

Chapter 3

Basic Feasible Solutions and The Fundamental Theorem of LP

Recall the Corner Point Theorem for LP:

- Thus, in principle, if the set of all corner points are identified, one can **Optimal solution of an LP occurs at a vertex (corner point)** of its feasible polyhedral.
 - evaluate the objective function at each of the corner points
 - Comparing these values, the point that yields the best value is an optimal solution.
- But plotting a feasible set (polyhedral) is possible only in two dimension (two decision variables)
- In this chapter, we will establish **algebraic representation of corner points** of a feasible polyhedral irrespective of the number of variables

3.1 Linear Programs in Standard Form

- A linear program is said to be in **standard form** if it is given as

$$\begin{aligned}
 &\max \quad c_1x_1 + c_2x_2 + \dots + c_nx_n \\
 &\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\
 &\quad \quad a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\
 &\quad \quad \vdots \\
 &\quad \quad a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\
 &\quad \quad x_j \geq 0, j = 1, \dots, n
 \end{aligned}$$

Max
problem
&
Constraints
:
system of
equations

Or :

$$\begin{aligned}
 &\max \quad cx \\
 &\text{s.t} \quad Ax = b \\
 &\quad \quad x \geq 0
 \end{aligned}$$

$c \in R^n$, - row vector
 A - $m \times n$ matrix
(Usually m

$< n$)

$b \in R^m$ - column vector

$x \in R^n$ - vector of

decision

- In fact, a general LP problem can be
 - ❖ either a maximization or minimization problem.
 - ❖ Some of the constraints can be inequalities
- However, any LP can be changed to the standard form as shown below.

1. Minimization problem $\min cx$ can be changed to its equivalent maximization problem $\max -cx$
2. Inequalities can be changed to equations using slack and/or surplus variables defined as follows.

A) Slack Variables:

The “ $\leq$ ” constraint, say

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i,$$

can be transformed into equality by introducing a variable **s**, called **slack variable**, where

$$s = b_i - (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)$$

Observe that, **$s \geq 0$** .

Hence, the given inequality is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s = b_i, \quad s \geq 0.$$

For instance,
is equivalent
to

$$x_1 + 2x_2 + 3x_3 \leq 10$$

$$x_1 + 2x_2 + 3x_3 + x_4 = 10,$$

$$x_4 \geq 0.$$

(**x_4** is slack

variable)

B) Surplus Variables:

The “ $\geq$ ” constraint, say

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

can be transformed into equality by introducing a variable s , called **surplus variable**, where

$$s = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i$$

Observe that, $s \geq 0$.

Hence, the given inequality is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - s = b_i, \quad s \geq 0.$$

For instance, $x_1 + 2x_2 + 3x_3 \geq 10$

is equivalent to $x_1 + 2x_2 + 3x_3 - x_4 = 10,$
 $x_4 \geq 0.$

(x_4 is surplus variable)

- Thus, any LP problem (min or max) with any form of inequality constraint can be converted into equivalent **standard LP form** (max problem with equality const.)

- Example**

Change the following LP problem into standard form

$$\text{min} - 3x_1 + x_2 - x_3$$

$$\text{s.t} \quad x_1 + 3x_2 + 5x_3 \leq 15$$

$$2x_1 - x_2 + 5x_3 \geq 10$$

$$5x_1 - 2x_2 + x_3 = 270$$

$$x_1, x_2, x_3 \geq 0$$

3.2 Basic Feasible Solutions

- ❖ If B is $m \times m$ non-singular matrix and $b \in \mathbb{R}^m$, then the system $Bx = b$ has a unique solution given by $x = B^{-1}b$
- ❖ Consider a system of m linear equations with n variables, $Ax = b$,
where A is an $m \times n$ matrix $b \in \mathbb{R}^m$, and $\text{rank}(A) = \text{rank}(A, b) = m < n$.
(Otherwise, omit redundant equations)
 $\Rightarrow A$ has m linearly independent columns (which forms a basis of $\mathbb{R}^m$)
- ❖ Any m linearly independent columns of A are called **basic columns**, and the remaining $n-m$ columns are called **non-basic columns**.
- ❖ The m variables attached to the basic columns are called **basic variables (BV)**, and the remaining $n-m$ variables are called **non-basic variables (NBV)**.

- The m linearly independent columns of A forms $m \times m$ invertible sub-matrix, called **basic submatrix** (or **basis**) and denoted by B .

