{yy}, and the second cross partial derivatives by Z_{xy} and Z_{yx}, the ***border Hessian determinant*** bordered with 0, g_x and g_y is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & Z_{xx} & Z_{xy} \\ -P_y & Z_{yx} & Z_{yy} \end{vmatrix} > o$$

d^2Z is referred to as positive definite subject to $dg = 0$ iff $\left|\overline{H}\right| < 0$,

d^2Z is referred to as negative definite subject to $dg = 0$ iff $\left|\overline{H}\right| > 0$.

Negative definiteness of d^2Z implies that the function achieves its relative maximum point where as a positive definite is a sufficient condition to satisfy the relative minimum of the objective function.

i) Maximization

Example

Given the utility function of the consumer who consumes two goods x and y as

U (x, y) = (x+ 2) (y+1)

If the price of good x is $P_x = 4$ birr, that of good y is $P_y = 6$ Birr and the consumer has a fixed budget of 130 birr. Determine the optimum values of x and y using the Lagrange multiplier method,

Solution

Maximize U (x, y) = x y + x+ 2y + 2

Subject to 4x + 6y = 130

Now we should formulate the Lagrange function to solve this problem. That is

$L(x, y, \lambda)$ = x y + x+ 2y + 2 + λ (130 - 4x - 6y) ---------- (9)

Necessary conditions for utility maximization are $\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial \lambda} = 0$

$$\frac{\partial L}{\partial x} = (y+1) - 4\lambda = 0$$

$y = -1 + 4\lambda$ ---------- (10)

$$\frac{\partial L}{\partial y} = (x+2) - 6\lambda = 0$$

$x = -2 + 6\lambda$ ---------- (11)

$$\frac{\partial L}{\partial \lambda} = 4x + 6y - 130 = 0$$

$4x+6y= 130$---------- (12)

Substituting the value of x and y explained in equation (10) and (11) in to (12) enables us to determine

$4(-2+6\lambda) + 6(-1+4\lambda) = 130$

$-8 + 24\lambda - 6 + 24\lambda = 130$

$48\lambda = 144$

$\lambda = 3$

Therefore, x = -2+6(3)

x = -2 + 18 = 16

y = -1 + 4 (3)

y = 11

Second order sufficient condition for utility maximization is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix}$$

The second partial derivatives of the objective function and the first partial derivatives of the constraint function are

$L_{xx} = \frac{\partial^2 L}{\partial x^2} = 0$, $L_{yy} = 0$, $L_{xy} = L_{yx} = 1$

$g_x = \frac{\partial g}{\partial x} = 4$, and $g_y = \frac{\partial g}{\partial y} = 6$

Therefore, the bordered Hessian determinant of this function is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = -4(0-6) + 6(4-0) = 48 > 0$$

The second order condition, i.e., $|\bar{H}| > 0$ which is negative definite, is satisfied for maximization.

Thus, the consumer maximizes utility when he consumes 11 units of good y and 16 units of good x. The maximum utility is U = (16+2) (11+1) = (18) (12) = 216 units which is similar to the value of the Lagrange function at these values of x, y and λ . The value of the Lagrange multiplier λ is 3. It indicates that a one unites increase (decrease) in the budget of the consumer increases (decreases) his total utility by 3 units.

2. Suppose the monopolist sells two products x and y and their respective demand is

$P_x = 100 - 2x$ and $P_y = 80 - y$

The total cost function is given as TC = 20x + 20y, when the maximum joint product of the two outputs 60 unit. Determine the profit maximizing level of each output and their respective price.

Solution

As it is known profit (π) = TR - TC, where TR represents total revenue and TC represents total cost.

$$TR = x P_x + y P_y = (100x - 2x^2) + (80y - y^2)$$

Thus $\pi = 100x - 2x^2 + 80y - y^2 - 20x - 20y$

$\pi = 80x + 60y - 2x^2 - y^2$

But this monopolist can maximize its profit subject to the production quota. Thus,

Maximize $\pi = 80x + 60y - 2x^2 - y^2$

Subject to $x + y = 60$

To solve this problem, we should formulate the Lagrange function,

$$L(x, y, \lambda) = 80x + 60y - 2x^2 - y^2 + \lambda (x + y - 60) \text{ ---------- (13)}$$

First order conditions for maximum profit are

$L_x = 80 - 4x + \lambda = 0$

$-4x = -80 - \lambda$

$\lambda = 20 + \frac{1}{4}\lambda$ ---------- ------- (14)

$L_y = 60 - 2y + \lambda = 0$

$-2y = -60 - \lambda$

$y = 30 + \frac{1}{2}\lambda$ ---------- (15)

$L_\lambda = x + y - 60 = 0$

$x + y = 60$ ---------- (16)

Substituting equation (14) and (15) in equation (16), we get

$20 + \frac{1}{4}\lambda + 30 + \frac{1}{2}\lambda = 60$

$50 + \frac{3}{4}\lambda = 60$

$\frac{3}{4}\lambda = 10$

$\lambda = \frac{40}{3}$

Thus, $x = 20 + \frac{1}{4}(\frac{40}{3})$ $\quad$ $y = 30 + \frac{1}{2}\frac{(40)}{3}$

$= 20 + 3.33$ $\quad$ $= 30 + 6.67$

$x = 23.33$ $\quad$ $y = 36.67$

Second order condition for maximum profit is

$$L_{XX} = -4, L_{YY} = -2, L_{XY} = L_{YX} = 0$$

$$g_X = 1 \& g_Y 1$$

Therefore, the bordered Hessian determinant of the given function is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1(-2-0) + 1(0+4) = 6 > 0$$

The second order condition is satisfied for maximization of functions.

$$P_x = 100 - 2(23.33) \qquad P_y = 80 - 36.67$$

$$P_x = 53.34 \qquad P_y = 43.33$$

Therefore, the monopolist maximizes its profit when it sells 23.33 of good x at a price of 53.34 birr per unit and 36.67 units of good y at a price of 43.33 birr per unit. $\lambda = \frac{40}{3}$ shows that a one unit increase in total expenditure on inputs increases total profit of the monopolist by $\frac{40}{3}$ units. In other words, if the constant of the constraint relaxes by one unit that is $x + y = 61$, then the value of the objective function increases by the value the Lagrange multiplier.

ii) Minimization

The firm can determine the least cost combination of inputs for the production of a certain level of output Q. Given the production function Q= f (L, K) and the cost function of the firm is $C = LP_L + KP_k$ Where L = labor, K = capital, Q = output. Suppose the price of both input to be exogenous, we can formulate the problem of minimizing the cost as

Minimizes $C = P_L L + P_k k$

Subject to $Q = f(L, K)$

To determine the amount of labor and capital that should be employed initially we should formulate the ***Lagrange function***. It is

$$L = LP_L + KP_K + \lambda(Q - f(L,K)) \text{---------- ---- (17)}$$

First order conditions for a minimum cost are

$$L_L = P_L - \lambda Q_L = 0$$

$$\lambda = \frac{P_L}{Q_L} = \frac{P_L}{MP_L} \text{---------- (18)}$$

$$L_K = P_K - \lambda Q_k = 0$$

$$\lambda = \frac{P_K}{Q_K} = \frac{P_K}{MP_K} \text{---------- (19)}$$

$$L_\lambda = Q - f(K,L) = 0 \text{ ---------- (20)}$$

Where Q_L and Q_k represents marginal product of labor and capital respectively.

From equation (17) and (18), we get

$$\lambda = \frac{P_L}{MP_L} = \frac{P_K}{MP_K} \text{ ---------- (21)}$$

Equation (21) indicates that, at the point of optimal input combination the input - price ratio and the marginal product ratio have to be the same for each input. This ratio shows the amount of expenditure per unit of the marginal product of the input under consideration. Thus, the interpretation the Lagrange multiplier is the marginal cost of product at the optimal condition. In other words, it indicates the effect of change in output on the total costs of production, i.e., it measures the comparative static - effect of the constraint constant on the optimal value of the objective function.

The first order condition indicated in equation (21) can be analyzed in terms of isoquants and isocosts as

$$\lambda = \frac{P_L}{P_k} = \frac{MP_L}{MP_k} \text{ ---------- (22)}$$

The $\frac{MP_L}{MP_K}$ represents the negative of the slope of the isoquant, which measures the marginal rate of technical substitution of labor to capital ($MRTS_{Lk}$).

The $\frac{P_L}{P_K}$ ratio shows the negative of the slope of the isocost. An isocost is a line which indicates the locus of input combinations which entail the same total cost. It is shown by the equation

$$C = P_L L + P_k K \quad \text{or} \quad K = \frac{C}{P_L} - \frac{P_L}{P_k} L$$

$\frac{P_L}{P_k} = \frac{MP_L}{MP_k}$ indicates the fact that the isocost and isoquant lines are tangent to each other at the point of optimal input combination.

Second order condition for minimization of cost.

A negative bordered Hessian determinant is sufficient to say the cost is at its minimum value. That is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & Q_L & Q_K \\ Q_L & L_{LL} & L_{LK} \\ Q_K & L_{KL} & L_{KK} \end{vmatrix} < 0$$

Example

Suppose a firm produces an output Q using labor L and capital K with production function $Q = 10K^{0.5}L^{0.5}$. If the output is restricted to 200 units, the price of labor is 10 birr per unit and

Price of capital is 40Birr per unit. Determine the amount of L and K that should be employed at minimum cost. Find the minimum cost.

The problem is Minimize C = 10 L + 40K

Subject to $200 = 10K^{0.5}L^{0.5}$

Formulating the Lagrange function

$$L(L,K,\lambda) = 10L + 40K + \lambda(200 - 10K^{0.5}L^{0.5}) \text{ ---------- (23)}$$

First order conditions

$$L_L = 10 - 5\lambda K^{0.5}L^{-0.5} = 0$$

$$\lambda = \frac{2L^{0.5}}{K^{0.5}} \text{ ---------- (24)}$$

$$L_K = 40 - 5\lambda K^{-0.5}L^{0.5} = 0$$

$$\lambda = \frac{8K^{0.5}}{L^{0.5}} \text{ ---------- (25)}$$

$$L_\lambda = 200 - 10K^{0.5}L^{0.5} = 0$$

$$10K^{0.5}L^{0.5} = 200 \text{ ---------- (26)}$$

From equation (24) and (25), we get

$$\frac{2L^{0.5}}{K^{0.5}} = \frac{8K^{0.5}}{L^{0.5}}$$

2L = 8K

L= 4K ---------- (27)

Substituting equation (27) in to (26) gives us

$$K^{0.5}(4K)^{0.5} = 20 \text{ ---------- (28)}$$

2K = 20

K = 10 and L = 4(10) = 40, $\lambda = 4$

Second order condition

Now we should check the second order condition to verify that cost of production is least at K = 10 and L = 40.

For cost minimization the determinant of the bordered Hessian matrix must be less than zero.

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & Q_L & Q_K \\ Q_L & L_{LL} & L_{LK} \\ Q_K & L_{KL} & L_{KK} \end{vmatrix} < 0$$

At L = 40 and K = 10

$$Q_L = \frac{\partial Q}{\partial L} = (5)\sqrt{\frac{K}{L}} = (5)\sqrt{\frac{10}{40}} = 2.5$$

$$Q_k = \frac{\partial Q}{\partial k} = (5)\sqrt{\frac{L}{K}} = (5)\sqrt{\frac{40}{10}} = 10$$

$$L_{LL} = 2.5\,\lambda K^{0.5} L^{-1.5} = 2.5(4)(10)^{0.5}(40)^{-1.5}$$
$$= 0.125$$

$$L_{kk} = 2.5\,\lambda K^{-1.5} L^{0.5} = 2.5(4)(10)^{-1.5}(40)^{0.5}$$
$$= 2$$

$$L_{KL} = L_{LK} = -2.5\lambda K^{-0.5} L^{-0.5} = -2.5(4)(10)^{-0.5}(40)^{-0.5}$$
$$= -0.5$$

Therefore, the determinant of the bordered Hessian matrix is

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 2.5 & 10 \\ 2.5 & 0.125 & -0.5 \\ 10 & -0.5 & 2 \end{vmatrix}$$

$$= -2.5\,(5+5) + 10(-1.25 - 1.25)$$
$$= -2.5\,(10) + 10\,(-2.5)$$
$$\left|\overline{H}\right| = -50 < 0$$

Thus, the firm can minimize its cost when it employs 10 units of capital and 40 units of labor in the production process and the minimum cost is

$$C = 10\,(40) + 40\,(10)$$
$$\text{Min. } C = 400 + 400 = 800 \text{ birr}$$

In this problem K, L and λ are endogenous. The Lagrange multiplier λ measures the responsiveness of the objective function to a change in the constant of the constraint function.

What happens to the value of the Lagrange function and the constrained function when total output increases from 200 to 201? What about the amount of L and K? Compare the value of the constrained function and that of the Lagrange function at this point. Interpret the value of λ.

Optimization of n - variable case

Given the objective function

$$\text{Optimize } Z = f(x_1, x_2, x_3, ..., x_n)$$
$$\text{Subject to } g(x_1, x_2, x_3, ..., x_n) = c$$

Similar to our earlier discussion we ought to first formulate the Lagrange function. That is

$$L = f(x_1, x_2, x_3, ..., x_n) + \lambda(c - g(x_1, x_2, x_3, ..., x_n))$$

The necessary condition for optimization of this function is that

$$L_\lambda = L_1 = L_2 = L_3 = L_4 = - - - = L_n = 0$$

The second order condition for optimization of this function depends on the sign of d^2L subject to $dg = g_1 dx_1 + g_2 dx_2 + g_3 dx_3 + ... + g_n dx_n = 0$ similar to our earlier discussion.

The positive or negative definiteness of d^2L involves the ***Bordered Hessian Determinate*** test. However, in this case the conditions have to be expressed in terms of the bordered principal minor of the Hessian. Given the bordered Hessian as

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 & . & . & g_n \\ g_1 & L_{11} & L_{12} & L_{13} & . & . & L_{1n} \\ g_2 & L_{21} & L_{22} & L_{23} & . & . & L_{2n} \\ g_3 & L_{31} & L_{32} & L_{33} & . & . & L_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ g_n & L_{n1} & L_{n2} & L_{n3} & . & . & L_{nn} \end{vmatrix}$$

The successive bordered principal minors are

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} \quad , \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix} \quad \text{etc}$$

However, $|\bar{H}| = |\bar{H}_n|$.

$|\bar{H}_2|$ Shows the second principal minor of the Hessian bordered with $0, g_1$ and g_2.

d^2L is positive definite subject to $dg = 0$ if and only if $|\bar{H}_2|, |\bar{H}_3|$,-----,$|\bar{H}_n| < 0$.

d^2L is negative definite subject to $dg = 0$ if and only if $|\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0$,---.

A positive definite d^2L is a sufficient condition for minimum value and negative definite d^2L is sufficient condition for maximization of the objective function.

In our analysis above $|\bar{H}_2|$ is the one which contains L_{22} as the last element of its principal diagonal. $|\bar{H}_3|$ is the one which includes L_{33} as the last element of its principal diagonal etc.

Optimization when there is more than one equality constraint

Let us consider the optimization problem involves three variables and two constraints.

Optimize $Z = f(x_1, x_2, x_3)$

Subject to $g^1(x_1, x_2, x_3) = c^1$

$g^2(x_1, x_2, x_3) = c^2$

As usual we should construct the Lagrange function by using the Lagrange multiplier λ .Since we have two constraint functions, we are required to incorporate two λ s, i.e., $\lambda_1 and \lambda_2$ in our analysis.

The Lagrange function is

$$L = f(x_1, x_2, x_3) + \lambda_1(c^1 - g^1(x_1, x_2, x_3)) + \lambda_2(c^2 - g^2(x_1, x_2, x_3))$$

First order conditions for optimization

$$L_1 = f_1 - \lambda_1 g_1^1 - \lambda_2 g_1^2 = 0$$
$$L_2 = f_2 - \lambda_1 g_2^1 - \lambda_2 g_2^2 = 0$$
$$L_3 = f_3 - \lambda_1 g_3^1 - \lambda_2 g_3^2 = 0$$
$$L_{\lambda 1} = c^1 - g^1(x_1, x_2, x_3) = 0$$
$$L_{\lambda 2} = c^2 - g^2(x_1, x_2, x_3) = 0$$

When there are n - variables and m - constraints, the Lagrange function becomes

$$L = f(x_1, x_2, x_3, ..., x_n) + \sum_{j=1}^{m} \lambda_j [c^j - g^i(x_1, x_2, x_3, ..., x_n)]$$

In this case we will have m+ n variables in the Lagrange function and we will have also m+ n simultaneous equations.

First order conditions are

$$L_i = f_i - \lambda_j g_j^i = \quad , \text{ (i = 1, 2, 3, ---, n) and (j= 1, 2, 3, --- m)}$$
$$L_{\lambda i} = c^j - g^i(x_1, x_2, x_3, ..., x_n) = 0$$

Second order conditions for optimization of three variables and two constraints problem are

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 0 & g_1^1 & g_2^1 & g_3^1 \\ 0 & 0 & g_1^2 & g_2^2 & g_3^2 \\ g_1^1 & g_1^2 & L_{11} & L_{12} & L_{13} \\ g_2^1 & g_2^2 & L_{21} & L_{22} & L_{23} \\ g_3^1 & g_3^2 & L_{31} & L_{32} & L_{33} \end{vmatrix}$$

In this case, $\left|\overline{H}_3\right| = \left|\overline{H}\right|$. Thus for a maximum value,

$\left|\overline{H}_2\right| > 0$, $\left|\overline{H}_3\right| < 0$.

For a minimum,

$\left|\overline{H}_2\right| < 0$, $\left|\overline{H}_3\right| < 0$.

With the existence of n - variables and m - constraints, the second order condition is explained as

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 0 & 0 & . & . & 0 & - & g_1^1 & g_2^1 & g_3^1 & . & . & g_n^1 \\ 0 & 0 & 0 & . & . & 0 & - & g_1^2 & g_2^2 & g_3^2 & . & . & g_n^2 \\ 0 & 0 & 0 & . & . & 0 & - & g_1^3 & g_2^3 & g_3^3 & . & . & g_n^3 \\ . & . & . & . & . & . & - & . & . & . & . & . & . \\ . & . & . & . & . & . & - & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & - & g_1^m & g_2^m & g_3^m & . & . & g_n^m \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ g_1^1 & g_2^1 & g_3^1 & . & . & g_n^1 & - & L_{11} & L_{12} & L_{13} & . & . & L_{1n} \\ g_1^2 & g_2^2 & g_3^2 & . & . & g_n^2 & - & L_{21} & L_{22} & L_{23} & . & . & L_{2n} \\ g_1^3 & g_2^3 & g_3^3 & . & . & g_n^3 & - & L_{31} & L_{32} & L_{33} & . & . & L_{3n} \\ . & . & . & . & . & . & - & . & . & . & . & . & . \\ . & . & . & . & . & . & - & . & . & . & . & . & . \\ g_1^m & g_2^m & g_3^m & . & . & g_n^m & - & L_{n1} & L_{n2} & L_{n3} & . & . & L_{nn} \end{vmatrix}.$$

Now we have divided the ***Bordered Hessian Determinant*** in to four parts. The upper left area includes zeros only and the lower right area is simply a ***plain Hessian***. The remaining two areas include the g_i^j derivatives. These derivatives have a mirror image relationship to each other considering the principal diagonal of the ***Bordered Hessian*** as a reference.

We can create several bordered principal minors from $\left|\overline{H}\right|$. It is possible to check the second order sufficient condition for optimization using the sing of the following bordered principal minors:

$\left|\overline{H}_{m+1}\right|$, $\left|\overline{H}_{m+2}\right|$,----------,$\left|\overline{H}_{n}\right|$

The objective function can sufficiently achieve its maximum value when the successive bordered principal minors alternate in sign. However, the sign of $\left|\overline{H}_{m+1}\right|$ is $(-1)^{m+1}$ where as for minimum value the sufficient condition is that all bordered principal minors have the same sign, i.e., $(-1)^{m}$. This indicates that if we have an odd number of constraints, then sign of all bordered principal minors will be negative and positive with even number of constraints.

Exercise

Solve the following questions based on the information above.

1. What is constrained function? ----------

2. Explain elasticity of substitution ----------

3. What does the Lagrange multiplier indicate? ----------

4. Suppose a firm faces the production function $Q = 120L + 200K - L^2 - 2K^2$ for positive values of Q. If it can buy L at 5 birr per unit, K at 8 birr per unit and has a budget of 70 Birr, determine the maximum output that it can produce using substitution method.----------

5.Suppose the prices of inputs K and L are 12 birr and 3 birr per unit respectively and the production function of the firm is $Q= 25K^{0.5} L^{0.5}$. Determine the minimum costs of producing 1,250 units of output using Lagrange multiplier method. ----------

6. Suppose a consumer has a utility function of $U = 40x^{0.5} y^{0.5}$. If the price of x is 20 birr per unit, price of y is 5 birr unit and the consumer has a budget of 600 birr. Determine the amount of x and y which maximize utility using the Lagrange multiplier method.

2.3 Inequality Constraints and Kuhn-Tucker Theorems, and Mixed Constraints

Nonlinear Programming

The problem of optimization of an objective function subject to certain restrictions or constraints is a usual phenomenon in economics. Mostly, the method of maximizing or minimizing a function includes equality constraints. For instance, utility may be maximized subject to a fixed income that the consumer has and the budget constraint is given in the form of equation. Such type of optimization is referred to as ***classical optimization***. But objective function subject to inequality constraints can be optimized using the method of ***mathematical programming***. If the objective function as well as the inequality constraints is linear, we will use a method of linear programming. However, if the objective function and the inequality constraints are nonlinear, we will apply the technique of nonlinear programming to optimize the function.

Maximization problem

$$\text{Maximize } \pi = f(x_1, x_2, x_3, ..., x_n)$$

$$\text{Subject to } g^1(x_1, x_2, x_3, ..., x_n) \le k_1$$
$$g^2(x_1, x_2, x_3, ..., x_n) \le k_2$$
$$g^3(x_1, x_2, x_3, ..., x_n) \le k_3$$
$$\vdots \quad \vdots \quad \vdots$$
$$g^m(x_1, x_2, x_3, ..., x_n) \le k_m \quad \text{and} \quad x_j \ge 0 \ , \ (j = 1,2,3....., n)$$

Minimization Problem

It can be expressed in the form of

$$\text{Minimize } C = f(x_1, x_2, x_3, ..., x_n))$$

$$\text{Subject to } g^1(x_1, x_2, x_3, ..., x_n) \ge k_1$$
$$g^2(x_1, x_2, x_3, ..., x_n) \ge k_2$$
$$g^3(x_1, x_2, x_3, ..., x_n) \ge k_3$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$g^m(x_1, x_2, x_3, ..., x_n) \ge k_m, \quad x_j \ge 0 \qquad (j = 1,2,3....., n)$$

Where C- represents total cost which is the objective function.

x_j - is the amount of output produced

k_i - is the constant of the constraint function

g^i - is the constraint function.

