College Algebra
CONCEPTS AND CONTEXTS

James Stewart
McMaster University
and
University of Toronto

Lothar Redlin
The Pennsylvania State University

Saleem Watson
California State University, Long Beach

Phyllis Panman

BROOKS/COLE
CENGAGE Learning™

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

BROOKS/COLE
CENGAGE Learning™

College Algebra: Concepts and Contexts
James Stewart, Lothar Redlin, Saleem Watson, Phyllis Panman
Acquisitions Editor: Gary Whalen
Developmental Editor: Stacy Green
Assistant Editor: Cynthia Ashton
Editorial Assistant: Guanglei Zhang
Media Editor: Lynh Pham
Marketing Manager: Myriah Fitzgibbon
Marketing Assistant: Angela Kim
Marketing Communications Manager: Katy Malatesta
Content Project Manager: Jennifer Risden
Creative Director: Rob Hugel
Art Director: Vernon Boes
Print Buyer: Judy Inouye
Rights Acquisitions Account Manager, Text: Roberta Broyer
Rights Acquisitions Account Manager, Image: Don Schlotman
Production Service: Martha Emry
Text Designer: Lisa Henry
Art Editor: Martha Emry
Photo Researcher: Bill Smith Group
Copy Editor: Barbara Willette
Illustrator: Jade Myers, Matrix Art Services; Network Graphics
Cover Designer: Larry Didona
Cover Images: *giant trees* (Cate Frost/Shutterstock.com 2009); *black-browed albatross* (Armin Rose/ Shutterstock .com 2009); *church* (Vladislav Gurfinkel/Shutterstock .com 2009); *student in chemistry lab* (Laurence Gough/Shutterstock 2009); *five skydivers performing formations* (Joggie Botma/Shutterstock.com 2009); *Shanghai at sunset* (David Roos/Shutterstock.com 2009); *combine harvester working on wheat crop* (Stephen Mcsweeny/Shutterstock.com 2009); *howler monkeys* (Christopher Marin/ Shutterstock. com 2009); *family making sand castle* (Magdalena Bujak/ Shutterstock.com 2009); *Easter Island* (Vladimir Korostyshevskiy/ Shutterstock.com 2009); *giardia* (Sebastian Kaulitzki/Shutterstock.com 2009); *female in handcuffs* (Jack Dagley Photography/Shutterstock.com 2009); *red eye tree frog* (Luis Louro/Shutterstock.com 2009); *humpback whale* (Josef78/Shutterstock.com 2009); *businessman in car* (Vladimir Mucibabic/Shutterstock.com 2009); *origami birds* (Slash331/Shutterstock.com 2009); *pine forest* (James Thew/Shutterstock.com 2009); *house finch* (Steve Byland/Shutterstock.com 2009); *mother with baby* (Lev Dolgachov/Shutterstock.com 2009); *bacteria* (Tischenko Irina/Shutterstock.com 2009); *Dos Amigos pumping plant* (Aaron Kohr/Shutterstock.com 2009); *polar bears* (Keith Levit/Shutterstock.com 2009); *combine harvester* (Orientaly/Shutterstock.com 2009); *streptococcus* (Sebastian Kaulitzki/Shutterstock.com 2009); *jumping girl* (Studio1One/Shutterstock.com 2009); *traffic* (Manfred Steinbach/Shutterstock.com 2009); *hybrid car* (Jonathan Larsen/Shutterstock.com 2009); *woman driving car* (Kristian Sekulic/Shutterstock .com 2009); *woman hiding money under mattress* (cbarnesphotography/Shutterstock.com 2009); *Mount Kilimanjaro* (Peter Zaharov/Shutterstock.com 2009)
Compositor: S4Carlisle Publishing Services

© 2011 Brooks/Cole, Cengage Learning

ALL RIGHTS RESERVED. No part of this work covered by the copyright herein may be reproduced, transmitted, stored, or used in any form or by any means graphic, electronic, or mechanical, including but not limited to photocopying, recording, scanning, digitizing, taping, Web distribution, information networks, or information storage and retrieval systems, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without the prior written permission of the publisher.

For product information and technology assistance, contact us at **Cengage Learning Customer & Sales Support, 1-800-354-9706.**

For permission to use material from this text or product, submit all requests online at **www.cengage.com/permissions.** Further permissions questions can be e-mailed to **permissionrequest@cengage.com.**

Library of Congress Control Number: 2009934974
ISBN-13: 978-0-495-38789-3
ISBN-10: 0-495-38789-4

Brooks/Cole
20 Davis Drive
Belmont, CA 94002-3098
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at **www.cengage.com/global.**

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Brooks/Cole, visit **www.cengage.com/brookscole** Purchase any of our products at your local college store or at our preferred online store **www.CengageBrain.com.**

Printed in Canada
1 2 3 4 5 6 7 13 12 11 10 09

CONTENTS

chapter **1** Data, Functions, and Models 1

1.1 Making Sense of Data 2

Analyzing One-Variable Data • Analyzing Two-Variable Data

1.2 Visualizing Relationships in Data 12

Relations: Input and Output • Graphing Two-Variable Data in a Coordinate Plane • Reading a Graph

1.3 Equations: Describing Relationships in Data 25

Making a Linear Model from Data • Getting Information from a Linear Model

1.4 Functions: Describing Change 35

Definition of Function • Which Two-Variable Data Represent Functions? • Which Equations Represent Functions? • Which Graphs Represent Functions? • Four Ways to Represent a Function

1.5 Function Notation: The Concept of Function as a Rule 52

Function Notation • Evaluating Functions—Net Change • The Domain of a Function • Piecewise Defined Functions

1.6 Working with Functions: Graphs and Graphing Calculators 64

Graphing a Function from a Verbal Description • Graphs of Basic Functions • Graphing with a Graphing Calculator • Graphing Piecewise Defined Functions

1.7 Working with Functions: Getting Information from the Graph 74

Reading the Graph of a Function • Domain and Range from a Graph • Increasing and Decreasing Functions • Local Maximum and Minimum Values

1.8 Working with Functions: Modeling Real-World Relationships 88

Modeling with Functions • Getting Information from the Graph of a Model

1.9 Making and Using Formulas 101

What Is a Formula? • Finding Formulas • Variables with Subscripts • Reading and Using Formulas

CHAPTER 1 Review 113

CHAPTER 1 Test 126

■ **EXPLORATIONS**

1 Bias in Presenting Data 128

2 Collecting and Analyzing Data 134

3 Every Graph Tells a Story 138

chapter **2** Linear Functions and Models 141

2.1 Working with Functions: Average Rate of Change 142

Average Rate of Change of a Function • Average Speed of a Moving Object • Functions Defined by Algebraic Expressions

2.2 Linear Functions: Constant Rate of Change 153

Linear Functions • Linear Functions and Rate of Change • Linear Functions and Slope • Using Slope and Rate of Change

2.3 Equations of Lines: Making Linear Models 165

Slope-Intercept Form • Point-Slope Form • Horizontal and Vertical Lines • When Is the Graph of an Equation a Line?

2.4 Varying the Coefficients: Direct Proportionality 177

Varying the Constant Coefficient: Parallel Lines • Varying the Coefficient of *x*: Perpendicular Lines • Modeling Direct Proportionality

2.5 Linear Regression: Fitting Lines to Data 189

The Line That Best Fits the Data • Using the Line of Best Fit for Prediction • How Good Is the Fit? The Correlation Coefficient

2.6 Linear Equations: Getting Information from a Model 201

Getting Information from a Linear Model • Models That Lead to Linear Equations

2.7 Linear Equations: Where Lines Meet 210

Where Lines Meet • Modeling Supply and Demand

CHAPTER 2 Review 219

CHAPTER 2 Test 228

EXPLORATIONS

1 When Rates of Change Change 229

2 Linear Patterns 233

3 Bridge Science 237

4 Correlation and Causation 239

5 Fair Division of Assets 242

chapter **3** Exponential Functions and Models 247

3.1 Exponential Growth and Decay 248

An Example of Exponential Growth • Modeling Exponential Growth: The Growth Factor • Modeling Exponential Growth: The Growth Rate • Modeling Exponential Decay

3.2 Exponential Models: Comparing Rates 261

Changing the Time Period • Growth of an Investment: Compound Interest

3.3 Comparing Linear and Exponential Growth 272

Average Rate of Change and Percentage Rate of Change • Comparing Linear and Exponential Growth • Logistic Growth: Growth with Limited Resources

3.4 Graphs of Exponential Functions 286

Graphs of Exponential Functions • The Effect of Varying *a* or *C* • Finding an Exponential Function from a Graph

3.5 Fitting Exponential Curves to Data 295

Finding Exponential Models for Data • Is an Exponential Model Appropriate? • Modeling Logistic Growth

CHAPTER 3 Review 303

CHAPTER 3 Test 311

■ **EXPLORATIONS**
1 Extreme Numbers: Scientific Notation 312
2 So You Want to Be a Millionaire? 315
3 Exponential Patterns 316
4 Modeling Radioactivity with Coins and Dice 320

chapter **4** Logarithmic Functions and Exponential Models 323

4.1 Logarithmic Functions 324
Logarithms Base 10 • Logarithms Base *a* • Basic Properties of Logarithms • Logarithmic Functions and Their Graphs

4.2 Laws of Logarithms 334
Laws of Logarithms • Expanding and Combining Logarithmic Expressions • Change of Base Formula

4.3 Logarithmic Scales 342
Logarithmic Scales • The pH Scale • The Decibel Scale • The Richter Scale

4.4 The Natural Exponential and Logarithmic Functions 350
What Is the Number *e*? • The Natural Exponential and Logarithmic Functions • Continuously Compounded Interest • Instantaneous Rates of Growth or Decay • Expressing Exponential Models in Terms of *e*

4.5 Exponential Equations: Getting Information from a Model 364
Solving Exponential and Logarithmic Equations • Getting Information from Exponential Models: Population and Investment • Getting Information from Exponential Models: Newton's Law of Cooling • Finding the Age of Ancient Objects: Radiocarbon Dating

4.6 Working with Functions: Composition and Inverse 377
Functions of Functions • Reversing the Rule of a Function • Which Functions Have Inverses? • Exponential and Logarithmic Functions as Inverse Functions

CHAPTER 4 Review 393

CHAPTER 4 Test 400

■ **EXPLORATIONS**
1 Super Origami 401
2 Orders of Magnitude 402
3 Semi-Log Graphs 406
4 The Even-Tempered Clavier 409

chapter **5** Quadratic Functions and Models 413

5.1 Working with Functions: Shifting and Stretching 414
Shifting Graphs Up and Down • Shifting Graphs Left and Right • Stretching and Shrinking Graphs Vertically • Reflecting Graphs

5.2 Quadratic Functions and Their Graphs 428
The Squaring Function • Quadratic Functions in General Form • Quadratic Functions in Standard Form • Graphing Using the Standard Form

5.3 Maxima and Minima: Getting Information from a Model 439
Finding Maximum and Minimum Values • Modeling with Quadratic Functions
5.4 Quadratic Equations: Getting Information from a Model 448
Solving Quadratic Equations: Factoring • Solving Quadratic Equations: The Quadratic Formula • The Discriminant • Modeling with Quadratic Functions
5.5 Fitting Quadratic Curves to Data 461
Modeling Data with Quadratic Functions
CHAPTER 5 Review 466
CHAPTER 5 Test 472
■ **EXPLORATIONS**
1 Transformation Stories 473
2 Toricelli's Law 476
3 Quadratic Patterns 478

chapter **6** Power, Polynomial, and Rational Functions 483

6.1 Working with Functions: Algebraic Operations 484
Adding and Subtracting Functions • Multiplying and Dividing Functions
6.2 Power Functions: Positive Powers 493
Power Functions with Positive Integer Powers • Direct Proportionality • Fractional Positive Powers • Modeling with Power Functions
6.3 Polynomial Functions: Combining Power Functions 504
Polynomial Functions • Graphing Polynomial Functions by Factoring • End Behavior and the Leading Term • Modeling with Polynomial Functions
6.4 Fitting Power and Polynomial Curves to Data 516
Fitting Power Curves to Data • A Linear, Power, or Exponential Model? • Fitting Polynomial Curves to Data
6.5 Power Functions: Negative Powers 527
The Reciprocal Function • Inverse Proportionality • Inverse Square Laws
6.6 Rational Functions 536
Graphing Quotients of Linear Functions • Graphing Rational Functions
CHAPTER 6 Review 546
CHAPTER 6 Test 553
■ **EXPLORATIONS**
1 Only in the Movies? 554
2 Proportionality: Shape and Size 557
3 Managing Traffic 560
4 Alcohol and the Surge Function 563

chapter **7** Systems of Equations and Data in Categories 567

7.1 Systems of Linear Equations in Two Variables 568

Systems of Equations and Their Solutions • The Substitution Method • The Elimination Method • Graphical Interpretation: The Number of Solutions • Applications: How Much Gold Is in the Crown?

7.2 Systems of Linear Equations in Several Variables 580

Solving a Linear System • Inconsistent and Dependent Systems • Modeling with Linear Systems

7.3 Using Matrices to Solve Systems of Linear Equations 590

Matrices • The Augmented Matrix of a Linear System • Elementary Row Operations • Row-Echelon Form • Reduced Row-Echelon Form • Inconsistent and Dependent Systems

7.4 Matrices and Data in Categories 602

Organizing Categorical Data in a Matrix • Adding Matrices • Scalar Multiplication of Matrices • Multiplying a Matrix Times a Column Matrix

7.5 Matrix Operations: Getting Information from Data 611

Addition, Subtraction, and Scalar Multiplication • Matrix Multiplication • Getting Information from Categorical Data

7.6 Matrix Equations: Solving a Linear System 619

The Inverse of a Matrix • Matrix Equations • Modeling with Matrix Equations

CHAPTER 7 Review 627

CHAPTER 7 Test 634

■ **EXPLORATIONS**

1 Collecting Categorical Data 635

2 Will the Species Survive? 637

■ **Algebra Toolkit A:** Working with Numbers T1

A.1 Numbers and Their Properties T1

A.2 The Number Line and Intervals T7

A.3 Integer Exponents T14

A.4 Radicals and Rational Exponents T20

■ **Algebra Toolkit B:** Working with Expressions T25

B.1 Combining Algebraic Expressions T25

B.2 Factoring Algebraic Expressions T33

B.3 Rational Expressions T39

■ **Algebra Toolkit C:** Working with Equations T47

C.1 Solving Basic Equations T47
C.2 Solving Quadratic Equations T56
C.3 Solving Inequalities T62

■ **Algebra Toolkit D:** Working with Graphs T67

D.1 The Coordinate Plane T67
D.2 Graphs of Two-Variable Equations T71
D.3 Using a Graphing Calculator T80
D.4 Solving Equations and Inequalities Graphically T85

ANSWERS A1

INDEX I1

1 Data, Functions, and Models



Joggie Botma/Shutterstock.com 2009

1.1 Making Sense of Data

1.2 Visualizing Relationships in Data

1.3 Equations: Describing Relationships in Data

1.4 Functions: Describing Change

1.5 Function Notation: The Concept of Function as a Rule

1.6 Working with Functions: Graphs and Graphing Calculators

1.7 Working with Functions: Getting Information from the Graph

1.8 Working with Functions: Modeling Real-World Relationships

1.9 Making and Using Formulas

EXPLORATIONS

1 Bias in Presenting Data

2 Collecting and Analyzing Data

3 Every Graph Tells a Story

Do you know the rule? We need to know a lot of rules for our everyday living—such as the rule that relates the amount of gas left in the gas tank to the distance we've driven or the rule that relates the grade we get in our algebra course to our exam scores. If we're more adventuresome, like the skydivers in the photo above, we may also need to know the rule that relates the distance fallen to the time we've been falling. Rules like these are expressed in algebra using functions; they are discovered by collecting data and looking for patterns in the data. You may collect data that describe how much gas your car uses at different speeds, how many calories you burn for different jogging times, or how far an object falls in a given time. Once a rule (or model) has been discovered, it allows us to *predict* how things will turn out—how far we can drive before running out of gas, how much weight we lose if we jog for so long, or how far a skydiver falls in a given length of time. Knowing this last rule allows us to enjoy skydiving . . . safely!

1.1 Making Sense of Data

- Analyzing One-Variable Data
- Analyzing Two-Variable Data

IN THIS SECTION... we learn about one-variable and two-variable data and the different questions we can ask and answer about the data.



From the first few minutes of life our brains are exposed to large amounts of data, and we must process and use the data to our advantage—sometimes even for our very survival. For example, after several small falls, a child begins to process the "falling down" data and concludes that the farther he falls, the more it hurts. The child sees the trend and reasons that "if I fall from a very great height, I'll be so badly hurt I may not survive." So as the child comes close to the edge of a 100-foot cliff for the first time, he's cautious of the height. Although the cliff is a new experience, the child is able to *predict* what would happen on the basis of the data he already knows. Fortunately, a child doesn't need to experience a 100-foot fall to know the probable result!

In general, in trying to understand the world around us, we make measurements and collect data. For example, a pediatrician may collect data on the heights of children at different ages, a scientist may collect data on water pressure in the ocean at different depths, or the weather section of your local newspaper may publish data on the temperature at different times of the day. Massive amounts of data are posted each day on the Internet and made available for research. In general, data are simply huge lists of numbers. To make sense of all these numbers, we need to look for patterns or trends in the data. Algebra can help us find and accurately describe hidden patterns in data. In this section we begin our study of data by looking at some of their basic properties.

Analyzing One-Variable Data

The ages of children in a certain group of preschoolers are

$$2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5$$

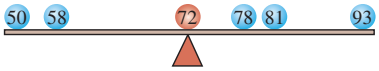
This list is an example of **one-variable data**—only one varying quantity (age) is listed. One way to make sense of all these numbers is find a "typical" number for the data or the "center" of the data. Any such number is called a measure of **central tendency**. One such number is the **average** (or **mean**) of the data. The average is simply the sum of the numbers divided by how many there are.

Average

The **average** of a list of n numbers is their sum, divided by n.

For example, if your scores on five tests are 50, 58, 78, 81, and 93, then your average test score is

$$\frac{50 + 58 + 78 + 81 + 93}{5} = 72$$



Intuitively, the average is where these numbers balance, as shown in the figure in the margin.

example 1 Average Age of Preschoolers

The ages of the children in a certain class of preschoolers are given in the table. Find the average age of a preschooler in this class.

Age (yr)	2	2	2	3	3	3	4	4	4	4	4	5

Solution

Since the number of children in the class is 12, we find the average by adding the ages of the children and dividing by 12:

$$\frac{2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 4 + 4 + 5}{12} = \frac{40}{12} \approx 3.3$$

So the average age of a preschooler in this class is about 3.3 years.

NOW TRY EXERCISE 15

Another measure of central tendency is the **median**, which is the "middle" number of a list of ordered numbers.

Median

Suppose we have an ordered list of n numbers.

(a) If n is odd, the **median** is the middle number.

(b) If n is even, the **median** is the average of the two middle numbers.



For example, if your scores on five tests are 50, 58, 78, 81, and 93, then your median test score is 78, the middle score—two test scores are above 78, and two test scores are below 78.

example 2 Average and Median Income

The yearly incomes of five college graduates are listed in the table below.

Income (thousands of dollars)	280	56	59	62	53

(a) Find the average income of these five college graduates.

(b) Find the median income of these five college graduates.

(c) How many data points are greater than the median? Than the average?

(d) Does the average income or the median income give a better description of the "central tendency" of these incomes?

Solution

(a) Since we are finding the average income of five college graduates, we find the sum of all the incomes and divide by 5.

$$\frac{280 + 56 + 59 + 62 + 53}{5} = 102$$

So the average income of the five graduates is \$102,000.

(b) To find the median income, we first order the list of incomes:

53, 56, 59, 62, 280

Since there are five numbers in this list and the number five is odd, the median is the middle number 59. So the median income of the five graduates is \$59,000.

(c) There are two data points above the median: 62 and 280. There is only one data point above the average, that is, 280.

(d) We can see from parts (a) and (b) that the average and the median can be very different. The average income of the five graduates is \$102,000, but this is not the typical income. In fact, not one of the graduates had an income close to \$102,000. However, the median income of \$59,000 is a more typical income for these five graduates. So the median income is a better description of the central tendency of the incomes in this case.

■ **NOW TRY EXERCISE 7** ■

IN CONTEXT ➤

In Example 2 the median seems to be a better indicator of "central tendency" than the average. This is because one of the incomes in the data is much greater than all the other incomes. In general, when the data have some "way out" numbers called **outliers**, the median is a better measure of central tendency than the average is. For this reason, for instance, the National Association of Realtors publishes yearly data on the *median* price of a home, and not the average price. Some homes are valued at tens of millions of dollars (like the house on the beach pictured here). It seems hardly fair to average the price of such a house with that of a typical \$250,000 home!



Ricardo Miguel/Shutterstock.com 2009

example 3 Median Price of a House

The table gives the selling prices of houses sold in 2007 in a neighborhood of Austin, Texas.

(a) Find the average selling price of a house in this neighborhood.

(b) Find the median selling price of a house in this neighborhood.

(c) Compare the average selling price and the median selling price.

Selling price (× $1000)	159	195	167	172	169	216	169	172

Solution

(a) Since we are finding the average of the selling prices of 8 houses, we find the sum of all the selling prices and divide by 8:

$$\frac{159 + 195 + 167 + 172 + 169 + 216 + 169 + 172}{8} \approx 177.4$$

So in 2007 the average selling price was about $177,400.

(b) To find the median selling price, we first order the list of selling prices:

$$159, 167, 169, 169, 172, 172, 195, 216$$

Since there are eight numbers in this list and the number eight is even, the median is the average of the middle two numbers

$$\frac{169 + 172}{2} = 170.5$$

So the median selling price was $170,500.

(c) The average and the median are different but are not as far apart as in the previous example because there are no outliers.

■ **NOW TRY EXERCISE 21** ■

■ Analyzing Two-Variable Data

Most real-world data involve two varying quantities. For example, we might want to match age with height, education with income, and so on. If we list the age and height of each child in a preschool class, we obtain data about the two variable quantities age and height. In general, data that involve two variables are called **two-variable data**. Our goal is to discover any relationships that may exist between the two variables.

example 4 Age and Height

The ages and heights of children in a preschool class are given in the table.

(a) Does being older necessarily imply being taller?

(b) Is the oldest student the tallest?



Age (yr)	2	2	2	3	3	3	4	4	4	4	4	5
Height (in.)	32	31	36	38	35	41	47	43	42	38	39	45

Solution

(a) No. The table shows several cases in which older does not necessarily imply taller. For example, there is a 2-year-old who is 36 in. tall and a 3-year-old who is only 35 in. tall.

(b) No. We see from the table that the oldest student is 5 years old and is 45 in. tall, but there is a 4-year-old who is 47 in. tall, so the oldest student is not the tallest.

NOW TRY EXERCISE 25

In Example 4(b) we found that a particular 2-year-old child may be taller than a particular 3-year-old. But in general, the trend is that older children tend to be taller. The process of finding such trends in data is discussed in the next section.

example 5 Time and Temperature

The table below gives the temperature in two hour intervals in Chemainus, British Columbia, on a pleasant June day.

(a) What is the highest temperature recorded?

(b) When was the lowest temperature recorded?

Time	Hours since 6:00 A.M.	Temperature (°F)
6:00 A.M.	0	59
8:00 A.M.	2	62
10:00 A.M.	4	68
12:00 P.M.	6	65
2:00 P.M.	8	58
4:00 P.M.	10	60
6:00 P.M.	12	62

Solution

(a) From the table we see that the highest temperature is 68°F.

(b) From the table we see that the lowest temperature occurred at 2:00 P.M.

NOW TRY EXERCISE 27

example 6 Depth and Pressure

Depth (ft)	Pressure (lb/in^2)
0	14.7
10	19.2
20	23.7
30	28.2
40	32.7
50	37.2
60	41.7

A deep sea diver measures the water pressure at different ocean depths. The results of her measurements are listed in the table.

(a) What is the pressure at a depth of 50 ft?

(b) At what depth is the pressure 28.2 lb/in^2?

(c) By how much does the pressure change as the depth changes from 0 ft to 10 ft? From 10 ft to 20 ft? From 20 ft to 30 ft?

(d) What pattern or trend do you see in these data?

Solution

(a) The data indicate that at a depth of 50 ft the pressure is 37.2 lb/in^2.

(b) The pressure is 28.2 lb/in^2 at a depth of 30 ft.

(c) From a depth of 0 ft to a depth of 10 ft, the pressure changes from 14.7 lb/in^2 to 19.2 lb/in^2, for a total increase of

$$19.2 - 14.7 = 4.5 \text{ lb/in}^2$$

From a depth of 10 ft to a depth of 20 ft, the pressure changes from 19.2 lb/in^2 to 23.7 lb/in^2, for a total increase of

$$23.7 - 19.2 = 4.5 \text{ lb/in}^2$$

From a depth of 20 ft to a depth of 30 ft, the pressure changes from 23.7 lb/in^2 to 28.2 lb/in^2, for a total increase of

$$28.2 - 23.7 = 4.5 \text{ lb/in}^2$$

You can check that the increase in pressure is 4.5 lb/in^2 for each successive 10-ft increase in depth shown in the data.

(d) From the table we see that the pressure seems to increase steadily as the depth increases.

■ **NOW TRY EXERCISE 29** ■

1.1 Exercises

CONCEPTS

Fundamentals

1. **(a)** What are one-variable data? Give examples.

(b) What is the difference between one-variable data and two-variable data? Give examples.

2. What is meant by *central tendency* for one-variable data? Measures of central tendency for one-variable data include the average and the _____.

3. To find the average of a list of n numbers, we first _____ all the numbers and then divide by _____.

4. To find the median of a list of numbers, we first rearrange the numbers to put them in _____. To find the median of a list of n ordered numbers, we do the following.

(a) If n is odd, the median is the _____ number in the list.

(b) If n is even, the median is the average of the two _____ numbers in the list.

Think About It

5. If you have a million data points (as may be available on the Internet), what technology would you need to help you find the average or the median of the data?

6. Find several examples of two-variable data that you may be able to collect from your classmates, for instance, height and shoe size.

SKILLS

7–10 ■ A table of one-variable data is given.

(a) Find the average of the data.

(b) Find the median of the data.

(c) How many data points are greater than the average? How many are greater than the median?

7.

A	113	21	16	16	19	29	21

8.

A	132	510	119	132	141	132	121

9.

A	69	71	74	73	72	73	69	69

10.

A	91	87	84	82	87	84	93	82

11–14 ■ A table of two-variable data is given. What pattern or trend do you see in these data?

11.

A	0	1	2	3	4	5
B	20	40	80	250	600	1000

12.

A	0	1	2	3	4	5
B	100	95	89	82	76	71

13.

A	0	1	2	3	4	5
B	3	10	25	26	25	25

14.

A	0	1	2	3	4	5
B	100	50	25	12	10	10

CONTEXTS

15. Basketball Stats The list below shows the number of points scored by Kobe Bryant of the Los Angeles Lakers in each basketball game in which he played in February 2007. What is the average number of points per game that Kobe Bryant scored in that month?

46, 30, 6, 11, 36, 33, 31, 29, 23, 41, 17, 21, 30, 21, 33

16. Lion Prides A wildlife biologist in the south Sahara Desert of Africa records the number of lions in the different prides in her area. What is the average number of lions in a pride?

18, 8, 14, 15, 16, 12, 17

17. Apgar Score Doctors use the Apgar score to assess the health of a newborn baby immediately after the child is born. The list below shows the Apgar scores of babies born in Memorial Hospital on March 11, 2007. What are the average and the median Apgar scores for these babies?

9, 8, 10, 3, 5, 8, 10

18. Weights of Sextuplets The Hanselman sextuplets were born three months premature on February 26, 2004, in Akron, Ohio, and they have all survived to this date. The list below shows the birth weights of each child in pounds. What are the average and the median birth weights of these sextuplets?

2.6250, 1.5625, 2.3750, 2.5000, 2.5000, 2.0625

19. Quiz Average The table below shows Jordan's scores on her first five algebra quizzes.

(a) What is her average quiz score?

(b) Jordan receives a score of 10 on her sixth quiz, so now what is her average quiz score?

Score	9	9	6	7	7

20. Quiz Average The table below shows Chad's scores on his first four geography quizzes.

(a) What is his average quiz score?

(b) Chad receives a score of 5 on his fifth quiz, so now what is his average quiz score?

Score	7	9	8	9

21. Investment Seminar The organizer of an investment seminar surveys the participants on their yearly income. The table below shows the yearly income of the participants.

(a) Find the average and median income of the participants.

(b) How many participants have an income above the average?

(c) A new participant joins the seminar, and her yearly income is $500,000. Now what are the average income and median income, and how many participants have an income above the average? Is the average or the median the better measure of central tendency?

Yearly income ($)	56,000	58,000	48,000	45,000	59,000	72,000	63,000

22. Dairy Farming A dairy farmer in Illinois records the weights of all his cows. The table below shows his data.

(a) Find the average and median weight of the cows on the farm.

(b) How many cows have a weight above the average?

(c) The farmer purchases a young calf that weighs only 420 pounds. Now what are the average weight and median weight of the cows on the farm, and how many cows have a weight above the average? In this case, is the average or the median the better measure of central tendency?

Weight (lb)	880	970	930	890	980	920	900

23. Home Sales A realty agency in Albuquerque, New Mexico, records the prices of homes sold in one neighborhood.

(a) What are the average and median home prices in this neighborhood? Which is the better indicator of central tendency?

(b) Is the house that sold for $329,000 above or below the average price? The median price?

(c) After these data were gathered, another home in the neighborhood sold for $2,860,000, so now what are the average and median home prices if we include the latest price? Now what is the better indicator of central tendency?

Price ($)	299,000	329,000	355,000	316,000	330,000

24. Home Sales A realty agency in Napa Valley, California, records the prices of homes sold in one neighborhood.

(a) What are the average and median home prices in this neighborhood? Which is the better indicator of central tendency?

(b) Is the house that sold for $2,319,000 above or below the average price? The median price?

(c) After these data were gathered, another home in the neighborhood sold for $300,000, so now what are the average and median home prices if we include the latest price? Now what is the better indicator of central tendency?

Price ($)	2,278,000	2,231,000	2,319,000	2,279,000	2,365,000	2,279,000	2,319,000

25. Pets per Household The home owners association of a small gated community conducts a survey to determine trends in the number of pets in their neighborhood. The

number of people in a household and the number of pets they own are recorded. The table below shows the results of the survey.

(a) What is the average number of pets in a household in this community?

(b) Does the largest family have the most pets?

(c) How many people are in the household with the most pets?

Number of people	2	1	5	3	2	3
Number of pets	5	11	2	0	15	1

26. Cars per Household The home owners association of Exercise 25 conducts another survey to determine trends in the number of cars owned by members of their community. The number of people in a household and the number of cars they own are recorded. The table below shows the results of this survey.

(a) What is the average number of cars in a household in this community?

(b) Does the largest family own the most cars?

(c) How many people are in the household with the most cars?

