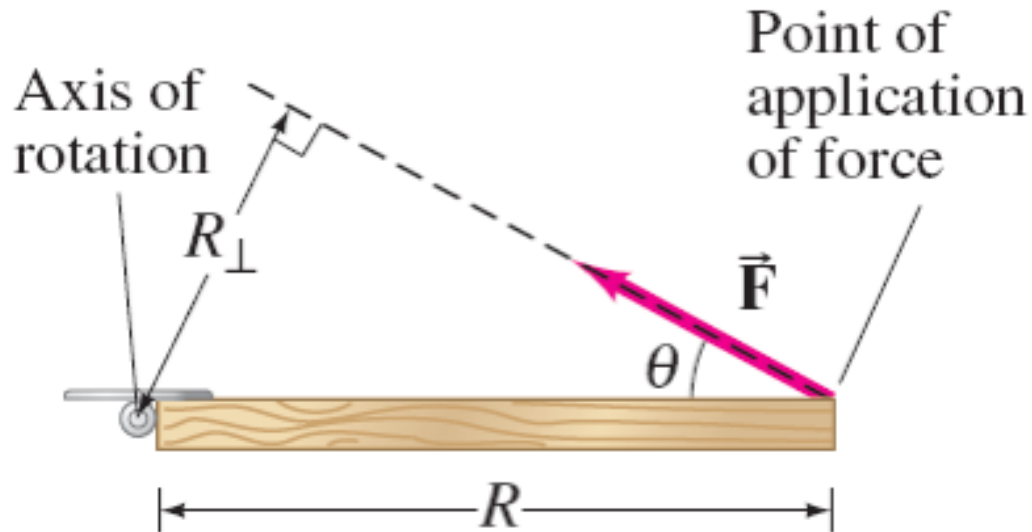


Chapter 10

Rotational Motion



10-4 Torque

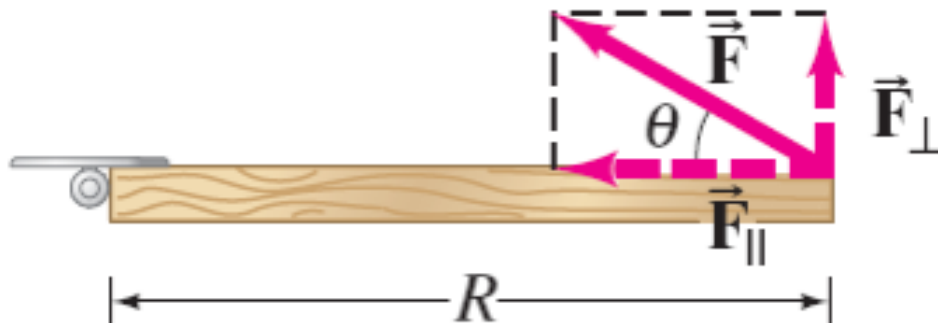


The torque is defined as:

$$\tau = R_{\perp} F$$

Equivalently, one can write:

$$\tau = R F_{\perp}$$

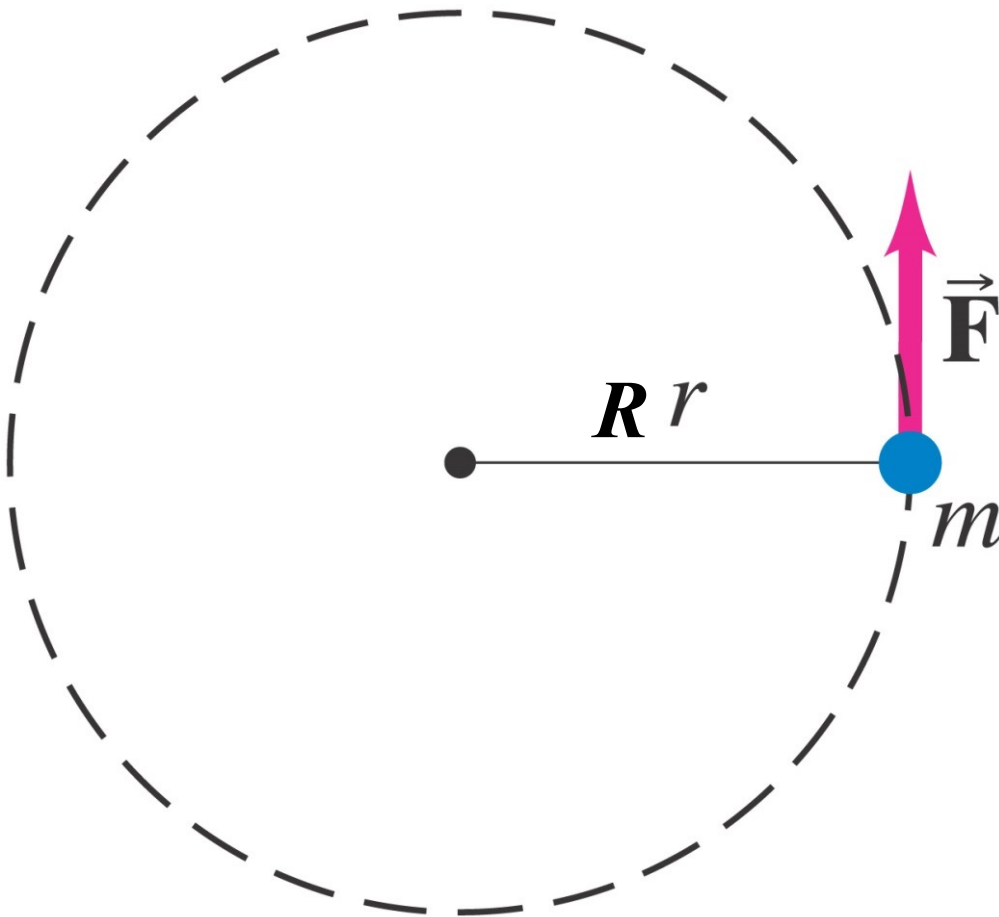


As drawn here:

$$\tau = R F \sin \theta$$

10-5 Rotational Dynamics; Torque and Rotational Inertia

Knowing that $F = ma$, we see that $\tau = mR^2\alpha$



This is for a single point mass; what about an extended object?

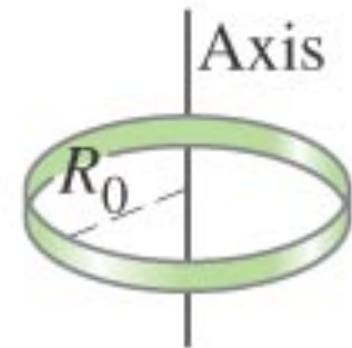
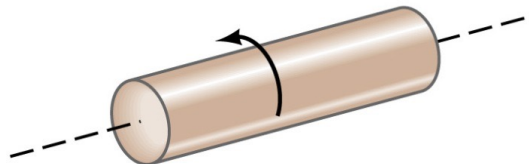
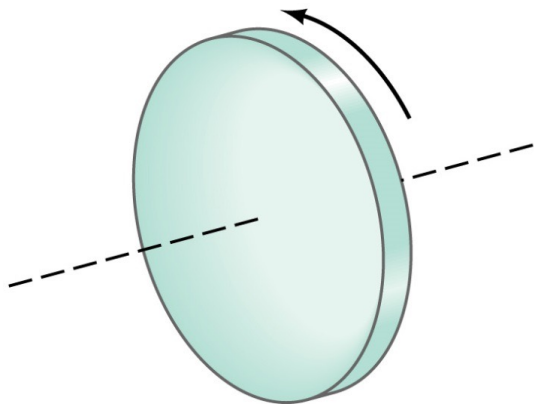
As the angular acceleration is the same for the whole object, we can write:

$$\Sigma\tau = (\Sigma mR^2)\alpha.$$

10-5 Rotational Dynamics; Torque and Rotational Inertia

The quantity $I = \sum m_i R_i^2$ is called the rotational inertia of an object with respect to a specific axis.

