

# Chapter 2

**Geometric  
Method**

**Solution**

# Graphical Solution Methods for LP

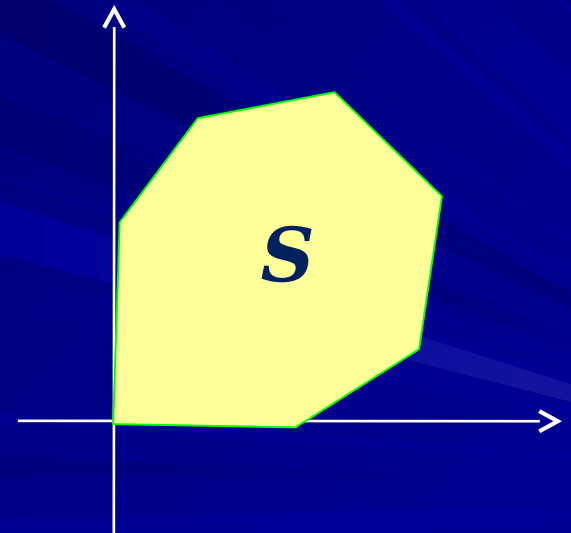
- In this section we consider the case when an LP problem involves only two decision variables. Thus, we consider

$$\max/\min f(x_1, x_2), \quad \text{such that } (x_1, x_2) \in S \subseteq R^2.$$

where  $f(x_1, x_2) = c_1x_1 + c_2x_2$

and the feasible set  $S$  is a polyhedral given by

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 (\leq \text{or } \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 (\leq \text{or } \geq) b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 (\leq \text{or } \geq) b_m \\ x_1, x_2 \geq 0 \end{array} \right.$$

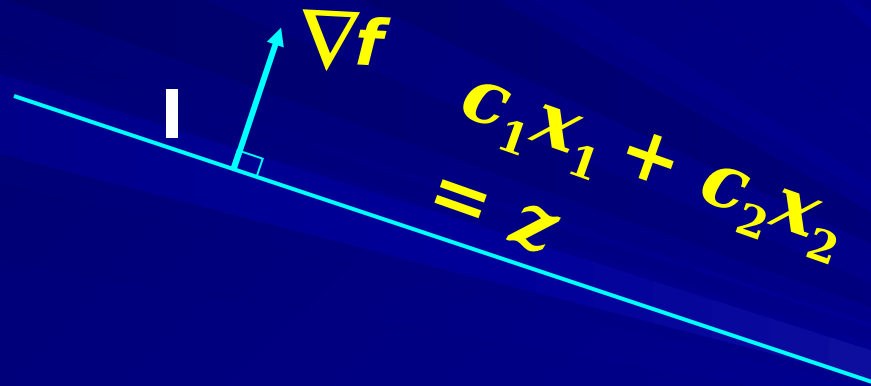


- The graphical solution method uses **level lines** of  $f$

- **Level line (contour)** of  $f$  is a set of all points where  $f$  has equal (a constant) value.