The remaining non-basic columns form an $m \times (n-m)$ submatrix denoted by N .

- Thus, rearranging if needed, A can be partitioned as $A = (B, N)$

- Let also x be partitioned as $x = (x_B, x_N)$, where

x_B is the m -vector of BVs attached to the basic columns in B ,

x_N is the $n-m$ vector of NBVs attached to the columns in N , $b \Rightarrow Bx_B + Nx_N = b$

$$\Leftrightarrow Bx_B = b - Nx_N$$

$$\Leftrightarrow x_B = B^{-1}b - B^{-1}Nx_N$$

(called **basis representation** of x

wrt B)

Hence, for any choice of values of x_N , say $x_N = u \in \mathbb{R}^{n-m}$,
 the values of x_B is uniquely determined by $x_B = B^{-1}b - B^{-1}Nu$

That is, if we choose $x_N = 0$, then $x_B = B^{-1}b$;

and

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

is the solution of $Ax = b$, called **basic solution** wrt B .

Remarks: Let $Ax = b$, where A is $m \times n$ & $\text{rank}(A) = m$.

- Any m columns of A can be selected to form an $m \times m$ submatrix B ; (Any such submatrix B is a basis if $\det(B) \neq 0$)

- There is exactly one basic solution, $(x_B, x_N) = (B^{-1}b, 0)$, w.r.t B .

- From the n columns, the nC_m ~~number of ways~~ to select m basic columns is at most $\binom{n}{m} \frac{n!}{m!(n-m)!}$

Hence, the system has at most nC_m basic solutions

- (finite) Two bases B_1 and B_2 of the system are said to be **adjacent** if they differ with only one column. In this case, the basic solutions wrt B_1 and B_2 are said to be **adjacent basic solutions**.

- Definition:** A basic solution $x = (x_B, 0)^T$ is called **basic feasible solution (BFS)** with respect to B if

$$x_B = B^{-1}b \geq 0. \quad (\text{nonnegative BS})$$

Thus, a BFS is a solution of $Ax = b, x \geq 0$.

🔴 **Example 1:** Find all basic solutions (BS) of

$$\begin{array}{rcl} x_1 + x_2 - 2x_3 + 3x_4 & = & 3 \\ x_1 + 2x_2 - 2x_3 & = & 4 \end{array}$$

(and also identify basic feasible solutions (BFS))

🔴 **Solution:** The coefficient matrix is $A = (A_1, A_2, A_3, A_4)$,

$$A_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

A has 6 sub-matrices of order 2:

$$\begin{array}{lll} B_1 = (A_1, A_2), & B_2 = (A_1, A_3), & B_3 = (A_1, A_4), \\ B_4 = (A_2, A_3), & B_5 = (A_2, A_4), & B_6 = (A_3, A_4) \end{array}$$

and

$$\begin{array}{lll} \det(B_1) = 1, & \det(B_2) = 0; & \det(B_3) = -3, \\ \det(B_4) = 2, & \det(B_5) = -6; & \det(B_6) = 6. \end{array}$$

⇒ 5 basic sub-matrices: B_1, B_3, B_4, B_5, B_6

• So, there can be **5 basic solutions** w.r.t each of the basic

sub-matrices. In

• $B_1 = (A_1, A_2)$, $X_{B1} = B_1^{-1}b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