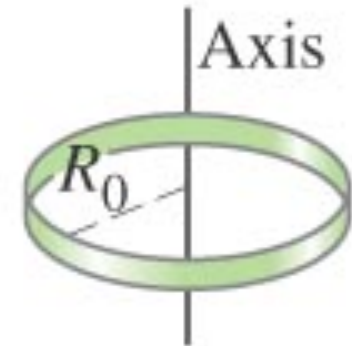
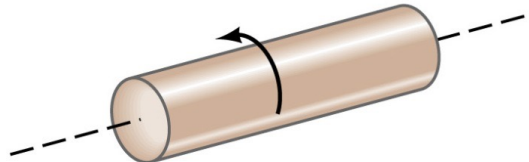
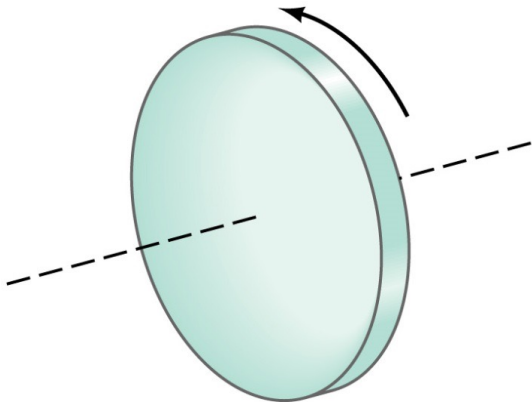
The distribution of mass matters here—these two objects have the same mass, but the one on the left has a greater rotational inertia, as so much of its mass is far from the axis of rotation.



10-5 Rotational Dynamics; Torque and Rotational Inertia

The quantity $I = \sum m_i R_i^2$ is called the rotational inertia of an object with respect to a specific axis.

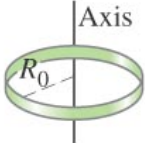
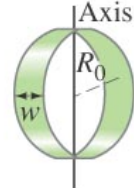
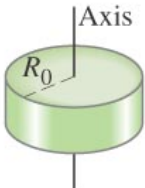
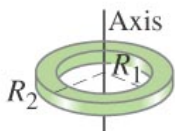
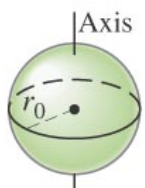
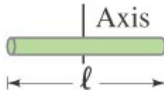
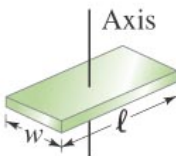
The distribution of mass matters here—these two objects have the same mass, but the one on the left has a greater rotational inertia, as so much of its mass is far from the axis of rotation.



Q: what is I for a thin hoop with mass M ?

10-5 Rotational Dynamics; Torque and Rotational Inertia

The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation—compare (f) and (g), for example.

Object	Location of axis		Moment of inertia
(a) Thin hoop, radius R_0	Through center		MR_0^2
(b) Thin hoop, radius R_0 width w	Through central diameter		$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c) Solid cylinder, radius R_0	Through center		$\frac{1}{2}MR_0^2$
(d) Hollow cylinder, inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e) Uniform sphere, radius r_0	Through center		$\frac{2}{5}Mr_0^2$
(f) Long uniform rod, length ℓ	Through center		$\frac{1}{12}M\ell^2$
(g) Long uniform rod, length ℓ	Through end		$\frac{1}{3}M\ell^2$
(h) Rectangular thin plate, length ℓ , width w	Through center		$\frac{1}{12}M(\ell^2 + w^2)$

10-6 Solving Problems in Rotational Dynamics

- 1. Draw a diagram.**
- 2. Decide what the system comprises.**
- 3. Draw a free-body diagram for each object under consideration, including all the forces acting on it and where they act.**
- 4. Find the axis of rotation; calculate the torques around it.**

10-6 Solving Problems in Rotational Dynamics

- 5. Apply Newton's second law for rotation. If the rotational inertia is not provided, you need to find it before proceeding with this step.**
- 6. Apply Newton's second law for translation and other laws and principles as needed.**
- 7. Solve.**
- 8. Check your answer for units and correct order of magnitude.**

10-7 Determining Moments of Inertia

If a physical object is available, the moment of inertia can be measured experimentally.

Otherwise, if the object can be considered to be a continuous distribution of mass, the moment of inertia may be calculated:

$$I = \int R_{\perp}^2 dM$$

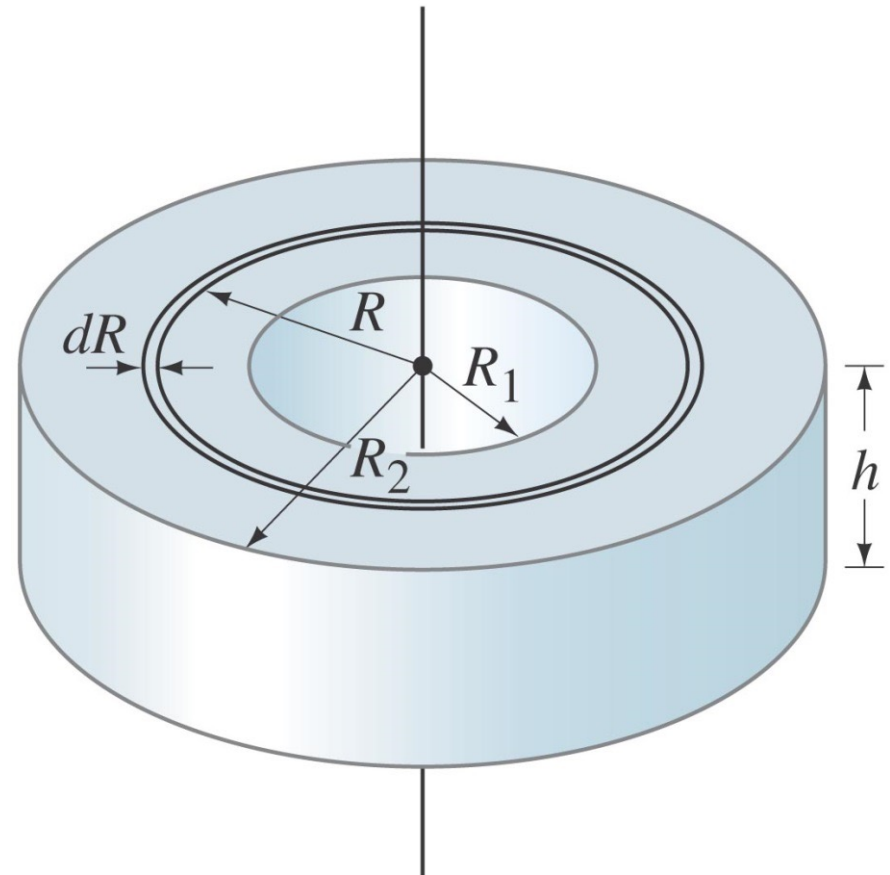


where the distance R is measured perpendicular to axis

10-7 Determining Moments of Inertia

Example 10-12: Cylinder, solid or hollow.

(a) Show that the moment of inertia of a uniform hollow cylinder of inner radius R_1 , outer radius R_2 , and mass M , is $I = \frac{1}{2} M(R_1^2 + R_2^2)$, if the rotation axis is through the center along the axis of symmetry. (b) Obtain the moment of inertia for a solid cylinder.

