

MYUNGSUNG MEDICAL COLLEGE(MMC)
Math 100: Mathematics for Natural Sciences
Exercises on the Real and Complex Number Systems

- (1) Two incandescent lamps are turned on at the same time, one blinks every 4 seconds and the other blinks every 6 seconds. In 60 seconds, how many times will they blink at the same time?
- (2) With the universal set $\mathbb{R}$, write each of the following sets as an interval or union of intervals.
 - (a) $A = \{x|x \neq 5 \wedge x \neq 8\}$.
 - (b) $B = \{x|x^2(x^3 - 1) \neq 0\}$.
 - (c) $C = \{x|x^3 - 2x \neq 0\}$.
 - (d) $D = \{x|(x - 2)(x + 3)(x + 4) = 0 \Rightarrow x \geq 2\}$.
- (3) Simplify each of the following sets and give the final result as an interval or union of intervals
 - (a) $([-3, 2] \cap (0, 5))'$.
 - (b) $((-3, 5.6] \cap (-5, 2.\bar{4}\bar{3})) \cup (0.\bar{3}, 11)$.
 - (c) $(2, 4]' \cap ((-\infty, 5)) \cup [0, \infty))$.
 - (d) $(-0.\bar{3}, 0.0\bar{3}) \cup ((-\infty, \frac{1}{30}) \cap [5, \infty))$.
- (4) Find the least upper bound (lub) and greatest lower bound (glb) of each of the following sets of real numbers:
 - (a) $S = \{2^{-p} + 3^{-q} + 5^{-r} | p, q, r \in \mathbb{N}\}$.
 - (b) $T = \{x | 3x^2 - 10x + 3 < 0\}$.
- (5) Find the smallest and the largest elements of each of the following sets and find also the lub and glb whenever they exist.
 - (a) $A = \{x | x = 5 - \frac{1}{n}; n \in \mathbb{N}\}$.
 - (b) $B = \{x | x \in (-1, 1) \cup \{\frac{1}{n} | n \in \mathbb{N}\}\}$.
 - (c) $C = \{x | x = \frac{n}{n+1}; n \in \mathbb{N}\}$.
 - (d) $D = \{x | x = \frac{n+1}{n}; n \in \mathbb{N}\}$.
 - e) $E = \{x | x = \frac{n^3}{2n^2+1}; n \in \mathbb{N}\}$.
- (6) Using the Principle of Mathematical Induction, prove that
 - (a) $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n-1)}{2}$ for $n \geq 1$.
 - (b) $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$ for $n \geq 1$.
 - (c) $2^{2n+1} + 7$ is a multiple of 3 for $n \geq 1$.
 - (d) $n^2 - n$ is even for $n \geq 1$.
 - (e) $6^n - 1$ is divisible by 5 for $n \geq 1$.
 - (f) $n! > 3^n$ for $n \geq 7$.
- (7) Consider the sequence of real numbers defined by the relations

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for } n \geq 1.$$

Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \geq 1$.

- (8) Prove that $\sqrt{2} + \sqrt{3}$ is irrational number.
- (9) If $\frac{a}{b} < \frac{c}{d}$ with $b > 0, d > 0$, prove that $\frac{a+c}{b+d}$ lies between the two fractions $\frac{a}{b}$ and $\frac{c}{d}$.
- (10) Write each of the following complex numbers in polar form,
- (a) $-2 - 4i$. (b) $1 - \sqrt{2}i$. (c) -27 . (d) $5i$.
- (11) Find the modulus and argument of each of the following complex numbers.
- (a) $(1 - \sqrt{2}i)^2$. (b) $(1 + i)^{11}$. (c) $(\frac{1}{2} - \frac{\sqrt{3}i}{2})^{15}$.
- (12) Find $z \in \mathbb{C}$ such that
- (a) $Re(z(1 + i)) + z\bar{z} = 0$; (b) $Re(z^2) + iIm(\bar{z}(1 + 2i)) = -3$;
(c) $Im(2 - i)z = 1$.
- (13) Express the following complex numbers in the form $a + bi$.
- (a) $(1 + i)^3$. (b) $\frac{2+3i}{3-4i}$. (c) $i^5 + i^{16}$. (d) $\frac{1}{2}(1+i)(1+i^{-8})$.
- (14) Compute the modulus and the conjugate of each of the following complex numbers.
- (a) $z = (1 + i)^6$. (b) $w = i^{17}$.
- (15) In each case, determine all real x and y which satisfy the given relation.
- (a) $x + iy = |x - iy|$. (b) $x + iy = (x - iy)^2$.
(c) $\sum_{k=0}^{100} i^k = x + iy$.
- (16) Solve the following equations.
- (a) $z^2 = 1 + i$. (b) $z^2 - 4z + 6 = 0$.
(c) $z^4 + 4z^2 + 16 = 0$. (d) $(\bar{z} - i)^3 = 1$. (e) $z^5 = 32$.
- (17) Compute
- (a) the square roots of $z = -1 - i$.
(b) the three cube roots of $z = -8$.
(c) the four fourth roots of $z = -1 + \sqrt{3}i$.
- (18) Write $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$. (Hint: use $(\cos \theta + i \sin \theta)^4$)
- (19) Find the set of points $z = x + iy$ that satisfy
- (a) $|z + 1| = |z - i|$. (b) $3|z| = |z + 3i|$.
(c) $|z - 1| = 2$. (d) $z^2 \cdot \bar{z} = z$.
- (20) If $\alpha = \cos \theta + i \sin \theta$ for $0 < \theta < 2\pi$, show that $\frac{1+\alpha}{1-\alpha} = i \cot \frac{\theta}{2}$.
- (21) If $z \neq -1$ and $Im(z) \neq 0$, show that $\frac{z}{(1+z)^2}$ is real iff $z \cdot \bar{z} = 1$.
- (22) If for $z = x + iy$, $f(z) = e^x(\cos y + i \sin y)$, then show that
- (a) $f(z_1 + z_2) = f(z_1)f(z_2)$. (b) $\overline{f(z)} = f(\bar{z})$
- (23) Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.