

DEPARTMENT OF MATHEMATICS EDUCATION

Mathematics 1 for teachers



Only study guide for **MTE1501**

Prof Zingi Jojo
Ms Masilo Motshidisi
Dr Joseph Dlamini
Ms Eva Makwakwa

UNIVERSITY OF SOUTH AFRICA, PRETORIA

© 2018 University of South Africa

All rights reserved

Printed and published by the
University of South Africa
Muckleneuk, Pretoria

MTE1501/1/2019–2019

70779503

InDesign

CONTENTS

	<i>Page</i>
Introduction	v
LEARNING UNIT 1: MATHEMATICS IN SOCIETY	1
1.1 INTRODUCTION	1
1.2 LEARNING OUTCOMES	1
1.3 DEFINING MATHEMATICS	1
1.3.1 The toolbox/instrumentalist view	2
1.3.2 The Platonist view	3
1.3.3 The system view	3
1.4 THE HISTORY OF MATHEMATICS	4
1.5 THE DEVELOPMENT OF MATHEMATICS	6
1.5.1 A history of mathematics from Mesopotamia to Modernity	6
1.5.2 Babylonian mathematics	7
1.5.3 African mathematics	9
1.5.4 Greek mathematics	11
1.5.4.1 The Ionian school	12
1.5.4.2 The Pythagorean school	12
1.5.4.3 The Eleatic school	12
1.5.4.4 The five Platonist ("Platonic") solids	13
1.5.5 Egyptians mathematics	14
1.5.6 Chinese Mathematics	15
1.5.7 Indian mathematics	18
1.6 20TH CENTURY MATHEMATICS	23
1.7 FEEDBACK ON ACTIVITIES	26
1.8 CONCLUSION	26
1.9 ADDITIONAL LEARNING EXPERIENCES	26
1.10 SELF-ASSESSMENT	26
1.11 REFERENCES	26
LEARNING UNIT 2: TEACHING AND LEARNING MATHEMATICS	28
2.1 INTRODUCTION	28
2.2 LEARNING OBJECTIVES	28
2.3 RATIONALE FOR DOING MATHEMATICS	28
2.3.1 What is mathematics?	29
2.3.2 Mathematics education	30
2.4 HISTORY OF MATHEMATICS	31
2.4.1 Mathematics education in South Africa including CAPS	31
2.5 THE MEANING OF DOING MATHEMATICS	32
2.6 MATHEMATICS AS A SCIENCE BASED ON ORDER AND PATTERN	32
2.7 HOW MATHEMATICS IS LEARNED	33
2.8 CONSTRUCTION OF MATHEMATICAL IDEAS	34

2.9	TEACHING FOR MEANINGFUL LEARNING	35
2.10	DEVELOPING UNDERSTANDING IN MATHEMATICS	35
2.11	TYPES OF MATHEMATICAL KNOWLEDGE	36
2.12	CONCEPTUAL AND PROCEDURAL KNOWLEDGE IN MATHEMATICS	37
2.12.1	Procedural knowledge of mathematics	37
2.12.2	Conceptual knowledge and understanding of mathematics	39
2.13	MATHEMATICAL PROFICIENCY	39
2.14	DEMONSTRATE THE ABILITY TO DESIGN BASIC MATHEMATICAL LEARNING TASKS	41
2.14.1	Plan a problem-based classroom focusing on different content areas	41
2.14.2	Utilise teaching aids effectively to enhance a problem based lesson for meaningful learning	43
2.14.3	Designing a lesson plan	47
2.15	SELF-ASSESSMENT	48
	REFERENCES	48
LEARNING UNIT 3:	THE CONCEPT OF NUMBERS	50
3.1	INTRODUCTION	50
3.2	LEARNING OUTCOMES	50
3.3	OVERVIEW OF THE NUMBER CONCEPT	50
3.3.1	Development of the number system	50
3.3.1.1	Number concept	50
3.3.1.2	Historical background of the number system	53
3.3.1.3	Types of numbers	58
3.3.1.4	The basic number operations	63
3.4	CONCLUSION	67
3.5	SELF-ASSESSMENT	68
3.6	REFERENCES	69
LEARNING UNIT 4:	DEVELOPMENT OF SPATIAL SENSE	70
4.1	INTRODUCTION	70
4.2	LEARNING OUTCOMES	70
4.3	THE VAN HIELE LEVELS OF GEOMETRIC THOUGHT	70
4.4	DIFFERENCES BETWEEN THE TWO-DIMENSIONAL SHAPES AND THREE DIMENSIONAL OBJECTS (FIGURES)	72
4.4.1	Two dimensional shape	72
4.4.2	Polygons	72
4.4.2.1	Quadrilaterals	73
4.4.2.2	Properties of polygons	75
4.4.3	Three-dimensional objects (figures)	78
4.4.3.1	Polyhedrons	78
4.4.3.2	Nets of polyhedra	79
4.4.3.3	Properties polyhedrons (Ministry of Education, 2008)	80
4.4.3.4	Non-polyhedrons	81
4.5	CONCLUSION	83
4.6	SELF-ASSESSMENT	83
4.7	REFERENCES	84

INTRODUCTION

Welcome to the module MTE1501, Orientation to Intermediate Phase Mathematics. This module is one of the two modules in the mathematics stream of the Higher Certificate Education and Training (Intermediate Phase).

If you had limited success in mathematics in high school, you should not feel threatened by having to study this module. If you work through the study guide meticulously, you might find that concepts become clearer, and that you actually start to enjoy doing mathematics.

In this module, we endeavour to move away from the traditional way of teaching mathematics. We will concentrate on understanding the basic concepts, which form the foundation of the learning of mathematics.

There are many activities in this study guide, and we advise you to buy an exercise book and make an effort to do all of them.

The purpose of Mathematics 1 for Teachers (MTE 1501) is to cultivate the knowledge and understanding of the following aspects of mathematics teaching at the Intermediate Phase:

- the nature of mathematics, the history of its development and its role in society
- how mathematics can be taught and learned.
- the development of number concepts; and
- the development of spatial sense.

CONTENTS OF THIS MODULE

The module consists of four learning units and focuses on the in-depth teaching as well as the development of special sense. We hope the module will not only motivate you to consider a career as a mathematics teacher but will also open new doors for you to do mathematics with understanding. The content is explained in such a way that, while you are working through the content, you are also introduced to some ideas of how to teach the content for understanding. In other words, we are building your pedagogical content knowledge. This means you will not only know WHAT to teach, but also HOW and WHY you teach in a particular way. As you progress through the study guide, you will gain more knowledge and confidence to teach. But there is not an easy solution. With hard work, commitment and a positive disposition you will succeed. Believe in yourself, and “carpe diem” (seize the day)!

THE INTERMEDIATE PHASE MATHEMATICS CURRICULUM

You will be expected to download a copy of the Intermediate Mathematics Curriculum, or the CAPS (Curriculum and Assessment Policy Statement) document Website: [www.education.gov.za/Curriculum/CurriculumAssessmentPolicyStatements\(CAPS\)/CAPSIntermediate.aspx](http://www.education.gov.za/Curriculum/CurriculumAssessmentPolicyStatements(CAPS)/CAPSIntermediate.aspx)

INTRODUCTION

The curriculum is divided into three main sections.

Section 1 deals with the general aims of the curriculum in the South African context.

Section 2 contains the specific aims of the Intermediate Mathematics Curriculum and skills that learners should develop while studying mathematics.

According to CAPS (DBE, 2013:8) mathematics is described as “a language that makes use of symbols and notations to describe numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem-solving that will contribute in decision-making.”

The aims and skills are given below:

AIMS

The teaching and learning of mathematics aims to develop

- a critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations
- confidence and competence to deal with any mathematical situation without being hindered by a fear of mathematics
- a spirit of curiosity and a love for mathematics
- an appreciation for the beauty and elegance of mathematics
- recognition that mathematics is a creative part of human activity
- deep conceptual understanding in order to make sense of mathematics
- specific knowledge and skills necessary for
 - the application of mathematics to physical, social and mathematical problems
 - the study of related subject matter (e.g. other subjects)
 - further study in mathematics

SKILLS

To acquire essential mathematical skills, the learner should

- develop the correct use of the language of mathematics
- develop number vocabulary, number concept and calculation and application skills
- learn to listen, communicate, think, reason logically and apply the mathematical knowledge gained
- learn to investigate, analyse, represent and interpret information
- learn to pose and solve problems
- build an awareness of the important role that mathematics plays in real-life situations including the personal development of the learner

In section 3, the curriculum is unpacked, with examples of what learners should be able to achieve.

On the next four pages, you will find a brief summary of the curriculum. It gives you an idea of what has to be taught in the four school terms.

In the Intermediate Phase Mathematics Curriculum, common fractions are covered throughout as can be seen in the attached time allocations for the different grades in the phase. Proportional reasoning (ratios and rate) is not a specific topic in CAPS, but is applied under whole numbers, measurement etc.

GRADE 4

TIME ALLOCATION PER TOPIC: GRADE 4							
Term 1		Term 2		Term 3		Term 4	
Topic	Time	Topic	Time	Topic	Time	Topic	Time
Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours	Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours
Whole numbers: counting, ordering, comparing, representing and place value (3-digit numbers)	2 hours	Whole numbers: counting, ordering, comparing, representing and place value (4-digit numbers)	1 hour	Capacity/volume	6 hours	Whole numbers: counting, ordering, comparing, representing and place value (4-digit numbers)	1 hour
Number sentences	3 hours	Whole numbers: addition and subtraction (4-digit numbers)	4 hours	Common fractions	5 hours	Whole numbers: addition and subtraction (4-digit numbers)	4 hours
Whole numbers: addition and subtraction (3-digit numbers)	8 hours	Common fractions	6 hours	Whole numbers: counting, ordering, comparing, representing and place value (4-digit numbers)	1 hour	Mass	6 hours
Numeric patterns	4 hours	Length	7 hours	Whole numbers: addition and subtraction (4-digit numbers)	4 hours	Properties of 3-D objects	4 hours
Whole numbers: multiplication and division (1-digit by 1 digit)	4 hours	Whole number: multiplication (2-digit by 2-digit)	6 hours	Viewing objects	2 hours	Common fractions	5 hours
Time	6 hours	Properties of 3-D objects	5 hours	Properties of 2-D shapes	4 hours	Whole numbers: division (3-digit by 1-digit)	3 hours
Data handling	10 hours	Geometric patterns	4 hours	Data handling	7 hours	Perimeter, area & volume	7 hours
Properties of 2-D shapes	5 hours	Symmetry	2 hours	Numeric patterns	4 hours	Position and movement	2 hours
Whole numbers: multiplication and division (2-digit by 1-digit)	5 hours	Whole numbers: addition and subtraction (4-digit numbers)	4 hours	Whole numbers: addition and subtraction (4-digit numbers)	4 hours	Transformations	3 hours
		Whole numbers: division (3-digit by 1-digit)	4 hours	Whole numbers: multiplication (2-digit by 2-digit)	5 hours	Geometric patterns	2 hours
				Number sentences	3 hours	Whole numbers: addition and subtraction (4-digit numbers)	3 hours
				Transformations	3 hours	Probability	2 hours
Revision	5 hours	Revision	4 hours	Revision	4 hours	Revision	5 hours
		Assessment (all subjects)	6 hours			Assessment (all subjects)	6 hours
Total: 60 hours		Total: 60 hours		Total: 60 hours		Total: 60 hours	

INTRODUCTION

GRADE 5

TIME ALLOCATION PER TOPIC: GRADE 5							
Term 1		Term 2		Term 3		Term 4	
Topic	Time	Topic	Time	Topic	Time	Topic	Time
Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours	Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours
Whole numbers: counting, ordering, comparing, representing and place value (4-digit numbers)	2 hours	Whole numbers: counting, ordering, comparing, representing and place value (6-digit numbers)	1 hour	Common fractions	5 hours	Whole numbers: counting, ordering, comparing, representing and place value (6-digit numbers)	1 hour
Number sentences	3 hours	Whole numbers: addition and subtraction (5-digit numbers)	5 hours	Mass	5 hours	Whole numbers: addition and subtraction (5-digit numbers)	5 hours
Whole numbers: addition and subtraction (5-digit numbers)	5 hours	Common fractions	5 hours	Whole numbers: counting, ordering, comparing, representing and place value (6-digit numbers)	1 hour	Properties of 3-D objects	5 hours
Numeric patterns	4 hours	Length	6 hours	Whole numbers: addition and subtraction	5 hours	Common fractions	5 hours
Whole numbers: multiplication (2-digit by 2-digit) and division (3-digit by 1-digit)	6 hours	Whole numbers: multiplication (3-digit by 2-digit)	7 hours	Viewing objects	3 hours	Whole numbers: division (3-digit by 2-digit)	7 hours
Time	6 hours	Properties of 3-D objects	6 hours	Properties of 2-D objects	4 hours	Area, perimeter & volume	7 hours
Data handling	10 hours	Geometric patterns	4 hours	Transformations	3 hours	Position and movement	2 hours
Properties of 2-D shapes	7 hours	Symmetry	2 hours	Temperature	2 hours	Transformations	4 hours
Capacity/volume	5 hours	Whole numbers: division (4-digit by 2 digit)	8 hours	Data handling	9 hours	Geometric patterns	2 hours
				Numeric patterns	5 hours	Number sentences	3 hours
				Whole numbers: multiplication (3-digit by 2-digit)	7 hours	Probability	2 hours
Revision	4 hours	Revision	3 hours	Revision	3 hours	Revision	4 hours
		Assessment (all subjects)	6 hours			Assessment (all subjects)	6 hours
TOTAL: 60 HOURS		TOTAL: 60 HOURS		TOTAL: 60 HOURS		TOTAL: 60 HOURS	

GRADE 6

TIME ALLOCATION PER TOPIC: GRADE 6							
Term 1		Term 2		Term 3		Term 4	
Topic	Time	Topic	Time	Topic	Time	Topic	Time
Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours	Mental Mathematics (10 minutes daily)	8 hours	Mental Mathematics (10 minutes daily)	7 hours
Whole numbers: counting, ordering, comparing, representing and place value (6-digit numbers)	2 hours	Whole numbers: counting, ordering, comparing, representing and place value (9-digit numbers)	1 hour	Mass	5 hours	Whole numbers: Counting, ordering, comparing, representing and place value (9-digit numbers)	1 hour
Number sentences	3 hours	Whole numbers: multiplication (4-digit by 2- digit)	5 hours	Whole numbers: counting, ordering, comparing, representing and place value (9-digit numbers)	1 hour	Whole numbers: Multiplication (4-digit by 3-digit)	5 hours
Whole numbers: addition and subtraction (5-digit numbers)	7 hours	Properties of 3-D objects	5 hours	Whole numbers: addition and subtraction (6-digit numbers)	8 hours	Common fractions	5 hours
Common fractions	10 hours	Geometric patterns	6 hours	Viewing objects	3 hours	Properties of 3-D objects	5 hours
Time	4 hours	Symmetry	2 hours	Properties of 2-D shapes	4 hours	Area, perimeter & volume	7 hours
Properties of 2-D shapes	8 hours	Whole numbers: division (4-digit by 2-digit)	8 hours	Transformations	3 hours	History	1 hour
Data handling	10 hours	Decimal fractions	10 hours	Temperature	1 hour	Whole numbers: Division (4-digit by 3-digit)	7 hours
Numeric patterns	4 hours	Capacity/volume	5 hours	Percentages	5 hours	Number sentences	3 hours
				Data handling	9 hours	Transformations	3 hours
				Numeric patterns	5 hours	Position and movement	2 hours
				Length	5 hours	Probability	2 hours
Revision	4 hours	Revision	5 hours	Revision	3 hours	Revision	6 hours
		Assessment (all subjects)	6 hours			Assessment (all subjects)	6 hours
TOTAL: 60 HOURS		TOTAL: 60 HOURS		TOTAL: 60 HOURS		TOTAL: 60 HOURS	

GRADE 7

TIME ALLOCATION PER TOPIC: GRADE 7							
TERM 1		TERM 2		TERM 3		TERM 4	
Topic	Time	Topic	Time	Topic	Time	Topic	Time
Whole numbers	9 hours	Common fractions	9 hours	Numeric and geometric patterns	6 hours	Integers	9 hours
Exponents	9 hours	Decimal fractions	9 hours	Functions and relationships	3 hours	Numeric and geometric patterns	3 hours
Construction of Geometric figures	10 hours	Functions and relationships	3 hours	Algebraic expressions	3 hours	Functions and relationships	3 hours
Geometry of 2D shapes	10 hours	Area and perimeter of 2D shapes	7 hours	Algebraic equations	3 hours	Algebraic expressions	3 hours
Geometry of straight lines	2 hours	Surface area and Volume of 3D objects	8 hours	Graphs	6 hours	Algebraic equations	4 hours
				Transformation geometry	9 hours	Collect, organize and summarize data	4 hours
				Geometry of 3D objects	9 hours	Represent data	3 hours
						Interpret, analyse and report data	3,5 hours
						Probability	4,5 hours
Revision/assessment	5 hours	Revision/assessment	9 hours	Revision/assessment	6 hours	Revision/assessment	8 hours
TOTAL: 45 hours		TOTAL: 45 hours		TOTAL: 45 hours		TOTAL: 45 hours	

CURRICULUM AND ASSESSMENT POLICY STATEMENT (CAPS)

The National Curriculum Statement

In the preliminaries of this document, the following is stated:

The National Curriculum Statement Grades R-12 aims to produce learners that are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively as individuals and with others as members of a team;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;
- use science and technology effectively and critically showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as a set of related systems by recognising that problem solving contexts do not exist in isolation. (DBE NCS 2011:5)

Specific aims

The aims of teaching and learning mathematics are to develop (Department of Basic Education 2011:8)

- a critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations
- confidence and competence to deal with any mathematical situation without being hindered by a fear of mathematics
- a spirit of curiosity and a love for mathematics
- an appreciation for the beauty and elegance of mathematics
- recognition that mathematics is a creative part of human activity
- deep conceptual understanding in order to make sense of mathematics
- specific knowledge and skills necessary for
 - the application of mathematics to physical, social and mathematical problems
 - the study of related subject matter (e.g. other subjects)
 - further study in mathematics

LEARNING UNIT 1

MATHEMATICS IN SOCIETY

1.1 INTRODUCTION

This learning unit provides insights on the history of mathematics. Firstly, various definitions of mathematics will be presented; secondly, we will look at the history of mathematics in relation to its origin; and thirdly, developments as it evolved in social and cultural contexts of the Babylonians, Egyptians, Greeks, Chinese, Arabs/ Islam, Indians, and Africans will be explored.

Please take note that you are required to study all the sections in this learning unit.

1.2 LEARNING OUTCOMES

On completion of this learning unit you should be able to:

- define mathematics
- demonstrate an awareness of how the history of Mathematics evolved in different social and cultural contexts
- identify, evaluate and solve real-life problems within real-life mathematical contexts

1.3 DEFINING MATHEMATICS

We all acknowledge the importance of mathematics as a school subject and in our everyday lives. However, very few people really understand what mathematics is about and how it was developed, yet the perceptions of the nature and role of mathematics held by society have a major influence on the development of school Mathematics curricula, instruction, and research.

Mathematics is often defined as a collection of “rules”, arithmetic computations, mysterious algebraic equations or geometric proofs that need to be learnt in order to pass an examination. Some people even think that mathematics was made up by somebody simply to complicate life. . Ptolemy (1952) defined Mathematics as something that “shows up quality with respect to forms and local motions, seeking figure, number, and magnitude, and also place, time, and similar things”.

Some of the general beliefs of what mathematics is about include the following:

- Mathematics requires a good memory.
- Mathematics is based on memorisation of facts, rules, formulas and procedures.
- You have to have a special brain to be able to do mathematics.

- Mathematics is not creative.
- There is a best way to solve a mathematics problem.
- Every mathematics problem has only one correct answer and the goal is to find the answer.
- Mathematics problems are meant to be solved as quickly as possible.
- Mathematics is all symbols and no words.
- School mathematics is useless.
- Mathematics is only for keeping the mind busy (Goodenough & Long, 2002).

Most of these restricted (even negative) views of mathematics stem from very authoritarian (which some people have called “traditional”) approaches to the teaching of mathematics. In such traditional teaching, the teacher tells learners about, or explains a mathematical concept or idea. Learners are not taught to see the importance of mathematics and why it is necessary. The teacher shows the learners how to use a mathematical idea in a certain way in order to get to the correct answer. Learners then practise the method and rely on the teacher to provide them with the correct answers.

This approach produces a follow-the-rules, computation-driven, answer-oriented view of mathematics. Learners exposed to this method of teaching accept that every problem has one solution only and that they cannot solve a problem without being told a method of solution beforehand. The “rules” often do not make sense to learners and there is nothing exciting about the lessons, particularly when they cannot remember the rule! These learners think mathematics was created only to make life a misery and is without purpose.

Thus, there are basically two views of teaching and learning mathematics. One view is traditional. The stereotypical traditional view emphasises procedures and the solving of routine problems without engaging learners in the origin, history and evolution of mathematics. It sees mathematics as a **static** discipline developed abstractly. Others see mathematics as a **dynamic** discipline, constantly changing as a result of new discoveries from experimentation and application. These contrasting views of the nature and source of mathematical knowledge have provided a continuum for conceptions of mathematics since the age of the Greeks.

Take note of the themes that were built from these two views as they are discussed in the next section.

Törner and Grigutsch (1994) considered mathematics as either a toolbox, a system or a process. The toolbox-view sees mathematics as a set of skills. The system-view sees mathematics as a logical and rigorous system. The process-view of mathematics states that it is a constructive problem-solving process (Törner & Grigutsch (1994). In addition, Ernest (1989) presents an instrumentalist view, which describes mathematics as “an accumulation of facts, rules and skills to be used in the pursuance of some external end” (p. 250). There is also the Platonist view, which states that mathematics is “a static but unified body of certain knowledge” (Plato, 1952: 250). Lastly there is a problem-solving view, describing mathematics as “a dynamic, continually expanding field of human creation and invention” (Plato, 1952: 250).

1.3.1 The toolbox/instrumentalist view

The way mathematics is taught and learned influences a person’s view of it. According to Beswick (2005) and other studies, this view is connected to

content-focused teaching where the emphasis is on performance and learning is seen as passive reception of knowledge. This view usually falls short of deep understanding, knowledge construction and learners have no active role in mathematics lessons. As a result, it is not experienced as interesting. The emphasis is mainly on ensuring the correctness of facts and rules.

1.3.2 The Platonist view

The Platonist view is often connected to an objectivistic worldview. This view involves understanding mathematics as a consistent, connected and objective structure. Mathematical objects are seen to be real and exist independently of human thinking. Mathematical statements are considered to be objectively true or false, and their truth-value is also seen to be independent from human understanding. In addition, mathematical knowledge is seen to be non-empirical.

This objectivistic view of knowledge positions the teacher in the classroom as an explainer and the learning is seen as a reception of knowledge. The Platonist view implies content-focused teaching, and the emphasis is on understanding and learning as an active construction of understanding.

1.3.3 The system view

This view endeavours to order results into a deductive system, for example the importance of proof and proving in mathematics. The building of an axiomatic system is seen as an essential goal. This view is useful in identifying inconsistencies, stated and non-stated assumptions, and it unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts.

In defining mathematics, you will be provided with definitions of modern mathematics; of what makes it important in, or irrelevant to, our everyday life; how mathematics came about in a clear, every day sense; and perhaps what is it like to be an active mathematician.

Mathematics is a branch of knowledge that allows individuals to discover properties of a world of ideas that human beings create, through the construction of proof and counterexamples (Hersh, 1997). Mathematics is part of human culture and mathematics acknowledge isn't infallible. It is a science; it advances by making mistakes, by correcting and re-correcting them; and it might have some gaps, depending on how one approaches it.

ACTIVITY 1.1

- (1) You have been provided with different definitions of mathematics. Read the definitions again and give your own definition of mathematics as it fits into your life.
 - (2) Write a short paragraph on your experiences as a learner in a mathematics class when you were at school. Write about at least one good experience and one bad experience.
-

1.4 THE HISTORY OF MATHEMATICS

Discussions on the nature of mathematics date back to the fourth century BCE. Among the first major contributors to the dialogue were **Plato** and his student, **Aristotle**.

Plato (1952) took the position that **the objects of mathematics had an existence of their own, beyond the mind, in the external world**. In doing so, he drew clear distinctions between the ideas of the mind and their representations perceived in the world by the senses. This caused Plato to **draw distinctions between arithmetic (the theory of numbers) and logistics (techniques of computation)**. Plato argued that the study of arithmetic has a positive effect on individuals; that it compels them to reason about abstract numbers. For example, in Plato's discussion of the five regular solids, he based his argument on mathematics as a **theory of an external, independent, unobservable body of knowledge**.

However, this was different for his student, Aristotle. He viewed mathematics as one of three genera into which knowledge could be divided: **the physical, the mathematical, and the theological**. Aristotle based his understanding of mathematics on experienced reality, where knowledge is obtained from experimentation, observation, and abstraction. In Aristotle's view, the construction of a mathematical idea comes through idealisations performed by the mathematician as a result of experience with objects. Aristotle attempted to understand mathematical relationships through the collection and classification of empirical results derived from experiments and observations, and then by deduction of a system to explain the inherent relationships in the data. By the Middle Ages, Aristotle's work became known for its contributions to logic and its use in substantiating scientific claims. It was from the two contrasting views or schools of mathematical thought that Francis Bacon, in the early 1500s, separated mathematics into pure and mixed mathematics.