Number of people	2	1	5	3	2	3
Number of cars	1	1	3	0	5	3

27. Snowfall The Sierra Nevada mountain range is well known for its large snowfalls, especially in the area of Lake Tahoe. This area usually gets over 200 inches of snow a year. The table below gives the snowfall for each month in 2006.

(a) What was the snowfall for the month of February 2006?

(b) What month had the highest snowfall? What month had 49 inches?

(c) Find the average monthly snowfall for 2006.

(d) Find the median monthly snowfall for 2006.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Snowfall (in.)	49	12	83	42	2	0	0	0	0	0	8	14

28. Precipitation The Olympic Peninsula in the state of Washington is home to three temperate rain forests and receives 80 or more inches of rainfall a year. The table below gives this area's monthly precipitation in 2007.

(a) How many inches of rainfall were recorded in May?

(b) What month had the highest rainfall? What month had 2 inches of rain?

(c) Find the average monthly precipitation for 2007.

(d) Find the median monthly precipitation for 2007.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall (in.)	15.6	10.8	9.8	4.9	3.2	2.1	2.0	1.3	2.7	8.3	14.0	14.6



29. Falling Watermelons The fifth grade class at Wilson Elementary School performs an experiment to determine the time it takes for a watermelon to fall to the ground. They have access to a five-story building for their experiment. The class drops a watermelon out of a window from each story of the building and records the time it takes for the watermelon to fall to the ground. The table below shows the results of their experiment.

(a) How long does it take for a watermelon to fall 5 ft from the first-story window?

(b) How much longer does it take a watermelon to fall from 15 ft than from 5 ft? From 45 ft than from 35 ft?

(c) What pattern or trend do you see in these data?

Story	Distance (ft)	Time (s)
First	5	0.60
Second	15	0.96
Third	25	1.26
Fourth	35	1.49
Fifth	45	1.68

30. Cooling Car Engine The operating temperature of Sofia's car is about 180°F. When Sofia returns home from work at 5:00 P.M., she parks her car in the garage. The table shows the temperature of the engine at half-hour intervals.

(a) Find the temperature of the engine at 5:30 P.M. and 7:30 P.M.

(b) When does the temperature reach 100°F?

(c) What pattern or trend do you see in these data?

Time (P.M.)	Hours since 5:00 P.M.	Temperature (°F)
5:00	0.0	180
5:30	0.5	145
6:00	1.0	100
6:30	1.5	83
7:00	2.0	76
7:30	2.5	72
8:00	3.0	71

31. Algebra and Alcohol The data in the Prologue (page P2) give the blood alcohol concentration following the consumption of different doses of alcohol. Consider the data for consumption of 15 mL.

(a) Find the blood alcohol concentration (mg/mL) after 0.2, 0.5, and 1.5 hours.

(b) When does the blood alcohol concentration drop below 0.1?

(c) What pattern or trend do you see in these data?

1.2 Visualizing Relationships in Data

- Relations: Input and Output
- Graphing Two-Variable Data in a Coordinate Plane
- Reading a Graph

IN THIS SECTION... we learn how to represent a set of two-variable data as a "relation." We also learn how graphs of two-variable data help us get information about the data.

GET READY... by reviewing how to plot points in a coordinate plane in **Algebra Toolkit D.1**. Test your readiness by doing the **Algebra Checkpoint** at the end of this section.

When business people, sociologists, or scientists analyze data, they look for a trend or pattern from which to draw a conclusion about the process they are studying. It is often hard to look at a list of numbers and see any kind of pattern; this is especially true when the lists are huge. One of the best ways to reveal a hidden pattern in data is to draw a graph, which we do in this section.

Relations: Input and Output

The following data give the height and weight of five college algebra students. We can also represent the data as *ordered pairs* of numbers.

Data

Height (in.)	Weight (lb)
72	180
60	204
60	120
63	145
70	184

Data as ordered pairs

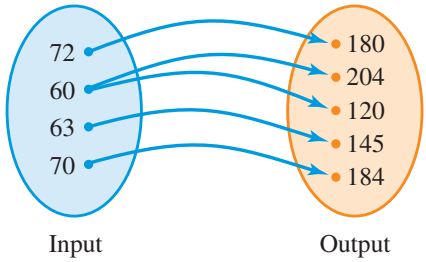
(height, weight)
(72, 180)
(60, 204)
(60, 120)
(63, 145)
(70, 184)

For example, the pair (72, 180) represents the student with height 72 inches and weight 180 pounds. In general, if we let x stand for height and y stand for weight, then the **ordered pair** (x, y) represents the student with height x and weight y. In mathematics any collection of ordered pairs is called a *relation*.

Relation

A **relation** is any set of ordered pairs.

Notice that by writing two numbers as an ordered pair, we are saying that the two numbers are linked together, or related to each other. We say that the first number is an **input** and the second is an **output** of the relation. So for the ordered pair (72, 180) the input is 72 and the output is 180. We can visualize this relation by drawing a diagram as follows.



The **domain** of a relation is the set of all inputs, and the **range** is the set of all outputs. In this example, the domain is the set of heights and the range is the set of weights:

Domain {60, 63, 70, 72}
Range {120, 145, 180, 184, 204}

Notice that the domain and range are sets, so each number is listed only once. Even though the number 60 occurs twice as an input, it is listed only once in the domain.

example 1 Two-Variable Data as a Relation

Score	Hours of sleep
92	8
80	5
70	1
82	9
91	7
80	7
70	3

The students in an algebra course gathered data on the final exam scores and the number of hours of sleep students had before the exam. The data are shown in the table.

(a) Express the data as a relation.

(b) Draw a diagram of the relation.

(c) What does the pair (80, 5) represent?

(d) Find the output(s) corresponding to the input 80.

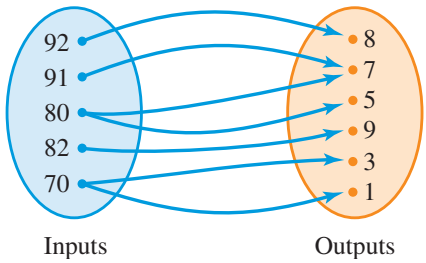
(e) Find the domain and range of the relation.

Solution

(a) The set of ordered pairs that defines this relation is

$$\{(92, 8), (80, 5), (70, 1), (82, 9), (91, 7), (80, 7), (70, 3)\}$$

(b) A diagram of the relation is shown below.



(c) The pair (80, 5) represents a student who received a score of 80 and had 5 hours of sleep before the exam.

(d) There are two outputs that correspond to the input 80; they are 5 and 7.

(e) The domain of this relation is the set {70, 80, 82, 91, 92}, and the range is the set {1, 3, 5, 7, 8, 9}.

■ NOW TRY EXERCISES 11 AND 35 ■

example 2 Domain and Range of a Relation

Day	Number of house finches
2	21
5	20
12	15
15	10
21	15
25	3

The data in the table give the number of house finches seen at a bird feeder on various days in February 2008.

(a) Express the data as a relation.

(b) What does the pair (12, 15) represent?

(c) Find the domain and range of the relation.

Solution

(a) The set of ordered pairs that defines this relation is

$$\{(2, 21), (5, 20), (12, 15), (15, 10), (21, 15), (25, 3)\}$$

(b) The pair (12, 15) tells us that 15 house finches were seen at the bird feeder on February 12.

(c) The domain of the relation is the set $\{2, 5, 12, 15, 21, 25\}$, and the range is the set $\{3, 10, 15, 20, 21\}$.

NOW TRY EXERCISE 37

Graphing Two-Variable Data in a Coordinate Plane

To review how we plot points in a coordinate plane, see Algebra Toolkit D.1, page T67.

Two-variable data consist of ordered pairs of numbers, so to graph such data, we graph the ordered pairs in a coordinate plane, with one variable plotted on the x-axis and the other on the y-axis. For instance, to plot the depth-pressure data tabulated in Example 6 of Section 1.1, we would plot the points (0, 14.7), (10, 19.2), (20, 23.7), and so on, where the first coordinate represents depth and the second represents the corresponding pressure. Such a graph of data is called a **scatter plot**. Graphs and scatter plots are so common in everyday life (you see them in magazines, in newspapers, on television news, and in advertising) that you may already have thought of making a graph of the data in the preceding examples.

example 3 Graphing Two-Variable Data

Draw a scatter plot of the time-temperature data and the depth-pressure data from Examples 5 and 6 of Section 1.1. Comment on any trends or patterns you observe from the graphs.

Hours since 6:00 A.M.	Temperature (°F)
0	59
2	62
4	68
6	65
8	58
10	60
12	62

Depth (ft)	Pressure (lb/in^2)
0	14.7
10	19.2
20	23.7
30	28.2
40	32.7
50	37.2
60	41.7

Solution

We plot the points given by the ordered pairs listed in the tables. The graphs are shown in Figures 1 and 2.

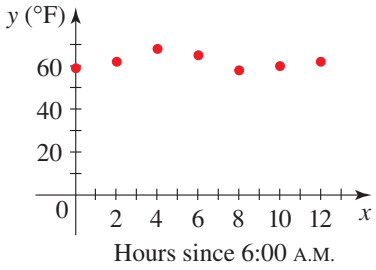


figure **1** Time and temperature

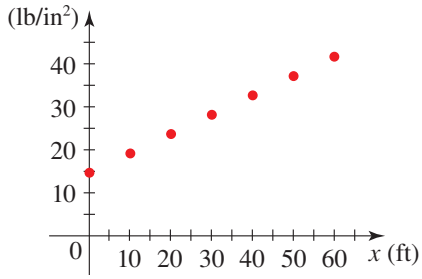


figure **2** Depth and pressure

In the time-temperature data we see that as time increases, the temperature first increases, then decreases, then increases again. In the depth-pressure data there appears to be a very precise trend: As the depth increases, so does the pressure. In fact, pressure appears to increase in proportion to the increase in depth.

NOW TRY EXERCISES 27 AND 39

IN CONTEXT ➤



Lev Dolgachov/Shutterstock.com 2009

Finding relationships in two-variable data is a fundamental activity in every branch of science. For example, scientists test different chemicals in the blood of expectant mothers to determine whether there is a relationship between the levels of these chemicals and the chances of the baby having birth defects. Knowing the potential for birth defects can sometimes give doctors a chance to intervene to repair the defect. One way to discover such relationships is to collect data on the levels of these chemicals for different expectant mothers. Graphing the data helps scientists visually discover any relationships that may exist. The next example illustrates this method.

example **4** **Levels of Enzymes in Expectant Mothers**

A biologist measures the levels of three different enzymes (call them A, B, and C) in 20 blood samples taken from expectant mothers. The data she obtains are given in the table, where enzyme levels are in milligrams per deciliter (mg/dL). The biologist wishes to determine whether there is a relationship between the levels of the different enzymes.

(a) Make a scatter plot of the levels of the pairs of enzymes A and B.

(b) Make a scatter plot of the levels of the pairs of enzymes A and C.

(c) What relationships do you see in the data?

Sample	A	B	C	Sample	A	B	C
1	1.3	1.7	49	11	2.2	0.6	25
2	2.6	6.8	22	12	1.5	4.8	32
3	0.9	0.6	53	13	3.1	1.9	20
4	3.5	2.4	15	14	4.1	3.1	10
5	2.4	3.8	25	15	1.8	7.5	31
6	1.7	3.3	30	16	2.9	5.8	18
7	4.0	6.7	12	17	2.1	5.1	30
8	3.2	4.3	17	18	2.7	2.5	20
9	1.3	8.4	45	19	1.4	2.0	39
10	1.4	5.8	47	20	0.8	2.3	56

Solution

Scatter plots for parts (a) and (b) are shown in Figures 3 and 4. Note that each point plotted represents the results for one sample; for instance, Sample 1 had 1.3 mg/dL of enzyme A and 1.7 mg/dL of enzyme B, so we plot the point (1.3, 1.7) to represent this data pair.

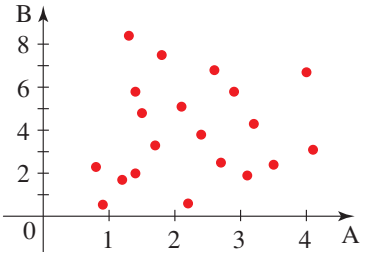


figure **3** Enzymes A and B

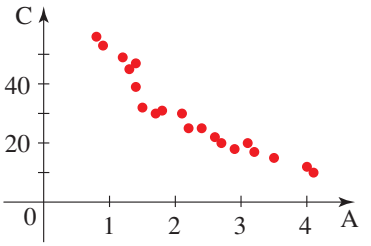


figure **4** Enzymes A and C

(c) From Figure 3 we see that there is no obvious relationship between the levels of enzymes A and B. From Figure 4 we see that when the level of enzyme A goes up, the level of enzyme C tends to goes down.

NOW TRY EXERCISES 29 AND 41

Reading a Graph

Data are often presented in the form of a graph. In the next example we read the data from a graph.

example 5 Reading a Graph

The average annual precipitation in Medford, Oregon is 21 inches. The graph in Figure 5 shows the total annual precipitation for the ten-year period 1996–2005. In the graph Year 1 corresponds to 1996, Year 2 to 1997, and so on.

(a) What was the precipitation in 2002?

(b) Which year(s) had a total precipitation of 29 in?

(c) Which year(s) had a total precipitation that was higher than average?

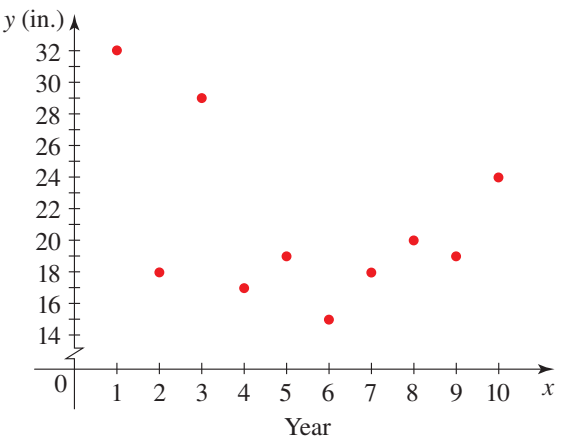
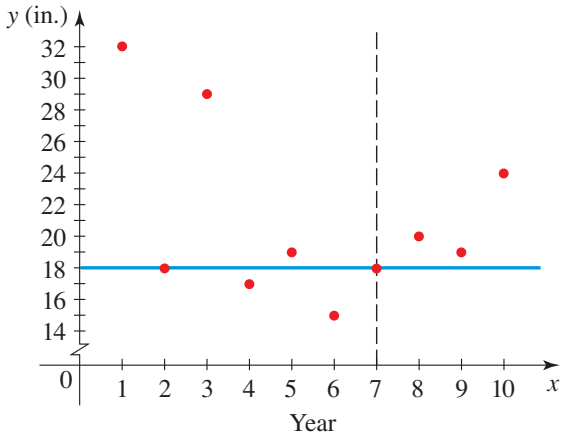


figure **5** Precipitation in Medford, Oregon

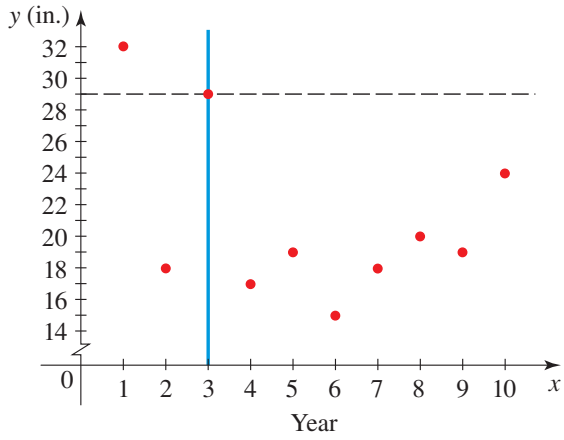
Solution

(a) The year 2002 is Year 7 on the graph. We need to find the height of the point in the graph above Year 7. From the graph in Figure 6(a) we see that the total rainfall was 18 in.

(b) Precipitation of 29 in. corresponds to the horizontal line in Figure 6(b). From the graph we see that only Year 3 had this level of precipitation, that is, the year 1998.



(a) Rainfall in Year 7



(b) Rainfall of 29 in.

figure **6**

(c) Since the average precipitation for this region is 21 in, we draw a horizontal line as in Figure 7. From Figure 7 we see that Years 1, 3, and 10 had more than 21 in. of precipitation. These years correspond to 1996, 1998, and 2005.

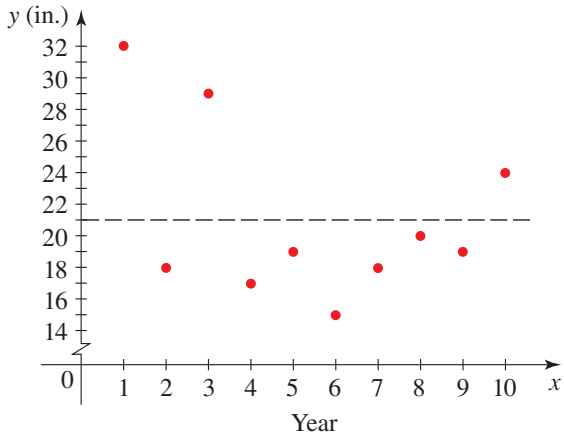
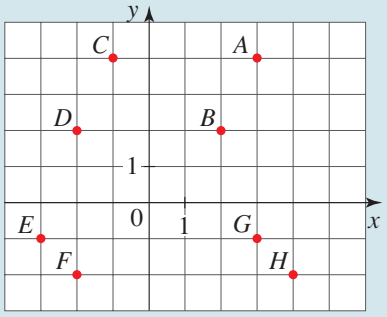


figure **7** Average rainfall is 21 in.

NOW TRY EXERCISES 15 AND 45

Algebra Checkpoint ✓



Test your knowledge of plotting points in a coordinate plane. You can review this topic in **Algebra Toolkit D.1** on page T67.

1. Plot the ordered pair in a coordinate plane.

(a) $(3, 4)$ **(b)** $(5, 1)$ **(c)** $(-2, 3)$ **(d)** $(-1, 1)$
(e) $(-2,-4)$ **(f)** $(5,-3)$ **(g)** $(-2, 5)$ **(h)** $(2, 1)$

2. Find the coordinates of the points shown in the figure to the left.

3–4 A set of points is given.

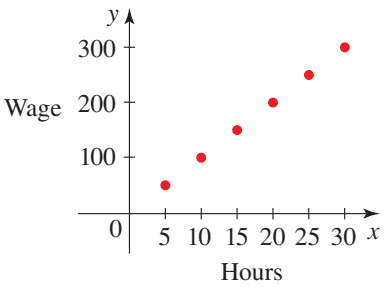
(a) Give a verbal description of the set.

(b) Graph the set in a coordinate plane.

3. $\{(x, y) \mid x = 2\}$ **4.** $\{(x, y) \mid y = -1\}$

1.2 Exercises

CONCEPTS



Fundamentals

1. The domain of a relation is the set of all _____, and the range of a relation is the set of all _____.

2. To express two-variable data as a relation, we represent the data as a set of _____ pairs.

3. To graph a relation, we graph each ordered pair by plotting the input on the _____-axis and the output on the _____-axis.

4. The scatter plot in the margin gives the relation between a worker's wages (in dollars) and the number of hours he works.

(a) What does the ordered pair (10, 100) represent?

(b) How many dollars does he earn when he works 20 hours?

(c) How many hours does he need to work to earn $100?

Think About It

5. Give examples of two-variable data where some inputs have more than one output.

6. Give examples of two-variable data where several inputs give the same output.

SKILLS

7–10 ■ A collection of ordered pairs defining a relation is given.

(a) Find the domain and range of the relation.

(b) Sketch a diagram of the relation.

7. $\{(1, 1), (2, 2), (3, 4), (4, 6)\}$

8. $\{(1, 0), (2, -1), (3, 0), (3, -1)\}$

9. $\{(5, 1), (3, 2), (-2, 5), (5, 5)\}$

10. $\{(4, 1), (4, 3), (5, 2), (6, 1)\}$

11–12 ■ Two-variable data are given.

(a) Express the data as a set of ordered pairs where x is the input and y is the output.

(b) Find the domain and range of the relation.

(c) Sketch a diagram of the relation.

(d) Find the output(s) corresponding to the input 5.

(e) Find the inputs(s) corresponding to the output 3.

11.

x	1	2	3	4	5	6
y	6	9	3	2	8	3

12.

x	1	2	3	5	5	6
y	7	2	8	3	10	10

13–14 ■ Two-variable data are given.

(a) Express the data as a relation where x is the input and y is the output.

(b) Find the domain and range of the relation.

(c) Sketch a diagram of the relation.

(d) Find the output(s) corresponding to the input 70.

(e) Find the inputs(s) corresponding to the output 20.

13.

x	10	20	30	40	70
y	20	20	20	70	20

14.

x	10	20	40	50	70
y	40	50	60	50	40

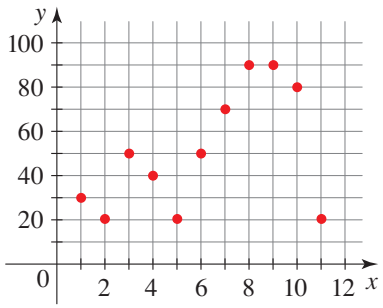
15–18 ■ The graph of a relation is given.

(a) List three ordered pairs in the relation.

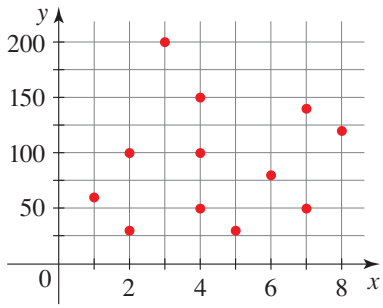
(b) Find the output(s) corresponding to the input 4.

(c) Find the input(s) corresponding to the output 50.

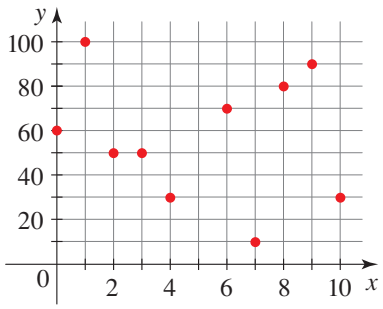
15.



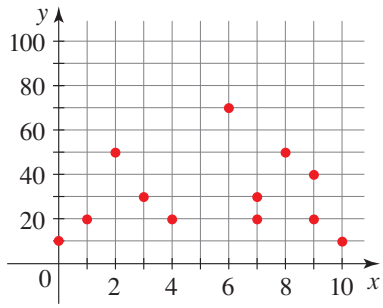
16.



17.



18.



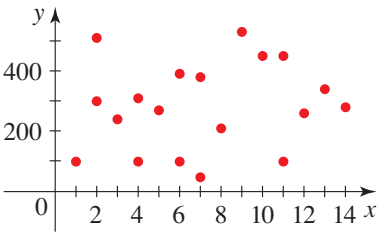
19–22 ■ The graph of a relation is given. Match the description of the relation to the appropriate graph.

19. The y values get larger as the x values get larger.

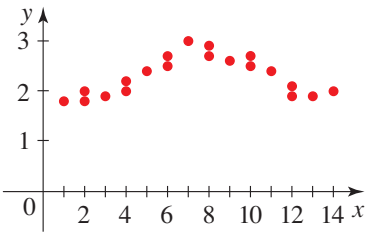
20. The y values get smaller as the x values get larger.

21. The y values get larger and then get smaller as the x values get larger.

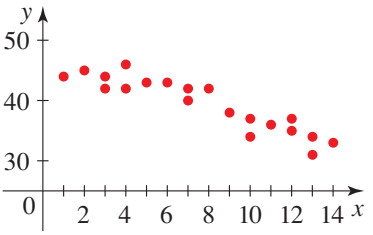
22. There is no obvious relationship between x and y.



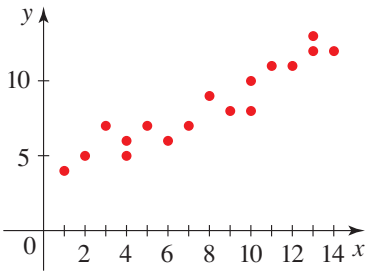
Graph A



Graph B



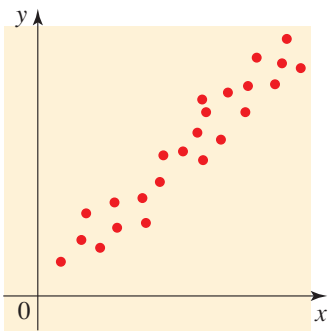
Graph C



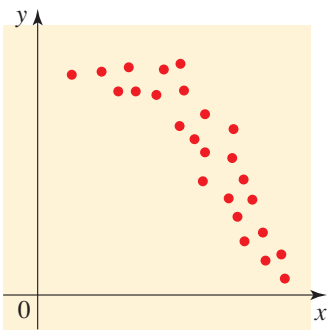
Graph D

23–26 ■ For each scatter plot, decide whether there is a relationship between the variables. If there is, describe the relationship.

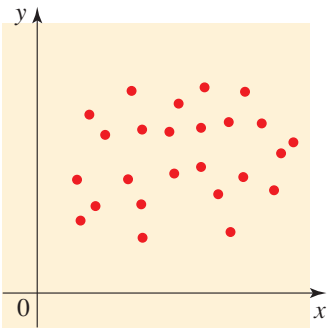
23.



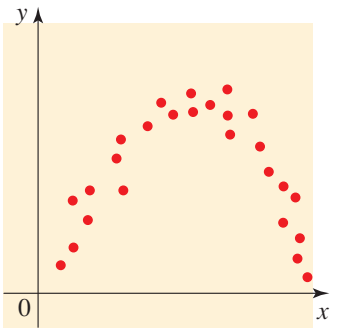
24.



25.



26.



27–28 ■ A table of data is given.

(a) Make a scatter plot of the data in the given table.

(b) Determine whether there is a relationship between the variables. If there is, describe the relationship.

27.

x	2	4	6	8	10	12	14
y	1	3	6	10	12	20	21

28.

x	2	4	3	1	3	5	2	1
y	10	6	7	10	9	2	1	5

29–30 ■ A table of data is given.

(a) Make three scatter plots: one for A and B, one for B and C, and one for A and C.

(b) From each scatter plot, determine whether there is a relationship between the variables. If there is, describe the relationship.

29.

A	B	C
6	8	40
4	8	50
5	3	45
7	2	20
8	5	10
6	4	25
6	7	30
5	3	40

30.

A	B	C
1	5	12
3	2	4
2	3	5
6	6	14
8	1	2
4	5	11
2	4	8
5	9	16

CONTEXTS

31–34 ■ Answer the following questions for the data in the indicated exercise from Section 1.1.

(a) Express the data as a relation.

(b) Draw a diagram of the data.

(c) Find the domain and range of the relation.

31. Exercise 25

32. Exercise 26

33. Exercise 27

34. Exercise 28

35. Income and Home Prices The median incomes and median home prices in several neighborhoods across the United States are shown in the table.

Median income	Median home price
80,000	400,000
30,000	250,000
70,000	300,000
55,000	250,000
80,000	450,000
60,000	300,000
55,000	300,000
150,000	1,500,000

(a) Express the data as a relation.

(b) Draw a diagram of the relation.

(c) What does the ordered pair (55,000, 250,000) represent?

(d) Find the output(s) corresponding to the input 80,000.

(e) Find the domain and range of the relation.

36. Student Directory The student directory at a university lists the last four digits of each student's ID number and the number of years the student has completed. The table on the next page contains some of this information.

(a) Express the data as a relation.

(b) Draw a diagram of the relation.

(c) What does the ordered pair (3371, 5) represent?

(d) Find the output(s) corresponding to the input 6731.

(e) Find the domain and range of the relation.

ID number	8271	4357	6731	3642	3371	5291	2273	1942
Year completed	1	2	1	3	5	1	4	2

37. Hybrid Car Sales With rising gas prices, there has been increasing interest in hybrid cars. The table below shows the number of hybrid cars sold in some recent years.

(a) Express the data as a relation.

(b) What does the pair (2005, 213) represent?

(c) Find the domain and range of the relation.

Year	2000	2004	2005	2006	2007
Hybrid cars sold (× 1000)	78	83	213	244	350

38. Women in Government Women did not have the right to vote in the United States until 1920. Since then, women have played an increasing role in government. The table shows the number of women who served in the U.S. Senate in a given year.

(a) Express the data as a relation.

(b) What does the pair (2003, 14) represent?

(c) Find the domain and range of the relation.

Year	**Women in the U.S. Senate**	**Year**	**Women in the U.S. Senate**
1975	0	1993	7
1977	3	1995	9
1979	2	1997	9
1981	2	1999	9
1983	2	2001	14
1985	2	2003	14
1987	2	2005	14
1989	2	2007	16
1991	4	2009	17

39. Christmas Bird Count For the past 100 years ornithologists have traditionally made a worldwide bird count at Christmas time. The table below shows the number of house finches observed in a Christmas bird count in California.

(a) Make a scatter plot of the data in the table.

(b) What trends do you detect in the house finch population of California?



Steve Byland/Shutterstock.com 2009

Year	**Years since 1960**	**Bird count**
1960	0	33,621
1965	5	44,787
1970	10	53,838
1975	15	86,507
1980	20	73,767
1985	25	91,659
1990	30	87,632
1995	35	107,190
2000	40	69,733
2005	45	61,053

Year	Average litter size	Coyotes killed
1	3	310
2	4	250
3	5	360
4	4	280
5	8	570
6	9	640
7	7	550

40. Coyote Reproduction Coyotes live throughout North America. They have become a nuisance for pet owners and ranchers, and often trapping and killing them seems to be the only solution. The table in the margin gives data in a certain county for the average number of coyote pups in a litter in a given year and the number of coyotes killed in that year.

(a) Make a scatter plot of the relation between litter size and number of coyotes killed.

(b) What trend do you detect from your graph?



41. Blood Pressure Data A man has his blood pressure and weight measured at his annual physical exam. The table below shows these data for a 10-year period.

(a) Make a scatter plot of the year and the systolic pressure. Describe any trends you detect from the graph.

(b) Make a scatter plot of the weight and systolic pressure. Describe any trends you detect from the graph.

Year	1	2	3	4	5	6	7	8	9	10
Weight (lb)	150	152	156	162	180	183	169	166	165	162
Systolic pressure (mm Hg)	110	114	125	132	151	160	124	124	122	125

42. Life Expectancy and Demographics Over the course of the past century both population and average life expectancy have increased in the United States. It is interesting to see how the age distribution of the population (demographics) is affected by these changes. The table below lists the percentage of the U.S. population that is between the ages of 30 and 39 years, the median age of the U.S. population, and the life expectancy at birth for a given year.

(a) Make a scatter plot of the data for the percentage of the U.S. population that is 30–39 and the life expectancy at birth. Describe any trends you detect from the graph.

(b) Make a scatter plot of median age and life expectancy of U.S. residents. Describe any trends you detect from the graph.