We have observed from the above expression that the nonlinear programming also includes three ingredients. These are

- the objective function
- a set of constraints (inequality)
- non - negativity restrictions on the choice variable

The objective function as well as the inequality constraints is assumed to be differentiable with respect to each of the choice variables. Like linear programming we apply the $\leq$ constraints for maximization and minimization problem involves only $\geq$ constraints.

Example1

1. Find the values of $\bar{x}$ and $\bar{y}$ of the following function graphically.

a) Minimize $C = x^2 + y^2$

Subject to $x\, y \geq 25$

$x, y \geq 0$

First we should convert the inequality constraint in to equality as $xy = 25$ and draw the graph of this constraint function on the xy plane.

x	1	2	3	4	5	6	7 25
y	25	12.5	8.3	8.3	5	4.6	3.57 1

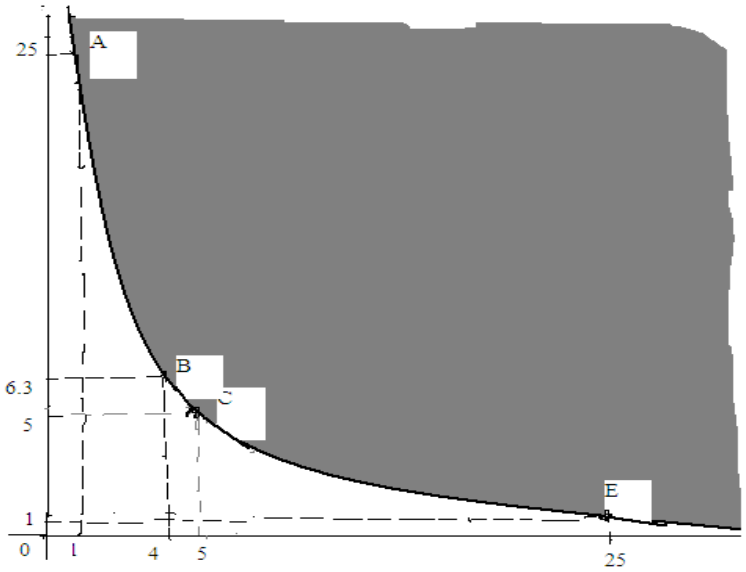


Fig. (a)

The shaded region in the above figure represents the feasible region. Let us evaluate the objective function C at points A, B, C, D and E on the graph.

At point a (1, 25), $C=1^2+ 25^2 = 1+ 625 = 626$
At point B (4, 6.3), $C = 4^2 + (6.3)^2 - 16+ 39.69 = 55.69$
At point c (5, 5), $C = 5^2 + 5^2 = 25 + 25 = 50$
At point d (6, 4.6) $C = 6^2 + (4.6)^2 = 36 + 21.16 = 57.16$
At point E (25, 1) , $C = (25)^2 + 1^2 = 625 + 1 = 626$

Therefore, the value of x and y which minimizes the objective function are 5 and 5 respectively. The minimum value is C = 50.

b) Maximize $\pi = x^2 + (y - 2)^2$
Subject to $5x + 3y \leq 15$
And $x, y \geq 0$

Solution
Similar to that of problem a, we should convert the inequality constraint in to equality constraint and draw its graph in the x y plane. It is $5x + 3y = 15$

X	0	1	2	3
Y	5	3.3	1.67	0

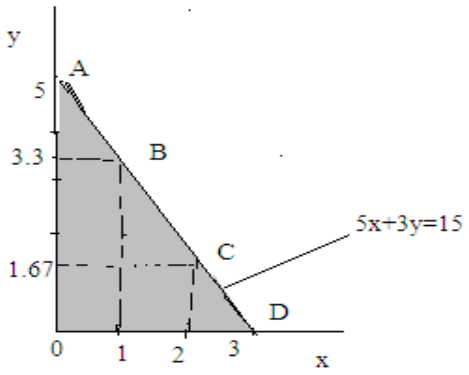


Fig.(b)

The shaded region of the above figure represents the feasible region as every point in this feasible region satisfies the inequality constraint $5 x + 3y \leq 15$.

Evaluating the objective function at points A, B, C and D of the above graph (fig. b),
At point A (0, 5), $\pi = 0^2 + (5 - 2)^2 = 0 + 9 = 9$
At point B (1, 3.3), $\pi = 1^2 + (3.3 - 2)^2 = 1+1.69 = 2.69$
At point C (2, 1.67), $\pi = 2^2 + (1.67 - 2)^2 = 4 + 0.1089 = 4.1089$
At Point D (3, 0), $\pi = 3^2 + 9 (0 - 2)^2 = 9+4 = 13$

Therefore, the objective function is maximized when x = 3 and y = 0. The maximum profit is π = 13

In general, we can distinguish the nonlinear programming from that of linear one based on the following points.

- In nonlinear programming the field of choice not necessarily locates at its extreme points.
- The number of constraints may not be the same with the choice variables.
- Following the same direction in a movement may not lead to a continually increasing or (decreasing) value of the objective function.
- The feasible region may not be a convex set.
- A local optimum may not be a global optimum.

Kuhn - Tucker Conditions

In the previous section of this chapter, we have discussed about optimization problems of the objective function with equality constraints and without explicitly restricting the sing of the choice variables. In this case, the first order condition is satisfied provided that the first order partial derivative of the Lagrange function with respect of each choice variable and with respect to the Lagrange multiplier is zero. For instance, in the problem

Maximize $\pi = f(x, y)$

Subject to $g(x, y) = k$

The Lagrange function is

$$L = f(x, y) + \lambda(k - g(x, y))$$

The first order condition states that

$$L_x = L_y = L_\lambda = 0$$

In non-linear programming, there is a similar first order condition which is referred to as **Kuhn-Tucker conditions**. As we discussed previously, in classical optimization process, the first order condition is a necessary condition. However, a certain condition should be fulfilled for the Kuhn - Tucker conditions to be necessary conditions.

Now let us discuss Kuhn-Tucker conditions in two steps for the purpose of making the explanation easy to understand.

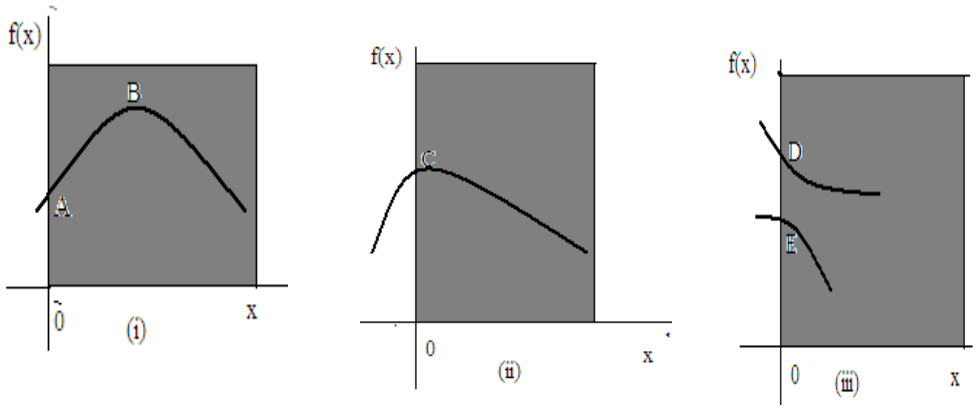
Step 1

In the first step, let us take a problem of optimizing the objective function with non negativity restrictions and with no other constraints. In economics, the most common inequality constraint is non negativity constraint.

Maximize π = f(x)

Subject to $x \geq 0$

provided that the function is supposed to be continuous and smooth. Based on the restriction $x \geq 0$, we may have three possible results. As shown in the following figures.



When the local maximum resides in side the shaded feasible region as shown above at point B of diagram (i), then we have an ***interior solution***. In this case, the first order condition is similar to that of the classical optimization process, i.e. $\frac{d\pi}{dx} = 0$.

Diagram (ii) shows that the local maximum is located on the vertical axis indicated by point C. At this point, the choice variable is 0 and the first order derivative is zero, i.e. $\frac{d\pi}{dx} = 0$, at point C we have a ***boundary solution***.

Diagram (iii) indicates that the local maximum may locate at point D or point E with in the feasible region. In this case, the maximum point is characterized by the inequality $\frac{d\pi}{dx} < 0$ because the curves are at their decreasing portion at these points.

From the above discussion it is clear that the following three conditions have to be met so as to determine the value of the choice variable which gives the local maximum of the objective function.

$f'(x) = 0$, and x > 0 (point B)
$f'(x) = 0$, and x = 0 (point C)
$f'(x) < 0$, and x = 0 (point D and E)

Combining these three condition in to one statement given us

$f'(x) \le 0 \quad x \ge 0$ and $xf'(x) = 0$

The first inequality indicates the information concerning $\frac{d\pi}{dx}$. The second inequality shows the non negativity restriction of the problem. The third part indicates the product of the two quantities x and $f'(x)$. The above statement which is a combination of the three conditions represents the

first order necessary condition for the objective function to achieve its local maximum provided that the choice variable has to be non negative.

If the problem involves n - choice variables like

Maximize $\pi = f(x_1, x_2, x_3, ...x_n)$

Subject to $x_i \geq 0$

The first order condition in classical optimization process is

$f_1 = f_2 = f_3 =$ -------$= f_n = 0$

The first order condition that should be satisfied to determine the value of the choice variable which maximizes the objective function is

$f_i \leq 0 \qquad x_i \geq 0$ and $x_i f_i = 0$ (i =1, 2, 3, -------, n)

Where f_i is the partial derivative of the objective function with respect to x_i, i.e.,

$$f_i = \frac{\partial \pi}{\partial x_i}.$$

Step 2

Now we continue to the second step. To do this, let us attempt to incorporate inequality constraints in the problem. In order to simplify our analysis, let us first discuss about maximization problem with three choice variables and two constraints as shown below.

Maximize $\pi = f(x_1, x_2, x_3)$

Subject to $g^1(x_1, x_2, x_3) \leq k_1$

$g^2(x_1, x_2, x_3) \leq k_2$

And $x_1, x_2, x_3 \geq 0$

Using the dummy variables s_1 and s_2 we can change the above problem in to

Maximize $\pi = f(x_1, x_2, x_3)$

Subject to $g^1(x_1, x_2, x_3) + s_1 = k_1$

$g^2(x_1, x_2, x_3) + s_2 = k_2$

$x_1, x_2, x_3 \geq 0 \;\&\; s_1, s_2 \geq 0$

We can formulate the Lagrange function using the classical method provided that the non negativity constraints of the choice variables are not existed as

$$L = f(x_1, x_2, x_3) + \lambda_1[k_1 - g^1(x_1, x_2, x_3) - s_1] + \lambda_2[k_2 - g^2(x_1, x_2, x_3) - s_2]$$

It is possible to derive the ***Kuhn- Tucker conditions*** directly from the Lagrange function. Considering the above 3-variable 2-constraints problem

The first order condition is

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial x_3} = \frac{\partial L}{\partial s_1} = \frac{\partial L}{\partial s_2} = \frac{\partial L}{\partial \lambda_2} = \frac{\partial L}{\partial \lambda_1} = 0$$

However, x_j and s_i variable are restricted to be non negative. As a result, the first order conditions on these variables ought to be modified as follows.

$$\frac{\partial L}{\partial x_j} \le 0 \qquad x_j \ge 0 \qquad \text{and } x_j \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial s_i} \le 0 \qquad s_i \ge 0 \qquad \text{and } s_i \frac{\partial L}{\partial s_i} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \qquad \text{Where (i = 1, 2 and j= 1, 2, 3)}$$

However, we can combine the last two lines and thereby avoid the dummy variables in the above first order condition as shown below. As $\frac{\partial L}{\partial s_i} = -\lambda_i$, the second line shows that

$$-\lambda_i \le 0,, \qquad s_i \ge 0 \quad \text{and} \quad -s_i \lambda_i = 0$$

or

$$\lambda_i \ge 0, \qquad s_i \ge 0 \quad \text{and} \quad s_i \lambda_i = 0$$

But, we know that $s_i = k_i - g^i(x_1, x_2, x_3)$. By substituting it in place of s_i, we can get

$$k_i - g^i(x_1, x_2, x_3) \ge 0 \ , \ \lambda_i \ge 0 \text{ and } \lambda_i \ [k_i - g^i(x_1, x_2, x_3)] = 0$$

Therefore, the first order condition without dummy variables in expressed as

$$\frac{\partial L}{\partial x_j} \le 0 \qquad x_j \ge 0 \qquad \text{and } x_j \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = k_i - g^i(x_1, x_2, x_3) \ge 0 \qquad \lambda_i \ge 0 \text{ and } \lambda_i \ [k_i - g^i(x_1, x_2, x_3)] = 0$$

These are the ***Kuhn - tucker conditions*** for the given maximization problem.

How can we solve minimization problem?

One of the methods to solve this problem is changing it in to maximization problem and then applies the same procedure with maximization. Minimizing C is similar to maximizing $(-C)$. However, keep in mined the fact that we have to multiply each constraint inequalities by (-1). We can directly apply the Lagrange multiplier method and determine the minimization version of ***Kuhn - Tucker condition*** instead of converting the inequality constraints into equality constraints using dummy variables as

$$\frac{\partial L}{\partial x_j} \ge 0 \qquad x_j \ge 0 \quad \text{and} \quad x_j \frac{\partial L}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial \lambda_i} \le 0 \qquad \lambda_i \ge 0 \qquad \text{and} \quad \lambda_i \frac{\partial L}{\partial \lambda_i} = 0 \qquad \text{(minimization)}$$

Example

2. Let us check whether the solutions of our example 1 satisfy the Kuhn - Tucker conditions or not

a) Minimize $C = x^{2+} y^2$

Subject to $x\, y \geq 25$

$x, y \geq 0$

The Lagrange function for this problem is

$$L = x^2 + y^2 + \lambda(25 - xy)$$

It is a minimization problem. Therefore, the appropriate conditions are

$$\frac{\partial L}{\partial x} = 2x - \lambda y \geq 0, \qquad x \geq 0 \quad \text{and} \quad x\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = 2y - \lambda x \geq 0, \quad y \geq 0 \quad \text{and} \quad y\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = 25 - xy \leq 0, \quad \lambda \geq 0 \quad \text{and} \quad \lambda\frac{\partial L}{\partial \lambda} = 0$$

Can we determine the non negative value λ which will satisfy all the above conditions together with the optimal solution x and y? The optimal solutions in our earlier discussion are x=5 and y=5, which are nonzero. Thus, the complementary slackness ($x\frac{\partial L}{\partial x} = 0$, $y\frac{\partial L}{\partial y} = 0$) shows that $\frac{\partial L}{\partial x} = 0$ and $\frac{\partial L}{\partial y} = 0$.

Thus, we can determine the value of λ by substituting the optimal values of the choice variables in either of these marginal conditions as

$$\frac{\partial L}{\partial x} = 2x - \lambda y = 0$$

$$2(5) - \lambda(5) = 0$$

$$10 - 5\lambda = 0$$

$$\lambda = 2 > 0$$

This value $\lambda = 2$, x = 5 & y = 5 imply that $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \lambda} = 0$ which fulfils the marginal conditions and the complementary slackness conditions. In other words, all the ***Kuhn - Tucker conditions*** are satisfied.

3. Maximize $Z = 10x - x^2 + 180y - y^2$

Subject to $x + y \leq 80$

$x, y \geq 0$

Solution

First we should formulate the Lagrange function assuming the equality constraint and ignoring the non negativity constraints.

$$L = 10x - x^2 + 180y - y^2 + \lambda(80 - x - y)$$

The first order conditions are

$$\frac{\partial L}{\partial x} = 10 - 2x - \lambda = 0 \Rightarrow \lambda = 10 - 2x \text{ ---------- (1)}$$

$$\frac{\partial L}{\partial y} = 180 - 2y - \lambda = 0 \Rightarrow \lambda = 180 - 2y \text{ ---------- (2)}$$

$$\frac{\partial L}{\partial \lambda} = 80 - x - y = 0 \Rightarrow x + y = 80 \text{ ---------- (3)}$$

Taking equation (1) and (2) simultaneously

$$10 - 2x = 180 - 2y$$

$$2y - 2x = 170$$

$$2y = 170 + 2x$$

$$y = 85 + x \text{ ---------- (4)}$$

If we substitute equation (4) in to (3), we get

$$x + 85 + x = 80$$

$$2x = -5 \Rightarrow x = -2.5$$

However, the value of the choice variables is restricted to be non negative. Thus $x^* = -2.5$ is infeasible. We must set x= 0 since it has to be non negative. Now we can determine the value of y by substituting zero in place of x in equation (3).

$$0 + y = 80$$

$$y^* = 80$$

Therefore, $\lambda^* = 180 - 2(80) = 20$

The possible solutions are $x^* = 0, y^* = 80, \lambda^* = 20$

However, we must check the inequality constraints and the complementary slackness conditions to decide whether these values are solutions or not.

1) Inequality constraints
   i) The non negativity restrictions are satisfied since $x = 0, y = 80, \lambda = 20 \geq 0$
   ii) Inequality constraints

$$x + y \leq 80$$

$$0 + 80 \leq 80$$

2) Complementary Slackness conditions
   i) $x\frac{\partial L}{\partial x} = 0, x = 0 \Rightarrow \frac{\partial L}{\partial x} < 0$ as the problem is maximization.

$$\frac{\partial L}{\partial x} = -10 < 0$$

   ii) $y\frac{\partial L}{\partial y} = 0, y = 80 \neq 0 \Rightarrow \frac{\partial L}{\partial y} = 0$

$$\frac{\partial L}{\partial y} = 180 - 2(80) - 20 = 0$$

iii) $\lambda \frac{\partial L}{\partial \lambda} = 0, \lambda = 20 \neq 0 \Rightarrow \frac{\partial L}{\partial \lambda} = 0$

$\frac{\partial L}{\partial \lambda} = 80 - 0 - 80 = 0$

All the Kuhn Tucker conditions are satisfied. Thus, the objective function is maximized when $x^* = 0, y^* = 80, \lambda = 20$.

Economic Application

Example

4. Given the revenue and cost conditions of a firm as $R = 32x - x^2$ and $C = x^2 + 8x + 4$, where x is output. Suppose the minimum profit is $\pi_0 = 18$. Determine the amount of the output which maximizes revenue with the given minimum profit. In this case, the revenue function is concave and the cost function is convex.

The Problem is

Maximize $R = 32x - x^2$

Subject to $x^2 + 8x + 4 - 32x + x^2 \leq -18$

And $x \geq 0$

Under these situations the Kuhn-Tucker conditions are necessary and sufficient conditions as all of the above three conditions, i.e., (1), (2), 4(3), are satisfied.

The Lagrange function of this problem is

$L = 32x - x^2 + \lambda(-22 - 2x^2 + 24x) - - - - - - - - - - - - - - - - - (1)$

Thus,

$\frac{\partial L}{\partial x} = 32 - 2x - 4\lambda x + 24\lambda = 0 - (2)$

$\frac{\partial L}{\partial \lambda} = -22 - 2x^2 + 24x = 0 - (3)$

From equation (3) $-22 - 2x^2 + 24x = 0$

$2x^2 - 24x + 22 = 0 - (4)$

Solving (4) we get, $x = 1$ or $x = 11$. $\lambda = \frac{-3}{2}$ 0r $\lambda = \frac{1}{2}$

However, we must check the inequality constraints and the complementary slackness conditions to decide whether these values are the solutions or not

$\frac{\partial L}{\partial x} \leq 0, \quad x \geq 0 \quad \text{and} \quad x\frac{\partial L}{\partial x} = 0,$ ----------(5)

$\frac{\partial L}{\partial \lambda} \geq 0, \quad \lambda \geq 0 \quad \text{and} \quad \lambda\frac{\partial L}{\partial \lambda} = 0,$ ----------(6)

At X=1

At this point $x > 0$ this implies that $\frac{\partial L}{\partial x} = 0$. Thus, $\frac{\partial L}{\partial x} = 30 + 20\lambda = 0 \Rightarrow \lambda = \frac{-3}{2}$. It does not satisfy equation (6).

At X=11, $x > 0$ this implies that $\frac{\partial L}{\partial x} = 0, \text{Thus} \frac{\partial L}{\partial x} = 10 - 20\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$. It satisfies both equation (5) and (6). This means, the ***Kuhn-Tucker conditions*** are fulfilled at $x = 11$. Therefore, revenue is maximized when the firm sells $x = 11$ units of output.

Mixed Constraints

An optimization problem with mixed constraints can be reformulated either as maximization or minimization problem. This procedure incorporates the following conditions.

i) Maximizing the objective function $Z(x)$ is equivalent to the problem of Minimizing $-Z(x)$ or vice versa-

ii) The constraint $g(x) \le c$ can be presented as $-g(x) \ge -c$.

iii) The constraint $g(x) \ge c$ is equivalent to the double constraint $-g(x) \le -c$

iv) The non negativity constraint $x \ge 0$ can be denoted by a new constraint $g(x) = -x \le 0$.

Exercise

Solve the following questions based on the information given above.

1. Describe non linear programming ----------

2. Explain the difference between linear programming and nonlinear programming.-------

3. What are the ingredients of non linear programming problems? ----------

4. Write the ***Kuhn - Tucker condition*** of the problem

$$\text{Maximize } U = U(x_1, x_2, x_3, ..., x_n)$$

$$\text{Subject to } \quad p_1x_1 + p_2x_2 + p_3x_3 + ...p_nx \le B$$

$$x_1, x_2, x_3, ...x_n \ge 0$$

When x_i represents goods consumed and p_i represent the respective price of these goods. --------

5. Check whether the ***Kuhn - Tucker conditions*** are satisfied or not for the problem given in example 1 (b) at the optimal values of x and y. ----------

6. Minimize $C = x^2 + y^2$

Subject to $x + y \ge 2$

$x, y \ge 0$

Write out the ***Kuhn - Tucker conditions*** and use them to find the optimal solution by trial and error, what are the values of x and y? ----------

7. Given the demand function of the firm is given as

$P = 12 - x$ And the cost function is $C = \frac{1}{2}x^2$ when the minimum profit is $\pi = 24$, then

Maximize $R = f(x)$

Subject to $C(x) - R(x) \leq -24$

$x \geq 0$

- Is the Kuhn - Tucker condition satisfied or not?
- Determine the value of x using trial and error.

