Addis Ababa Science and Technology University
College of Natural and Social Sciences

Department of Mathematics

Mathematics for Natural Science (Math 1011)

Questions on Propositional Logic and Set Theory

1. For statements p, q and r, use a truth table to show that each of the following pairs of statements is logically equivalent.
   a. $(p \wedge q) \Longleftrightarrow p$ and $p \Longrightarrow q$.
   b. $p \Longrightarrow (q \vee r)$ and $q \Longrightarrow (p \vee r)$.
   c. $(p \vee q) \Longrightarrow r$ and $(p \Longrightarrow q) \wedge (q \Longrightarrow r)$.
   d. $p \Longrightarrow (q \vee r)$ and $(r) \Longrightarrow (p \Longrightarrow q)$.
   e. $p \Longrightarrow (q \vee r)$ and $((r) \wedge p) \Longrightarrow q$.
2. For statements p, q, and r, show that the following compound statements are tautology.
   a. $p \Longrightarrow (p \vee q)$.
   b. $(p \wedge (p \Longrightarrow q)) \Longrightarrow q$.
   c. $((p \Longrightarrow q) \wedge (q \Longrightarrow r)) \Longrightarrow (p \Longrightarrow r)$.
3. For statements p and q, show that $(p \wedge q) \wedge (p \wedge q)$ is a contradiction.
4. Write the contrapositive and the converse of the following conditional statements.
   a. If it is cold, then the lake is frozen.
   b. If Solomon is healthy, then he is happy.
   c. If it rains, Tigist does not take a walk.
5. Let p and q be statements. Which of the following implies that $p \vee q$ is false?
   a. $p \vee q$ is false.
   b. $p \vee q$ is true.
   c. $p \wedge q$ is true.
   d. $p \Longrightarrow q$ is true.
   e. $p \wedge q$ is false.
6. Suppose that the statements $p, q, r,$ and s are assigned the truth values $T, F, F,$ and T, respectively. Find the truth value of each of the following statements.
   a. $(p \vee q) \vee r$.
   b. $p \vee (q \vee r)$.
   c. $r \Longrightarrow (s \wedge p)$.
   d. $p \Longrightarrow (r \Longrightarrow s)$.
   e. $p \Longrightarrow (r \vee s)$.
   f. $(p \vee r) \Longleftrightarrow (r \wedge s)$.
   g. $(s \Longleftrightarrow p) \Longrightarrow (p \vee s)$.
   h. $(q \wedge s) \Longrightarrow (p \Longleftrightarrow s)$.
   i. $(r \wedge s) \Longrightarrow (p \Longrightarrow (q \vee s))$.
   j. $(p \vee q) \vee r \Longrightarrow (s \wedge s)$.
7. Suppose the value of $p \Longrightarrow q$ is T; what can be said about the value of $p \wedge q \Longleftrightarrow p \vee q$?
8. a. Suppose the value of $p \Longleftrightarrow q$ is T; what can be said about the values of $p \Longleftrightarrow q$ and $p \Longleftrightarrow q$?
   b. Suppose the value of $p \Longleftrightarrow q$ is F; what can be said about the values of $p \Longleftrightarrow q$ and $p \Longleftrightarrow q$?