Year	Years since 1980	Percentage age 30–39	Median age	Life expectancy
1980	0	6.1	34.2	73.7
1990	10	7.9	34.9	75.4
1995	15	8.4	35.2	75.8
1998	18	8.3	35.3	76.7
1999	19	8.2	34.0	76.7
2000	20	8.0	36.5	76.7
2001	21	7.8	35.6	77.2
2002	22	7.6	35.7	77.3
2003	23	7.4	35.9	77.6
2004	24	7.1	36.0	77.9

43. Women in the Work Force The disparity between the median incomes of men and women has narrowed significantly in the last two decades. The table on the next page shows the median income of men and women over a 40-year period.

(a) On the same graph, make scatter plots of the data for the yearly median income of men and the yearly median income of women.

(b) In what year did the median income of men go down and the median income of women go up?

(c) Describe any trends you detect from the graphs.

Year	Median income of men (× $1000)	Median income of women (× $1000)
1965	28.599	9.533
1975	33.148	12.697
1985	42.847	27.720
1995	39.186	27.990
2005	41.386	31.858

44. Health Care Coverage The table shows the percentage of U.S. residents that were unemployed and the percentage that did not have health insurance from 1999 to 2004.

(a) On the same graph, make a scatter plot of the yearly percentages of unemployed and the yearly percentages of uninsured Americans.

(b) In what year(s) did the percentage of uninsured Americans go up while unemployment went down?

(c) Describe any trends you detect from the graph in part (a).

Year	Percentage unemployed	Percentage with no health insurance
1999	4.2	13.1
2000	4.0	13.3
2001	4.7	14.0
2002	5.8	14.7
2003	6.0	15.2
2004	5.5	15.3

45. Pan Evaporation and Climate Change For decades, water evaporation rates have been measured worldwide by using a system called *pan evaporation*. Pan evaporation is the measurement of the amount of water that evaporates from a standard pan in a given period of time. Historically (and still today), pan evaporation data were used in planning irrigation schedules for crops. The availability of long-term pan evaporation data makes it possible to detect global climate trends. The following graph plots the yearly pan evaporation (in./year) for Fresno, California, from 1940 to 2000.



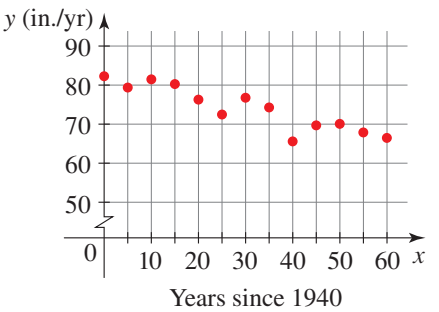
NOAA

(a) What was the pan evaporation in 1940?

(b) What year had a pan evaporation of 65 inches?

(c) What years had a pan evaporation above 80 inches?

(d) What trend do you detect in your graph?



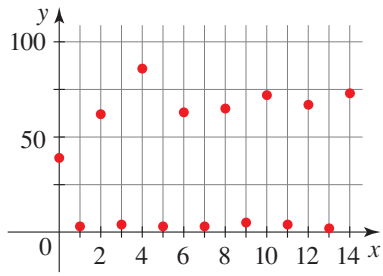
46. Wintering Habits of the Common Redpoll The common redpoll is one of those species of birds that shift from their typical winter region when there is a lack of food in their wintering grounds. They "irrupt" from their normal winter habitat in Canada into areas where food is more plentiful; their irruptions range as far south as the Middle Atlantic states. The graph below shows the percentage of feeders visited by common redpolls in the Mid-Atlantic states in the years between 1990 and 2005.

(a) What percentage of feeders was visited in 2002?

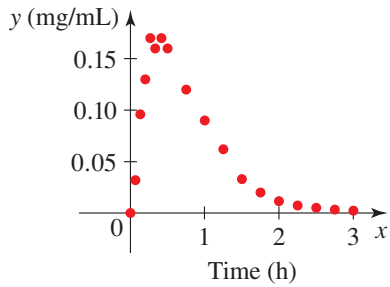
(b) In what year(s) were over 75% of the feeders visited?

(c) In what years did the common redpoll irrupt from its wintering grounds?

(d) Do you detect a trend in the common redpoll wintering habits? If so, describe the trend.



47. Algebra and Alcohol A scatter plot of the data in the Prologue (page P2) for the average concentration of alcohol in an individual after a 15-mL dose is shown in the graph. What pattern or trend do you see in these data? Compare with your answer to Exercise 31 of Section 1.1.



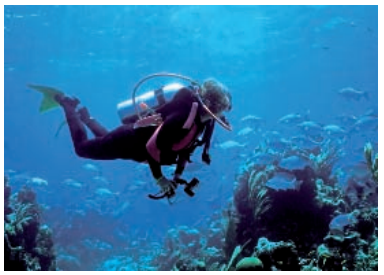
1.3 Equations: Describing Relationships in Data

- **Making a Linear Model from Data**
- **Getting Information from a Linear Model**

IN THIS SECTION... we learn how to model data by an equation and how the equation allows us to *predict* data points whose input is outside the domain of our data. This begins our study of modeling—a theme that we encounter throughout this book.

GET READY... by reviewing how to graph two-variable equations in **Algebra Toolkit D.2**. Test your graphing skills by doing the **Algebra Checkpoint** at the end of this section.

In the preceding sections we described patterns in data using words or graphs. If we use letters to represent the variables, we can sometimes find an *equation* that describes or "models" the data precisely.


Dennis Sabo/Shutterstock.com 2009

For example, an ocean diver observes that the deeper she dives, the higher the water pressure—she can feel the water pressing on her ears. How deep can she dive before the pressure becomes dangerously high? To answer this question, she must be able to predict what the pressure is at different depths without having to endanger her life by diving to these depths. So she begins by diving to safe depths and measuring the water pressure. The data she obtains help her find a model (or equation) that she can use to *predict* the pressure at depths to which she cannot possibly dive. This situation is summarized as follows.

- **The data** give the water pressure at different depths.
- **The model** is an equation that represents the data.
- **Our goal** is to use the model to predict the pressure at depths that are not in the data.

The pressure-depth data and model are given below. Note that the single equation $P = 14.7 + 0.45d$ contains *all* the data and more. For instance, we can use this equation to predict the pressure at a depth of 200 ft, a value that is not available from the data.

Data

Depth (ft)	Pressure (lb/in^2)
0	14.7
10	19.2
20	23.7
30	28.2
200	?

Making a model →

← Using the model

Model

$P = 14.7 + 0.45d$

In this section we learn how to make such models for data. The depth-pressure model is obtained in Example 3.

Making a Linear Model from Data

A **model** is a mathematical representation (such as an equation) of a real-world situation. **Modeling** is the process of finding mathematical models. Once a model is found, it can be used to answer questions about the thing being modeled.

Many real-world data start with an initial value for the output variable, and then a fixed amount is added to the output variable for each unit increase in the input variable. For example, the production cost for manufacturing a certain number of cars consists of an initial *fixed cost* for setting up the equipment plus an additional *unit cost* for manufacturing each car. In these cases we use a *linear* model to describe the data.

Linear Models

Linear equations are studied in more detail in Section 2.2.

A **linear model** is an equation of the form $y = A + Bx$. In this model, A is the **initial value** of y, that is, the value of y when x is zero, and B is the constant amount by which y changes (increases or decreases) for each unit increase in x.

$$y = \overbrace{A}^{\text{Initial value of } y} + \overbrace{Bx}^{\text{Add } B \text{ for each unit change in } x}$$

In the next three examples we find some linear models from data.

example 1 Data, Equation, Graph

table 1

x (chairs)	C (dollars)
0	80
1	92
2	104
3	116
4	128

A furniture maker collects the data in Table 1, giving his cost C of producing x chairs.

(a) Find a linear model for the cost C of making x chairs.

(b) Draw a graph of the equation you found in part (a).

(c) What does the shape of the graph tell us about his cost of making chairs?

Solution

(a) The initial cost (the cost of producing zero chairs) is \$80. We can see from the table that each chair produced costs an additional \$12. That is, the unit cost is \$12. So an equation that models the relationship between C and x is

$$C = \overbrace{80}^{\text{Initial cost (or fixed cost)}} + \overbrace{12x}^{\text{Add \$12 for each chair produced}}$$

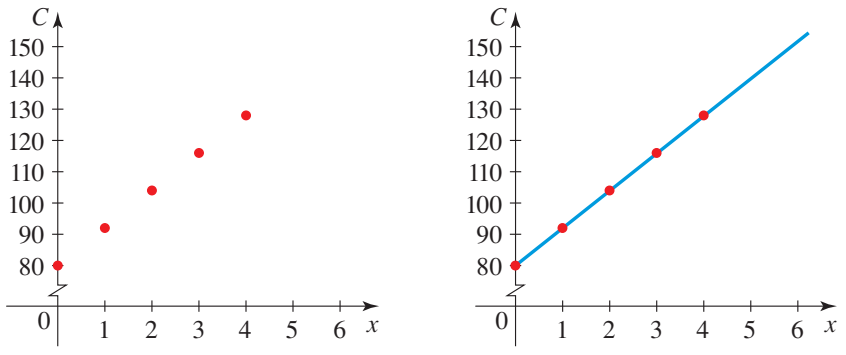
✓ CHECK To check that this equation correctly models the data, let's try some values for x. If four chairs are produced, then we can use the equation to calculate the cost.

$$\begin{aligned} C &= 80 + 12x && \text{Model} \\ C &= 80 + 12(4) && \text{Replace } x \text{ by } 4 \\ &= 128 && \text{Calculate} \end{aligned}$$

This matches the cost given in the table for making four chairs. You can check that the other values in the table also satisfy this equation.

Graphing equations is reviewed in Algebra Toolkit D.2, page T71.

(b) The ordered pairs in the table are solutions of the equation, so we plot them in Figure 1(a). We can see that the points lie on a straight line, so we complete the graph of the equation by drawing the line containing the plotted points as in Figure 1(b).



(a) Graph from table (b) Graph of equation

figure 1

(c) From the graph, it appears that cost increases steadily as the number of chairs produced increases.

■ NOW TRY EXERCISES 7 AND 19 ■

How can we tell whether a given set of data has a linear model? Let's consider data sets whose inputs are *evenly spaced*. For instance, the inputs 0, 1, 2, 3, . . . in Example 1 are evenly spaced—successive inputs are one unit apart. For data with evenly spaced inputs, there is a linear model for the data if the outputs increase (or decrease) by a constant amount between successive inputs. We can test whether data satisfy this condition by making a table of **first differences**. Each entry in the first difference column of the table is the difference between an output and the immediately preceding output.

First Differences

For data with evenly spaced inputs:

- The **first differences** are the differences in successive outputs.
- If the first differences are constant, then there is a linear model for the data.

The next example illustrates how we make and use a first difference table.

example 2 A Model for Temperature and Elevation

table **2**

Elevation (km)	Temperature (°C)
0	20
1	10
2	0
3	−10
4	−20
5	−30

A mountain climber knows that the higher the elevation, the colder is the temperature. Table 2 gives data on the temperature at different elevations above ground level on a certain day, gathered by using weather balloons.

(a) Show that a linear model is appropriate for these data.

(b) Find a linear model for the relationship between temperature and elevation.

(c) Draw a graph of the equation you found in part (b).

Solution

(a) We first observe that the inputs for these data are evenly spaced. To see whether a linear model is appropriate, let's make a table of first differences. The entries in the first difference column are obtained by subtracting from each output the preceding output.

$10 - 20 = -10$

$0 - 10 = -10$

$-10 - 0 = -10$

$-20 - (-10) = -10$

$-30 - (-20) = -10$

Elevation (km)	Temperature (°C)	First difference
0	20	—
1	10	−10
2	0	−10
3	−10	−10
4	−20	−10
5	−30	−10

We see that the first differences are constant (each is −10), so there is a linear model for these data.

(b) The linear model we seek is an equation of the form

$$T = A + Bh$$

where T represents temperature and h elevation.

When h is zero (ground level), the temperature is 20°C, so the initial value A is 20. The first differences are the constant -10, so the number B in the model is -10. We can now express the model as

$$T = \overbrace{20}^{\text{Temperature at elevation 0}} - \overbrace{10h}^{\text{Subtract 10 for each kilometer of elevation}}$$

Notice how this equation fits the data. From the data we see that for each 1-km increase in elevation, the temperature *decreases* by 10°C. So at an elevation of h km we must subtract $10h$ degrees from the ground temperature.

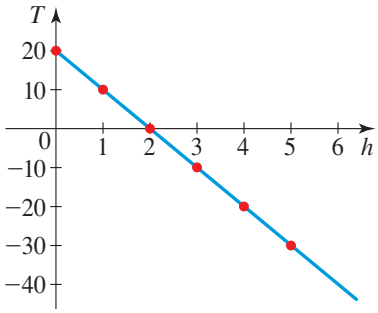


figure **2**
Graph of $T = 20 - 10h$

✓ CHECK To check that this equation correctly models the data, we try some values for h from the table. For example, if the elevation is 5 km, we replace h by 5 in the model:

$$T = 20 - 10h \qquad \text{Model}$$
$$T = 20 - 10(5) \qquad \text{Replace } h \text{ by } 5$$
$$T = -30 \qquad \text{Calculate}$$

This matches the temperature given in the table for an elevation of 5 km. You can check that the other values in the table also satisfy this equation.

(c) We plot the points in the table and then complete the graph by drawing the line that contains the plotted points. (See Figure 2.)

■ **NOW TRY EXERCISES 11 AND 21** ■

example **3**

A Model for Depth-Pressure Data

table **3**

Depth (ft)	Pressure (lb/in^2)
0	14.7
10	19.2
20	23.7
30	28.2
40	32.7
50	37.2

A scuba diver obtains the depth-pressure data shown in Table 3.

(a) Show that a linear model is appropriate for these data.

(b) Find a linear model that describes the relationship between depth and pressure.

Solution

(a) The inputs for these data are evenly spaced. We make a table of first differences.

Depth (ft)	Pressure (lb/in^2)	First differences
0	14.7	—
10	19.2	4.5
20	23.7	4.5
30	28.2	4.5
40	32.7	4.5
50	37.2	4.5

We see that the first differences are constant (each is 4.5), so there is a linear model for these data.

(b) The linear model we seek is an equation of the form

$$P = A + Bd$$

where P represents pressure and d represents depth.

When the depth d is 0, the pressure is 14.7 lb/in^2, so the initial value A is 14.7. From the first difference column in the table we see that pressure increases by 4.5 lb/in^2 for each 10-ft increase in depth. So for each 1-ft increase in depth the pressure increases by

$$\frac{4.5}{10} = 0.45 \text{ lb/in}^2$$

So the number B in the model is 0.45. We can now express the model as

$$P = \overbrace{14.7}^{\text{Pressure at depth 0}} + \overbrace{0.45d}^{\text{Add 0.45 lb/in}^2\text{ for each foot of depth}}$$

If d is 0, then $P = 14.7 + 0.45(0) = 14.7$.
If d is 10, then $P = 14.7 + 0.45(10) = 19.2$.
If d is 30, then $P = 14.7 + 0.45(30) = 28.2$.

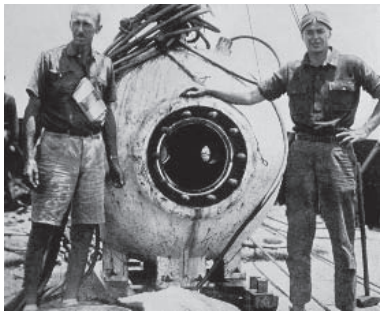
✓ CHECK Let's check whether this model fits the data. For instance, when d is 20, we get $P = 14.7 + 0.45(20) = 23.7$, which agrees with the table. In the margin we check the model against other entries in the table.

NOW TRY EXERCISE 23

Getting Information from a Linear Model

The point of making a model is to use it to predict conditions that are not directly observed in our data.

IN CONTEXT ➤



© Ralph White/CORBIS

In the next example we use the depth-pressure model of Example 3 to find the pressure at depths to which no human can dive unaided. This illustrates the power of the modeling process: It allows us to explore properties of the real world that are beyond our physical experience. In the first-ever attempt to explore the ocean depths, Otis Barton and William Beebe built a steel sphere (see the photo) with a diameter of 4 ft 9 in., which they called the bathysphere (*bathys* is the Greek word for deep). They needed to build their craft to be strong enough to withstand the crushing water pressure at the great depths to which they planned to descend. They used the depth-pressure model to estimate the pressure at those depths and then built the bathysphere accordingly. On August 15, 1934, they successfully descended to a depth of 3028 ft below the surface of the Atlantic. From the bathysphere's portholes they observed exciting new marine species that had never before been seen by humans.

example 4 Using the Depth-Pressure Model

The bathysphere described above is lowered to the bottom of a deep ocean trench. Use the depth-pressure model $P = 14.7 + 0.45d$ to predict the pressure at a depth of 3000 ft.

Solution

Since the depth is 3000 ft, we replace d by 3000 in the model and solve for P:

$P = 14.7 + 0.45d$	Model
$P = 14.7 + 0.45(3000)$	Replace d by 3000
$P = 1364.7$	Calculate

So the pressure is 1364.7 lb/in^2.

NOW TRY EXERCISE 25

Algebra Checkpoint ✓

Test your skill in graphing equations in two variables. You can review this topic in **Algebra Toolkit D.2** on page T71.

1. An equation is given. Determine whether the given point (x, y) is a solution of the equation.

(a) $y = 5x - 3$; $(3, 12)$ **(b)** $5x - 2y = 4$; $(2, 3)$

(c) $y = 25 - 3x^3$; $(-2, 1)$

2. An equation is given. Determine whether the given point (x, y) is on the graph of the equation.

(a) $y = 3x - 7$; $(2, -1)$ **(b)** $3y - x = -6$; $(2, 0)$

(c) $y = 2x^2 + 1$; $(-1, 3)$

3. An equation is given. Complete the table and graph the equation.

(a) $y = 2 + x$

x	0	1	2	3	4	5
y	2					

(b) $y = 1 - 3x$

x	-2	-1	0	1	2	3
y	7					

(c) $y = -7 + 4x$

x	0	1	2	3	4	5
y	-7					

(d) $y = -3 - 2x$

x	-3	-2	-1	0	1	2
y	3					

4. Find the x- and y-intercepts of the given equation.

(a) $y = x - 2$ **(b)** $2y = 3x + 2$ **(c)** $2x - 6y = 12$ **(d)** $4x - 5y = 8$

5. Graph the given equation, and find the x- and y-intercepts.

(a) $y = 2x$ **(b)** $y = 6 + 2x$ **(c)** $y = -6 + 3x$ **(d)** $3y = 6 - 2x$

1.3 Exercises

CONCEPTS

x	y	First difference
0	45	—
1	39	
2	33	
3	27	
4	21	
5	15	

Fundamentals

1. **(a)** What is a model? Give some examples of models you use every day.

(b) If you work for \$15 an hour, describe the relation between your pay and the number of hours you work, using (i) a table, (ii) a graph, (iii) an equation.

2. For data with evenly spaced inputs, if the first differences are _____, then a linear model is appropriate for the data. In the data shown in the margin, x represents the input and y represents the output. What are the first differences? Is a linear model appropriate?

3. The equation $L = 4S$ is a linear model for the total number of legs L that S sheep have. Using the model, we find that 12 sheep have $L = 4 \times \square = \square$ legs.

4. Suppose digital cable service costs \$49 a month with an initial installation fee of \$110. We make a linear model for the total cost C of digital cable service for x months by writing the equation _____.

Think About It

5. What is the purpose of making a model? Support your answer by examples.

6. Explain how data, equations, and graphs work together to describe a real-world situation. Give an example of a real-world situation that can be described in these three ways.

SKILLS

7–10 ■ A set of data is given.

(a) Find a linear model for the data.

(b) Use the model to complete the table.

(c) Draw a graph of the model.

7.

x	y
0	5
1	12
2	19
3	26
4	
5	
6	

8.

u	v
0	110
1	98
2	86
3	74
4	
5	
6	

9.

A	B
0	55
1	52
2	49
3	46
4	
5	
6	

10.

a	b
0	−10
1	−4
2	2
3	8
4	
5	
6	

11–14 ■ A set of data is given.

(a) Find the first differences.

(b) Is a linear model appropriate? If so, find a linear model for the data.

(c) If there is a linear model, use it to complete the table.

11.

x	y	First difference
0	205	
1	218	
2	231	
3	244	
4		
5		
6		

12.

x	y	First difference
0	60	
1	54	
2	48	
3	42	
4		
5		
6		

13.

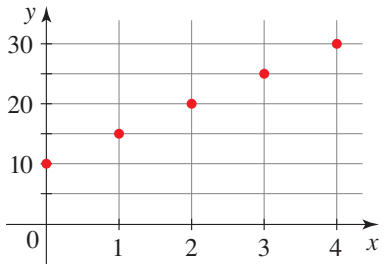
x	y	First difference
0	23	
1	19	
2	16	
3	11	
4		
5		
6		

14.

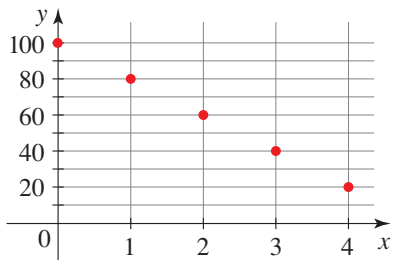
x	y	First difference
0	17	
1	38	
2	59	
3	80	
4		
5		
6		

15–18 ■ Find a linear model for the data graphed in the scatter plot.

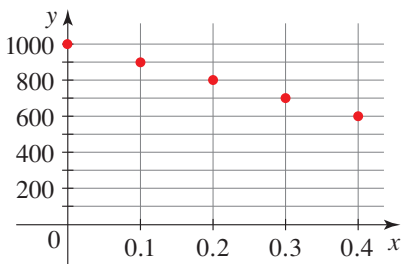
15.



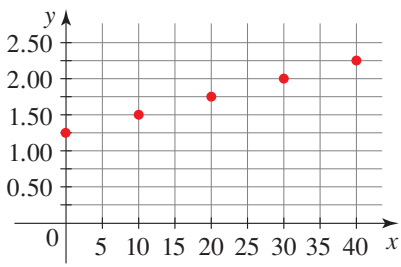
16.



17.



18.



CONTEXTS

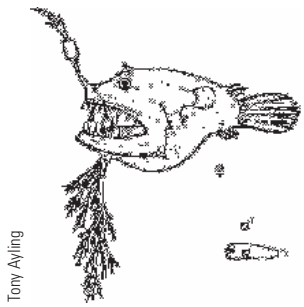
19. Truck Rental A home improvement store provides short-term truck rentals for their customers to take large items home. The store charges a base rate of $19 plus a time charge for every half hour that the truck is used. The table gives rental rate data for different rental periods.

(a) Find a linear model for the relation between the rental cost and rental period (in hours).

(b) Draw a graph of the equation you found.

(c) Use the model to predict the rental cost for 5 hours.

Period (h)	Rental cost ($)
0.0	19.00
0.5	24.00
1.0	29.00
1.5	34.00
2.0	39.00
2.5	44.00
3.0	49.00



Tony Ayling

Male and female angler fish

20. Aquatic Life in the Midnight Zone In Example 6 we used the model $P = 14.7 + 0.45d$ to find the pressure (lb/in^2) at various ocean depths. The deepest ocean trenches plunge to an astounding 7 miles below sea level, far too deep for sunlight to penetrate. Yet our planet is so teeming with life that even at these depths there are living creatures. Anglerfish can live at depths up to 11,000 ft and are characterized by luminescent appendages, which they use to lure their prey. The scientific name of the bizarre-looking anglerfish shown here, *linophryne arborifera*, means, roughly, "toad that fishes with a tree-like net."

(a) Use the model to predict the pressure at the 11,000-ft depth where anglerfish can live.

(b) One atmosphere (atm) is defined as a pressure of 14.7 lb/in^2, which is the normal air pressure we experience at sea level. Convert your answer in part (a) to atmospheres. How many times greater is the pressure under which anglerfish live than the pressure under which we live?

21. Boiling Point Most high-altitude hikers know that cooking takes longer at higher elevations. This is because the atmospheric pressure decreases as the elevation increases, causing water to boil at a lower temperature, and food cooks more slowly at that lower

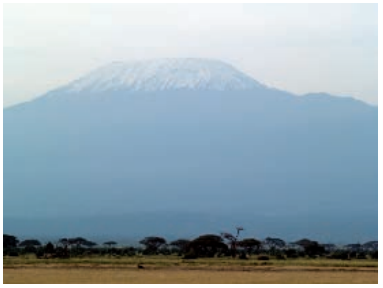
temperature. The table below gives data for the boiling point of water at different elevations.

(a) Use first differences to show that a linear model is appropriate for the data.

(b) Find a linear model for the relation between boiling point and elevation.

(c) Use the model to predict the boiling point of water at the peak of Mount Kilimanjaro, 19,340 ft above sea level.

Elevation (× 1000 ft)	Boiling point (°F)	First difference
0	212.0	—
1	210.2	
2	208.4	
3	206.6	
4	204.8	
5	203.0	



Peter Zaharov/Shutterstock.com 2009

Mount Kilimanjaro

22. Temperatures on Mount Kilimanjaro Mount Kilimanjaro is the highest mountain in Africa. Its snow-covered peak rises 4800 m above the surrounding plain. It is located in northern Tanzania near the equator, and conditions on the mountain vary from equatorial, to tropical, to arctic, because of the steadily decreasing temperature as the altitude increases. The table below gives data for the temperature on a typical day on Kilimanjaro at various elevations above the base of the mountain.

(a) Use first differences to show that a linear model is appropriate for the data.

(b) Find a linear model for the relation between the temperature on Kilimanjaro and the elevation above the base of the mountain.

(c) Use the model to predict what the temperature will be on a typical day at the peak of Kilimanjaro.

Elevation above base (m)	Temperature (°C)	First difference
0	30	—
400	28	
800	26	
1200	24	
1600	22	
2000	20	

23. Chocolate-Powered Car Two British entrepreneurs, Andy Pag and John Grimshaw, drove 4500 miles from England to Timbuktu, Mali, in a truck powered by chocolate. They used an ethanol that is made from old, unusable chocolate, and it took about 17 pounds of chocolate to make 1 gallon of ethanol. The table in the margin gives data for the relationship between the amount of chocolate used and the number of miles driven.

(a) Use first differences to show that a linear model is appropriate for the data.

(b) Find a linear model for the relation between the amount of chocolate used and the number of miles driven.

(c) Use the model found in part (b) to predict how many pounds of chocolate it took to drive from England to Timbuktu.

Miles driven	Pounds of chocolate used
0	0
20	17
40	34
60	51
80	68
100	85

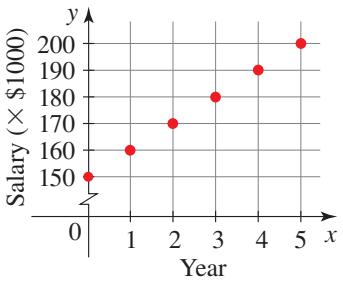
Number	Profit ($)
0	−80.00
10	−40.00
20	0.00
30	40.00
40	80.00
50	120.00

24. Profit An Internet company sells cell phone accessories. The table in the margin gives the profit they make on selling battery chargers. (Note that negative numbers in the table represent a loss.) Because of storage costs, the company needs to sell at least 20 chargers before they begin to make a profit.

(a) Find a linear model for the relation between profit and the number of battery chargers sold.

(b) Draw a graph of the equation you found.

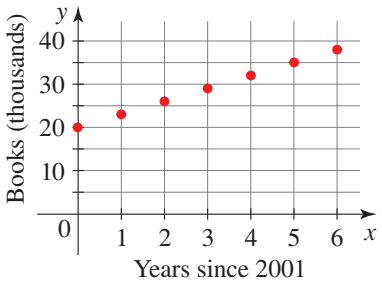
(c) Use the model to predict the profit from selling 150 battery chargers.



25. Salary A woman is hired as CEO of a small company and is offered a salary of $150,000 for the first year. In addition, she is promised regular salary increases. The graph in the margin shows her potential salary (in thousands of dollars) for the first few years that she works for the company.

(a) Find a linear model for the relation between her salary and the number of years she works for the company.

(b) Use the model to predict what her salary will be after she has worked 10 years for the company.



26. Library Book Collection A city library remodeled and expanded in the year 2001 and increased its maximum capacity to about 100,000 books. In 2001 the library held about 20,000 books, and each subsequent year the library adds a fixed number of books to its collection. The graph in the margin plots the number of books (in thousands) the library held each year from 2001 to 2007.

(a) Find a linear model for the relation between the number of books in the library and the number of years since 2001.

(b) Use the model to predict the number of books in the library after 25 years (in 2026).

1.4 Functions: Describing Change

- Definition of Function
- Which Two-Variable Data Represent Functions?
- Which Equations Represent Functions?
- Which Graphs Represent Functions?
- Four Ways to Represent a Function

IN THIS SECTION... we begin our study of functions. There are four basic ways to represent functions: words, data, equations, and graphs. The concept of function is a versatile tool for modeling the real world. In succeeding chapters we study more properties of functions; each new property provides a new modeling tool.

GET READY... by reviewing how to solve equations in **Algebra Toolkit C.1**. Test your skill by doing the **Algebra Checkpoint** at the end of this section.

In Sections 1.1–1.3 we saw how analyzing two-variable data can reveal relationships between the variables. Such relationships can be seen from the data themselves, from a graph, or from an equation. But this is not the whole story; in many real-world situations we are interested in how a *change* in one variable results in a *change* in the other variable. We'll study these types of relations using the concept of *function*. Equipped with this concept, we will be able make a great leap in our understanding of our ever-changing world.

Definition of Function

We use the term *function* to describe the dependence of one changing quantity on another. For example, we say that the height of a child is a *function* of the child's age, the weather is a *function* of the date, the cost of mailing a package is a *function* of its weight, and so on. Situations like these, which involve change, have the property that each input (or each value for the first variable) results in exactly one output (exactly one value for the other variable). For instance, mailing a package does not result in two different costs. This leads us to the following definition of function.

Definition of Function

A **function** is a relation in which each input gives exactly one output.

We can easily tell whether a relation is a function by making a diagram, as we did in Section 1.2. The diagram in Figure 1(a) represents a function because for each input there is exactly one output. But the relation described by the diagram in Figure 1(b) is not a function—the input 2 corresponds to two different outputs, 20 and 30.

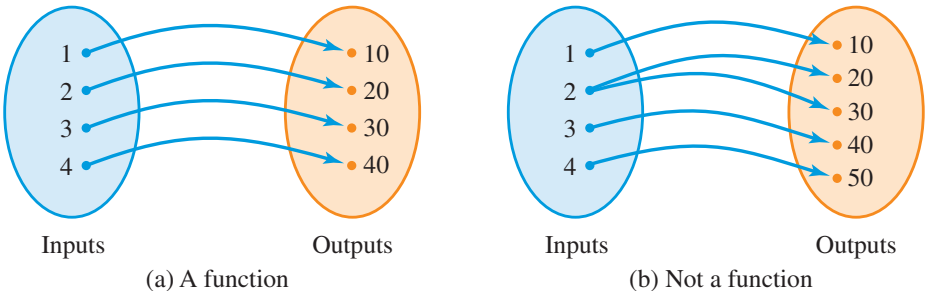


figure **1** When is a relation a function?

example 1 Which Relations Are Functions?