UNIT Three

COMPARATIVE STATICS ANALYSIS

Introduction

The change in equilibrium states in response to a change in exogenous variables leads to a type of analysis known as ***comparative static*** (equilibrium) analysis where as the question of attainability and stability of equilibrium lies on the field of dynamic analysis. But this unit will be emphasized on the nature of comparative static (equilibrium) and its application on economic models (particularly on market and national income models) and its limitations.

First of all you should have good understanding about the meaning of equilibrium in order to know the concept of comparative static analysis.

Equilibrium is "a constellation of selected interrelated variables so adjusted one another that no inherent tendency to change prevails in the model which they constitute."[1]
In this definition there are some words, which require particular consideration. These are,
The word "***selected***" gives emphasis to the reality that there are other variables that have not been incorporated in the model. It is clear that the equilibrium which we are dealing with can be functional based on certain assumptions such as other things being equal. The reason is that, if other variables are included in the model, the existing equilibrium state will not persist.

The word "***interrelated***" indicates that all variables in the model have to be in a state of rest at the same time for achieving equilibrium condition.

The word "***inherent***" shows that this state of rest considered should depend only on the balancing of internal forces of the model. It is supposed that there are no changes in external factors, for a change in these factors will result in a new equilibrium

5.1.Nature of Comparative Static

As its name indicates the term ***comparative static*** is emphasized on the comparison of different equilibrium states, which are related, with different set of values of parameters and exogenous variables. First of all, it is usual to suppose a given initial equilibrium state so as to carry out this comparison. For instance, this initial equilibrium will be indicated by a define price $\overline{P}$ and its corresponding quantity $\overline{Q}$ in a ***closed–market model.***

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In the simple national income model a determinate income $\overline{Y}$ and a corresponding consumption $\overline{C}$ will represent the initial equilibrium. This initial equilibrium will be disturbed provided that there is a change in exogenous variable or certain parameters in the model. Consequently, several endogenous variables have to be subjected to some adjustment. Supposing that some of the new equilibrium state can be defined and attained, the problem faced by comparative static analysis is comparing the new equilibrium with the old one.

We don't pay attention for the process of adjustments in comparative static analysis rather we compare the old equilibrium with the new equilibrium state. In this analysis, we assume that the new equilibrium has to be attainable as we did for the old one.

The nature of comparative static may be either ***qualitative*** or ***quantitative***. The analysis will be qualitative provided that our emphasis is only to determine the direction of the change. On the other hand, quantitative analysis is concerned on determining the magnitude of the change in the endogenous variables resulting from a given change in some parameters or exogenous variables to the model. However, it is clear that we can get the direction of the change from the algebraic sign of the quantitative solution. This means quantitative analysis often incorporate qualitative analysis.

5..2 Differentiation and its Application to Comparative Static Analysis

As we know the main problem in comparative static analysis is finding the rate of change of the dependent variable with respect to a small change in the independent variables. For instance, in a function

$$y = f(x)$$

Where y is dependent variable

x is an independent variable

The rate of change of y with respect to x is denoted by

$$\frac{dy}{dx}$$

I think you have studied about the concept of differentiation and its rules in quantitative method for economists I and in the first unit of this course. Therefore, go to these sources and check what you have studied and discuss with your friends in order to avoid the redundancy.

Have you checked and grasp the concept of differentiation and its rules? Ok. Good. By now, therefore, we have the ability to solve certain problems we face in comparative static analysis such as determining the equilibrium values of the endogenous variables when there is a change in any of the exogenous variables or parameters. Let us apply the mathematical method of derivative on the following economic models.

Market Model

Let us take a simple-one commodity market model to observe the application of techniques of derivative to determine equilibrium quantity and equilibrium price. There are three variables in this model as it is a one commodity model. These are:

- quantity demanded of a good (Q_d)
- quantity supplied of a good (Q_s)
- price of a good (P)

It is clear that we should put assumptions considering the working of the market.

Assumptions

- It is assumed that market equilibrium will be achieved when quantity demanded is equal to quantity supplied, i.e. when the market is cleared.
- Quantity demanded is supposed to be a decreasing linear function of price. This means, as price decreases, quantity demanded of a good increase, *citrus paribus* and vice versa.
- Quantity supplied is assumed to be an increasing function of price, other things being equal. In this case no quantity will be supplied if the price is not greater than a certain positive value.

The model includes:-

- One equilibrium condition.
- Two behavioral equations.

Algebraically, the model can be expressed as

$$Q_d = Q_s$$

$$Q_d = \alpha_1 - \beta_1 P \qquad (\alpha_1,\ \beta_1 > 0) \text{ ---------- (1)}$$

$$Q_s = -\alpha_2 + \beta_2 P \qquad (\alpha_2,\ \beta_2 > 0)$$

All of the four parameters $\alpha_1,\ \alpha_2,\ \beta_1,$ and β_2 are supposed to be positive.

Equating Q_d and Q_s results in

$$Q_d = Q_s = Q$$

And the model can be written as a reduced form of two equations in two variables:

$$\left.\begin{aligned} Q &= \alpha_1 - \beta_1 P \\ Q &= -\alpha_2 + \beta_2 P \end{aligned}\right\} \text{ ---------- (2)}$$

The model can be further reduced to a single equation in a single variable as

$$\alpha_1 - \beta_1 P = -\alpha_2 + \beta_2 P$$

Rearranging this equation gives us

$$\alpha_1 + \alpha_2 = \beta_1 P + \beta_2 P$$

$$\alpha_1 + \alpha_2 = P(\beta_1 + \beta_2)$$

Dividing both sides of the equation by $(\beta_1 + \beta_2)$ gives us the solution value of P, i.e. $\overline{P}$.

$$\overline{P} = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2}\ ,\ \beta_1 + \beta_2 \neq 0 \text{ ---------- (3)}$$

Equation 3 indicates the fact that the determinate value, i.e., equilibrium price, is expressed in terms of the parameters. The equilibrium price, i.e. $\overline{P}$, is positive because we put the restriction that all the four parameters are positive.

To get the equilibrium quantity of the market which corresponds to $\overline{P}$ we should substitute equation (3) in to one of the equations of equation (2). Substituting it in the supply equation gives us

$$\overline{Q} = -\alpha_2 + \beta_2\left(\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2}\right)$$

$$\overline{Q} = -\alpha_2 + \frac{\alpha_1\beta_2+\alpha_2\beta_2}{\beta_1+\beta_2}$$

$$\overline{Q} = \frac{-\alpha_2(\beta_1+\beta_2)+(\alpha_1\beta_2+\alpha_2\beta_2)}{\beta_1+\beta_2}$$

$$= \frac{-\alpha_2\beta_1-\alpha_2\beta_2+\alpha_1\beta_2+\alpha_2\beta_2}{\beta_1+\beta_2}$$

$$\overline{Q} = \frac{\alpha_1\beta_2-\alpha_2\beta_1}{\beta_1+\beta_2}, \quad \beta_1+\beta_2 \neq 0 \text{ ---------- (4)}$$

Similarly in equation (4) we have observed that equilibrium quantity is expressed in terms of the parameters. We know that quantity never be negative. Thus, this situation requires that the numerator $(\alpha_1\beta_2 - \alpha_2\beta_1)$ should be positive. This means $\alpha_1\beta_2 > \alpha_2\beta_1$.

Now we can observe the effect of a small change in one of the parameters either on $\overline{P}$ or $\overline{Q}$. To do this, we must partially differentiate each variable with respect to each of the parameters. Through observing the sign of the partial derivative of the variables, for example $\overline{P}$, with respect to the parameters, we can point out the direction in which $\overline{P}$ will move when the parameter changes. If the magnitude is known, it will constitute quantitative analysis. Let us see the effect of change in the parameters on $\overline{P}$,

$$\frac{\partial\overline{P}}{\partial\alpha_1} = \frac{1}{(\beta_1+\beta_2)}$$

$$\frac{\partial\overline{P}}{\partial\beta_1} = \frac{0(\beta_1+\beta_2)-1(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)^2}$$

$$= \frac{-(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)^2} \text{----------Quotient rule)}$$

$$\frac{\partial\overline{P}}{\partial\alpha_2} = \frac{1}{\beta_1+\beta_2} = \frac{\partial\overline{P}}{\partial\alpha_1}$$

$$\frac{\partial\overline{P}}{\partial\beta_2} = \frac{-(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)^2} = \frac{\partial\overline{P}}{\partial\beta_1}$$

We have put a restriction that all parameters are positive in this model. As a result, we have the ability to conclude that

$$\frac{\partial \overline{P}}{\partial \alpha_1} = \frac{\partial \overline{P}}{\partial \alpha_2} > 0$$

It indicates that as the parameter α_1 increases, there will be an upward shift in the demand curve which leads to an increase in equilibrium price, i.e. $\overline{P}$, given the supply curve. However, the slope of the demand curve is the same. Similarly, $\frac{\partial \overline{P}}{\partial \alpha_2} > 0$ shows that an increase in the parameter α_2 leads to a parallel downward shift in the supply curve which in turn results in an increase in equilibrium price, i.e. $\overline{P}$ given the demand curve. In this case the slope of the supply curve is constant.

In addition to this $\frac{\partial \overline{P}}{\partial \beta_1} = \frac{\partial \overline{P}}{\partial \beta_2} < 0.$

$\frac{\partial \overline{P}}{\partial \beta_1} < 0$ reflects the fact that as β_1 increases, the slope of the demand curve will be steeper and it will rotate to the left (inward) at the point of the intercept. Therefore, the equilibrium price decreases given the supply curve. Similarly, $\frac{\partial \overline{P}}{\partial \beta_2} < 0$ shows that as the slope of the supply curve increases, the original supply curve will be rotated to the left about the point of its intercept which decreases the equilibrium prices given the original demand curve. It is in line with the negative sign of the above derivative.

Have you done the above exercise? Ok that is good. Now try to relate your answer with the following analysis. We know that

$$\overline{Q} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_1 + \beta_2}$$

From this equation one can see the effect of change in each of the parameters using the technique of partial differentiation as follows.

$$\frac{\partial \overline{Q}}{\partial \alpha_1} = \frac{\beta_2}{\beta_1 + \beta_2} > 0$$

$$\frac{\partial \overline{Q}}{\partial \beta_1} = \frac{-\alpha_2(\beta_1 + \beta_2) - 1(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\beta_1 + \beta_2)^2} \text{ ---------- (Quotient rule)}$$

$$= \frac{-\alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_1\beta_2 + \alpha_2\beta_1}{(\beta_1 + \beta_2)^2}$$

$$\frac{\partial \overline{Q}}{\partial \beta_1} = \frac{-(\alpha_2\beta_2 + \alpha_1\beta_2)}{(\beta_1 + \beta_2)^2} < 0$$

$$\frac{\partial \overline{Q}}{\partial \alpha_2} = \frac{-\beta_1}{\beta_1 + \beta_2} < 0$$

$$\frac{\partial \overline{Q}}{\partial \beta_2} = \frac{\alpha_1(\beta_1+\beta_2)-1(\alpha_1\beta_2-\alpha_2\beta_1)}{(\beta_1+\beta_2)^2}$$

$$= \frac{\alpha_1\beta_1+\alpha_1\beta_2-\alpha_1\beta_2+\alpha_2\beta_1}{(\beta_1+\beta_2)^2}$$

$$\frac{\partial \overline{Q}}{\partial \beta_2} = \frac{\alpha_1\beta_1+\alpha_2\beta_1}{(\beta_1+\beta_2)^2} > 0$$

Try to interpreter the above result. ----------

Example

1. Given the demand function Q_d = 24 -2p

The supply function Q_s = - 5 + 7 P

Find the equilibrium price and equilibrium quantity of the commodity. In our example α_1 = 24, $\alpha_2 = 5, \beta_1 = 2$ and $\beta_2 = 7$. Equilibrium is achieved where quantity demanded is equal to quantity supplied of a commodity, i.e.

$Q_d = Q_s = Q$

Therefore, at the point of equilibrium

$$24- 2p= -5 +7p$$
$$24+5 = 7p+2p$$
$$29=9p$$
$$\overline{P} = \frac{29}{9}$$

$\overline{P}$ represents the equilibrium price (market price) of the commodity. Substituting the value of equilibrium price either in the demand or the supply equation gives us the equilibrium quantity. When we substitute it in the demand equation

$$\overline{Q} = 24- 2(\frac{29}{9})$$

$$\overline{Q} = 24 - \frac{58}{9}$$

$$\overline{Q} = \frac{216-58}{9}$$

$$\overline{Q} = \frac{158}{9}$$

What happens to the equilibrium price $\overline{P}$ when α_1 increases from 24 to 25 other things being equal?

It becomes

$$25-2p = -5+7p$$
$$25+5 = 9p$$
$$30= 9p$$
$$\overline{P} = \frac{30}{9}$$

The equilibrium price increases from $\frac{29}{9}$ to $\frac{30}{9}$ when the parameter α_1 increases from 24 to 25 units.

What happens to the equilibrium price, $\overline{P}$ when the parameter β_2 increases from 7 to 8?

Please, try to observe the effect of change in the remaining parameters on equilibrium price and equilibrium quantity of the commodity.

2. Given

$$Q_d = 51 - 3P$$
$$Q_s = 6P - 10$$

Find equilibrium price and quantity of the commodity. ----------

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National Income Model

Now let us consider a national income model with three endogenous variables namely

- national in come (Y)
- consumption (C)
- taxes (T)

The model becomes $Y = C + I_0 + G_0$

$C = a + b(Y - T)$, Where a >0; 0<b<1 ---------- (5)

$T = c + dY$, where c > 0, 0<d<1

As it is shown above, the equilibrium condition of the national income is reflected in the first equation. But the second and the third equations indicate the way to determine consumption (C) and taxes (T) respectively.

In this model, a, b, c and d are parameters. Their values are restricted and they are described as follows.

The parameter a represents ***autonomous consumption***, i.e. consumption without income. Its value is positive. It indicates that the consumer may consume from past saving, family remittance or borrowing with zero income. The parameter ***b*** shows marginal propensity to consume (MPC). It is the rate at which consumption changes when disposable income changes by one.

Algebraically, MPC = b = $\frac{\Delta C}{\Delta(Y - T)}$

Its value lies in between zero and one. MPC = b= 0.6 means that as disposable income increases (decreases) by one unit consumption increases (decreases) by 0.6 units.

The parameter c represents tax revenue without income. It is clear that government can have positive tax revenue from the tax base other than income. As a result c is positive.

The exogenous variables in the model namely investment (I_0) and government expenditure (G_0) are supposed to be nonnegative. Similar to the parameters, these exogenous variables are assumed to be independent of one another.

In this model, it is possible to solve the equilibrium income $\overline{Y}$ in such a way that substituting the third equation of the model in to the second one and then substituting the resulting equation in to the first equation.

The second equation of the mode is

$$C = a + b(Y - T)$$

Substituting the third equation it gives us

$$C = a + b[Y - (c + dY)]$$

$$C = a + bY - bc - bdY$$

Substituting this resulting equation in the first equation of the model, we get

$$Y = a + bY - bc - bdY + I_0 + G_0$$

Rearranging this equation,

$$Y - bY + bdY = a - bc + I_0 + G_0$$

$$Y(1 - b + bd) = a - bc + I_0 + G_0$$

Thus, the solution value of Y (the equilibrium national income, $\overline{Y}$) in reduced form is

$$\overline{Y} = \frac{a - bc + I_0 + G_0}{1 - b + bd} \quad \text{---------- (6)}$$

From this equation we can get six comparative static derivatives. What are they? Please, partially differentiate $\overline{Y}$ with respect to a, b, c ,d, I_0 and G_0, and thereby answer this question .----------

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Have you answered this question? Ok, good. Now try to relate your answer with the following analysis and correct it if there you have done a mistake.

$$\frac{\partial \overline{Y}}{\partial a} = \frac{1}{1 - b + bd}$$

$$\frac{\partial \overline{Y}}{\partial b} = \frac{2(1 - d) - c(1 - b + bd)}{(1 - b + bd)^2}$$

$$\frac{\partial \overline{Y}}{\partial c} = \frac{-b}{1 - b + bd}$$

$$\frac{\partial \overline{Y}}{\partial d} = \frac{-b(a - bc + I_0 + G_0)}{1 - b + bd)^2} = \frac{-b\overline{Y}}{1 - b + bd}$$

$$\frac{\partial \overline{Y}}{\partial I_0} = \frac{1}{1 - b + bd} \qquad \text{(Investment multiplier)}$$

$$\frac{\partial \overline{Y}}{\partial G_0} = \frac{1}{1 - b + bd} \qquad \text{(Government expenditure multiplier)}$$

From these partial derivatives, the following three are very crucial from policy point of view.[2]

$$\frac{\partial \overline{Y}}{\partial c} = \frac{-b}{1-b+bd} < 0 \quad \text{---------- (7)}$$

$$\frac{\partial \overline{Y}}{\partial d} = \frac{-b(a-bc+I_0+G_0)}{(1-b+bd)^2} = \frac{-b\overline{Y}}{1-b+bd} < 0 \text{ ---------- (8)}$$

$$\frac{\partial \overline{Y}}{\partial G_0} = \frac{1}{1-b+bd} > 0 \text{ ---------- (9)}$$

The partial derivative shown in equation (7) above indicates the non- income tax multiplier. This equation explains how the change in government revenue from non- income tax sources affects equilibrium income. It is negative in the model as the denominator is positive and the numerator is negative as shown above.

The partial derivative indicated in equation (8) represents an income-tax rate multiplier which is negative for any positive equilibrium income. As tax rate increases, then equilibrium income decreases.

Finally, equation (9) shows the partial derivative of equilibrium income with respect to government expenditure. It is referred to as government expenditure multiplier. The value is positive which indicates that increase (decrease) in government expenditure increases (decreases) equilibrium national income.

Example

Given the following national income mode,

$Y = C + I_0 + G_0$

$C = 50 + 0.6 (Y-T)$ ---------- (10)

$T = 30 + 0.3Y$

$I_0 = 20$

$G_0 = 15$

Find $\overline{Y}$, $\overline{C}$ and $\overline{T}$

Solution

From the given model, substituting the third equation in to the second equation given C = 50+ 0.6 [Y- (30 + 0.3Y)]

$= 50 + 0.6 (Y-30-0.3Y)$

$= 50+ 0.6 (0.7Y -30)$

$= 50-18+0.42Y$

$\overline{C} = 32 + 0.42Y$ ---------- (11)

Substituting equation (11) in to the first equation of equation 10, and the value of I_0 and G_0 in this equation, we get

[2] Chiang A.C., fundamental Methods of Mathematical Economics, 3rd ed. P.182

$Y = 32 + 0.42Y + 20 + 15$

$Y = 67 + 0.42\,Y$

$Y - 0.42Y = 67$

$0.58Y = 67$

$$\bar{Y} = \frac{67}{0.58}$$

$\bar{Y} = 115.5$ Equilibrium national income

Therefore,

$\bar{C} = 32 + 0.42\,(115.5)$

$= 32 + 48.51$

$\bar{C} = 80.51$ or

We can get the value of $\bar{C}$ by using its reduced form

$$\bar{C} = \frac{a - bc + b(1 - d(1 - d)(I_0 + G_0)}{1 - b + bd}$$

In our example $a = 50$, b = 0.6, c = 30, d = 0.3

$I_0 = 20$, $G_0 = 15$

Thus, $$\bar{C} = \frac{50 - 0.6(30) + 0.6(1 - 0.3)(20 + 15)}{1 - 0.6 + (0.6)(0.3)}$$

$$= \frac{50 - 18 + (0.42)(35)}{0.4 + 0.18}$$

$\bar{C} = 80.52$

Similarly,

$\bar{T} = 30 + 0.3\ \bar{Y}$

$= 30 + 0.3(115.5)$

$= 30 + 34.65$

$\bar{T} = 64.65$

Or

$$\bar{T} = \frac{(c - bc) + d(a + I_0 + G_0)}{1 - b + bd}$$

$$= \frac{30 - (0.6)(30) + 0.3(50 + 20 + 15)}{1 - 0.6 + (0.6)(0.3)}$$

$$= \frac{30 - 18 + 0.3(85)}{0.58}$$

$$= \frac{12 + 25.5}{0.58}$$

$\bar{T} = 64.65$

What happens to equilibrium national income when government expenditure increases by 5 units? What about consumption and taxes at this point?

$$\Delta Y = \Delta G(\frac{1}{0.58}) = \frac{5}{0.58} = 8.62$$

$$\overline{Y} = 115.5 + 8.62 = 124.12$$

It shows that an increase in government expenditure increases equilibrium national income.

Try to determine $\overline{C}$ and $\overline{T}$ at this new level of equilibrium income.

Input - Output Model

I think you have studied about the concept of input-output model in your Quantitative Method for Economists II course. Therefore, please try to revise what you studied from these sources in order to understand the forthcoming discussion properly.

As you know the solution of the open input - output model is described in the form of matrix equation as follows.

$$\overline{X} = (I - A)^{-1} d$$

Where $\overline{X}$ represents the output vector in the model

I - is an identity Matrix

A- Technical coefficient Matrix

(I-A) is Leontief Matrix (technology Matrix)

d- is the final demand vector

Representing the inverse of the technology Matrix $(I - A)^{-1}$ by C= [C_{ij}], the solution of the a simple three industry economy can be described as

$\overline{X}$ = C d

or in matrix form

$$\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Based on the above matrix we can determine the rate at which the solution values (the output of each industry) $\overline{X}_j$ changes with respect to change in the exogenous final demand, d_1, d_2, d_3.