W.r.t

$\Rightarrow (x_1, x_2, x_3, x_4) = (2, 1, 0, 0)$ is a BS, which is also **BFS**

• $B_3 = (A_1, A_4)$

W.r.t

$X_{B3} = B_3^{-1}b = \begin{pmatrix} 4 \\ -1/3 \end{pmatrix}$

$\Rightarrow (x_1, x_2, x_3, x_4) = (4, 0, 0, -1/3)$ is a BS, but NOT a **BFS**

• Similarly, w.r.t B_4 , B_5 & B_6 we obtain,

res $(x_1, x_2, x_3, x_4) = (0, 1, -1, 0)$ a BS, but NOT **BFS**

$(x_1, x_2, x_3, x_4) = (0, 4, 0, 1/3)$ a BS, and also **BFS**

$(x_1, x_2, x_3, x_4) = (0, 0, -2, -2/3)$ a BS, but NOT **BFS**

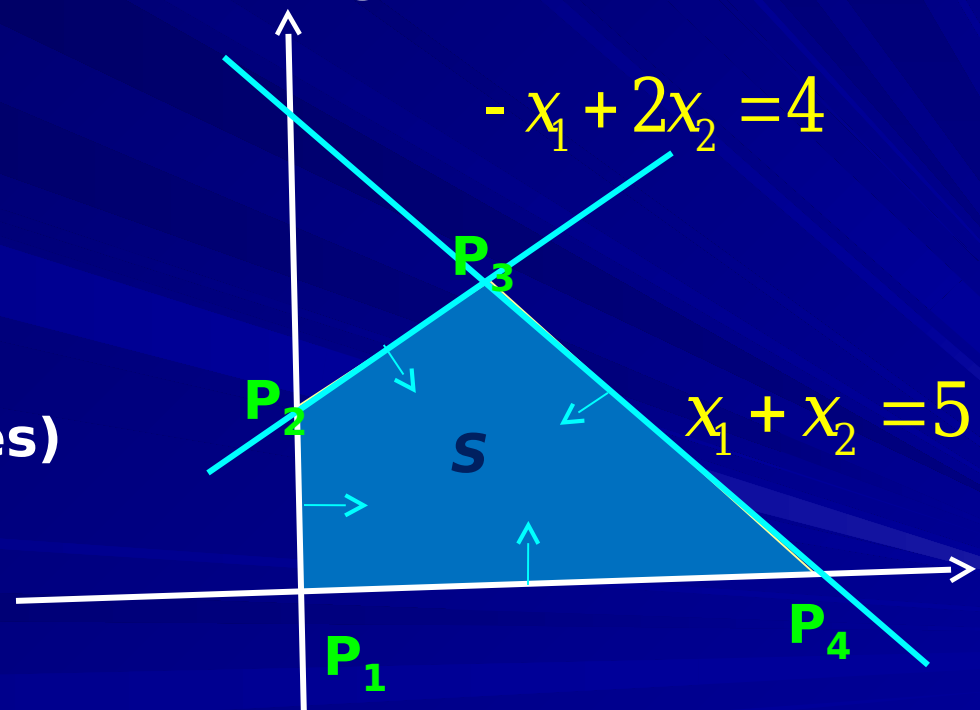
3.3 Algebraic Analysis of the Constraint Set

- From Chapter 2: the maximum or minimum of an LP (if it exists) occurs at a **corner (vertex)** point of the feasible polyhedral set.
- Next, we will show that a corner point is BFS .
- To see this, consider the following constraints:

$$\begin{aligned}-x_1 + 2x_2 &\leq 4 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0\end{aligned}$$

- Corner points (vertices) of S:

$P_1, P_2, P_3,$ & P_4



- Now, introducing two slack variables $x_3, x_4 \geq 0$, the above two inequalities are equivalent to

$$\begin{aligned} -x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 + x_4 &= 5 \end{aligned}$$

BFS is obtained by setting 0 to two variables

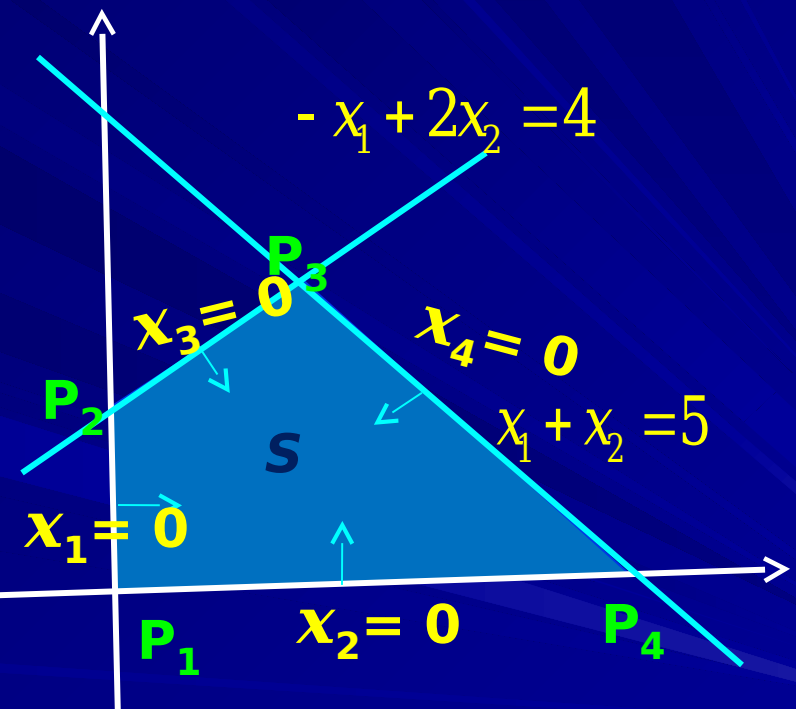
- Note that one of the variables is 0 on an edge of the polyhedral S .
- A corner (vertex) corresponds to point where two variables are 0, i.e., non basic, are:

$$P_1 : x_1 = 0, x_2 = 0,$$

$$P_2 : x_1 = 0, x_3 = 0,$$

$$P_3 : x_3 = 0, x_4 = 0,$$

$$P_4 : x_2 = 0, x_4 = 0,$$



- Thus, the corner points are basic feasible solutions

- This suggests that an optimal solution of an LP occurs at its BFS.
- In fact, as proved later, this is true for any LP.

Hence, we have the following important Theorem.

Theorem 1 (Fundamental Theorem of LP):

$$\begin{array}{ll} \max/\min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Consider an LP

where **A** is an **$m \times n$** matrix.

An optimal solution of the LP, if it exists, will occur at a BFS of its constraint equations.

- Based on the Fundamental of LP, one may solve a **bounded** LP problem, $\max \{cx \mid Ax=b, x \geq 0\}$, using the following procedure:

- Step 1.** Find all BFS of the constraint system $Ax=b$
- Step 2.** Evaluate the objective function cx at each BFS
- Step 3.** Comparing the values in Step 2; a BFS at which cx

Example:
Consider
solution.

attain

$$\begin{aligned} \max & x_1 + 4x_2 - 5x_3 + 3x_4 \\ \text{s.t. } & x_1 + x_2 - 2x_3 + 3x_4 = 3 \\ & x_1 + 2x_2 - 2x_3 = 4 \\ & x_i \geq 0 \end{aligned}$$

is an optimal

- We have seen that (sec. 3.2), the BFS of the constraints are

$$X_1 = (2, 1, 0, 0) \quad \text{and} \quad X_2 = (0, 2, 0, 1/3)$$

- The objective function at X_1 is 6, and at X_2 it is 9.

Hence (assuming it is bounded) optimal solution is

3.4 Optimality Condition for a BFS of LP

- Based on the fundamental theorem of LP, the procedure in previous section attempts to find optimal solution of LP by computing BFSs of the LP and selecting the one at which cx has the largest value .
- The drawback of the procedure are
 - 1, It needs computations of all BFSs (can be too many)
- We may avoid computations of all BFSs if we have a condition that tells us optimality of a given BFS.

The following theorem provides such condition.

Theorem 2 - Optimality Condition for LP:

Consider, a feasible LP in standard

$$\begin{array}{ll} \text{form } \max & cx \\ \text{s.t } & Ax = b \\ & x \geq 0 \end{array} \quad \begin{array}{l} c \in R^n, \\ A \text{ is } m \times n, \\ b \in R^m \end{array}$$

Suppose B is a basis with a BFS $\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$

Let $A = (B, N)$ and $c = (c_B, c_N)$ $c_N - c_B B^{-1}N \leq 0$,

then $\bar{x}$ is optimal BFS with optimal value

$$z^* = c_B B^{-1}b$$

Proof: (Use $x_B = B^{-1}b - B^{-1}Nx_N$ in $z = c_B x_B + c_N x_N$)

Definition: A basic submatrix B of a standard LP is called **optimal basis** if it satisfies the optimality condition, i.e., if

Definition: $c_N - c_B B^{-1}N \leq 0$ is called vector of **reduced costs** wrt B .