A relation is given by a table. The input is in the first column, and the output in the second column. Is the relation a function?

(a) Table 1 gives the number of women in the U.S. Senate between 1997 and 2007.

(b) Table 2 gives the ages of women in a certain neighborhood and the number of children each woman has.

table **1**
Women in U.S. Senate

Year	Number
1997	9
1999	9
2001	14
2003	14
2005	14
2007	16

table **2**
Number of children

Age	Number
31	3
32	0
24	1
35	1
31	2
22	0

Solution

(a) This relation is a function because each input (year) corresponds to exactly one output (the number of women in the Senate that year).

(b) This relation is not a function because the input 31 gives two different outputs (3 and 2).

NOW TRY EXERCISES 7 AND 9

Notice the difference between the two relations in Example 1. The first relation is a function, so the year *determines* the number of women in the senate. The second relation is not a function—the age of a woman does not determine how many children she has; women of the same age can have different numbers of children.

Which Two-Variable Data Represent Functions?

From the definition of function we see that two-variable data represent a function if to each value of the input variable there is exactly one value for the output variable. Since the output *depends* entirely on the input, we call the output variable the *dependent variable* and the input variable the *independent variable*.

Dependent and Independent Variables

1. **A variable y is a function of a variable x** if each value of x (the input) corresponds to exactly one value of y (the output). In this case we simply say

   "y is a function of x"

2. If y is a function of x, then the input variable x is called the **independent variable**, and the output variable y is called the **dependent variable**.

example 2 Independent and Dependent Variables

x	y
1	22
2	22
3	28
4	31
5	34
6	37

Two-variable data are given in the table in the margin.

(a) Is the variable y a function of the variable x? If so, which is the independent variable and which is the dependent variable?

(b) Is the variable x a function of the variable y? If so, which is the independent variable and which is the dependent variable?

Solution

(a) The variable y is a function of the variable x because each value of x corresponds to exactly one value of y. Since y is a function of x, the variable x is the independent variable (the input), and the variable y is the dependent variable (the output).

(b) The variable x is not a function of the variable y because when the input y is 22, there are two different outputs (1 and 2).

NOW TRY EXERCISE 11

Once we have determined that a variable y is a function of a variable x we can find the change in y as x changes. We find the **net change** in the variable y between the inputs a and b by subtracting the value of y at the input a from the value of y at the input b (where $a \le b$). Note that the net change is the change between two particular values of y; between these two values y could increase, then decrease, then increase again. But subtracting the value of y at the input a from the value of y at the input b gives the *net* change between these two points. The next example illustrates this concept.

example 3 Net Change in the Dependent Variable

x (year)	y (dollars)
1996	1.32
1997	1.33
1998	1.16
1999	1.36
2000	1.66
2001	1.64
2002	1.51
2003	1.83
2004	2.12
2005	2.17
2006	2.81

The table in the margin gives the annual average California gasoline price from 1996 to 2006, where x is the year and y is average price.

(a) Show that the variable y is a function of the variable x.

(b) Find the net change in the average California gasoline price from 1996 to 1998.

(c) Find the net change in the average California gasoline price from 1996 to 2006.

Solution

(a) The variable y is a function of the variable x because each value of x corresponds to exactly one value of y.

(b) Since the average California gasoline price was \$1.32 in 1996 and \$1.16 in 1998, the net change from 1996 to 1998 is

$$1.16 - 1.32 = -0.16$$

The negative sign means that there was a net *decrease* in the price of gas from 1996 to 1998.

(c) Since the average California gasoline price was \$1.32 in 1996 and \$2.81 in 2006, the net change from 1996 to 2006 is

$$2.81 - 1.32 = 1.49$$

This means that there was a net *increase* in the price of gas from 1996 to 2006.

■ NOW TRY EXERCISE 15 ■

Which Equations Represent Functions?

An equation in two variables defines a relation between the two variables—namely, the relation consisting of the ordered pairs that satisfy the equation. For example, some of the ordered pairs (x, y) in the relation determined by the equation $y = x^2$ are

$$(1, 1), (2, 4), (2.5, 6.25), (3, 9), (-2, 4)$$

Does this equation define the variable y as a function of the variable x? Since each value of the variable x corresponds to exactly one value of the variable y (namely, x^2), the equation does define a function. In general, we have the following.

Equations That Represent Functions

An equation in x and y defines y as a function of x if each value of x (the independent variable) corresponds via the equation to exactly one value of y (the dependent variable).

Notice that in the equation $y = x^2$ the variable x is not a function of the variable y because certain values of y determine more than one value of x. For example, $y = 4$ determines $x = 2$ and $x = -2$. The diagram below shows the difference.



y* is a function of *x ***x* is not a function of *y***

In other words, if we view x as the independent variable and y as the dependent variable, then the equation defines a function. On the other hand, if we choose to view y as the independent variable and x as the dependent variable then the equation does not define a function.

If an equation determines a function, we can write the equation in **function form**—that is, with the dependent variable alone on one side of the equation. For example, the following equation determines y as a function of x:

$$x^2 - y = 0 \qquad \text{Equation}$$

We express this equation in function form as

$$y = x^2 \qquad \text{Equation in function form}$$

In general, to determine whether an equation defines a function, we try to put the equation in function form. The next example illustrates this procedure.

example 4 Deciding Whether an Equation Defines a Function

Consider the equation $5z + 2w^2 = 8$.

(a) Does the equation define z as a function of w?

(b) Does the equation define w as a function of z?

Solution

Solving for one variable in terms of another is reviewed in Algebra Toolkit C.1, page T47.

(a) To answer this question, we write the equation in function form with the dependent variable z alone on one side:

$$\begin{aligned} 5z + 2w^2 &= 8 && \text{Equation} \\ 5z &= 8 - 2w^2 && \text{Subtract } 2w^2 \\ z &= \frac{8 - 2w^2}{5} && \text{Divide by 5} \end{aligned}$$

We see that z is a function of w, because for each value of w we can use the equation $z = \dfrac{8 - 2w^2}{5}$ to calculate exactly one corresponding value for z.

(b) We try to write the equation in function form with the dependent variable w alone on one side:

$$\begin{aligned} 5z + 2w^2 &= 8 && \text{Equation} \\ w^2 &= \frac{8 - 5z}{2} && \text{Subtract } 5z\text{, then divide by 2} \\ w &= \pm\sqrt{\frac{8 - 5z}{2}} && \text{Take the square root} \end{aligned}$$

We see from the last equation that w is not a function of z. For example, if z is 0, we get two corresponding values for w, namely,

$$w = \sqrt{\frac{8 - 5(0)}{2}} = 2 \qquad \text{and} \qquad w = -\sqrt{\frac{8 - 5(0)}{2}} = -2$$

NOW TRY EXERCISE 23

example 5 Deciding Whether an Equation Defines a Function

When the wind blows with speed v km/h, a windmill generates P watts of power according to the equation

$$P = 15.6v^3$$

(a) Show that P is a function of v.

(b) Find the net change in the power P as the wind speed v changes from 7 to 10 km/h.

(c) If possible, express v as a function of P.

(d) Find the net change in the wind speed v if the power generated changes from 1000 W to 10,000 W.

Solution

(a) P is a function of v because for each value of v the equation gives exactly one corresponding value for P.

(b) When v is 7, the power P is $15.6(7)^3 = 5350.8$; and when v is 10, the power P is $15.6(10)^3 = 15{,}600$. So the net change in the power is

$$15{,}600 - 5350.8 = 10{,}249.2 \text{ W}$$

(c) We need to rewrite the equation with v alone on one side:

$$P = 15.6v^3 \qquad \text{Equation}$$

$$\frac{P}{15.6} = v^3 \qquad \text{Divide by 15.6}$$

$$v = \sqrt[3]{\frac{P}{15.6}} \qquad \text{Take the cube root, and switch sides}$$

This equation does define v as a function of P because each real number has exactly one cube root.

(d) When P is 1000, the equation we got in part (c) gives

$$v = \sqrt[3]{\frac{P}{15.6}} = \sqrt[3]{\frac{1000}{15.6}} \approx 4.00$$

When P is 10,000, the equation we got in part (c) gives

$$v = \sqrt[3]{\frac{P}{15.6}} = \sqrt[3]{\frac{10{,}000}{15.6}} \approx 8.62$$

So the net change in the speed of the wind when the power P changes from 1000 W to 10,000 W is

$$8.62 - 4.00 = 4.62 \text{ km/h}$$

NOW TRY EXERCISE 57

Which Graphs Represent Functions?

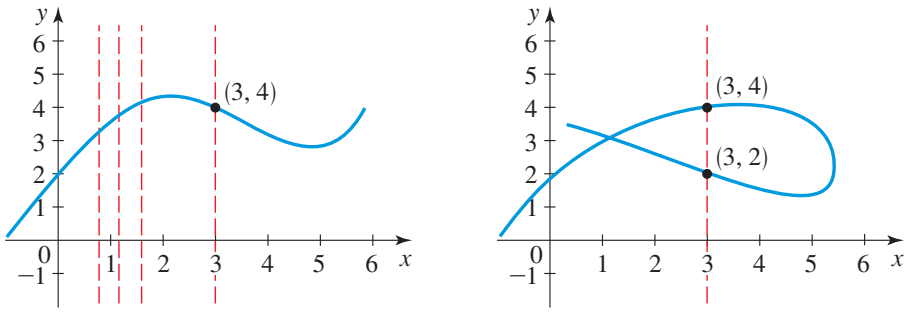
For a relation to be a function, each input must correspond to exactly one output. What does this mean in terms of the graph? It means that every vertical line intersects the graph in at most one point. Of course, we are using the convention that the inputs are graphed on the horizontal axis and the outputs are graphed on the vertical axis.

Vertical Line Test

A graph is the graph of a function if and only if no vertical line intersects the graph more than once.

So if any vertical line crosses the graph more than once, the graph is not the graph of a function. This is illustrated in Figure 2.

The graph in Figure 2(a) is the graph of a function because we can see that each input corresponds to exactly one output; for instance, the input 3 corresponds to the output 4. But the graph in Figure 2(b) is not the graph of a function; for instance, the input 3 corresponds to two different outputs: 2 and 4.



(a) Graph of a function (b) Not a graph of a function

figure **2** Vertical Line Test

example **6**

Cooling Coffee

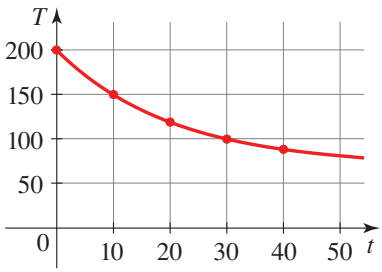


figure **3** Cooling coffee

A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70°F. The graph in Figure 3 shows the temperature T of the coffee after t minutes.

(a) Use the Vertical Line Test to show that T is a function of t.

(b) Find the net change in temperature during the first 10 minutes the coffee is cooling.

Solution

(a) We see that every vertical line intersects the graph exactly once, so the graph defines T as a function of t.

(b) From the graph we see that the temperature is 200°F at time 0 and 150°F 10 minutes later, so the net change in the temperature is $150 - 200 = -50$. The negative sign indicates that there was a net *decrease* in the temperature in the first 10 minutes.

■ **NOW TRY EXERCISE 29** ■

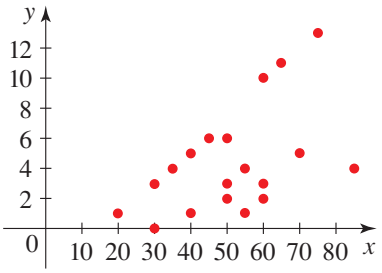


figure **4** Voter survey

example 7 Voter Survey

A local campaign in the Midwest surveyed voters on their frequency of voting. The graph in Figure 4 shows some of the survey data. Displayed is the age x of the voter surveyed and the number y of elections in which they have participated.

(a) Use the graph to find the outputs that correspond to the input 40.

(b) List the ordered pairs on the graph with input 40.

(c) Use the Vertical Line Test to show that the graph does not determine y as a function of x.

Solution

(a) From the graph we see that the input 40 corresponds to the outputs 1 and 5.

(b) The ordered pairs are (40, 1) and (40, 5).

(c) A vertical line at $x = 40$ intersects the graph at more than one point, so the graph is not the graph of a function.

NOW TRY EXERCISE 63

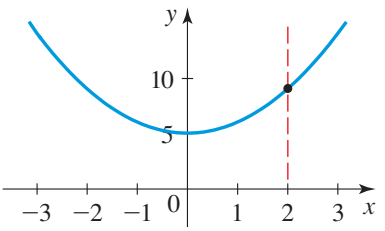
Notice that if a relation is not a function, it doesn't make sense to talk about net change. In Example 6 the age of a voter does not determine how many elections the person voted in, so a change in age does not determine a change in voting.

Another way to determine whether an equation defines a function is to graph the equation and then use the Vertical Line Test. For example, the graph of the equation $x^2 + y^2 = 25$ is a circle (see **Algebra Toolkit D.2**). You can easily check that a circle does not statisfy the Vertical Line Test, so this equation does not define a function. Here are some other examples.

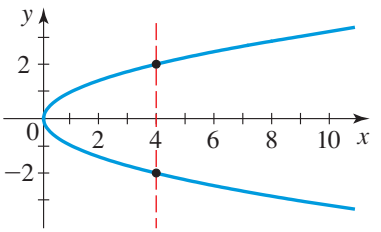
example 8 Using the Vertical Line Test

An equation and its graph are given. Use the Vertical Line Test to decide whether the equation defines y as a function of x. If the equation does not define a function, use the graph to help find an input that corresponds to more than one output.

(a) $y = x^2 + 5$



(b) $y^2 = x$



Solution

(a) We see from the figure that every vertical line intersects the graph exactly once, so the equation $y = x^2 + 5$ defines y as a function of x.

(b) The vertical line shown in the figure intersects the graph at two different points, so the equation $y^2 = x$ does not define y as a function of x. The input 4 corresponds to two different outputs. To find these outputs, we use

the equation. If x is 4, then the equation tells us that $y^2 = 4$, and so y is 2 or -2.

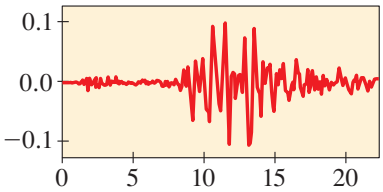
NOW TRY EXERCISE 33

Four Ways to Represent a Function

In Section 1.3 we represented relations in four different ways. Since functions are relations, they also can be represented in these ways.

- **Verbally** using words
- **Numerically** by a table of two-variable data
- **Symbolically** by an equation
- **Graphically** by a graph in a coordinate plane

Any function can be represented in all four ways, but some are better represented in one way rather than another. For example, a seismograph reading is a graph of the intensity of an earthquake as a function of time (see the figure in the margin). It would be difficult to find an equation that describes this function. We now give an example of a function that can be represented in all four ways.



Seismograph of Loma Prieta earthquake, 1989

example **9** **Representing a Function in Four Ways**

A pizza parlor charges \$10 for a plain cheese pizza and \$1.25 for each additional topping. The cost of a pizza is a function of the number of toppings. Express this function verbally, symbolically, numerically, and graphically.

Solution

- **Verbally** We can express this function verbally by saying "the cost of a pizza is \$10 plus \$1.25 for each additional topping."
- **Symbolically** If we let y be the cost of a pizza and x be the number of toppings, then we can express the function symbolically by

$$y = 10 + 1.25x$$

Here the independent variable is x, the number of toppings, and the dependent variable is y, the cost of the pizza.

- **Numerically** We make a table of values that gives a numerical representation of the function.

x	0	1	2	3	4
y	10.00	11.25	12.50	13.75	15.00

- **Graphically** We plot the data in the table to get the graphical representation shown in Figure 5.

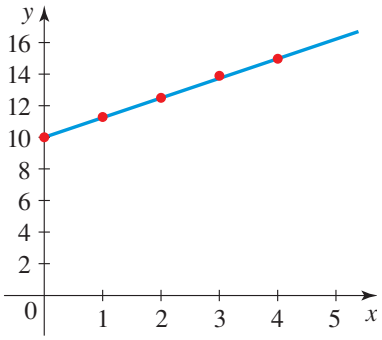


figure **5**
Graph of $y = 10 + 1.25x$

NOW TRY EXERCISE 55

Algebra Checkpoint ✓

Test your skill in solving equations. You can review this topic in **Algebra Toolkit C.1** on page T47.

1–4 Solve the equation for x.

1. $3x - 5 = 4$

2. $7 + 17x = 3x$

3. $7(x - 2) = 5x + 6$

4. $8 = 3(x + 2) - 12$

5–8 An equation in the variables x and y is given.

(a) Find the value(s) of y when x is 3.

(b) Find the value(s) of x when y is 2.

5. $y = 5 + 3x$

6. $x = y^2 - 1$

7. $y^3 = x^2 + 18$

8. $3x - y^2 = 0$

9–18 An equation in two variables is given. Solve the equation for the indicated variable.

9. $y = 5 + 3x; \quad x$

10. $P = 6T; \quad T$

11. $r - 2z = z; \quad z$

12. $8x - y^3 = 0; \quad y$

13. $\frac{2}{t} = 3x; \quad t$

14. $\frac{1}{r} = \frac{4}{p}; \quad p$

15. $st + 4t = 3s; \quad t$

16. $wh = 3h + 2; \quad h$

17. $y^2 - 9x = 0; \quad y$

18. $p - R^2 = -8p; \quad R$

1.4 Exercises

CONCEPTS

Fundamentals

1. (a) A relation is a function if each input corresponds to exactly one _____.

(b) If two-variable data are given in a table and the variable y is a function of the variable x, then _____ is the independent variable and _____ is the dependent variable.

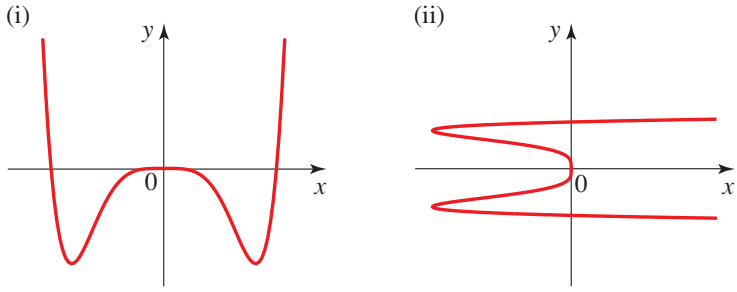
2. (a) To find out whether an equation in x and y determines y as a function of x, we first solve the equation for _____. If an equation defines y as a function of x, then how many y values correspond to each x value?

(b) The equation $y = x^2 + x + 1$ defines y as a function of x, so _____ is the independent variable and _____ is the dependent variable.

3. The equation $y = x^2 + 2$ defines y as a function of x. Then the net change of the variable y from $x = 0$ to $x = 4$ is _____ − _____ = _____.

4. (a) To determine whether a graph defines y as a function of x, we use the _____ _____ Test.

(b) Which of the following are graphs of functions of x?



Think About It

5. True or false?

(a) All relations are functions.

(b) All functions are relations.

(c) All equations in x and y determine y as a function of x.

(d) Some functions can be defined by an equation.

6. In this section we represented functions in four different ways. Think of a function that can be represented in these four ways, and write down the four representations.

SKILLS

7–10 ■ A set of ordered pairs defining a relation are given. Is the relation a function?

7. $\{(1, 2), (-1, 2), (3, 5), (-3, 5), (5, 8), (-5, 8)\}$

8. $\{(1, 4), (2, 6), (3, 1), (3, 7)\}$

9. $\{(3, 4), (4, -1), (3, 5), (-1, 5), (3, 9), (-2, 7)\}$

10. $\{(-2, -1), (-1\ , -1), (0, -1), (1, -1), (2, -1), (3, -1)\}$

11–14 ■ Two-variable data are given in a table.

(a) Is the variable y a function of the variable x? If so, which is the independent variable and which is the dependent variable?

(b) Is the variable x a function of the variable y? If so, which is the independent variable and which is the dependent variable?

11.

x	1	2	2	4	4
y	6	9	3	2	8

12.

x	0	1	2	3	4	5	6	7
y	6	6	6	6	6	6	6	6

13.

x	1	2	3	4	5	6
y	10	9	10	15	10	21

14.

x	0	1	1	9	9	81	81
y	0	−1	1	−3	3	−9	9

15–18 ■ Two-variable data are given in the table.

(a) Explain why the variable y is a function of the variable x.

(b) Find the net change in the variable y as x changes from 0 to 5.

(c) Find the net change in the variable y as x changes from 3 to 5.

15.

x	0	1	2	3	4	5
y	2	0	3	4	2	1

16.

x	0	1	2	3	4	5	6	7
y	6	6	6	6	5	5	5	5

17.

x	0	1	2	3	4	5	6
y	3	4	5	6	5	4	3

18.

x	0	1	2	3	4	5	6
y	2	4	7	10	6	3	1

19–22 ■ An equation is given in function form.

(a) What is the independent variable and what is the dependent variable?

(b) What is the value of the dependent variable when the value of the independent variable is 2?

19. $y = 3x^2 + 2$

20. $x = 2y + 1$

21. $w = 5l^2 + 3l^3$

22. $z = 5(y + 2)^2$

23–28 ■ An equation is given.

(a) Does the equation define y as a function of x?

(b) Does the equation define x as a function of y?

23. $2y + 3x = 0$

24. $2y + 3x^2 = 0$

25. $4x^3 + 2 = y^2$

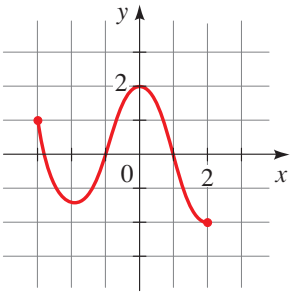
26. $(y + 3)^2 + 1 = x^3$

27. $(y + 3)^3 + 1 = 2x$

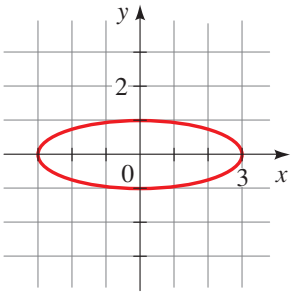
28. $\frac{x^3}{4} + 3y = 0$

29–32 ■ Use the Vertical Line Test to determine whether the graph determines y as a function of x. If so, find the net change in y from $x = -1$ to $x = 2$.

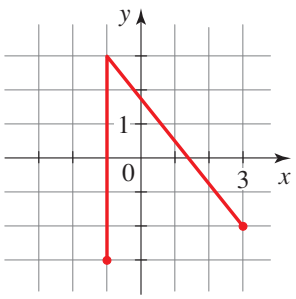
29.



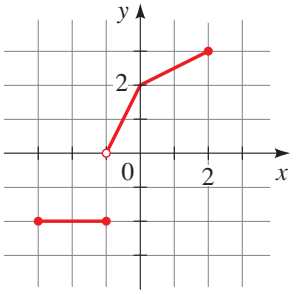
30.



31.

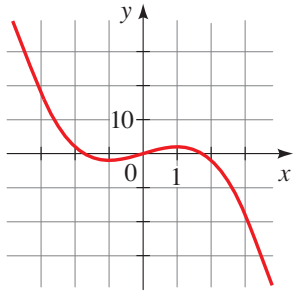


32.

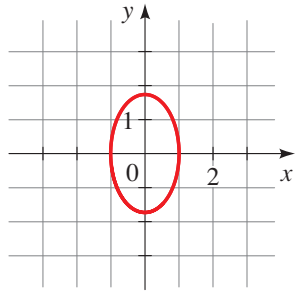


33–36 ■ An equation and its graph are given. Use the Vertical Line Test to determine whether the equation defines y as a function of x.

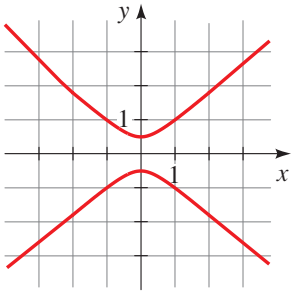
33. $y = 3x - x^3$



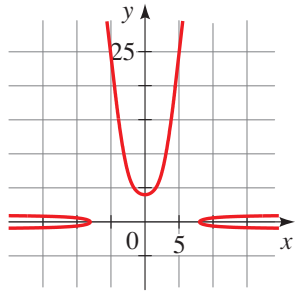
34. $x^2 + \frac{y^2}{3} = 1$



35. $4y^2 - 3x^2 = 1$

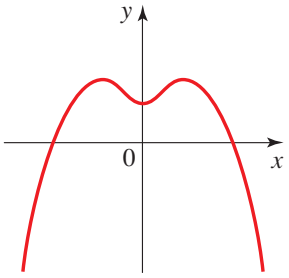


36. $x^2 + y^3 - x^2y^2 = 64$

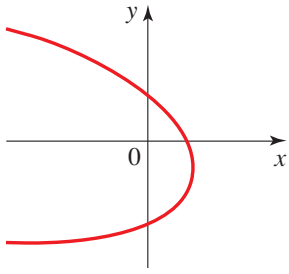


37–42 ■ Use the Vertical Line Test to determine whether the graph determines y as a function of x.

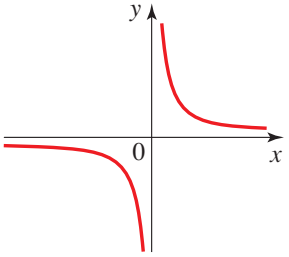
37.



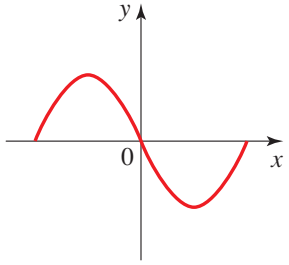
38.



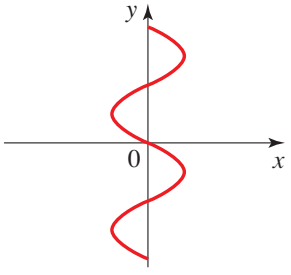
39.



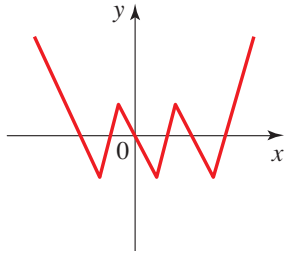
40.



41.



42.



43–48 ■ An equation is given.

(a) Make a table of values and sketch a graph of the equation.

(b) Use the Vertical Line Test to see whether the equation defines y as a function of x. If so, put the equation in function form.

43. $y = 2x^2$

44. $x = 2y^2$

45. $(y - 2)^2 = 2x$

46. $(y + 3)^2 = x$

47. $3y^3 - x = 0$

48. $5y = x^3$

49–52 ■ A verbal description of a function is given. Find **(a)** algebraic, **(b)** numerical, and **(c)** graphical representations for the function.

49. "Multiply the input by 3 and add 2 to the result."

50. "Subtract 4 from the input and multiply the result by 3."

51. The amount of sales tax charged in Lemon County on a purchase of x dollars. To find the tax (the output), take 8% of the purchase price (the input).

52. The volume of a sphere of diameter d. To find the volume (the output), take the cube of the diameter (the input), then multiply by π and divide by 6.

CONTEXTS

53. Big Box Retail Stores The number of "big box" retail stores has increased nationwide in recent years. The following table shows the number of existing big box retail stores and the median home price for a certain region for the years 1997–2006.

(a) Show that the variable z is not a function of the variable y.

(b) Show that the variable y is a function of the variable x. Find the net change in the number of big box stores from 2003 to 2005 and from 1997 to 2006.

(c) Show that the variable z is a function of the variable x. Find the net change in the median home price from 1997 to 2006.

x Year	y Number of big box retail stores	z Median home price (dollars)
1997	27	145,000
1998	30	150,000
1999	32	162,000
2000	38	187,000
2001	38	190,000
2002	42	199,000
2003	46	195,000
2004	46	200,000
2005	46	210,000
2006	49	230,000

54. World Consumption of Energy The table on the following page shows the yearly world consumption of petroleum and coal from 1995 to 2005.

(a) Show that the variable z is not a function of the variable y.

(b) Show that the variable y is a function of the variable x.

(c) Find the net change in the world consumption of coal from 1997 to 1999 and from 1995 to 2005.

(d) Show that the variable z is a function of the variable x.

(e) Find the net change in the world consumption of petroleum from 1995 to 2005.

x Year	y World consumption of coal (millions of tons)	z World consumption of petroleum (quadrillions of BTUs)
1995	88	142
1996	90	146
1997	92	149
1998	90	150
1999	92	153
2000	94	155
2001	95	157
2002	98	158
2003	107	161
2004	116	167
2005	123	169

55. Deliveries A feed store charges \$25 for a bale of hay plus a \$15 delivery charge. The cost of a load of hay is a function of the number x of bales purchased. Express this function verbally, symbolically, numerically, and graphically.

56. Cost of Gas Simone rents a compact car that gets 35 miles per gallon. When she rents the car, the price of a gallon of gas is \$3.50. The cost C of the gas used to drive the car is a function of the number of miles x that the car is driven. Express this function verbally, symbolically, numerically, and graphically.

57. Profit A university music department plans to stage the opera *Carmen*. The fixed cost for the set, costumes, and lighting is \$5000, and they plan to charge \$15 a ticket. So if they sell x tickets, then the profit P they will make from the performance is given by the equation

$$P = 15x - 5000$$

(a) Show that P is a function of x.

(b) Find the net change in the profit P when the number of tickets sold increases from 100 to 200.

(c) Express x as a function of P.

(d) Find the net change in the number of tickets sold when the profit changes from \$0 to \$5000.

58. Flower Bed Mulch Susan manages a landscape design business that operates in a suburb of Philadelphia. In the fall, her customers need mulch for all of their flower beds. If Susan has x square feet of flower beds to mulch, then the number y of cubic yards of mulch she needs is given by the equation

$$y = \frac{1}{81}x$$

(a) Show that y is a function of x.

(b) Find the net change in the amount of mulch y when the area of the flower beds increases from 10 to 50 square feet.

(c) Express x as a function of y.

(d) Find the net change in the area x of flower beds mulched when the amount of mulch increases from 2 to 20 cubic yards.

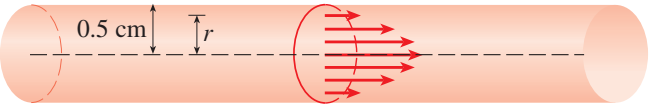
59. Blood Flow As blood moves through a vein or an artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure). The equation that relates v and r is called the *Law of Laminar Flow*. For an artery with radius 0.5 cm, the relationship between v (in cm/s) and r is given by the equation

$$r^2 = 0.25 - \frac{v}{24}$$

(a) Express v as a function of r.

(b) What is the velocity v when the distance r from the central axis is 0, 0.1, and 0.5 cm?