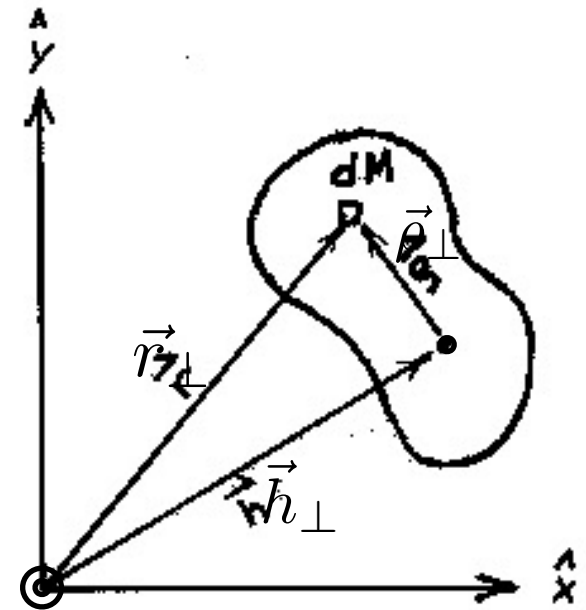


10-7 Determining Moments of Inertia

The parallel-axis theorem gives the moment of inertia about any axis parallel to an axis that goes through the center of mass of an object:

$$\begin{aligned} I &= \int_V r_{\perp}^2 dm = \int_V \left(\vec{h}_{\perp} + \vec{\rho}_{\perp} \right)^2 dm \\ &= \int_V \left(\vec{h}_{\perp} \cdot \vec{h}_{\perp} + 2\vec{h}_{\perp} \cdot \vec{\rho}_{\perp} + \vec{\rho}_{\perp} \cdot \vec{\rho}_{\perp} \right) dm \\ &= \vec{h}_{\perp} \cdot \vec{h}_{\perp} \int_V dm + 2\vec{h}_{\perp} \cdot \int_V \vec{\rho}_{\perp} dm + \int_V \vec{\rho}_{\perp} \cdot \vec{\rho}_{\perp} dm \\ &= Mh^2 + 0 + I_{CM} \end{aligned}$$

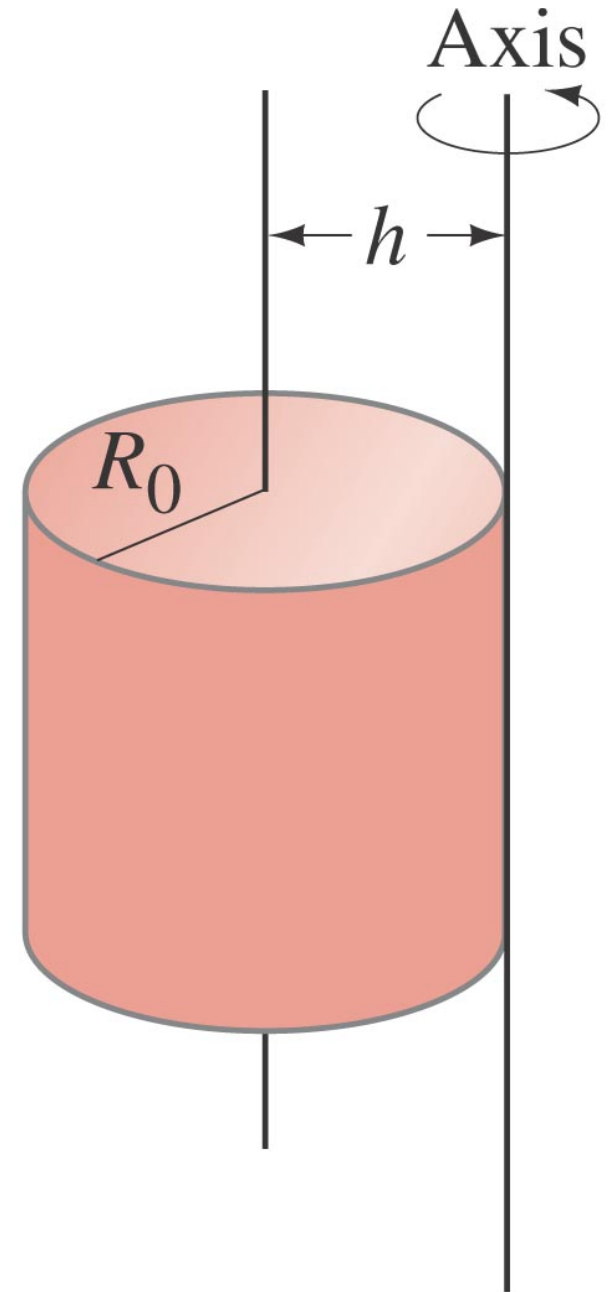
$$I = Mh^2 + I_{CM}$$



10-7 Determining Moments of Inertia

Example 10-13: Parallel axis.

Determine the moment of inertia of a solid cylinder of radius R_0 and mass M about an axis tangent to its edge and parallel to its symmetry axis.



10-7 Determining Moments of Inertia

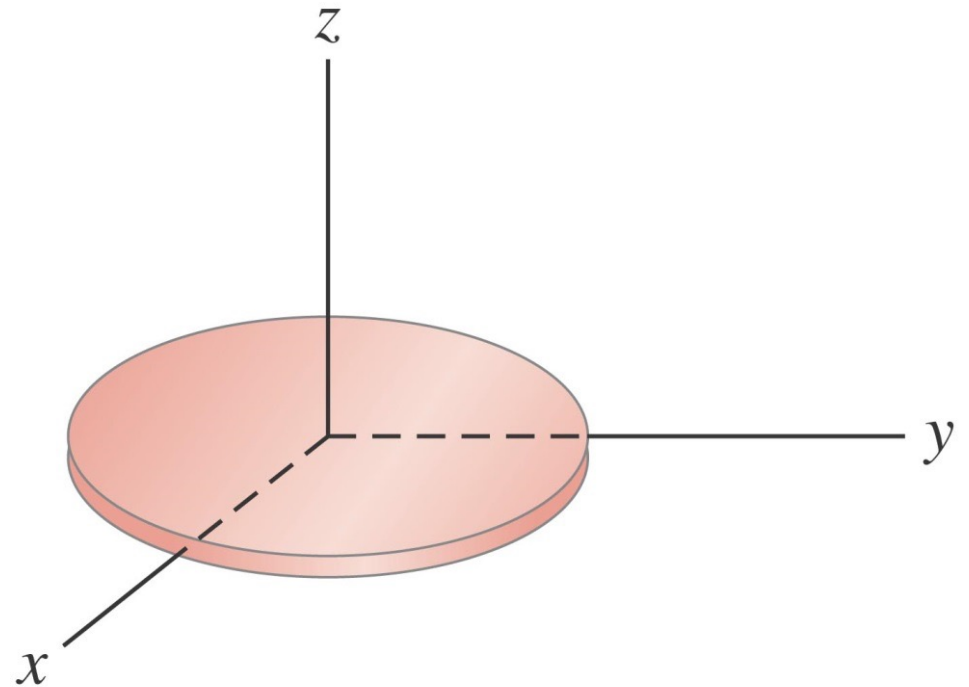
The perpendicular-axis theorem is valid only for flat objects.

$$I_z = I_x + I_y.$$

$$I_x = \int_V y^2 dV$$

$$I_y = \int_V x^2 dV$$

$$\rightarrow I_x + I_y = \int_V (x^2 + y^2) dV = I_z$$



10-8 Rotational Kinetic Energy

The kinetic energy of a rotating object is given by

$$K = \Sigma\left(\frac{1}{2}mv^2\right).$$

By substituting the rotational quantities, we find that the rotational kinetic energy can be written:

$$\text{rotational } K = \frac{1}{2}I\omega^2.$$

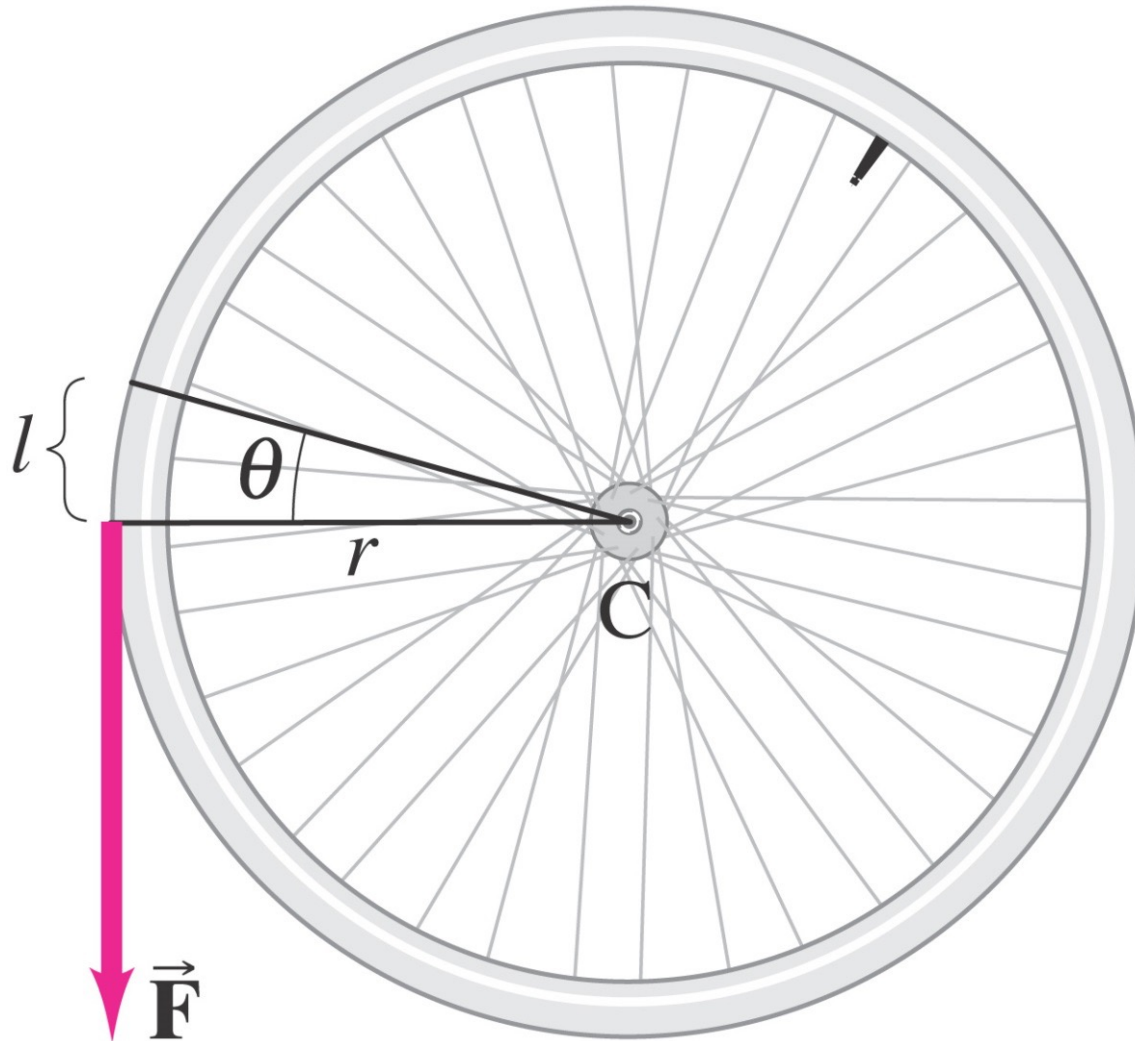
A object that both translational and rotational motion also has both translational and rotational kinetic energy:

$$K = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2.$$

10-8 Rotational Kinetic Energy

The torque does work as it moves the wheel through an angle θ :

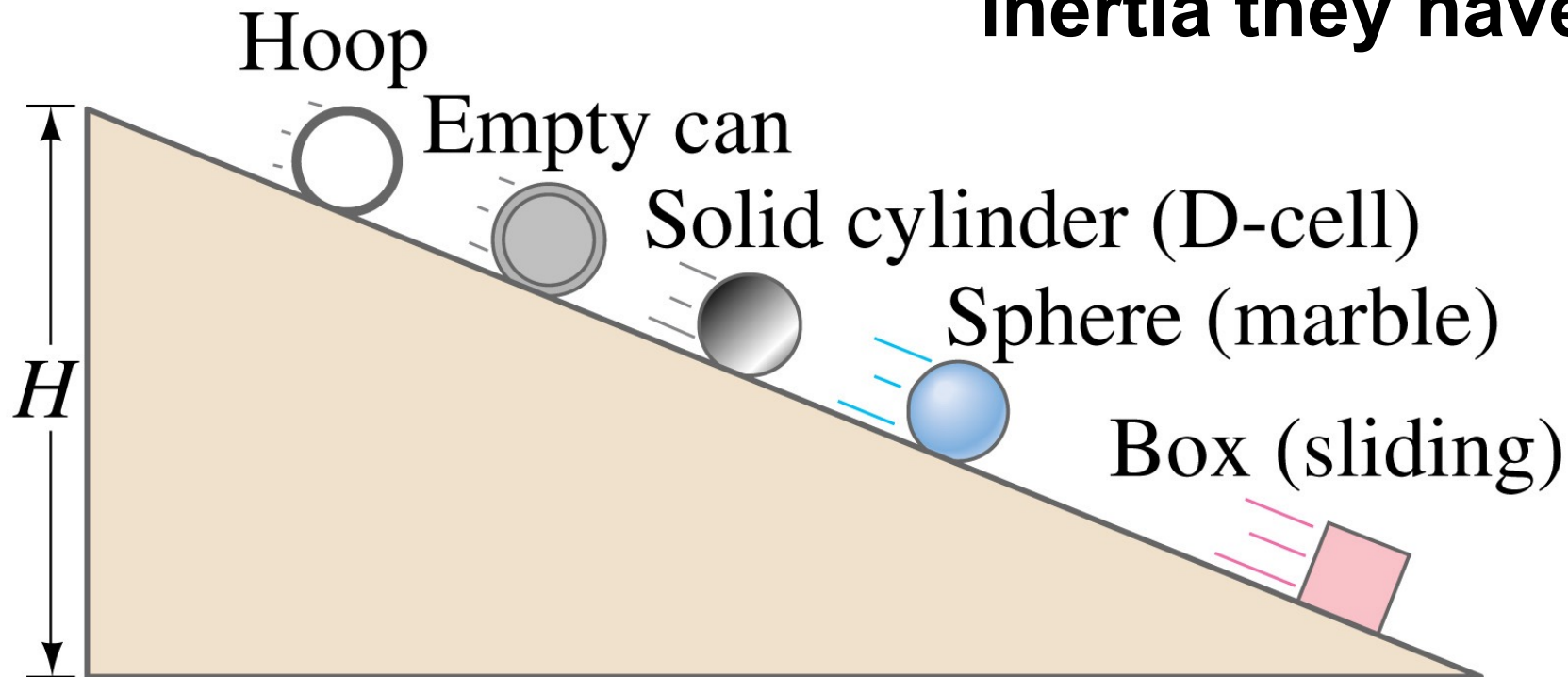
$$W = \tau \Delta \theta.$$



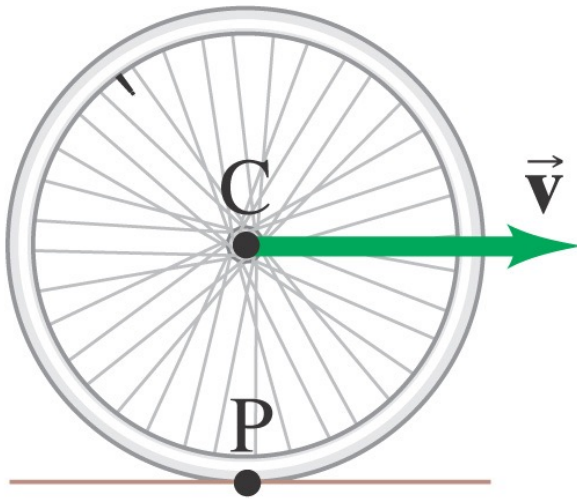
10-8 Rotational Kinetic Energy

When using conservation of energy, both rotational and translational kinetic energy must be taken into account.