9. Construct the truth table for each of the following statements.
   a. $p \Longrightarrow (p \Longrightarrow q)$.
   b. $(p \vee q) \Longleftrightarrow (q \vee p)$.
   c. $p \Longrightarrow (q \wedge r)$.
   d. $(p \Longrightarrow q) \Longleftrightarrow (p \vee q)$.
   e. $(p \Longrightarrow (q \wedge r)) \vee (\neg p \wedge q)$
      $(p \Longrightarrow (q \wedge r)) \vee (p \wedge q)$.
   f. $(p \wedge q) \Longrightarrow ((q \wedge q) \Longrightarrow (r \wedge q))$.
10. For each of the following determine whether the information given is sufficient to decide the truth value of the statement. If the information is enough, state the truth value. If it is insufficient, show that both truth values are possible.
    a. $(p \Longrightarrow q) \Longrightarrow r$, where $r = T$.
    b. $p \wedge (q \Longrightarrow r)$, where $q \Longrightarrow r = T$.
    c. $p \vee (q \Longrightarrow r)$, where $q \Longrightarrow r = T$.
    d. $(p \vee q) \Longleftrightarrow (p \wedge q)$, where $p \vee q = T$.
    e. $(p \Longrightarrow q) \Longrightarrow (q \Longrightarrow p)$, where $q = T$.
    f. $(p \wedge q) \Longrightarrow (p \vee s)$, where $p = T$ and $s = F$.
11. In each of the following, two open statements $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \Longrightarrow Q(x, y)$ for the given values of x and y.
    a. $P(x, y): x^2 - y^2 = 0$. and $Q(x, y): x = y$. $(x, y) \in \{(1, -1), (3,4), (5,5)\}$.
    b. $P(x, y): |x| = |y|$. and $Q(x, y): x = y$. $(x, y) \in \{(1,2), (2, -2), (6,6)\}$.
    c. $P(x, y): x^2 + y^2 = 1$. and $Q(x, y): x + y = 1$. $(x, y) \in \{(1, -1), (-3,4), (0, -1), (1,0)\}$.
12. Let O denote the set of odd integers and let $P(x): x^2 + 1$ is even, and $Q(x): x^2$ is even. be open statements over the domain O. State $(\forall x \in O) P(x)$ and $(\exists y \in O) Q(x)$ in words.
13. State the negation of the following quantified statements.
    a. For every rational number r, the number $\frac{1}{r}$ is rational.
    b. There exists a rational number r such that $r^2 = 2$.
14. Let $P(n): \frac{5n-6}{3}$ is an integer. be an open sentence over the domain $\mathbb{Z}$. Determine, with explanations, whether the following statements are true or false:
    a. $(\forall n \in Z) P(n)$.
    b. $(\exists n \in Z) P(n)$.
15. Determine the truth value of the following statements.
    a. $(\exists x \in R)(x^2 - x = 0)$.
    b. $(\forall x \in N)(x + 1 \geq 2)$.
    c. $(\forall x \in R)(\sqrt{x^2} = x)$.

d. $(\exists x \in Q)(3x^2 - 27 = 0)$.
e. $(\exists x \in R)(\exists y \in R)(x + y + 3 = 8)$.
f. $(\exists x \in R)(\exists y \in R)(x^2 + y^2 = 9)$.
g. $(\forall x \in R)(\exists y \in R)(x + y = 5)$.
h. $(\exists x \in R)(\forall y \in R)(x + y = 5)$

16. Consider the quantified statement

For every $x \in A$ and $y \in A$, $xy - 2$ is prime.

where the domain of the variables x and y is $A = \{3,5,11\}$.

a. Express this quantified statement in symbols.
b. Is the quantified statement in (a) true or false? Explain.
c. Express the negation of the quantified statement in (a) in symbols.
d. Is the negation of the quantified in (a) true or false? Explain.

17. Consider the open statement $P(x, y): \frac{x}{y} < 1$. where the domain of x is $A = \{2,3,5\}$ and the domain of y is $B = \{2,4,6\}$.

a. State the quantified statement $(\forall x \in A)(\exists y \in B)P(x, y)$ in words.
b. Show quantified statement in (a) is true.

18. Consider the open statement $P(x, y): x - y < 0$. where the domain of x is $A = \{3,5,8\}$ and the domain of y is $B = \{3,6,10\}$.

a. State the quantified statement $(\exists y \in B)(\forall x \in A)P(x, y)$ in words.
b. Show quantified statement in (a) is true.