(c) What is the net change in velocity v when the distance r changes from 0 to 0.1 cm and when the distance r changes from 0.1 to 0.5 cm?



60. Relativity According to the Theory of Relativity, the length of an object depends on its velocity v with respect to an observer. For an object whose length at rest is 10 m, its observed length L satisfies the equation

$$c^2L^2 + 100v^2 = 100c^2$$

where c is 300,000 km/s, the speed of light.

(a) Express L as a function of v.

(b) What is the observed length L of a meteor that is 10 m long at rest and that is traveling past the earth at 10,000 km/s?

(c) What is the observed length L of a satellite that is 10 m long at rest and that is traveling at 5000 km/s?

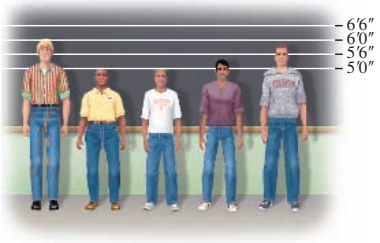
(d) What is the net change in the observed length L of an object whose velocity changes from 5000 km/s to 10,000 km/s?

61. Women and Cancer Linda knows quite a few women who have been diagnosed with breast cancer. She decides to make a table of these women's current ages and the age at which they were diagnosed with breast cancer.

(a) Graph the relation between current age and age at diagnosis.

(b) Use the Vertical Line Test to determine whether the variable y is a function of the variable x.

x Current age	y Age at diagnosis
63	50
72	65
45	35
62	60
62	54
71	66
59	54



62. Data A college algebra class collected some data from their classmates. The table at the top of page 51 lists some of these data.

(a) Graph the relation (x, y), where x is height and y is weight. Use the Vertical Line Test to determine whether the variable y is a function of the variable x.

(b) Graph the relation (x, y), where x is age and y is year of graduation. Use the Vertical Line Test to determine whether the variable y is a function of the variable x.

(c) If x is ID number and y is age, is the relation (x, y) a function?

ID	Age	Height	Weight	Graduation year
54-6514	21	72 in.	170 lb	2008
25-9778	18	67 in.	204 lb	2010
60-5213	20	63 in.	120 lb	2009
94-3256	21	67 in.	150 lb	2008
69-4781	30	65 in.	145 lb	2010

63. Body Mass Index During the last two decades the prevalence of obesity has increased considerably in the United States, despite the popularity of diets and health clubs. A health survey conducted over the course of forty years (1963–2003) used the Body Mass Index (BMI) to measure obesity. The BMI is defined by the formula

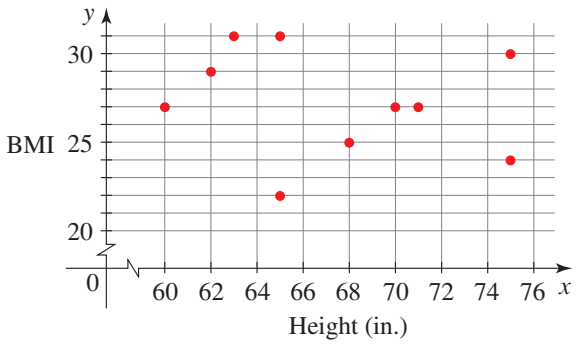
$$\text{BMI} = \frac{W}{H^2}$$

where W is the weight in kilograms and H is the height in meters. The survey found that over this 40-year period the average BMI of adults increased from 25 to 28. The graph displays the height (in inches) and BMI of several of the subjects in the survey.

(a) Make a table of the ordered pairs in the relation given by the graph.

(b) List the ordered pairs with input 65.

(c) Use the Vertical Line Test to determine whether the relation is a function.

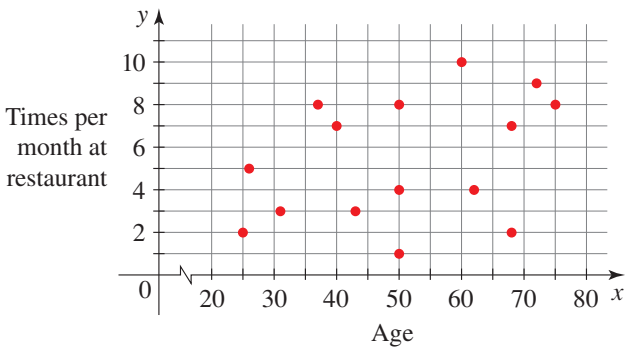


64. Restaurant Survey A restaurant in Rolla, Missouri, surveys its customers to help improve the service. The graph shows some of the survey data. Displayed are the ages of the customers surveyed and the numbers of times per month they eat in the restaurant.

(a) Make a table of the ordered pairs in the relation given by the graph.

(b) List the ordered pair(s) whose input is 50 and the ordered pair(s) with input 72.

(c) Use the Vertical Line Test to determine whether the relation is a function.



65. Algebra and Alcohol The first two columns of the table in the Prologue (page P2) give the alcohol concentration at different times following the consumption of 15 mL of alcohol.

(a) Do these data define the alcohol concentration as a function of time?

(b) Confirm your answer to part (a) by using the scatter plot of the data given in Exercise 47 of Section 1.2.

(c) Explain why data obtained by the real-world process of measuring the alcohol concentration in a person's blood at different times must define a function.

1.5 Function Notation: The Concept of Function as a Rule

- Function Notation
- Evaluating Functions—Net Change
- The Domain of a Function
- Piecewise Defined Functions

IN THIS SECTION... we introduce function notation. This notation associates each input value with its output and, at the same time, describes the rule that relates the input and output. This notation is immensely useful and will be used throughout this book.

Recall from Section 1.4 that a function is a relation in which each input gives exactly one output. In real-world applications of functions we need to know *how* the output is obtained from the input. In other words, we need to know the *rule* or *process* that acts on the input to produce the corresponding output. So we can define a function as the rule that relates the input to the output. Viewing a function in this way leads to a new and very useful notation for expressing functions, which we study in this section.

Function Notation

We previously used letters such as $x, y, a, b, \ldots$, to represent numbers; here we use the letter f to represent a rule.

A function is a relation between two changing quantities. Our goal is to discover the *rule* that relates these changing quantities. To describe such a rule symbolically, we use **function notation**. In this notation we give the function a name. If we use the letter f for the name of the function (or rule), then starting with an *input* and applying the rule f, we get the desired *output*.

$$\text{input} \xrightarrow{\text{Apply the rule } f} \text{output}$$

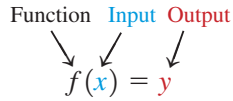
We express this process as an equation by writing

$$f(\text{input}) = \text{output}$$

Note that in this notation the letter f stands for a rule, not a number.

Function Notation

A **function** f is a rule that assigns to each input exactly one output. If we write x for the input and y for the output, then we use the following notation to describe f:



The symbol $f(x)$ is read "f of x" or "f at x" and is called **the value of f at x** or the **image of x under f**. This notation emphasizes the dependence of the output on the corresponding input—namely, the output $f(x)$ is the result of applying the rule f to the input x:

$$x \longrightarrow f(x)$$

The following examples illustrate the meaning of function notation.

example 1 Function Notation

Consider the function $f(x) = 8x$.

(a) What is the name of the function?

(b) What letter represents the input? What is the output?

(c) What rule does this function represent?

(d) Find $f(2)$. What does $f(2)$ represent?

Solution

(a) The name of the function is f.

(b) The input is x, and the output is $8x$.

(c) The rule is "Multiply the input by 8."

(d) $f(2) = 8 \cdot 2 = 16$. So 16 is the value of the function at 2.

■ **NOW TRY EXERCISE 9** ■

example 2 Writing Function Notation

Express the given rule in function notation.

(a) "Multiply by 2, then add 5."

(b) "Add 3, then square."

Solution

(a) First we need to choose a letter to represent this rule. So let g stand for the rule "Multiply by 2, then add 5." Then for any input x, multiplying by 2 gives $2x$, then adding 5 gives $2x + 5$. Thus we can write

$$g(x) = 2x + 5$$

Note that the input is the number x and the corresponding output is the number $2x + 5$.

(b) Let's choose the letter h to stand for the rule "Add 3, then square." Then for any input x, adding 3 gives $x + 3$, then squaring gives $(x + 3)^2$. Thus we have

$$h(x) = (x + 3)^2$$

Note that the input is the number x and the corresponding output is the number $(x + 3)^2$.

NOW TRY EXERCISE 13

example 3 Expressing a Function in Words

Give a verbal description of the given function.

(a) $f(x) = x - 5$

(b) $g(x) = \dfrac{x + 3}{2}$

Solution

(a) The rule f is "Subtract 5 from the input."

(b) The rule g is "Add 3 to the input, then divide by 2."

NOW TRY EXERCISE 17

example 4 Dependent and Independent Variables

(a) Express the equation $y = x^2 + 2x$ in function notation, where x is the independent variable and y is the dependent variable.

(b) Express the function $f(x) = 5x + 1$ as an equation in two variables. Identify the dependent and independent variables.

Solution

Although we chose the letter x for the independent variable, any letter can be chosen. The function (or rule) is the same regardless of the letter we choose to describe it.

(a) The equation determines a function in which each input x gives the output $x^2 + 2x$. If we call this rule g, then we can write this function as

$$g(x) = x^2 + 2x$$

(b) If we write y for the output $f(x)$, then we can write the function as

$$y = 5x + 1$$

The dependent variable is y, and the independent variable is x.

NOW TRY EXERCISES 23 AND 29

Evaluating Functions—Net Change

In function notation the input x plays the role of a placeholder. For example, the function $f(x) = 3x^2 + x$ can be thought of as

$$f(\square) = 3(\square)^2 + \square$$

So any letter can be used for the input of a function. Both of the following are the same function:

$$f(x) = 3x^2 + x$$
$$f(z) = 3z^2 + z$$

We can see that each of these represent the *same* rule applied to the input.

example 5 Evaluating a Function

Let $f(x) = 3x^2 + x$. Evaluate the function at the given input.

(a) $f(-2)$ **(b)** $f(0)$ **(c)** $f(4)$ **(d)** $f\left(\frac{1}{2}\right)$

Solution

To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3\cdot(-2)^2 + (-2) = 10$

(b) $f(0) = 3\cdot(0)^2 + 0 = 0$

(c) $f(4) = 3\cdot(4)^2 + 4 = 52$

(d) $f\left(\frac{1}{2}\right) = 3\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$

NOW TRY EXERCISE 39

Function notation is a useful and practical way of describing the rule of a function; we'll see many examples of the advantages of function notation in the coming chapters. Here we'll see how function notation allows us to write a concise formula for the net change in the value of a function between two points.

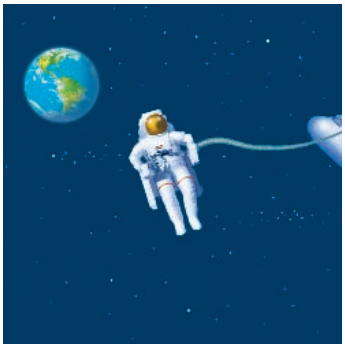
Net Change in the Value of a Function

The **net change in the value of a function** f as x changes from a to b (where $a \le b$) is given by

$$f(b) - f(a)$$

In the next example we find the net change in the weight of an astronaut at different elevations above the earth.

example 6 Net Change in the Weight of an Astronaut



If an astronaut weighs 130 pounds on the surface of the earth, then her weight when she is h miles above the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960 + h}\right)^2$$

(a) Evaluate $w(100)$. What does your answer mean?

(b) Construct a table of values for the function w that gives the astronaut's weight at heights from 0 to 500 miles. What do you conclude from the table?

(c) Find the net change in the weight of the astronaut from an elevation of 100 miles to an elevation of 400 miles. Interpret your result.

Solution

(a) We want the value of the function w when h is 100; substituting 100 for h in the definition of w, we get

$$w(h) = 130\left(\frac{3960}{3960 + h}\right)^2 \qquad \text{Function}$$

$$w(100) = 130\left(\frac{3960}{3960 + 100}\right)^2 \qquad \text{Replace } h \text{ by } 100$$

$$\approx 123.67 \qquad \text{Calculator}$$

This means that at a height of 100 mi, she weighs about 124 lb.

h	$w(h)$
0	130
100	124
200	118
300	112
400	107
500	102

(b) The table gives the astronaut's weight, rounded to the nearest pound, at 100-mile increments. The values in the table in the margin are calculated as in part (a). The table indicates that the higher the astronaut ascends, the less she weighs.

(c) The net change in the weight of the astronaut between these two elevations is $w(400) - w(100)$. We use the entries already calculated in the table in part (b) to evaluate this net change.

$$w(400) - w(100) \approx 107 - 124 \qquad \text{From the table}$$

$$= -17$$

The net change is -17 lb. The negative sign means that the astronaut's weight *decreased* by about 17 lb.

NOW TRY EXERCISES 35 AND 61

The Domain of a Function

The **domain** of a function is the set of all inputs for the function. The domain may be stated explicitly. For example, if we write

$$f(x) = x^2 \qquad 0 \le x \le 5$$

then the domain is the set of real numbers x for which $0 \le x \le 5$. If the domain of a function is not given explicitly, then by convention the domain is the set of all real number inputs for which the output is defined. For example, consider the functions

$$f(x) = \frac{1}{x - 4} \qquad g(x) = \sqrt{x}$$

The function f is not defined when x is 4, so its domain is $\{x \mid x \neq 4\}$. The function g is not defined for negative x, so its domain is $\{x \mid x \ge 0\}$.

example 7 Finding the Domain of a Function

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x(x - 2)}$ **(b)** $g(x) = \sqrt{x - 1}$

Solution

(a) The function is not defined when the denominator is 0, that is, when x is 0 or 2. So the domain of f is $\{x \mid x \neq 0 \text{ and } x \neq 2\}$.

(b) The function is defined only when $x - 1 \geq 0$. This means that $x \geq 1$. So the domain of g is $\{x \mid x \geq 1\}$.

Solving inequalities is reviewed in Algebra Toolkit C.3, page T62.

■ NOW TRY EXERCISES 49 AND 51 ■

Piecewise Defined Functions

A **piecewise defined function** is a function that is defined by different rules on different parts of its domain. The rules for such functions cannot be expressed as a single algebraic equation. This is one of the many reasons that we need function notation.

example **8** **Cell Phone Plan**

A cell phone plan has a basic charge of $39 per month. The plan includes 400 minutes and charges 20 cents for each additional minute of usage. The monthly charges are a function of the number x of minutes used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \leq x \leq 400 \\ 39 + 0.20(x - 400) & \text{if } x > 400 \end{cases}$$

Find **(a)** $C(100)$, **(b)** $C(400)$, and **(c)** $C(480)$.

Solution

Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input x. If $0 \leq x \leq 400$, then the value of $C(x)$ is 39. On the other hand, if $x > 400$, the value of $C(x)$ is $39 + 0.20(x - 400)$.

(a) Since $100 \leq 400$, we have $C(100) = 39$.

(b) Since $400 \leq 400$, we have $C(400) = 39$.

(c) Since $480 > 400$, we have $C(480) = 39 + 0.20(480 - 400) = 55$. Thus, the plan charges $39 for 100 minutes, $39 for 400 minutes, and $55 for 480 minutes.

■ NOW TRY EXERCISES 55 AND 67 ■

IN CONTEXT ➤



Aaron Kohr/Shutterstock.com 2009

The California aqueduct

Some of the fastest-growing cities in the United States are located in areas where water is in short supply. In the Southwest, water is brought to many cities from distant rivers and lakes via aqueducts. City planners must regulate growth carefully to ensure that adequate water sources exist for new developments. On average, each U.S. resident uses between 40 and 80 gallons of water daily. In arid areas people tend to use less water. In Arizona, for example, people conserve water in many ways, including the use of xeriscaping (landscaping with desert-friendly plants).

To discourage excessive water use in arid areas, cities charge different rates that depend on the amount of water used. In the next example we model the domestic cost of water using a piecewise defined function.

example 9 Water Rates

To discourage excessive water use, a city charges its residents \$0.008 per gallon for households that use less than 4000 gallons a month and \$0.012 for households that use 4000 gallons or more a month.

(a) Find a piecewise defined function C that gives the water bill for a household using x gallons per month.

(b) Find $C(3900)$ and $C(4200)$. What do your answers represent?

Solution

(a) Since the cost of x gallons of water depends on the usage, we need to define the function C in two pieces: for $x < 4000$ and for $x \geq 4000$. For x gallons the cost is $0.008x$ if $x < 4000$ and $0.012x$ if $x \geq 4000$. So we can express the function C as

$$C(x) = \begin{cases} 0.008x & \text{if } x < 4000 \\ 0.012x & \text{if } x \geq 4000 \end{cases}$$

(b) Since $3900 < 4000$, we have $C(3900) = 0.008(3900) = 31.20$. Since $4200 > 4000$, we have $C(4200) = 0.012(4200) = 50.40$. So using 3900 gallons costs \$31.20; using 4200 gallons costs \$50.40.

■ NOW TRY EXERCISE 69 ■

1.5 Exercises

CONCEPTS

Fundamentals

1. If a function f is given by the equation $y = f(x)$, then the independent variable is _____, the dependent variable is _____, and $f(a)$ is the _____ of f at a.

2. If $f(x) = x^2 + 1$, then $f(2) =$ _____ and $f(0) =$ _____.

3. The net change in the value of the function f from a to b is the difference ____ − ____. So if $f(x) = x^2 + 1$, then the net change in the value of the function f when x changes from 0 to 2 is the difference ____ − ____ = ____.

4. For a function f, the set of all possible inputs is called the _____ of f, and the set of all possible outputs is called the _____ of f.

5. **(a)** Which of the following functions have 5 in their domain?

$$f(x) = x^2 - 3x \qquad g(x) = \frac{x - 5}{x} \qquad h(x) = \sqrt{x - 10}$$

(b) For the functions from part (a) that *do* have 5 in their domain, find the value of the function at 5.

6. A function is given algebraically by the formula $f(x) = (x - 4)^2 + 3$. Fill in the table in the margin to give a numerical representation of f.

x	0	2	4	6
$f(x)$	19			

Think About It

7. How would you describe the quantity $f(b) - f(a)$ *without* using function notation? For example, how would you describe the net change between −2 and 5 for the function $y = x^2 - 2$ without function notation?

8. True or false?

(a) If the net change of a function f from a to b is zero, then the function must be constant between a and b.

(b) If the net change of a function f from a to b is positive, then the values of the function must steadily increase from a to b.

SKILLS

9–12 ■ A function is given.

(a) What is the name of the function?

(b) What letter represents the input? What is the output?

(c) What rule does this function represent?

(d) Find $f(10)$. What does $f(10)$ represent?

9. $f(x) = 2x + 1$ **10.** $g(w) = w^3 - 1$ **11.** $h(z) = (z - 2)^2$ **12.** $A(r) = \dfrac{r^2}{5} - 7$

13–16 ■ Express the given rule in function notation.

13. "Add 2, then multiply by 5."

14. "Divide by 3, then add 2."

15. "Square, add 7, then divide by 4."

16. "Subtract 5, square, then subtract 1."

17–22 ■ Give a verbal description of the function.

17. $f(x) = \dfrac{x}{2} + 7$

18. $g(x) = \dfrac{x + 7}{2}$

19. $h(w) = 7w^2 - 5$

20. $k(w) = (w + 1)^2 - 9$

21. $A(r) = 5(r + 3)^3$

22. $V(r) = \dfrac{2}{r} + 1$

23–28 ■ An equation is given.

(a) Does the equation define a function with x as the independent variable and y as the dependent variable? If so, express the equation in function notation with x as the independent variable.

(b) Does the equation define a function with y as the independent variable and x as the dependent variable? If so, express the equation in function notation with y as the independent variable.

23. $y = 5x$ **24.** $y = 3x^3$ **25.** $x = 4y^3$

26. $x = y^3 - 1$ **27.** $y = 3x^2$ **28.** $x = 4y^2$

29–34 ■ A function is given. Express the function as an equation. Identify the independent and dependent variables.

29. $f(x) = 3x^2 - 1$ **30.** $g(x) = 2x^3 - 2x$ **31.** $f(w) = 3w^2 - 1$

32. $g(z) = 2z^3 - 2z$ **33.** $S(r) = 4\pi r^2$ **34.** $V(r) = \frac{4}{3}\pi r^3$

35–38 ■ A function is given.

(a) Complete the table of values for the function.

(b) Find the net change in the value of the function when x changes from 0 to 2.

35. $f(x) = 2x^2 - 7$

x	-2	-1	0	1	2
$f(x)$					

36. $g(s) = 3s - 5$

s	-4	-2	0	1	2
$g(s)$					

37. $h(z) = (z + 3)^3 - 2$

z	-3	-1	0	1	2
$h(z)$					

38. $k(t) = \dfrac{1}{t + 1}$

t	0	1	2	3	4
$k(t)$					

39–42 ■ Evaluate the function at the indicated values.

39. $f(x) = x^2 + 2x$: **(a)** $f(0)$ **(b)** $f(\sqrt{2})$ **(c)** $f(-2)$ **(d)** $f(a)$

40. $g(x) = x^3 - 4x^2$: **(a)** $g(2)$ **(b)** $g(\frac{1}{2})$ **(c)** $g(-1)$ **(d)** $g(b)$

41. $r(x) = \sqrt{2x - 1}$: **(a)** $r(2)$ **(b)** $r(5)$ **(c)** $r(3)$ **(d)** $r(c)$

42. $s(x) = \dfrac{1}{\sqrt{x + 1}}$: **(a)** $s(0)$ **(b)** $s(3)$ **(c)** $s(4)$ **(d)** $s(a)$

43. The surface area S of a sphere is a function of its radius r given by $S(r) = 4\pi r^2$.

(a) Find $S(2)$ and $S(3)$.

(b) What do your answers in part (a) represent?

(c) Find the net change in the value of the function S when r changes from 1 to 4.

44. The volume V of a can that has height two times the radius is a function of its radius r given by $V(r) = 2\pi r^3$.

(a) Find $V(4)$ and $V(9)$.

(b) What do your answers in part (a) represent?

(c) Find the net change in the value of the function V when r changes from 1 to 4.

45–54 ■ Find the domain of the function.

45. $f(x) = 2x$

46. $f(x) = x^2 + 1$

47. $g(x) = 2x, \quad -1 \le x \le 5$

48. $g(x) = x^2 + 1, \quad 0 \le x \le 5$

49. $h(x) = \dfrac{1}{x - 3}$

50. $h(x) = \dfrac{1}{3x - 6}$

51. $k(x) = \sqrt{x - 5}$

52. $k(x) = \sqrt{x + 9}$

53. $r(x) = \dfrac{3}{\sqrt{x - 4}}$

54. $r(x) = \dfrac{\sqrt{2 + x}}{3 - x}$

55–60 ■ Evaluate the piecewise defined function at the indicated values.

55. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$

(a) $f(-1)$ **(b)** $f(0)$ **(c)** $f(-2)$ **(d)** $f(1)$

56. $g(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x \ge 0 \end{cases}$

(a) $g(-1)$ **(b)** $g(0)$ **(c)** $g(-2)$ **(d)** $g(1)$

57. $h(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x + 1 & \text{if } x \ge 1 \end{cases}$

(a) $h(-3)$ **(b)** $h(0)$ **(c)** $h(3)$ **(d)** $h(1)$

58. $k(x) = \begin{cases} 5 & \text{if } x \le 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$

(a) $k(2)$ **(b)** $k(0)$ **(c)** $k(-3)$ **(d)** $k(7)$

59. $F(x) = \begin{cases} x^2 + 2x & \text{if } x \le -1 \\ 2x & \text{if } x > -1 \end{cases}$

(a) $F(2)$ **(b)** $F(-1)$ **(c)** $F(0)$ **(d)** $F(-2)$

60. $G(x) = \begin{cases} 3x & \text{if } x < 0 \\ x + 1 & \text{if } 0 \le x \le 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

(a) $G(-2)$ **(b)** $G(-1)$ **(c)** $G(0)$ **(d)** $G(3)$

CONTEXTS

61. How Far Can You See? Because of the curvature of the earth, the maximum distance D that one can see from the top of a tall building or from an airplane at height h is modeled by the function

$$D(h) = \sqrt{7920h + h^2}$$

where D and h are measured in miles.

(a) Find $D(0.1)$. What does this value represent?

(b) How far can one see from the observation deck of Toronto's CN Tower, 1135 ft above the ground? (Remember that one mile is 5280 ft.)

(c) Commercial aircraft fly at an altitude of about 7 mile. How far can the pilot see?

(d) Find the net change of the distance D as one climbs the CN Tower from a height of 100 ft to a height of 1135 ft.

62. Torricelli's Law A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** models the volume V (in gallons) of water remaining in the tank after t minutes as

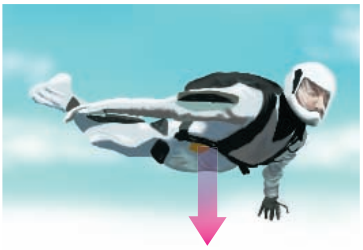
$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20$$

(a) Find $V(0)$ and $V(20)$. What do these values represent?

(b) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.

(c) Find the net change of the volume of water in the tank when t changes from 0 to 10.





$h(t) = 13{,}000 - 16t^2$

63. A Falling Sky Diver When a sky diver jumps out of an airplane from a height of 13,000 ft, her height h (in feet) above the ground after t seconds is given by the function

$$h(t) = 13{,}000 - 16t^2$$

(a) Find $h(10)$ and $h(20)$. What do these values represent?

(b) For safety reasons a sky diver must open the parachute at a height of about 2500 ft (or higher). A sky diver opens her parachute after 24 seconds. Did she open her parachute at a safe height?

(c) Find the net change in the sky diver's height from 0 to 25 seconds.

64. Path of a Ball A baseball is thrown across a playing field. Its path is given by the function

$$h(x) = -0.005x^2 + x + 5$$

where x is the distance the ball has traveled horizontally and $h(x)$ is its height above ground level, both measured in feet.

(a) Find $h(10)$ and $h(100)$. What do these values represent?

(b) What is the initial height of the ball?

(c) Find the net change in the height of the ball as the horizontal distance x changes from 10 to 100 ft.

65. Movie Theater Revenue Box office revenues for movie theaters have remained high for the last decade, in spite of the recent popularity of DVD players and home movie theaters. The function R represented by the table in the margin shows the annual U.S. box office revenue (in billions).

x (year)	$R(x)$ (billions of dollars)
2000	7.66
2001	8.41
2002	9.52
2003	9.49
2004	9.54
2005	8.99
2006	9.49

(a) Find $R(2000)$ and $R(2006)$. What do these values represent?

(b) For what values of x is $R(x) = 9.49$?

(c) Find the net change in box office revenues from 2000 to 2006.

66. Biotechnology The biotechnology industry is responsible for hundreds of medical, environmental, and technological advances; one such advance is DNA fingerprinting. In the table below, the function f gives annual biotechnology revenue and the function g gives annual research and development (R&D) revenue, in billions of dollars, between 1995 and 2005.

(a) Find $f(1995)$, $f(2005)$, $g(1995)$, and $g(2005)$. What do these values represent?

(b) Find x and y so that $f(x) = f(y)$.

(c) Find the net change in biotechnology revenues and in R&D expenses from 1995 to 2005.

x (year)	$f(x)$ (billions of dollars)	$g(x)$ (billions of dollars)
1995	12.7	7.7
1996	14.6	7.9
1997	17.4	9.0
1998	20.2	10.6
1999	22.3	10.7
2000	26.7	14.2
2001	29.6	15.7
2002	29.6	20.5
2003	39.2	17.9
2004	43.8	19.6
2005	50.7	19.8

67. Income Tax In a certain country, income tax T is assessed according to the following function of income x (in dollars):

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10{,}000 \\ 0.08x & \text{if } 10{,}000 < x \le 20{,}000 \\ 1600 + 0.15(x - 20{,}000) & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $T(5000)$, $T(12{,}000)$, and $T(25{,}000)$. What do these values represent?

(b) Find the tax on an income of \$20,000.

(c) If a businessman pays \$25,000 in taxes in this country, what is his income?

68. Interest Income Greg earned $5000 over the summer working on a construction site. He deposits his earnings in a money market account that offers a graduated interest rate based on the balance of the account. If the balance is between $2500 and $20,000, the account earns 2.5% interest per annum; if the balance is above $20,000, the account earns 3.68%; and if the balance falls below $2500, the bank pays no interest but charges a $13 fee per month. The interest i earned in 1 month on a balance of x dollars is modeled by the function

$$i(x) = \begin{cases} -13 & \text{if } 0 \le x < 2500 \\ \dfrac{0.025}{12}x & \text{if } 2500 \le x \le 20{,}000 \\ \dfrac{0.0368}{12}x & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $i(10{,}000)$ and $i(20{,}000)$. What do these values represent?

(b) Find the interest Greg earns in the first month (on his initial $5000 deposit).

(c) Find the balance Greg needs to have in his account to earn $70 interest in one month.

69. Internet Purchases An Internet bookstore charges $15 shipping for orders under $100 but provides free shipping for orders of $100 or more. The cost C of an order is a function of the total price x of the books purchased.

(a) Express C as a piecewise defined function:

$$C(x) = \begin{cases} _____ & \text{if } x < 100 \\ _____ & \text{if } x \ge 100 \end{cases}$$

(b) Find $C(75)$, $C(100)$, and $C(105)$. What do these values represent?

70. Cost of a Hotel Stay A hotel chain charges $75 each night for the first two nights and $50 for each additional night's stay. The total cost T is a function of the number of nights x that a guest stays.

(a) Express T as a piecewise defined function:

$$T(x) = \begin{cases} _____ & \text{if } 0 \le x \le 2 \\ _____ & \text{if } 2 < x \end{cases}$$

(b) Find $T(2)$, $T(3)$, and $T(5)$. What do these values represent?

71. Speeding Tickets In a certain state the maximum speed permitted on freeways is 65 mi/h, and the minimum is 40 mi/h. The fine for violating these limits is $15 for every mile above the maximum or below the minimum speed. So the fine F is a function of the driving speed x (mi/h) on the freeway.

(a) Express F as a piecewise defined function:

$$F(x) = \begin{cases} _____ & \text{if } 0 < x < 40 \\ _____ & \text{if } 40 \le x \le 65 \\ _____ & \text{if } 65 < x \end{cases}$$

(b) Find $F(30)$, $F(50)$, and $F(75)$. What do these values represent?

(c) A driver is assessed a fine of $225. What are the two possible speeds at which he could have been driving when he was caught?

72. Utility Charges A utility company charges a base rate of 10 cents per kilowatt hour (kWh) for the first 350 kWh and 15 cents per kilowatt hour for all additional electricity usage. The amount E that the utility company charges is a function of the number x of kilowatt hours used.

(a) Express E as a piecewise defined function:

$$E(x) = \begin{cases} _____ & \text{if } 0 \le x \le 350 \\ _____ & \text{if } 350 < x \end{cases}$$

(b) Find $E(300)$, $E(350)$, and $E(600)$. What do these values represent?

(c) One bill for electric usage is $65.67. How many kilowatt hours are covered by this bill?