$\frac{\partial \overline{X}_j}{\partial d_k}$ = C_{jk}, where (j, k=1, 2, 3)

To understand this fact, let us multiply the technology matrix with the final demand, and show the solution as

$$\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{bmatrix} = \begin{bmatrix} C_{11}d_1 + C_{12}d_2 + C_{13}d_3 \\ C_{21}d_1 + C_{22}d_2 + C_{23}d_3 \\ C_{31}d_1 + C_{32}d_2 + C_{33}d_3 \end{bmatrix}$$

Partially differentiating this system of three equations gives us a total of nine comparative- static derivatives. These derivatives are

$$\frac{\partial \overline{X}_1}{\partial d_1} = C_{11} \qquad \frac{\partial \overline{X}_1}{\partial d_2} = C_{12} \qquad \frac{\partial \overline{X}_1}{\partial d_3} = C_{13}$$

$$\frac{\partial \overline{X}_2}{\partial d_1} = C_{21} \qquad \frac{\partial \overline{X}_2}{\partial d_2} = C_{22} \qquad \frac{\partial \overline{X}_2}{\partial d_3} = C_{23}$$

$$\frac{\partial \overline{X}_3}{\partial d_1} = C_{31} \qquad \frac{\partial \overline{X}_3}{\partial d_2} = C_{32} \qquad \frac{\partial \overline{X}_3}{\partial d_3} = C_{33}$$

These nine derivatives can be summarized in matrix form as shown below.

$$\frac{\partial \overline{X}}{\partial d} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

It is a compact way of representing all of the comparative static derivatives of our open input - output model. It is clear that comparative static derivatives of the input – output model are beneficial for economic planning as they offer a means of solving the problem of changing the output target of the industries due to a change in the final demand of consumers.

Numerical Example

Suppose a simple economy has three inter-industrial sector such as agriculture, manufacturing, and service, and one final demand sector, i.e., Household sector. It is given that a birr wroth of agricultural product requires 20 cents worth of agriculture, manufacturing and service sectors product each respectively as an input, a birr worth of manufacturing product needs 5 cents, 20 cents and 10 cents worth of agriculture, manufacturing and service sector's output respectively as input and that of a birr worth of service sector's product requires 10 cents, 20 cents and 40 cents worth of agricultural, manufacturing and service sector's output respectively as inputs.

Based on the given above determine the technical coefficient matrix in the order of agriculture, manufacturing and service sector.

$$A = \begin{bmatrix} 0.2 & 0.05 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}$$

Determine the technology matrix

$$(I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.05 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}$$

$$(I-A)=\begin{bmatrix} 0.8 & -0.05 & -0.1 \\ -0.2 & 0.8 & -0.2 \\ -0.2 & -0.1 & 0.6 \end{bmatrix}$$

With this technology matrix, the open input - out put system can be expressed in the form (I- A) X = d as follows

$$\begin{bmatrix} 0.8 & -0.05 & -0.1 \\ -0.2 & 0.8 & -0.2 \\ -0.2 & -0.1 & 0.6 \end{bmatrix}\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \overline{X}_3 \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

By inverting the 3x3 technology matrix (I - A), the solution of the above system can be determined as

$$\overline{X}=(I-A)^{-1}d$$

To put it in matrix form, we ought to find the determinant of the technical coefficient matrix and the inverse of the technology matrix, i.e., $(I-A)^{-1}$.

$$(I-A)^{-1}=\frac{1}{|I-A|}adj.(I-A)$$

$$|I-A|=0.8\begin{vmatrix} 0.8 & -0.2 \\ -0.1 & 0.6 \end{vmatrix}+0.05\begin{vmatrix} -0.2 & -0.2 \\ -0.2 & 0.6 \end{vmatrix}-0.1\begin{vmatrix} -0.2 & 0.8 \\ -0.2 & -0.1 \end{vmatrix}$$

$$|I-A|=0.8(0.48-0.02)+0.05(-0.12-0.04)-0.1(0.02.0.16)$$

$$|I-A|=0.8(0.46)+0.05(-0.16)-0.1(0.18)$$

$$|I-A|=0.368-0.008-0.018$$

$$|I-A|=0.342$$

To get the adj.(I- A), first we should construct the cofactor matrix of (I - A).
The cofactor matrix C of (I - A) is

$$C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Therefore,

$$C_{11} = (-1)^2 \begin{vmatrix} 0.8 & -0.2 \\ -0.1 & 0.6 \end{vmatrix} \qquad C_{12} = (-1)^3 \begin{vmatrix} -0.2 & -0.2 \\ -0.2 & 0.6 \end{vmatrix} \qquad C_{13} = (-1)^4 \begin{vmatrix} -0.2 & 0.8 \\ -0.2 & -0.1 \end{vmatrix}$$

$$C_{11} = 0.48 - 0.02 \qquad C_{12} = (-1)(-0.12 - 0.04) \qquad C_{13} = (0.02 + 0.16$$

$$C_{11} = 0.46 \qquad C_{12} = 0.16 \qquad C_{13} = 0.18$$

$$C_{21} = (-1)^3 \begin{vmatrix} -0.05 & -0.1 \\ -0.1 & 0.6 \end{vmatrix} \qquad C_{22} = (-1)^4 \begin{vmatrix} 0.8 & -0.1 \\ -0.2 & 0.6 \end{vmatrix} \qquad C_{23} = (-1)^5 \begin{vmatrix} 0.8 & -0.05 \\ -0.2 & -0.1 \end{vmatrix}$$

$$C_{21} = (-1)(-0.03 - 0.01) \qquad C_{22} = 0.48 - 0.02 \qquad C_{23} = (-1)(-0.08 - 0.01)$$

$$C_{21} = 0.04 \qquad C_{22} = 0.46 \qquad C_{23} = 0.09$$

$$C_{31} = (-1)^4 \begin{vmatrix} -0.05 & -0.1 \\ 0.8 & -0.2 \end{vmatrix} \qquad C_{32} = (-1)^5 \begin{vmatrix} 0.8 & -0.1 \\ -0.2 & -0.2 \end{vmatrix} \qquad C_{33} = (-1)^6 \begin{vmatrix} 0.8 & -0.05 \\ -0.2 & 0.8 \end{vmatrix}$$

$$C_{31} = (0.01 + 0.08) \qquad C_{32} = (-1)(-0.16 - 0.02) \qquad C_{33} = (0.64 - 0.01)$$

$$C_{31} = 0.09 \qquad C_{32} = 0.18 \qquad C_{33} = 0.43$$

Thus, the cofactor matrix of the technology matrix is

$$C = \begin{bmatrix} 0.46 & 0.16 & 0.18 \\ 0.04 & 0.46 & 0.09 \\ 0.09 & 0.18 & 0.63 \end{bmatrix}$$

And the ad joint matrix is the transpose of the cofactor matrix. As a result, the ad joint matrix of the technology matrix is

$$Adj.(I - A) = C' = \begin{bmatrix} 0.46 & 0.04 & 0.09 \\ 0.16 & 0.46 & 0.18 \\ 0.18 & 0.09 & 0.63 \end{bmatrix}$$

We know that

$$(I - A)^{-1} = \frac{1}{(I - A)}.adj(I - A)$$

Therefore,

$$(I - A)^{-1} = \frac{1}{0.342} \begin{bmatrix} 0.46 & 0.04 & 0.09 \\ 0.16 & 0.46 & 018 \\ 0.18 & 0.09 & 0.63 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.35 & 0.12 & 0.26 \\ 0.42 & 1.35 & 0.53 \\ 0.53 & 0.26 & 1.83 \end{bmatrix}$$

Consequently,

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix} = \begin{bmatrix} 1.35 & 0.12 & 0.26 \\ 0.42 & 1.35 & 0.53 \\ 0.53 & 0.26 & 1.83 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\bar{X}_1 = 1.35d_1 + 0.12d_2 + 0.26d_3$$
$$\bar{X}_2 = 0.42d_1 + 1.35d_2 + 0.53d_3$$
$$\bar{X}_3 = 0.53d_1 + 0.26d_2 + 1.83d_3$$

Determine the nine comparative static derivatives based on the above system of equations ----------

If the final demand for agriculture, manufacturing and service sector's output of this simple economy is valued 40 Birr, 80 Birr and 140 Birr respectively, find the gross output (equilibrium output level) of each sector for the given final demand. Putting it in matrix form

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix} = \begin{bmatrix} 1.35 & 0.12 & 0.26 \\ 0.42 & 1.35 & 0.53 \\ 0.53 & 0.26 & 1.83 \end{bmatrix} \begin{bmatrix} 40 \\ 80 \\ 140 \end{bmatrix}$$

Then, the equilibrium levels of output are

$$\bar{X}_1 = 1.35\ (40) + 0.12(80) + 0.26\ (140)$$
$$= 54 + 9.6 + 36\ .4$$
$$\bar{X}_1 = \mathbf{100}$$
$$\bar{X}_2 = 0.42\ (40) + 1.35\ (80) + 0.53\ (140)$$
$$\bar{X}_2 = 16.8 + 108 = 74.2$$
$$\bar{X}_2 = \mathbf{199}$$
$$\bar{X}_3 = 0.53\ (40) + 0.26\ (80) + 1.83\ (140)$$
$$\bar{X}_3 = 21.\ 2 + 20.8 + 256.2$$
$$\bar{X}_3 = \mathbf{298.2}$$

By now you have completed the first section of this unit. Therefore, try to do the following self - test questions in order to examine how you have understood this section.

Exercise

Try to do the following questions based on the above analysis.

1. What is comparative static analysis? ----------

2. Comparative static analysis may be either qualitative or quantitative in nature. Explain their difference ----------

3.Discuss the different rules of differentiation.----------

4. Given the total cost function $C=Q^3-5Q^2+14Q+75$, write out the variable cost function. Find the derivative of the variable cost function and interpret the economic meaning of that derivative-

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5. Given the demand function $Q_d = 30 - 2p$

the supply function $Q_s = -6+5P$

Find equilibrium price $\overline{P}$ and quantity $\overline{Q}$ ----------

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6. Given the national income model

$$Y= C+ I_0 + G_0$$

$$C = 25+0.6Y$$

$$I_0 = 16$$

$$G_0 = 14$$

Find the equilibrium national income $\overline{Y}$ and consumption $\overline{C}$ ----------

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Have you answered these questions? If your answer is yes, go to the next section. If no, please re-read this section and try to answer these questions.

5.2 Jacobean Determinant, Hessian Determinants and Limitation of Comparative Static Analysis

5.2.1 Jacobean Determinant

A system of equations can have unique solution if there is the same number of equations as unknowns. However, it is not a sufficient condition .In addition to this; the equations are to be consistent and functionally independent. There exists a systematic method for testing whether there is a unique solution or not for a system.

Simultaneous Equation Models

For simultaneous linear equation models, we can use the concept of determinant of a matrix. Given a linear equation system $BX = d$ where B is an n x n coefficient matrix the determinant of the coefficient matrix B, i.e. $|B| \neq 0$, implies that

- there is row (column) independence in matrix B
- B^{-1} exists and
- a unique solution $\bar{x} = B^{-1} d$ exists

This means, the determinant of the matrix enables us to decide whether there is unique solution in a system or not. If the determinant of a matrix vanishes, i.e. zero, a system of the equation does not have a unique solution. If the determinant of a matrix is different from zero, these systems of

the equations can have a unique solution. However, the determinant criterion does not give the algebraic sign of the solution value although we know that there is unique solution.

Let us see the following example in order to make this the analysis clear.

Example

Given the systems of sum ' tenuous linear equations

$$7x_1 - 3x_2 - 3x_3 = 7$$
$$2x_1 + 4x_2 + x_3 = 0$$
$$0\,x_1 - 2x_2 - x_3 = 2$$

Determine whether there is a unique solution or not. To do this, we should use the determinant of the coefficient matrix of the system.

The coefficient matrix of the system is

$$A = \begin{bmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

The determinant of this matrix is

$$|A| = \begin{vmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & 1 \end{vmatrix} = -8 \neq 0$$

Therefore, the equation system possesses unique solution as the determinant does not vanish.

However, the technique of ***partial derivative*** and a special type of determinant known as ***Jacobean determinant*** is needed in order to identify the existence of a unique solution provided that the system of equations are ***non - linear models***. This means, this type of determinant enables us to determine whether there is functional dependency among a set of n - functions in n- variables or not.

If the value of the Jacobian determinant is zero, for all values of $x_1, ..., x_n$, then the functions are (linearly or non - linearly) dependent of one another. In this case, there is no unique solution. On the other hand, the functions are said to independent and there exists a unique solution if the value of the Jacobian determinant is different from zero.

For the sake of understanding this concept, let us consider the following examples.

Example

Given the functions

$$y_1 = 3x_1^2 + 2x_2^2$$

$$y_2 = 5x_1 + 1$$

Partially differentiating these two functions gives us

$$\frac{\partial y_1}{\partial x_1} = 6x_1, \qquad \frac{\partial y_1}{\partial x_2} = 4x_2$$

$$\frac{\partial y_2}{\partial x_1} = 5, \qquad \frac{\partial y_2}{\partial x_2} = 0$$

If we arrange these partial derivatives in to a square matrix in a prescribed order we can get the ***Jacobian matrix***, which is represented by J.

$$\mathrm{J} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

Its determinant is denoted by $|J|$.

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 6X_1 & 4x_2 \\ 5 & 0 \end{vmatrix}$$

$$|J| = 6x_1(0) - 5(4x_2)$$

$$|J| = -20x_2 \neq 0$$

Therefore, the above two functions are functionally independent of each others if $x_2 \neq 0$. Consequently, there is unique solution for this system of equations.

2. Given

$$y_1 = 3x_1^2 + x_2$$

$$y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$$

If we partially differentiate these two functions, we get

$$\frac{\partial y_1}{\partial x_1} = 6x_1, \qquad \frac{\partial y_1}{\partial x_2} = 1$$

$$\frac{\partial y_2}{\partial x_1} = 36x_1^3 + 12x_1x_2 + 48x_1 \ , \qquad \frac{\partial y_2}{\partial x_2} = 6x_1^2 + 2x_2 + 8$$

Putting these partial derivatives in the form of Jacobian matrix

$$\mathrm{J} = \begin{bmatrix} 6x_1 & 1 \\ 36x_1^3 + 12x_1x_2 + 48x_1 & 6x_1^2 + 2x_2 + 8 \end{bmatrix}$$

And its determinant is

$$|J| = \begin{vmatrix} 6x_1 & 1 \\ 36_1^3 + 12x_1x_2 + 48x_1 & 6x_1^2 + 2x_2 + 8 \end{vmatrix}$$

$$|J| = 36x_1^3 + 12x_1x_2 + 48x_1 - (36x_1^3 + 12x_1x_2 + 48x_1$$

$$|J| = 0$$

Therefore, these two functions are dependent of each other. As a result, there is no unique solution for the system of equations.

Try to test the existence of functional dependence between these two functions and the existence of unique solution using the ***Jacobian determinant***.

$$y_1 = 2x_1 + 3x_2$$

$$y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2$$

5.2.2 Hessian Determinant

Do you remember the necessary and sufficient conditions of a function with two or more independent variables to achieve its optimum value?

What are these conditions for a function? Z = *f* (x, y)?

Have you answered this question? Ok, Good. Try to relate your answer with the following analysis.

Given the objective function with two independent variables Z=*f*(x, y), to get a maximum or minimum of Z, it is necessary that the total differential of this function to be zero ($dZ = 0$) for an arbitrary values of dx and dy where both are different from zero

$$dZ = f_x dx + f_y dy$$

Since dx and dy are a very small change in the independent variables ($dx \neq 0$ and $dy \neq 0$), the only situation which leads to a zero dZ is that

$$f_x = f_y = 0$$

This means, the necessary condition for a function to achieve its minimum or maximum value is that both of the first order partial derivatives of the function must be equal to zero.

The sufficient condition to have the extremum value of the function is related with the positive and negative definiteness of the second - order differential of the given function, d^2z.

If d^2Z is ***positive definite***, it is a sufficient condition for a stationary value to be a minimum value of a function whereas ***the negative definiteness*** of d^2Z suffice a stationary value to be a maximum value of the given function. The second order total differential of the above function is

$d^2Z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$ It is in a quadratic form.

d^2Z is positive definite iff $f_{xx} > 0$ and $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

d^2Z is negative definite iff $f_{xx} < 0$ and $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

The determinant $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ is referred to as the second order Hessian determinant, $|H_2|$ or simply Hessian.

And f_{xx} is the first order Hessian determinant, i.e. $|H_1|$. According to the Young's theorem, $f_{xy} = f_{yx}$.

Example

Find the extreme value of the function

$$Z = x^2 + xy + 2y^2 + 3$$

To solve this problem let us first find the first order partial derivatives and the second order partial derivatives of the function. These are

$$\frac{\partial Z}{\partial x} = f_x = 2x + y \ , \quad \frac{\partial Z}{\partial y} = f_y = x + 4y$$

$$f_{xx} = 2 \ , f_{yy} = 4 \ , f_{xy} = f_{yx} = 1$$

Setting the first order partial derivates equal to zero gives us

$$2x + y = 0 \text{---------- (12)}$$
$$x+4y = 0 \text{---------- (13)}$$

Taking these two equations simultaneously, multiplying equation (13) by 2 and deducting it from equation (12) enables us to find the value of x and y.

$$2\begin{cases} 2x + y = 0 \\ \underline{x+4y = 0} \end{cases}$$

$$2x + y = 0$$
$$\underline{2x + 8y = 0}$$
$$0-7y = 0$$
$$y = 0$$

Substituting zero in place of y in either of the two equations, we can find the value of x

$2x + 0 = 0$

$$x = 0$$

Thus, the critical points are at x = 0 and y = 0. Now let us see the second order partial derivates at the critical point. $f_{xx} = 2$, $f_{yy} = 4$, $f_{xy} = f_{yx} = 1$. They are independent of the critical points. Arranging the second order partial derivates in the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

The determinant of this matrix is

$$|H| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 2(4) - (1)^2 = 7 > 0$$

At x = 0 and y = 0, $f_{xx} = 2 > 0$ and $(f_{xx})(f_{yy}) > (f_{yx})^2 = 7 > 0$ which implies that d^2Z is positive definite. Thus, the function achieves its minimum value at x = 0 and y = 0 and the minimum value is

$$Z = (0)^2 + 0(0) + 2(0)^2 + 3$$
$$Z = 3$$

If the given function is with three independent (choice) variables,

$$Z = f(x_1, x_2, x_3)$$

The necessary and sufficient conditions that should be satisfied to achieve its extreme value are similar to that of a function with two choice variables. That is the necessary condition is which

$$f_1 = f_2 = f_3 = 0$$

The sufficient condition is related with the positive and negative definiteness of the second order total differential of the function.

$$d^2z = \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3 =$$

$$\frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1 + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2 + \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3$$

$$= f_{11}dx_1^2 + f_{12}dx_1dx_2 + f_{13}dx_1dx_3 + f_{21}dx_1dx_2 + f_{22}dx_2^2 + f_{23}dx_2dx_3 +$$
$$f_{31}dx_3dx_1 + f_{32}dx_3dx_2 + f_{33}dx_3^2$$

In order to determine the positive and / or negative definiteness of d^2Z we should construct the Hessian matrix and then its determinant based on the second order partial derivatives of the function as follows.

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{32} & f_{32} & f_{33} \end{bmatrix}$$

Then, the Hessian determinant is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

The successive principal minors of this determinant are

$$|H_1| = f_{11} \qquad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad |H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = |H|$$

The second-order sufficient condition for an optimum value of the given function can be described as

- Z is at its maximum point if and only if $|H_1| < 0, |H_2| > 0, |H_3| < 0$ (in this case d^2Z is negative definite).
- Z is at its minimum if and only if $|H_1| > 0, |H_2| > 0, |H_3| > 0$ (d^2Z is positive definite).

Example

Find the extremum values of

$$Z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$$

The first order condition for extremum constitutes

$$f_1 = 2x_1 - 3x_2 = 0 \text{----------(14)}$$
$$f_2 = -3x_1 + 6x_2 + 4x_3 = 0 \text{---------(15)}$$
$$f_3 = 4x_2 + 12x_3 = 0 \text{----------(16)}$$

The determinant of the coefficient of these three linear equations is

$$|A| = \begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$$
$$|A| = 2(36-16) + 3(-36) = -68 \neq 0$$

It is different from zero which implies that all of the equations are independent to each other. As a result, there exists a single solution

$$x_1 = x_2 = x_3 = 0$$

From equation (14), we got

$$2x_1 = 3x_2$$

$$\boxed{x_1 = \frac{3}{2}x_2}$$

From equation (16), we get

$$4x_2 + 12x_3 = 0$$
$$4x_2 = -12x_3$$

$$\boxed{x_3 = \frac{-1}{3}x_2}$$

Substituting these two results in equation (15) gives us

$$(-3(\frac{3}{2}x_2) + 6x_2 + 4(\frac{-1}{3}x_2) = 0$$

$$\frac{-9}{2}x_2 + 6x_2 - \frac{4}{3}x_2 = 0$$

$$\frac{-27x_2 + 36x_2 - 8x_2}{6} = 0$$

$$\frac{-35x_2 + 36x_2 = 0}{6}$$

$$-35x_2 + 36x_2 = 0$$

$$x_2 = 0$$

This means,

$$x_1 = \frac{3}{2}(0) = 0$$

$$x_3 = \frac{-1}{3}(0) = 0$$

Thus the critical values are $x_1 = x_2 = x_3 = 0$.

As you know the sufficient condition is determined by using the Hessian determinant. Thus, the Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$$

The successive principal minors are

$$|H_1| = 2\ , \quad |H_2| = \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 12 - 9 = 3 \quad |H_3| = |H| = 2\ (72\ -16) + 3\ (-36) = 4$$

The result indicates that all of the principal minors are greater than zero. This implies that $d^2 Z$ is ***positive definite***. Thus, the function achieves its minimum value at $x_1 = x_2 = x_3 = 0$ and the minimum value is

$$Z = 0^2 + 3(0)^2 - 3(0)(0) + 4(0)(0) + 6(0)^2$$

$$Z = 0$$

For functions of n-choice variables such as

$$Z = f(x_1, x_2, x_3, x_4, ...x_n)$$

The necessary condition for the extremum of this function is that all of the n-first order partial derivatives must be equal to zero. The second order condition can be described by using the sign of ***Hessian determinant*** as we have discussed in unit three.

Bordered Hessian Determinant, $|\bar{H}|$

We have discussed the use of the Hessian determinant in order to expresses the second order sufficient condition of achieving free extremum. We can also use a determinant form of expressing this condition for the problem of constrained optimization. In this case, we will apply what is known as the bordered Hessian determinant, $|\bar{H}|$ rather than the Hessian determinant, $|H|$ .The bar on the top of the bordered Hessian indicates the border.

Given the function $Z = f(x, y)$ Subject to $g(x, y) = c$, the bordered Hessian determinant is

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix}$$

We can determine the sign definiteness of d^2Z using $|\bar{H}|$ as follows:

d^2Z is positive definite subject to $dg = 0$ iff $|\bar{H}| < 0$ (Z is minimized). d^2Z is negative definite subject to $dg = 0$ iff $|\bar{H}| > 0$ (Z is maximized). Given the stationary values of Z= f(x, y) + λ [c- g (x, y)], if d^2Z is positive definite, it is sufficient for a function Z to achieve its relative minimum where as a negative definite d^2Z is a sufficient condition to satisfy the relative maximum of the objective function.