From the Platonic school of thought, there was an outgrowth that **set out to show that ideas of mathematics could be viewed as a subset of the ideas of logic**. This was the first school founded by the German mathematician Gottlob Frege in 1884. It was called the school of **logicism**. The proponents of logicism set out to show that mathematical propositions could be expressed as completely general propositions whose truth followed from their form rather than from their interpretation in a specific contextual setting. **Logicists argued that the ideas of mathematics can be viewed as a subset of the ideas of formal logic**. They thought that mathematics consists of objects and an external (outside the senses) structure of generalised statements.

The followers of the Dutch mathematician, L.E.J. Brouwer, on the other hand, did not accept the existence of any idea of classical mathematics unless it could be **constructed via a combination of clear inductive steps from first principles**. Members from this school of thought were called **intuitionists**.

Unlike the logicists, who accepted the contents of classical mathematics, **the intuitionists accepted only the mathematics that could be developed from the natural numbers forward through the mental activities of constructive proofs**. This conception portrayed mathematics as the objects resulting from "valid" demonstrations. Mathematical ideas existed only insofar as they were constructible by the human mind.

The insistence on construction placed the mathematics of the intuitionists within the Aristotelian tradition. This view took logic to be a subset of mathematics. The intuitionists' labours resulted in a set of theorems and conceptions different from those of classical mathematics.

The third conception of mathematics emerged near the beginning of the 20th century and was called **formalism**. This school was molded by the German mathematician, David Hilbert. Hilbert's views were more in line with the Aristotelian tradition than with Platonism. He saw mathematics as arising from **intuition based on objects that could at least be considered as having concrete representations in the mind**. Formalism was grounded in the attempts to characterise mathematical ideas in terms of formal axiomatic systems. This attempt to free mathematics from contradictions was built around the construction of a set of axioms for a branch of mathematics that allowed for the topic to be discussed in a first-order language.

All three of those schools of thought tended to view the contents of mathematics as products. In **logicism**, the contents were the elements of the body of classical mathematics, its definitions, its postulates, and its theorems. In **intuitionism**, the contents were the theorems that had been constructed from first principles via "valid" patterns of reasoning. In **formalism**, mathematics was made up of the formal axiomatic structures developed to rid classical mathematics of its shortcomings. The influence of the Platonic and Aristotelian notions still ran as a strong undercurrent through these theories. The origin of the "product", either as a pre-existing external object or as an object created through experience from sense perceptions or experimentation, remained an issue.

The **modern view** of mathematics proposes that mathematics must be accepted as a human activity; an activity not strictly governed by any one school of thought (logician, formalist, or constructivist). Mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be represented or suggested by physical objects). The main properties of mathematical activity or mathematical knowledge are listed as:

- Mathematical objects are invented or created by humans.
- They are created not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life. This implies that there is mathematics in everything that humans have created and also in everything that humans have not created.
- Once created, mathematical objects have properties which are well determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them (Herse, 1997, p. 22). In school children have to learn formulas, exact proofs or formalised definitions. Without real life connections this kind of maths learning may restrict the talk about maths into formal mathematics.

The philosophy of the nature of mathematics should call for experiences that help mathematicians, teachers, and students to experience the invention of mathematics. It should call for experiences that allow for the mathematisation (construction of mathematical ideas), or modelling of ideas and events. According to Steinbring (2006, 136) mathematical knowledge cannot be revealed by **a mere reading of mathematical signs, symbols, and principles**. The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge – one cannot understand these signs without any presuppositions. Such implicit knowledge, as well as attitudes and ways of

using mathematical knowledge, are essential within a culture. Therefore, the learning and understanding of mathematics requires a cultural environment.

Always remember that when children make actions, marks, draw, model and play; they make personal meaning. It is the child's own meanings that should be the focus of the developing interest, rather than the child's outcome of an adult's planned piece of work, such as copied writing or representing a person "correctly".

ACTIVITY 1.2

Two major contrasting themes concerning the nature of mathematics have been discussed in the section above. Identify those themes and give clear distinctions between them. Please give examples in each case.

1.5 THE DEVELOPMENT OF MATHEMATICS

Before you read about the origin, development and evolution of mathematics, do the following activity.

ACTIVITY 1.3

When did mathematics begin? Write a short paragraph and indicate the first people you believe who started mathematics. Why were they doing this mathematics? What prompted them to start this idea? What kind of mathematics?

Try to think of your own answers to these questions. Even if you are not correct, keep them as your focus goal.

FEEDBACK

This activity focuses mainly on the use of mathematics by earlier people before it was written down. For instance, how were they separating in order to hunt for food, or how they used to divide amongst themselves, and so on.

1.5.1 A history of mathematics from Mesopotamia to Modernity

Many people used mathematics for many reasons such as to fulfil their daily and real-life needs. According to Hodgkin (2005), many cultures before and since the Greeks have used mathematical operations. People used simple counting and measuring to solve problems of differing degrees of difficulty. The question is how one draws the line to demarcate when mathematics proper started, or if indeed it is worth drawing.

As we shall see, the early history of Greek mathematics is hard to reconstruct with certainty. In contrast, the history of the much more ancient civilisations of Iraq (Sumer, Akkad & Babylon) in the years from 2500 to 1500 BCE provides a quite detailed, if still patchy record of different stages along a route which leads to mathematics of a kind. Without retracing the whole history in detail, in

this unit we look at some of these stages as illustrations of the problem raised by our initial question(s). Mathematics of what kind, and what for? And what are the conditions which seem to have favoured its development?

The next sections give the different societies' history of mathematics and some reasons why they used mathematics.

1.5.2 Babylonian mathematics

Various civilizations with different names followed each other in the region now known as Iraq, from about 4000 to 300 BCE (the approximate date of the Greek conquest). Evidence about them is entirely archaeological – The artefacts and records that they left have been excavated and studied by scholars. From a very early date, for whatever reason, they had, as the quotation from Bottéro describes, developed a high degree of hierarchy, slave or semi-slave labour, and obsessive bureaucracy in the service of a combination of kings, gods, and their priests.

“Priority in the development of mathematics belongs to Babylon, where ancient land numeration, algebra, and geometry methods existed at least from the Hammurabi dynasty, around 1700 BCE, although neither Egypt nor Rome advanced beyond the level of elementary practical arithmetic and mensuration” Beswick (2005, p40). These ancient civilizations developed mostly practical mathematics, but their effect, even upon modern mathematics, is still enormous.

Marxist Karl advanced the thesis that mathematics was born out of the need of the ancient Oriental states of Egypt and Iraq to control their irrigation. There is little doubt that mathematics arose from necessity. The annual flooding of the Nile valley, for example, forced the ancient Egyptians to develop some system of re-establishing land boundaries. Increased barter increased the need for early practical arithmetic. **The need for a calendar**, if a basic one, led to development in mathematics; “the astronomy of the old Babylonian period was just adequate for maintaining the calendar on which the irrigation system supporting the civilization depended” Struik (1948, p183). Civilization and mathematics are inseparable, i.e., “mathematics beyond primitive counting originated within the evolution of advanced form: its society” (Struik, 1948, p183?).

As Aristotle once pointed out, it is necessary for a civilization to separate the thinking class from the working class. It is necessary because some people think civilization is an advanced stage in the progression of human cultural evolution. But, when historians or anthropologists use the term civilization, they mean a society has many different, interconnected parts. So, rather than thinking about different forms of social organization as completely separate models, it's helpful to think in terms of a spectrum of complexity. The development of counting-symbols seems to take place at a time when the things counted, e.g., different types of pigs were described by pictures rather than any phonetic system of writing.

There were several features that subsequent scholars felt could be identified as truly “mathematical”:

- (1) the use of a sophisticated system for writing numbers
- (2) the ability to deal with quadratic (and sometimes, if rather by luck, higher order) equations

- (3) the “uselessness” of problems, even if they were framed in apparently useful language

Today we write our numbers in a “place-value” system, derived from India, using the symbols 0, 1, ..., 9; so that the figure “3” appearing in a number means 3, 30, 300, etc. (i.e. 3×100 , 3×101 , 3×102 , ...) depending on where it is placed. The Babylonians used a similar system, but the base was 60 instead of 10 (“sexagesimal” not “decimal”), and they therefore based it on signs corresponding to the numbers 1, ..., 59 – without a “zero” sign. The signs were made by combining symbols for “ten” and “one”.

Examples

- “1, 30” means, then, what we would call $[(1 \times 60) + 30] = 90$
- “2, 30, 30” means $(2 \times 60^2) + (30 \times 60) + 30 = 7\,200 + 1\,800 + 30 = 9\,030$

Note that in the Babylonian system the base 60 plays the role which is played by base 10 in our decimal system. In this system, a number x would be presented as a reciprocal $\frac{1}{x}$ of 60. To get the Babylonian system from the decimal system, you need to divide 60 by the given number x . The table below illustrates the numbers in the Babylonian system.

TABLE 1.1
Babylonian system

The number x	Babylonian System
2	30
3	20
4	15
5	12
6	10
7	8
8	7,30
9	6,40

The only problem noticed with this system was the absence of zero.

ACTIVITY 1.4

Work out $(1,40)/(8)$ using the above table, given that the reciprocal of 8 is 7,30. (Check that this is indeed the reciprocal, and verify that you have the right answer, given that $1,40 = 100$ in the decimal system.)

Early mathematics consisted almost exclusively of trial and error. Early Egyptian mathematics was geometry. The Egyptians also developed formulas for the areas and volumes of many shapes, but used trial and error rather than proofs, so they were not entirely correct in their formulas.

It is also traditional to consider Babylonian mathematics more "serious" than Egyptian, because its number system was more sophisticated, and the problems solved more difficult. The Babylonians were only more advanced than the Egyptians in using the number system. The Babylonians were interested in number relations beyond the merely practical mathematics, i.e. "an old Babylonian text (1700 BC) investigates triples of Pythagorean numbers; that is, numbers satisfying the relationship $a^2 + b^2 = c^2$. The facility of numerical computation, as the result of the place value notation, is visible everywhere in Babylonian mathematics. The Babylonians had a sort of combination between base 10 and base 60 with a zero.

ACTIVITY 1.5

- (1) What do you think the Babylonians used Mathematics for? Or, what made them to start mathematics?
 - (2) From the reader, can you identify the approximate time when written mathematics began and provide some evidence thereof.
-

1.5.3 African mathematics

The idea of purely Eurocentric origins of mathematics can no longer be upheld. The oldest (35 000 BC) mathematical object was found in Swaziland. (The oldest fossils of modern man are nearly 150 000 years old. Perhaps, on a rainy day, a human walked in wet sand near what is now known as Border Cave in the Lebombo Mountains between South Africa and Swaziland. In the 1970s, during the excavations of Border Cave, a small piece of the fibula of a baboon, the Lebombo bone, was found, marked with 29 clearly defined notches, and, at 37 000 years old, it ranks with the oldest mathematical objects known. The bone is dated approximately 35 000 BCE and resembles the calendar sticks still in use by Bushmen clans in Namibia.) The oldest example of arithmetic (6000 BC) was found in Zaire. Although the area is sparsely populated today, approximately 25 000 (update from 9 000) years ago, on the shores of the lake, lived a small community that fished, gathered, and grew crops. The settlement only existed a few hundred years before being buried in a volcanic eruption. The place where their remains were found (in 1960) now has a name given to these people – Ishango. Among their remains is the second oldest mathematical object in Africa. Some say that the Ishango Bone is the oldest table of prime numbers. Marshack later concluded, on the basis of his microscopic examination, that it represented a six-month lunar calendar, prime numbers or menstrual calendar.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

FIGURE 1.

All the numbers not crossed are the prime numbers between 1 and 100

The 4000 year old so-called Moscow papyrus contains geometry from the Middle Kingdom of Egypt, the consequence of the formula for the volume of a truncated square pyramid. From Herodotus (~450 BC) to Proclus (~400 BC) to Aristotle (~350 BC), Egypt was the cradle of mathematics (astronomy and surveying too). From the earliest, the great Greek mathematicians, including Pythagoras (~500 BC), Thales (~530 BC), and Exodus (the teacher of Aristotle) all learned much of their mathematics from Egypt (Mesopotamia, and possibly India) – even the concept of zero.

It is true that a zero placeholder was not used (or needed) in the Egyptian hieroglyphic or hieratic numerals, because these numerals did not have positional value. But the zero concept has many other applications.

Generalisations about the area of a circle, or the volume of a truncated square pyramid are most evident in Egyptian mathematics. Checking the correctness of a division by a subsequent multiplication, or verifying the solutions of different types of equation by the method of substitution are found from a time before the Greeks “existed”. A method used in Europe until the 19th century for solving linear equations, is generally known as the method of false position. This method was commonly used in Egyptian mathematics to solve practical problems such as finding the potency of beer, or optimal feed mixtures for cattle and poultry.

A century before US slavery was ended, slaves and even ordinary African slave traders demonstrated mathematical abilities more sophisticated than the European buyers. Some 250 years prior to Newton and Liebnitz, a 15th century Indian mathematician, Madhava of Kerala, derived infinite series for sequences and for some trigonometric functions. Here we end our discussions of African ancient mathematics, and we will not discuss the extremely significant results due to Indian, Chinese, Babylonian, South American and other non-European groups.

ACTIVITY 1.6

By looking at the African history of mathematics, write down the country, the year, the mathematical evidence found during that particular year, and the use for that particular evidence.

1.5.4 Greek mathematics

The ancient Greek numeral system, known as Attic or Herodianic numerals, was fully developed by about 450 BCE, and in regular use possibly as early as the 7th Century BCE. It was a base 10 system similar to the earlier Egyptian one (and even more similar to the later Roman system), with symbols for 1, 5, 10, 50, 100, 500 and 1,000 repeated as many times needed to represent the desired number. Addition was done by totalling separately the symbols (1s, 10s, 100s, etc) in the numbers to be added, and multiplication was a laborious process based on successive doublings (division was based on the inverse of this process).

One feature of mathematics which has remained fairly constant from the earliest times to the present day, is a general view that its aim is to use “numbers” to solve problems which arise in the world. Plato, a philosopher, whose dates are usually given as roughly 427–348 BCE, and who was mostly writing in the early fourth century, is one of the central figures in the history of Greek mathematics. There are a number of reasons for this. A simple one is that Plato dealt in some detail with mathematical questions in his works, and, while mathematics had supposedly been practised for 200 years before his time, his *Dialogues* are the earliest first-hand documents that we have.

Almost equally important is that Plato defined a particular view of what mathematics was, or should be. A rough characterisation is that real mathematics is more abstract – numbers are no longer numbers of “things” or measurements of length, area, or time, but have an independent existence as objects that one reasons with.

One of Plato’s longest and clearest mathematical discussions often referred to, is in the dialogue called the *Meno*. Plato defined a particular view of what mathematics was, or should be. A rough characterisation is that real mathematics is more abstract – numbers are no longer numbers of ‘things’ or measurements of length, area, or time, but have an independent existence as objects which you reason with **the** concept of **time** as a way to **measure the** ... This view doesn’t mean that **time** does **not** exist, **but that time has more** to do with space than with **the** ... “Einstein said, ‘**Time has no independent existence** apart from **the** ... that **the number 1** comes before **the number 2** in **the** numerical order

Socrates said earlier in the same *Dialogue as that of Plato*. The ordinary arithmetician, surely, operates with unequal units; his “two” may be two armies or two cows or two anything from the smallest thing in the world to the biggest; while the philosopher will have nothing to do with him, unless he consents to make every single instance of his unit equal to every other of its infinite number of instances. (Plato, *Philebus*, cited in Fauveland Gray 2.E.4, p. 75). The Greeks wrote histories of mathematics:

- histories of arithmetic, geometry and astronomy
- a history of physics
- the Mathematical Collection, an account of classical mathematics from Euclid to Ptolemy which forms the basis of Euclidean Geometry
- Treasury of Analysis, a collection of the Greek works themselves
- the Commentary, treating Book I of Euclid

The Greeks had three schools: The Ionian, Pythagorean and the Eleatic schools.

1.5.4.1 *The Ionian school*

This school was founded by Thales who discovered the five basic theorems in plane geometry. One of them was that every triangle inscribed in a semicircle is a right triangle. Another result, that the diameter bisects a circle, appears in *The Elements of Euclid* as a definition.

1.5.4.2 *The Pythagorean school*

This school was founded by Pythagoras whose contributions included:

- philosophy
- the study of proportion
- the study of plane and solid geometry

1.5.4.3 *The Eleatic school*

This school was founded by Zeno of Elea who proposed that:

- any unit has magnitude
- any unit is infinitely divisible
- any unit is indivisible

The three classical problems were also discovered by the Greeks.

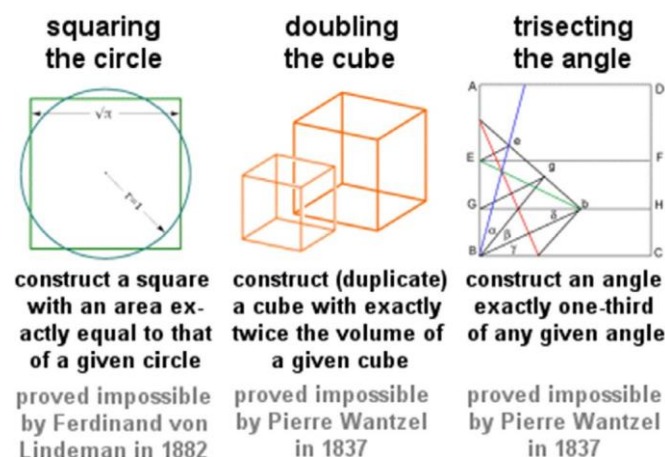


FIGURE 1.2

The Three Classical Problems

1.5.4.4 The five regular ("Platonic") solids

There are precisely five regular convex polyhedra, namely the Platonic solids: **the tetrahedron, cube, octahedron, icosahedron and dodecahedron**. A regular polyhedron has its volume enclosed by flat faces so that:

- for every vertex, the number and nature of the faces meeting at that vertex, and the angles between the edges incident with the vertex, are all equal; and
- all faces are congruent regular polygons (all equilateral triangles, all squares, all regular pentagons, etc).

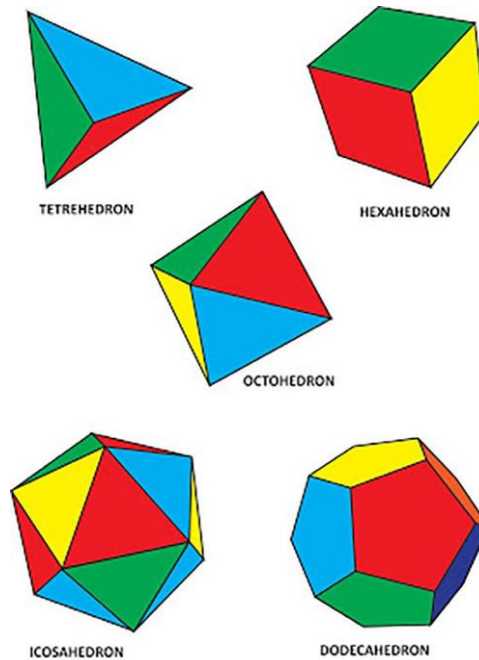
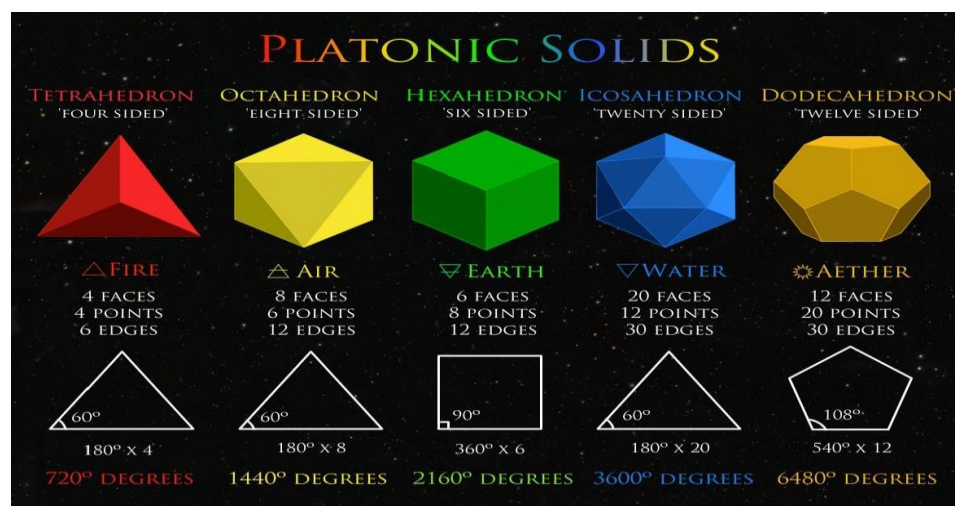


FIGURE 1.3

The five regular ("Platonic") solids

The literature on ancient Greek mathematics (see 1.5.4) is large. The primary sources for the period we are considering – up to and including Euclid, around 300 BCE – are the works of Plato, Aristotle, and Euclid himself.

To see how "classical" Greek mathematics works, it is best to start with Euclid's *Elements*. This is a strange and complex work – some say a composite, or a cut-and-paste compilation of previous works – but it is the most read and com-

mented on of all mathematical works in history, and thus deserves a central position in any account. For that reason, we shall privilege it over the harder works of Archimedes and Apollonius, the other main classics. The works of Euclid and his successors have been “classics” for three major civilizations – their own, the medieval Islamic, and the modern European.

ACTIVITY 1.7

Name and define the five Platonic solids and their characteristics.

1.5.5 Egyptians mathematics

It is thought that the Egyptians introduced the earliest fully-developed base-10 numeration system at least as early as 2700 BCE (but probably much earlier). Written numbers were depicted with a stroke for units, a heel-bone symbol for tens, a coil of rope for hundreds and a lotus plant for thousands, as well as other hieroglyphic symbols for higher powers of ten up to a million. However, there was no concept of place value, so larger numbers were rather unwieldy (e.g. although a million required just one character, a million minus one required fifty-four characters).

Multiplication, for example, was achieved by a process of repeated doubling of the number to be multiplied on one side, and of one on the other; essentially a kind of multiplication of binary factors similar to that used by modern computers (see figure 1.4 below). These corresponding blocks of counters could then be used as a kind of multiplication reference table: first, the combination of powers of two that add up to the number to be multiplied by was isolated, and then the corresponding blocks of counters on the other side yielded the answer. This effectively made use of the concept of binary numbers, over 3000 years before Leibniz introduced it into the West, and many more years before the development of the computer was to fully explore its potential.

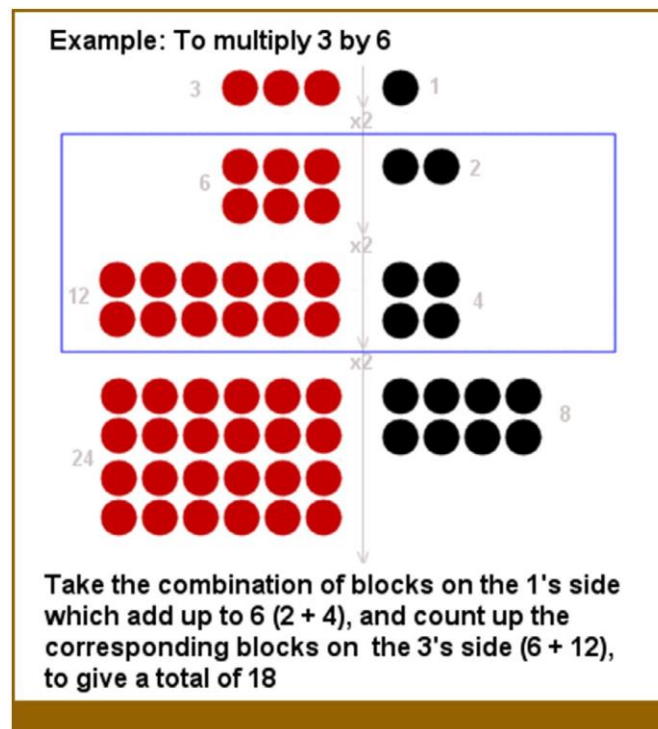


FIGURE 1.4

Ancient Egyptian method of multiplication

Practical problems of trade and market led to the development of a notation for fractions. Papyrus found demonstrate the use of unit fractions based on the symbol of the Eye of Horus, where each part of the eye represented a different fraction, each half of the previous one (i.e. half, quarter, eighth, sixteenth, thirty-second, sixty-fourth), so that the total was one-sixty-fourth short of a whole, the first known example of a geometric series. Unit fractions could also be used for simple division sums. For example, if they needed to divide 3 loaves among 5 people, they would first divide two of the loaves into thirds and the third loaf into fifths, then they would divide the left over third from the second loaf into five pieces. Thus, each person would receive one third plus one fifth plus one fifteenth (which totals three fifths, as we would expect).

The Egyptians approximated the area of a circle by using shapes with known areas. They observed that the area of a circle of diameter nine units, for example, was very close to the area of a square with sides of eight units, so that the area of circles of other diameters could be obtained by multiplying the diameter by $\frac{8}{9}$ and then squaring it. This gives an effective approximation of π accurate to within less than one per cent.