That is, a **level line** of  $f$  is a line given by

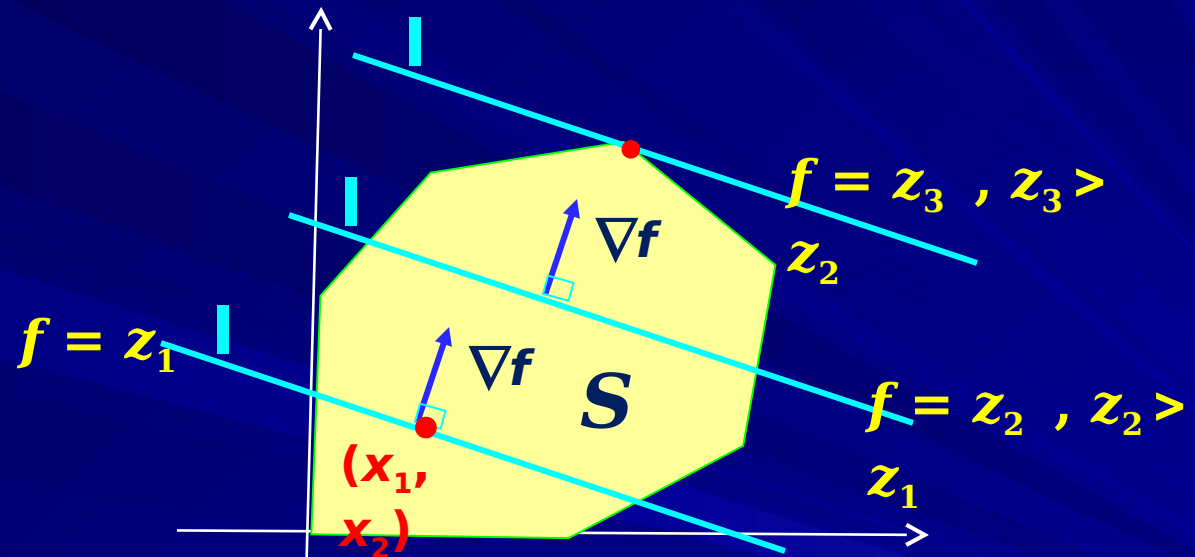
- $c_1x_1 + c_2x_2 = z$ , where  $z$  is a **constant**. Recall from calculus that the gradient,  $\nabla f = (c_1, c_2)^T$ , is orthogonal (perpendicular) to the level line of  $f$ .



**Recall:**

$\nabla f$  points toward the direction of maximum rate of increase of  $f$ ; and  $-\nabla f$  towards minimum rate of decrease of  $f$ .

- ❖ Thus, to solve LP, take any point  $(x_1, x_2) \in S$ ,
  - determine a level line  $l$  of  $f$  through this point, and use the fact that  $f$  increases if we move from a point  $(x_1, x_2)$  on  $l$  to any other point in  $S$  in the direction of  $\nabla f$  (for maximization).



- ❖ So, the level line  $l$  are kept moving parallel to it self in  $S$  in the direction of improving value of  $f$  as far as possible.
- ❖ The last point of intersection of  $l$  and  $S$  (where  $l$  is tangent to  $S$ ) is the optimal solution.

- Thus, to graphically solve  $\max f(x_1, x_2) = c_1x_1 + c_2x_2$  on a polyhedral set  $S$ , follow the following steps.

1. Plot the feasible set,  $S$ , on the coordinate plane.
2. Select an  $x_0 \in S$  and get the value  $f(x_0)$ . Let  $z = f(x_0)$ .
3. Plot the level line of  $f$  given by  $c_1x_1 + c_2x_2 = z$ , through  $x_0$ .
4. Translate (move) the level line parallel to itself in the direction of  $\nabla f$  within  $S$  until it is tangent to  $S$  at some point  $x^*$  on the boundary of  $S$ . This  $x^*$  is an optimal solution.