Example

Suppose a consumer uses two goods with utility function $U = xy + x$ subject to the budget constraint 6x +2y =110. Determine the amount of x and y which maximize utility and check the second order condition using the ***bordered Hessian determinant***.

Solution

Formulating the Lagrange function

$$L = xy + x + \lambda(110 - 6x - 2y) \quad \text{.....(1)}$$

The first order conditions are

$$Lx = y + 1 - 6\lambda = 0 \quad \text{.....(2)}$$

$$Ly = x - 2\lambda = 0 \quad \text{.....(3)}$$

$$L\lambda = 110 - 6x - 2y = 0 \quad \text{.....(4)}$$

The matrix form is

$$\begin{vmatrix} 0 & 1 & -6 \\ 1 & 0 & -2 \\ -6 & -2 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ \lambda \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ -110 \end{vmatrix}$$

Using ***Cramer's rule***

The determinant of the coefficient matrix, i.e., $|A|$ is 24.

$$|A_1| = \begin{vmatrix} -1 & 1 & -6 \\ 0 & 0 & -2 \\ -110 & -2 & 0 \end{vmatrix} = 224, \quad |A_2| = \begin{vmatrix} 0 & 1 & -6 \\ 1 & 0 & -2 \\ -6 & -110 & 0 \end{vmatrix} = 648$$

$$|A_3| = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -6 & -2 & -110 \end{vmatrix} = 112$$

Therefore, $x = \dfrac{|A_1|}{|A|} = \dfrac{224}{24}, \quad y = \dfrac{|A_2|}{|A|} = \dfrac{648}{24} = 27$

$$\lambda = \frac{|A_3|}{|A|} = \frac{112}{24}$$

Taking the second partial of L with respect to x and y and the first order partial derivative of the constraint function with respect to x and y, we can form Hessian bordered as follows.

$$L_{xx} = 0, \quad L_{yy} = 0, \qquad L_{xy} = L_{yx} = 1 \quad \text{(Young's theorem)}$$

$$g_x = 6, \qquad g_y = 2$$

Thus, $\left|\overline{H}\right| = \begin{vmatrix} 0 & 6 & 2 \\ 6 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -6(-2) + 2(6) = 24 > 0$

Thus d^2L is negative definite and total utility is maximized when

$$x = \frac{224}{4} \quad and \quad y = 27$$

5. 2. 3 Limitations of Comparative Static Analysis

We have discussed that ***comparative static analysis*** is concerned with the analysis of different equilibrium points. Comparative static is helpful in finding how the disequilibrating change in a parameter or exogenous variable will affect the equilibrium state of the model. However, comparative static's has the following shortcomings.

- It neglects the process of adjustment from the old equilibrium to the new equilibrium
- It ignores the time element contained in the adjustment process
- It assumes that the new equilibrium can be defined and attained after a disequilibrating change in the exogenous variable or a parameter.

By now you have completed that last section of this Unit. Therefore, try to do the following question to evaluate how you have understood the main point in this section.

Exercise

Do the following questions based on the information given above.

1. Describe what the ***Jacobian determinant*** is ----------

----------2. Given the following functions determine whether there is functional dependence or not using Jacobian determinant.

a) $Y_1 = 2x_1 + 6x_2$

$Y_2 = -x_1 + 5x_2$

b) $Y_1 = 5x_1 - 3x_2$

$Y_2 = -2x_1 + 4x_2$

3. If the total cost function of the firm is $C = 5Q_1^2 + 2Q_1Q_2 + 3Q_2^2 + 800$ and a production quota represented by $Q_1 + Q_2 = 39$ is imposed on a firm.

Minimize $C = 5Q_1^2 + 2Q_1Q_2 + 3Q_2^2 + 800$

Subject to $Q_1 + Q_2 = 39$

Check the second order sufficient condition using the bordered ***Hessian determinant***.

4. Suppose the demand functions of the product sold in two separate markets are given as

$Q_1 = 24 - 0.2P_1; Q_2 = 10 - 0.05P_2$

$C = 35 + 40Q, where, Q = Q_1 + Q_2$

Find the output levels which maximize profit and check the second order sufficient condition using the Hessian determinant----------

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Have you done these questions? If your answer in no reread this section and try to do them. If yes, go to the next section..



Answers for Self Test Questions

Self Test 5.1

4) $TVC = Q^3 - 5Q^2 + 14Q, \frac{dTVC}{dQ} = 3Q^2 - 10Q + 14$

5) $\overline{P} = \frac{36}{7}, \overline{Q} = \frac{138}{7}$

6) $\overline{Y} = 137.5 - C = 107.\overline{5}$

Self Test 5.2

2) a. $|J| = 16 > 0$ b. $|J| = 14 > 0$ Thus, the functions are linearly independent in both cases.

3) $Q_1 = 13, Q_2 = 26, \lambda = 182$ $|H| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 10 & 2 \\ 1 & 2 & 6 \end{vmatrix}$, $|H_2| = 56 > 0; |H_3| = |H| = -12 < 0$

4) $Q_1 = 8, Q_2 = 4$ $|H| = \begin{vmatrix} -10 & 0 \\ 0 & -40 \end{vmatrix} = 400 > 0, |H_1| = -10 < 0$ The second order condition is satisfied.

CHAPTER FOUR

DIFFERENTIAL EQAUATIONS

4.1. Partial Derivatives

In many problems involving functions of two variables, the goal is to find the rate of change of the function with respect to one of its variables when the other is held constant. That is, the goal is to differentiate the function with respect to the particular variable in question while keeping the other variable fixed. This process is known as partial differentiation, and the resulting derivative is said to be a partial derivative of the function.

For example, suppose a manufacturer finds that $Q(x, y) = 5x^2 + 7xy$ units of a certain commodity will be produced when x skilled workers and y unskilled workers are employed. Then if the number of unskilled workers remains fixed, the production rate with respect to the number of skilled workers is found by differentiating Q (x, y) with respect to x while holding y constant. We call this the partial derivative of Q with respect to x and denote it by $Q_x(x, y)$; thus, $Q_x(x, y) = 10x + 7y$

Similarly, if the number of skilled workers remains fixed, the production rate with respect to the number of unskilled workers is given by the partial derivative of Q with respect to y, which is obtained by differentiating $Q(x, y)$ with respect to y, holding x constant; that is, by $Q_y(x, y) = 7x$

Partial Derivatives: Suppose $z = f(x, y)$. The partial derivative of z with respect to x is denoted by $\frac{\partial z}{\partial x}$ or $f_x(x, y)$ and is the function obtained by differentiating z with respect to x treating
y as a constant. The partial derivative of z with respect to y is denoted by $\frac{\partial z}{\partial y}$ or $f_y(x, y)$ and is the function obtained by differentiating z with respect to y treating x as a constant.

Example

1. Find the partial derivatives f_x and f_y if $f(x,y) = x^2 + 2xy^2 + \frac{2y}{3x}$, to simplify thecomputation, begin by rewriting the function as $f(x,y) = x^2 + 2xy^2 + \frac{2yx^{-1}}{3}$

To compute f_x, think of f as a function of x and differentiate the sum term by term, treating y as a constant to get $f_x(x,y) = 2x + 2y^2 - \frac{2y}{3x^2}$. To compute f_y, think of f as a function of y and differentiate term by term, treating x as a constant to get $f_y(x,y) = 4xy + \frac{2}{3x}$

2. Find the partial derivatives f_x and f_y if $f(x,y) = xe^{-2xy}$

Solution

From the product rule, $f_x(x,y) = e^{-2xy} + x(-2ye^{-2xy}) = e^{-2xy}(1 + -2xy)$ and from the constant multiple rule, $f_y(x,y) = x(-2xe^{-2xy}) = -2x^2e^{-2xy}$.

4.1.1 Higher Order Partial Derivatives

Partial derivatives can themselves be differentiated. The resulting functions are called second-order partial derivatives.Here is a summary of the definition and notation for the four possible second-order partial derivatives of a function of two variables.

Second-Order Partial Derivatives: If $z = f(x,y)$, the partial derivative of f_x with respect to x is $f_{xx} = (f_x)$ or $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$
The partial derivative of f_x with respect to y is $f_{xy} = (f_x)_y$ or $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$
The partial derivative of f_y with respect to x is $f_{yx} = (f_y)_x$ or $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$
The partial derivative of f_y with respect to y is $f_{yx} = (f_y)_y$ or $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

Examples

Compute the four second-order partial derivatives of the function

$f(x,y) = xy^3 + 5xy^2 + 2x + 1$

Solution

Since $f_x = y^3 + 5y^2 + 2$ it follows that $f_{xx} = 0$ and $f_{xy} = 3y^2 + 10y$

Since $f_y = 3xy^2 + 10xy$ it follows that $f_{yy} = 6xy + 10x$ and $f_{yx} = 3y^2 + 10y$

Note that $f_{xy} = f_{yx}$. This means you will get the same answer if you first differentiate f(x, y) with respect to x and then differentiate the resulting function with respect to y as you would if you performed the differentiation in the reverse order.

4.2 Differentials and total derivatives

Given a function $y = f(x)$, a specific Δx will call forth a corresponding Δy, and we can use the difference quotient $\frac{\Delta y}{\Delta x}$ to represent the rate of change of y with respect to x. Since it is true that $\Delta y = \left(\frac{\Delta y}{\Delta x}\right)\Delta x$. The magnitude of Δy can be found, once the rate of change $\frac{\Delta y}{\Delta x}$ and the variation in x are known.When Δx is infinitesimal, Δy will also be infinitesimal, and the difference quotient $\frac{\Delta y}{\Delta x}$ will turn into the derivative $\frac{dy}{dx}$. Thus, $\Delta y = \left(\frac{\Delta y}{\Delta x}\right)\Delta x = \left(\frac{dy}{dx}\right)dx = dy$.

Symbols of dy and dx are called differential of y and x, respectively.

Example

Given $y = 3x^2 + 7x - 5$, find dy.

The derivative of the function is $\frac{dy}{dx} = 6x + 7$; thus, the desired differential $dy = (6x + 7)dx$.

4.2.1 Total differentials

The concept of differentials can easily be extended to a function of two or more independent variables. Consider a saving function $S = f(Y, i)$ where S is savings, Y is national income and i is interest rate. The partial derivative $S_Y(Y, i)$ measures the rate of change of S with respect to

an infinitesimal change in Y (or it signifies the marginal propensity to save). As a result, the change in S due to change in Y may be represented by $\left(\frac{\partial S}{\partial Y}\right)dY$. By the same token, the change in S due to infinitesimal change in i can be denoted as $\left(\frac{\partial S}{\partial i}\right)di$. The change in S will be then equal to $dS = \left(\frac{\partial S}{\partial Y}\right)dY + \left(\frac{\partial S}{\partial i}\right)di$. The expression dS, being the same of the changes from both sources, is called the total differential of the saving function. The more general case of a function of n independent variables can be exemplified by, say, a utility function in the general form

$U = f(x_1, x_2, x_3, ..., x_n)$

The total differential of this function can be written as

$$dU = \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \frac{\partial U}{\partial x_3}dx_3 + ... + \frac{\partial U}{\partial x_n}dx_n$$

Example

Find the total differential, given:

a. $z = 3x^2 + xy - 2y^3$ b. $y = \frac{2x_1x_2}{x_1 + x_2}$

Solution

a. $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \Rightarrow dz = (6x + y)dx + (x - 6y^2)dy$

b. $dy = \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2$. Applying the quotient rule gives us,

$$dy = \left(\frac{2x_2(x_1 + x_2) - 2x_1x_2(1)}{(x_1 + x_2)^2}\right)dx_1 + \left(\frac{2x_1(x_1 + x_2) - 2x_1x_2(1)}{(x_1 + x_2)^2}\right)dx_2$$

Rules of Differentials

Let k be a constant and u and v be two functions of the variables x_1 *and* x_2. Then the following rules are valid:

Rule I: $dk = 0$ (constant function rule)

Rule II: $d(cu^n) = cnu^{n-1}$ (power function rule)

Rule III: $d(u \pm v) = du \pm dv$ (sum-difference rule)

Rule IV: $d(uv) = vdu + udv$ (product rule)

Rule V: $d\left(\frac{u}{v}\right) = \frac{1}{v^2}(vdu - udv)$ (quotient rule)

Example 1 Find the total differential dy of the function

$y = 5x_1^2 + 3x_2$

The straightforward method calls for the evaluation of the partial derivatives $f_1 = 10x_1$ and $f_2 = 3$, which will then enable us to write $dy = f_1 dx_1 + f_2 dx_2 = 10x1dx_1 + 3dx_2$.

Example 2 Find the total differential of the function

$y = 3x_1^2 + x_1x_2^2$

Since $f_1 = 6x_1 + x_2^2$ and $f_2 = 2x_1x_2$, the desired differential is

$dy = (6x_1 + x_2^2)dx_1 + (2x_1x_2)dx_2$

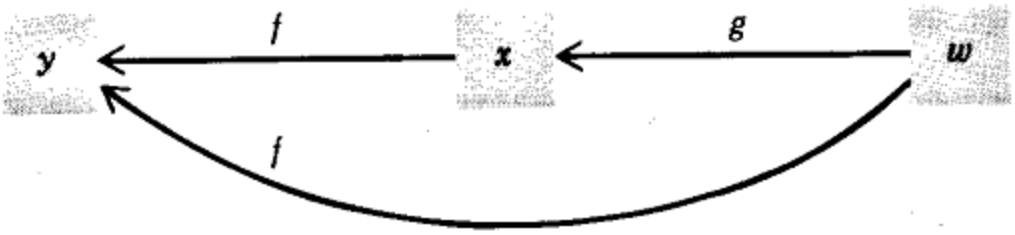
Exercise: Find the total differential of the following functions

a. $y = \frac{x_1 + x_2}{2x_1^2}$ **c.** $y = \frac{x_1}{x_1 + x_2}$

b. $z = 3x^2 + xy - 4y^3$ **d.** $y = \frac{2x_1x_2}{x_1 + x_2}$

Total Derivatives

How to find the rate of change of $f(x, w)$ with respect to w, when x and w are related? The answer for this question lies on the concept of total derivatives. Unlike a partial derivative, a total derivative does not require the argument x to remain constant as w varies, and can thus allow for the postulated relationship between the two variables.



Finding the total derivatives

To carry discussion in a more general framework, let us consider any function $y = f(x, w)$ where $x = g(w)$ With three variables $y, x,$ and w related to one another as in Figure above. In this figure, it is clearly seen that w can affect y through two channels: (1) indirectly, via the function g and then f (the straight arrows), and (2) directly, via the function f (the curved arrow). Whereas the partial derivative f_w is adequate for expressing the direct effect alone, a total derivative is needed to express both effects jointly.

To obtain this total derivative, we first differentiate y totally, to get the total differential $dy = f_x dx + f_w dw$. When both sides of this equation is divided by the differential dw, the result is

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw} = \underbrace{\frac{\partial y}{\partial x}\frac{dx}{dw}}_{\substack{\textit{measures}\text{ the indirect}\\ \text{effect of w on y}}} + \underbrace{\frac{\partial y}{\partial w}}_{\substack{\textit{measures}\text{ the direct}\\ \text{effect of w on y}}} \qquad \left[\frac{dw}{dw} = 1\right]$$

The process of finding the total derivative $\frac{dy}{dw}$ is referred to as total differentiation of y with respect to w.

Examples

Find the total derivative $\frac{dy}{dw}$, given the function

$y = f(x, w) = 3x - w^2$ where $x = g(w) = 2w^2 + w + 4$

$$y = f(x,w) = 3x - w^2 \quad \text{where} \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$$

Solution

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw} \Rightarrow \frac{dy}{dw} = 3(4w+1) + (-2w) = 10w + 3$$

The situation of total derivative is only slightly more complicated when we have

$y = f(x,w) = 3x - w^2 \quad \text{where} \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases}$. This time, variable w can affect y through three channels: (1) indirectly, via the function g and then f, (2) again indirectly, via the function h and then f, and (3) directly via f.

Then the total derivative $\frac{dy}{dw}$, is given by:

$$\frac{dy}{dw} = f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw}$$

Example

Let the production function be $Q = Q(K,L,t)$ where, aside from the two inputs K and L, there is the third variable t, denoting time. The presence of the t argument indicates that the production function can shift over time in reflection of technological changes. Since capital and labor, too, can change over time, we may write $K = K(t) \qquad and \qquad L(t)$

Then the rate of change of output with respect to time can be expressed, in line with the total derivative formula as $\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t}$

For example, find $\frac{dz}{dt}$, given: $z = x^2 - 8xy - y^3,$ where $x = 3t$ and $y = 1 - t$

Solution

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_t \Rightarrow \frac{dz}{dt} = (2x - 8y)(3) + (-8x - 3y^2)(-1) + \frac{\partial Q}{\partial t}$$

4.3 Partial Derivative of Implicit Functions

If the equation $F(y, x_1, x_2, ..., x_m) = 0$ can be solved for y, we can explicitly write out the function $y = f(x_1, x_2, ..., x_m)$, and find its derivatives by the methods learned before. But what if $F(y, x_1, x_2, ..., x_m) = 0$, cannot be solved for y explicitly? In this case, if under the terms of the implicit-function theorem an implicit function is known to exist, we can still obtain the desired derivatives without having to solve for y first. The concept of total differentials can also enable us to find the derivatives of implicit functions.

The total differential of the function $F(y, x_1, x_2, ..., x_m) = 0$ is given by

$$F_y dy + F_1 dx_1 + F_2 dx_2 + ... + F_m dx_m = 0$$

Suppose that only y and x_1 are allowed to vary (only dy and dx_1 are not set equal to zero). Then, the above equation reduces to for $F_y dy + F_1 dx_1 = 0$. Upon dividing through by dx_1, and solving for $\frac{dy}{dx_1}$, we get $\left.\frac{dy}{dx_1}\right|_{\text{other variables constatnt}} \equiv \frac{\partial y}{\partial x_1} = -\frac{F_x}{F_y}$

By similar means, we can derive all other partial derivatives of implicit function f. Generally, given $F(y, x_1, x_2, ..., x_m) = 0$, if an implicit function $y = f(x_1, x_2, ..., x_m)$ exists, then the partial derivatives of f are

$$\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y} \qquad (i = 1, 2, 3, ..., m)$$

Example 1: Find $\frac{dy}{dx}$ for the function $F(x, y) = y - 3x^4$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = 12x^3$$

Example 2: Find $\frac{dy}{dx}$ for the function $F(x, y) = x^2 + y^2 - 9$.

$F_x = 2x$ and $F_y = 2y \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ (y ≠ 0)

Example 3: Find $\frac{\partial y}{\partial x}$ for the function $F(x, y, w) = y^3x^2 + w^2 + yxw - 3 = 0$.

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}.$$

CHAPTER FIVE

DIFFERENCE EQUATIONS

5. 1 First-Order-Linear Differential Equations: Continuous Time

Definitions and Concepts

A differential equation is an equation which expresses an explicit or implicit relationship between a function $y = f(t)$ and one or more of its derivatives or differentials. Examples of differential equations include

$$\frac{dy}{dt} = 5t + 7 \qquad y' = 12y \qquad \text{and} \quad y'' - 4y' + 9 = 0$$

Equations involving a single independent variable, such as those above, are called ordinary differential equation (ODE). The solution or integral of a differential equation is any equation, without derivative or differential, which is defined over an interval and satisfies the differential equation for all the values of the independent variable(s) in the interval.

Example: Solve $y''(t) = 7$

Solution: to solve this

$$\int y''(t) = y'(t) = \int 7dt = 7t + c_1$$

$$\int y'(t) = y(t) = \int (7t + c_1)\, dt = \frac{7}{2}t^2 + c_1 t + c$$

This is called a general solution which indicates that when c is unspecified, a differential equation has an indefinite number of possible solutions. If c can be specified, the differential equation has a particular or definite solution which alone of all possible solutions is relevant.

The order of a differential equation is the order of the highest derivative in the equation. The degree of a differential equation is the highest power to which the derivative of the highest order is raised.

Example

1. $\frac{dy}{dt} = 2x + 10$ first-order, first degree
2. $\left(\frac{dy}{dt}\right)^3 - 4t^5 = 0$ first-order, fourth degree
3. $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^4 + x^3 = 0$ second-order, first degree
4. $\left(\frac{d^2y}{dt^2}\right)^9 + \left(\frac{d^3y}{dt^3}\right)^4 = 60y$ third-order, fouth degree

General Formula for First-Order Linear Differential Equations

For a first-order linear differential equation, $\frac{dy}{dt}$ and y must be of the first degree, and no product of $y\left(\frac{dy}{dt}\right)$ may occur. For such an equation

$$\frac{dy}{dt} + ay = b$$

where a and b may be constants or functions of time, the formula for a general solution is

$y(t) = Ae^{-at}$ if $b = 0$ [General solution]

or

$y(t) = y(0)e^{-at}$ [specific solution] (a)

$y(t) = Ae^{-at} + \frac{b}{a}$ if $b \neq 0$ & $a \neq 0$ [General solution]

or

$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$ [specific solution] (b)

where A is an arbitrary constant. A solution (b) is composed of two parts: Ae^{-at} is called the *complementary function*, and $\frac{b}{a}$ is called *particular integral*. The particular integral y_p equals the inter-temporal equilibrium level of $y(t)$; the complementary function y_c represents the deviation

from the equilibrium. For $y(t)$ to be dynamically stable, y_c must approach zero as t infinity (that is, a in e^{-at} must be negative).