- Based on the above theorems one may try to solve the LP $\max \{cx \mid Ax=b, x \geq 0\}$ using the following procedure:
- Step 1.** Choose a basic matrix B that yields a BFS.
- Step 2.** Consider the corresponding partitions: $A = (B, N)$

$$c = (c_B, c_N) \quad x = (x_B, x_N)$$

- Step 3. Optimality Test:**

If $c_N - c_B B^{-1}N \leq 0$, then $x^* = (x_B, x_N) = (B^{-1}b, 0)$

is optimal solution (so,

stop)

Example:

Consider **otherwise** construct a new (adjacent) basis B and

$$\max 3x_1 - x_2 + 4x_3 - 2x_4$$

s.t.

$$x_1 + 3x_2 + x_3 + 2x_4 = 3$$

$$x_1 + 2x_2 + 2x_3 + x_4 = 4$$

repeat from **Step 2**

$$x_1, \dots, x_4 \geq 0$$

• Take $B = (A_1, A_2) \Rightarrow x_B = (x_1, x_2), x_N = (x_3, x_4)$

$$= (4, -2) \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, N = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, B^{-1}N = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

$$\Rightarrow c_B B^{-1}N = (11, -4)$$

Optimality Test: $c_N - c_B B^{-1}N = (4, -2) - (11, -4) = (-7, 2)$

[not ≤ 0]

• Next Take $B = (A_1, A_3) \Rightarrow x_B = (x_1, x_3), x_N = (x_2, x_4)$

$$(-1, -2) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, N = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}, B^{-1}N = \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}.$$

$$\Rightarrow c_B B^{-1}N = (7, 5)$$

Optimality Test: $c_N - c_B B^{-1}N = (-1, -2) - (7, 5) = (-8, -7) \leq (0, 0)$

Thus, $x_B = B^{-1}b = (2, 1) \Rightarrow x_1 = 2, x_3 = 1, x_2 = 0, x_4 = 0$
 satisfied optimality condition ; with optimal objective value $z^* = 10$.

- **The drawback of the above procedure are**
 - **May need too many checks (optimality tests) before reaching optimal basis**
 - **Direct computation of B^{-1} could be costly**
 - **It Does not know unbounded LP unless all BFS fail to satisfy the optimality condition.**

Hence, we still need to look for a better solution method (Next chapter)

3.5 Computation of Basic Solutions

- **Notation:** e_i represents a vector whose **i-th component is 1** and all other components are **0** (i-th standard unit vector in a given space)

$e_1, e_2, \dots, e_m$ forms the standard basis of $\mathbb{R}^m$;

- We consider a linear system $Ax = b$ where $I = (e_1, e_2, \dots, e_m)$ is $m \times m$ Identity matrix such that A is $m \times n$ matrix such that $\text{rank}(A) = m < n$.
- If B is a basis (i.e, $m \times m$ invertible sub-matrix) of A , then $(x_B, x_N) = (B^{-1}b, 0)$ is the basic solution (BS) of the system wrt B . But, as finding B^{-1} could be computationally costly, we usually use the **Gauss-Jordan elimination method (GJEM)** to compute a BS.

That is, a BS of a system $Ax = b$ can be found by applying the following **Gauss-Jordan elimination method** on $(A:b)$

For $i = 1, 2, \dots, m$

➤ Take a column of A , say A_j , whose i -th entry $a_{ij} \neq 0$.
(The column A_j is called **pivot column** and a_{ij} called **pivot element**)

➤ Applying elementary row operations (ERO) convert A_j to e_i .

• Thus, (i.e. 1 at pivot element and 0 for all other elements in the column) rearranging if needed, the **GJM** converts the system into equivalent system whose basis is $m \times m$ identity matrix:

$$Ix_B + Nx_N =$$

$\Rightarrow x_B = b', x_N = 0$, is a basic solution of the given system.

Example: Find BS of the following systems wrt the given basis

1.
$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 3 \\ x_1 + 2x_2 - 2x_3 &= 4 \end{aligned}$$
, i. $B = (A_1, A_2)$, ii. $B = (A_4, A_2)$

2.
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 5 \\ x_1 + 2x_2 + x_3 + x_4 + 3x_5 &= 9 \\ x_2 + x_3 + x_4 - x_5 &= 5 \end{aligned}$$
 $\approx$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & -2 & 3 \end{array} \right]$$

$B = (A_1, A_3, A_4)$

$\Rightarrow x_1=4, x_3=2, x_4=3; x_2=x_5=0$

Note: If B is a basis and A_j is a nonbasic column, then A_j can enter the basis in place of a basic column that contains e_r only if $a_{rj} \neq 0$.

In this case, one can apply ERO with a_{rj} as pivot element to convert the j -th column to e_r . This yields a

- In the given basis B of Example 2, can A_2 replace A_3 ? A_4 ? This is called **basis exchange**.

END