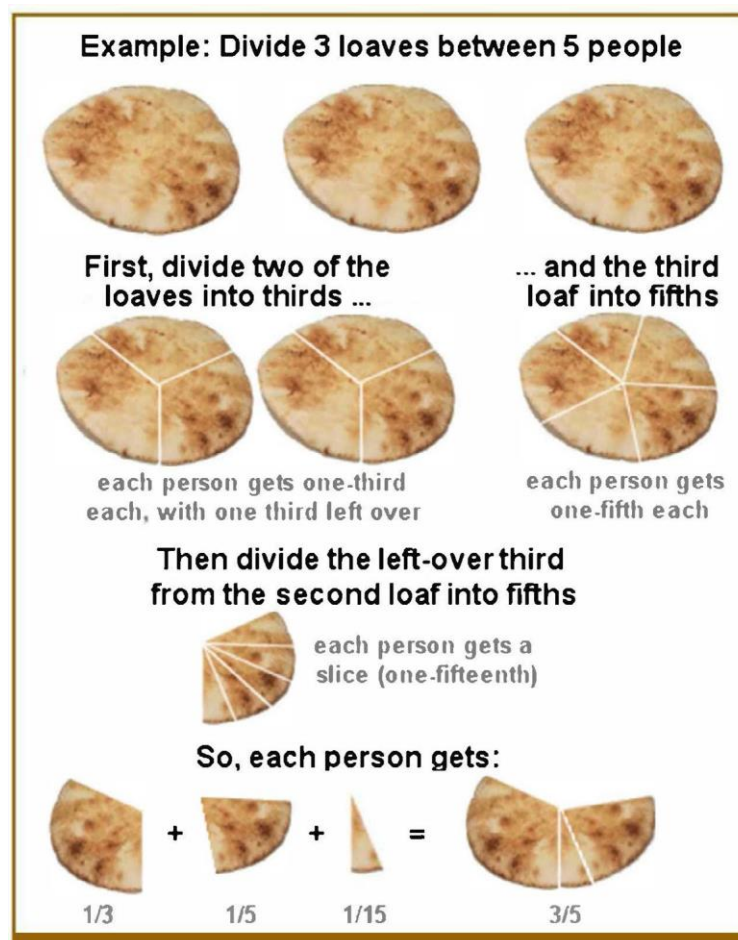


FIGURE 1.5

Ancient Egyptian method of division.

1.5.6 Chinese Mathematics

Even as mathematical developments in the ancient Greek world were beginning to falter during the final centuries BCE, the burgeoning trade empire of China was leading Chinese mathematics to ever greater heights.

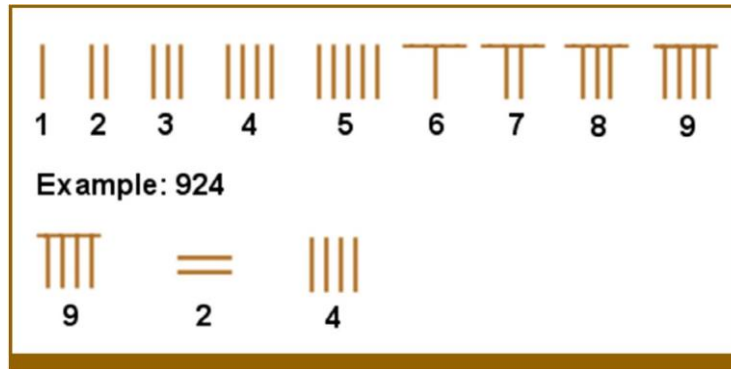


FIGURE 1.6
Ancient Chinese number system

The simple but efficient ancient Chinese numbering system, which dates back to at least the second millennium BCE, used small bamboo rods arranged to represent the numbers 1 to 9, which were then placed in columns representing units, tens, hundreds, thousands, etc. It was therefore a decimal place value system, very similar to the one we use today – indeed, it was the first such number system, adopted by the Chinese over a thousand years before it was adopted in the West – and it made even quite complex calculations very quick and easy.

Written numbers, however, employed the slightly less efficient system of using a different symbol for tens, hundreds, thousands, etc. This was largely because there was no concept or symbol of zero, and it had the effect of limiting the usefulness of the written number in Chinese.

The use of the abacus is often thought of as a Chinese idea, although some type of abacus was in use in Mesopotamia, Egypt and Greece, probably much earlier than in China (the first Chinese abacus, or “suanpan” we know of, dates to about the second century BCE).

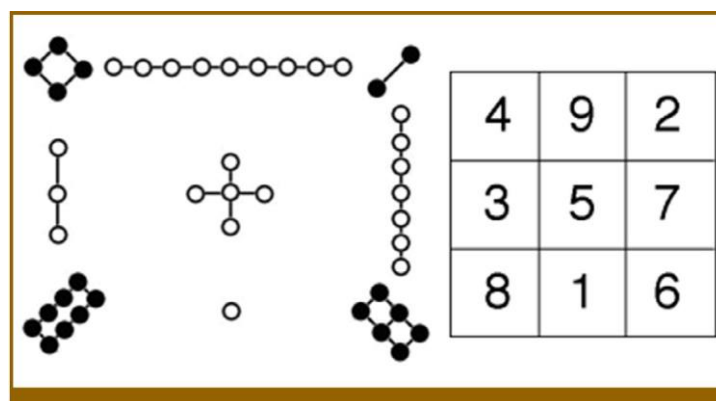


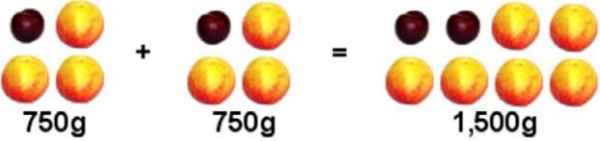
FIGURE 1.7
The Lo Shu magic square, with its traditional graphical representation.

There was a pervasive fascination with numbers and mathematical patterns in ancient China, and different numbers were believed to have cosmic significance. In particular, magic squares – squares of numbers where each row, column and diagonal added up to the same total – were regarded as having great spiritual and religious significance.

The Lo Shu Square, an order-three square where each row, column and diagonal adds up to 15, is perhaps the earliest of these, dating back to around 650 BCE (the legend of Emperor Yu's discovery of the square on the back of a turtle is said to have taken place in about 2800 BCE). But soon, bigger magic squares were being constructed, with even greater magical and mathematical powers, culminating in the elaborate magic squares, circles and triangles of Yang Hui in the 13th century (Yang Hui also produced a triangular representation of binomial coefficients identical to the later Pascal's triangle, and was perhaps the first to use decimal fractions in the modern form).

Problem:
If one plum and three peaches weigh a total of 750g, and two plums and one peach weigh a total of 500g, how much does a single peach and plum weigh?

First, double the contents of the first scale:



750g + 750g = 1,500g

Subtract from this the contents of the second set of scales:



1,500g - 500g = 1,000g

Therefore, a single peach must weigh **200g** (1,000 ÷ 5).

Then, take the peach off the second scale:



500g - 200g = 300g

Therefore, a single plum must weigh **150g** (300 ÷ 2)

FIGURE 1.8

Early Chinese method of solving equations

But the main thrust of Chinese mathematics developed in response to the empire's growing need for mathematically competent administrators. A textbook called *Jiuzhang Suanshu* or *Nine Chapters on the Mathematical Art* (written over a period of time from about 200 BCE onwards, probably by a variety of authors) became an important tool in the education of such a civil service, covering hundreds of problems in practical areas such as trade, taxation, engineering and the payment of wages.

It was particularly important as a guide on how to solve equations – the deduction of an unknown number from other known information – using a sophisticated matrix-based method which did not appear in the West until Carl Friedrich Gauss rediscovered it at the beginning of the 19th century (and which is now known as Gaussian elimination).

Among the greatest mathematicians of ancient China was Liu Hui, who produced a detailed commentary on the *Nine Chapters* in 263 CE, was one of

the first mathematicians known to leave roots unevaluated, giving more exact results instead of approximations. By an approximation using a regular polygon with 192 sides, he also formulated an algorithm which calculated the value of π as 3.14159 (correct to five decimal places), as well as developing a very early forms of both integral and differential calculus.

1.5.7 Indian mathematics

Despite developing quite independently of Chinese (and probably also of Babylonian mathematics), some very advanced mathematical discoveries were made at a very early time in India. The diagram illustrated as figure 1.9 indicates the origins of Indian Mathematics.

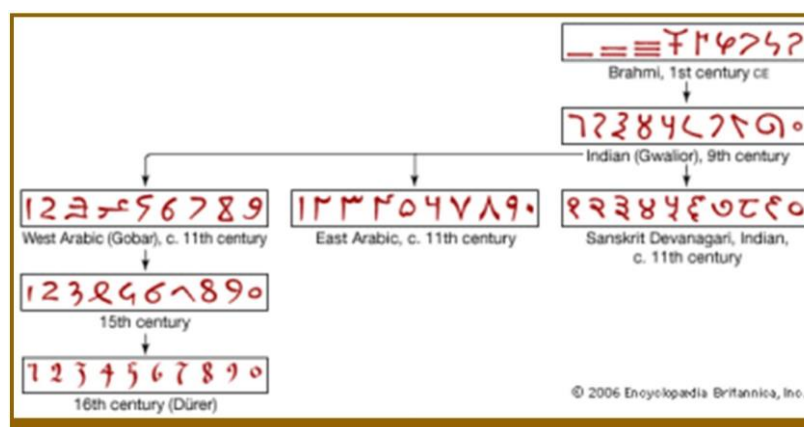


FIGURE 1.9
The evolution of Hindu-Arabic Numerals

Mantras from the early Vedic period (before 1000 BCE) invoke powers of ten from a hundred all the way up to a trillion, and provide evidence of the use of arithmetic operations such as addition, subtraction, multiplication, fractions, squares, cubes and roots. A fourth century CE Sanskrit text reports Buddha enumerating numbers up to 10^{53} , as well as describing six more numbering systems over and above these, leading to a number equivalent to 10^{421} . Given that there are an estimated 10^{80} atoms in the whole universe, this is as close to infinity as any in the ancient world came. It also describes a series of iterations in decreasing size, in order to demonstrate the size of an atom, which comes remarkably close to the actual size of a carbon atom (about 70 trillionths of a metre).

As early as the eight century BCE, long before Pythagoras, a text known as the “Sulba Sutras” (or “Sulva Sutras”) listed several simple Pythagorean triples, as well as a statement of the simplified Pythagorean theorem for the sides of a square and for a rectangle (indeed, it seems quite likely that Pythagoras learned his basic geometry from the “Sulba Sutras”). The Sutras also contain geometric solutions of linear and quadratic equations in a single unknown, and give a remarkably accurate figure for the square root of 2, obtained by adding $1 + \frac{1}{3} + \frac{1}{(3 \times 4)} - \frac{1}{(3 \times 4 \times 34)}$, which yields a value of 1.4142156, correct to

five decimal places.

As early as the third or second century BCE, Jain mathematicians recognised five different types of infinities: infinite in one direction, in two directions, in area, infinite everywhere and perpetually infinite. Ancient Buddhist literature also

demonstrates a prescient awareness of indeterminate and infinite numbers, with numbers deemed to be of three types: countable, uncountable and infinite.

Like the Chinese, the Indians discovered the benefits of a decimal place value number system early, and were certainly using it before about the third century BCE. They refined and perfected the system, particularly the written representation of the numerals, creating the ancestors of the nine numerals that (thanks to its dissemination by medieval Arabic mathematicians) we use across the world today, and it is sometimes considered one of the greatest intellectual innovations of all time.

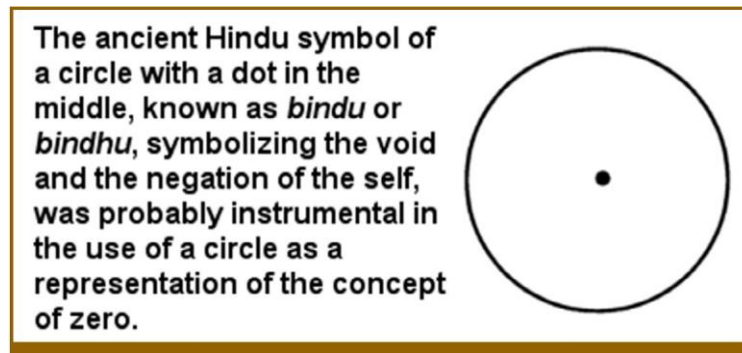


FIGURE 1.10

The earliest use of a circle character for the number zero was in India

The Indians were also responsible for another hugely important development in mathematics. The earliest recorded usage of a circle character for the number zero is usually attributed to a ninth century engraving in a temple in Gwalior in central India. But the brilliant conceptual leap to include zero as a number in its own right (rather than merely as a placeholder, a blank or empty space within a number, as it had been treated until that time) is usually credited to the seventh century Indian mathematicians Brahmagupta – or possibly another Indian, Bhaskara I – although it may well have been in practical use for centuries before that. The use of zero as a number that can be used in calculations and mathematical investigations, would revolutionise mathematics.

Brahmagupta established the basic mathematical rules for dealing with zero: $1 + 0 = 1$; $1 - 0 = 1$; and $1 \times 0 = 0$ (the breakthrough that would make sense of the apparently nonsensical operation $1 \div 0$ would also fall to an Indian, the 12th century mathematician Bhaskara II). Brahmagupta also established rules for dealing with negative numbers, and pointed out that quadratic equations could in theory have two possible solutions, one of which could be negative. He even attempted to write down these rather abstract concepts, using the initials of the names of colours to represent unknowns in his equations, one of the earliest intimations of what we now know as algebra.

The so-called Golden Age of Indian mathematics can be said to extend from the fifth to 12th centuries, and many of its mathematical discoveries predated similar discoveries in the West by several centuries, which has led to some claims of plagiarism by later European mathematicians, at least some of whom were probably aware of the earlier Indian work. Certainly, it seems that Indian contributions to mathematics have not been given due acknowledgement until very recently in modern history.

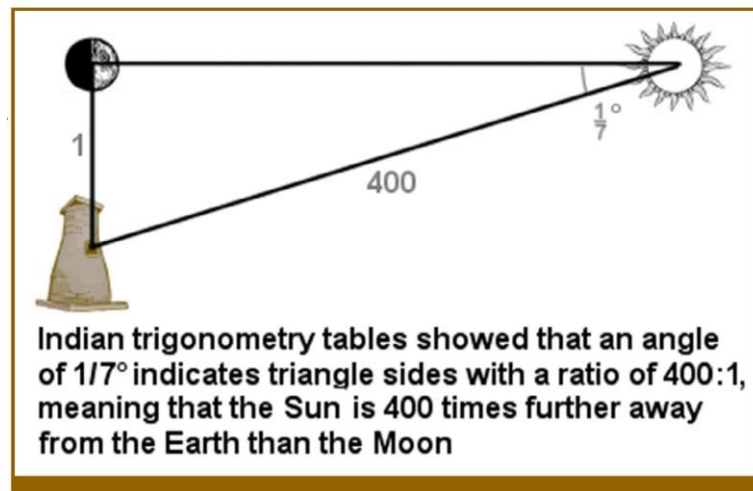


FIGURE 1.11

Indian astronomers used trigonometry tables to estimate the relative distance of the Earth to the Sun and Moon

Golden Age Indian mathematicians made fundamental advances in the theory of trigonometry, a method of linking geometry and numbers first developed by the Greeks. They used ideas like the sine, cosine and tangent functions (which relate the angles of a triangle to the relative lengths of its sides) to survey the land around them, navigate the seas and even chart the heavens. For instance, Indian astronomers used trigonometry to calculate the relative distances between the earth and the moon and the earth and the sun. They realised that, when the moon is half full and directly opposite the sun, then the sun, moon and earth form a right-angled triangle, and were able to accurately measure the angle as $1/7^\circ$. Their sine tables gave a ratio for the sides of such a triangle as 400:1, indicating that the sun is 400 times further away from the earth than the moon.

Although the Greeks had been able to calculate the sine function of some angles, the Indian astronomers wanted to be able to calculate the sine function of any given angle. A text called the *Surya Siddhanta*, by unknown authors and dating from around 400 BCE, contains the roots of modern trigonometry, including the first real use of sines, cosines, inverse sines, tangents and secants.

As early as the sixth century BCE, the great Indian mathematician and astronomer Aryabhata produced categorical definitions of sine, cosine, versine and inverse sine, and specified complete sine and versine tables, in 3.75° intervals from 0° to 90° , to an accuracy of 4 decimal places. Aryabhata also demonstrated solutions to simultaneous quadratic equations, and produced an approximation for the value of π equivalent to 3.1416, correct to four decimal places. He used this to estimate the circumference of the earth, arriving at a figure of 24,835 miles, only 70 miles off its true value. But, perhaps even more astonishing, he seems to have been aware that π is an irrational number, and that any calculation can only ever be an approximation, something not proved in Europe until 1761.

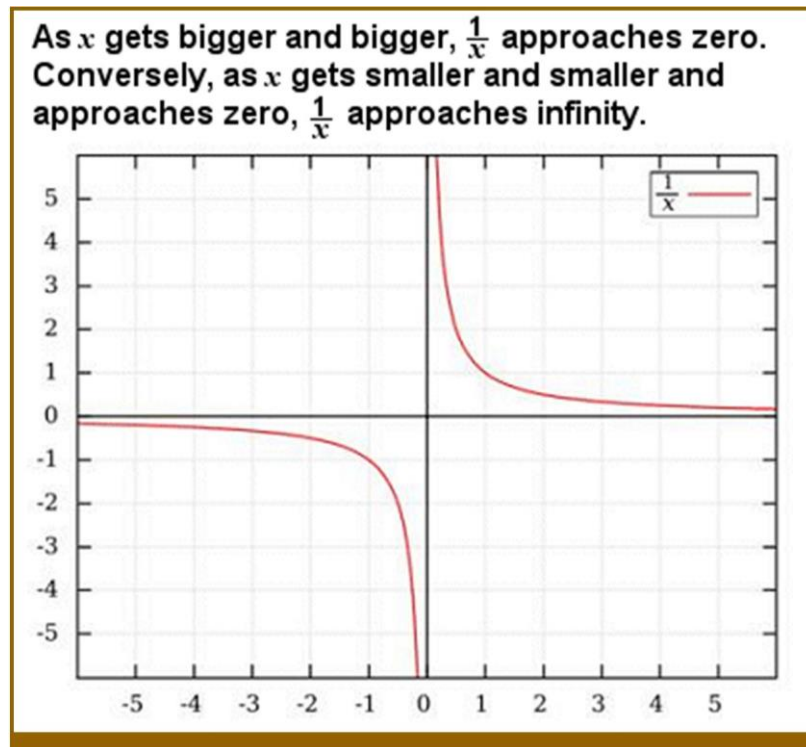


FIGURE 1.12

Illustration of infinity as the reciprocal of zero

Bhaskara II, who lived in the 12th century, was one of the most accomplished of all India's great mathematicians. He is credited with explaining the previously misunderstood operation of division by zero. He noticed that dividing one into two pieces yields a half, so $1 \div \frac{1}{2} = 2$. Similarly, $1 \div \frac{1}{3} = 3$. So, dividing

1 by smaller and smaller fractions yields a larger and larger number of pieces. Ultimately, therefore, dividing one into pieces of zero size would yield infinitely many pieces, indicating that $1 \div 0 = \infty$ (the symbol for infinity).

However, Bhaskara II also made important contributions to many different areas of mathematics from solutions of quadratic, cubic and quartic equations (including negative and irrational solutions) to solutions of Diophantine equations of the second order to preliminary concepts of infinitesimal calculus and mathematical analysis to spherical trigonometry and other aspects of trigonometry. Some of his findings predate similar discoveries in Europe by several centuries, and he made important contributions in terms of the systemisation of (then) current knowledge and improved methods for known solutions.

The Kerala School of Astronomy and Mathematics was founded in the late 14th century by Madhava of Sangamagrama, sometimes called the greatest mathematician-astronomer of medieval India. He developed infinite series approximations for a range of trigonometric functions, including π , sine, etc. Some of his contributions to geometry and algebra and his early forms of differentiation and integration for simple functions may have been transmitted to Europe via Jesuit missionaries, and it is possible that the later European development of calculus was influenced by his work to some extent.

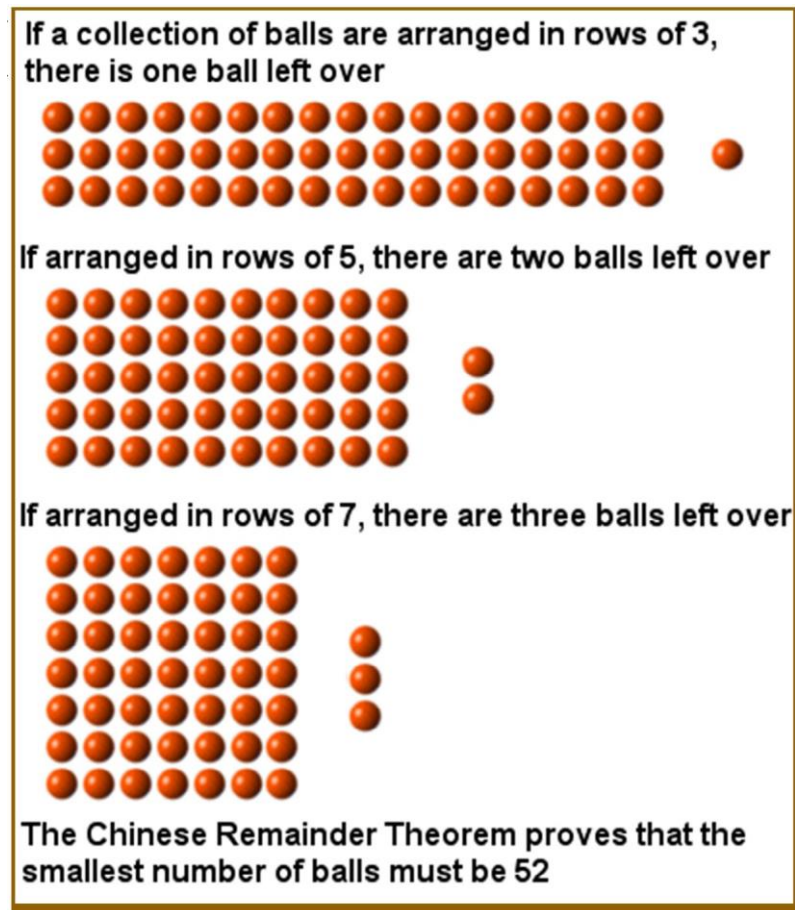


FIGURE 1.13
The Chinese Remainder Theorem

The Chinese went on to solve far more complex equations using far larger numbers than those outlined in the *Nine Chapters*, though. They also started to pursue more abstract mathematical problems (although usually couched in rather artificial practical terms), including what became known as the Chinese Remainder Theorem. This uses the remainders after dividing an unknown number by a succession of smaller numbers, such as 3, 5 and 7, in order to calculate the smallest value of the unknown number. A technique for solving such problems, initially posed by Sun Tzu in the third century BCE and considered one of the jewels of mathematics, was being used to measure planetary movements by Chinese astronomers in the sixth century AD, and even today it has practical uses, such as in internet cryptography.

By the 13th century, in the Golden Age of Chinese mathematics, there were over 30 prestigious mathematics schools scattered across China. Perhaps the most brilliant Chinese mathematician of this time was Qin Jiushao, a rather violent and corrupt imperial administrator and warrior, who explored solutions to quadratic and even cubic equations using a method of repeated approximations very similar to that later devised in the West by Sir Isaac Newton in the 17th century. Qin even extended his technique to solve (albeit approximately) equations involving numbers up to the power of ten, extraordinarily complex mathematics for its time.

The usual warnings to avoid thinking of ancient mathematics in modern terms seem quite unnecessary in the Chinese case. Certainly the Chinese, like the

Greeks, recognised a particular study called mathematics (suanshu). This included nine chapters on the mathematical art which became an important tool in the education of a civil service, covering hundreds of problems in practical areas such as trade, taxation, engineering and the payment of wages. As a component of education, they seem often to have given it a rather subordinate role, in which it was particularly important as a guide to how to solve equations including the deduction of an unknown number from other known information—using a sophisticated matrix-based method which was essential for the standard preoccupations of irrigation, public works, and taxation.

1.6 20TH CENTURY MATHEMATICS

The 20th century continued the trend of the 19th century towards increasing generalisation and abstraction in mathematics, in which the notion of axioms as “self-evident truths” was largely discarded in favour of an emphasis on such logical concepts as consistency and completeness.

It also saw mathematics become a major profession, involving thousands of new PhDs each year and jobs in both teaching and industry, and the development of hundreds of specialised areas and fields of study, such as group theory, knot theory, sheaf theory, topology, graph theory, functional analysis, singularity theory, catastrophe theory, chaos theory, model theory, category theory, game theory, complexity theory and many more.

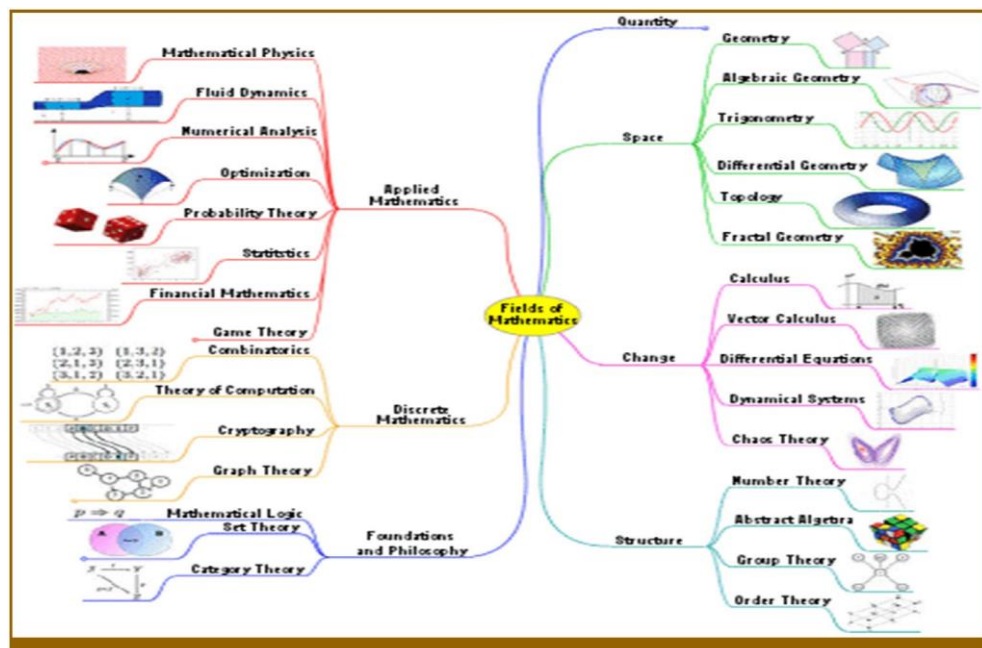


FIGURE 1.14
Fields of Mathematics

ACTIVITY 1.8

Based on the above and some research on the internet, compare by time which countries were earliest to use mathematics and which ones were the earliest to write and use written.