Example: Find the general solution for the following differential equations

a. $\frac{dy}{dt} + 4y = 12$

b. $\frac{dy}{dt} + 3t^2 y = t^2$

c. $\frac{dy}{dt} + 2y = 6; y(0) = 10$

d. $\frac{dy}{dt} = 2; y(0) = 5$

e. $\frac{dy}{dt} - 2y = 0; y(0) = 9$

f. $2\frac{dy}{dt} + 4y = 6; y(0) = 4$

Solution

a. Since $a = 4$ and $b = 12$, substituting in (a) gives

$$y(t) = Ae^{-4t} + \frac{12}{4} = e^{-4t} + 3$$

As $t \to \infty, y_c = Ae^{-4t} \to 0$ and $y(t)$ approaches $y_p = 3$, the inter-temporal equilibrium level implying that $y(t)$ is dynamically stable.

b. In the equation, $a = 3t^2$ and $b = t^2$. Substituting it into the general solution formula, we get

$$y(t) = Ae^{-(3t^2)t} + \frac{t^2}{3t^2} = e^{-3t^3} + \frac{1}{3} \qquad \text{(time path of } y(t)\text{)}$$

As $t \to \infty, y_p = Ae^{-t^3} \to 0$ and $y_c \to \frac{1}{3}$. The equilibrium is dynamically stable.

c. Substituting $a = 2, b = 6$ and $y_0 = 10$ into the specific formula, we get

$$y(t) = \left[10 - \frac{6}{2}\right]e^{-2t} + \frac{6}{2} = 7e^{-2t} + 3$$

As $t \to \infty, y_p = 7e^{-2t} \to 0$ and $y_c \to 3$. The equilibrium is dynamically stable.

d. The general solution when $a = 0$ is given by $y(t) = A + bt$ and the specific solution is $y(t) = y_0 + bt$. Accordingly, solution for a differential equation under question is $y(t) = 5 + 2t$.

e. $y(t) = 9e^{2t}$ and it is not dynamically stable.

f. $2\frac{dy}{dt} + 4y = 6 \Rightarrow \frac{dy}{dt} + 2y = 3$. Solving the equation yields

$$y(t) = \left[4 - \frac{3}{2}\right]e^{-2t} + \frac{3}{2} = \frac{5}{2}e^{-2t} + \frac{3}{2}$$

As $t \to \infty, y_p = \frac{5}{2}e^{-2t} \to 0$ and $y_c \to 3/2$. The equilibrium is dynamically stable.

5.2 First-Order-Linear Difference Equations: Discrete Time
Definitions and Concepts

A difference equation expresses a relationship between a dependent variable and a lagged independent variable (or variables) which changes at discrete intervals of time, for example, $I_t = f\left(Y_{t-1}\right)$, where I and Y are measured at the end of each year. The order of a difference equation is determined by the greatest number of periods lagged. A first-order difference equation expresses a time lag of one period; a second-order, two periods; etc. the change in y as t changes from t to $t+1$ is called the first difference of y. It is written as

$$\frac{\Delta y}{\Delta t} = \Delta y_t = y_{t+1} - y_t$$

where Δ is an operator replacing $\frac{dy}{dt}$ that is used in to measure continuous change in differential equations. The solution of difference equation defines y for every value of t and does not contain a difference equations.

Example: Each of the following is a difference equation of the order indicated.

$I_t = a(Y_{t-1} - Y_{t-2})$ order two

$Q_s = a + bp_{t-1}$ order one

$y_{t+3} - 9y_{t+2} + 2y_{t+1} + 6y_t = 8$ order three

$\Delta y_t = y_t$ oder one

$\Rightarrow y_{t+1} - y_t = y_t \Rightarrow y_{t+1} = 2y_t$

Given that the initial value of y is y_0, in the difference equation

$$y_{t+1} = by_t \qquad \text{(a)}$$

the solution is found as follows:

By successive substitutions of $t = 0,1,2,3,$ etc. in (a), we get

$y_1 = by_0$ $\qquad$ $y_3 = by_2 = b(b^2 y_0) = b^3 y_0$

$y_2 = by_1 = b(by_0) = b^2 y_0$ $\qquad$ $y_4 = by_3 = b(b^3 y_0) = b^4 y$

Thus, for any period t,

$y_t = b^t y_0$.

Given a first-order difference equation which is linear (i.e., all the variables are raised to the first power and there are no cross products),

$$y_t = by_{t-1} + a$$

where a and b are constants, the general formula for a definite solution is

$$y_t = \left(y_0 - \frac{a}{1-b}\right)b^t + \frac{a}{1-b} \quad \text{where } b \neq 1 \qquad \text{(b)}$$

$$y_t = y_0 + at \qquad \text{where } b = 1 \qquad \text{(c)}$$

If no initial condition is given, an arbitrary constant A is used for $y_0 - \frac{a}{1-b}$ in (b) and for y_0 in (b). This is called a **general solution**.

The stability condition of depends on the base b. If $|b| > 1$, the time path will explode and move farther and farther away from equilibrium; if $|b| < 1$, the time path will be damped and move toward equilibrium. If $b < 0$, the time path will oscillate between positive and negative values; if $b > 0$, the time path will be non-oscillating.

Example: Solve the difference equations given below

a. $y_t = -7y_{t-1} + 16$ and $y_0 = 5.$

b. $y_t = 6y_{t-1}$

c. $x_t + 3x_{t-1} + 8 = 0$ and $y_0 = 16$

d. $y_t = \frac{-1}{4} y_{t-1} + 60$ and $y_0 = 8$

Solution

a. In the equation, $b = -7$ and $a = 16$. Since $b \neq 1$, it is solved by using (b), as follows:

$y_t = \left(5 - \frac{16}{1+7}\right)(-7)^t + \frac{16}{1+7} = 3(-7)^t + 2$ (the time path is oscillating and unstable)

b. Since $a = 0$ and $b = 6$, we have $y_t = (y_0 - 0)(6)^t + 0 = A(6)^t$ where A, as generally used unspecified constant, replaces y_0.

c. Rearranging $x_t + 3x_{t-1} + 8 = 0$ to get $x_t = -3x_{t-1} - 8$. Thus, $b = -3$ *and* $a = -8$ and substituting in the formula, we obtain

$y_t = \left(16 + \frac{16}{1+3}\right)(-3)^t - \frac{8}{1+3} = 18(-3)^t - 2$ (the time path is oscillating and unstable)

d. $y_t = \left(8 - \frac{60}{1+\frac{1}{4}}\right)\left(-\frac{1}{4}\right)^t + \frac{60}{1+\frac{1}{4}} = -40\left(-\frac{1}{4}\right)^t + 48$ (the time path is oscillating and stable)

5.3 Economic Applications of Linear Differential and Difference Equations

5.3.1 Economic Applications of Linear Differential Equations

Differential equations serve many functions in economics. They are used to determine the conditions for dynamic stability in microeconomic models of market equilibria and to trace the time path growth under various conditions in macroeconomics. Given the growth rate of a function, differential equations enable the economist to find the function whose growth is described; from pint elasticity, they enable the economist to estimate the demand function.

Example: Dynamics of Market Price

Suppose that, for a particular commodity, the demand and supply functions are as follows

a

At equilibrium, $\alpha - \beta p = -\gamma + \delta p \Rightarrow \alpha + \gamma = (\beta + \delta) p$

$\bar{p} = \frac{\alpha + \gamma}{\beta + \delta}$ (= some positive constant)

If $p(0) = \bar{p}$, the market will clearly be already, and no dynamic analysis will be needed. However, if $p(0) \neq \bar{p}$, $\bar{p}$ is attainable only after a due process of adjustment, during which not only price change over time but Q_d and Q_s, being functions of P, must change over time as well. In this regard, the price and quantity variables can all be taken as functions of time.

The rate of changein price isa function of excess demand. That is, if demand is larger than what isrequired by suppliers, market price increases and if it is lower than whatis needed by suppliers price will decrease. Hence, without any complexity, we can model the rate of change in price as follows.

$\frac{dp}{dt} = m(Q_d - Q_s)$ where $m > 0$ & represents adjustment coefficient

$\frac{dp}{dt} = m(\alpha - \beta p + \gamma - \delta p) = m(\alpha + \gamma) - m(\beta + \delta) p$

$\frac{dp}{dt} + m(\beta + \delta) p = m(\alpha + \gamma)$ (similar with differential equation presented before)

Applying the solution formula to the above equation, the time path of price is give as

$$p(t) = \left[p(0) - \frac{\alpha + \gamma}{\beta + \delta} \right] e^{-m(\beta+\delta)t} + \frac{\alpha + \gamma}{\beta + \delta}$$
$$= [p_0 - \bar{p}] e^{-at} + \bar{p} \qquad (a = m(\beta + \delta))$$

Note that the model is dynamically stable if $a = m(\beta + \delta) > 0$.

5.3. 2 Economic Applications First-Order Linear Difference Equations

1. Lagged Income Determination Model

Assume that consumption is a function of the previous period's income so that

$C_t = C_0 + cY_{t-1}$ $\quad Y_t = C_t + I_t$ where $I_t = I_0$

Thus, $Y_t = C_0 + cY_{t-1} + I_0$ and rearranging it, we obtain

$\Rightarrow Y_t = cY_{t-1} + C_0 + I_0$ where $b = c$ and $a = C_0 + I_0$. Substituting these values in the solution formula, since the marginal propensity to consume c cannot be equal 1, and assuming $Y_t = Y_0$ at $t = 0$,

$$Y_t = \left[Y_0 - \frac{C_0 + I_0}{1 - c} \right] c^t + \frac{C_0 + I_0}{1 - c}$$

Since $0 < MPC < 1, |c| < 1$ the time path will converge. The equilibrium is stable, and as $t \to \infty, Y_t \to \frac{C_0 + I_0}{1-c}$, which is the inter-temporal equilibrium level of income.

Example: Given $Y_t = C_t + I_t, C_t = 200 + 0.9Y_{t-1}, I_t = 100$, and $Y_0 = 4500$. Find time path of Y_t. Solving for Y_t,

$$Y_t = 200 + 0.9Y_{t-1} + 100 = 0.9Y_{t-1} + 300$$

$$Y_t = \left[4500 - \frac{300}{1-0.9}\right](0.9)^t + \frac{300}{1-0.9} = 1500(0.9)^t + 3000 \quad \text{(dynamically stable path)}$$

2. The Cobweb Model

For many products, such as agricultural commodities, which are planted a year before marketing, current supply depends on last year's price. This poses interesting stability question. If

$Q_{dt} = \alpha + \beta p_t$ and $Q_{st} = \gamma + \delta p_{t-1}$

at equilibrium, $\alpha + \beta p_t = \gamma + \delta p_{t-1} \Rightarrow \beta p_t = \delta p_{t-1} + \gamma - \alpha$

$$p_t = \frac{\delta}{\beta} p_{t-1} + \frac{\gamma - \alpha}{\beta}$$

Since $\beta < 0$ and $\delta > 0$ under normal demand and supply conditions, $\frac{\delta}{\beta} \neq 1$. Hence,

$$p_t = \left[p_0 - \frac{(\gamma - \alpha)/\beta}{1 - \frac{\delta}{\beta}} \right]\left(\frac{\delta}{\beta}\right)^t + \frac{(\gamma - \alpha)/\beta}{1 - \frac{\delta}{\beta}}$$

$$= \left[p_0 - \frac{\gamma - \alpha}{\beta - \delta} \right]\left(\frac{\delta}{\beta}\right)^t + \frac{\gamma - \alpha}{\beta - \delta}$$

When the model is in equilibrium, $p_t = p_{t-1}$. Substituting p_e for p_t & p_{t-1} in $p_t = \frac{\delta}{\beta} p_{t-1} + \frac{\gamma - \alpha}{\beta}$, we get

$p_e = \frac{\gamma - \alpha}{\beta - \delta}$ and substituting it in solution for p_t,

$$p_t = [p_0 - p_e]\left(\frac{\delta}{\beta}\right)^t + p_e$$

For the equilibrium to be stable, $\left|\frac{\delta}{\beta}\right| < 1$ which implies that $|\delta| < |\beta|$. In other words, the supply curve must be flatter than the demand curve for stability.

Example: Given $Q_{dt} = 86 - 0.8 p_t$ and $Q_{st} = -10 + 0.2 p_{t-1}$, the market price p_t for any time period and the equilibrium price p_e can be found as follows. At equilibrium,

$$86 - 0.8 p_t = -10 + 0.2 p_{t-1} \Rightarrow p_t = -0.25 p_{t-1} + 120$$

Thus, $p_t = \left[p_0 - \frac{120}{1 + 0.25} \right](-0.25)^t + \frac{120}{1 + 0.25} = (p_0 - 96)(-0.25)^t + 96$ & $p_e = 96$

Since $b = -0.25$, which is negative and less than 1, the time path oscillates and converges. The equilibrium is stable, and p_t will converge to $p_e = 96$ as $t \to \infty$.

Exercise

Given demand and supply for the cobweb as follows, find the inter-temporal equilibrium price, and determine whether the equilibrium is stable:

a. $Q_{dt} = 18 - 3 p_t$ and $Q_{st} = -3 + 4 p_{t-1}$

b. $Q_{dt} = 22 - 3 p_t$ and $Q_{st} = -2 + p_{t-1}$

c. $Q_{dt} = 19 - 6 p_t$ and $Q_{st} = -5 + 6 p_{t-1}$

UNIT SIX

DYNAMIC EQUILIBRIUM ANALYSIS

Introduction

So far we have discussed about comparative static analysis which is emphasized on comparison of different equilibrium situations. For instance, in the theory of supply and demand analysis, an increase in demand for a product increases equilibrium price and quantity, other things being equal. However, nothing is said about the process by which price and quantity are being adjusted to the new equilibrium values. The study of the process of adjustment over time is the concern of dynamic analysis. In this unit you will deal about discrete time and continuous time dynamic analysis and their economic application. In the former case there will be a change in variable only once with in a period of time where as in the later case there may be a change in variables at each point of time.

6.1 Continuous Time Dynamic Analysis

Differential equation is an equation that shows an implicit or explicit relationship between the function $y = f(t)$ and one or more of its derivatives. For instance,

$\frac{dy}{dt}$ = 6t + 10, where t represents time

$\frac{dy}{dt}$ = 15 y, $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 19 = 0$

The highest power attained by the derivative in the equation is known as ***degree of the differential equation*** where as the order of a differential equation is the order of the highest derivative in the equation.

Example

a) $\frac{dy}{dt} = 3x + 7$ (first degree, first order)

b) $\frac{d^3y}{dt^3} + (\frac{dy}{dt})^4 + 2x^2 = 0$ (third order, first degree)

6.1.1. First -Order Linear Differential Equations

In this case, if the derivative, $\frac{dy}{dt}$ and y have to be in first degree and if no product of $y(\frac{dy}{dt})$ exist, then the equation is referred to as ***first order linear differential equation***. This equation will generally take a form

$$\frac{dy}{dt} + vy = z \quad \text{or} \quad \frac{dy}{dt} + v(t)\, y = z(t) \text{ ---------- (1)}$$

Where v and z are constants or functions of time, t, like y. When v and z are constants and z is zero, then equation (1) will be

$$\frac{dy}{dt} + \alpha y = 0 \text{ ---------- (2)}$$

Where α is constant which represents v. Such type of differential equation is referred to as ***homogenous*** as each term in the equation is in first degree in terms of y and $\frac{dy}{dt}$.

Rearranging equation (2), we get

$$\frac{1}{y}\frac{dy}{dt} = -\alpha \text{ ---------- (3)}$$

Now we can obtain the solution of equation (3) by integrating both sides of the equation with respect to t. By substitution rule and log rule, the left side of the equation gives us

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int \frac{dy}{y} = \ln|y| + c_1, \qquad (y \neq 0)$$

The right side gives us

$$\int -\alpha dt = -\alpha t + c_2, \text{ (as } -\alpha \text{ is constant)}$$

By equating the above two integrals and combining the two constants, we get

$$\ell n|y| = -\alpha t + C$$

Using the method of antilogarithm

$$e^{\ell n|y|} = e^{(-\alpha t + C)}$$

$$|y| = e^{(-\alpha t}e^{c}$$

$$y(t) = Ae^{-\alpha t}, \text{ where } A = e^{c} \text{ ---------- (4)}$$

There is an arbitrary constant A in equation (4). Therefore, it is a ***general solution***.

If we substitute a particular value for A, then it will be a ***definite solution***. We have seen two situations in the solution of differential equation. These are.

- The solution is given in function form, not in number.
- The solution is free of any derivative. As result, we can determine the value of y if a specific value is assigned for t.

When we put a nonzero constant β in equation (2) in place of z, we get the equation,

$$\frac{dy}{dt} + \alpha y = \beta, \text{ where } \beta \text{ is constant ---------- (5)}$$

Such type of equation is known as ***non-homogenous linear differential equation***. Here, the problem is how can we determine the solution? The solution of this equation will contain two different terms. One is ***complementary function*** represented by y_c and the other is ***particular integral*** represented by y_p as we will see it later.

Let us observe the solution of the homogenous equation even if we are dealing about the way to solve non homogenous equation. For its convenient let us consider the homogenous equation as reduced equation and the non homogenous one as complete equation. Thus, the solution of the reduced equation represents y_c and y_p represents particular solution of the complete equation. This means,

$y_c = Ae^{-\alpha t}$, from equation (4)

We have said that y_p denotes a particular solution of the complete function. Thus, let us first try to obtain the simplest type of solution taking y as some constant k. Then, $\frac{dy}{dt} = 0$ when y is constant. As a result equation (5)

$$0 + \alpha y = \beta$$

$$y_p = \frac{\beta}{\alpha}, \alpha \neq 0$$

Then the general solution of the complete function is

$$y(t) = y_c + y_p = Ae^{-\alpha t} + \frac{\beta}{\alpha}, (\alpha \neq 0) \text{ ---------- } (6)$$

$$y(0) = Ae^{-\alpha(0)} + \frac{\beta}{\alpha}$$

When t = 0, $y(0) = A + \frac{\beta}{\alpha}$

$$A = y(0) - \frac{\beta}{\alpha}$$

Thus, $y(t) = [y(0) - \frac{\beta}{\alpha}]e^{-\alpha t} + \frac{\beta}{\alpha}$, (definite solution, $\alpha \neq 0$) ---------- (7)

Example

1. Find the definite solution of the equation $\frac{dy}{dt}$ +4y=12, y(0)=0. In this case, $\alpha = 4, \beta = 12$, thus

$$y(t) = [0 - 3]e^{-4t} + 3$$

$$y(t) = -3e^{4t} + 3$$

2. Determine the general solution of the equation $3\frac{dy}{dt}$ + 6y = 5

Dividing both sides by 3, we get the equation

$$\frac{dy}{dt} + 2y = \frac{5}{3}$$

$$y(t) = Ae^{-2t} + \frac{5}{6}$$

6.1.2 Economic Applications

Differential equations are beneficial to determine the dynamic stability condition of the market equilibrium. When the rate of growth of a function is given, differential equation enables us to get the function whose growth is explained.

Example

1. Assume that the demand and supply functions of a particular product are given as

$$Q_d = a - bP, (a, b, > 0) - (8)$$

$$Q_s = -c + dP, (c, d > 0)$$

As we remember from comparative static analysis, equilibrium Price, $\overline{P}$ is

$$\overline{P} = \frac{a+c}{b+d} \quad \text{---------- (9)}$$

Assuming that the rate of change of price in the market, $\frac{dP}{dt}$ is a positive linear function of excess demand, $Q_d - Q_S > 0$,

$$\frac{dP}{dt} = k(Q_d - Q_s) \ , \ k \quad \text{is constant, } k > 0 \text{ ---------- (10)}$$

Given sufficient time for adjustment process, under what condition $P(t)$ converges to the equilibrium price $\overline{P}$ as t $\rightarrow \infty$ ----------?

We know that $\frac{dp}{dt} = 0$ if and only if $Q_d = Q_S$

Substituting the given parameters for Q_d and Q_s in equation (10)

$$\frac{dP}{dt} = k[(a - bP) - (-c + dP)] = k(a + c) - k(b + d)P$$

$$\frac{dP}{dt} + k(b + d)P = k(a + c) - - - - - - - - - - - - - - - - - - (11)$$

We have seen that equation (11) is expressed in the form of differential equation. As the coefficient P is different form zero, it is possible to apply the solution formula and present the solution as

$$P(t) = [P(0) - \frac{(a+c)}{(b+d)}]e^{-k(b+d)t} + \frac{(a+b)}{(b+d)}$$

$$P(t) = [P(0) - \overline{P}]e^{-k(b+d)t} + \overline{P} - - - - - - - - - - - - - - - - - (12)$$

As P (0) and $\overline{P}$ are constants whether P (t) $\rightarrow \overline{P}$ as t $\rightarrow \infty$ or not depends on the exponential expression $e^{-k(b+d)t}$. If k (b+ d) > 0, the exponential expression approaches to zero as t approaches

to infinity. As a result, the time path will really direct the price towards equilibrium position on the assumption of our model. In this case, the equilibrium is refereed to as **dynamically stable**.

2. If $\hat{C}, \hat{I} and \hat{Y}$ are deviations of consumption, investment and income from their respective equilibrium values $\overline{C}$, $\bar{I}$ and $\overline{Y}$, determine the ***dynamic stability*** conditions for the two sector income determination model.

$$\hat{C} = C(t) - \overline{C}, \hat{Y} = Y(t) - \overline{Y} \text{ and } \hat{I} = I(t) - \bar{I}$$

Income changes at a rate proportional to excess demand $(C + I - Y)$ and $\hat{C} = g\hat{Y}(t), \hat{I}(t) = b\hat{Y}(t)$

, $\frac{d\hat{Y}(t)}{dt} = a(\hat{C} + \hat{I} - \hat{Y}),\ 0 < a, b, g < 1$

Using substitution, $\frac{d\hat{Y}}{dt} = a(g\hat{Y} + b\hat{Y} - \hat{Y}) = a(g + b - 1)\hat{Y}$

$$\frac{d\hat{Y}}{\hat{Y}} = a(g + b - 1)dt \text{ ----------} (13)$$

When we integrate this function, we get

$$\ln \hat{Y} = a(g + b - 1)t + C \text{ ----------} (14)$$

Using the method of antilogarithm

$$e^{\ell n\, \hat{y} =} \ell^{a(g+b-1)t+c}$$

$$\hat{Y} = e^{c} e^{a(g+b-1)t}$$

$$\hat{Y} = ke^{a(g+b-1)t}, if, k = e^{c}$$

As t = 0, ŷ = y (0) - ŷ = k

As a result, $\hat{Y} = [Y(0) - \overline{Y})]e^{a(g+b-1)t}$. However, $\hat{Y} = Y(t) - \overline{Y}, Y(t) = \hat{Y} + \overline{Y}$

Thus, $Y(t) = [Y(0) - \overline{Y}]e^{a(g+b-1)t} + \overline{Y}$ ----------(15)

This solution shows that as $t \to \propto, Y(t) \to \overline{Y}$ if and only if $g + b < 1$.

3. It is clear that a change in the rate of investment will affect both aggregate demand and productive capacity of the economy. The Domar model wants to determine the time path along which an economy can grow while maintaining full utilization of its productive capacity. Given the marginal propensity to save s and the marginal capital- output ratio k as constant, determine the desired investment function for the growth to be needed.