1.6 Working with Functions: Graphs and Graphing Calculators

- Graphing a Function from a Verbal Description
- Graphs of Basic Functions
- Graphing with a Graphing Calculator
- Graphing Piecewise Defined Functions

IN THIS SECTION... we continue studying properties of functions by analyzing their graphs. We use graphing calculators as a convenient way of obtaining graphs quickly.

GET READY... by learning how your own graphing calculator works. Review the material on graphing calculators in **Algebra Toolkit D.3**. Test your graphing calculator skills by doing the **Algebra Checkpoint** at the end of this section.

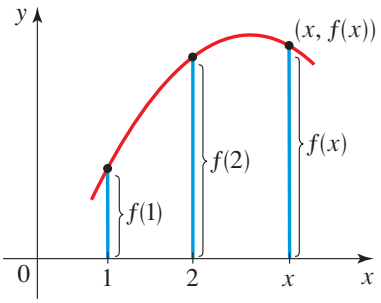


figure **1** The height of the graph above x is the value of $f(x)$

We graph functions in the same way we graphed relations in Section 1.2: by plotting the ordered pairs in the relation. So the **graph of a function** f is the set of all ordered pairs (x, y) where $y = f(x)$, plotted in a coordinate plane. This means the value $f(x)$ is the height of the graph above the point x, as shown in Figure 1.

We can sketch the graph of a function from a verbal, numerical, or algebraic description of the function. In this section we examine graphs of some basic functions. In subsequent sections we use these basic functions to model real-world phenomena.

Graphing a Function from a Verbal Description

Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph. Consider the following example.

example **1** **Graphing a Function from Verbal and Numerical Descriptions**



When you turn on a hot water faucet, the temperature of the water depends on how long the water has been running. Let T be the function defined by

$$T(x) = \text{“Temperature of the water from the faucet at time } x\text{”}$$

where x is measured in minutes.

(a) Draw a rough graph of the function T.

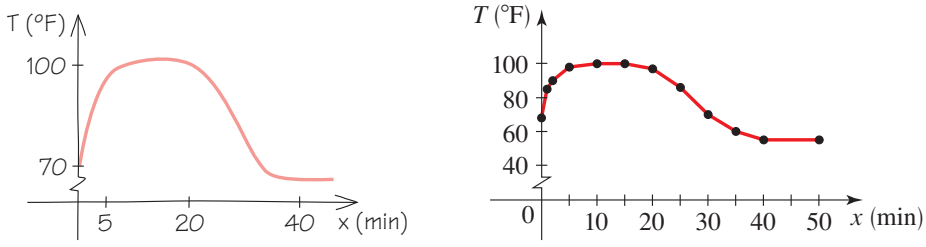
(b) To get a more accurate graph, the following data were gathered from a particular faucet. Draw a graph of the function T based on these data.

x (min)	0	1	2	5	10	15	20	25	30	35	40	50
T (°F)	68	85	90	98	100	100	97	86	70	60	55	55

Solution

(a) When the faucet is turned on, the initial temperature of the water is close to room temperature. When the water from the hot water tank reaches the faucet, the water's temperature T increases quickly. In the next phase, T is constant at the temperature of the water in the tank. When the tank is drained, T decreases to the temperature of the cold water supply. Figure 2(a) shows a rough graph of the temperature T of the water as a function of the time t that has elapsed since the faucet was turned on.

(b) We make a scatter plot of the data and connect the points with a smooth curve as in Figure 2(b). Notice that the height of the graph is the value of the function; in other words, the height of the graph is the temperature of the water at the given time.



(a) Rough graph (b) Graph from data

figure **2** Graph of water temperature as a function of time

NOW TRY EXERCISE 49

It would be nice to have a precise rule (function) that gives the temperature of the water at any time. Having such a rule would allow us to predict the temperature of the water at any time and perhaps warn us about getting scalded. We don't have such a rule for this situation, but as we study more functions with different graphs, we may be able to find a function that models this situation.

Graphs of Basic Functions

When the rule of a function is given by an equation, then to graph the function, we first make a table of values. We then plot the points in the table and connect them by a smooth curve. Let's try graphing some basic functions.

In the next example we graph a **constant function**, that is, a function of the form

$$f(x) = c$$

where c is a fixed constant number. Notice that a constant function has the same output c for every value of the input.

example 2 Graph of a Constant Function

Graph the function $f(x) = 3$.

Solution

We first make a table of values. Then we plot the points in the table and join them by a line as in Figure 3. Notice that the graph is a horizontal line 3 units above the x-axis.

x	$f(x) = 3$
−3	3
−2	3
−1	3
0	3
1	3
2	3
3	3

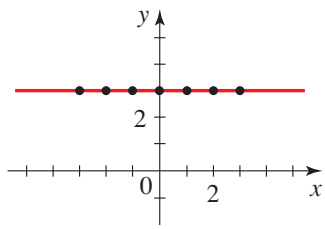


figure **3** Graph of $f(x) = 3$

NOW TRY EXERCISE 7

In the next example we graph the function $f(x) = x$. For each input, this function gives the same number as output. This function is called the **identity function** (the output is identical to the input). We also graph $f(x) = x^2$, whose graph has the shape of a parabola.

example 3 Graphs of Basic Functions

Graph the function.

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

Solution

We first make a table of values. Then we plot the points in the table and join them by a line or smooth curve as in Figures 4, 5, and 6.

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

x	$f(x) = x$
-3	-3
-2	-2
-1	-1
0	0
1	1
2	2
3	3

x	$f(x) = x^2$
-2	4
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
2	4

x	$f(x) = x^3$
$-\frac{3}{2}$	$-\frac{27}{8}$
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
$\frac{3}{2}$	$\frac{27}{8}$

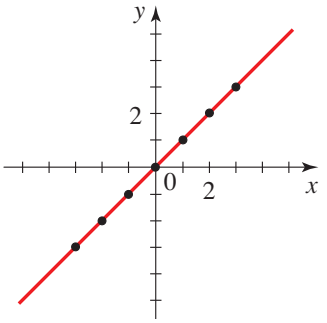


figure **4**

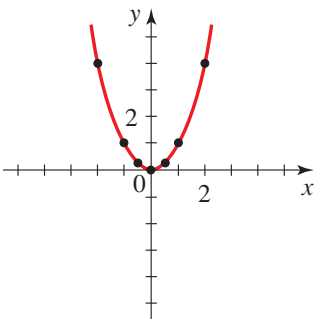


figure **5**

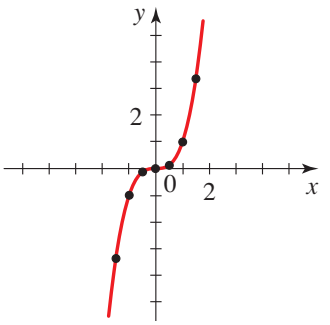


figure **6**

■ **NOW TRY EXERCISES 15, 19, AND 23** ■

Another basic function is the **square root function** $f(x) = \sqrt{x}$, which we graph in the next example.

example 4 Graph of the Square Root Function

Graph the function.

(a) $f(x) = \sqrt{x}$ **(b)** $g(x) = \sqrt{x} + 3$

Solution

We first make a table of values. Since the domain of f is $\{x \mid x \geq 0\}$, we use only nonnegative values for x. Then we plot the points in the table and join them by a line or smooth curve as in Figures 7 and 8.

(a) $f(x) = \sqrt{x}$

x	$f(x) = \sqrt{x}$	Decimal
0	0	0
$\frac{1}{4}$	$\frac{1}{2}$	0.5
1	1	1.0
2	$\sqrt{2}$	1.4
3	$\sqrt{3}$	1.7
4	2	2.0
5	$\sqrt{5}$	2.2

(b) $g(x) = \sqrt{x} + 3$

x	$g(x) = \sqrt{x} + 3$	Decimal
0	$0 + 3$	3.0
$\frac{1}{4}$	$\frac{1}{2} + 3$	3.5
1	$1 + 3$	4.0
2	$\sqrt{2} + 3$	4.4
3	$\sqrt{3} + 3$	4.7
4	$2 + 3$	5.0
5	$\sqrt{5} + 3$	5.2

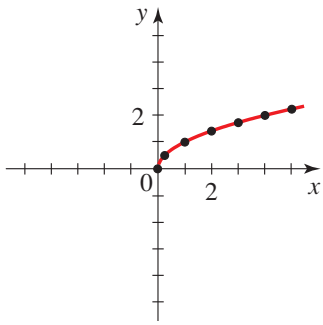


figure **7**

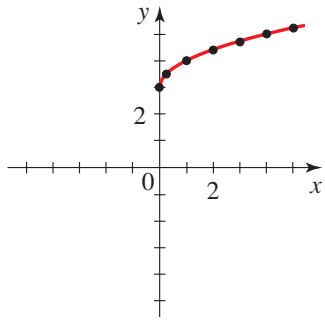


figure **8**

NOW TRY EXERCISE 33

In Example 4 notice how the graph of $g(x) = \sqrt{x} + 3$ has the same shape as the graph of $f(x) = \sqrt{x}$ but is shifted up 3 units. We will study shifting of graphs more systematically in Chapter 4.

Graphing with a Graphing Calculator

Algebra Toolkit D.3, page T80, gives guidelines on using a graphing calculator as well as advice on avoiding some common graphing calculator pitfalls.

A graphing calculator graphs a function in the same way you do: by making a table of values and plotting points. Of course, the calculator is fast and accurate; it also frees us from the many tedious calculations needed to get a good picture of the graph. But the graph that is produced by a graphing calculator can be misleading. A graphing calculator must be used with care, and the graphs it produces must be interpreted appropriately.

example **5** **Graphing a Function**

Graph the function $f(x) = x^3 - 49x$ in an appropriate viewing rectangle.

Solution

To graph the function $f(x) = x^3 - 49x$, we first express it in equation form

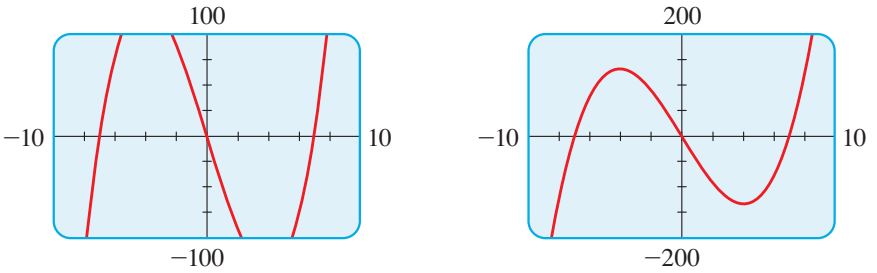
$$y = x^3 - 49x$$

We now graph the equation using a graphing calculator. We experiment with different viewing rectangles. The viewing rectangle in Figure 9(a) gives an incomplete graph—we need to see more of the graph in the vertical direction. So we choose the larger viewing rectangle $[-10, 10]$ by $[-200, 200]$ by choosing

$$\text{Xmin} = -10 \qquad \text{Ymin} = -200$$

$$\text{Xmax} = 10 \qquad \text{Ymax} = 200$$

The graph in this larger viewing rectangle is shown in Figure 9(b). This graph appears to show all the main features of this function. (We will confirm this when we study polynomial functions in Chapter 5.)



(a) (b)

figure **9** Graphing $f(x) = x^3 - 49x$

NOW TRY EXCEPTION 35

example **6** **Where Graphs Meet**

Consider the functions $f(x) = x^2 - 1$ and $g(x) = x^3 - 2x - 1$.

(a) Graph the functions f and g in the viewing rectangle $[-2, 3]$ by $[-3, 6]$.

(b) Find the points where the graphs intersect in this viewing rectangle.

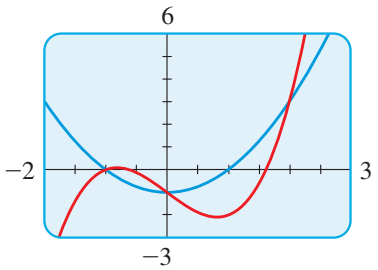


figure **10** Finding where graphs meet

Solution

The graphs are shown in Figure 10. The graphs appear to meet at three different points. Zooming in on each point, we find that the points of intersections are

$$(-1, 0) \quad (0, -1) \quad (2, 3)$$

You can check that each of these points satisfies both equations.

NOW TRY EXCEPTION 41

Graphing Piecewise Defined Functions

Recall that a piecewise defined function is a function that is defined by different rules on different parts of its domain. As you might expect, the graph of such a function consists of separate "pieces," as the following example shows.

example 7 Graphing a Piecewise Defined Function

Graph the function

$$f(x) = \begin{cases} x & \text{if } x \le 2 \\ x + 1 & \text{if } x > 2 \end{cases}$$

Solution

The rule f is given by $f(x) = x$ for $x \le 2$ and $f(x) = x + 1$ for $x > 2$. We use this information to make a table of values and plot the graph in Figure 11.

x	$f(x)$
−2	−2
−1	−1
0	0
1	1
2	2
3	4
4	5
5	6

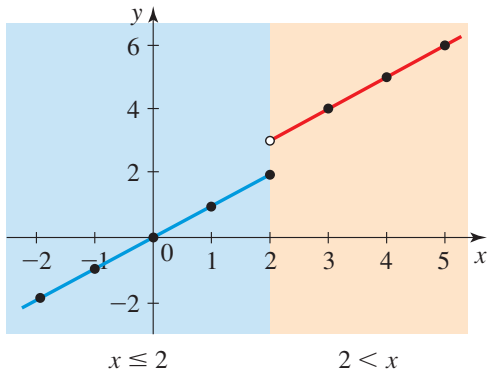


figure **11** Graph of the piecewise defined function f

On many graphing calculators the graph in Figure 11 can be produced by using the logical functions in the calculator. For example, on the TI-83 the following equation gives the required graph:

Y_1=(X≤2)*X+(X>2)*(X+1)

(To avoid the extraneous vertical line between the two parts of the graph, put the calculator in Dot mode.)

Notice the closed and open circles on the graph in Figure 11. The closed circle at the point (2, 2) indicates that this point is on the graph. The open circle above it at (2, 3) indicates that this point is not on the graph.

NOW TRY EXERCISE 45

example 8 Water Rates

A city charges its residents \$0.008 per gallon for households that use less than 4000 gallons a month and \$0.012 for households that use 4000 gallons or more a month (see Example 9 in Section 1.5). The function

$$C(x) = \begin{cases} 0.008x & \text{if } x < 4000 \\ 0.012x & \text{if } x \ge 4000 \end{cases}$$

gives the cost of using x gallons of water per month.

(a) Graph the piecewise defined function C.

(b) What does the break in the graph represent?

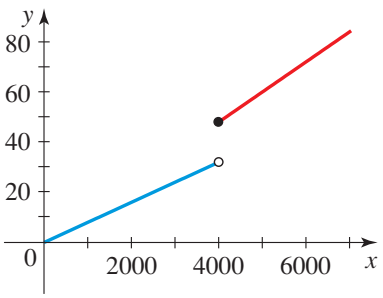


figure **12** Graph of the cost function C

Solution

(a) We graph the function C in two pieces. For $x < 4000$ we graph the equation $y = 0.008x$, and for $x \ge 4000$ we graph the equation $y = 0.012x$. The graph is shown in Figure 12.

(b) From the graph we see that at 4000 gallons there is a jump in the cost of water. This corresponds to the jump in the price from \$0.008 to \$0.012 per gallon.

NOW TRY EXERCISE 55

The **absolute value function** $f(x) = |x|$ is a piecewise defined function:

$$f(x) = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

This function leaves positive inputs unchanged and reverses the sign of negative inputs.

example **9** **Graph of the Absolute Value Function**

Sketch the graph of $f(x) = |x|$.

Solution

To graph $f(x) = |x|$ on the TI-83, enter the function as `Y1=abs(X)`.

We make a table of values and then sketch the graph in Figure 13.

x	$f(x) = \lvert x \rvert$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3

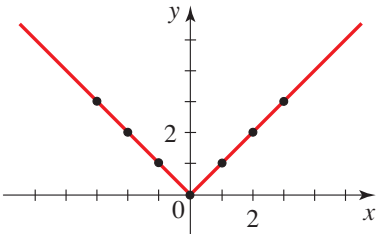


figure **13** Graph of $f(x) = |x|$

■ **NOW TRY EXERCISE 31** ■

Algebra Checkpoint ✓

Test your skill in using your graphing calculator. Review the guidelines on graphing calculators in **Algebra Toolkit D.3** on page T80.

1–4 Use a graphing calculator or computer to decide which viewing rectangle (i)–(iv) produces the most appropriate graph of the equation.

1. $y = x^4 + 2$

(i) $[-2, 2]$ by $[-2, 2]$+
(ii) $[0, 4]$ by $[0, 4]$
(iii) $[-8, 8]$ by $[-4, 40]$
(iv) $[-40, 40]$ by $[-80, 800]$

2. $y = x^2 + 7x + 6$

(i) $[-5, 5]$ by $[-5, 5]$
(ii) $[0, 10]$ by $[-20, 100]$
(iii) $[-15, 8]$ by $[-20, 100]$
(iv) $[-10, 3]$ by $[-100, 20]$

3. $y = 10 + 25x - x^3$

(i) $[-4, 4]$ by $[-4, 4]$
(ii) $[-10, 10]$ by $[-10, 10]$
(iii) $[-20, 20]$ by $[-100, 100]$
(iv) $[-100, 100]$ by $[-200, 200]$

4. $y = \sqrt{8x - x^2}$

(i) $[-4, 4]$ by $[-4, 4]$
(ii) $[-5, 5]$ by $[0, 100]$
(iii) $[-10, 10]$ by $[-10, 40]$
(iv) $[-2, 10]$ by $[-2, 6]$

5. Graph the equation in an appropriate viewing rectangle.

(a) $y = 50x^2$
(b) $y = x^3 - 2x - 3$
(c) $y = x^4 - 5x^2$
(d) $y = \sqrt{4 + x - x^2}$

1.6 Exercises

CONCEPTS

Fundamentals

1. To graph the function f, we plot the points $(x,$ _____$)$ in a coordinate plane. To graph $f(x) = x^3 + 2$, we plot the points $(x,$ _____$)$. So the point $(2,$ _____$)$ is on the graph of f. The height of the graph of f above the x-axis when $x = 2$ is _____.

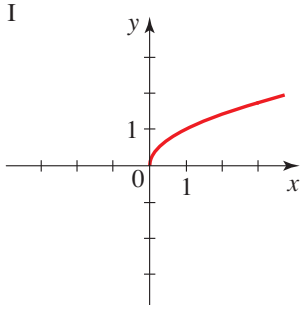
2. If $f(2) = 3$, then the point $(2,$ _____$)$ is on the graph of f.

3. If the point $(1, 5)$ is on the graph of f, then $f(1) =$ _____.

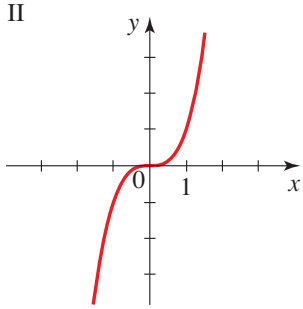
4. Match the function with its graph.

(a) $f(x) = x^2$ **(b)** $f(x) = x^3$ **(c)** $f(x) = \sqrt{x}$ **(d)** $f(x) = |x|$

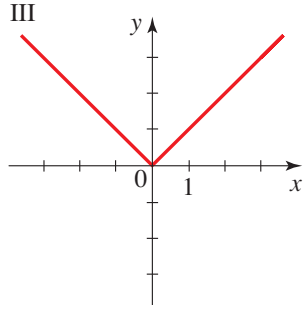
I



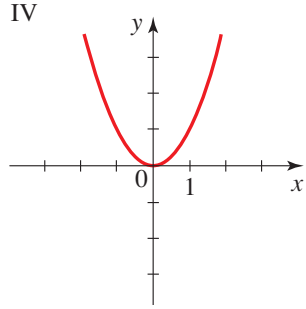
II



III



IV



Think About It

5. In what ways can a graph produced by a graphing calculator be misleading? Explain using an example.

6. A student wishes to graph the following functions on the same screen:

$$y = x^{1/3} \qquad \text{and} \qquad y = \frac{x}{x + 4}$$

He enters the following information into the calculator:

$$Y_1 = X \wedge 1/3 \qquad \text{and} \qquad Y_2 = X/X + 4$$

The calculator graphs two lines instead of the information he wanted. What went wrong?

SKILLS

7–12 ■ A function is given. Complete the table and then graph the function.

7. $f(x) = 5$ **8.** $g(x) = 2x - 4$ **9.** $h(x) = 2x^2 - 3$

x	$f(x)$
−3	
−2	
−1	
0	
1	
2	
3	

x	$g(x)$
−3	
−2	
−1	
0	
1	
2	
3	

x	$h(x)$
−3	
−2	
−1	
0	
1	
2	
3	

10. $k(x) = x^3 + 8$

x	$k(x)$
-3	
-2	
-1	
0	
1	
2	
3	

11. $F(x) = \sqrt{x + 4}$

x	$F(x)$
-4	
-2	
0	
2	
4	
8	

12. $G(x) = 2|x + 1|$

x	$G(x)$
-4	
-3	
-2	
-1	
0	
1	

13–28 ■ Sketch the graph of the function by first making a table of values.

13. $f(x) = 8$

14. $f(x) = -1$

15. $g(x) = x - 4$

16. $g(x) = 3x - 7$

17. $h(x) = -x + 3, \quad -3 \le x \le 3$

18. $h(x) = 3 - x, \quad -2 \le x \le 2$

19. $k(x) = -x^2$

20. $k(x) = 4 - x^2$

21. $F(x) = x^2 - 4$

22. $F(x) = (x - 3)^2$

23. $G(x) = x^3 - 8$

24. $G(x) = (x + 2)^3$

25. $A(x) = \sqrt{x} + 1$

26. $A(x) = \sqrt{x + 4}$

27. $r(x) = |2x|$

28. $r(x) = |x + 1|$

29–34 ■ Graph the given functions on the same coordinate axes.

29. $f(x) = x^2, \quad g(x) = x^2 - 3, \quad h(x) = \frac{1}{4}x^2$

30. $f(x) = x^2, \quad g(x) = (x + 5)^2, \quad h(x) = (x - 5)^2$

31. $f(x) = |x|, \quad g(x) = 2|x|, \quad h(x) = |x + 2|$

32. $f(x) = |x|, \quad g(x) = |3x|, \quad h(x) = |x| + 4$

33. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{4x}, \quad h(x) = \sqrt{x} - 4$

34. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x - 9}, \quad h(x) = \sqrt{x - 1} + 2$

35–40 ■ Draw a graph of the function in an appropriate viewing rectangle.

35. $f(x) = 4 + 6x - x^2$

36. $f(x) = \sqrt{12x - 17}$

37. $f(x) = x^4 - 4x^3$

38. $f(x) = 0.1x^3 - x^2 + 1$

39. $f(x) = \left|\frac{x}{2} + 7\right|$

40. $f(x) = 2x - |x^2 - 5|$

41–42 ■ Do the graphs of the two functions intersect in the given viewing rectangle? If they do, how many points of intersection are there?

41. $f(x) = 3x^2 + 6x - \frac{1}{2}, \quad g(x) = \sqrt{7 - \frac{7}{12}x^2}; \quad [-4, 4]$ by $[-1, 3]$

42. $f(x) = 6 - 4x - x^2, \quad g(x) = 3x + 18; \quad [-6, 2]$ by $[-5, 20]$

43–48 ■ Sketch the graph of the piecewise defined function.

43. $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

44. $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

45. $f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ 5 & \text{if } x \geq -2 \end{cases}$

46. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

47. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 2 \\ x & \text{if } x > 2 \end{cases}$

48. $f(x) = \begin{cases} 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

CONTEXTS

49. Filling a Bathtub A bathtub is being filled by a constant stream of water from the faucet. Sketch a rough graph of the water level in the tub as a function of time.

50. Cooling Pie You place a frozen pie in an oven and bake it for an hour. Then you take the pie out and let it cool before eating it. Sketch a rough graph of the temperature of the pie as a function of time.

51. Christmas Card Sales The number of Christmas cards sold by a greeting card store depends on the time of year. Sketch a rough graph of the number of Christmas cards sold as a function of the time of year.

52. Height of Grass A home owner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period beginning on a Sunday.

r	$T(r)$
5	
6	
7	
8	
9	
10	

53. Weather Balloon As a weather balloon is inflated, the thickness T of its latex skin is related to the radius of the balloon by

$$T(r) = \frac{0.5}{r^2}$$

where T and r are measured in centimeters. Complete the table in the margin and graph the function T for values of r between 5 and 10.

54. Gravity near the Moon The gravitational force between the moon and an astronaut in a space ship located a distance x above the center of the moon is given by the function

$$F(x) = \frac{350}{x^2}$$

where F is measured in newtons (N) and x is measured in megameters (Mm). Graph the function F for values of x between 2 and 8.

55. Toll Road Rates The toll charged for driving on a certain stretch of a toll road depends on the time of day. The amount of the toll charge is given by

$$T(x) = \begin{cases} 5.00 & \text{if } 0 \leq x < 7 \\ 7.00 & \text{if } 7 \leq x \leq 10 \\ 5.00 & \text{if } 10 < x < 16 \\ 7.00 & \text{if } 16 \leq x \leq 19 \\ 5.00 & \text{if } 19 < x < 24 \end{cases}$$

where x is the number of hours since 12:00 A.M.

(a) Graph the function T.

(b) What do the breaks in the graph represent?

56. Postage Rates The domestic postage rate depends on the weight of the letter. In 2009, the domestic postage rate for first-class letters weighing 3.5 oz or less was given by

$$P(x) = \begin{cases} 0.44 & \text{if } 0 \leq x \leq 1 \\ 0.61 & \text{if } 1 < x \leq 2 \\ 0.78 & \text{if } 2 < x \leq 3 \\ 0.95 & \text{if } 3 < x \leq 3.5 \end{cases}$$

where x is the weight of the letter measured in ounces.

(a) Graph the function P.

(b) What do the breaks in the graph represent?

1.7 Working with Functions: Getting Information from the Graph

- Reading the Graph of a Function
- Domain and Range from a Graph
- Increasing and Decreasing Functions
- Local Maximum and Minimum Values

IN THIS SECTION... we use the graph of a function to get information about the function, including where the values of the function increase or decrease and where the maximum or minimum value(s) of the function occur.

GET READY... by reviewing interval notation in **Algebra Toolkit A.2**. Test your skill in working with interval notation by doing the **Algebra Checkpoint** at the end of this section.

The graph of a function allows us to "see" the behavior, or life history, of the function. For example, we can see from the graph of a function the highest or lowest value of the function or whether the values of the function are rising or falling. So if a function represents cost, the lowest point on its graph tells where the minimum cost occurs. If a function represents profit, its graph can tell us where profit is increasing or decreasing. In this section we examine how to obtain these and other types of information from the graph of a function.

Reading the Graph of a Function

If a function models a real-world situation, such as the weight of a person, its graph is usually easy to interpret. For example, suppose the weight of Mr. Hector (in pounds) is given by the function W, where the independent variable x is his age in years. So

$$W(x) = \text{"weight of Mr. Hector at age } x\text{"}$$

The graph of the function W in Figure 1 gives a visual representation of how his weight has changed over time. Note that Mr. Hector's weight $W(x)$ at age x is the height of the graph above the point x.



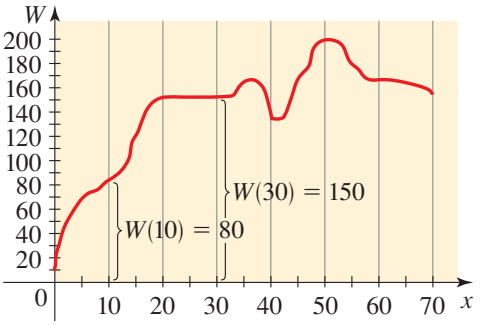


figure 1 Graph of Mr. Hector's weight

example 1 Verbal Description from a Graph

Answer the following questions about the function W graphed in Figure 1.

(a) What was Mr. Hector's weight at age 10? At age 30?

(b) Did his weight increase or decrease between the ages of 40 and 50? Between the ages of 50 and 70?

(c) How did his weight change between the ages of 20 and 30?

(d) What was his minimum weight between the ages of 30 and 50?

(e) What was his maximum weight between the ages of 30 and 70?

(f) What is the net change in his weight from the age of 30 to 50?

Solution

(a) His weight at age 10 is $W(10)$. The value of $W(10)$ is the height of the graph above the x-value 10. From the graph we see that $W(10) = 80$. Similarly, from the graph, $W(30) = 150$.

(b) From the graph we see that the values of the function W were increasing between the x-values 40 and 50, so Mr. Hector's weight was increasing during that period. However, the graph indicates that his weight was decreasing between the ages of 50 and 70.

(c) From the graph we see that Mr. Hector's weight was constant between the ages of 20 and 30. He maintained his weight at 150 lb during that period.

(d) From the graph we see that the minimum value that W achieves between the x-values of 30 and 50 is 130. So Mr. Hector's minimum weight during that period was 130 lb.

(e) From the graph we see that the maximum value that W achieves between the ages of 30 and 70 is 200 lbs. So Mr. Hector's maximum weight during that period was 200 lb.

(f) From the graph we see that at age 30 Mr. Hector weighed 150 lb and at age 50 he weighed 200 lb. We have $W(50) - W(30) = 200 - 150 = 50$, so the net change in his weight between those two ages is 50 lb.

■ **NOW TRY EXERCISE 41** ■

A complete graph of a function contains all the information about the function, because the graph tells us which input values correspond to which output values. To analyze the graph of a function, we must keep in mind that *the height of the graph is the value of the function*. So we can read the values of a function from its graph.

example 2 Finding the Values of a Function from a Graph

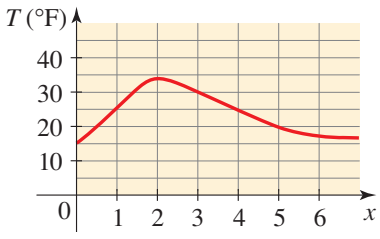


figure 2

The function T graphed in Figure 2 gives the temperature between noon and 6:00 P.M. at a certain weather station.

(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) Which is larger, $T(2)$ or $T(4)$?

(c) Find the value(s) of x for which $T(x) = 25$.

(d) Find the values of x for which $T(x) \geq 25$.

(e) Find the net change in temperature between 3:00 and 5:00 P.M.

Solution

(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at the x-value 1. Thus $T(1) = 25$. Similarly, $T(3) = 30$ and $T(5) = 20$.

(b) Since the graph is higher at the x-value 2 than at the x-value 4, it follows that $T(2)$ is greater than $T(4)$.

(c) The height of the graph is 25 when x is 1 and when x is 4. So $T(x) = 25$ when x is 1 and when x is 4.

Interval notation is reviewed in Algebra Toolkit A.2, page T7.

(d) The graph is higher than 25 for x between 1 and 4. So $T(x) \geq 25$ for all x-values in the interval $[1,4]$.