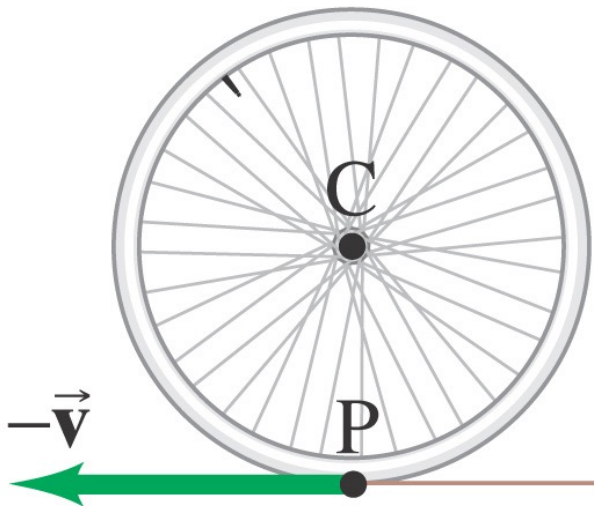
All these objects have the same potential energy at the top, but the time it takes them to get down the incline depends on how much rotational inertia they have.



10-9 Rotational Plus Translational Motion; Rolling



In (a), a wheel is rolling without slipping. The point P, touching the ground, is instantaneously at rest, and the center moves with velocity $\vec{v}$.

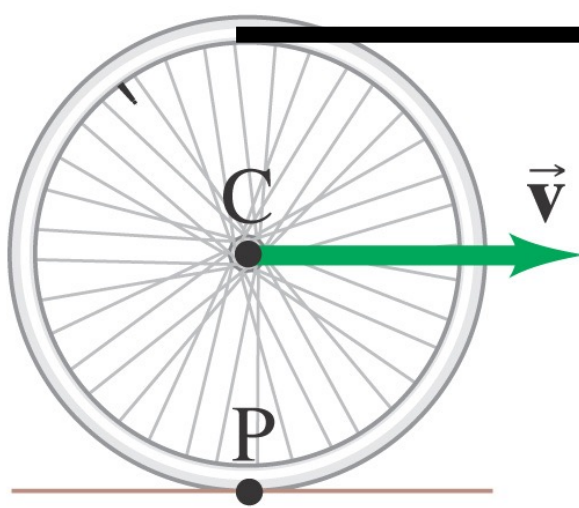


In (b) the same wheel is seen from a reference frame where C is at rest. Now point P is moving with velocity $-\vec{v}$.

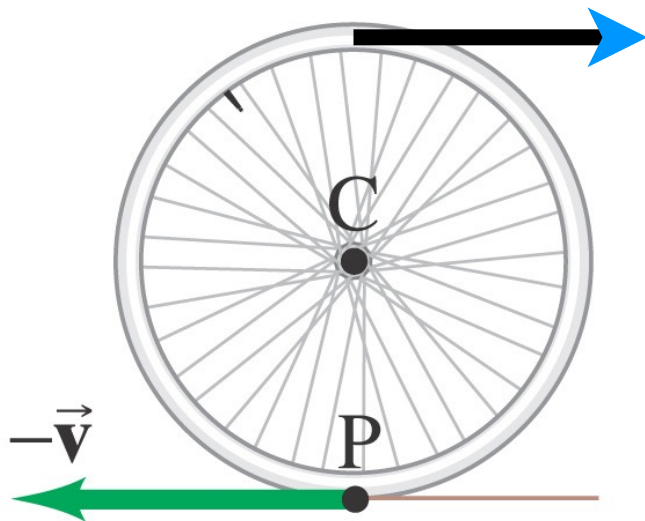
The linear speed of the wheel is related to its angular speed:

$$v = R\omega.$$

10-9 Rotational Plus Translational Motion; Rolling



In (a), a wheel is rolling without slipping. The point P, touching the ground, is instantaneously at rest, and the center moves with velocity $\vec{v}$.



In (b) the same wheel is seen from a reference frame where C is at rest. Now point P is moving with velocity $-\vec{v}$.

The linear speed of the wheel is related to its angular speed:

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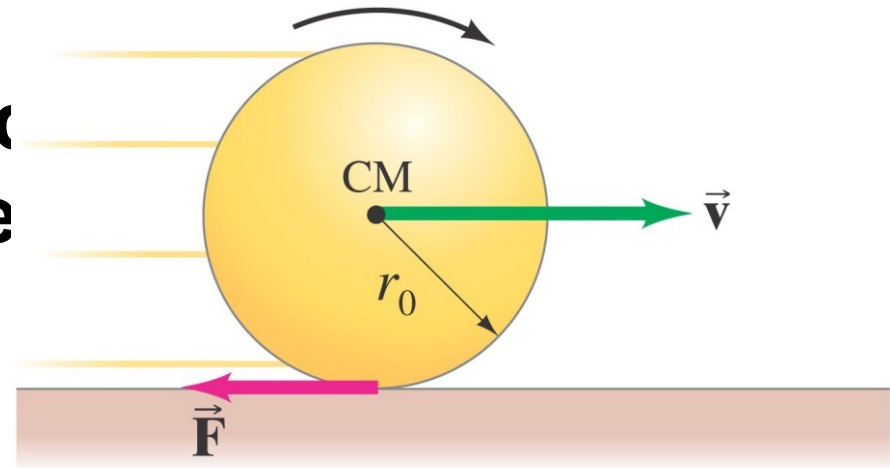
Q: what is speed for a point at the top of the wheel?

10-10 Why Does a Rolling Sphere Slow Down?

A rolling sphere will slow down and stop rather than roll forever. What force would cause this?

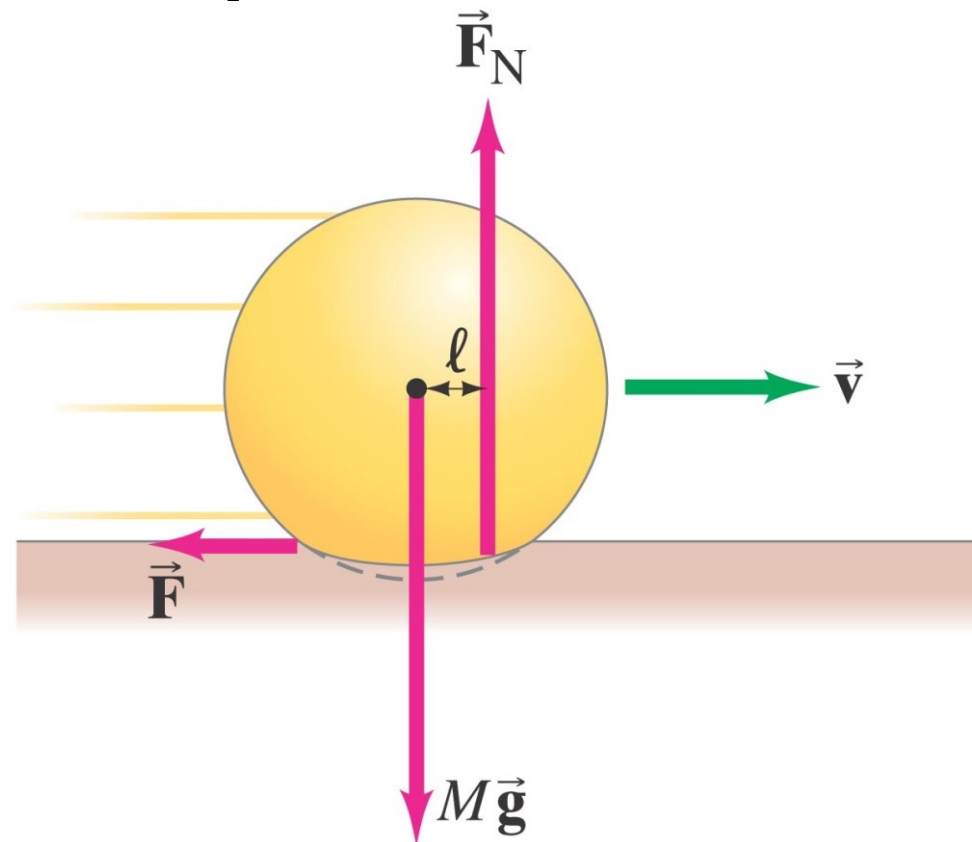
If we say “friction”, there are problems:

- **The frictional force has to act at the point of contact; this means the angular speed of the sphere would increase**
- **Gravity and the normal force both act through the center of mass, and cannot create a torque.**



10-10 Why Does a Rolling Sphere Slow Down?