As we know change in aggregate demand is equal to change in investment multiplied by the multiplier $\frac{1}{s}$.

$$\frac{dy}{dt} = \frac{dI}{dt}(\frac{1}{s}) \text{ ----------} (16)$$

Change in productive capacity is equal to the change in capital stock multiplied by the reciprocal of marginal capital- output ratio,

$$\frac{dQ}{dt} = \frac{1}{k}\frac{dK}{dt} = \frac{1}{k}\text{I}, \quad \text{as } \frac{dK}{dt} = \text{I} \text{ ----------} (17)$$

Equating equation (16) and (17), gives us

$$\frac{1}{s}\left(\frac{dI}{dt}\right) = \frac{1}{k}\mathrm{I}$$

$$\frac{1}{s}dI = \frac{1}{k}Idt \text{ ---------- (18)}$$

Rearranging this equation, we get

$$\frac{dI}{I} - \frac{s}{k}dt = 0 \text{ ---------- (19)}$$

Applying the method of integration on equation (19), it becomes

$$\ln I - \frac{s}{k}t = C$$

$$\ln I = \frac{s}{k}t + C$$

Using antilogarithm

$$e^{\ln I} = e^{[(\frac{s}{k})t+c]}$$

$$I(t) = e^{c}e^{(\frac{s}{k})t}$$

$$I(t) = Ae^{(\frac{s}{k})t}, if, A = e^{c} \text{ ----------(20)}$$

Equation (20) is the desired investment function for the growth to be needed which indicates that investment has to grow at a constant rate determined by $\frac{s}{k}$ (saving rate divided by capital output ratio).

6.1.3 Second - Order Linear Differential Equations

Functions of the form $\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2y = b$ where a a_1, a_2, and b are constants is referred to as ***second-order linear differential equations***. ---------- (21)

If the tem b is zero, the given function is ***homogenous***. However, if b is different from zero, the equation is known as non homogenous equation.

The solution of the given equation has two components. These are complementary function (y_c) and particular integral (y_p).

$$y(t) = y_c + y_p$$

Similar to the previous analysis y_c represents the deviation of y (t) from equilibrium for every point of time path where as y_p denotes the equilibrium value of the variable y in intertemporal sense.

Particular integral (y_p)

Let us try the simplest possible way, i.e., setting y = a, constant, to find the particular integral y_p.

- If $y = a$ which is constant, then the above second order linear differential equation becomes $\frac{dy}{dt} = \frac{d^{2y}}{dt^2} = 0$, As a result, the equation $\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2 y = b$ will be reduced to $a_2 y = b$ which gives us $y = \frac{b}{a_2}$.

Thus, the particular integral is $y_p = \frac{b}{a_2}$, $(a_2 \neq 0)$ ----------(22)

- If $a_2 = 0$, we have to find some non constant form of a solution. Setting $y = kt$ which is the simplest possibility, the second order differential equation (21) is reduced to $a_1 k = b$ .This gives us the value of $k = \frac{b}{a_1}$ which is the particular integral .Then the particular integral is

  $y_p = \frac{b}{a_1}t, a_1 \neq 0, a_2 = 0$, ---------- (23)

- If $a_1 = 0$, the solution with a form $y = kt$ fails since it is undefined. consequently, we have to find other solution of the form $y = kt^2$ when $a_1 = a_2 = 0$, then equation (21) is reduced to $\frac{d^2y}{dt^2} = b$ ----------(24)

$y = kt^2$ Implies that, $\frac{dy}{dt} = 2kt, \frac{d^2y}{dt^2} = 2k$. Then the differential equation indicated in (24) can be presented as 2k = b which implies that $k = \frac{b}{2}$ and then the particular integral is

$$y_p = \frac{b}{2}t^2, (a_1 = a_2 = 0) \text{----------} (25)$$

Example

Find the particular integral of the equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 2$$

Solution: $y_p = \frac{2}{5}$

If $a_2 = 0$, $y_p = \frac{-2}{2}t = -t$ (using equation (23)). But, if $a_1 = 0, y_p = \frac{2}{2}t^2 = t^2$.

The complementary function (y_c)

The complementary function of equation (21) denotes the general solution of the equation

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2 y = 0 \text{ ----------} (26)$$

As we have discussed in equation (4), $y = e^{rt}$. Adopting this solution, $\frac{dy}{dt} = rAe^{rt}$ and

$$\frac{d^2y}{dt^2} = r^2Ae^{rt}$$

Substituting these terms in equation (26), we get

$$r^2Ae^{rt} + a_1rAe^{rt} + a_2Ae^{rt} = 0$$

$$Ae^{rt}(r^2 + a_1r + a_2) = 0 \text{ ----------} (27)$$

Equation (27) shows that $A e^{rt} = 0$ or $(r^2 + a_1r + a_2) = 0$

We know that the value of A is determined by the use of the initial condition of the problem. Consequently, we cannot set $A e^{rt} = 0$. Thus, it is necessary to observe the value of r which satisfies the equation $(r^2 + a_1r + a_2) = 0$. This equation is referred to as ***characteristic equation*** of the homogenous function expressed in equation (26). This characteristic equation has a form of quadratic equation which will result in two solutions (roots). These roots are known as ***characteristic roots.*** Using quadratic formula,

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \text{ ----------} (28)$$

r_1 and r_2 are the only values we can put for r in the solution $y = A e^{rt}$. Since there are two roots r_1 and r_2, then we will have two solutions. These are,

$y_1 = A_1e^{r_1t}$ and $y_2 = A_2e^{r_2t}$, where A_1 and A_2 are constants and r_1 and r_2 are characteristic roots. The general solution is the summation of the above two solutions. This means,

$$y_c = y_1 + y_2 \text{ ----------} (29)$$

The characteristic equation may have three different types of roots, namely distinct real roots, reaped real roots and complex roots.

Case 1 When $a_1^2 > 4a_2$, the square root in equation (28) will be a real number and the two roots r_1 and r_2 will be distinct real numbers. In this case $y_c = y_1 + y_2 = A_1e^{r_1t} + A_2e^{r_2t}$, $(r_1 \neq r_2)$ ---------- (30)

Case 2 When $a_1^2 = 4a_2$, the term in the square root of equation (28) becomes zero. As a result the two roots will be equal. The complementary function will be

$$y_c = (A_1 + A_2)e^{rt}$$

Case 3 When $a_1^2 < 4a_2$ the term in the square root of equation (28) becomes negative number which is referred to as ***imaginary number***. For the time being let us ignored this case.

For the first two cases it is possible to determine the dynamic stability of equilibrium condition based on the sign of the characteristic roots. The condition (i.e., $y_c \to 0$ as $t \to \infty$) is fulfilled iff both r_1 and r_2 have negative sign.

Example

Find y_c, y_p, the general solution and the definite solution of the equation

$$y''(t) + 9y'(t) + 14y = 7 \text{ ----------(32)}$$

Since the value of a_2 is different from zero, $y_p = \frac{1}{2}$

To determine the complementary function y_c, we should first find the value of r_1 and r_2

$$r_1,\ r_2 = \frac{-9 \pm \sqrt{81-56}}{2} = \frac{-9 \pm 5}{2},\ r_1 = -2,\ r_2 = -7$$

We know that $y_c = y_1 + y_2 = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ ---------- (33)

Subsisting the value of r_1 and r_2 in equation (33) gives us

$$y_c = A_1 e^{-2t} + A_2 e^{-7t}$$

Therefore, the general solution of the equation is

$$y(t) = A_1 e^{-2t} + A_2 e^{-7t} + \frac{1}{2} \text{ ----------(34)}$$

When y (0) $= \frac{5}{2}$ and $y'(0) = 6$, then the definite solution becomes,

When t = 0, $y(0) = A_1 + A_2 + \frac{1}{2} = \frac{5}{2}$

$$A_1 + A_2 = 2 \text{ ----------(35)}$$

$$y'(0) = -2A_1 e^{-2t} - 7A_2 e^{-7t} = -2A_1 - 7A_2$$

$$y'(0) = -2A_1 - 7A_2 = 6$$

$$2A_1 + 7A_2 = -6 \text{ ----------(36)}$$

Taking equation (35) and (36) simultaneously, multiplying (35) by 2 and deduct it from equation (36) gives $A_2 = -2$. Substituting the value of A_2 in equation (35), we get

$$A_1 - 2 = 2$$

$$A_1 = 4$$

Therefore, $y(t) = 4e^{-2t} + (-2)e^{-7t} + \frac{1}{2}$ is the definite solution when y (0)$= \frac{5}{2}$ and $y'(0) = 6$

6.1.4 Economic Applications

It is clear that future price expectation affects supply and demand decisions. The price trend existing at the moment in continuous time context can be found in $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$. Therefore, we ought to incorporate these two derivatives in the demand and supply functions as additional components to consider the price trend in our analysis.

Example

1.Suppose the demand and supply functions of a product are given as

$$Q_d = 42 - 4P - 4\frac{dP}{dt} + \frac{d^2P}{dt^2}$$

$$Q_s = -6 + 8P \text{ ----------(37)}$$

With initial condition $P(0) = 6$ and $P'(0) = 4$. Assuming market clearance at every point of time, find the time path $P(t)$.

As we know market equilibrium is achieved when $Q_d = Q_s$. Thus, at the point of equilibrium

$$42 - 4P - 4\frac{dP}{dt} + \frac{d^2P}{dt^2} = -6 + 8P \text{ ----------(38)}$$

Rearranging this equation, we get

$\frac{d^2P}{dt^2}$ - $4\frac{dP}{dt}$-12P = -48 which is second order differential equation in form. Thus, $P(t) = P_p + P_c$ where P_p represents the particular integral and P_c denotes the complementary function. Thus, to determine the general solution $P(t)$, we should find P_p and P_c.In this equation, $a_1 = -4, a_2 = -12, b = -48$, since $a_2 \neq 0$, then $P_p = \frac{-48}{-12} = 4$.

To determine, we should find r_1 and r_2. Using the formula presented in equation (28),

$$r_1, r_2 = \frac{4 \pm \sqrt{16 + 48}}{2} = \frac{4 \pm 8}{2}$$

$r_1, r_2 = 6, -2$, i.e., $r_1 = 6$ and $r_2 = -2$

This means $P_c = A_1 e^{6t} + A_2 e^{-2t}$

Therefore, the general solution is

$$P(t) = A_1 e^{6t} + A_2 e^{-2t} + 4 \text{ ----------(39)}$$

Considering the initial conditions p (0) = 6 and $P'(0) = 4$,

$$P(0) = A_1 + A_2 + 4 = 6$$

$$A_1 + A_2 = 2 \text{ ----------(40)}$$

$$P'(0) = 6A_1 e^0 + (-2)A_2 e^0 = 4$$

$$6A_1 - 2A_2 = 4 \text{ ----------(41)}$$

Taking equation (40) and (41) simultaneously, multiplying equation (40) by 2 and adding it on (41), we get $A_1 = 1$ and $A_2 = 1$. Therefore, the definite solution is

$$P(t) = e^{6t} + e^{-2t} + 4$$

Since $r_1 = 6$, $P(t)$ cannot converge to the intertemporal equilibrium. It is ***dynamically unstable.***

By now you have completed the first section of this unit. Therefore, try to do the following self - test questions to examine how you understand this section.

Exercise

Do the following questions based on the information given above.

1. Describe dynamic equilibrium analysis----------
---

2. What is differential equation explain it briefly. ----------
---

3. Describe the difference between homogenous and non homogenous differential equation. ------
---
---

4. Explain the difference between general solution and definite solution of an equation ----------
---

5. Given demand and supply functions of a product as

$$Q_d = 125 - 2P$$

$$Q_s = -45 + 8P$$

Determine whether the equilibrium price , $\bar{P}$ is dynamically stable over time or not as there is sufficient time for adjustment using the formula presented in equation (12) ----------
---

6. Suppose the demand and supply functions of a product are given as:

$$Q_d = \frac{d^2P}{dt^2} + 3\frac{dP}{dt} - 2P + 10$$

$$Q_s = 2P - 30$$

With P (0) = 4 and $P'(0) = 9$, determine the price path, assuming market clearance at every point of time. Is the time path convergent? ----------
---

Dear colleague! Have you answered these questions? If your answer is no, please reread this section and answer them. If your answer is yes, go to the next section.

6.2 Discrete Time Dynamic Analysis

In the previous section we have discussed the continuous time dynamic analysis. In that case, the magnitude of change in time is very small. As result, the change in y which is emanated from this change in time is expressed in the form of derivative. However, this section is emphasized on ***discrete time dynamic analysis.*** In this case, time is considered as discrete variable and the dynamic adjustment process between equilibriums is observed as a step-by-step process rather than that of continual adjustment. Thus the change in y has to be explained by ***"differences"*** rather than derivatives or differentials. Accordingly, we can calculate the different values of the variables which are adjusting to new equilibrium levels using

- The mathematical concept of difference equation and
- Spreadsheet

.***Difference equation*** is an equation which expresses the relationship between the dependent variable and a lagged independent variable that changes at discrete intervals of time. The largest number of periods lagged represents the order of difference equation.

6.2.1 First Order Linear Difference Equations

First order difference equation represents a time lag of one period. The change in y which is resulted from change in time t from (t) *to* $(t+1)$is referred to as the first difference of y which is represented by

$$\Delta y/\Delta t = \Delta y = y_{t+1} - y_t \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This equation is first-order linear difference equation. Given the first-order linear difference equation

$$\Delta y = 3 = y_{t+1} - y_t$$

$$y_{t+1} = y_t + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let us find the solution of this equation by using ***iterative method*** first and then the ***general method***.

A. Iterative Method

Given equation (2), it is not difficult to find the value of y_1 provided that the initial value of y, i, e, y_0 is given. We can also find the value of y_2 once we have determined the value of y_1

EXAMPLE

1. Find the solution of the difference equation $y_{t+1} = y_t + 3$ provided that $y_0 = 20$ using iteration method.

$$y_1 = y_o + 3$$
$$y_2 = y_1 + 3 = (y_o + 3) + 3 = Y_0 + 2(3)$$
$$y_3 = y_2 + 3 = (y_1 + 3) + 3 = (y_o + 3)3 + 3 = y_0 + 3(3)$$
$$y_4 = y_3 + 3 = y_0 + 4(3)$$
$$y_t = y_0 + t(3)$$

For any period t, the solution of equation (2) with $y_0 = 20$ is

$$\boxed{y_t = 20 + 3t} \quad \text{----------} (3)$$

2. Find the solution of the difference equation $y_{t+1} - 2y_1 = 0$. In this case, the initial value is unspecified simply represented by y_0. using the method of iteration,

$$y_1 = 2y_0$$
$$y_2 = 2y_1 = 2(2y_0) = (2)^2 y_0$$
$$y_3 = 2y_2 = (2)^3 y_0$$
$$y_t = (2)^t y_0$$

The solution of $y_{t+1} - 2y_0 = 0$ at any period t is

$$\boxed{y_t = (2)^t y_0} \quad \text{----------} (4)$$

B) General method

Given the first order linear difference equation

$$y_{t+1} + by_t = a \quad \ldots\ldots(5)$$

Where a and b are constants, the general solution s a combination of the particular integral y_p and the complementary function y_c. y_p denotes the intertemporal equilibrium level of y where as y_c represents the deviation of the time path from this equilibrium. Incorporating the initial condition we can determine the definite solution.

Let us first determine the complementary function which is the solution of the reduced equation $y_{t+1} + by_t = 0$. Based on equation (4) above,

$$y_t = (-b)^t y_0 \quad \ldots\ldots(6)$$

Putting the constant A in place of y_0, we get

$$y_t = A(-b)^t$$

Thus, the complementary function is

$$y_c = A(-b)^t \quad \ldots\ldots(7)$$

Now let us find the particular integral y_p

Taking the simplest form $y_t = k$ where k is constant, we have to have $y_{t+1} = k$. If we substitute these n to equation (5), we will get k+ b k= a

$$k(1+b) = a \Rightarrow k = \frac{a}{1+b}, b \neq -1$$

Then the particular integral becomes,

$$y_p = k = \frac{a}{1+b}, (b \neq -1) \quad \ldots\ldots(8)$$

Therefore, the general solution of the equation is

$$y_t = y_c + y_p$$

$$y_t = A(-b)^t + \frac{a}{1+b}, b \neq -1 \quad \ldots\ldots(9)$$

If b=-1 the general solution will be

$$y_t = A + at$$

Setting the initial condition that $y_t = y_0$ when t =0,

$$y_0 = A(-b)^0 + \frac{a}{1+b} = A + \frac{a}{1+b}, b \neq -1,$$

$$A = y_0 - \frac{a}{1+b}$$

Are a result, the definite solution is

$$\boxed{y_t = \left(y_0 - \frac{a}{1+b}\right)(-b)^t + \frac{a}{1+b},} \qquad b \neq -1 \quad \ldots\ldots(10)$$

Example

Solve the first order difference equation

$$y_{t+1} + 3y_t + 8 = 0 \Rightarrow y_{t+1} + 3y_t = -8, \text{ when } y_0 = 16$$

$$y_{t+1} + 3y_t = -8$$

Using the formula in equation (10) , a =-8 , b 3, $y_0 = 16$

$$y_t = \left(16 - \left(\frac{-8}{1+3}\right)\right)(-3)^t - \frac{8}{1+3}$$

$$y_t = (16+2)(-3)^t - 2 = 18(-3)^t - 2$$

The given equation s referred to as dynamically stable or convergent provided that the complementary function approaches to zero as t approaches to infinity. The convergence of the term b^t depend the absolute value of b. If $|b| > 1$, the time path will move away from equilibrium. If $|b| < 1$, the time path will move towards to equilibrium, i.e., convergent. In our example above, b=-3, $|b| = |-3| > 1$. Therefore, the equilibrium is dynamically unstable. In other words, the time path is divergent.

6.2.2 Second Order Linear Difference Equations

Given the second order difference equation

$y_{t+2} + a_1 y_{t+1} + a_2 y_t = C$.....(11) , the general solution is composed of that of particular integral and complementary function. We know that the particular integral is any solution of the complete equation. Thus, setting the simplest solution $y_t = k,$ Equation (11) becomes

$$k = \frac{c}{1 + a_1 + a_2}, \quad (a_1 + a_2 \neq -1)$$

Therefore, $\boxed{y_p = k = \frac{c}{1 + a_1 + a_2}}$, $(a_1 + a_2 \neq -1)$.....(13)

If $a_1 + a_2 = -1,$ equation (13) will be undefined. Thus, we have to set another value for $y_{t.}$ .Let $y_t = kt,$ then equation (11) becomes,

$$k(t+2) + a_1 k(1+t) + a_2 kt = c$$.....(14)

Rearranging this equation,

$$k[t + a_1 t + a_2 + a_1 + 2] + c,$$

$$k[t(1 + a_1 + a_2) + a_1 + 2] = c, \quad as \quad a_1 + a_2 = -1$$

This equation becomes

$$k(a_1+2)=c \quad k=\frac{c}{a_1+2}, a_1 \neq -2$$

Thus, $\boxed{y_p=\frac{c}{a_1+2}t}$, $(a_1+a_2=-1, a_1 \neq -2)$.....(15)

When $a_1+a_2=-1$ *and* $a_1=-2$ then equation (15) will be undefined. Therefore, we should set another value for y_t. Let $y_t=kt^2$, then equation (11) becomes,

$$k(t+2)^2+a_1k(t+1)^2+a_2k(t)^2=c.....(16)$$

$$k\left[t^2+4t+4+a_1\left(t^2+2t+1\right)+a_2t^2\right]=c$$

$$k\left[t^2(1+a_1+a_2)+t(4+2a_1)+4+a_1\right]=c, as \quad a_1+a_2=-1 \quad and \quad a_1=-2$$

This equation becomes,

$$k\left[t^2(0)+t(o)+4-2\right]=c$$

$$k(2)=c \Rightarrow k=\frac{c}{2}, a_1+a_2=-1 \quad and \quad a_1=-2.$$

Therefore, $\boxed{y_p=\frac{c}{2}t^2}$, $(a_1+a_2=-1 \quad and \quad a_1=-2)$.....(17)

As complementary function is the solution of the reduced equation, we should emphasize on the equation

$y_{2+t}+a_1y_{t+1}+a_2y_t=0$.....(18) to determine the complementary function of the given equation we know that $y_t=Ab^t$. Thus the above reduced equation becomes

$$Ab^{(2+t)}+a_1Ab^{(t+1)}+a_2Ab^t=0.....(19)$$

$$Ab^t\left(b^2+a_1b+a_2\right)=0$$

Ignoring the nonzero common factory gives us,

$$b^2+a_1b+a_2=0.....(20)$$

This equation is referred to as characteristic equation of equation (18)
The solution of this equation using the quadratic formula

$$b_1,b_2=\frac{-a_1\pm\sqrt{a_1^2-4a_2}}{2}.....(21)$$

We face three possible phenomenon based on the term in the square root of equation (21).