(e) From the graph we know that $T(5)$ is 20 and $T(3)$ is 30. We have

$$\begin{aligned} T(5) - T(3) &= 20 - 30 \\ &= -10 \end{aligned}$$

So the net change in temperature is -10°F.

■ **NOW TRY EXERCISES 7 AND 43** ■

example 3 Where Graphs of Functions Meet

Use a graphing calculator to draw graphs of the functions $f(x) = 5 - x^2$ and $g(x) = 3 - x$ in the same viewing rectangle.

(a) Find the value(s) of x for which $f(x) = g(x)$.

(b) Find the values of x for which $f(x) \geq g(x)$.

(c) Find the values of x for which $f(x) < g(x)$.

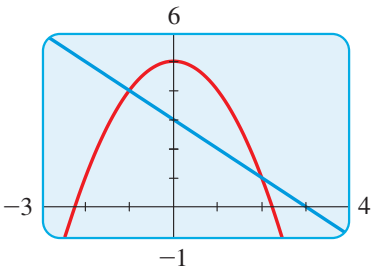


figure **3** Graphs of f and g

Solution

We graph the equations

$$y_1 = 5 - x^2 \qquad \text{and} \qquad y_2 = 3 - x$$

in the same viewing rectangle in Figure 3.

(a) Recall that the value of a function is the height of the graph. So $f(x) = g(x)$ at the x-values where the graphs of f and g meet. From Figure 3 we see that the graphs meet when x is -1 and when x is 2. So $f(x) = g(x)$ when x is -1 and when x is 2.

(b) We need to find the x-values where $f(x) \geq g(x)$. These are the x-values where the graph of f is above the graph of g. From Figure 3 we see that this happens for x between -1 and 2. So $f(x) \geq g(x)$ for x in the interval $[-1, 2]$.

(c) We need to find the x-values where $f(x) < g(x)$. These are the x-values where the graph of f is below the graph of g. From Figure 3 we see that this happens for x strictly less than -1 and x strictly bigger than 2, that is, $x < -1$ and $x > 2$. (We don't include the points -1 and 2 because of the strict inequality.) So $f(x) < g(x)$ for x in the intervals $(-\infty, -1)$ and $(2, \infty)$.

■ **NOW TRY EXERCISE 11** ■

Domain and Range from a Graph

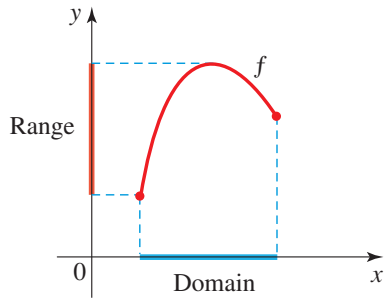
Recall from Section 1.4 that for a function of the form $y = f(x)$ we have the following:

Domain	Range
Inputs	Outputs
Independent variable	Dependent variable
x-values	y-values

So since the graph of f consists of the ordered pairs (x, y), the domain and range of the function can be obtained from the graph as follows.

Domain and Range from a Graph

The **domain** and **range** of a function f are represented in the graph of the function as shown in the figure.



For the function W graphed in Figure 1 on page 74, the domain is the interval $[0, 70]$ and the range is the interval $[10, 200]$. Note that the domain consists of all inputs (ages of Mr. Hector) and the range consists of all outputs (weights of Mr. Hector).

example 4 Domain and Range from a Graph

Let f be the function defined by $f(x) = \sqrt{4 - x^2}$.

(a) Use a graphing calculator to draw a graph of f.

(b) Find the domain and range of f from the graph.

Solution

(a) The graph is shown in Figure 4.

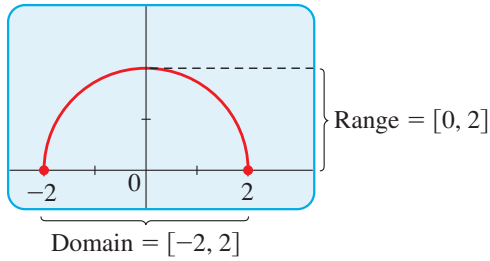


figure 4

(b) From the graph we see that the domain is the interval $[-2, 2]$ and the range is the interval $[0, 2]$.

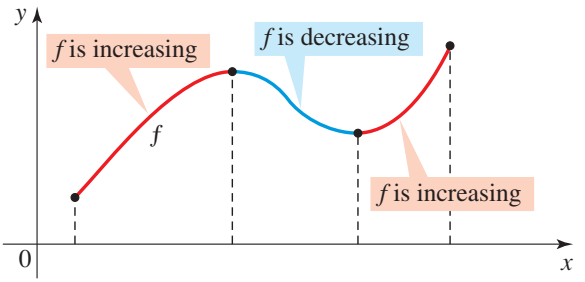
NOW TRY EXERCISE 19

Increasing and Decreasing Functions

At the beginning of this section we saw that the graph of Mr. Hector's weight rises when his weight increases and falls when his weight decreases. In general, a function is said to be *increasing* when its graph rises and *decreasing* when its graph falls.

Increasing and Decreasing Functions

- The function f is **increasing** if the values of $f(x)$ increase as x increases. That is, f is increasing on an interval I if $f(a) < f(b)$ whenever $a < b$ in I.
- The function f is **decreasing** if the values of $f(x)$ decrease as x increases. That is, f is decreasing on an interval I if $f(a) > f(b)$ whenever $a < b$ in I.



example 5 Increasing and Decreasing Functions

Use a graphing calculator to draw a graph of $f(x) = x^3 - 3x^2 + 2$.

(a) Find the intervals on which f is increasing.

(b) Find the intervals on which f is decreasing.

Solution

Using a graphing calculator, we draw a graph of the function f as shown in Figure 5.

(a) From the graph we see that f is increasing on $(-\infty, 0]$ and on $[2, \infty)$ (represented in red in Figure 6).

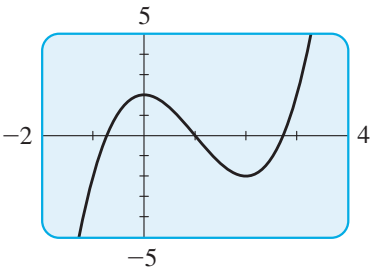


figure **5**

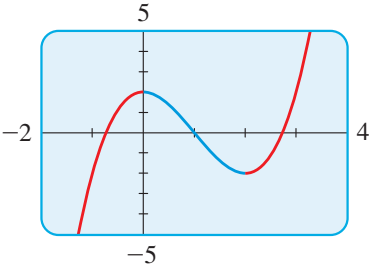


figure **6**

(b) From the graph we see that f is decreasing on $[0, 2]$ (represented in blue in Figure 6).

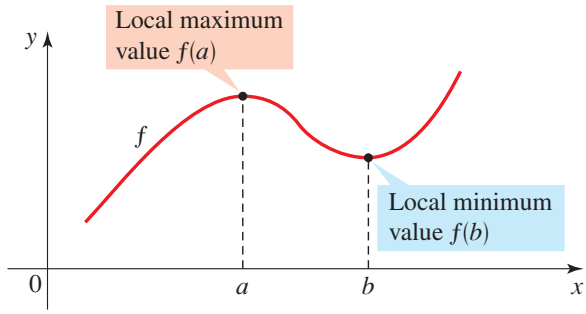
NOW TRY EXERCISE 27

Local Maximum and Minimum Values

Finding the largest or smallest values of a function is important in many applications. For example, if a function represents profit, then we are interested in its maximum value. For a function that represents cost, we would be interested in its minimum value. We can find these values from the graph of a function. We first define what we mean by a local maximum or minimum.

Local Maximum and Minimum Values

- The function value $f(a)$ is a **local maximum value** of f if
$$f(a) \geq f(x) \quad \text{for values of } x \text{ near } a$$
- The function value $f(a)$ is a **local minimum value** of f if
$$f(a) \leq f(x) \quad \text{for values of } x \text{ near } a$$



Intervals are studied in Algebra Toolkit A.2, page T7.

The statement "$f(a) \geq f(x)$ for values of x near a" means that $f(a) \geq f(x)$ for all x in some open interval containing a. Similarly, the statement "$f(a) \leq f(x)$ for values of x near a" means that $f(a) \leq f(x)$ for all x in some open interval containing a.

example 6 Local Maximum and Minimum Values of Functions

Find the local maximum and minimum values of $f(x) = x^3 - 3x^2 + 2$.

Solution

The graph of f is shown in Figure 7 on the next page. From the graph we see that f has a local maximum value 2 at the x-value 0. In other words, $f(0) = 2$ is a local maximum

value (represented by the red dot on the graph in Figure 7). Similarly, $f(2) = -2$ is a local minimum value (represented by the blue dot on the graph in Figure 8).

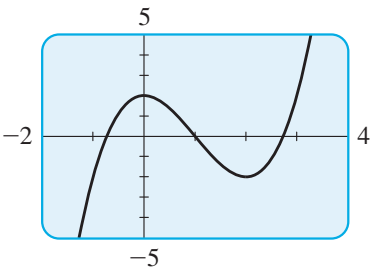


figure **7**

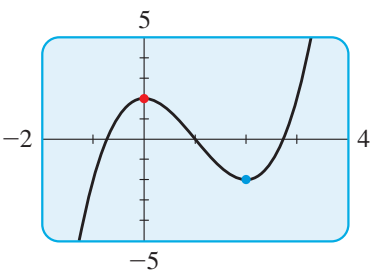


figure **8**

NOW TRY EXERCISE 37

IN CONTEXT ➤



Manfred Steinbach/Shutterstock.com 2009

Highway engineers use mathematics to study traffic patterns and relate them to different road conditions. One feature that they are interested in is the *carrying capacity* of a road—that is, the maximum number of cars that can safely travel along a certain stretch of highway. If the cars drive very slowly past a given point on the road, only a few can pass by every minute. On the other hand, if the cars are zooming quickly past that point, safety concerns require them to be spaced much farther apart, so again not many can pass by every minute. Between these two extremes is an optimal speed at which these two competing tendencies balance to allow as many cars as possible to drive down this stretch of road.

In the next example we use a graphing calculator to analyze the graph of a function developed by an engineer to model the carrying capacity of a highway. (See Exploration 3 on page 560 to learn how this model is obtained.) The model assumes that all drivers observe the "safe following distance" guidelines; in reality, the majority of drivers do not, resulting in traffic congestion and accidents.

example 7 Highway Engineering

A highway engineer develops a formula to estimate the number of cars that can safely travel a particular highway at a given speed. She assumes that each car is 17 feet long, travels at a speed of x mi/h, and follows the car in front of it at the safe following distance for that speed. She finds that the number N of cars that can pass a given point per minute is modeled by the function

$$N(x) = \frac{88x}{17 + 17\left(\frac{x}{20}\right)^2}$$

Graph the function in the viewing rectangle [0, 100] by [0, 60].

(a) Where is the function N increasing? Decreasing?

(b) What is the local maximum value of N? At what x-value does this local maximum occur?

(c) At what speed is the maximum carrying capacity of the road achieved?

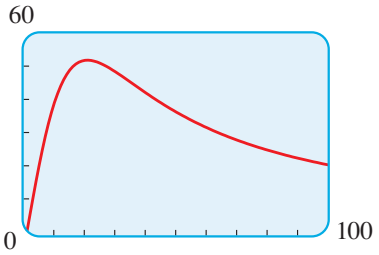


figure **9** Highway capacity at speed x

Solution

The graph is shown in Figure 9.

(a) From the graph we see that the function N is increasing on $[0, 20]$ and decreasing on $[20, 100]$.

(b) From the graph we see that N has a local maximum value of about 52 at the x-value 20. So the highway can accommodate more cars at about 20 mi/h than at higher or lower speeds.

(c) Since N has a local maximum value at the x-value 20, the maximum carrying capacity of the road is achieved at 20 mi/h.

NOW TRY EXERCISE 49

Algebra Checkpoint ✓

Test your skill in working with interval notation. Review this topic in **Algebra Toolkit A.2** on page T7.

1–4 A set of numbers is given.

(a) Give a verbal description of the set.
(b) Express the set in interval notation.
(c) Graph the set on the number line.

1. $\{x \mid 1 \le x \le 4\}$

2. $\{x \mid -3 \le x < 2\}$

3. $\{x \mid -10 < x < -3\}$

4. $\{x \mid x \ge 0\}$

5–8 An interval is given.

(a) Give a verbal description of the interval.
(b) Express the interval in set-builder notation.
(c) Graph the interval on the number line.

5. $(2, 6)$ **6.** $(-4, 3]$ **7.** $(-\infty, 2)$ **8.** $[-1, \infty)$

9–12 The graph of an interval is given.

(a) Give a verbal description of the interval.
(b) Express the interval in set-builder notation.
(c) Express the graphed interval in interval notation.

9. 

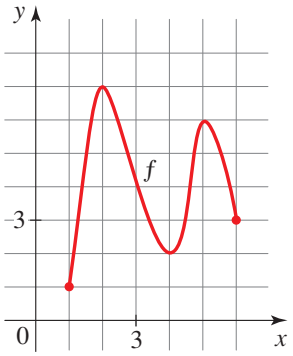
10. 

11. 

12. 

1.7 Exercises

CONCEPTS



Fundamentals

1–4 ■ These exercises refer to the graph of the function f shown at the left.

1. To find a function value $f(a)$ from the graph of f, we find the height of the graph above the x-axis at $x =$ _____. From the graph of f we see that $f(3) =$ _____.

2. The domain of the function f is all the _____-values of the points on its graph, and the range is all the corresponding _____-values. From the graph we see that the domain of f is the interval _____ and the range of f is the interval _____.

3. **(a)** If f is increasing on an interval, then the y-values of the points on the graph _____ (increase/decrease) as the x-values increase. From the graph we see that f is increasing on the intervals _____ and _____.

(b) If f is decreasing on an interval, then the y-values of the points on the graph _____ (increase/decrease) as the x-values increase. From the graph we see that f is decreasing on the intervals _____ and _____.

4. **(a)** A function value $f(a)$ is a local maximum value of f if $f(a)$ is the _____ value of f on some interval containing a. From the graph we see that one local maximum value of f is _____ and that this value occurs when x is _____.

(b) A function value $f(a)$ is a local minimum value of f if $f(a)$ is the _____ value of f on some interval containing a. From the graph we see that one local minimum value of f is _____ and that this value occurs when x is _____.

Think About It

5. In Example 7 we saw a real-world situation in which it is important to find the maximum value of a function. Name several other everyday situations in which a maximum or minimum is important.

6. Draw a graph of a function f that is defined for all real numbers and that satisfies the following conditions: f is always decreasing and $f(x) > 0$ for all x.

SKILLS

7. The graph of a function h is given.

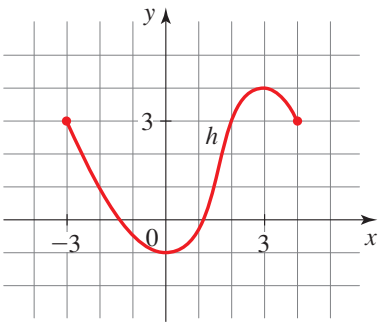
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.

(b) Find the domain and range of h.

(c) Find the values of x for which $h(x) = 3$.

(d) Find the values of x for which $h(x) \le 3$.

(e) Find the net change in the value of h when x changes from -2 to 4.



8. The graph of a function g is given.

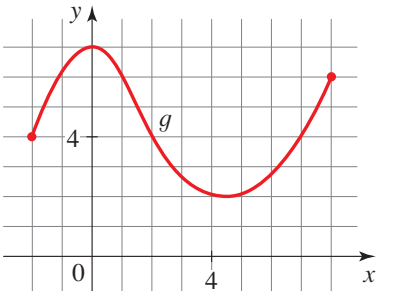
(a) Find $g(-2)$, $g(0)$, and $g(7)$.

(b) Find the domain and range of g.

(c) Find the values of x for which $g(x) = 4$.

(d) Find the values of x for which $g(x) > 4$.

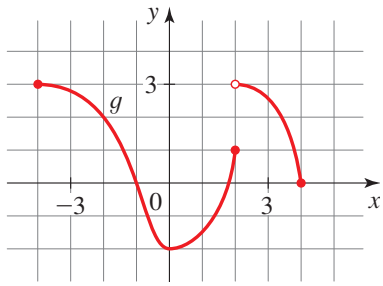
(e) Find the net change in the value of g when x changes from 2 to 7.



9. The graph of a function g is given.

(a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

(b) Find the domain and range of g.

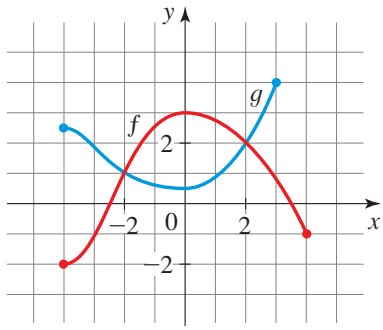


10. Graphs of the functions f and g are given.

(a) Which is larger, $f(0)$ or $g(0)$?

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For which values of x is $f(x) = g(x)$?



11–14 ■ Graph the functions f and g with a graphing calculator. Use the graphs to find the indicated values or intervals; state your answer correct to two decimal places.

(a) Find the value(s) of x for which $f(x) = g(x)$.

(b) Find the values of x for which $f(x) \geq g(x)$.

(c) Find the values of x for which $f(x) < g(x)$.

11. $f(x) = x^2 - 5x + 1, \quad g(x) = -3x + 4$

12. $f(x) = -2x^2 + 3x - 1, \quad g(x) = 3x - 9$

13. $f(x) = 2x^2 + 3, \quad g(x) = -x^2 + 3x + 5$

14. $f(x) = 1 - x^2, \quad g(x) = x^2 - 2x - 1$

15–22 ■ A function f is given.

(a) Use a graphing calculator to draw the graph of f.

(b) Find the domain and range of f from the graph.

15. $f(x) = x - 1$

16. $f(x) = 4$

17. $f(x) = -x^2$

18. $f(x) = 4 - x^2$

19. $f(x) = \sqrt{16 - x^2}$

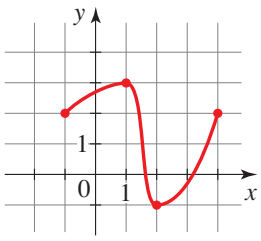
20. $f(x) = -\sqrt{25 - x^2}$

21. $f(x) = \sqrt{x - 1}$

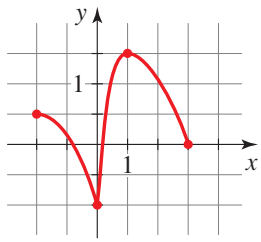
22. $f(x) = \sqrt{x + 2}$

23–26 ■ The graph of a function is given. Determine the intervals on which the function is **(a)** increasing and **(b)** decreasing.

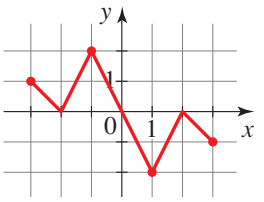
23.



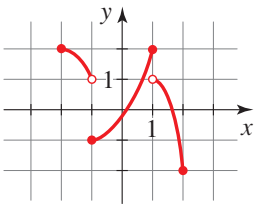
24.



25.



26.



27–32 ■ A function f is given.

(a) Use a graphing device to draw the graph of f.

(b) State approximately the intervals on which f is increasing and on which f is decreasing.

27. $f(x) = x^2 - 5x$

28. $f(x) = x^3 - 4x$

29. $f(x) = 2x^3 - 3x^2 - 12x$

30. $f(x) = x^4 - 16x^2$

31. $f(x) = x^3 + 2x^2 - x - 2$

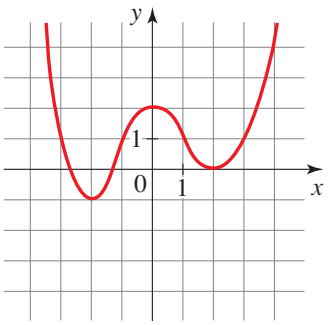
32. $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$

33–36 ■ The graph of a function is given.

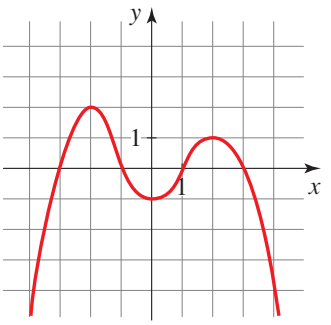
(a) Find all the local maximum and minimum values of the function and the value of x at which each occurs.

(b) Find the intervals on which the function is increasing and on which the function is decreasing.

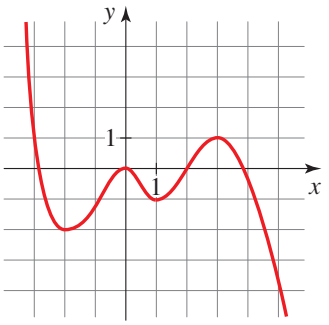
33.



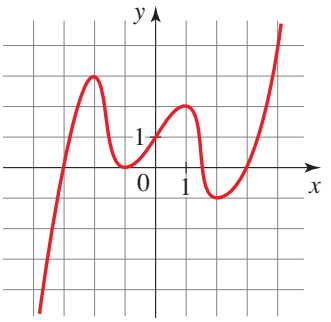
34.



35.



36.



37–40 ■ A function is given. Use a graphing calculator to draw a graph of the function.

(a) Find all the local maximum and minimum values of the function and the value of x at which each occurs. State each answer correct to two decimals.

(b) Find the intervals on which the function is increasing and on which the function is decreasing. State each answer correct to two decimal places.

37. $f(x) = x^3 - x$

38. $g(x) = 3 + x + x^2 - x^3$

39. $F(x) = x\sqrt{6 - x}$

40. $G(x) = x\sqrt{x - x^2}$

CONTEXTS

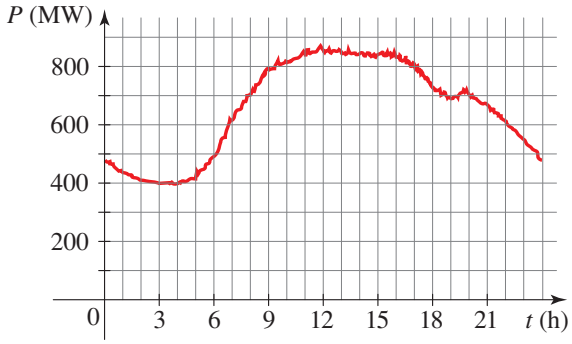
41. Power Consumption The figure shows the power consumption in San Francisco for September 19, 1996. (P is measured in megawatts; t is measured in hours starting at midnight.)

(a) What was the power consumption at 6:00 A.M.? At 6:00 P.M.?

(b) When was the power consumption a maximum?

(c) When was the power consumption a minimum?

(d) What is the net change in the values of P as the value of x changes from 0 to 12?



Source: Pacific Gas & Electric.

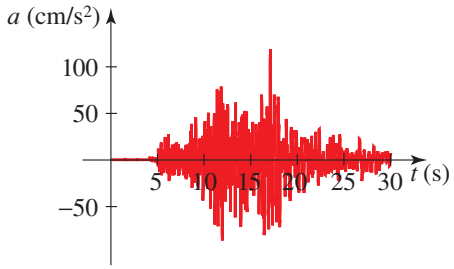
42. Earthquake The graph shows the vertical acceleration of the ground from the 1994 Northridge earthquake in Los Angeles, as measured by a seismograph. (Here t represents the time in seconds.)

(a) At what time t did the earthquake first make noticeable movements of the earth?

(b) At what time t did the earthquake seem to end?

(c) At what time t was the maximum intensity of the earthquake reached?

(d) What is the approximate net change in the intensity of the earthquake as the value of t changes from 5 to 30?



Source: California Department of Mines and Geology.

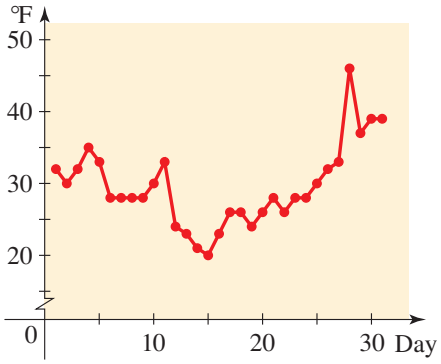
43. Low Temperatures In January 2007 the state of California experienced remarkably cold weather. Many crops that usually thrive in California were lost because of the frost. Orange crops just ripening in Tulare County, California, were frozen on the trees. The table and graph on the next page show the daily low temperatures T in Tulare County for the month of January 2007.

(a) Find $T(1)$ and $T(15)$.

(b) Which is larger, $T(15)$ or $T(19)$?

(c) On what day(s) was the daily low temperature below 32°F? On what day was it the lowest?

(d) Find the net change in the daily low temperatures from January 1 to January 31.



Day	Daily low temperature (°F)	Day	Daily low temperature (°F)	Day	Daily low temperature (°F)
1	32	12	24	23	28
2	30	13	23	24	28
3	32	14	21	25	30
4	35	15	20	26	32
5	33	16	23	27	33
6	28	17	26	28	46
7	28	18	26	29	37
8	28	19	24	30	39
9	28	20	26	31	39
10	30	21	28		
11	33	22	26		

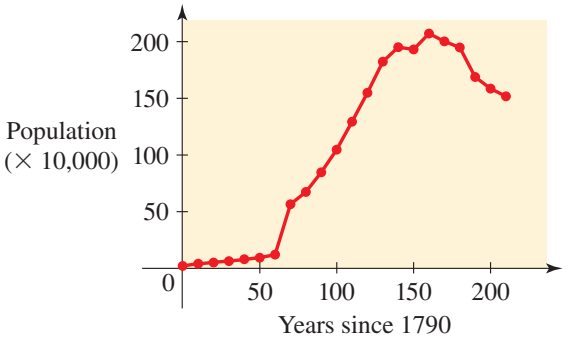
44. Population of Philadelphia The table and graph below show the history of the population P of the city of Philadelphia from 1790 to 2000.

(a) Find $P(100)$ and $P(200)$.

(b) Which is larger, $P(160)$ or $P(210)$?

(c) In what year did Philadelphia have its largest population? In what year(s) did Philadelphia have a population over 1.8 million?

(d) Find the net change in the population of Philadelphia from 1950 to 2000.

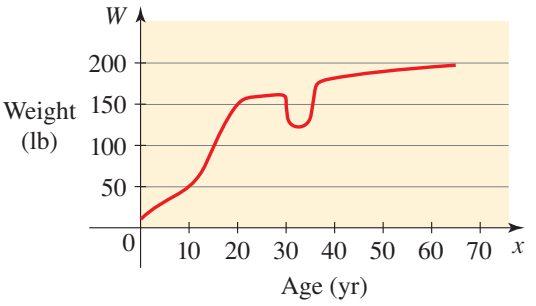


Years since 1790	Population (× 10,000)	Years since 1790	Population (× 10,000)
0	2.85	110	129.37
10	4.12	120	154.90
20	5.37	130	182.34
30	6.38	140	195.10
40	8.05	150	193.13
50	9.37	160	207.16
60	12.14	170	200.25
70	56.55	180	194.86
80	67.40	190	168.82
90	84.72	200	158.56
100	104.70	210	151.76

45. Weight Function The graph gives the weight W of a person at age x.

(a) Determine the intervals on which the function W is increasing and on which it is decreasing.

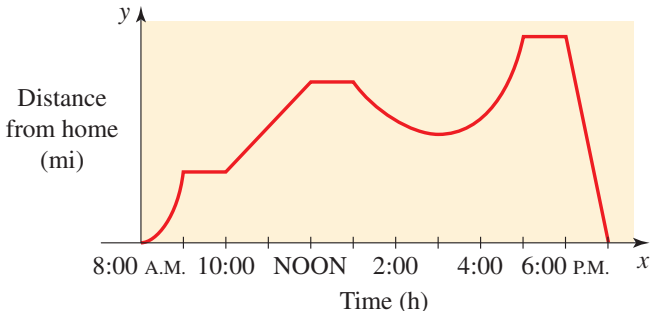
(b) What do you think happened when this person was 30 years old?

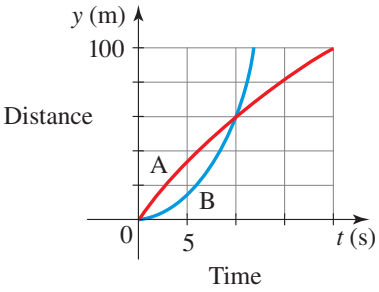


46. Distance Function The graph gives a sales representative's distance from his home as a function of time on a certain day.

(a) Determine the time intervals on which his distance from home was increasing and on which it was decreasing.

(b) Describe in words what the graph indicates about his travels on this day.





47. Running Race Two runners compete in a 100-meter race. The graph in the margin depicts the distance run as a function of time for each runner.

(a) Did each runner finish the race? Who won the race?

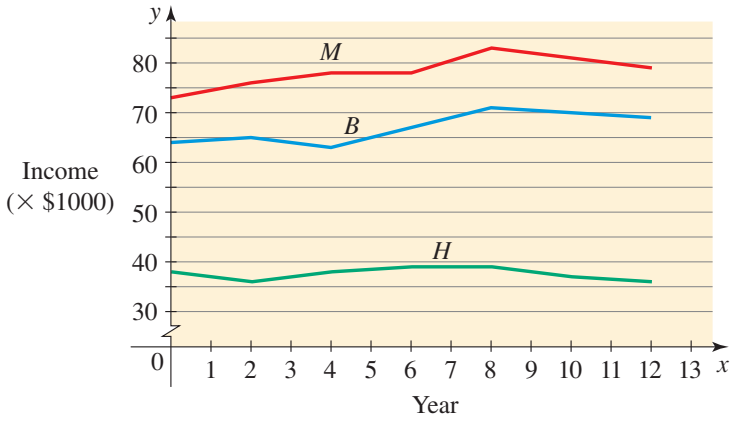
(b) At what time did one runner overtake the other?

(c) On what time interval was Runner A leading?

48. Education and Income The graph shows the yearly median income H for Americans with a high school diploma, the yearly median income B of Americans with a bachelor's degree, and the yearly median income M of Americans with a master's degree, all in the time period 1991 to 2003 (with $x = 0$ corresponding to 1991).

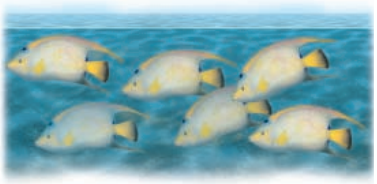
(a) Find the interval on which all three functions are decreasing.

(b) Find the net change of each function from 1995 to 1999. Which function had the most net change in this time period?



49. Migrating Fish Suppose a fish swims at a speed v relative to the water, against a current of 5 mi/h. Using a mathematical model of energy expenditure, it can be shown that the total energy E required to swim a distance of 10 mi is given by

$$E(v) = 2.73v^3 \frac{10}{v - 5} \qquad 5.1 \le v \le 10$$



Biologists believe that migrating fish try to minimize the total energy required to swim a fixed distance.

(a) Graph the function E in the viewing rectangle [5.1, 10] by [4000, 13,000].

(b) Where is the function E increasing and where is the function E decreasing?