19. Use the truth table method to show that the following argument forms are valid.

i. $p \Longrightarrow q, q \vdash p$.
ii. $p \Longrightarrow p, p, r \Longrightarrow q \vdash r$.
iii. $p \Longrightarrow q, r \Longrightarrow q \vdash r \Longrightarrow p$.
iv. $r \vee s, (s \Longrightarrow p) \Longrightarrow r \vdash p$.
v. $p \Longrightarrow q, p \Longrightarrow r, r \Longrightarrow s \vdash q \Longrightarrow s$.

20. For the following argument given a, b and c below:

vi. Identify the premises.
vii. Write argument forms.
viii. Check the validity.

a. If he studies medicine, he will get a good job. If he gets a good job, he will get a good wage. He did not get a good wage. Therefore, he did not study medicine.
b. If the team is late, then it cannot play the game. If the referee is here, then the team is can play the game. The team is late. Therefore, the

referee is not here.

c. If the professor offers chocolate for an answer, you answer the professor's question. The professor offers chocolate for an answer. Therefore, you answer the professor's question

21. Give formal proof to show that the following argument forms are valid.
    a. $p \Longrightarrow q, q \vdash p.$
    b. $p \Longrightarrow q, p, r \Longrightarrow q \vdash r.$
    c. $p \Longrightarrow q, r \Longrightarrow q \vdash r \Longrightarrow p.$
    d. $r \wedge s, (s \Longrightarrow p) \Longrightarrow r \vdash p.$
    e. $p \Longrightarrow, p \Longrightarrow r, r \Longrightarrow s \vdash q \Longrightarrow s.$
    f. $p \vee q, r \Longrightarrow p, r \vdash q.$
    g. $p \wedge q, (q \vee r) \Longrightarrow p \vdash r.$
    h. $p \Longrightarrow (q \vee r), r, p \vdash q.$
    i. $q \Longrightarrow p, r \Longrightarrow p, q \vdash r.$

22. Prove the following are valid arguments by giving formal proof.
    a. If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes.
    b. If the team is late, then it cannot play the game. If the referee is here then the team can play the game. The team is late. Therefore, the referee is not here.

23. Which of the following are sets?
    d. 1,2,3
    e. {1,2},3
    f. {{1},2},3
    g. {1,{2},3}
    h. {1,2,a,b}.

24. Which of the following sets can be described in complete listing, partial listing and/or set-builder methods? Describe each set by at least one of the three methods.
    a. The set of the first 10 letters in the English alphabet.
    b. The set of all countries in the world.
    c. The set of students of Addis Ababa University in the 2018/2019 academic year.
    d. The set of positive multiples of 5.
    e. The set of all horses with six legs.

25. Write each of the following sets by listing its elements within braces.
    a. $A = \{x \in Z : -4 < x \leq 4\}$
    b. $B = \{x \in Z : x^2 < 5\}$

c. $C=\{x \in N : x^3<5\}$

d. $D=\{x \in R : x^2-x=0\}$

e. $E=\{x \in R : x^2+1=0\}$.

26. Let A be the set of positive even integers less than 15. Find the truth value of each of the following.
    a. $15 \in A$
    b. $-16 \in A$
    c. $\phi \in A$
    d. $12 \subset A$
    e. $\{2,8,14\} A$
    f. $\{2,3,4\} \subseteq A$
    g. $\{2,4\} \in A$
    h. $\phi \subset A$
    i. $\{246\} \subseteq A$

27. Find the truth value of each of the following and justify your conclusion.
    a. $\phi \subseteq \phi$
    b. $\{1,2\} \subseteq \{1,2\}$
    c. $\phi \in A$ for any set A
    d. $\{\phi\} \subseteq A$, for any set A
    e. $5,7 \subseteq \{5,6,7,8\}$
    f. $\phi \in \{\{\phi\}\}$
    g. For any set $A, A \subset A$
    h. $\{\phi\}=\phi$

28. For each of the following set, find its power set.
    a. $\{ab\}$
    b. $\{1,1.5\}$
    c. $\{a,b\}$
    d. $\{a,0.5,x\}$

29. How many subsets and proper subsets do the sets that contain exactly $1,2,3,4,8,10$ and 20 elements have?

30. If n is a whole number, use your observation in Problems 6and 7 to discover a formula for the number of subsets of a set with n elements. How many of these are proper subsets of the set?