The solution: No real sphere is perfectly rigid. The bottom will deform, and the normal force will create a torque that slows the sphere.



(also: the “flat” surface may similarly deform)

Summary of Chapter 10

- **Angles are measured in radians; a whole circle is 2π radians.**
- **Angular velocity is the rate of change of angular position.**
- **Angular acceleration is the rate of change of angular velocity.**
- **The angular velocity and acceleration can be related to the linear velocity and acceleration.**
- **The frequency is the number of full revolutions per second; the period is the inverse of the frequency.**

Summary of Chapter 10, cont.

- The equations for rotational motion with constant angular acceleration have the same form as those for linear motion with constant acceleration.
- Torque is the product of force and lever arm.
- The rotational inertia depends not only on the mass of an object but also on the way its mass is distributed around the axis of rotation.
- The angular acceleration is proportional to the torque and inversely proportional to the rotational inertia.

Summary of Chapter 10, cont.

- An object that is rotating has rotational kinetic energy. If it is translating as well, the translational kinetic energy must be added to the rotational to find the total kinetic energy.
- Angular momentum is $L = I\omega$. Chapter 11
- If the net torque on an object is zero, its angular momentum does not change. Chapter 11

Chapter 11

Angular Momentum; General Rotation



Units of Chapter 11

- **Angular Momentum—Objects Rotating About a Fixed Axis**
- **Vector Cross Product; Torque as a Vector**
- **Angular Momentum of a Particle**
- **Angular Momentum and Torque for a System of Particles; General Motion**
- **Angular Momentum and Torque for a Rigid Object**

Units of Chapter 11

- **Conservation of Angular Momentum**
- **The Spinning Top and Gyroscope**
- **Rotating Frames of Reference; Inertial Forces**
- **The Coriolis Effect**

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

The rotational analog of linear momentum is angular momentum, L :

$$L = I\omega.$$

Then the rotational analog of Newton's second law is:

$$\Sigma \tau = \frac{dL}{dt}.$$

This form of Newton's second law is valid even if I is not constant.

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

In the absence of an external torque, angular momentum is conserved:

$$\frac{dL}{dt} = 0 \text{ and } L = I\omega = \text{constant.}$$

More formally,

the total angular momentum of a rotating object remains constant if the net external torque acting on it is zero.

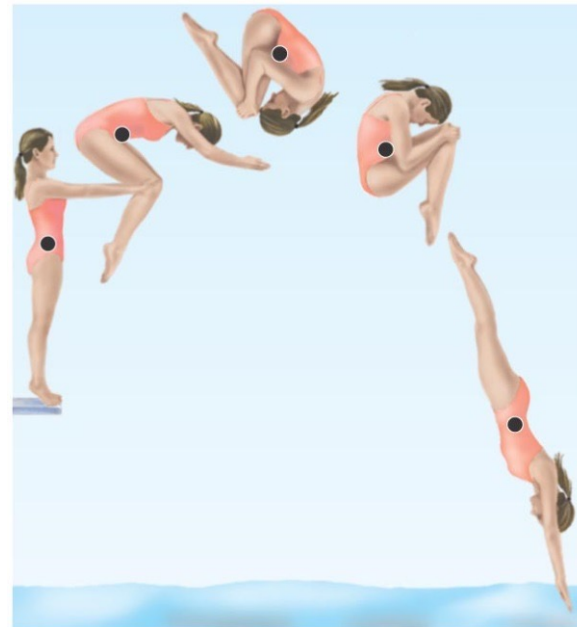
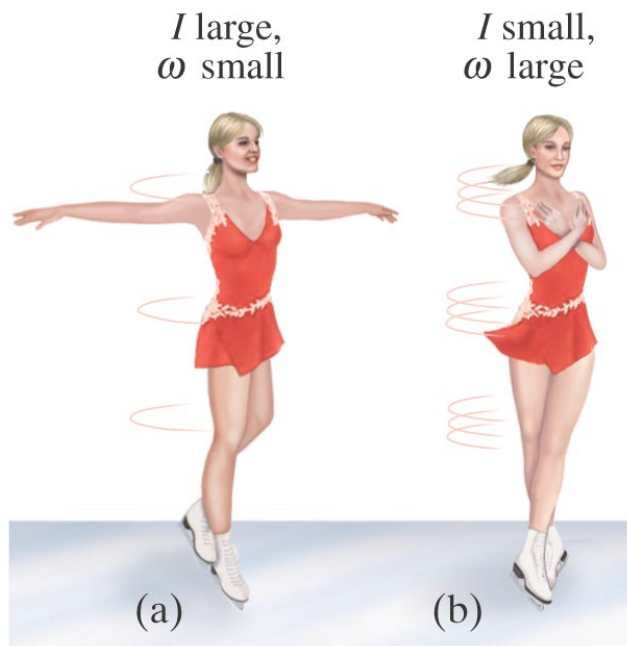
demo

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

This means:

$$I\omega = I_0\omega_0 = \text{constant.}$$

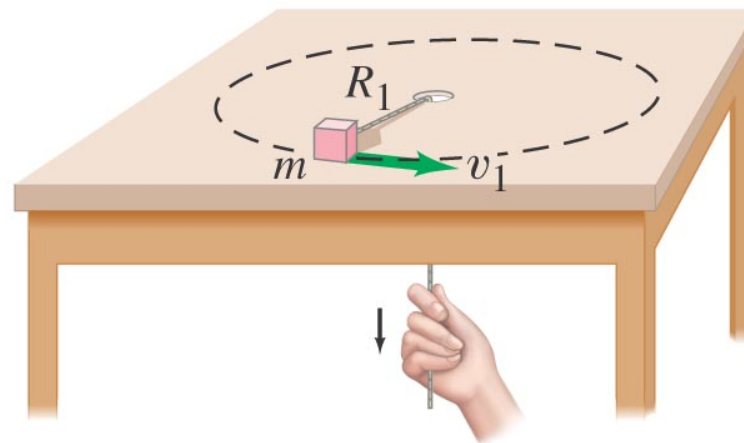
Therefore, if an object's moment of inertia changes, its angular speed changes as well.



11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-1: Object rotating on a string of changing length.

A small mass m attached to the end of a string revolves in a circle on a frictionless tabletop. The other end of the string passes through a hole in the table. Initially, the mass revolves with a speed $v_1 = 2.4$ m/s in a circle of radius $R_1 = 0.80$ m. The string is then pulled slowly through the hole so that the radius is reduced to $R_2 = 0.48$ m. What is the speed, v_2 , of the mass now?