Case1, When $a,^2>4a_{2,}$ the characteristic equation will have two distinct roots, b_1 *and* $b_{2,}$ In this case, the complementary solution will be

$$y_c=A_1b_1^t+A_2b_2^t.....(22)$$

Case 2, When $a,^2 = 4a_{2,}$ the characteristic equation will have only one root, this means,

$$b_1 = b_2 = b = \frac{-a_1}{2}$$

The complementary function becomes

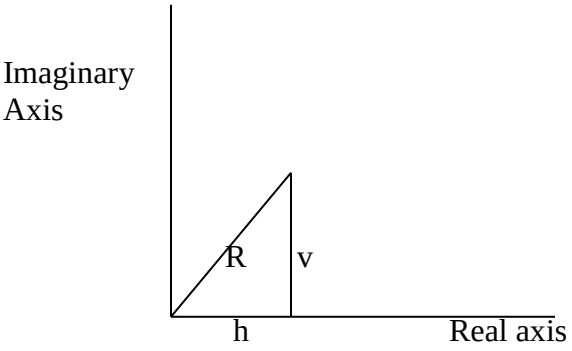
$$y_c = A_1 b^t + A_2 b^t = (A_1 + A_2) b^t = A_3 b^t$$

$$y_c = A_3 b^t + A_4 t b^t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

Case 3, When $a_1^2 < 4a_{2,}$ the term in the square root will be negative which an imaginary number is. There is no characteristic root for the equation provided that we restrict our domain to the real number system. Thus, we are forced to include numbers outside the real number system. Ideally, we can define $i = \sqrt{-1}$ which a square root of negative one is. It is not real number rather it is known as imaginary number. Therefore, we can write

$\sqrt{-25} = (\sqrt{25})(\sqrt{-1}) = 5i, \sqrt{-10} = \sqrt{10}\sqrt{-1} = \sqrt{10}i$

Given the diagrams



$$\sin\theta = \frac{v}{R}, \qquad \cos\theta = \frac{h}{R}$$

$$v = R\sin\theta, \quad h = R\cos\theta \qquad R = \sqrt{h^2 + v^2}$$

Then the conjugate complex number

$$(h \pm vi) = R\cos\theta \pm RiSin\theta$$

According to **De Moivers theorem** $(h \pm vi)^n = R^n(\cos\theta n \pm Ri\sin\theta n)$.....(24)

If $a,^2 - 4a_2 < 0$ in equation (21), then the roots will tae the form

$$b_1, b_2 = h \pm vi, \quad where \quad h = \frac{-a_1}{2}, v = \frac{\sqrt{a_1^2 - 4a_2}}{2}, i = \sqrt{-1}$$

$$\mathrm{R} = \sqrt[2]{\frac{a_1^2 + 4a_2 - a_1^2}{4}} = \sqrt{a_2}$$

The complementary function becomes

$$y_c = A_1 b_1^t + A_2 b_2^t$$
$$= A_1 (h+vi)^t + A_2 (h-vi)^t$$
$$= A_1 R^t (\cos t\theta + i \sin t\theta) + A_2 R^t (\cos t\theta - i \sin t\theta)$$
$$y_c = R^t [(A_1 + A_2)\cos t\theta + (A_1 - A_2) i \sin t \quad \theta]$$
$$If A_1 + A_2 = A_5 \quad and \quad (A_1 - A_2) i = A_6, then$$
$$y_c = R^t (A_5 \cos t\theta + A_6 \sin t\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

In al of the above three cases, the equilibrium will be dynamically stable provided that the absolute value of every root is less than one whatever the initial condition may be

Table 6 1. Values of sin θ *and* cos θ

θ	0	$\frac{1}{2}\Pi$	Π	$\frac{3}{2}\Pi$	2Π	$\Pi/6$	$\Pi/4$	$\Pi/3$	$\frac{3}{4}\Pi$
$\sin\theta$	0	1	0	-1	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\sqrt{3/2}$	$\frac{1}{\sqrt{2}}$
$\cos\theta$	1	0	-1	0	1	$\sqrt{3/2}$	$1/\sqrt{2}$	$\frac{1}{2}$	$-1/\sqrt{2}$

Example

Given the equation $y_{t+2} + 4y_t = 15, y_0 = 12, y_1 = 11,$

a. Determine the general and definite solution of the equation

b. Is the time path convergent or not?

Solution

$$a_1 = 0, a_2 = 4, \quad C = 15 \qquad y_p = \frac{15}{5} = 3$$

$$R = \sqrt{4} = 2, \qquad \cos\theta = \frac{-a_1}{2\sqrt{a_2}} = \frac{o}{2(2)} = 0$$

$$\sin\theta = \sqrt{1 - \frac{a_1^2}{4a_2}}$$

$$= \sqrt{1 - \frac{0}{16}} = \sqrt{1}$$

$$\sin\theta \quad = 1$$

$Sin\theta = 1 \; and \; \cos\theta = 0$ .Thus, as it is shown in table 1, $\theta = \pi/2$ when we substitute this value in equation (25), the complementary function will be

$$y_c = 2^t \left[A_5 \cos\left(\frac{\pi}{2} t\right) + A_6 \sin\left(\frac{\pi}{2} t\right) \right]$$

Then the general solution becomes

$$y_t = 2^t\left[A_5\cos\left(\frac{\pi}{2}t\right)+A_6\sin\left(\frac{\pi}{2}t\right)\right]+3 \quad \text{.....(26)}$$

When $t=0$, $y_0 = 2^0[A_5\cos 0 + A_6\sin 0]+3=12$

$$[A_5(1)+A_6(0)]+3=12$$

$$A_5+3=12$$

$$A_5=12-3=9$$

When t=1, $y_1 = 2^1\left[A_5\cos\pi/2 + A_6\sin\pi/2\right]+3=11$

$$2[A_5(0)+A_6(1)]+3=11$$

$$2A_6+3=11$$

$$2A_6=8$$

$$A_6=4$$

Therefore, the definite solution be comes

$$y_t = 2^t\left[9\cos\frac{\pi}{2}t+4\sin\frac{\pi}{2}t\right]+3 \quad \text{.....(27)}$$

6.2.3 Economic Applications

In this sub topic you will learn about the application of first order and second order linear difference equations on economic models.

A. Application of First Order Linear Difference Equation

Cobweb Model

In certain markets particularly agricultural markets, if demand for a product increase, supply cannot immediately increase to satisfy the prevailing demand. Crops must be planted and growth and livestock takes time to raise. Some manufactured products may also take certain time when customers order that products suddenly. The Cobweb model considers this lagged in supply side of the market by supposing that the present quantity supplied is based on the ruling price in the previous time period. That is,

$$Q_t^s = f(P_{t-1})$$

But the consumers demand for the same product depends on the prevailing price that is

$$Q_t^d = f(P_t)$$

This is a logical situation of several agricultural markets. The quantity supplied this year is based on the price of that product in last year. The Cobweb model assume that

- The market is perfectly competitive.
- The demand and supply functions are linear.

Given these assumptions, the Cobweb model can still give a fair idea of how price and quantity adjust in many markets with a lagged supply. Based on the above assumptions, the demand and supply functions of a product can be expressed in the form

$$Q_t^d = a + b\,P_t \qquad (demand\ function).....$$

$$Q_t^s = c + d\,P_{t-1} \qquad (supply\ function)$$

As you know, the market achieves its equilibrium when quantity demanded is equal to quantity supplied.

In other words, $Q_t^d = Q_t^s$

$$a + bP_t = c + dP_{t-1}$$

$$bP_t = c - a + dP_{t-1}$$

$$\boxed{P_t = \frac{c-a}{b} + \frac{d}{b}P_{t-1}} \quad \text{.....(28)}$$

This equation is the ***Cobweb difference equation.*** Thus, solving the Cobweb difference equation means setting it in the format $P_t = f(t)$. As a result; we can calculate the value of P_t at any time period. The solution of the ***Cobweb difference equation*** includes two parts. There are:

a) Long run equilibrium price
b) a complementary function which indicates how much price diverges from this equilibrium level at different points in time.
c)

Now let us first find the long run equilibrium price $\overline{P}$. In the log run, $P_t = P_{t-1} = \overline{P}$. Therefore

$$P_t = \frac{c-a}{b} + \frac{d}{b}P_{t-1}$$

$$\overline{P} = \frac{c-b}{b} + \frac{d}{b}\overline{P} \Rightarrow \overline{P} = \frac{a-c}{d-b}, b \neq d \text{.....(29)}$$

To determine the complementary function, we can rewrite equation (28) as

$$P_{t+1} - \frac{d}{b}P_t = \frac{c-a}{b} \text{.....(30)}$$

Using the method of determining the complementary function described in equation (7) above,

$$P_c = P_0\left(d/b\right)^t$$

If we substitute, A, a Constance in place of P_0, we get

$$P_c = A\left(\frac{d}{b}\right)^t \text{.....(31)}$$

In this case the value of A can be determined if a certain value of P_t is known for specific value of t.

The complete solution of the Cobweb difference equation, therefore, becomes

$$P_t = \left(\frac{a-c}{d-b}\right) + A\left(\frac{d}{b}\right)^t$$

$$\boxed{P_t = \overline{P} + A\left(\frac{d}{b}\right)^t} \quad \text{.....(32)}$$

From this equation

1. If $\left|d/b\right| < 1, \left|\frac{d}{b}\right|^t$ approaches to zero when t approach to infinity. In this case the market is stable. Price converges to the long run equilibrium price $\overline{P}$.
2. If $\left|\frac{d}{b}\right| > 1, then \left|\frac{d}{b}\right|^t \to \infty \; when\, t \to \infty.$ In this case the market is unstable. Price diverges from the equilibrium level.
3. If $\left|\frac{d}{b}\right| = 1, then \left|\frac{d}{b}\right|^t = 1, \; as\, t \to \infty.$ In this case, price neither return to its equilibrium nor 'explode'.

Example

1. Given the demand and supply for the Cobweb model as

$$Q_t^d = 400 - 20P_t \text{..(33)}$$

$$Q_t^s = -50 + 10P_{t-1}$$

- Determine the long run (intertemporal) equilibrium price $\left(\overline{P}\right)$, the complementary function and the complete solution.
- Determine whether the equilibrium is stable or not.

Solution

a =400, b =-20, c =-50, d=10

$$\overline{P} = \frac{a-c}{d-b} = \frac{400+50}{10+20} = \frac{450}{30}$$

$\overline{P} = 15$ (Inter temporal equilibrium).

The complementary function is

$$P_c = A\left(\frac{10}{-20}\right)^t$$

Therefore, the complete solution is

$$P_t = A(-0.5)^t + 15 \text{...(34)}$$

Since $\left|\frac{d}{b}\right| = \left|-1/2\right| = 1/2 < 1,$ the equilibrium is dynamically stable.

What happens to the market if quantity supplied suddenly changes to 160?

The initial stock output level is $Q_0 = 160$ so that price will adjust until all output is sold. Then P_0 can be calculated from the demand function by substituting $Q_0 = 160.$

Therefore, $160 = 400 - 20P_0$.

$$P_0 = 12$$

By now we can determine the value of A.

$$P_0 = A(0.5)^0 + 15 = 12$$
$$A + 15 = 12$$
$$A = -3$$

Thus, the definite solution is

$$P_t = -3(-0.5)^t + 15 \quad \ldots\ldots(35)$$

Using this equation we can calculate the price of the product at any time period and there by we can check whether the price converges to the equilibrium level or not.

2. In a market where the assumptions of the cobweb model hold,

$$Q_t^d = 200 - 8P_t$$

$$Q_t^s = -43 + 8.2P_{t-1} \quad \ldots\ldots(36)$$

The long run equilibrium is disturbed when quantity suddenly changes to 90.
What happens to price in the following time period?

Solution

In the long run equilibrium, $\overline{Q} = Q_t^d = Q_t^s$ & $\overline{p} = p_t = p_{t-1}$ thus,

$$200 - 8\overline{P} = -43 + 8.2\overline{P}$$
$$16.2\overline{P} = 243$$
$$\overline{P} = 15$$

The complementary function is $p_c = A\left(\frac{8.2}{-8}\right)^t = A(-1.025)^t \quad \ldots\ldots(37)$

Thus, the general solution is

$$P_t = A(-1.025)^t + 15 \quad \ldots\ldots(38)$$

When

$$Q_0 = 90,$$
$$90 = 200 - 8P_0$$
$$8P_0 = 110$$
$$P_o = 13.75$$

$$If \quad P_o = 13.75, the \quad value \quad of \quad A \quad will \quad be,$$

$$13.75 = A(-1.025)^0 + 15$$

$$13.75 = A + 15$$

$$A = -1.25$$

Thus, the definite solution is

$$p_t = (-1.25)(-1.025)^t + 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

The Keynesian Macroeconomic Model

In the Keynesian model of national income determination, if foreign trade and government expenditure and tax are not included, the model becomes,

And

$$Y = C + I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

$$C = a + bY \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Where Y is national income, C is consumption, I is investments a is autonomous consumption and b is marginal propensity to consume. Then equilibrium level of national income is

$$Y = a + bY + I$$

$$Y - bY = a + I$$

$$Y(1-b) = a + I$$

$$\bar{Y} = \frac{a+I}{1-b}, (b \neq 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

$$, Y = \bar{Y} at \quad equilibrim$$

This equation shows that this national equilibrium income $(\bar{Y})$ can be evaluated for a given values of a, b and I . When there is a disturbance from this equilibrium, there will not be an immediate adjustment to the new equilibrium. As the consumer expenditure may not adjust immediately to the new level of income, a lagged effect may be introduced in the consumption function, as follows,

$$C_t = a + bY_{t-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

This means, consumer's expenditure at present depends on the income of the previous year. But national income is still determined by the sum of all current expenditure

$$Thus, \quad Y_t = C_t + I_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

Substituting (43) in to (44) gives us

$$Y_t = a + bY_{t-1} + I_t$$

$$Y_t = bY_{t-1} + a + I_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

Equation (45) is a first order linear difference equation,

The general solution of this equation includes two parts, namely intertemporal equilibrium and complementary function (Y_c). At the point of equilibrium,

$$\bar{Y} = Y_t = Y_{t-1}$$

Thus,

$$\bar{Y} = b\bar{Y} + a + I_t$$

$$\bar{Y} - b\bar{Y} = a + I_t$$

$$\bar{Y} = \frac{a + I_t}{1 - b} \qquad (\text{int}\, ertemporal \quad equilibrium).....(46)$$

The complementary function as referred to equation ()7

$$Y_c = Ab^t .....(47)$$

Thus, the general solution becomes

$$Y_t = Ab^t + \frac{a + I_t}{1 - b} .....(48)$$

This equation shows that the equilibrium will be dynamically stable provided that $|b| < 1.$

Example

Given the Keynesian model

$$y_t = C_t + I_t$$

$$C_t = 650 + 0.5Y_{t-1}$$

$$And \quad I_t = 300$$

Then it suddenly increases to 420

What will be the actual level of Y six time periods after this change?----------

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Solution

$a = 650, b = 0.5, I_t = 300$

The initial equilibrium level of income is

$$\bar{Y} = \frac{650 + 300}{1 - 0.5} = 1{,}900$$

Thus, consumption C_o is

$$C_0 = 650 + 0.5(1.900) = 1{,}600$$

Thus, the value of Y immediately after the shock is

$$Y_o = C_0 + I_0$$

$$Y_o = 1600 + 420 = 2{,}020$$

The new equilibrium level is

$$\hat{Y} = \frac{650 + 420}{1 - 0.5} = 2{,}140$$

Substituting this value in general solution above, we will get

$$Y_t = \hat{Y} + \left(Y_o - \hat{Y}\right)b^t$$
$$Y_t = 2{,}140 + (2{,}020 - 2{,}140)(0.5)^t$$
$$= 2{,}140 - 120(0.5)^t$$
$$Y_6 = 2{,}140 - 120(0.5)^6$$
$$= 2{,}140 - 1.875$$
$$Y_6 = 2{,}138.125$$

How much time period will it take y to reach 2,130? ----------

$$2{,}130 = 2{,}140 - 120(0.5)^t$$
$$-10 = -120(0.5)^t$$
$$\frac{10}{120} = (0.5)^t$$

Putting this equation in it's logarithm form

$$\log\frac{10}{120} = \log 0.5t$$
$$\log\frac{10}{120} = t\log 0.5t$$
$$t = \frac{\log\frac{10}{120}}{\log 0.5} = 3.585$$
$$t = 3.585$$

The equilibrium national income will reach 2,130 units after 3.585 time period.

Samuel Son's Multiplier-Acceleration Interaction Model

To observe the application of second order difference equations in economies, let us consider Samuel son's classic interaction model. This model examines the dynamic process of income determination if the acceleration principle works with the Keynesian multiplier. The model describes the situation that the interaction of the multiplier and the accelerator results in cyclical fluctuation endogenously.

Suppose the national income Y_t is a combination of consumption C_t, investment I_t and government expenditure G_t. It is also assumed that current consumption is a function of last year's income (Y_{t-1}), Investment I_t is function of difference in consumption $\Delta C_{t-1} = C_t - C_{t-1}$. suppose I_t is s fixed proportion of ΔC_{t-1}. G_t is assumed to be exogenous. Thus, the model becomes

$$Y_t = C_t + I_t + G_0$$
$$C_t = bY_{t-1}\ , 0 < b < 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$
$$I_t = a(C_t - C_{t-1}), a > 0$$

, Where b is marginal propensity to consume and a is accelerator coefficient. Now we can express I_t in terms of income as

$$I_t = a(bY_{t-1} - bY_{t-2})$$
$$I_t = ab(Y_{t-1} - Y_{t-2}) \quad \ldots\ldots(50)$$

If we substitute (50) and C_t in equation (49), we get

$$Y_t = bY_{t-1} + ab(Y_{t-1} - Y_{t-2}) + G_0$$
$$I_t = b(1+a)Y_{t-1} + abY_{t-2} = G_0$$

This means

$$Y_{t+2} - b(1+a)Y_{t+1} + abY_t = G_0 \quad \ldots\ldots(51)$$

This equation is a ***second order linear difference equation.*** As a result, we can solve it using the method discussed in equation (11).

Solution

The general solution will include the particular integral Y_p and the complementary function (y_c).

The particular integral Y_p is

$$Y_p = \frac{G_0}{1 - b(1+a) + ab} = \frac{G_0}{1-b}$$

Complementary function

As far as the complementary function is concerned, we are faced with three possible phenomena

Case 1 If $b^2(1+a)^2 > 4ab \quad or \quad b > \frac{4a}{(1+a)^2}$,

Equation (51) with characteristic equation $r^2 - b(1+a)r + ab = 0$ will have two roots,

$$r_1, r_2 = \frac{b(1+a) \pm \sqrt{b^2(1+a)^2 - 4ab}}{2} \quad \ldots\ldots(52)$$

As we know whether the equilibrium is dynamically stable or not depends on r_1 and r_2 which in turn are based on the value of a and b, we can express the dynamic stability in terms of $a \quad and \quad b.$

Case 2 If $b^2(1+a)^2$ =4ab, we will have only one root,

$$r = \frac{b(1+a)}{2}$$

Case 3 If $b^2(1+a)^2 < 4ab,$ then we should use the method explained in equation (25)

$$r_1, r_2 = h \pm vi, \quad Where \quad h = \frac{b(1+a)}{2}, v = \frac{\sqrt{b^2(1+a)^2 - 4ab}}{2}$$
$$i = \sqrt{-1} \qquad , R = \sqrt{ab}$$

Then the complementary function becomes

$$Y_c = A_1 r_1^t + A_2 r_2^t$$
$$= A_1(h+vi)^t + A_2(h-vi)^t$$
$$Y_c = R^t[(A_1 + A_2)\cos t\theta + (A_1 - A_2)i \sin t\theta]$$
$$If A_1 + A_2 = 5 \quad and \quad (A_1 - A_2)i = A_6,$$

$$Y_c = R^t(A_5 \ Cost\theta + A_6 \ \sin t\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

Then the general solution in case 3 becomes,

$$Y_t = R^t(A_5 \ Cost\theta + A_6 \ \sin t\theta) + \frac{G_o}{1-b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

Now try to determine the general solution of the equation in ***case 1*** and ***case 2***.

In all of the three cases, the equilibrium will be dynamically stable iff a b < 1, that is, the product of the ***accelerator*** and ***marginal propensity to consume*** must be less than one.

By now you have completed the second section of this limit. Thus, try to do the following question to evaluate how you understand this section.

Exercise

Solve the following questions based on the information above.

1. What is difference equation? ---
2. What is the difference between discrete and continuous variable-------------------------------------
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3. Solve the difference equation $5y_{t+1} + 2y_t = 140, y_0 = 30$--
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4. Given the second order linear difference equation $y_{t+2} + 7y_{t+1} + 6y_t = 42$ Find the general solution. ---
5. assuming the usual Cobweb assumption, derive the difference equation which will calculate price in the time periods following the unexpected decrease in equilibrium quantity to 75 given

$$Q_t^d = 160 - 20P_t$$
$$Q_t^s = -80 + 40P_{t-1}$$

6. A Keynesian macroeconomic model with a single time period lag on the consumption function described below is initially in equilibrium with the level of $I_t = 500$.

$$Y_t = C_t + I_t$$
$$C_t = 750 + 0.5Y_{t-1}$$

If it is increased to 650, find the value of y in the fourth time period after this disturbance.

7. If the accelerator is 0.5 and the marginal propensity to consume is 0.6, what will happen to the time path? Answer this question using equation (51).

Have you answered these questions? If your answer is no, re read this section and try to do them. If your answer is yes, good. Go to the following check list.

Assignment Questions

1. A form produces two types of products X and Y passing through to two machines M_1 and M_2. Each unit of X requires 2 hours of time in M_1 and 8 hours of time in M_2 where as each unit of Y requires 12 hours of time in M_1 and 4 hrs of time in M_2.The total time available in M_1 is 40 hours and M_2 is 28 hours. Determine the amount of X and Y produced using gauss - Jordan elimination method. ----------

2. Find the price elasticity of demand if the demand function is
$Q = 40 - 5P + 3P^2$

3. Find the point elasticity of demand if the demand function is given as

$Q = \frac{50}{pk}$ where k is constant. ----------

4. Prove that $\sqrt{x+1} \approx 1 + \frac{1}{2}x$ as x is close to zero

5. Find the Taylor's seines of the function $f(x) = 4x^2 + 6x + 3$ around $x_{0=2}$ ----------

6. Given the Cobb - Douglas production function $Q = A K^{\alpha} L^{\beta}$ Where Q represents total product, K is capital and L is labor. Find the marginal product of each input and determine whether the law of diminishing marginal productivity holds for each input as far as $\alpha < 0$ & $\beta < 1$. ----------

7. Given the production function $Q = 32K^{0.5} L^{0.25} R^{0.4}$, derive the nine second - order partial derivatives and show that the two cross partial derivatives with respect to each possible pair of independent variables will be equal to each other. ----------

8. Suppose the total cost faction of the monopolist is given by $TC = 2Q^2 + 10Q + 15$ and the demand function is given by $P = 10 - 3Q$. Find
   i) The level of output which maximize profit
   ii) Determine the elasticity of demand at the point of equilibrium.

9. Discuss the difference between comparative static analysis and dynamic analysis briefly.

10. Go to section 5.1.2 observe the effect of change in all the parameters on equilibrium quantity and interpret the result.

11. Explain the limitation of comparative static analysis.

12. Considering the following national- income model with tax ignored

$Y - C(Y) - I(i) - G_0 = 0, (0 < C' < 1; I' < 0)$

$kY + L(i) - M_{s0} = 0$

(Where k is positive constant; $L' < 0$)

Analyze the comparative static effect of the model when money supplies changes and government expenditure changes.

13. Suppose that the monopolistic firm has the following average revenue functions in three separated markets

$P_1 = 63 - 4Q_1$

$P_2 = 105 - 5Q_2$ and the cost function is $C = 20 + 15Q$

$P_3 = 75 - 6Q_3$

Determine the price that should be charged in each market to maximize profit. Check the second order sufficient conditions using the Hessian determinant.

14. Given the demand and supply functions be

$Q_d = 9 - P + P' + 3P''$

$Q_s = -1 + 4P - P' + 5P''$ with $P(0) = 4 \& P'(0) = 4$

Find the price path assuming market clearance at every point of time.

Is the time path convergent or not?

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