TABLE 1.2

List of important mathematicians

Date	Name	Nationality	Major Achievements
35000 BCE		African	First notched tally bones
3100 BCE		Sumerian	Earliest documented counting and measuring system
2700 BCE		Egyptian	Earliest fully-developed base 10 number system in use
2600 BCE		Sumerian	Multiplication tables, geometrical exercises and division problems
2000–1800 BCE		Egyptian	Earliest papyrus showing numeration system and basic arithmetic
1800–1600 BCE		Babylonian	Clay tablets dealing with fractions, algebra and equations
1650 BCE		Egyptian	Rhind Papyrus (instruction manual in arithmetic, geometry, unit fractions, etc)
1200 BCE		Chinese	First decimal numeration system with place value concept
1200–900 BCE		Indian	Early Vedic mantras invoke powers of ten from a hundred all the way up to a trillion
800–400 BCE		Indian	<i>Sulba Sutra</i> lists several Pythagorean triples and simplified Pythagorean theorem for the sides of a square and a rectangle, quite accurate approximation to $\sqrt{2}$
650 BCE		Chinese	Lo Shu order is three (3 x 3) “magic squares” in which each row, column and diagonal sums to 15
624–546 BCE	Thales	Greek	Early developments in geometry, including work on similar and right triangles
570–495 BCE	Pythagoras	Greek	Expansion of geometry, rigorous approach building from first principles, square and triangular numbers, Pythagoras’ theorem
500 BCE	Hippasus	Greek	Discovered potential existence of irrational numbers while trying to calculate the value of $\sqrt{2}$
490–430 BCE	Zeno of Elea	Greek	Describes a series of paradoxes concerning infinity and infinitesimals
470–410 BCE	Hippocrates of Chios	Greek	First systematic compilation of geometrical knowledge, Lune of Hippocrates

Date	Name	Nationality	Major Achievements
460–370 BCE	Democritus	Greek	Developments in geometry and fractions, volume of a cone
428–348 BCE	Plato	Greek	Platonic solids, statement of the Three Classical Problems, influential teacher and populariser of mathematics, insistence on rigorous proof and logical methods
410–355 BCE	Eudoxus of Cnidus	Greek	Method for rigorously proving statements about areas and volumes by successive approximations
384–322 BCE	Aristotle	Greek	Development and standardisation of logic (although not then considered part of mathematics) and deductive reasoning
300 BCE	Euclid	Greek	Definitive statement of classical (Euclidean) geometry, use of axioms and postulates, many formulas, proofs and theorems including Euclid's Theorem on infinitude of primes
287–212 BCE	Archimedes	Greek	Formulas for areas of regular shapes, "method of exhaustion" for approximating areas and value of π , comparison of infinities
276–195 BCE	Eratosthenes	Greek	"Sieve of Eratosthenes" method for identifying prime numbers
262–190 BCE	Apollonius of Perga	Greek	Work on geometry, especially on cones and conic sections (ellipse, parabola, hyperbola)
200 BCE		Chinese	<i>Nine Chapters on the Mathematical Art</i> , including guide to how to solve equations using sophisticated matrix-based methods
190–120 BCE	Hipparchus	Greek	Develop first detailed trigonometry tables
36 BCE		Mayan	Pre-classic Mayans developed the concept of zero by at least this time
10–70 CE	Heron (or Hero) of Alexandria	Greek	Heron's formula for finding the area of a triangle from its side lengths; Heron's method for iteratively computing a square root
90–168 CE	Ptolemy	Greek/ Egyptian	Develop even more detailed trigonometry tables

Note: This is for enrichment. You do not need to memorise the names in the above table, but it is always important to know and associate each mathematics concepts with its original founder.

1.7 FEEDBACK ON ACTIVITIES

A detailed feedback will be provided and sent to you via the announcements on myUnisa after the due date for submission of each assignment.

1.8 CONCLUSION

In this learning unit you were introduced to the various definitions of mathematics. Second, the history of mathematics in relation to its origin and development as it evolved in the social and cultural contexts of the Babylonians, Egyptians, Greeks, Chinese, Arabs/Islam, Indians, and Africans.

We hope that your knowledge of how mathematics evolved has been developed.

1.9 ADDITIONAL LEARNING EXPERIENCES

In this section we provide you with additional links that will assist you with additional learning experiences. Please click on these links and learn from the presentations provided. Please take note that this section is NOT COMPULSORY, it is provided to enhance your learning experiences.

1.10 SELF-ASSESSMENT

This section contains questions that will test if you have understood the learning outcomes covered in this learning unit. Please take note that these are self-reflection questions that will assist you to evaluate your understanding in this unit.

1.11 REFERENCES

- Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Educational Research journal*, 17(2), 39–68.
- Dossey, JA. (1992). The nature of mathematics: its role and influence. (In Grouws, DA, ed. *Handbook of research on mathematics teaching and learning*. Reston, VA: National Council of Teachers of Mathematics. p. 334–370).
- Ernest, P. (1989). Philosophy, mathematics and education. *International journal of Mathematics Education in Science and Technology*, 20, 555–559.
- Goodnough, K and Long, R. (2002), 'Mind Mapping: A Graphic Organizer for the Pedagogical Toolbox'. *Science Scope*, Vol. 25, No. 8, pp 20–24.
- Hersh, R. (1997). *What Is Mathematics, Really?* Oxford Univ. Press.
- Hodgkin, L. (2005). *A history of mathematics: from Mesopotamia to modernity*, Oxford Univ. Press.
- Plato. (1952). *The republic*. In RM Hutchins (Ed.), *Great books of the western world: Vol. 7 Plato* (pp. 295–441). Chicago: Encyclopaedia Britannica, Inc.

- Ptolemy. (1952). The almagest. In RM. Hutchins (Ed.), Great books of the western world: Vol. 16. Ptolemy, Copernicus, & Kepler (pp. 1–478). Chicago: Encyclopaedia Britannica, Inc.
- Steinbring, H. (2006). What Makes a Sign a Mathematical Sign? – An Epistemological Perspective on Mathematical Interaction. Educational Studies in Mathematics, vol. 61, 1–2, pp. 133–162.
- Struik, DJ. 1948. Marx and Mathematics Science and Society 12, 181–196
- Törner, G & Grigutsch, S. (1994). 'Mathematische Weltbilder' bei Studienanfänger-eine Erhebung. Journal für Mathematik-Didaktik, 15(3/4),S.211–251.

UNIT 2

TEACHING AND LEARNING MATHEMATICS

2.1 INTRODUCTION

Methods for teaching mathematics are often underestimated. Some teachers overlook the fact that a teacher must understand these methods, namely, there is theory behind every mathematical concept learned and taught. In this unit you are going to explore how mathematics is learned, what is “doing mathematics”, and eventually you will be able to design mathematical learning tasks to illustrate the use of teaching aids for teaching mathematics. As you work through this study unit, you will be challenged to construct your own understanding of what it means to know and do mathematics, so that your teaching of mathematics will be exciting and effective and ultimately contribute towards meaningful learning. Doing mathematics (mathematisation) will be eventful, compelling and creative.

2.2 LEARNING OBJECTIVES

On completion of this learning unit, you should be able to:

- provide the rationale for doing mathematics
- discuss how mathematics is learned
- design basic mathematical learning tasks to illustrate the use of teaching aids
- effectively teach mathematics and contribute towards meaningful learning

KEY CONCEPTS

Mathematics	Theoretical perspective
Mathematics Education	Instrumental understanding
History of Mathematics	Relational understanding
Science	Conceptual understanding
Problem solving	Procedural Knowledge
Content area	

2.3 RATIONALE FOR DOING MATHEMATICS

In this section you will learn what mathematics is, the meaning of mathematics education, the history of mathematics, conceptions and perspectives about

mathematics, mathematics as a science, the verbs of doing mathematics and the process of mathematics.

2.3.1 What is mathematics?



Mathematics is a discipline that has been existing for centuries. In the 18th and 19th centuries the need for mathematics was intensely realised when urban populations started applying basic mathematics such as numeracy and arithmetic, for example, counting money, telling time and so on. Therefore, studying mathematics was formalised, and a curriculum was developed. In the 20th century most of the developed countries adopted mathematics and included mathematics in their curriculum. The adoption of mathematics in diverse countries and cultures was aimed at achieving diverse objectives, including:

- basic numeracy to citizens
- practical mathematics in order to equip those studying mathematics to follow a trade or craft
- advanced mathematics

The word mathematics is defined from the Latin, French and Greek background as follows:

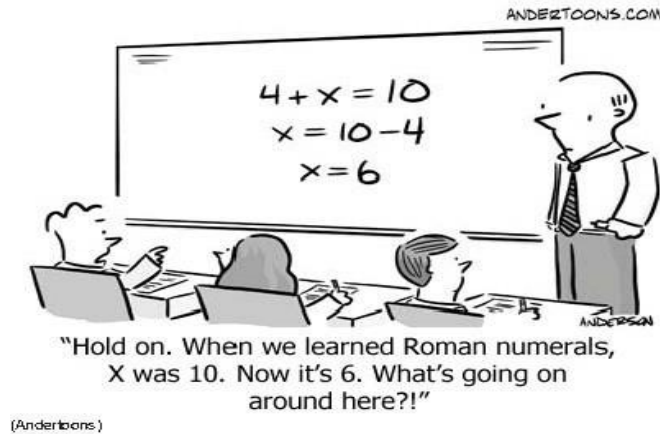
Greek	Latin	French
<p>Greek –</p> <ul style="list-style-type: none"> • mathematike tekhnē referring to mathematical science • mathematikós – relating to mathematics • máthema – science knowledge; mathematical knowledge; a lesson; that which is learnt • manthanein – means to learn 	<ul style="list-style-type: none"> • mathematica; mathematicus 	<ul style="list-style-type: none"> • Mathématique from Latin mathematicus

etymology dictionary – www.etymonline.com

Today, the term ‘mathematics’ refers to a specific body of knowledge and it involves the study of quantity, structure, patterns, space and change. Mathematics content evolved through the use of abstraction and logical reasoning, counting, calculation, measurements, study of shape and motion of physical objects. Mathematical concepts within the mathematical content are explored by those who study mathematics (Mathematicians) in order to hypothesise and through conjecturing, establish truth. Mathematicians establish truth by making deductions from selected theorems, axioms and definitions.

Ziegler (2010) defines mathematics as a study of science that developed from the investigation of figures and computing the numbers. It is a science that investigates abstract structures that is created itself for their properties and patterns.

Mathematics relates with education based on the fact that human activity through teaching must be advanced in order to critically assist others to learn and participate in studying quantity, structure, patterns, space and shape. Education is involved in Mathematics through teaching, learning and researching the content of mathematics. Therefore, Mathematics Education as a practice comes into picture.



2.3.2 Mathematics education

Mathematics education is the study of the practices and methods of teaching and learning mathematics. The term Mathematics education does not only refer the practices in classrooms, but it also refers to an academic discipline. Consider the statement by Dorfler (2003: 147–148) below:

Mathematics education as a scientific discipline is considered here in a rather broad view as the study of how people learn and do mathematics of any kind, and of how this learning and doing can be influenced and fostered among others by teaching, by the use of media, by different representations, or by the social organisation of mathematical activity. Thus the object of mathematics education clearly is a certain area of human activities whose content, object and goal is mathematics at different levels and in different forms ... the role (professional) mathematicians currently take in relation to mathematics education ... they themselves consider as their main task to develop mathematical theories and/or to apply them in various contexts. This should/could give a reason for mathematics education to investigate the mathematical activity of mathematicians, its conditions, forms, means, goals, intentions, etc.

This study unit orientates you to teachers' and learners' classroom practices rather than investigation or research of the mathematical activity as carried out beyond the classroom. Within the context of mathematics classroom practices, it is essential to comprehend the origin of mathematics and mathematics curriculum.

Curriculum refers to content and lessons taught in a school, district, province, country, etc. It also points at materials and processes that teachers and learners will adhere to in order to achieve generally prescribed outcomes. In simple terms, curriculum means a set of activities identified and prescribed for general application in teaching and learning.

2.4 HISTORY OF MATHEMATICS

Mathematics is a discipline that has been existing for centuries. In the 18th and 19th centuries the need for mathematics was intensely realised when urban populations started applying basic mathematics such as numeracy and arithmetic, for example, counting money, telling time and so on. Therefore, studying mathematics was formalised and a curriculum was developed. In the 20th century most of the developed countries adopted mathematics and included mathematics in their curriculum. The adoption of mathematics in diverse countries and cultures was aimed at achieving diverse objectives, including:

- basic numeracy to citizens
- practical mathematics in order to equip those studying mathematics to follow a trade or craft
- advanced mathematics

2.4.1 Mathematics education in South Africa including CAPS

There have been various curriculum reforms in South Africa resulting in the current implemented Curriculum and Assessment Statement (CAPS). As a result of these reforms, school mathematics have moved out of the old system and mathematics is now offered as pure mathematics and mathematical literacy in the Grades 10 to 12.

ACTIVITY 2.1

- (1) Refer to readers 2.1.
- (2) Consider the following CAPS documents.



- (i) Write a short paragraph to reflect on the origin of mathematics and what mathematics is to you.
- (ii) Briefly discuss Mathematics Education.
- (iii) Considering the NCS (CAPS) documents attached, write a short paragraph to summarise important issues you have learned about the *National curriculum statement for intermediate and senior phase*.

FEEDBACK

You should be able to:

- describe the difference between mathematics and mathematics education
 - outline the origin of mathematics and specific functions of mathematics as explained in the history of mathematics
 - explain the issues around the CAPS curriculum
 - define curriculum as a general term
 - outline the aim of the South African curriculum – the specific aims and skills conveyed by the mathematics content in South Africa, and the content covered for specific grades
 - specify and clarify the content of the curriculum as well as assessment
-

2.5 THE MEANING OF DOING MATHEMATICS

The teaching and learning of mathematics in most schools is based on a practice labelled by most sources as traditional approach to teaching. The practice is defined by characteristics such as the teacher being the only knowledgeable person in the classroom; students remain passive and take everything said by the teacher; student copy what the teacher is writing on the chalkboard; problem solving means striving to arrive at only one correct answer, students and teacher get information and activities from one prescribed textbook; teacher is always right; the prescribed book is always correct and it is the source of all information

The outlined practice has over the years contributed to contrasting perceptions and conceptions about mathematics teaching and learning. Further, the practise contributed to short term retention of mathematical knowledge and errors in problem solving. In addition, the traditional practice of teaching and learning contributes to a belief that (1) mathematics is based on rules, formulas and procedures; (2) there is only one correct answer. However, doing mathematics currently involves open methods of teaching and learning that are contrary to the beliefs stated.

ACTIVITY 2.2

- (1) Would you agree with the conceptions listed above? Write a short paragraph to motivate your answer. Contrary to these conceptions, list your own conceptions about teaching and learning mathematics.
- (2) Discuss the open-ended approaches of teaching mathematics.

FEEDBACK

- (1) Answer yes or no. Thereafter, motivate your answer in a paragraph. In your motivation, state why do you agree or not agree with the conceptions.
 - (2) Think of more conceptions and list additional conceptions about the mathematics as a content; and the teaching and learning of mathematics in a table format.
-

2.6 MATHEMATICS AS A SCIENCE BASED ON ORDER AND PATTERN

According to Van de Walle et al (2004:13; 2016:37) mathematics is a science of things that have a pattern of regularity and logical order. Finding and exploring this regularity or order and then making sense of it, is what doing mathematics is all about.

In Van de Walle's idea of mathematics as a science, many verbs erupt showing actions of doing mathematics. For example, the following verbs appear: finding, exploring, making sense. In addition to Van de Walle's verbs of doing mathematics the following verbs (Table 2.1) are evident more often when people are engaged in the action of doing mathematics.

TABLE 2.1
Action verbs evident when engaged in mathematical activities

solve	construct	explain
justify	use	investigate
predict	explore	represent
discover	verify	formulate
conjecture	develop	determine

The verbs listed in table 2.1 contribute to processes (expressed in action or “doing verbs”), products (expressed by nouns) and at the end, acquiring of skills and knowledge.

Processes of mathematics include:

- generalising
- proving
- solving
- computing
- conjecturing/hypothesising/assuming

Products of Mathematics include:

- definition
- formula
- theorem
- concepts

During the process of mathematising, students acquire skills such as computational skills, problem-solving skills, investigatory skills, analysis skills, exploratory skills, logical thinking skills, formulating mathematical models, estimation and evaluation skills.

Knowledge and understanding is evaluated in terms of specific outcomes as outlined based on the content covered in each content area.

There are five content areas as outlined generally in the mathematics curriculum, and the content areas are fundamental to mathematics knowledge.

Mathematics content areas

1. Number, operations and relationships
2. Patterns, functions and algebra
3. Space and shape (geometry)
4. Measurement
5. Data handling

2.7 HOW MATHEMATICS IS LEARNED

Individuals learn and acquire knowledge in different ways. It is essential that teachers know how learning occurs in order for them to determine suitable teaching strategies. Learning theories are concerned with principles explaining how individuals learn or how individuals acquire new knowledge. Further,

learning theories provide principles that teachers may use to create solutions to a variety of classroom challenges (Newby et al. 2011).

In this unit we will look at behaviourism and constructivism theories.

TABLE 2.3
Contrasting constructivist and behaviourist practices of learning

Constructivism	Behaviourism
Demonstrations of reality or real world setting; concrete settings	Pre-determined sequences of instruction; abstract setting
Emphasis on knowledge construction	Emphasis on knowledge consumption
Thoughtful reflection on pre-knowledge and experience	Pre-knowledge or experience is not applied
Collaborative construction of knowledge	Individual attempt to construct knowledge; there is competition among students for recognition
Teacher-student and social interaction; teacher-student and student-student discussions	Teacher is the only knowledgeable; students listen passively

2.8 CONSTRUCTION OF MATHEMATICAL IDEAS

Inductive and deductive reasoning are processes engaged in constructing mathematical ideas. During construction of mathematical ideas, firstly, inductive reasoning is applied in generalising or formulating conjectures, and hypothesising about observed patterns. Secondly, deductive reasoning is applied in reaching essential conclusions from conjectures or given facts. Look at the example of inductive and deductive reasoning in table 2.4.

TABLE 2.4
An example of inductive and deductive reasoning

Prove that the following number trick always results in 5:

1. Choose any number.
2. Multiply the chosen number by 2.
3. Add 10 to the answer above.
4. Divide by 2.
5. Subtract the number chosen in 1.

Instead of choosing any number, a variable is used to substitute an unknown number.

1. Choose a number ____ (generalising, basing judgement on opinion). Steps two to five are based on deductive reasoning as they are depend on existing facts of mathematics

Instead of choosing any number, a variable is used to substitute an unknown number.

1. Choose a number ____ x (generalising, basing judgement on opinion). Steps two to five are based on deductive reasoning as they are depend on existing facts of mathematics
2. Multiply by 2 ____ $2x$
3. Add 10 ____ $2x + 10$
4. Divide by 2 ____ $\frac{2x + 10}{2} = x + 5$
5. Subtract the number chosen in step 1 ____ $x + 5 - x = 5$

Deductive reasoning was applied to prove that the number trick will end up with 5. Conclusions of deductive reasoning becomes certain if the assumptions are correct.

(from Sonnabend 2010)

2.9 TEACHING FOR MEANINGFUL LEARNING

Teaching should consider the following in order to ensure meaningful learning and proper construction of mathematical ideas:

- usage of activities that build on students' experiences
- using activities that are interesting and relevant to students
- develop and apply correct mathematical language
- provide feedback for regularly for all activities
- encourage co-operative work or collaborative learning among the students

2.10 DEVELOPING UNDERSTANDING IN MATHEMATICS

Understanding relates to connecting ideas together, rather than simply knowing isolated facts. The question "does the learner know it?" must be replaced with "how well does the learner understand it?" The first question refers to **instrumental** understanding and the second leads to **relational** understanding. Memorisation of formulae and rules as well as application of prescribed methods in computations are **knowing** the idea. **Understanding** is evident when a student connects a network of ideas to form a new idea and arrive at a solution. Relational understanding is essential as there is less to remember when it is applied.

Benefits of relational understanding include assisting students to:

- learn new concepts and procedures
- enhance memory
- improve attitude and belief
- improve problem-solving abilities

In essence, mathematics teachers should promote learning with understanding, that is, relational understanding. Learning with understanding contributes towards meaningful learning and enables students to cope with problem-solving processes.

ACTIVITY 2.3

- (1) Does deductive reasoning always work? Write a short paragraph to elaborate on your answer.
- (2) Can inductive and deductive reasoning be applied simultaneously, or do they need to be applied separately? Write a short paragraph and use mathematical examples to elaborate on this answer.
- (3) Relate reasoning and understanding, thereafter, discuss the following statements:
 - Inductive reasoning contributes to relational understanding of mathematical concepts.
 - Deductive reasoning contributes to instrumental understanding of mathematical concepts
 - Instrumental understanding complements relational understanding.

FEEDBACK

- Explain deductive reasoning and its importance when solving problems. From your own experience of doing mathematics, explain whether deductive reasoning can lead to meaningful learning and effective mathematisation.
- Discuss the importance and need of combining inductive and deductive reasoning in solving mathematics problems. Choose any content area and a mathematics problem that needs both inductive and deductive reasoning when solved.
- Outline the properties of inductive and deductive reasoning as well as those of instrumental and reflective understanding. Thereafter, relate reasoning and understanding using properties, and outline which reasoning contributes to which type of understanding. Further, outline whether relational and instrumental understanding can apply simultaneously.

2.11 TYPES OF MATHEMATICAL KNOWLEDGE

Piaget distinguished three types of knowledge, that is, social, physical and logico-mathematical knowledge.

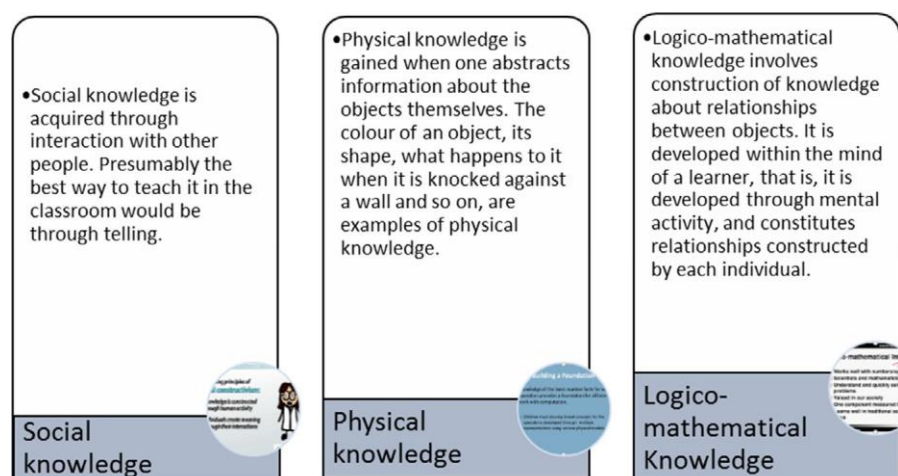


FIGURE 2.1

Types of mathematical knowledge Adapted from Njisane (1992)

Understanding the three types of knowledge modelled by Piaget assist teachers to decide when is it necessary to tell and when is it necessary to let the students puzzle the problem out for themselves.

Essentially, social and physical knowledge are fundamental to the acquisition of logico-mathematical knowledge. In most cases mathematics teachers neglect the mathematical ideas that are basic and concrete (that is, social and physical knowledge). Therefore, when teaching mathematics, teachers should consider concrete knowledge, since **relational understanding** depends on the integration of concrete ideas into abstract networks of ideas. It is worth noting how mental representations are constructed, that is through effective interaction and doing mathematics.

2.12 CONCEPTUAL AND PROCEDURAL KNOWLEDGE IN MATHEMATICS

As you read on, it is important for you to understand why the connections between procedural knowledge and the underlying conceptual knowledge and relationships are vital for the construction of relational understanding in mathematics. You will also see that the ability to make connections plays a very important role in “doing” mathematics.

2.12.1 Procedural knowledge of mathematics

Procedural knowledge of mathematics refers to memorising rules and procedures for application in routine mathematical tasks. Mathematics is represented by memorised symbols.

You could therefore infer that knowledge of mathematics consists of more than concepts. Step-by-step procedures exist for performing tasks such as:

56×74	(Multiplying two digit numbers)
$1\,932 \div 28$	(Long division)
$\frac{3}{8} + \frac{5}{6}$	(Adding fractions)
$0,85 \times 0,25$	(Multiplying decimal numbers)

Concepts are represented by special words and mathematical symbols (such as $+$, $=$, $<$, $>$, $//$, $\div$, $ABC = 45$ and so on). These procedures and symbols can be connected to, or supported by concepts, but very few cognitive relationships are needed to have knowledge of a procedure (since these could be diligently memorised through drill and practice).

In mathematics, we use a number of different symbols that indicate procedures which need to be followed. For example, when we write $(12 + 6) \div 3 + 10$, it means a different procedure has to be followed than if we write this as $12 + 6 \div (3 + 10)$. We get different answers when we follow different procedures.