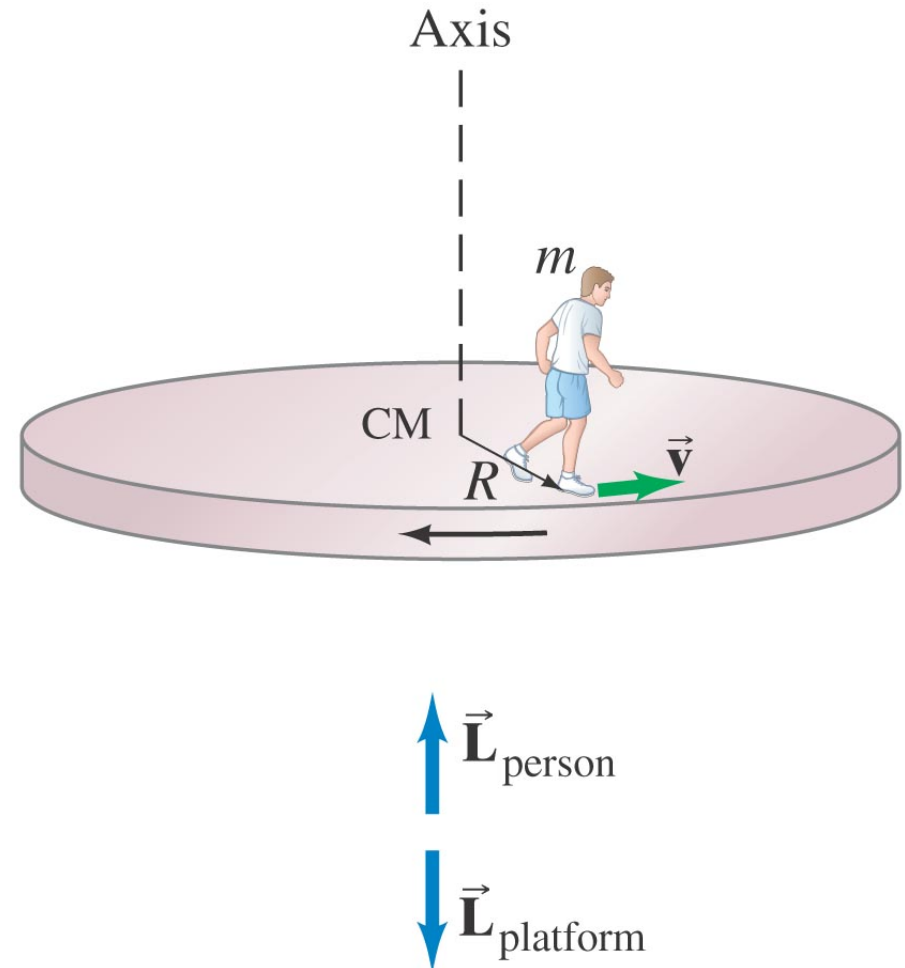
11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-3: Neutron star.

Astronomers detect stars that are rotating extremely rapidly, known as neutron stars. A neutron star is believed to form from the inner core of a larger star that collapsed, under its own gravitation, to a star of very small radius and very high density. Before collapse, suppose the core of such a star is the size of our Sun ($r \approx 7 \times 10^5$ km) with mass 2.0 times as great as the Sun, and is rotating at a frequency of 1.0 revolution every 100 days. If it were to undergo gravitational collapse to a neutron star of radius 10 km, what would its rotation frequency be? Assume the star is a uniform sphere at all times, and loses no mass.

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Angular momentum is a vector; for a symmetrical object rotating about a symmetry axis it is in the same direction as the angular velocity vector.



11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-4: Running on a circular platform.

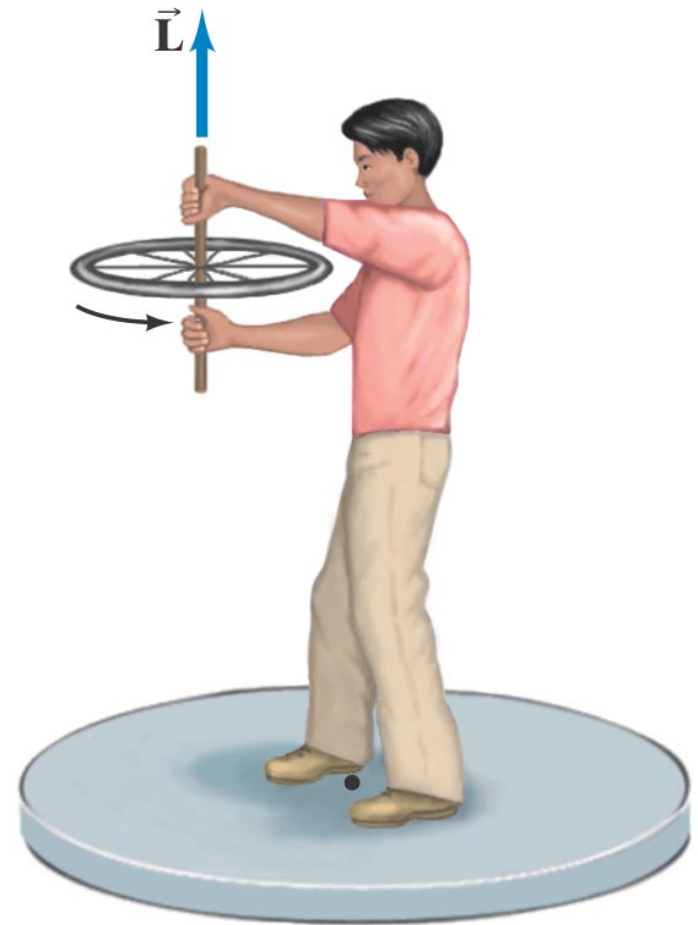
Suppose a 60-kg person stands at the edge of a 6.0-m-diameter circular platform, which is mounted on frictionless bearings and has a moment of inertia of $1800 \text{ kg}\cdot\text{m}^2$. The platform is at rest initially, but when the person begins running at a speed of 4.2 m/s (with respect to the Earth) around its edge, the platform begins to rotate in the opposite direction. Calculate the angular velocity of the platform.

demo

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Conceptual Example 11-5: Spinning bicycle wheel.

Your physics teacher is holding a spinning bicycle wheel while he stands on a stationary frictionless turntable. What will happen if the teacher suddenly flips the bicycle wheel over so that it is spinning in the opposite direction?

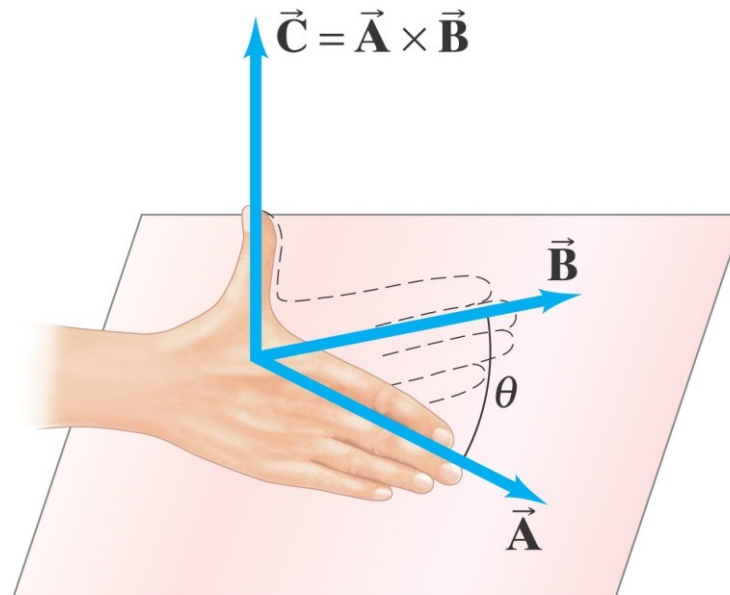


11-2 Vector Cross Product; Torque as a Vector

The vector cross product is defined as:

$$C = |\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = AB \sin \theta.$$

The direction of the cross product is defined by a right-hand rule:



11-2 Vector Cross Product; Torque as a Vector

The cross product can also be written in determinant form:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}}.$$