**Note:** The procedure is the same for minimization problem

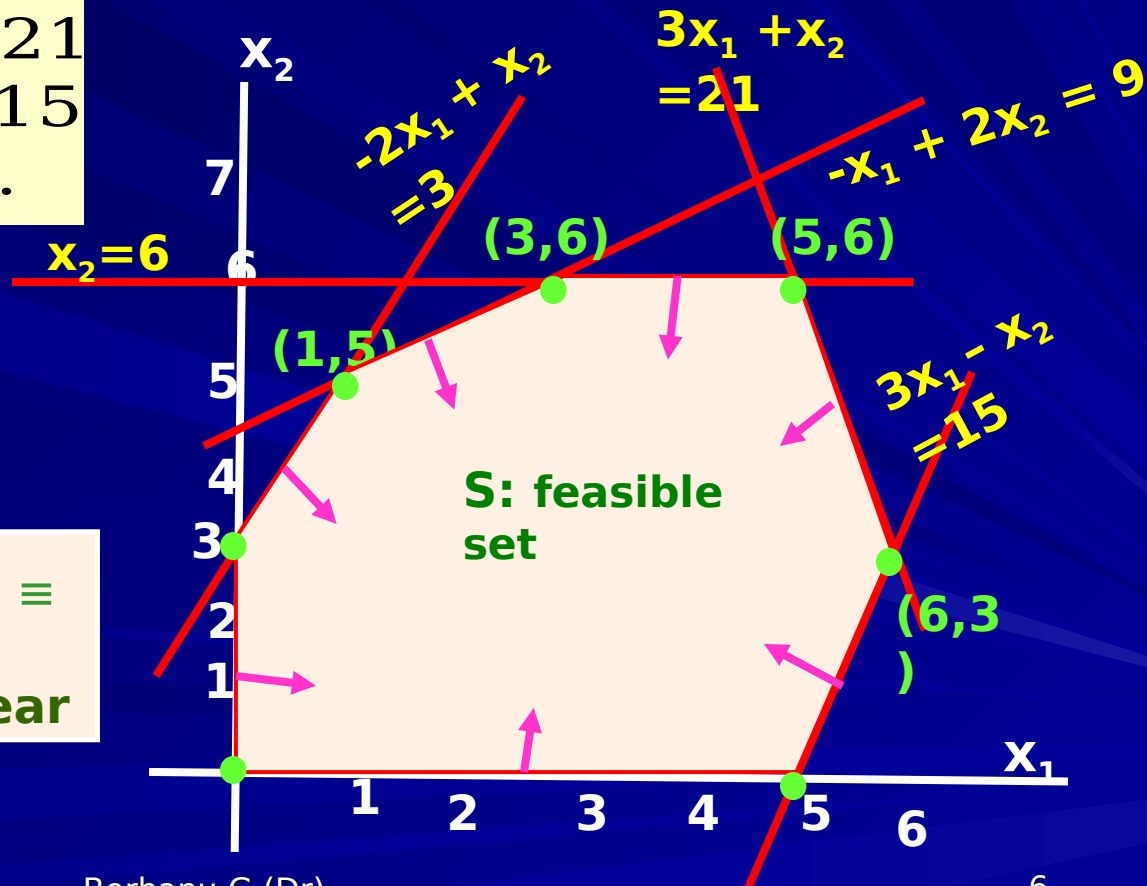
except the direction of the motion is  $-\nabla f$ .

**Example:** we consider maximizing/minimizing  
 $f(x_1, x_2) = c_1x_1 + c_2x_2$  on  $S$ , where  $S$  is  
 given by the  
 following system of inequalities.