However, the meaning we attach to symbolic knowledge depends on how it is understood – what concepts and other ideas we connect to the symbols. The procedure applied in advancing procedural knowledge refers to the step-by-step routines learned to accomplish some task – like a computation in the classroom situation. Further, the procedure applied in computations to advance procedural knowledge are referred to as **algorithm**. An example of an algorithm is a set of steps used to solve the following problems: $\frac{2}{3}$; $\frac{7}{16}224 \div 1$.

Algorithmic procedures help us to do routine tasks easily and thus free our minds to concentrate on more important tasks such as thinking of problem-solving strategies. Symbolism (which is part of procedural knowledge) is a powerful mechanism for conveying mathematical ideas to others and for manipulating an idea as we do mathematics. Efficient use of the procedures and symbolism of mathematics does not necessarily imply an understanding of these things.

For example, think of the endless long-division and long-multiplications exercises in the classroom. Will these algorithmic exercises help the learner understand what division and multiplication mean? Carrying out the step-by-step computation does not necessarily translate into understanding the underlying concepts and relationships. In fact, learners who are skilful with a particular procedure are often reluctant to attach meaning to it after the fact.

Why the focus on concepts and relationships?

Remember that we said earlier that learning and thinking cannot be separated.

If the focus of learning is on the acquisition of specific skills, facts, procedures and the memorisation of information and rules, then thinking is suppressed. The learner requires concepts and information in order to think and he or she will think according to the knowledge already at his or her disposal. Procedural knowledge with little or no attached meaning results in inflicting on the learner the manipulation of symbols according to a number of rote memorised rules, which makes learning much harder to remember than an integrated conceptual structure – a network of connected ideas.

Procedural knowledge is what is used widely in mathematics classroom and it relates to what is referred as the traditional way of acquiring knowledge; conveyed through routine problem-solving strategies and the rules-based approach of teaching.

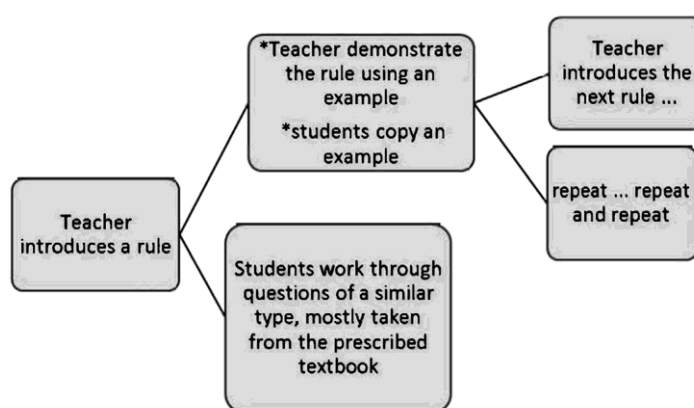


FIGURE 2.2

Rules-based teaching and learning approach

To construct and understand a new idea (or concept) requires active thinking about it. Passive learning disables the students from understanding mathematical ideas, therefore, it is difficult for a teacher to force understanding when learners are inactive or passive. Learners must be engaged in “doing mathematics” and remain mentally active for learning to take place. In the classroom, the learners must be encouraged to grapple with new ideas, work at fitting them into existing networks, and to challenge their own ideas and those of others.

Simply put, constructing knowledge requires reflective thought, actively thinking about or mentally working on an idea – all this to overcome the acquisition of procedural knowledge without relational understanding.

To acquire conceptual knowledge and understanding the following learner characteristics apply:

- active involvement in doing mathematics
- construction of new ideas
- reflective thinking and engagement in non-routine problem solving

2.12.2 Conceptual knowledge and understanding of mathematics

Conceptual knowledge refers to knowledge that is understood. Conceptual knowledge of mathematics consists of logical relationships constructed internally and existing in the mind as a part of the greater network of ideas. It is the type of knowledge Piaget referred to as logico-mathematical knowledge, that is knowledge made up of relationships between objects that are not integral to the objects themselves, but are introduced through mental activity. Conceptual knowledge means applying critical, creative and self-regulated thought for application in non-routine mathematical tasks.

2.13 MATHEMATICAL PROFICIENCY

Mathematical proficiency, that is a high degree of skill or expertise in learning, emanates from an interconnection of strands which are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick et al, 2001). In order to enable learners to achieve high mathematical proficiency, teaching must consider the strands.

TABLE 2.5

Mathematical teaching and learning proficiencies

Five Strands or components of Mathematical proficiency:	In the context of teaching, proficiency requires:
(1) conceptual understanding – comprehension of mathematical concepts, operations, and relations (2) procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (3) strategic competence – ability to formulate, represent, and solve mathematical problems (4) adaptive reasoning – capacity for logical thought, reflection, explanation, and justification (5) productive disposition – habitual inclination to see mathematics accessible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy	<ul style="list-style-type: none"> • conceptual understanding of the core knowledge required in the practice of teaching • fluency in carrying out basic instructional routines • strategic competence in planning effective instruction and solving problems that arise during instruction • adaptive reasoning in justifying and explaining one's instructional practices and in reflecting on those practices so as to improve them • productive disposition toward mathematics, teaching, learning, and the improvement of practice

Source: Kilpatrick et al (2001)

ACTIVITY 2.4

- (1) Richard R Skemp states that *“mathematics is not a collection of facts which can be demonstrated and verified in the physical world, but a structure of closely related concepts, arrived at by a process of pure thought”*.

Do you agree with the statement? Write a paragraph to reflect on Skemp's statement. In your argument consider how concepts and logical relationships are constructed internally and exist in the mind as part of a network of ideas.

- (2) Draw a table to distinguish between procedural and conceptual knowledge.
(3) Create your classroom where procedural and conceptual understanding coexist. Revisit your content areas and choose a problem to solve and demonstrate how procedural and conceptual knowledge can be linked in teaching and learning.
(4) Scrutinise the following problems:

PROBLEM 1

Brian visits the Game-on. He is interested in playing two electronic games, that is “Squad cars” and “Space invaders”.

It costs R5,00 to play “Squad cars” and R2,00 to play “Space invaders”. Brian has R20,00 to use in playing games.

Find out if Brian will be able to play 3 games each of “Squad cars” and “Space invaders”.

Find all possible combinations of games that Brian can play.

PROBLEM 2

Solution for x and y :

$$y = x + 4 \text{ and}$$

$$y = 2x^2 + 6x - 8$$

Classify the problem as either routine and non-routine based on the procedure and skills required to solve the problems. Write brief notes to explain your classification of problems 1 and 2.

FEEDBACK

- (1) Collect facts about conceptual knowledge and understanding, then argue in favour or contrary to Skemp's statement.
(2) Read about procedural and conceptual knowledge. Use a table to show the differences between the two based on learner involvement, teacher's instructional strategies, how content is acquired and how problem solving is advanced.
(3) Use mathematical problems (choose any problem) to show that both procedural and conceptual knowledge complement each other. Show all the steps of how you would solve the problem and indicate which part is conceptual and which is procedural. Write brief notes of what you observe in the application of both procedural and conceptual knowledge.

- (4) Study the given problems. Do not show how to solve the problem. By just scrutinising them, tell whether the problems are routine and non-routine. That is, will the problems need procedural or conceptual processes to be solved; will each problem require procedural or conceptual skills and knowledge?
-

2.14 DEMONSTRATE THE ABILITY TO DESIGN BASIC MATHEMATICAL LEARNING TASKS

2.14.1 Plan a problem-based classroom focusing on different content areas

It is very important that you give adequate thought to the planning of your lesson. Remember that every class is different and choices of tasks and how they are presented must be made daily to best fit the needs of the learners and the defined outcomes. Further, remember that every content area is different. A problem-based lesson has three parts:

- (1) before the lesson;
- (2) during, and
- (3) after the lesson.

Van de Walle (2004) recommends the following steps for planning a problem-based lesson:

STEP 1: Begin with the mathematics!

- Articulate clearly the ideas you want learners to construct – something new or unfamiliar.
- Describe the mathematics, not the behaviour.
- For skills as intended outcomes – identify the underlying concepts and relationships.

STEP 2: Consider your students

- Consider what the learners already know or understand about the topic. Are there any background ideas that they need to still develop?
- Be sure that your objectives are not out of reach.
- For learning to take place, there must be some challenge – some new idea within the grasp of learners.

STEP 3: Decide on a task

- Keep it simple! Good tasks need not be elaborate.
- Build a task bank – from resource books, journals, workshops, in-service programmes and conferences.

STEP 4: Predict what will happen

- Predict, don't hope! Use the information about what learners know to predict all of the things your learners are likely to do with this task.
- If they flounder, provide **hints** or modifications in the tasks for different learners.

- Think about whether your 41 learners will work alone, in pairs, or in groups.
- Revisit the task if you find it inappropriate at this stage – modify the task if necessary.
- These first four decisions define the heart of your lesson. The next four decisions define how you will carry out the plan in your classroom.

STEP 5: Articulate learner responsibilities

- For nearly every task, you want learners to be able to tell you:
 - what they did to get the answer
 - why they did it that way
 - why they think the solution is correct
- Plan how learners would supply this information. They could write individually or prepare a group presentation – in their journals, worksheet, chart-paper and so on. There is an option of no writing – learners could report or discuss their ideas.

STEP 6: Plan the “before” portion of the lesson

- Prepare the learners by working quickly through an easier related task or some related warm-up exercise.
- Articulate what is required of the learners in terms of their **responsibilities**.
- Present the task and “let go”! They could brainstorm ideas or estimate ideas.
- Consider how the task can be presented – written on paper, taken from the text and so on.

STEP 7: Think about the “during” portion of the lesson

- What hints can you plan in advance for learners who may be stuck?
- Think of **extended questions or challenges** you can pose to learners who finish quickly.
- Tell learners in advance how much time they have to complete the task but be flexible.

STEP 8: Think about the “after” portion of the lesson

- How will you begin the discussion? List the options and then return to individual learners or groups to **explain** their solutions and **justify** their answers.
- For oral reports, think about how you would record on the board what is being said.
- Plan an adequate amount of time for your discussion.

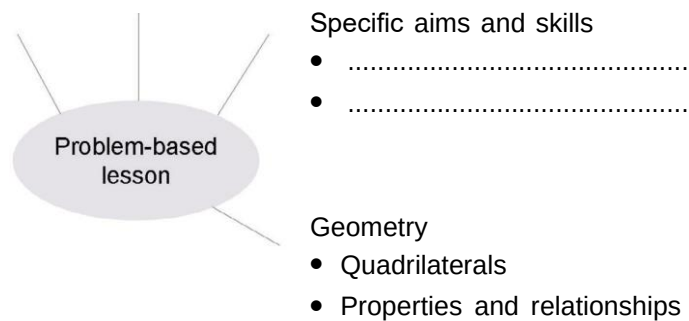
STEP 9: Write your lesson plan

- The outline here is a possible format of the critical decisions:
 - The mathematics or goals
 - The task and expectations
 - The before activities
 - The “during” hints and extensions for early finishers
 - The “after”-lesson discussion format
 - Assessment notes (whom you want to assess and how).

ACTIVITY 2.5

Reflect analytically on the nine steps given above for the planning of a problem-based lesson.

Choose a content area and a topic. Use a mind map/concept map and show your planning for a problem-based lesson. Base your concept map on explaining what you would do in your lesson in each of the nine steps.



FEEDBACK

Use a mind map to collect all ideas and concepts that you will include in your lesson plan. Describe everything, for example, resources needed, time, teacher and student activities, etc.

2.14.2 Utilise teaching aids effectively to enhance a problem based lesson for meaningful learning

Teaching aids play a major role in teaching and learning mathematics. It is with the aid of resources that teaching and learning can adhere to a combination of both concrete (inductive) to abstract (deductive). Read the following statement by Masilo (2015):

Learning in this digital era, that is, in the 21st century, has attained a gradual improvement and students are best motivated through incorporating multi-media and technology into the text material ... Teacher familiarity with advanced multiple teaching media resources, as well as skills to use technology to plan and create multiple resources is of utmost importance to ensure that students' learning needs are fulfilled. In addition, teacher knowledge of diverse teaching media can assist the teacher to ensure that the learning tasks are adequately addressed, and that diverse students' characteristics and learning styles are catered for. The knowledge of planning and coordinating multiple resources is also a need to enable teachers to organise and keep well-resourced classrooms to ensure effective learning in mathematical problem solving.

Diverse learning contents need diverse materials in order to effectively teach problem solving and mathematical reasoning. A teacher should consider choosing accessible, affordable resources that are linked to the objectives of the lesson in order to cater for diverse needs of students.

ACTIVITY 2.6

Choose any content area and topic in order to answer the following questions:

- What is your view on utilisation of resources in the 21st century mathematics classroom?
- What resources do you think would be appropriate in such a classroom?

FEEDBACK

- Base your argument on the 21st century era where students are exposed to diverse technology and material in their everyday life out of the classroom. How can you bring their out of classroom context into the mathematics classroom based on multi-media? State the essence of multiple teaching resources in teaching and learning mathematics in the 21st century. In addition, state multiple ways in which technology can be used by a teacher to create teaching resources, e.g. retrieving pictures from internet for display and exploration in the classroom. You need to consider getting more resources from the web or library to learn more about utilisation of teaching and learning resources in the 21st century.
-

Table 2.6 outlines diverse material that can be utilised in various situations of doing mathematics.

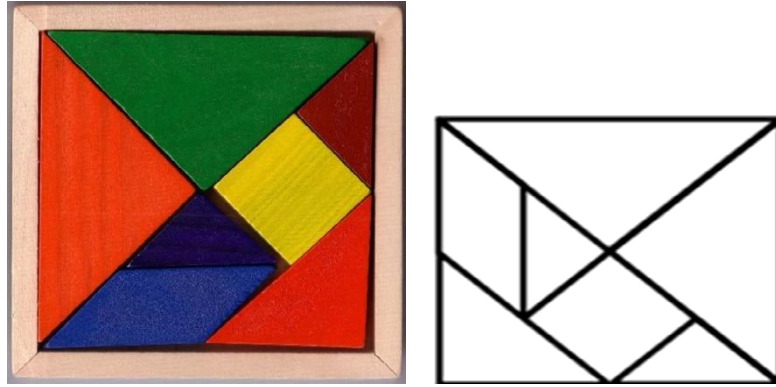
TABLE 2.6

Type of resources	Example
Text material	Textbooks; worksheets
Audio visual material	Video tape; television
Audio material	Radio, CD player, voice recorder
Visual resources	Projector, display board; chalkboard, charts, tangram, posters magazine, newspaper
Digital resources	Laptop, computer software and programmes such as geogebra and more
Mathematical Virtual manipulatives	Cuisenaire rods, geometric planes, blocks, tangram, solid figures and more

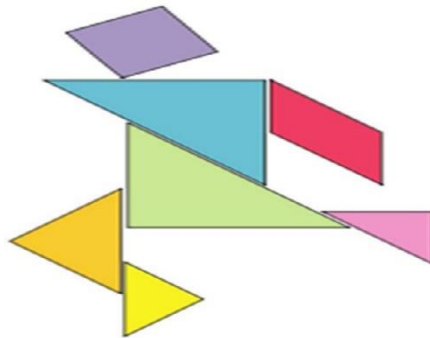
Adapted from Masilo (2016)

From the list of teaching and learning resources (table 2.6), let us explore a tangram.

Does the figure on next page look familiar? It is a tangram, that is, a Chinese geometrical puzzle consisting of a square cut into seven pieces that can be arranged to create other different shapes.



Seven pieces of a tangram



- Five triangles – two small triangles, one medium triangle, and two large triangles
- One square
- One parallelogram

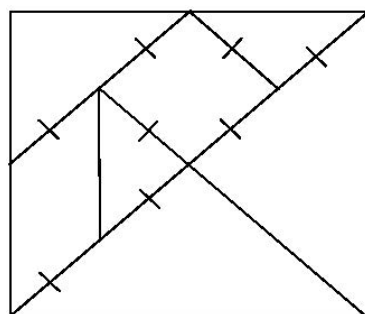


What can tangrams do?

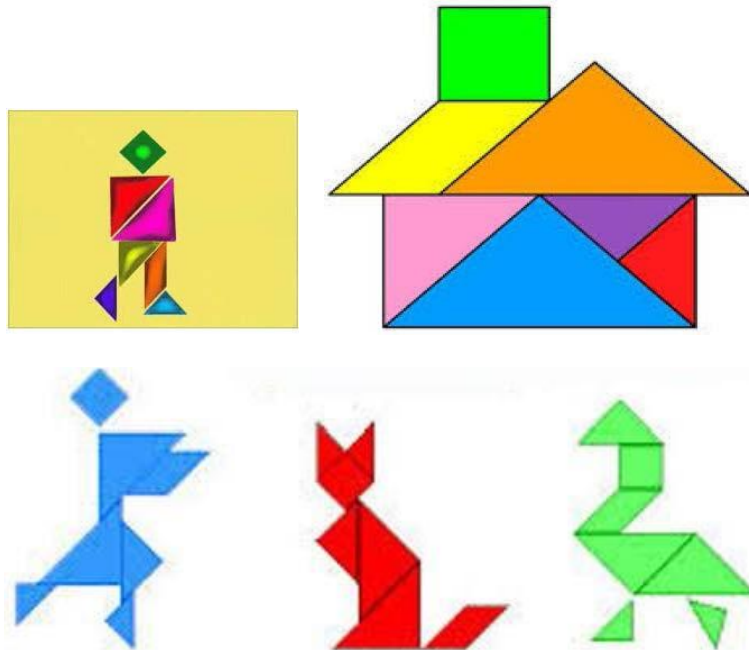
- Tangram are essential resource in developing spatial reasoning
- They are also important in assisting learners to explore a variety of geometric concepts such as shape, size, properties of polygons, similarity and more.

Making your own tangram

You need a paper and pair of scissors. Cut the paper into a square with size of sides being 20cm. Look at the the sketch of a tangarm below and create your own.



Cut out the pieces. Explore the relationships between various shapes. The seven pieces you have in front of you can be manipulated to form numerous other shapes such as examples below:



ACTIVITY 2.7

Consider this statement:

Teacher can provide tangrams in a mathematics classroom and allow students to work independently, either individually or in a group in order to explore with a tangram in order to develop their spatial reasoning before or during the actual lesson on measurement or geometry.

- (1) Read the statement above. Argue in favour or against the given statement in a paragraph of five lines.
- (2) Another instrument that is essential in developing learners' spatial sense or reasoning is a **geoboard**. Carry out an investigation and discuss what a geoboard is. Use the library internet resources to get more ideas in order to complete your investigation. The following questions will guide you in your investigation:
 - What is a geoboard?
 - How is it important in the mathematics classroom?
 - How many types of a geoboard are there?
 - How many content areas and topics would need a geoboard as a teaching and learning resource. Which content areas and topics are these
 - How can a geoboard develop spatial reasoning
 - Other than buying a geoboard, how can you make one? Specify the material needed, size and use a drawing to show the type you would make.

FEEDBACK

- (1) Write a paragraph of five lines. Elaborate on how the statement can be true or not, for example, tell about the possibility or impossibility of group/indi-

vidual work using a tangram; explain how one can or cannot explore with a tangram; speculate skills that support or do not support spatial reasoning in measurement and geometry when exploring a tangram.

- (2) Do your own search of information about a geoboard. The questions out-lined will help you to find relevant information to carry out the investigation.
-

2.14.3 Designing a lesson plan

A teacher needs to have a plan that will help him or her to carry out the teaching work for any lesson. It is a description of the teaching and learning trajectory in a lesson. A lesson plan can be developed daily, weekly, monthly or per topic by the teacher. A lesson plan for the mathematics lesson should cover and specify the following scope: Grade; time to be taken to conduct a lesson; content area; topic in the specified content area; specific aims and skills related to the content area and topic as specified by the policy document; context, that is, specification of the classroom environment such as working in a mathematics laboratory, in a computer centre, classroom; formal informal setting; as well as factors such as group work, individual work and other contextual factors that the lesson operate under. More elements of a lesson plan are: teacher activity, learners' activity; content range and assessment that is, how the lesson will be assessed based on methods of assessment, types of assessment and methods of collecting assessment evidence. Content range means which parts of the topic will be taught in a lesson. For example, out of all sub-topics which few topics will be covered in the lesson. It is not everything in the content area and topic that will be taught, therefore, the content range must be specified. Complete activity 2.8 about writing a lesson plan

ACTIVITY 2.8

Writing a lesson plan

Instructions:

- Refer to the CAPS documents attached in Activity 2.1 when choosing the content area, relevant specific skills and outcomes.
- Plan a lesson in any grade from 7 to 12. Indicate the grade for which the lesson is planned. Consistency is necessary in points 1–10 of 1.1. Ensure that there is cohesion between 1.1 and 1.2 of section B. Further, consider the concept map you have drawn in Activity 2.4 when completing your lesson.

Background information for the lesson

In your lesson plan specify the following:

- (1) Grade
- (2) Total time for the lesson
- (3) Content area
- (4) Topic
- (5) Specific aims applicable
- (6) Specific skills appropriate
- (7) Context that is applicable
- (8) Teacher's activity (how to facilitate learning including teaching methods)
- (9) Learners' activities (what will the learners do)
- (10) Content range that is applicable, explained in terms of concepts and skills

- (11) Teaching and learning support material necessary
 - (12) Assessment (types of assessment, methods of assessment and methods of collecting assessment evidence; also specify what will be assessed and how it will be assessed).
 - (13) Reflection (what to consider when reviewing your lesson).
-

2.15 SELF-ASSESSMENT

Tick the boxes to assess whether you have achieved the outcomes for unit 2. If you cannot tick the box, you should go back and work through the activity that you find challenging.

	Criteria	Tick ✓
1	I can: Provide the rationale for doing mathematics	
2	Discuss how mathematics is learned	
3	Design a problem-based lesson	
4	Identify teaching and learning resources that can be utilised in various situations of doing mathematics	
5	Write a lesson plan	

REFERENCES

- Andrew, R. (2017). Procedural vs conceptual knowledge in Mathematics Education: Preparing to embrace a conceptual approach to teaching mathematics. In Learn Implement Share: Creating Engaged Learners. Retrieved from www.learnimplementshare.com.
- Dorfler, W. (2003). Mathematics and Mathematics Education: content and people, relation and difference. *Educational Studies in Mathematics* 54:147–170, 2003.
- Kilpatrick, J, Swafford, J & Findell, B. (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.
- Masilo, MM. (2015). Keeping pace with advances in multimedia to improve mathematical problem solving skills in classrooms. ISTE Conference Proceedings. 25–29 October 2015. Kruger National Park, Mopani Camp, Phalaborwa, Limpopo Province, South Africa.
- Masilo, MM. (2016). Teachers' perspectives on a multimedia supported mathematical problem solving: Focus on grade 8 algebra in the Mount Ayliff District. ISTE Conference. 23–27 October 2016. Kruger National Park, Mopani Camp, Phalaborwa, Limpopo Province, South Africa.
- Masilo, MM. (2015). Keeping pace with advances in multimedia to improve mathematical problem solving skills in classrooms. ISTE Conference Proceedings. 25–29 October 2015. Kruger National Park, Mopani Camp, Phalaborwa, Limpopo Province, South Africa.
- Masilo, MM. (2016). Teachers' perspectives on a multimedia supported mathematical problem solving: Focus on grade 8 algebra in the Mount Ayliff

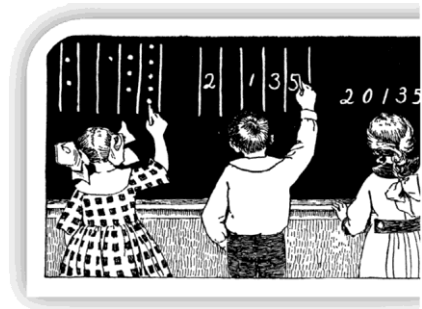
- District. ISTE Conference. 23–27 October 2016. Kruger National Park, Mopani Camp, Phalaborwa, Limpopo Province, South Africa.
- Newby, TJ, Stepich, DA, Lehman, JD, Russel, JD & Ottenbreich-Leftwich, A. (2011). *Educational Technology for teaching and learning* (Fourth edition). Boston: Pearson Education.
- Sonnabend, T. (2010). *Teaching Mathematics: An interactive approach for grades K-8*. (Fourth edition). USA: Brooks/Cole, Cengage Learning.
- The Attic Learning Community. (2016). Types of knowledge. www.The-attic.org.
- Van de Walle, JA, Karp, KS & Bay-Williams, JM. (2004). *Elementary and Middle School Mathematics: Teaching Developmentally* (9th edition). England: Pearson Education.
- Ziegler, GM. (2010). What is Mathematics. Retrieved from <https://www.springer.com/cda/content/document/cda.../9783642195327-p1.pdf?...>
- Andertoons. (n.d). Retrieved from https://www.google.com/search?q=what+is+mathematics+cartoons&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwjui7yJ2_3aAhWhJMAKHfBfBGcQsAQIJw&biw=1920&bih=970#imgsrc=uG-UlICU7o-TVM:&spf=1526043853238.

LEARNING UNIT 3

THE CONCEPT OF NUMBERS

3.1 INTRODUCTION

This learning unit aims to provide you with knowledge and understanding of the number concept. Knowledge presented here will help to build a basis for future learning of more advanced and complex mathematical concepts. (In early grade levels, the section of *numbers*, *operations* and *relationship* forms an important content area in the Curriculum and Assessment Policy Statement (CAPS).) Specifically, this learning unit will help you to understand the development of number sense, the types of numbers, the relationships among numbers, the number systems and operations.



3.2 LEARNING OUTCOMES

At the end of the learning unit you should be able to:

- design suitable mathematical activities to explain the development of number concepts and explore its historical background
- explore different types of numbers
- identify and perform various mathematical operations involving whole numbers, such as counting and calculating

3.3 OVERVIEW OF THE NUMBER CONCEPT

3.3.1 Development of the number system

Mathematics is generally considered to be a science of numbers requiring one to have a good intuition or awareness of number concept.