31. Is there a set A with exactly the following indicated property?

a. Only one subset
b. Only one proper subset
c. Exactly 3 proper subsets
d. Exactly 4 subsets
e. Exactly 6 proper subsets
f. Exactly 30 subsets
g. Exactly 14 proper subsets
h. Exactly 15 proper subsets

32. How many elements does A contain if it has:
    a. 64 subsets?
    b. 31 proper subsets?
    c. No proper subset?
    d. 255 proper subsets?
33. Find the truth value of each of the following.
    a. $\phi \in P(\phi)$
    b. *For any set* A, $\phi \subseteq P(A)$
    c. *For any set* A, $A \in P(A)$
    d. *For any set* A, $A \subset P(A)$.
34. For any three sets A, B and C, prove that:
    a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
    b. If $A \subset B$ and $B \subset C$, then $A \subset C$.
35. If $B \subseteq A$, $A \cap B' = \{1,4,5\}$ and $A \cup B = \{1,2,3,4,5,6\}$, find B.
36. Let $A = \{2,4,6,7,8,9\}$,
    $B = \{1,3,5,6,10\}$ and
    $C = ¿\{x : 3x+6=0 \text{ or } 2x+6=0\}$. Find
    a. $A \cup B$.
    b. Is $(A \cup B) \cup C = A \cup (B \cup C)$?
37. Suppose $U = ¿$ The set of one digit numbers and
    $A = ¿\{x : x$ is an even natural number less than or equal to 9$\}$
    Describe each of the sets by complete listing method:
    a. A'.
    b. $A \cap A'$.
    c. $A \cup A'$.
    d. $(A')'$
    e. $\phi - U$.
    f. ϕ'
    g. U'

38. Suppose $U =$¿ The set of one digit numbers and
$A =$¿$\{x : x$ is an even natural number less than or equal to 9$\}$
Describe each of the sets by complete listing method:
    h. A'.
    i. $A \cap A'$.
    j. $A \cup A'$.
    k. $(A')'$
    l. $\phi - U$.
    m. ϕ'
    n. U'

39. Use Venn diagram to illustrate the following statements:
    a. $(A \cup B)' = A' \cap B'$.
    b. $(A \cap B)' = A' \cup B'$.
    c. If $A \not\subseteq B$, then A¿$\neq \phi$.
    d. $A \cup A' = U$.

40. Let $A = \{5,7,8,9\}$ and $C = \{6,7,8\}$. Then show that $(A$¿$)$¿$= A\,(B$¿¿.

41. Perform each of the following operations.
    a. $\phi \cap \{\phi\}$
    b. $\{\phi, \{\phi\}\} - \{\{\phi\}\}$
    c. $\{\phi, \{\phi\}\} - \{\phi\}$
    d. $\{\{\{\phi\}\}\} - \phi$

42. Let $U = \{2,3,6,8,9,11,13,15\}$,
$A =$¿$\{x \vee x$ is a positive prime factor of 66$\}$
$B =$¿$\{x \in U \vee$¿ x is composite number $\}$ and $C = \{x \in U \vee x - 5 \in U\}$. Then find each of the following.
$A \cap B, (A \cup B) \cap C, (A - B) \cup C, (A - B) - C, A - (B - C), (A - C) - (B - A), A \cap B \cap C$

43. Let $A \cup B = \{a,b,c,d,e,x,y,z\}$ and $A \cap B = \{b,e,y\}$.
    a. *If* $B - A = \{x,z\}$, then $A =$¿¿
    b. *If* $A - B = \phi$, then $B =$¿¿
    c. *If* $B = \{b,e,y,z\}$, then $A - B =$¿¿