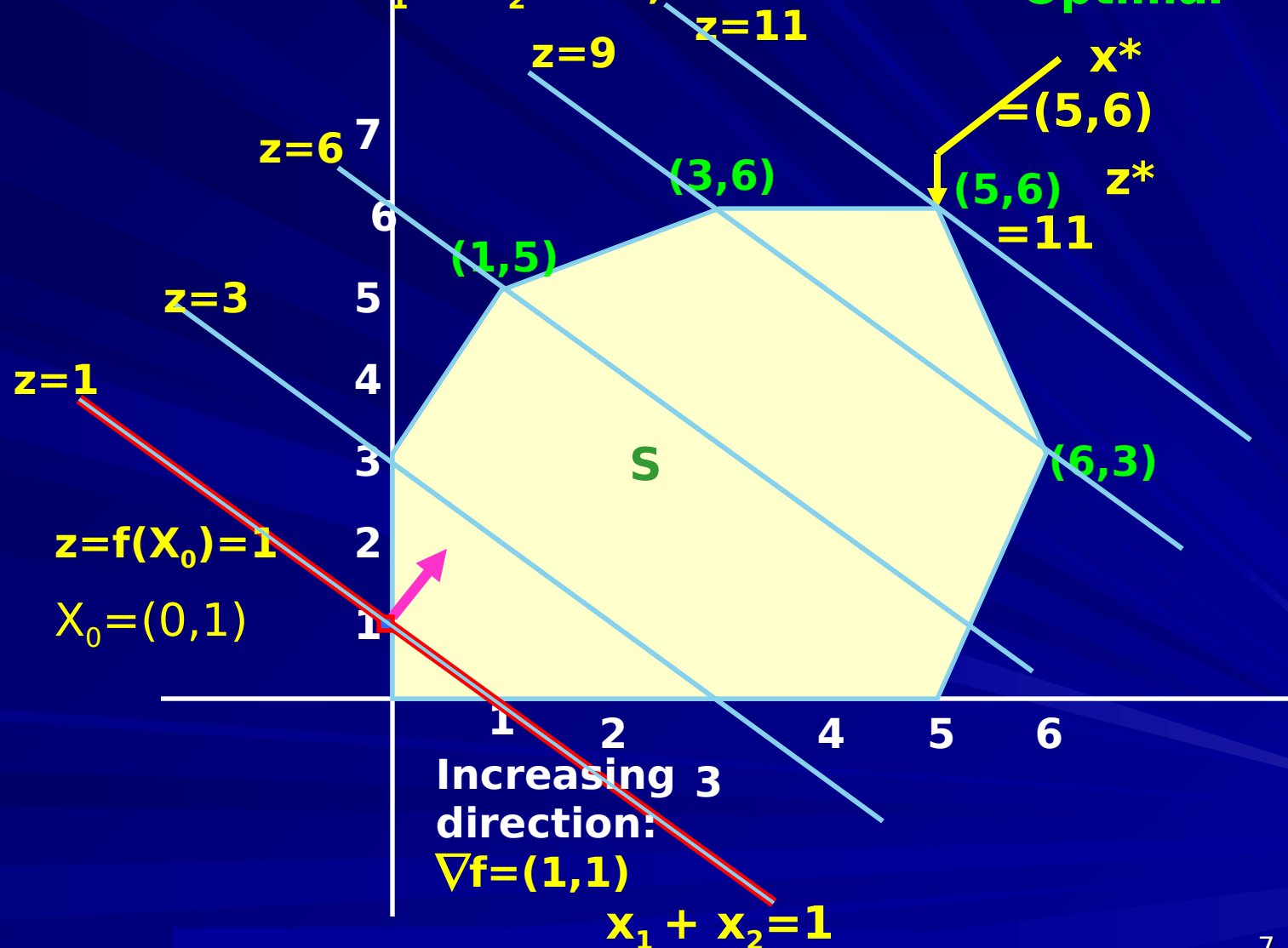
$$\begin{cases} -2x_1 + x_2 \leq 3 \\ -x_1 + 2x_2 \leq 9 \\ x_2 \leq 6 \\ 3x_1 + x_2 \leq 21 \\ 3x_1 - x_2 \leq 15 \\ x_1, x_2 \geq 0. \end{cases}$$

First draw the  
 feasible set  $S$  as  
 shown here:

$S$  is a polyhedral  $\equiv$   
 Set of all points  
 satisfying all linear  
 constraints