3.3.1.1 *Number concept*

The word “concept” has been used in this learning unit to imply “an idea or an intuition”, “a general understanding or a notion” that one has in relation to numbers. If you have a good understanding of numbers, you can be described as someone who has a good sense of numbers and their relationships. One’s understanding of numbers may represent one’s state of number conceptualisation or one’s sense and awareness of numbers, normally referred to as *number sense*.

Among other things, number sense helps you to understand the place value of a number and relationships between numbers. Van de Walle and Lovin (2006) say that your awareness of the concepts of “more”, “less” and “same” helps you to understand important and basic relationships in number concepts. Number sense develops gradually as you work with numbers, as you visualise them in different contexts, and as you learn to relate them.



How will a teacher know that her learners have a good number sense?

At the primary school level, number sense may include skills such as:

- *Recognising* all 10 numerals (from 0 to 9).
- *Matching*: A skill to group together items that are similar.
- *Counting*: A skill needed to demonstrate ability to observe quantities. Counting helps you to assign a numeric value to an object.
- *Sorting*: A skill to classify objects.
- *Ordering*: A skill associated with arranging items or objects, etc.

A number sense could include your understanding of whole numbers, related number operations and number relationships (Dyson, Jordan & Gluttin, 2011:167). Other authors have emphasised that early number sense is a strong predictor of future success in school mathematics (Dyson *et al.*, 2011; Starr, Libertus & Brannon, 2013).



What kind of mathematics activities can a teacher construct to develop learners' number concept?

You may consider some of the following mathematical activities to develop and assess your learners' sense of numbers.

While learners are attempting to do the activities in figure 3.1 the teacher may allow them to work as individuals or to work in groups. In either way, a teacher should be present at all times to assist learner(s), and to monitor how they are attempting to demonstrate number sense-related skills.



- An example of a **sorting** activity

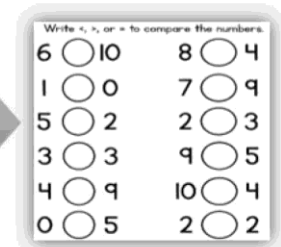
In this activity a teacher provides learners with three representations of a number that should then be sorted in a way that indicate that three representations stand for the same number.


- An example of a **matching** activity

In this activity a teacher provides two columns, one for diagrams and the other for number letters. The learner must then match each diagram or picture to its respective number.


- An example of a **comparing** activity

In this activity the learner is expected to compare two numbers in terms of size. The words "same", "bigger" and "smaller" could be used to facilitate comparison. Alternatively, the symbols "=", ">" and "<" may be used.


- An example of **counting** activity

In this activity the teacher can ask learners to do the counting of numbers in the chart verbally while in the process associating a number to a diagram below it.



FIGURE 3.1
Examples of mathematics activities for number sense

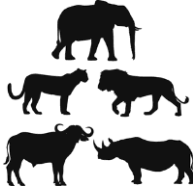
ACTIVITY 3.1

Imagine yourself to be a teacher. How would you formulate or pose your activity questions in order to instruct your learners to perform the mathematical skills shown in figure 3.1? Construct the question for each number sense activity (sorting, matching, comparing and counting).

Learners conceptualise numbers differently as a result of their varied everyday interaction with numbers. Numbers are represented in various ways. It means that a particular number may be associated with certain events or objects, i.e. number representation is one's conceptualisation of a number (see activity 3.2). Number representation may also be thought of as a way of attaching meaning to a number. What experience does a certain number bring out to you?

ACTIVITY 3.2

Describe how a number “five” is represented in each diagram:

Diagram	Explain number representation
	
	
five	
	
	

3.3.1.2 *Historical background of the number system*

When you study the history of numbers, the word *numeration* always comes up. Numeration refers to the process of calculating or assigning a number to something.

It is difficult to know exactly when the numeration system started. In the early thousands of years, human beings did not have a system of numbers to represent the numerical concepts of “one”, “two” or “three”. However, humans have always had some sense of a number, at least to the extent of recognising MORE or LESS when objects were added to or taken away from a group. For instance, the tribe had to know the number of its members, or a man had to have a sense in some way that his flock was increasing or decreasing. Hence the answer to the question “*How many ...?*” became very important.

ACTIVITY 3.3

Use your current knowledge of numbers to explain the following concepts: enumeration; numeration; number and numeral.

Suppose the answer to the question “How many ...?” is “ten boys”. We can say:

- The word “ten” is a *number name*
- The symbol “10” is a *numeral*
- The concept “ten” is a *number*

Number representation

Number concept

You can see that written symbols such as 2 or 5 are *numerals*. Using numerals to represent numbers greater than 9 requires a numeration system. Numbers from 0 to 9 are generally easy to handle. Early history of numeration system is associated with the ancient nations of Babylon, Egypt and the Hindu-Arabic.

3.3.1.2.1 The Babylonian numeration system

In almost 3000 BC the Babylonians were able to formulate wedge-shaped marks by means of a stylus in soft clay. A stylus is a hard point, typically like the edges of a diamond. Babylonians used the symbol () to represent the numbers from 1 to 9. The Babylonian numeration system from 1 to 59 is represented in figure 3.2.

1 	11  	21   	31    	41     	51 
2  	12   	22    	32     	42      	52 
3   	13    	23     	33      	43       	53 
4    	14     	24      	34       	44       	54 
5     	15      	25       	35        	45       	55 
6      	16       	26        	36         	46       	56 
7       	17        	27         	37          	47       	57 
8        	18         	28          	38           	48       	58 
9	19	29	39	49	59
10	20	30	40	50	

3.3.1.2.2 The Egyptian numeration system

The Egyptian numeration system goes back to the time of 3400 BC. In the ancient times the Egyptian numeration system:

- made use of concept of “base 10”
- did not have symbols
- was not positional
- numerals were referred to as *hieroglyphics* (pictures or symbols represented numbers)

The Egyptian numeration system developed and adopted some of the symbols to represent numbers. The symbols were written on stones, papyrus, wood and pottery. The writing was from right to left. Egyptians used tally marks (“||||”) to represent numbers from 1 to 9. You may read a piece of history in the embed-

ded picture to get a sense of how the Egyptian system of numbers developed, and also how some of the aspects of this numerical development have been preserved. The following figure shows some of the symbols used by the Egyptians to represent their numeration system.

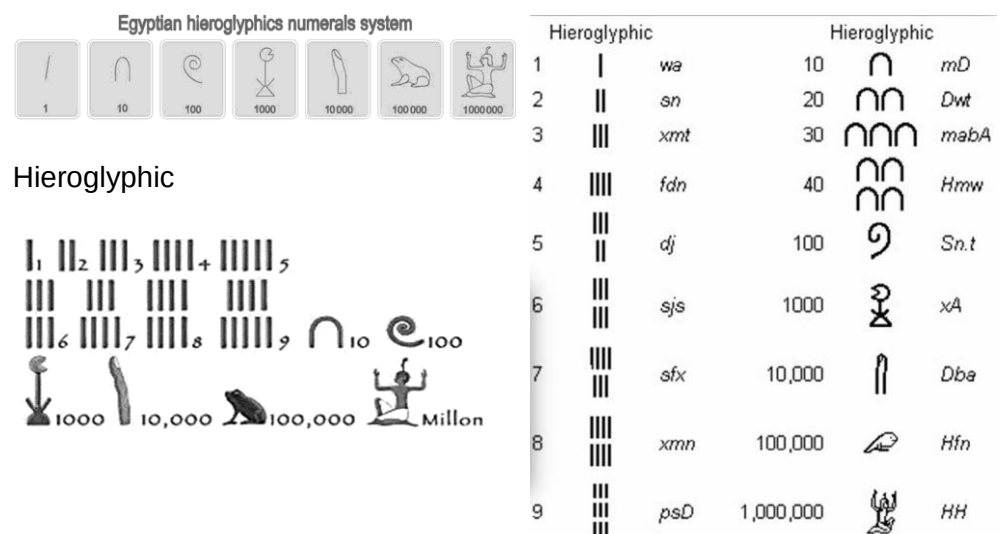


FIGURE 3.3

Pictures of the ancient Egyptian numeration system

The symbols in figure 3.3 were used by Egyptians to compute basic quantities.

ACTIVITY 3.5

What would you say were the ancient Egyptian's general conceptualisation of a number (number concept)?

3.3.1.2.3 The Hindu-Arabic numeral system

The Hindu-Arabic numeral system originated in India and is the most commonly used number system in the world today. This numerical system is based on ten different glyphs (symbols), namely, 0, 1, 2, 3, 4, 5, ..., 9.

The following are some of the key attributes of the Hindu-Arabic numeral system:

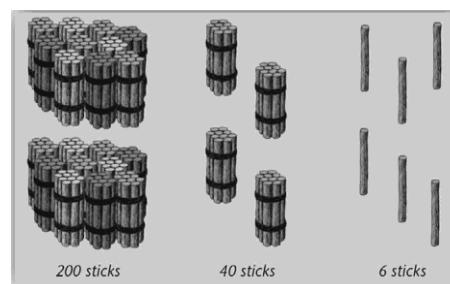
- Ten numerical symbols are used to form and represent all possible numbers.
- It classifies and group numbers by 10, and is known as a base-10 system.
- It uses the place value system, which starts from right to left.

3.3.1.2.4 A place value

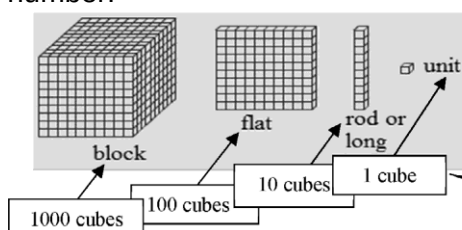
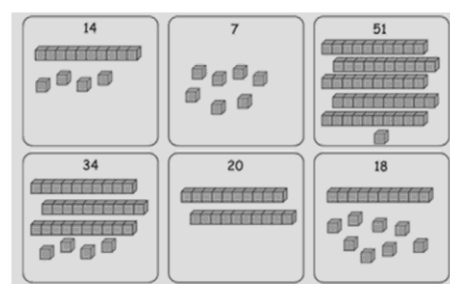
A place value represents the value of a digit in a number, which is based on the location of the digit. A place value is established when parts (digits) of a number are considered. Van de Walle, Karp and Bay-Williams (2016:246) say a number sense is linked to a complete understanding of place value. Human, Olivier, Lampen, Le Roux, Long and Human (2016) use an idea of a stick or sticks to illustrate the part(s) of a number. The authors talk about:

- “One stick”
- “A bundle of ten sticks”
- “A heap of hundred sticks”, and so on...

The number represented with sticks here is $246 = 200 + 40 + 6$, implying that there are 2 hundreds, 4 tens and 6 units in the number 246. Number 246 has THREE digits (parts), and we have now established the place value of each digit (Human et al. 2016).



Another system uses blocks (base-10 blocks) to represent parts of a number to illustrate the place value in a number.



The notion of “cubes/ blocks” will help you to understand the concept of “parts of a number” when working with place value.

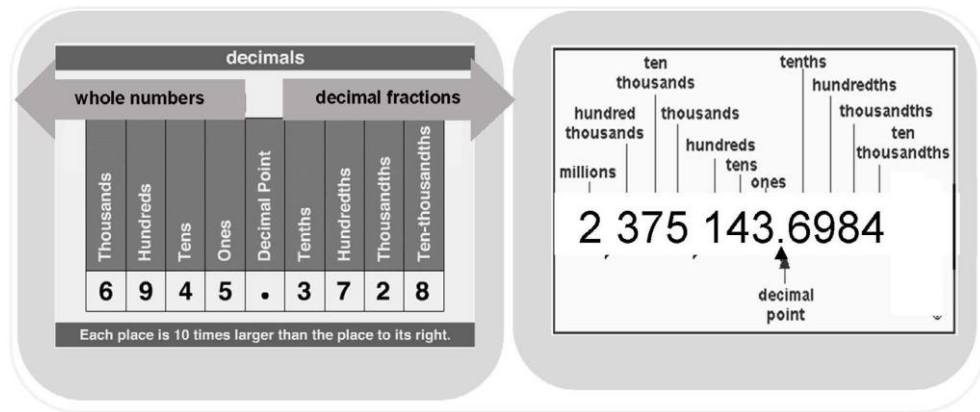
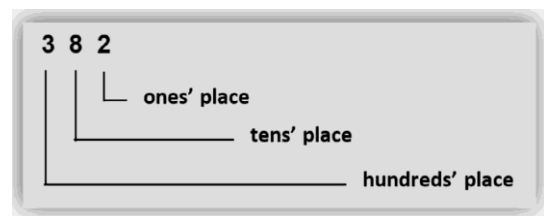


FIGURE 3.4

Examples of charts illustrating the placing and values of different digits

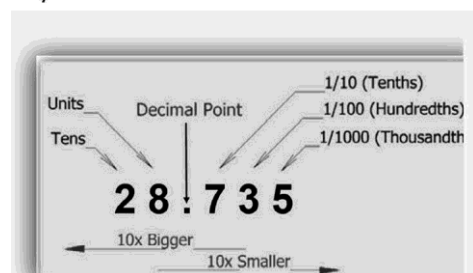
The number 382 has three digits or parts that constitute different place values. The first digit (3) is called the hundreds' place. It tells you how many sets of one hundred are in the number; 382 has three hundreds.



The middle digit (8) is the tens' place. It tells you that there are eight tens in addition to the three hundreds. The last or right digit (2) is the ones' place. Therefore, there are 3 sets of 100, plus 8 sets of 10, plus 2 ones in the number 382. Each digit (part) has its own unique place value in the number 382. See examples in figure 3.4.

In figure 3.4 the number in the first chart could be written as 6945.3728. In the second chart the place value of each digit is accurately explained. In both charts you can see that digits are either placed on the right side of the decimal point, or on the left side of the decimal point. Digits on the right are expressed in terms of $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$; etc Using these numerical notations we can say "7" occupies the place value of *ten thousandths* or is placed $\frac{1}{10\,000}$ in 1 342 365.1427 (see examples below).

Example 1: Representation and explanation of a decimal number



Example 2: Using the knowledge gained from the three figures we can now say:

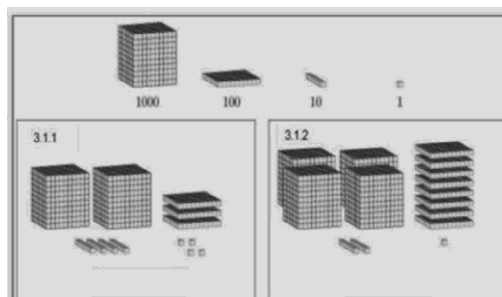
1. The place value of 8 in 8531 is thousands.
2. The place value of 9 in 3405.296 is hundredths.

ACTIVITY 3.6

- (1) What is the meaning of a *place value* of a number?
- (2) Use "stick" to write/represent the following numbers:

- (a) 53
- (b) 167
- (c) 2 345

- (3) Name the two numbers represented by the base-10 blocks alongside.



- (4) Write the place value of numbers that are underlined:

- (a) 34.73 (b) 342.853 (c) 4127.6519

- (5) Fill in the missing parts in the diagram below:

27,582 =thousands hundreds tens.....ones
 8,214 =thousands hundreds tens.....ones
 135,634 =thousands hundreds tens.....ones

3.3.1.3 Types of numbers

The number system within which different types of numbers exist is known as the *real number system*. The real number system is the one that is popularly used in schools. It has evolved and expanded in order to accommodate new and emerging numbers. The following figure is the representation of the types of numbers that form the set of real numbers.

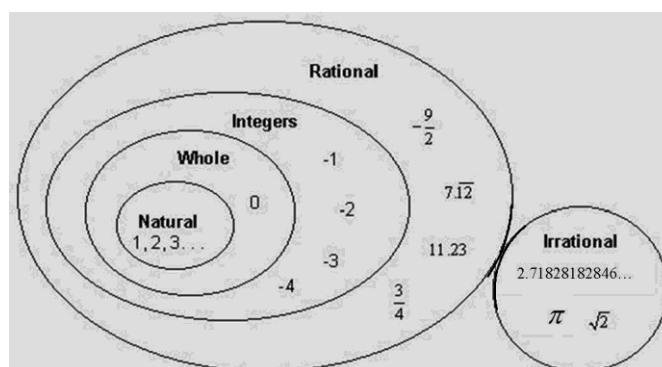


FIGURE 3.5

A representation of a real number system

3.3.1.3.1 Natural numbers

In the early days, numbers were used largely for counting purposes. For instance, a man would want to know the number of cows he owns, and in that way a counting process would be initiated. Natural numbers (**N**), also called *counting numbers*, represent a set of first numbers that humans use for counting. The set of natural numbers can be represented as follows:

$$\text{Natural numbers} = \mathbf{N} = \{1; 2; 3; 4; \dots\}$$

The three dots at the end of the set is a common mathematical notation to indicate that the set is continuous and infinite, meaning, it goes on for ever. In figure 3.5 natural numbers are placed at the beginning of the real number system.

3.3.1.3.2 Whole numbers

At some point in history the idea of a “zero” came to be recognised as representing a particular number or as needing to occupy a particular position in the set of counting numbers or natural numbers. A zero was then introduced in the developing real number system, which at a time only consisted of counting numbers.

This innovation gave birth to whole numbers. For instance, in the early days a household that did not have cows would be counted as having “no cows”, which was then equivalently expressed as “zero cows” in counting terms. There was a need in earlier times to infuse a zero number into the existing list or set of counting numbers.

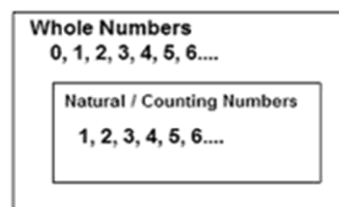
The emerging set of whole numbers was then represented as follows (see also figure 3.5):

$$\text{Whole numbers} = \mathbf{N_0} = \{0; 1; 2; 3; 4; \dots\}$$

ACTIVITY 3.7

Whole numbers and natural numbers

- (1) Explain why *counting numbers* are also called *natural numbers*?
- (2) What is *zero*? Would you say zero is “nothing” or “something”?
- (3) Would you say a zero is a number? Explain your answer.
- (4) In figure 3.5 the small circle that represent the natural numbers seems to be sitting inside the bigger circle representing the whole numbers. What does this diagrammatical placing of two number sets represent, and how should one understand and explain this representation mathematically? (Also refer to the figure below in which rectangular shapes have been used.)



3.3.1.3.3 Integers

In the previous section you have been introduced to the idea of a zero, which resulted in the formation of whole numbers from natural numbers. Let us reflect back to the analogue of owning cows used in earlier discussions: We may have a case of a household that does not have a cow(s), but what if this household

is owing a cow to a neighbour? The household does not have a cow, which is a zero quantity, but also owes a cow. How do you describe the last condition?

In terms of a number system this condition (of owing) introduces a new concept of negative numbers. If you extend the set of whole numbers by including the negative numbers, you will get a new set of numbers called *integers*, with a symbol **Z**. Integers represent a combination of whole numbers and negative numbers. The set of integers is represented as follows:

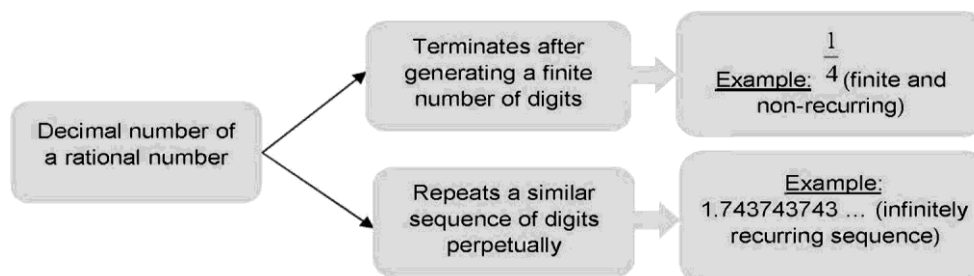
$$\text{Integers} = \mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

3.3.1.3.4 Rational numbers

Rational numbers represent a set of number that can be written in the ratio form involving two integers. This means that a rational number can be expressed as a quotient or fraction ($\frac{a}{b}$), where a and b are both integers and b is non-zero. When the denominator of a fraction is equal to zero then the fraction is undefined.

Given this explanation you should be able to see that “3” is a ($\frac{3}{1}$), in which case “3” and “1” are integers, and $1 \neq 0$.

All integers are rational numbers because all of them can be written as a fraction with a denominator of “1”, where $1 \neq 0$. A rational number is a fraction that can be converted into a decimal number, but not all decimal numbers are rational numbers. The symbol used to represent a set of rational numbers is **Q**.



You must always remember that a rational number can be converted into a decimal number that is either recurring (repeating decimal) or is a decimal number with a finite number of digits after the coma.

ACTIVITY 3.8

Explain if the following numbers are rational numbers or not:

Number	Rational?: Yes/ No	Explanation
7		
$\frac{\sqrt{9}}{3}$		
1.5		

Number	Rational?: Yes/ No	Explanation
0.111 ...		
$\sqrt{2}$		

3.3.1.3.5 Irrational numbers

In activity 1.6, $\sqrt{2}$ cannot be written as a fraction, that is in the form where a and b will represent a set of integers and $b \neq 0$. Therefore, $\sqrt{2}$ is not a rational number because it does not meet the requirements of the integer definition.

What type of a number does $\sqrt{2}$ represent?

The $\sqrt{2}$ is an example of a new set of numbers that are called irrational numbers. How does an irrational number differ from a rational number? Two questions arise:

- Can an irrational number be expressed as a fraction consisting of integers?
- When converted into a decimal number, does a decimal part terminate, and also become periodic (that is, recurring)?

In order to answer the questions above let us look at the number $\sqrt{8}$. Can $\sqrt{8}$ be expressed as a fraction of two integers?

$$\sqrt{8} = \frac{\sqrt{8}}{1} = \frac{2\sqrt{2}}{1}$$

In above, the numerator is not an integer, and therefore $\sqrt{8}$ is not a rational number.



- A famous irrational number is a *pi* (π).
- $\pi = \frac{22}{7} = 3.14159265389...$
(and so on). This number is not accurate because it has continuous dots on its tail
- It is not possible to locate an irrational number on a number line.
- It is not possible to write the value of a *pi* (π) as a simple fraction.



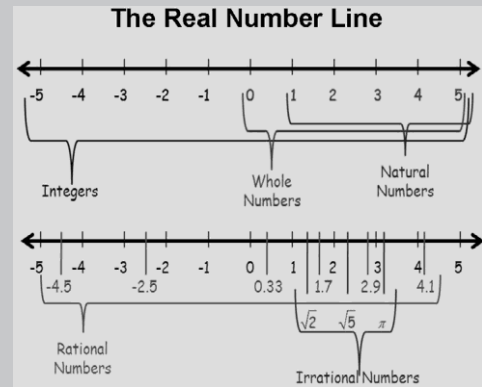
Irrational numbers do not have specific points on the number line to represent them.

Thus, one of the characteristics of irrational numbers is that the decimal component keeps going on and on, and is not recursive or repetitive. The common symbol used to represent the irrational numbers is **Q'**.

SUMMARY

We can use a number line to summarise our discussions of a real number system. The following are some of the facts to consider:

- A real number system is made up of rational and irrational numbers.
- The number line is an ideal model to demonstrate the real number system.
- Each set of numbers in a number line is a subset of the real number system.
- The union of all sets of numbers forms the real number system.



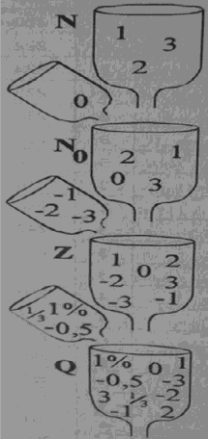
ACTIVITY 3.9

Find the values of the following famous irrational numbers:

Name of irrational number	The estimated value
Euler's number	
Golden ratio	

ACTIVITY 3.10

Study the accompanying figure and answer related questions:



- (1) Use the analogue of pouring bottles alongside to explain the development of the real number system. Begin your explanation from the top bottle.
- (2) Explain why do you think all bottles are faced upside down?
- (3) All bottles are open at the bottom. In particular, explain why the last bottle is also open.

ACTIVITY 3.11

Complete the following worksheet to demonstrate your knowledge of real numbers:

Real Number system		
Definition		Facts/ characteristics
Example	Natural Numbers	Non-example
Definition		Facts/ characteristics
Example	Whole Numbers	Non-example
Definition		Facts/ characteristics
Example	Integers	Non-example
Definition		Facts/ characteristics
Example	Rational Numbers	Non-example
Definition		Facts/ characteristics
Example	Irrational numbers	Non-example

3.3.1.4 The basic number operations

In mathematics there are four basic operations, namely, addition, subtraction, multiplication and division.

ADDITION

Addition is based on the idea of summing up or finding a total count of objects or values.

Knowledge of a place value system (based-10 system) should assist you to perform addition with efficiency. For instance, when adding two whole numbers:

$$134 + 583$$

Procedure: (1) Start by adding the “ones’ place” digits, that is, “4+3”
 (2) Then add the “tens’ place” digits, and so on.

There could be instances where the sum of adding the “ones’ place” digits would represent a “carrying over” into the “tens’ place”, for example: $7 + 5 = 12$. Now we go back to our initial task of adding 134 to 583, or vice versa.

Here the sum (addition) of 30 and 80 creates a situation where 110 is a number with “a one-hundred” as a part to it, but sitting in a “tens’ place”. We should then carry over the “100” (as a part of 110) to the “hundreds’ place”, and only a “10” will be left as a part in the “tens’ place”.