11-2 Vector Cross Product; Torque as a Vector

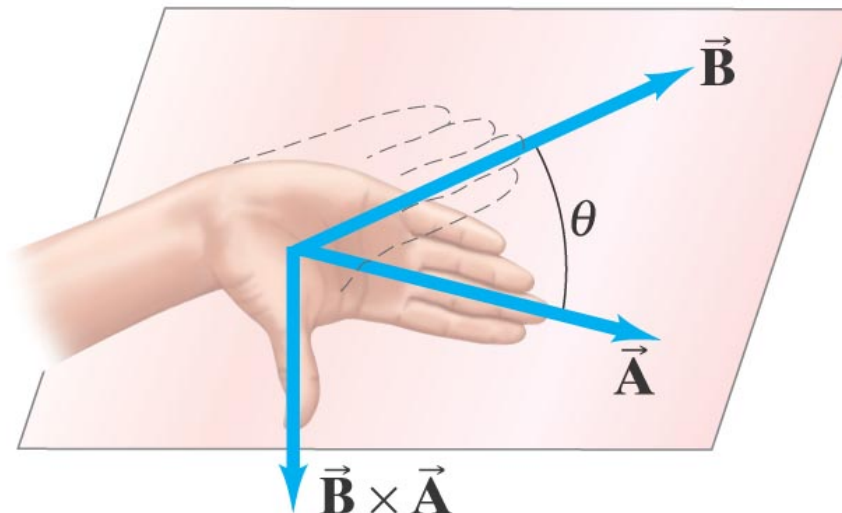
Some properties of the cross product:

$$\vec{\mathbf{A}} \times \vec{\mathbf{A}} = 0$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) + (\vec{\mathbf{A}} \times \vec{\mathbf{C}})$$

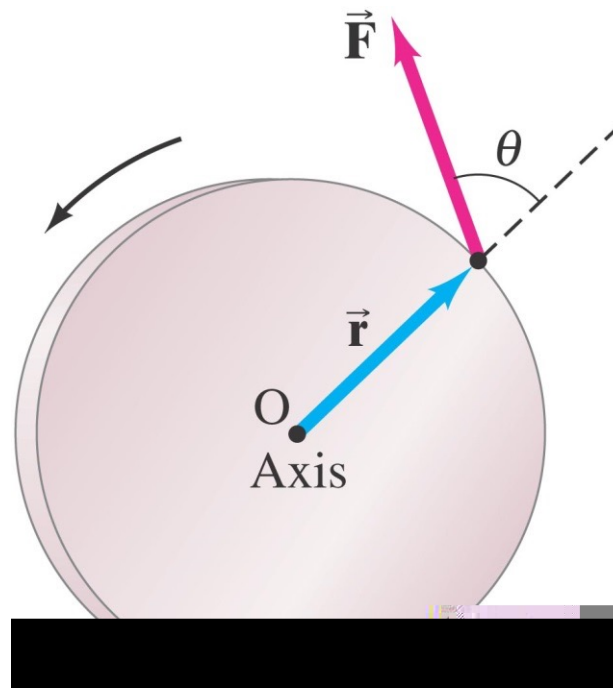
$$\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}.$$



11-2 Vector Cross Product; Torque as a Vector

Torque can be defined as the vector product of the force and the vector from the point of action of the force to the axis of rotation:

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

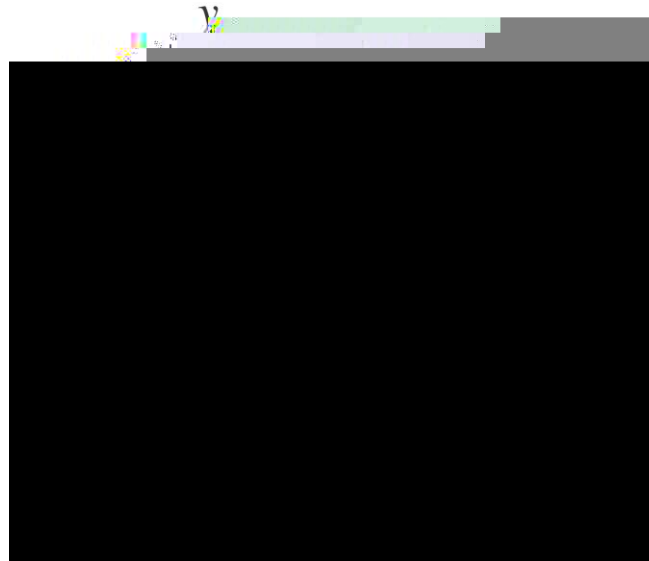


11-2 Vector Cross Product; Torque as a Vector

For a particle, the torque can be defined around a point O:

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

Here, $\vec{r}$ is the position vector from the particle relative to O.



11-2 Vector Cross Product; Torque as a Vector

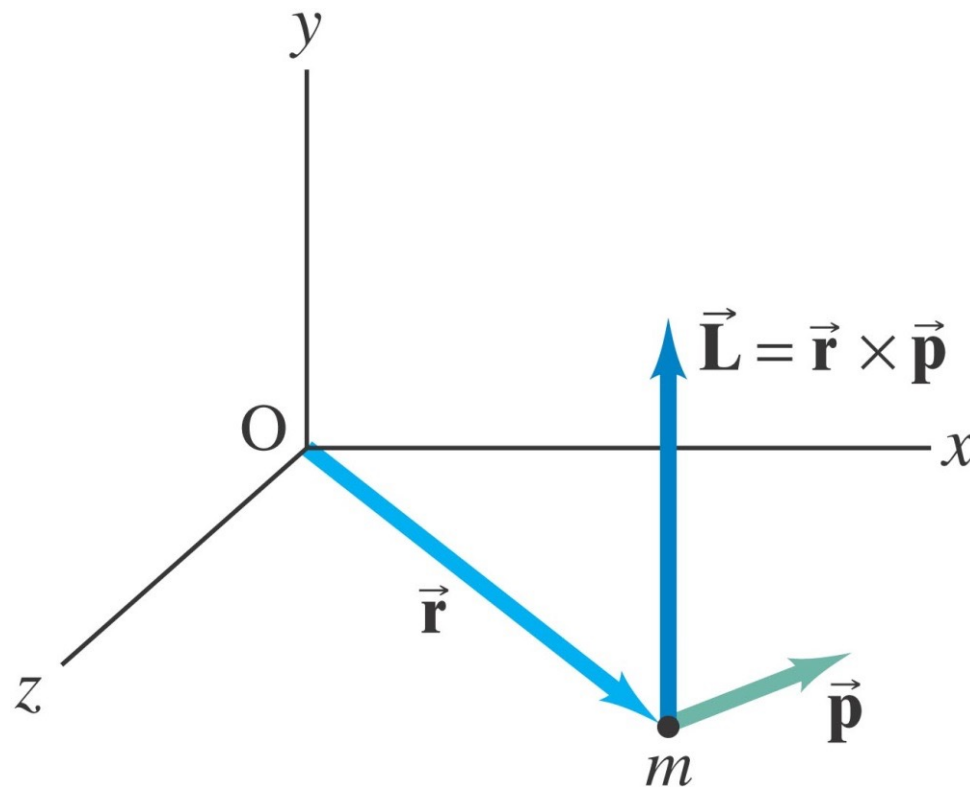
Example 11-6: Torque vector.

Suppose the vector $\vec{r}$ is in the xz plane, and is given by $\vec{r} = (1.2 \text{ m})\hat{i} + (1.2 \text{ m})\hat{k}$. Calculate the torque vector $\vec{\tau}$ if $\vec{F} = (150 \text{ N})\hat{i}$.

11-3 Angular Momentum of a Particle

The angular momentum of a particle about a specified axis is given by:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}.$$



11-3 Angular Momentum of a Particle

If we take the derivative of $\vec{\mathbf{L}}$, we find:

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}.$$

Since $\vec{\mathbf{r}} \times \Sigma \vec{\mathbf{F}} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \frac{d\vec{\mathbf{L}}}{dt},$

we have: $\Sigma \vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt}.$

11-3 Angular Momentum of a Particle

Conceptual Example 11-7: A particle's angular momentum.

What is the angular momentum of a particle of mass m moving with speed v in a circle of radius r in a counterclockwise direction?