44. Let $U = \{1,2,\ldots,10\}, A = \{3,5,6,8,10\}, B = \{1,2,4,5,8,9\}$,
$C = \{1,2,3,4,5,6,8\}$ and $D = \{2,3,5,7,8,9\}$. Verify each of the following.
    a. $(A \cup B) \cup C = A \cup (B \cup C)$.
    b. $A \cap (B \cup C \cup D) = (A \cap B) \cup (A \cap C) \cup (A \cap D)$.
    c. $(A \cap B \cap C \cap D) = A \cup B \cup C \cup D$.
    d. $C - D = C \cap D$.

e. $A \cap (B \cap C) = (A - B) \cup (A - C)$.

45. Depending on question No. 10 find.
    a. $A \Delta B$.
    b. $C \Delta D$.
    c. $(A \Delta C) \Delta D$.
    d. $(A \cup B)(A \Delta B$¿.

46. For any two subsets A and B of a universal set U, prove that:
    a. $A \Delta B = B \Delta A$.
    b. $A \Delta B = (A \cup B) - (A \cap B)$.
    c. $A \Delta \phi = A$.
    d. $A \Delta A = \phi$.

47. Draw an appropriate Venn diagram to depict each of the following sets.
    a. U = The set of high school students in Addis Ababa.
       A = The set of female high school students in Addis Ababa.
       B = The set of high school anti-AIDS club member students in Addis Ababa.
       C = The set of high school Nature Club member students in Addis Ababa.
    b. U = The set of integers.
       A = The set of even integers.
       B = The set of odd integers.
       C = The set of multiples of 3.
       D = The set of prime numbers.

48. Use the truth table method to show that the following argument forms are valid.
    ix. $p \Longrightarrow q, q \vdash p$.
    x. $p \Longrightarrow p, p, r \Longrightarrow q \vdash r$.
    xi. $p \Longrightarrow q, r \Longrightarrow q \vdash r \Longrightarrow p$.
    xii. $r \vee s, (s \Longrightarrow p) \Longrightarrow r \vdash p$.
    xiii. $p \Longrightarrow q, p \Longrightarrow r, r \Longrightarrow s \vdash q \Longrightarrow s$.

49. For the following argument given a, b and c below:
    xiv. Identify the premises.
    xv. Write argument forms.
    xvi. Check the validity.
        a. If he studies medicine, he will get a good job. If he gets a good job, he will get a good wage. He did not get a good wage. Therefore, he did not study medicine.
        b. If the team is late, then it cannot play the game. If the referee is here, then the team is can play the game. The team is late. Therefore, the referee is not here.

c. If the professor offers chocolate for an answer, you answer the professor's question. The professor offers chocolate for an answer. Therefore, you answer the professor's question

50. Give formal proof to show that the following argument forms are valid.

j. $p \Longrightarrow q, q \vdash p.$

k. $p \Longrightarrow q, p, r \Longrightarrow q \vdash r.$

l. $p \Longrightarrow q, r \Longrightarrow q \vdash r \Longrightarrow p.$

m. $r \wedge s, (s \Longrightarrow p) \Longrightarrow r \vdash p.$

n. $p \Longrightarrow, p \Longrightarrow r, r \Longrightarrow s \vdash q \Longrightarrow s.$

o. $p \vee q, r \Longrightarrow p, r \vdash q.$

p. $p \wedge q, (q \vee r) \Longrightarrow p \vdash r.$

q. $p \Longrightarrow (q \vee r), r, p \vdash q.$

r. $q \Longrightarrow p, r \Longrightarrow p, q \vdash r.$

51. Prove the following are valid arguments by giving formal proof.

c. If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes.

d. If the team is late, then it cannot play the game. If the referee is here then the team can play the game. The team is late. Therefore, the referee is not here.