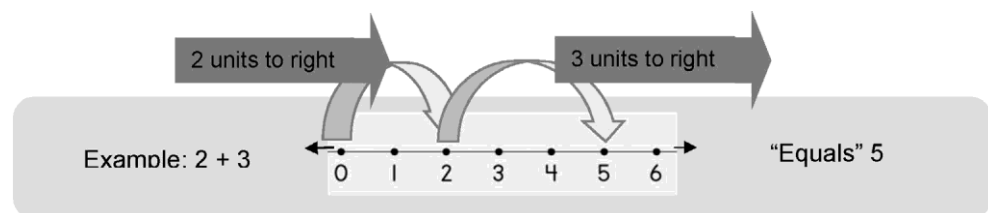
	Hundreds	Tens	Units
134	100	30	4
583	500	80	3
	600	110	7

Sum is the answer we get when adding two, three, and more numbers.

We now see that $134 + 583 = 600 + 110 + 7 = \underline{700 + 10 + 7 = 717}$

o

Another important tool to use when doing simple numericals is a *number line*. A number is a line on which real numbers can be represented in intervals.



SUBTRACTION

When you subtract you are actually taking away something from a group or items. Addition and subtraction are complementary.

In previous discussion we said addition and subtraction are complementary. This also means that addition and subtraction are *inverse operations*. Mathematically, the latter means: If, for example:

$$40 - 6 = 34, \text{ then } 40 = 34 + 6$$

Subtraction answers questions such as: In a class there are 76 learners of which 25 are boys. How many girls are in the class?

Subtraction can also be represented in a number by moving some units backwards.

The answer we get when we subtract is called a **difference**.

MULTIPLICATION

Multiplication is repeated addition
 $3 \text{ (multiplier)} \times 5 \text{ (multiplicand)} = 15$

Product is the answer we get when we multiply numbers. The numbers “3” and “5” above are also called the factors (of “15”). Repeated addition is: $3 + 3 + 3 + 3 + 3 = 15$.

There are many computational strategies that can be used to do multiplication. In early stages of learning about mathematics operations a concept of “arrays” may be used to assist learners to understand multiplication. Arrays are useful because they make use of visual representations to explain the multiplication process. An array makes use of the concepts of rows and columns. Figure 3.6 below consists of TWO rows and SIX columns.

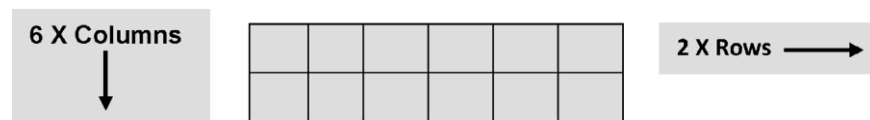
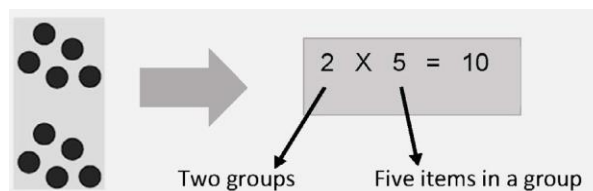


FIGURE 3.6

Multiplication by columns and rows

Rows in each array must be seen as being equal groups. For instance, $2 \times 5 = 10$ may be represented as follows:



DIVISION

Division is an act of almost partitioning. Loosely explained, the word partition may be associated to phrases like “to demarcate”, “to divide into subsections”, “to split”, and so on.

Division is the inverse, or opposite, of multiplication. An example of division using manipulatives can be illustrated as follows (meaning “20” could be partitioned or divided into FIVE groups, each of which with FOUR items):

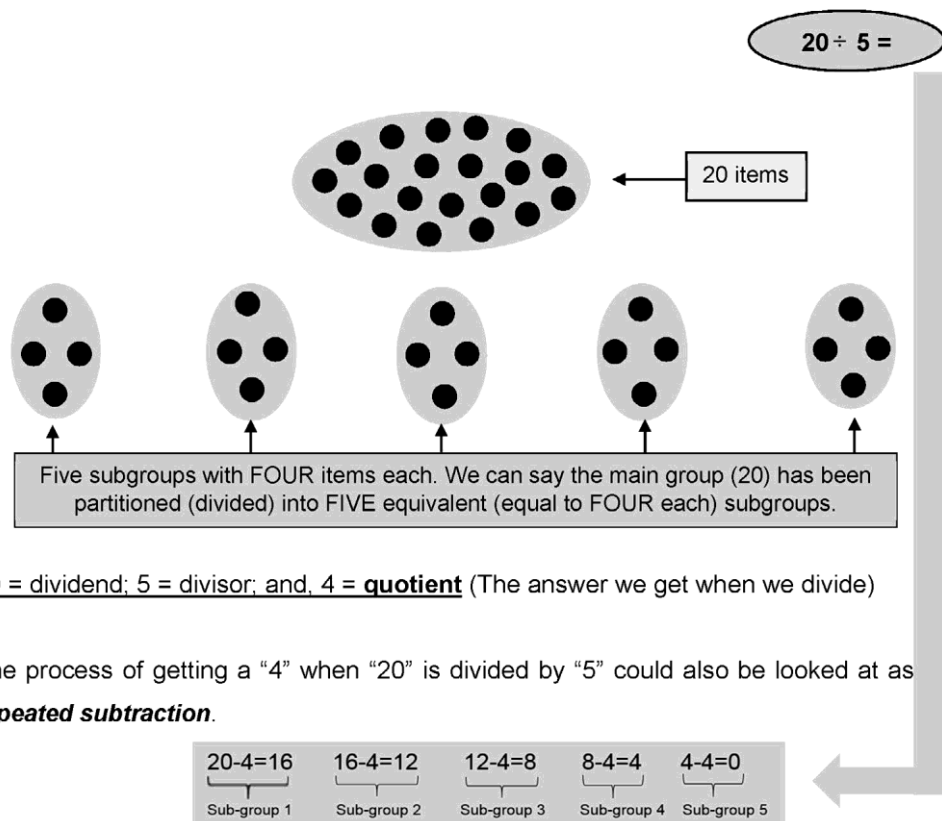


FIGURE 3.7
Division by using manipulatives

3.3.1.4.1 Whole numbers and number operations

We have already talked about whole numbers. So far we have confined our discussion to four basic operations of mathematics and only to whole numbers.



Do the four mathematical operations relate to each other?

3.3.1.4.1.1 Communicative property

Addition and multiplication can use the commutative property. Commutative property implies that numbers can be switched around when adding or multiplying without affecting or changing the answer.

Examples: $3 + 8 = 8 + 3$addition
 $3 \times 8 = 8 \times 3$multiplication

3.3.1.4.1.2 Associative property

Associative property is observed when numbers are grouped, and again, this property only applies to ADDITION and MULTIPLICATION.

$$(8 + 3) + 5 = 16 \text{ is the same as } 8 + (3 + 5) = 16$$

$$(5 \times 3) \times 2 = 30 \text{ is the same as } 5 \times (3 \times 2) = 30$$

3.3.1.4.1.3 Distributive property

Distributive property is concerned with the behaviour of numbers when a mathematics expression has multiplication together with addition and subtraction. In this property, anything inside the brackets (parenthesis) must be multiplied first by the multiplier.

$$\text{Example: } 3(5 + 2) = (3 \times 5) + (3 \times 2) = 15 + 6 = 21; \text{ same as } 3 \times 7 = 21$$

$$3(5 - 2) = (3 \times 5) - (3 \times 2) = 15 - 6 = 9; \text{ same as } 3 \times 3 = 9$$

ACTIVITY 3.12

Number operations:

- (1) Do addition and subtraction in the following problems

$\begin{array}{r} 804 \\ - 188 \\ \hline \end{array}$	$\begin{array}{r} 851 \\ - 195 \\ \hline \end{array}$	$\begin{array}{r} 939 \\ + 336 \\ \hline \end{array}$	$\begin{array}{r} 740 \\ + 797 \\ \hline \end{array}$	$\begin{array}{r} 242 \\ + 119 \\ \hline \end{array}$	$\begin{array}{r} 973 \\ + 186 \\ \hline \end{array}$	$\begin{array}{r} 165 \\ - 104 \\ \hline \end{array}$
$\begin{array}{r} 419 \\ - 385 \\ \hline \end{array}$	$\begin{array}{r} 910 \\ + 639 \\ \hline \end{array}$	$\begin{array}{r} 182 \\ + 509 \\ \hline \end{array}$	$\begin{array}{r} 629 \\ - 220 \\ \hline \end{array}$	$\begin{array}{r} 747 \\ + 434 \\ \hline \end{array}$	$\begin{array}{r} 507 \\ - 504 \\ \hline \end{array}$	$\begin{array}{r} 468 \\ - 164 \\ \hline \end{array}$
$\begin{array}{r} 380 \\ + 369 \\ \hline \end{array}$	$\begin{array}{r} 676 \\ + 380 \\ \hline \end{array}$	$\begin{array}{r} 470 \\ - 366 \\ \hline \end{array}$	$\begin{array}{r} 439 \\ + 747 \\ \hline \end{array}$	$\begin{array}{r} 551 \\ - 368 \\ \hline \end{array}$	$\begin{array}{r} 100 \\ + 502 \\ \hline \end{array}$	$\begin{array}{r} 619 \\ - 579 \\ \hline \end{array}$

- (2) Do multiplication and division in the following problems

$\begin{array}{r} 7 \\ \times 2 \\ \hline \end{array}$	$\begin{array}{r} 49 \\ \div 7 \\ \hline \end{array}$	$\begin{array}{r} 84 \\ \div 7 \\ \hline \end{array}$	$\begin{array}{r} 10 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 4 \\ \times 7 \\ \hline \end{array}$
$\begin{array}{r} 84 \\ \div 7 \\ \hline \end{array}$	$\begin{array}{r} 7 \\ \times 4 \\ \hline \end{array}$	$\begin{array}{r} 9 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 63 \\ \div 9 \\ \hline \end{array}$	$\begin{array}{r} 7 \\ \times 7 \\ \hline \end{array}$
$\begin{array}{r} 42 \\ \div 6 \\ \hline \end{array}$	$\begin{array}{r} 14 \\ \div 2 \\ \hline \end{array}$	$\begin{array}{r} 14 \\ \div 2 \\ \hline \end{array}$	$\begin{array}{r} 21 \\ \div 7 \\ \hline \end{array}$	$\begin{array}{r} 77 \\ \div 11 \\ \hline \end{array}$
$\begin{array}{r} 63 \\ \div 9 \\ \hline \end{array}$	$\begin{array}{r} 7 \\ \times 2 \\ \hline \end{array}$	$\begin{array}{r} 12 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 21 \\ \div 7 \\ \hline \end{array}$	$\begin{array}{r} 7 \\ \times 7 \\ \hline \end{array}$

- (3) Choose any TWO multiplication problems in (2) and re-do them by making use of the following strategies:
- Repeated addition
 - Array representations

3.4 CONCLUSION

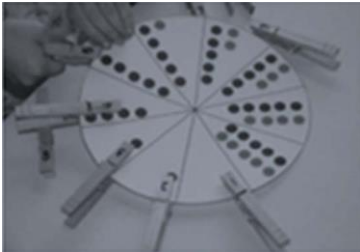
In this learning unit you have learnt about numbers, and the following topics were discussed:

- the number concept and number sense
- the history on the numeration system
- the types of numbers
- the place value of a digit in a number
- the four mathematics operations in relation to whole numbers

3.5 SELF-ASSESSMENT

This section contains questions that will test whether you have understood the Learning Outcomes covered in this learning unit.

The following questions are meant to provide you with an overview revision of this learning unit:

- (1) Distinguish between the phrases “*number concept*” and “*number sense*”.
- (2) Explain the meaning of the phrase “*number representation*”; and give three examples in which a number “2” may be represented by your learners.
- (3) Use a diagram alongside as a reference to develop a mathematical task that will help you to assess the number sense in your learners. Explain all the steps you will undertake in constructing the task. Explain the material/manipulatives you will use for the task. 
- (4) Why do you think the Hindu-Arabic numerical system is currently the most used number system in the world?
- (5) The Hindi-Arabic numerical system is also known as a “*base-10 number system*”. Explain your understanding of the phrase “*base-10 number system*”.
- (6) Explain why the word “*natural numbers*” is used when referring to counting numbers.
- (7) Complete: Any number which is either rational or irrational is called
- (8) Which of the following statements are *True* or *False*? Where it is FALSE, provide explanation for your answer.
 - (a) No integer is an irrational number.
 - (b) Every integer is a rational number.
 - (c) Every integer is a whole number.
- (9) What is the difference between rational numbers and irrational numbers?
- (10) Draw a *number line* to graph the following numbers: -4 ; 2.5 ; -0.6 ; $-\frac{5}{2}$; $\sqrt{2}$; 3 ; $\frac{3}{4}$.
- (11) What number is represented by the base-10 blocks alongside? 
- (12) Complete the diagram alongside by filling up each place value space, and also writing the actual number. 
- (13) Use the idea of “sticks” to explain the number in the diagram alongside, assuming that 1 *bundle* = 10 *sticks*. 

(14) What are the place values of underlined numbers below

23.61' 423.658; 23.062; 10.958265; 986 234.8390123

3.6 REFERENCES

- Dyson, NI, Jordan, NC & Glutting, J. 2011. A Number Sense Intervention for Low-Income Kindergartners at Risk for Mathematics Difficulties. *Journal of Learning Disabilities*, 46(2), 166–181.
- Starr, A, Libertus, EL & Brannon, EM. 2013. Number sense in infancy predicts mathematical abilities in childhood. *PNAS*, 110(45), 18116–18120.
- Van de Walle, JA & Lovin, LH. 2006. *Teaching student-centered mathematics: Grades K-3*. Boston: Pearson, Allyn and Bacon.

LEARNING UNIT 4

DEVELOPMENT OF SPATIAL SENSE

4.1 INTRODUCTION

In this unit you will learn about geometric shapes and objects. This is what is called spatial sense.

Spatial sense can be defined as an awareness about shapes and the relationships between them. It includes the ability to mentally visualise objects and spatial relationships.

Please note that you are required to study all sections in this unit.

4.2 LEARNING OUTCOMES

At the end of the learning unit you should be able to:

- demonstrate the ability to act on the first two Van Hiele levels of spatial development
- use appropriate mathematical terminology to demonstrate an informed understanding of concepts involving two-dimensional shapes and three-dimensional objects

4.3 THE VAN HIELE LEVELS OF GEOMETRIC THOUGHT

Pierre van Hiele and Dina van Hiele-Geldof explored the development of geometric thinking in children and adults (Van de Walle et al., 2016). They proposed a five-level hierarchical model of understanding geometric thinking that describes how individuals think and what types of geometric ideas they think about at each level of development.

According to Van de Walle et al. (2016) progression from one level to the next is more dependent on the amount of a student's experience with geometric thought than on the student's age and level of maturation. The levels are:

- Level 0: Visualisation

Students at level 0 recognise and identify two-dimensional shapes and three-dimensional object by their appearance as a whole. Students do not describe properties (defining characteristics) of shapes and figures.

- Level 1: Analysis

Students at level 1 recognise the properties of two-dimensional shapes and three-dimensional objects. They understand that all shapes or objects within a class share common properties (e.g. all rectangles have four sides, with opposite sides parallel and congruent).

- Level 2: Informal deduction

Students use informal, logical reasoning to deduce properties of two-dimensional shapes and three-dimensional figures (e.g. if one pair of opposite sides of a quadrilateral is parallel and congruent, then the other sides must be parallel and congruent). Students at this level can use this informal, logical reasoning to describe, explore, and test arguments about geometric forms and their properties. Level 2 represents the geometric thinking required in mathematics programs at the intermediate and secondary levels.

- Level 3: Deduction

As students continue to explore the relationships between and among the properties of geometric forms, they use deductive reasoning to make conclusions about abstract geometric principles.

- Level 4: Rigour

Students compare different geometric theories and hypotheses. Level 4 represents the geometric thinking required in advanced mathematics courses. (Note, check the mathematics in unit 1.)

For the purpose of this module, we will focus on the first two levels of Van Hiele, thus, level 0 and level 1.

ACTIVITY 4.1

Sorting and classifying

Classify the shapes in figure 4.1 according to the following criteria:

- (1) Shapes with curved edges
- (2) Three sides
- (3) Four sides
- (4) Opposite sides “go the same way” (parallelograms)
- (5) Shapes with “dents” (concave)
- (6) In what level of Van Hiele do questions 1 to 5 fall into?
- (7) Give a reason(s) for the answer you provided in question 6.

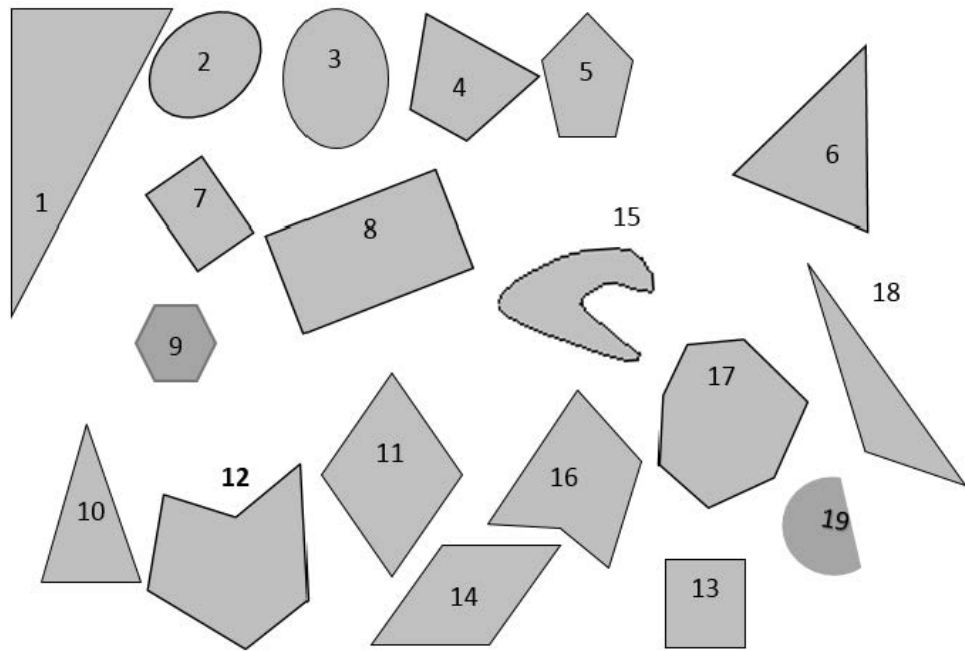


FIGURE 4.1

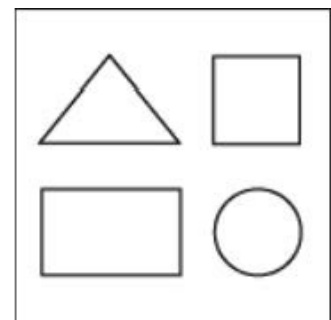
A collection of shapes for sorting

4.4 DIFFERENCES BETWEEN THE TWO-DIMENSIONAL SHAPES AND THREE DIMENSIONAL OBJECTS (FIGURES)

Dimension is a property of shapes that tells whether the shape has height (or depth). There are many dimensions, however, in mathematics (spatial sense and geometry) we only talk about two-dimensional shapes and three-dimensional objects.

4.4.1 Two dimensional shape

Two-dimensional shape refers to a flat or plane (see unit 1 as well) shape. Flat shapes are shapes that lie flat. You can put it flat on a desk, and it will not stick up in space, and it will rotate in a plane. They have two dimensions namely length and width. Polygons are examples of two-dimensional shapes.



4.4.2 Polygons

A polygon is a **closed plane** (flat) shape made up of line segments. These line segments must touch only once at their end points. The naming of polygons follows the number of sides it has. The following are some of the names:

TABLE 4.1
Names of polygons

Number of sides	Polygons
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
20	Icosagon

The affix “gon” means “sides”. So a decagon is a polygon with 10 sides. It is based on “deca” = ten and “gon” = sides

4.4.2.1 *Quadrilaterals*

As we have seen in the table 4.1, a quadrilateral is a four-sided polygon. There are many classifications of quadrilaterals and it is important for you to know that quadrilaterals belong to more than one category. For example, a square is classified as a parallelogram, a rectangle and a rhombus. In addition, quadrilaterals can be classified as more than one shape. For example, a rhombus is a parallelogram, as it has two sets of parallel sides. It is also a kite, because it has two pairs of congruent adjacent sides (Ministry of Education, 2008). Figures 4.2 and 4.3 represent a classification of quadrilaterals.

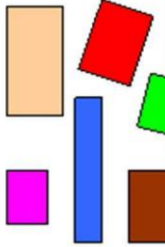
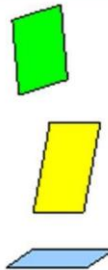
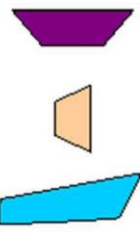
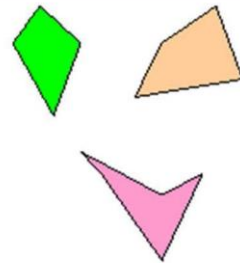
Quadrilaterals				
Parallelograms			Other shapes	
Rectangles	Rhombuses	Others	Trapeziums	Kites and others
				

FIGURE 4.2
Quadrilaterals

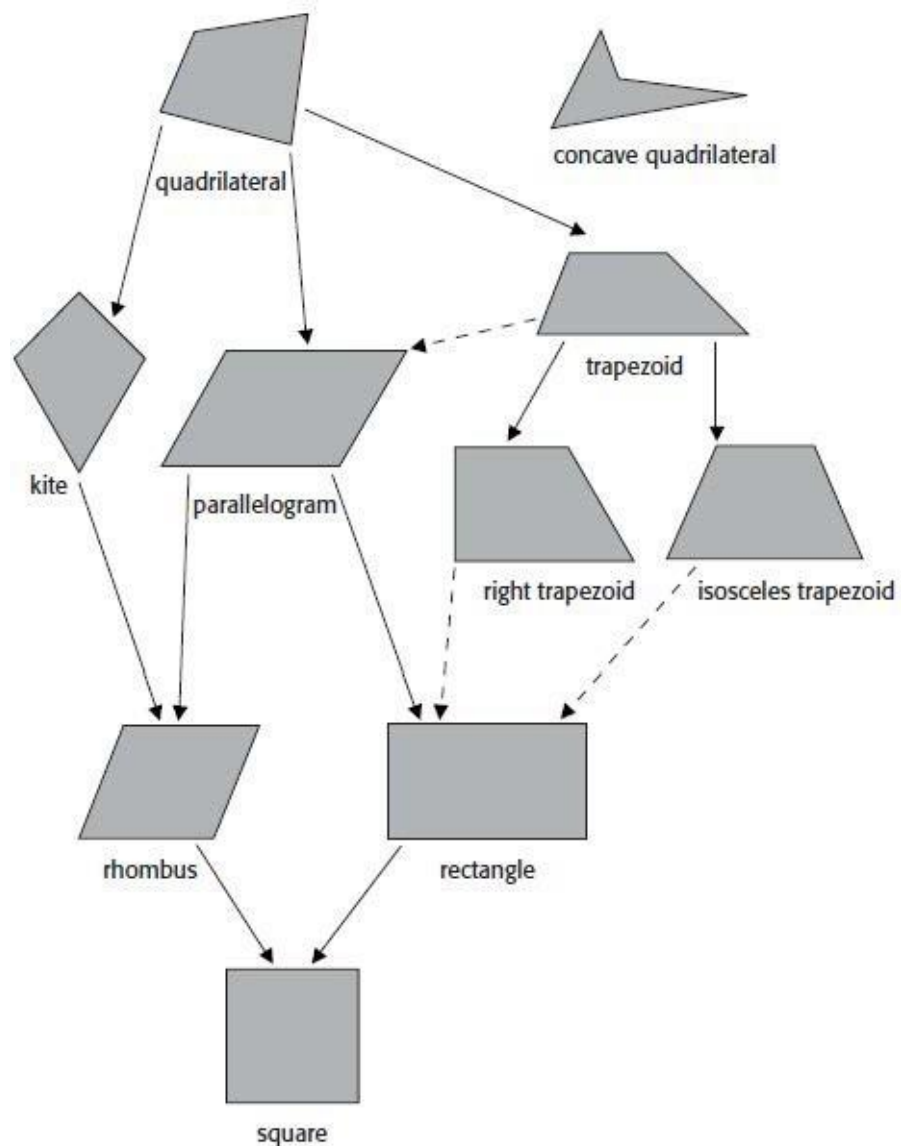
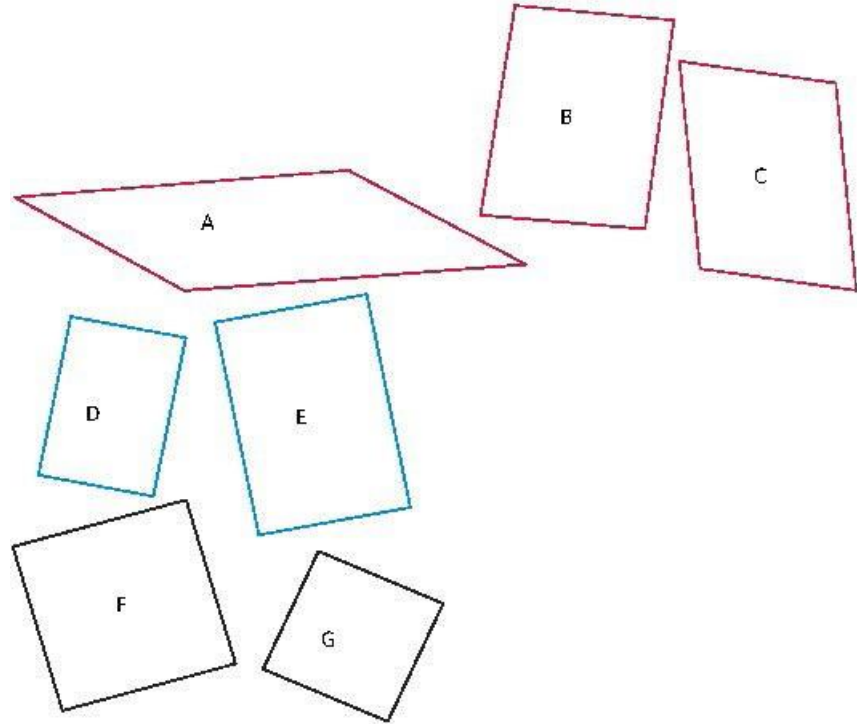


FIGURE 4.3
Quadrilaterals (Ministry of Education, 2008)

ACTIVITY 4.2

(1) How do figures D & E and Figures A, B & C differ from figures F & G?