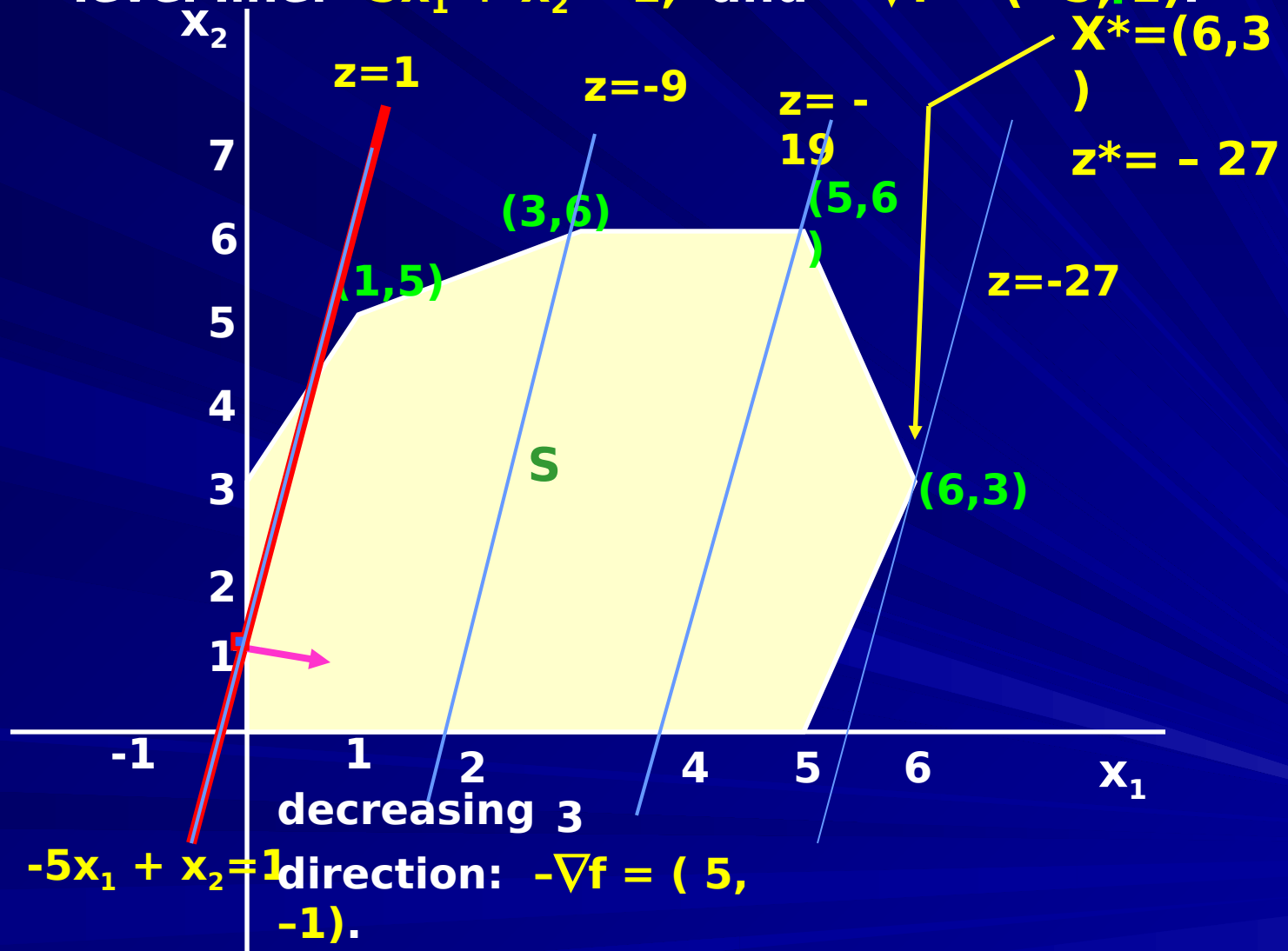


- Now let,  $f(x_1, x_2) = x_1 + x_2$  and **maximize  $f$  on  $S$**   
 $\Rightarrow$  Level line  $x_1 + x_2 = z$ ,  $z$  a constant **Optimal**



Next let  $f(x_1, x_2) = -5x_1 + x_2$  and **minimize**  $f$  on  $S$ .

- level line:  $-5x_1 + x_2 = z$ , and  $\nabla f = (-5, 1)$ .



For the above examples, optimal solutions occurred at the boundary of the polyhedral feasible sets , particularly at a vertex.

- This is always the case for LP since  $\nabla f$  is a nonzero constant vector and  $f$  keeps increasing in the direction of  $\nabla f$  (or keeps decreasing in the direction of  $-\nabla f$ ) unless the progress is stopped by the boundary of the feasible set.

- In fact, the farthest (outermost) boundary point of a polyhedral in a direction of improvement is a vertex (corner point) of the polyhedral. Hence, we have the following Theorem.

### Theorem (Corner Point Theorem)

An optimal solution (max or min) of an LP (if exist) occurs at a vertex (corner point) of its feasible polyhedral.

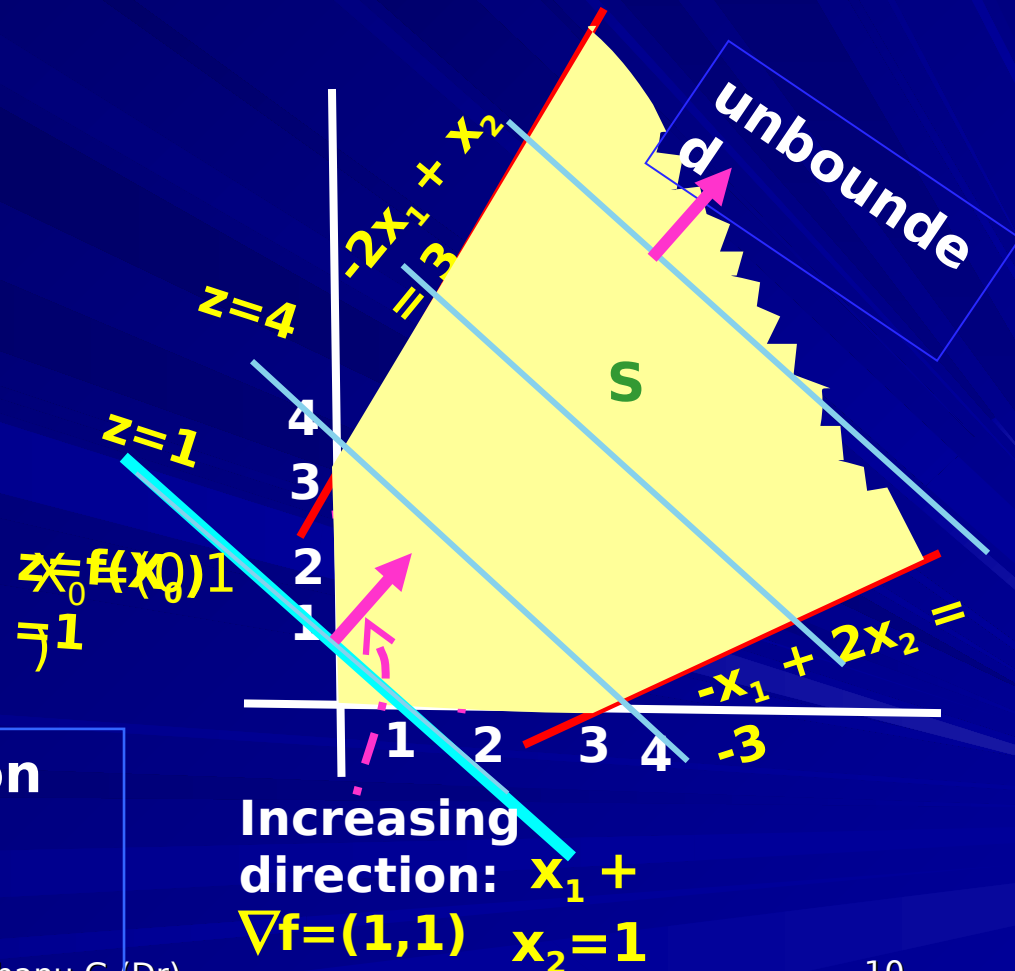
- If the polyhedral set is unbounded in the direction of improvement, then the LP is unbounded.

## Example: Consider

$$\begin{array}{ll}\max & x_1 + x_2 \\ \text{s.t.} & -2x_1 + x_2 \leq 3 \\ & -x_1 + 2x_2 \geq -3 \\ & x_1, x_2 \geq 0.\end{array}$$

- **1<sup>st</sup> Draw the feasible set  $S$ .**
- **The zigzag at a side of the polyhedral  $S$  indicates the polyhedral is unbounded in that side.**
- **The objective function of this problem keeps increasing in  $S$  with no bound.**

**Note:** If the objective function in the example is  $x_1 - 5x_2$  then it has max value (not unbounded)



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