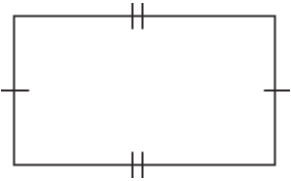
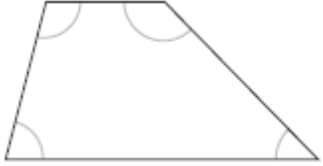
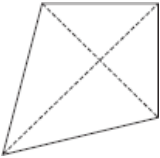
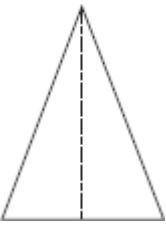


- (2) Which of the above figures are squares? Why?
(3) Which of the above figures are rectangles? Why?

4.4.2.2 Properties of polygons

TABLE 4.2
Properties of polygons (Ministry of Education, 2008)

Number of sides	One of the first properties you learn to consider is the number of sides a shape has. This information allows you to identify triangles, quadrilaterals, pentagons, hexagons, heptagons, octagons, and so forth.	
Number of vertices	You should recognise that the point at which two lines meet is called a vertex . Note that the number of vertices in a polygon is the same as the number of sides.	

Length of sides	You should realise that the length of sides is an important property of many two-dimensional shapes. You recognise that the sides of squares are of equal length, and that pairs of opposite sides are equal for all other parallelograms . ¹	
Size of angles	You learn to consider angles as they identify classes and subclasses of two-dimensional shapes. For example, you can determine that an isosceles triangle ² has two identical angles, and that an isosceles right triangle has one 90° angle and therefore two 45° angles. Both interior and exterior angles of polygons can be considered when identifying and classifying polygons.	
Parallel lines	You will remember that parallel lines are always the same distance apart. You will recognise parallelism as an essential property in describing the sides of trapezoids ³ and parallelograms .	
Diagonals	A diagonal is a line connecting two nonadjacent ⁴ vertices. You will recognise that a polygon can be <u>classified</u> by the number and nature of its <u>diagonals</u> .	
Lines of symmetry	You can identify or classify a polygon by the number of lines of symmetry it has. For example, squares have four lines of symmetry; isosceles triangles have one line of symmetry.	

1 **Parallelogram** is a simple quadrilateral with two pairs of parallel sides. The opposite or facing sides of a parallelogram are of equal length and the opposite angles of a parallelogram are of equal measure.

2 **Isosceles triangle** is a triangle that has two sides of equal length.

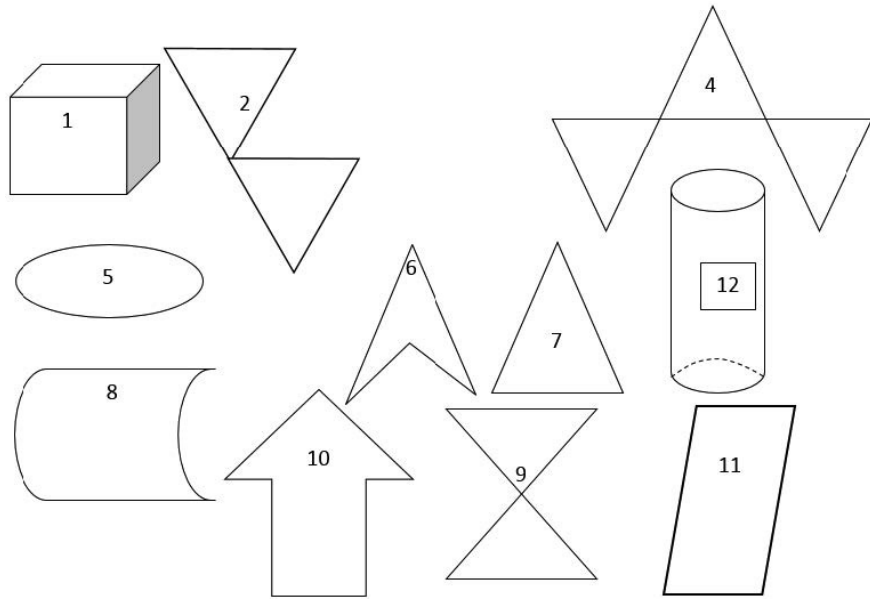
3 **Trapezoids** a quadrilateral with no sides parallel.

4 **Nonadjacent** is when the side, edges, vertices of a graph does not share the common side, edges or vertices.

ACTIVITY 4.3

Use the definition of polygons to answer the following activity:

(1) Which of the following are polygons?



Based on the quadrilaterals in section 4.4.2.1 answer the following questions:

- (2) What is the main/common property that all quadrilaterals have?
- (3) Draw the following polygons:
- An equilateral triangle
 - An isosceles triangle
 - A scalene triangle
 - A right-angled triangle
- (4) Mention the properties of each of the polygons you drew in question 3.
- (5) Complete the following table

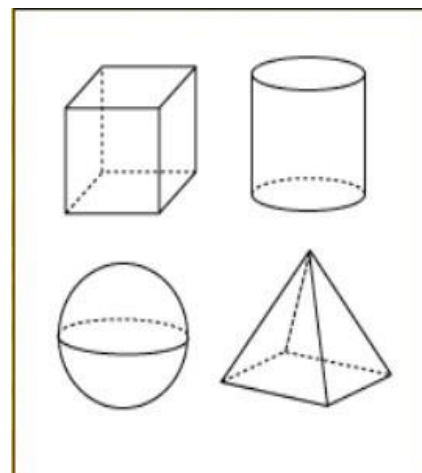
Quadrilaterals	Properties		
	Sides	Angles	Diagonals
Rectangle			
Square			
Kite			
Rhombus			
Parallelogram			

FEEDBACK

A polygon is a closed plane (flat) shape made up of line segments. These line segments must touch only once at their end points. The naming of polygons follows the number of sides it has.

4.4.3 Three-dimensional objects (figures)

Three-dimensional objects are the objects we see in our daily lives, the ones we can pick up, touch, and can rotate in a space. They “stand up” or “stick out”; they “take up space”. They have three dimensions namely length, width, and depth (height). Examples of three-dimensional objects (figures) are pyramids, prisms, cone, sphere and cylinder. The three-dimensional figures can be classified as polyhedrons and non-polyhedrons



4.4.3.1 Polyhedrons

A polyhedron (plural polyhedra or polyhedrons) is a three-dimensional object or figure with faces made up of polygons, straight edges and sharp corners or vertices. We name polyhedral according to the number of faces they have. We use the same prefixes as for polygons, but the names end in the word “hedron” (penta–5, hexa–6, hepta–7, octa–8, nona–9 and deca–10). The smallest polyhedron is tetrahedron with four faces. Examples of polyhedrons are cubes, prisms, or pyramid.

A **prism** is a solid geometric figure whose two ends (surfaces or bases) are parallel and congruent polygons. Lines joining corresponding points on the bases are always parallel. The sides of prisms are always parallelograms.

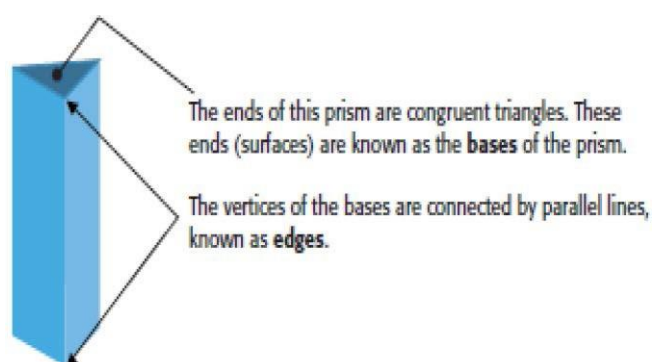


FIGURE 4.4

A prism (Ministry of Education, 2008)

We name prisms according to the shape of their bases. Examples of prisms include rectangular prisms, triangular prisms, hexagonal prisms, and so forth. A **cube** is a special rectangular prism, with square faces congruent to the base.

A **pyramid** is a polyhedron whose base is a polygon and whose other faces are triangles that meet at a common vertex.

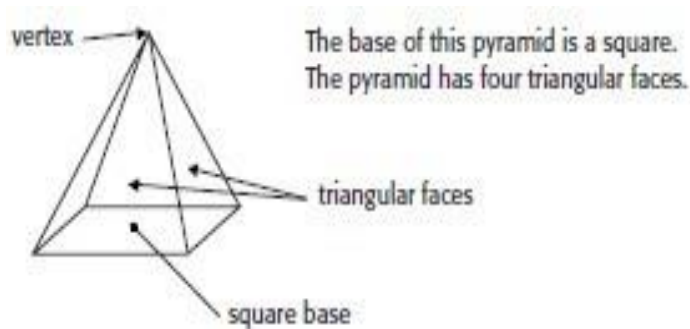
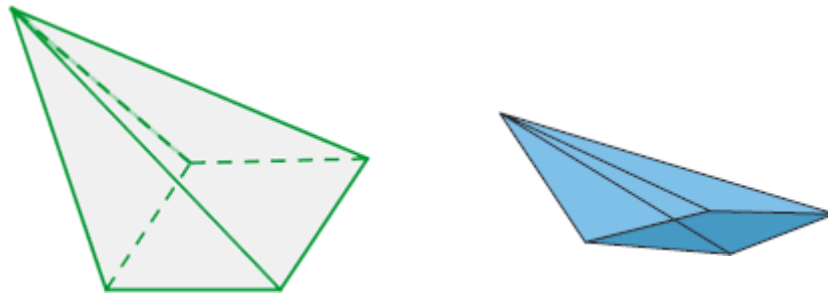


FIGURE 4.5

A pyramid (Ministry of Education)

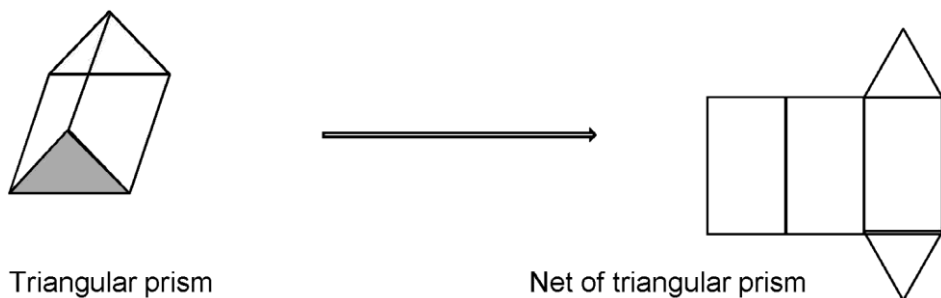
Also, note that pyramids can have a common vertex that lies outside the base.

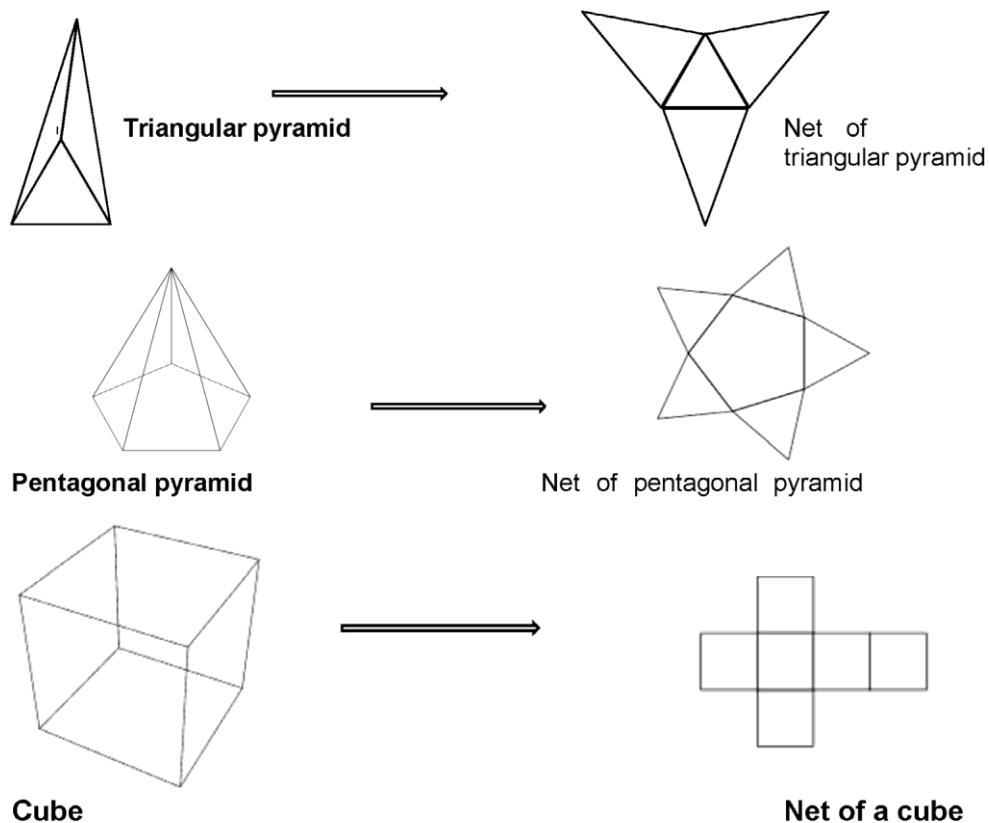


Like prisms, we name pyramids according to the shape of their base. Examples of pyramids include square pyramids, hexagonal pyramids, and octagonal pyramids.

4.4.3.2 Nets of polyhedra

A net is a fold-out (flat) shape that can be folded up into a three-dimensional figure. A net must include all of the two-dimensional faces of the figure. For example, a net for a triangular prism must have two triangles and three rectangles; a net for a pentagonal pyramid must have one pentagon and five triangles; a net of a cube must have six squares; and a net of triangular pyramid must have four triangles.





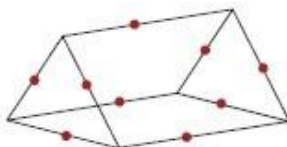
ACTIVITY 4.4

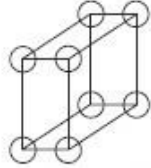
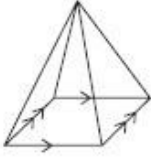
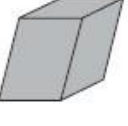
Draw the nets of the five platonic solids.

FEEDBACK

A net is a pattern that can be folded into a three-dimensional figure. A **platonic solid** is a special type of polyhedra, in which each face is exactly the same, and the same number of faces meet at each corner, or **vertex** (Note, check the mathematics in unit 1, section 1.5.4.4).

4.4.3.3 Properties polyhedrons (Ministry of Education, 2008)

Number or type of faces	The face of a polyhedron is a flat surface. You describe polyhedra by the number or shapes of their faces.	 <p>A cube has 6 faces.</p>
Number of edges	An edge is the line where two faces meet. You should explore the relationships between the number of edges and faces in categories of polyhedra.	 <p>A triangular prism has 9 edges.</p>

Number of vertices	A vertex is a point at which two or more edges meet. You will explore the relationship between the number of edges, vertices, and faces of various polyhedra.	 <p>A rectangular prism has 8 vertices.</p>
Parallelism	Faces and edges of a figure (object) can be described as being parallel or nonparallel to other faces and edges.	 <p>Only the edges of the base are parallel in pyramids.</p>
Perpendicularity	Perpendicularity is a concept that develops in the later junior grades. Edges and faces can often be described as “at right angles” to other faces and edges.	<div style="display: flex; justify-content: space-around; align-items: flex-start;"> <div style="text-align: center;">  <p>The base of this rectangular prism is perpendicular to the vertical faces.</p> </div> <div style="text-align: center;">  <p>The base of this prism is not perpendicular to its vertical faces.</p> </div> </div> <p>If the vertical edges of the prism are perpendicular to the edges of the base, the prism is described as a right prism. When the edges do not run perpendicular to each other, the prism is said to be an oblique prism.</p>

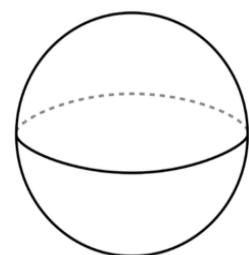
To summarise we describe a polyhedral by face, edge and vertex. Thus:

- faces are the flat surfaces (they are all polygonal regions)
- edges are where the faces meet (they are all straight lines)
- vertices are where the edges meet (they are all points)

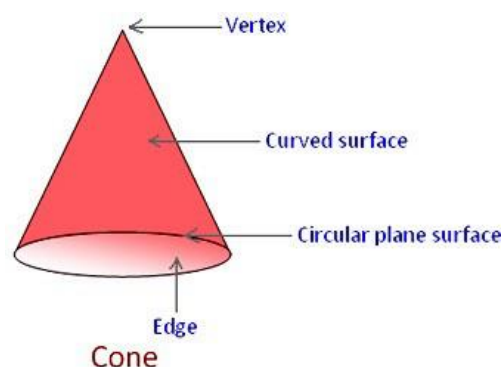
4.4.3.4 Non-polyhedrons

A non-polyhedron is a three-dimensional figure with faces made up of curves (i.e. sides that are not polygons). Examples of non-polyhedrons are cones, spheres, and cylinders.

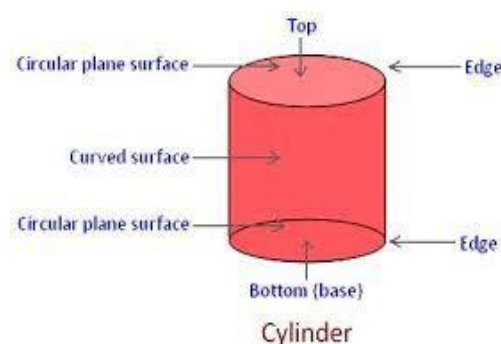
A sphere is a three-dimensional object shaped like a ball. Every point on the surface is the same distance from the center.



A **cone** is a three-dimensional object that has a circular base (surface) and a single vertex. If the vertex is over the center of the base, it is called a right **cone**. If it is not, it is called an oblique **cone**. An object that is shaped like a **cone** is said to be “conical”.

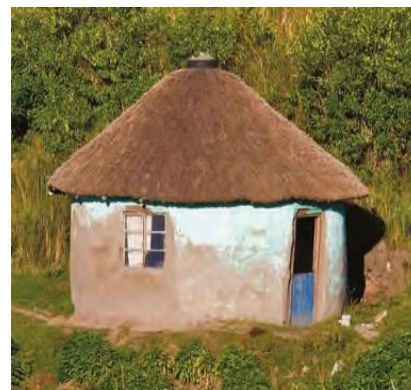


A **cylinder** is a three-dimensional object with two identical flat ends that are circular and one curved side. It has the same cross-section from one end to the other.



ACTIVITY 4.5

South Africans were using and still use clay, soil and thatch roofs to build rondavel houses. People from all over the country and world visit Mpumalanga province to see the natural three rondavels. The following pictures represent the man-made and natural rondavels. Identify them as either cones, cylinders or pyramids.



Take a photo of real objects at your home that are either cones, cylinders and or pyramids other than the ones indicated here.

FEEDBACK

You should provide pictures of different real objects for this activity other than the ones presented in this unit. You may use your own cellphone to capture these objects.

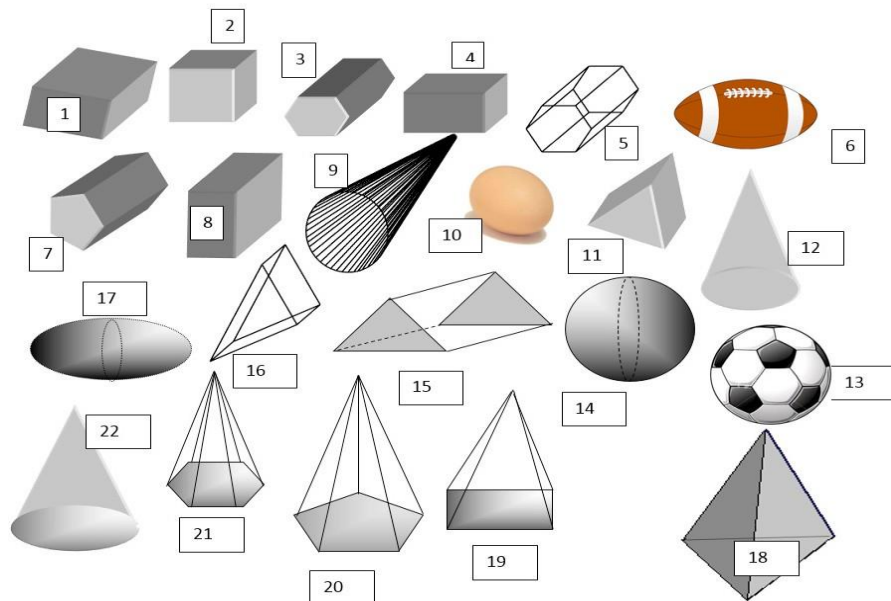
4.5 CONCLUSION

In this learning unit we described the Van Hiele level of geometric thinking. We differentiated between and classified the two-dimensional shapes and three-dimensional objects (figures) by their appearance and as well with their properties.

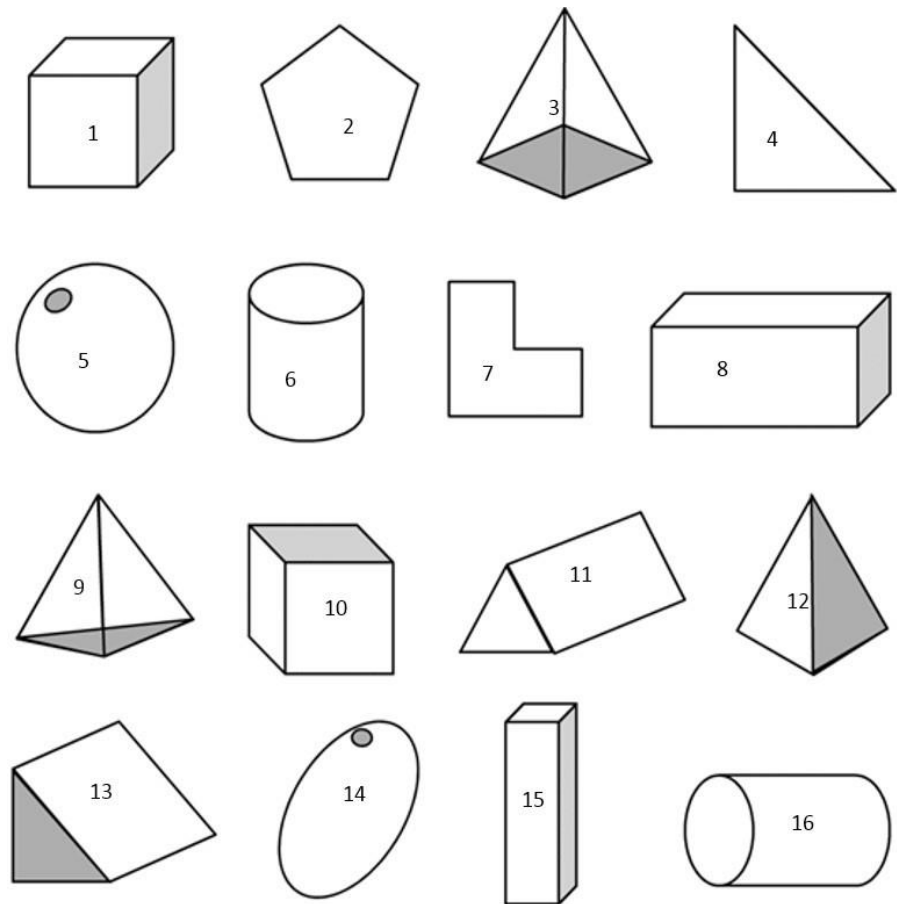
4.6 SELF-ASSESSMENT

This section contains self-assessment activities that you can use to test your knowledge of the learning unit.

- (1) Draw the following rectangular prisms:
 - The object has six faces. All faces are squares.
 - The object has six faces. Two opposite faces are squares. All other faces are rectangles that are not squares.
- (2) Suppose they tell you that certain object has six faces.
 - Is it possible that the object is a rectangular prism?
 - Can you be sure that it is actually a rectangular prism? If you think you cannot be sure, explain why.
- (3) Answer the same two questions in question 2 in each of the following cases.
 - All you know about the object is that it has rectangular faces.
 - You only have information about two faces of the object, and what you know is that these two faces are rectangular.
 - You only have information about three faces of the object, and what you know is that these three faces are rectangular.
 - You only have information about four faces of the object, and what you know is that these four faces are rectangular.
- (4) The following figures are a mixture of polyhedrons and non-polyhedrons solid objects. Identify the polyhedrons and non-polyhedrons objects and classify them.



- (5) The following figures are a mixture of two-dimensional and three-dimensional solid objects. Identify the two-dimensional and three-dimensional objects and classify them.



4.7 REFERENCES

Van de Walle, JA, Karp, KS & Bay-Williams, JM. (2016). *Elementary and Middle School Mathematics-teaching developmentally*, 9th edition. New Jersey: Pearson Edition

Ministry of Education (2008). *Geometry and Spatial Sense, Grades 4 to 6 – A guide to Effective Instruction in Mathematics, Kindergarten to Grade 6*. Ontario