(c) Find the local minimum value of E. For what velocity v is the energy E minimized? [*Note:* This result has been verified; migrating fish swim against a current at a speed 50% greater than the speed of the current.]

50. Coughing When a foreign object that is lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward, causing an increase in pressure in the lungs. At the same time, the trachea contracts, causing the expelled air to move faster and increasing the pressure on the foreign object. According to a mathematical model of coughing, the velocity v of the air stream through an average-sized person's trachea is related to the radius r of the trachea (in centimeters) by the function

$$v(r) = 3.2(1 - r)r^2 \qquad \tfrac{1}{2} \le r \le 1$$

(a) Graph the function v.

(b) Where is the function v increasing and where is the function v decreasing?

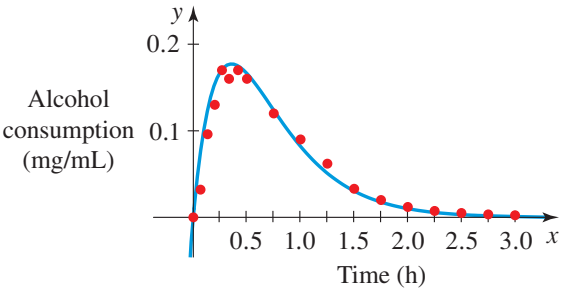
(c) What is the local maximum value of v and at what value of r does the local maximum occur?

51. Algebra and Alcohol The first two columns of the table in the Prologue (page P2) give the alcohol concentration at different times in a three-hour interval following the consumption of 15 mL of alcohol. The data are modeled by the function A, which is graphed below together with a scatter plot of the data. (In Chapter 5 we learn how to find an algebraic expression for the function A.)

(a) Use the graph of A to determine at what time the alcohol concentration is at its maximum level.

(b) On what time interval is the alcohol concentration increasing?

(c) On what time interval is the alcohol concentration decreasing?



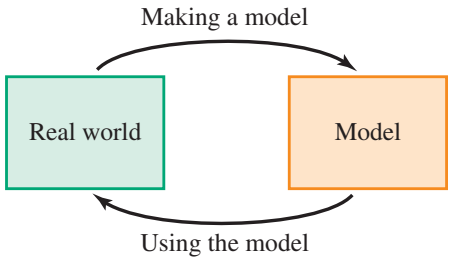
1.8 Working with Functions: Modeling Real-World Relationships

- **Modeling with Functions**
- **Getting Information from the Graph of a Model**

IN THIS SECTION... we make real-world models, not from data but from a verbal description. This includes models for cost, revenue, or depletion of resources and models that depend on some geometrical property. Once a model has been found, we use it to get information about the thing being modeled.

In Section 1.3 we learned how to find equations that model real-world data. In Sections 1.5–1.7 we used functions to model the dependence of one changing quantity on another. In this section we explore the process of making a model from a verbal description of a real-world situation.

Recall that a **model** is a mathematical representation (such as an equation or a function) of a real-world situation. **Modeling** is the process of making mathematical models. Why make a mathematical model? Because we can use the model to answer questions and make predictions about the thing being modeled. The process is described in the following diagram.



Modeling with Functions

We observed in Section 1.3 that many real-world situations can be described by linear models. It is useful to express a linear model in function form; this helps us to obtain information more easily from the model. We begin by creating a linear model for a situation that is described verbally.

example 1 A Model for Cost



A company manufactures baseball caps with school logos. The company charges their customers a fixed fee of $500 for setting up the machines and $8 for each cap produced.

(a) Find a linear model for the cost of purchasing any number of caps from this company. Express the model in function form.

(b) Use the model to find the cost of purchasing 225 caps.

Solution

(a) To find a linear model, we need to assign letters to the quantities involved. Let n represent the number of caps to be purchased. The model we want is a function C that gives the cost of purchasing n caps. From the information given, we can write

$$C = \overbrace{500}^{\text{Fixed cost}} + \overbrace{8n}^{\text{Add \$8 for each cap}}$$

We put this model in function form:

$$C(n) = 500 + 8n$$

(b) To find the cost of purchasing 225 caps, we need to find $C(225)$:

$$\begin{aligned} C(n) &= 500 + 8n && \text{Model} \\ C(225) &= 500 + 8(225) && \text{Replace } n \text{ by } 225 \\ &= 2300 && \text{Calculator} \end{aligned}$$

The cost is $2300.

■ NOW TRY EXERCISE 17 ■

IN CONTEXT ➤

The state of the environment has played an important role in the rise and fall of civilizations throughout history. For example, deforestation is thought to have played a significant role in the decline of the once-flourishing Easter Island culture in the South Pacific. The decline of the world's rain forests is a controversial topic frequently discussed in the news. Fortunately, wise forest management practices are now common in many parts of the world, promising a healthier environment for generations to come. The next example illustrates the impact that paper usage has on our forests.



James Thew/Shutterstock.com 2009

Long-leaf pine forest



Vladimir Korostyshevskiy/Shutterstock.com

Deforested Easter Island

example 2 A Function Model for Paper Consumption

The average U.S. resident uses 650 lb of paper a year. The average pine tree produces 4130 lb of paper.

(a) Find a function N that models the number of trees used for paper in one year by x U.S. residents.

(b) The city of Cleveland Heights, Ohio, had a population of about 49,000 in 2003. Use the model to find the number of trees required to make the paper used by the residents of Cleveland Heights in 2003.

Solution

(a) Since each resident uses an average of 650 lb of paper a year and each tree produces 4130 lb of paper, the number of trees each person uses per year is

$$\text{trees used by each resident} = \frac{\text{amount of paper used by each resident (lb)}}{\text{amount of paper each tree produces (lb/tree)}}$$

$$= \frac{650 \text{ lb}}{4130 \text{ lb/tree)}}$$

$$\approx 0.157 \text{ tree}$$

So each resident uses about 0.157 tree per year. Thus, x residents use $0.157x$ tree per year.

The model we want is a function N that gives the number of trees used by x residents. We can now describe this function as follows:

$$N(x) = 0.157x \qquad \text{Model}$$

(b) For Cleveland Heights the number of residents x is 49,000. Using the model, we have

$$\begin{aligned} N(x) &= 0.157x && \text{Model} \\ N(49{,}000) &= 0.157(49{,}000) && \text{Replace } x \text{ by } 49{,}000 \\ &= 7693 && \text{Calculator} \end{aligned}$$

So the Cleveland Heights residents used about 7693 trees in 2003.

NOW TRY EXERCISE 19

example 3 Irrigating a Garden

A gardener waters his vegetable plot using a drip irrigation system. Water flows slowly from a 1200-gallon tank through a perforated hose network to keep the soil appropriately moist. During the spring planting season, the garden requires 80 gallons of water per day.

(a) Find a function W that gives the amount of water in the tank x days after it has been filled.

(b) Use the function W to find the water remaining in the tank after 3 days and after 12 days.

(c) Calculate $W(20)$. What does your answer tell you?

(d) How many days will it take for the tank to empty?

(e) The gardener prefers not to let the tank empty completely. Instead, he decides to refill it when the level has dropped to 200 gallons. How many days will it take for the water level to drop to this level?

Thinking About the Problem

Let's try a simple case. If the garden has been watered for 10 days after the tank has been filled, how much water is left in the tank? Since the garden requires 80 gal of water per day, the number of gallons used in 10 days is

$$(80 \text{ gal/day})(10 \text{ days}) = 800 \text{ gal}$$

So the number of gallons left in the tank is

$$1200 \text{ gal} - (80 \text{ gal/day})(10 \text{ days}) = 1200 - 800 = 400 \text{ gal}$$

So the amount left in the tank is the number of gallons in a full tank minus the amount of water used per day times the number of days since the tank was filled.

Solution

(a) We need to find the rule W that takes the input x (the number of days since the tank was filled) and gives as output the number of gallons of water $W(x)$ remaining in the tank. We know that the gardener starts out with 1200 gallons and uses 80 gallons per day, so we must subtract the total water usage after x days from the initial amount in the tank to find the number of gallons left in the tank. Let's express these quantities in symbols.

In Words	In Algebra
Days since fill-up	x
Gallons used each day	80
Gallons used since fill-up	$80x$
Gallons left in tank	$1200 - 80x$

So the function W that models the amount of water left in the tank is

$$W(x) = 1200 - 80x \quad \text{Model}$$

(b) To find the water level in the tank after 3 days and after 12 days, we evaluate the function W at 3 and at 12:

$$W(3) = 1200 - 80(3) = 960 \quad \text{Replace } x \text{ by } 3$$

$$W(12) = 1200 - 80(12) = 240 \quad \text{Replace } x \text{ by } 12$$

So the water level after 3 days is 960 gal, and after 12 days is 240 gal.

(c) When x has value 20, the value of the function W is

$$W(20) = 1200 - 80(20) = -400 \quad \text{Replace } x \text{ by } 20$$

We get a negative value for $W(20)$. It is impossible for the tank to contain a negative amount of water, so this must mean that the gardener would run out of water before 20 days have passed.

Solving equations is reviewed in Algebra Toolkit C.1, page T47.

(d) An empty tank means that the water level has dropped to zero; that is, $W(x) = 0$.

$$W(x) = 1200 - 80x \quad \text{Model}$$

$$1200 - 80x = 0 \quad \text{Replace } W(x) \text{ by 0 and switch sides}$$

$$1200 = 80x \quad \text{Add } 80x \text{ to each side}$$

$$x = \frac{1200}{80} \quad \text{Divide by 80 and switch sides}$$

$$x = 15 \quad \text{Calculator}$$

The tank will empty after 15 days.

(e) We need to find how many days x it takes for the water level $W(x)$ to drop to 200.

$$W(x) = 1200 - 80x \quad \text{Model}$$

$$1200 - 80x = 200 \quad \text{Replace } W(x) \text{ by 200 and switch sides}$$

$$1000 - 80x = 0 \quad \text{Subtract 200 from each side}$$

$$1000 = 80x \quad \text{Add } 80x \text{ to each side}$$

$$x = \frac{1000}{80} \quad \text{Divide by 80, switch sides}$$

$$x = 12.5 \quad \text{Calculator}$$

The gardener needs to refill the tank every $12\frac{1}{2}$ days.

NOW TRY EXERCISE 21

Getting Information from the Graph of a Model

We now model real-world phenomena by a function, then use the graph of the function to get information about the thing being modeled.

example 4 Modeling the Volume of a Box

A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 3 times its depth, and its height is 5 times its depth.

(a) Find a function that models the volume of the box in terms of its depth, and graph the function.

(b) Find the volume of the box if the depth is 1.5 in.

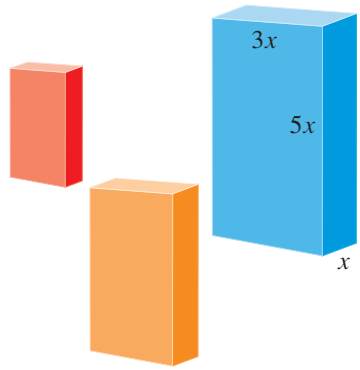
(c) For what depth is the volume 90 in^3?

(d) For what depth is the volume greater than 60 in^3?

Thinking About the Problem

Let's experiment with the problem. If the depth is 1 in., then the width is 3 in. and the height is 5 in. So in this case the volume is $V = 1 \times 3 \times 5 = 15$ in^3. The table gives other values. Notice that all the boxes have the same shape, and the greater the depth, the greater the volume.

Depth	Volume
1	$1 \times 3 \times 5 = 15$
2	$2 \times 6 \times 10 = 120$
3	$3 \times 9 \times 15 = 405$
4	$4 \times 12 \times 20 = 960$



Solution

(a) To find the function that models the volume of the box, we first recall the formula for the volume of a rectangular box.

$$\text{volume} = \text{depth} \times \text{width} \times \text{height}$$

There are three varying quantities: depth, width, and height. Because the function we want depends on the depth, we let

$$x = \text{depth of the box}$$

Then we express the other dimensions of the box in terms of x.

In Words	In Algebra
Depth	x
Width	$3x$
Height	$5x$

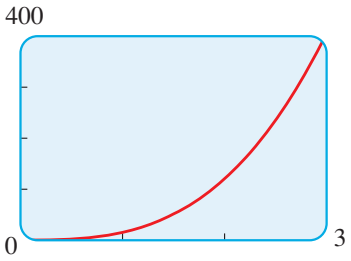


figure **1** $V(x) = 15x^3$

The model is the function V that gives the volume of the box in terms of the depth x.

$$\text{volume} = \text{depth} \times \text{width} \times \text{height}$$

$$V(x) = x \cdot 3x \cdot 5x$$

$$V(x) = 15x^3$$

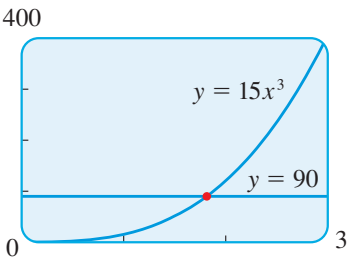
So the volume of the box is modeled by the function $V(x) = 15x^3$. The function V is graphed in Figure 1.

(b) If the depth is 1.5 in., the volume is $V(1.5) = 15(1.5)^3 = 50.625$ in^3.

(c) We need to solve the equation $V(x) = 90$. The function $V(x) = 15x^3$ and the line $y = 90$ are graphed in Figure 2. Using the TRACE feature on a graphing calculator, we find that the two graphs intersect when $x \approx 1.82$, so the volume is 90 in^3 when the depth is about 1.82 in.

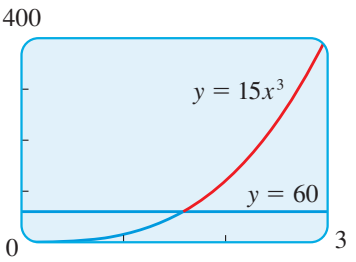
(d) We need to solve the inequality $V(x) \geq 60$. The function $V(x) = 15x^3$ and the line $y = 60$ are graphed in Figure 3. Using the TRACE feature on a graphing calculator, we find that the two graphs intersect when $x \approx 1.59$ and hence that $V(x) \geq 60$ when $x \geq 1.59$ (as shown in red in Figure 3). So the volume is greater than 60 in^3 when the depth is greater than 1.59 in.

We can also solve this equation algebraically. (See Algebra Toolkit C.1, page T47.)



$15x^3 = 90$

figure **2**



$15x^3 \geq 60$

figure **3**

NOW TRY EXERCISE 27

In the next example we find a model that allows us to maximize the area that can be enclosed by a fence of fixed length.

example 5 Fencing a Garden

A gardener has 140 ft of fencing to fence a rectangular vegetable garden.

(a) Find a function that models the area of the garden she can fence.

(b) For what range of widths is the area greater than 825 ft^2?

(c) Can she fence a garden with area 1250 ft^2?

(d) Find the dimensions of the largest area she can fence.

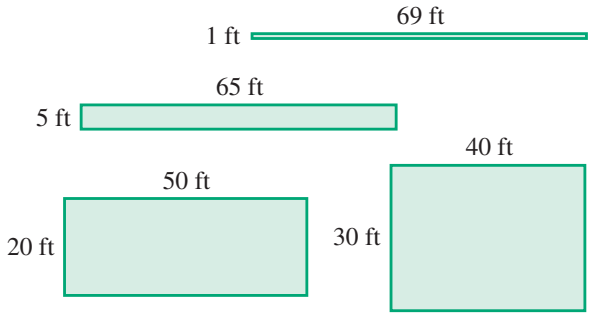
Thinking About the Problem

If the gardener fences a plot with width 10 ft, then the length must be 60 ft, because $10 + 10 + 60 + 60 = 140$. So the area is

$$A = \text{width} \times \text{length} = 10 \cdot 60 = 600 \text{ ft}^2$$

The table shows various choices for fencing the garden. We see that as the width increases, the area of the garden first increases and then decreases.

Width	Length	Area
10	60	600
20	50	1000
30	40	1200
40	30	1200
50	20	1000
60	10	600



Solution

(a) The model that we want is a function that gives the area she can fence. So we begin by recalling the formula for the area of a rectangle.

$$\text{area} = \text{width} \times \text{length}$$

There are two varying quantities: width and length. Because the function we want depends on only one variable, we let

$$x = \text{width of the garden}$$

Then we must express the length in terms of x. The perimeter is fixed at 140 ft, so the length is determined once we choose the width. If we let the length be l, as in Figure 4, then $2x + 2l = 140$, so $l = 70 - x$. We summarize these facts.

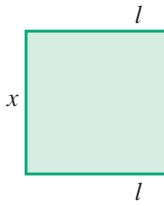


figure **4**

In Words	In Algebra
Width	x
Length	$70 - x$

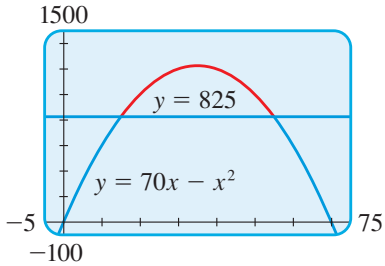


figure **5**

The model we want is the function A that gives the area of the garden for any width x.

$$\text{area} = \text{width} \times \text{length}$$

$$A(x) = x(70 - x)$$

$$A(x) = 70x - x^2$$

The area that she can fence is modeled by the function $A(x) = 70x - x^2$.

(b) We need to solve the inequality $A(x) \geq 825$. To solve graphically, we graph $y = 70x - x^2$ and $y = 825$ in the same viewing rectangle (see Figure 5). We see that the graph of A is higher than the graph of $y = 825$ for $15 \leq x \leq 55$.

(c) From Figure 6 we see that the graph of $A(x)$ always lies below the line $y = 1250$, so an area of 1250 ft^2 is never attained. Hence, the gardener cannot fence an area of 1250 ft^2.

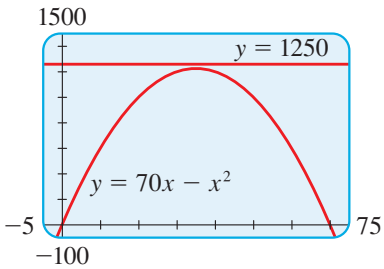


figure **6**

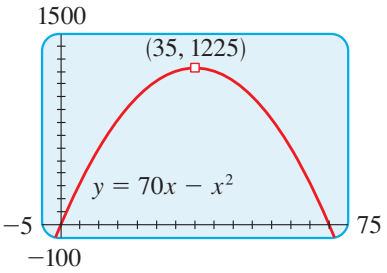


figure **7**

(d) We need to find where the maximum value of the function $A(x) = 70x - x^2$ occurs. The function is graphed in Figure 7. Using the TRACE feature on a graphing calculator, we find that the function achieves its maximum value at $x = 35$. So the maximum area that she can fence occurs when the garden's width is 35 ft and its length is $70 - 35 = 35$ ft. Then the maximum area is $35 \times 35 = 1225$ ft^2.

NOW TRY EXERCISE 29

1.8 Exercises

SKILLS

1–6 ■ Find a function that models the quantity described.

1. The number N of days in w weeks.

2. The number N of cents in q quarters.

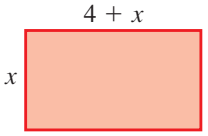
3. The sum S of two consecutive integers, the first integer being n.

4. The sum S of a number n and its square.

5. The product P of a number x and twice that number.

6. The product P of a number y and one and a half times that number.

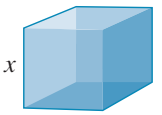
7–12 ■ Find a function that models the quantity described. You may need to consult the formulas for area and volume listed on the inside back cover of this book.



7. The area A of a rectangle whose length is 4 ft more than its width x.

8. The perimeter P of a rectangle whose length is 4 ft more than its width x.

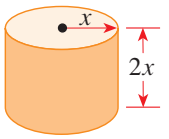
9. The volume V of a cube of side x.



10. The volume B of a box with a square base of side x and height $2x$.

11. The area A of a triangle whose base is twice its height h.

12. The volume V of a cylindrical can whose height is twice its radius, as shown in the figure.



13–16 ■ In these problems you find a function that models a real-life situation and then use the graphing calculator to graph the model and answer questions about the situation. Exercise 13 shows the steps involved in solving these problems.

13. Consider the following problem: Find two numbers whose sum is 19 and whose product is as large as possible.

(a) Experiment with the problem by making a table like the one in the margin, showing the product of different pairs of numbers that add up to 19. On the basis of the evidence in your table, estimate the answer to the problem.

(b) Find a function f that models the product $f(x)$ in terms of one of the numbers x.

(c) Use a graphing calculator to graph the model and solve the problem. Compare with your answer to part (a).

First number	Second number	Product
1	18	18
2	17	34
3	16	48
⋮	⋮	⋮

14. Find two positive numbers whose sum is 100 and the sum of whose squares is a minimum.

15. Find two numbers whose sum is -24 and whose product is a maximum.

16. Among all rectangles that have a perimeter of 20 ft, find the dimensions of the one with the largest area.

CONTEXTS

17. Tee Shirt Cost A tee shirt company makes tee shirts with school logos. The company charges a fixed fee of $200 to set up the machines plus $3.50 per tee shirt.

(a) Find a function C that models the cost of purchasing x tee shirts.

(b) Use the model to find the cost of purchasing 600 tee shirts.

18. Rental Cost A flea market charges vendors a fixed fee of $60 a month plus 75 cents per square foot for renting a space.

(a) Find a function C that models the cost for one month's rental of a space with area x square feet.

(b) Use the model to find the cost of one month's rental of a space with area 150 square feet.

19. Gas Cost The cost of driving a car depends on the number of miles driven and the gas mileage of the car. Kristi owns a Honda Accord that gets 30 miles to the gallon.

(a) Find a function C that models the cost of driving Kristi's car x miles if the cost of gas is $3.20 per gallon.

(b) Use the model to find the cost of driving Kristi's car 500 miles.

(c) Kristi's budget for gas is $250 a month. Use the model to find the number of miles Kristi can drive each month without exceeding her monthly gas budget.

20. Exchange Rate Jason travels from his home in Connecticut to Germany to visit his grandparents. At the time the euro/dollar exchange rate was 1.5532, which means that each euro cost 1.5532 U.S. dollars.

(a) Find a function A that models the number of U.S. dollars required to purchase x euros.

(b) Jason bought a vase in Hamburg for his grandmother for 153.00 euros. Use the model to find the price of the vase in U.S. dollars.

(c) The day before returning home, Jason found that he had 200 U.S. dollars worth of traveler's checks left. He decided to convert these to euros to spend in Germany. Use the model to find how many euros he received for his $200.

21. Cost of Wedding Sherri and Jonathan are getting married. They have a budget of $5000. They are planning the reception and choose a reception hall that costs $700, a DJ that costs $300, a caterer that charges $18.50 a plate, and a wedding cake that costs $1.50 per guest.

(a) Complete the table for the cost of the reception for the given number of guests.

Number of guests	Cost of hall	Cost of DJ	Cost of caterer	Cost of wedding cake	Total cost of reception
10	$700	$300	$185	$15	$1200
20					
30					
40					
50					

(b) Find a function C that models the cost of the reception when x guests attend.

(c) Determine how much the reception would cost if 75 people attend; that is, find the value of $C(75)$.

(d) Determine how many people can attend the reception if Sherri and Jonathan spend their total budget of \$5000; that is, find the value of x when $C(x) = 5000$.

22. Cost of Reception A business group is hosting a reception for local dignitaries. The group chooses to hold the event at an exclusive country club that charges a \$2000 rental fee. In addition, they choose a caterer that charges \$21.00 a plate, gifts that cost \$5.00 per guest, and decorations that cost \$1500.

(a) Find a function C that models the cost of hosting the reception when x guests attend.

(b) Determine how much the reception would cost if 200 people attend; that is, find the value of $C(200)$.

(c) Determine how many people can attend the reception if the business group's budget for the reception is \$10,000.

23. Discounts An art supply store has a sale on picture frames, advertising "Buy One, Get the 2nd for One Penny."

(a) Complete the table for the total cost of purchasing the indicated number of picture frames.

Number of frames	Number that are \$10 each	Number that are 1 cent each	Total cost
2	1	1	\$10.01
4			
6			
8			
10			

(b) Find a function C that models the cost of purchasing x frames that normally cost \$10 each. (Assume that x is an even number.)

(c) Aaron needs 20 picture frames for all his childhood pictures. Use the model to find Aaron's cost of getting these frames, if all the frames he gets normally cost \$10 each.

24. Discounts A competitor of the art supply store in Exercise 23 offers a 35% discount on all frames.

(a) Find a function C that models the cost of purchasing x frames that normally cost \$10 each.

(b) Use the model to find Aaron's cost of getting 20 frames with the competitor's sale, if all the frames normally cost \$10. Is this a better deal than the one in Exercise 23?

25. Volume of Cereal Box A breakfast cereal manufacturer packages cereal in boxes that are 4 inches taller than they are wide and always have a depth of 3 inches.

(a) Complete the table for the dimensions and volume of a cereal box.

Width (in.)	Height (in.)	Depth (in.)	Volume (in^3)
3	7	3	63
4			
5			
6			
7			

(b) Find a function V that models the volume of a cereal box that is x inches wide.

(c) Use the model to find the volume of a cereal box that is 10 in. wide.

(d) The manufacturer makes a box of wheat bran cereal with a volume of 300 in^3. Use a graphing calculator to find the width of this box, as in Example 4.

26. Profit of Fund-Raiser A land conservancy in California organizes several fund-raisers every year. One year, the board of directors for the conservancy suggests raising money by offering tours of their nature preserve for a price of $50 per person. They believe they can attract more people if they offer group discounts of $1 per person. So if two people go on the tour, they will charge $49 per person (for a total of $98); if three people go on the tour, they will charge $48 per person (for a total of $144), and so on.

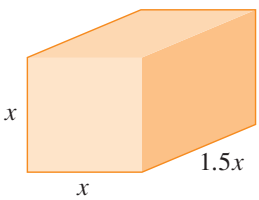
(a) Complete the table for the revenue from a tour with the given number of people.

Number of people in tour	Price per person	Revenue
1	$50	$50
2	$49	$98
3	$48	$144
4		
5		
6		

(b) Find a function R that models the revenue when x people take the tour.

(c) Find the revenue if 10 people go on a tour; that is, find the value of $R(10)$.

(d) Use a graphing calculator to find the number of people that must go on the tour in order for the conservancy to raise $650.



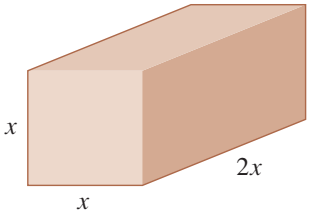
27. Volume of a Container A Florida orange grower ships orange juice in rectangular plastic containers that have square ends and are one and a half times as long as they are wide. (See the figure.)

(a) Find a function V that models the volume of a container of width x.

(b) Use the model to find the volume of a container of width 10 in.

(c) Graph the function V. Use the graph to find the width of the plastic container that has a volume of 315 in^3.

(d) Use the graph from part (c) to find the widths for which the container has volume greater than 450 in^3. Express your answer in interval notation.



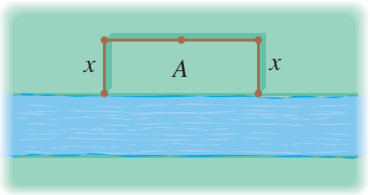
28. Volume of Box A shipping company uses boxes that have square ends and are twice as long as they are wide. (See the figure.)

(a) Find a function S that models the surface area of a box whose square end is x in. wide.

(b) The company uses boxes that are 8 in. wide to ship cans of beans. Use the model to find the area of the material used to make each box.

(c) A box that ships a dozen cans of soup has a surface area of 330 in^2. Graph the function S to find the width of that box.

(d) To ensure that the boxes are strong enough to safely hold their contents, they should have a surface area no larger than 550 in^2. Use the graph from part (c) to find all possible widths for the shipping box. Express your answer in interval notation.

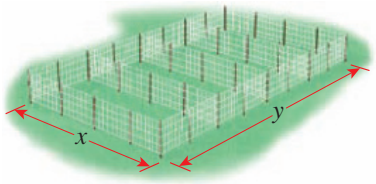


29. Fencing a Field Consider the following problem: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river (see the figure). What are the dimensions of the field of largest area that he can fence?

(a) Experiment with the problem by drawing several diagrams illustrating the situation. Calculate the area of each configuration, and use your results to estimate the dimensions of the largest possible field.

(b) Find a function A that models the area of a field in terms of one of its sides x.

(c) Use a graphing calculator to find the dimensions of the field of largest area. Compare with your answer to part (a).



30. Dividing a Pen A rancher with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Show that the total area of the four pens is modeled by the function

$$A(x) = \frac{x(750 - 5x)}{2}$$

(b) Use a graphing calculator to find the largest possible total area of the four pens.

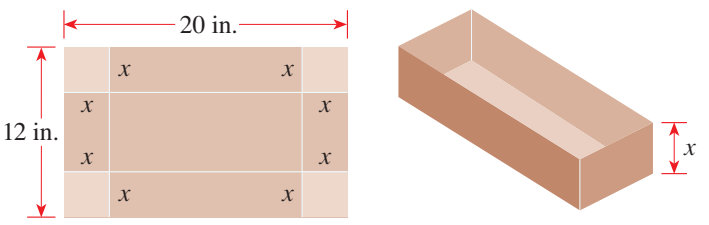
31. Volume of a Box A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides (see the figure).

(a) Show that the volume of the box is modeled by the function

$$V(x) = x(12 - 2x)(20 - 2x)$$

(b) Use a graphing calculator to find the values of x for which the volume is greater than 200 in^3.

(c) Use a graphing calculator to find the largest volume that such a box can have.



32. Area of a Box A box with an open top and a square base is to have a volume of 12 ft^3.

(a) Find a function S that models the surface area of the box.

(b) Use a graphing calculator to find the box dimensions that minimize the amount of material used.

1.9 Making and Using Formulas

- What Is a Formula?
- Finding Formulas
- Variables with Subscripts
- Reading and Using Formulas

IN THIS SECTION... we learn about formulas. Formulas are the fundamental way in which algebra is used in everyday life, including in science and engineering courses. In this section we learn to read and use formulas as well as make formulas.

We are already familiar with many formulas. For example, you certainly know the formula for the area of a circle, $A = \pi r^2$. You may remember the formula $P = kT/V$ from your science courses; this formula relates the pressure P, volume V, and temperature T of a gas. No doubt you've heard of Einstein's famous formula relating energy and mass,

$$E = mc^2$$

where E is energy, m is mass, and c is the speed of light (186,000 mi/s). In this section we study formulas and how they are used to model real-world phenomena.

What Is a Formula?

A **formula** is simply an equation involving variables. The term *formula* is employed when an equation is used to calculate specific quantities (such as the area of a circle) or when it is used to describe the relationship between real-world quantities (such as the formula that relates pressure, volume, and temperature). Formulas provide a compact way of describing relationships between real-world quantities. Many of the equations we encountered in the preceding sections, such as the equation $d = 16t^2$, which relates the distance d an object falls to the time t it has been falling, are also called formulas. In this section we study formulas that involve *several variables*. We learn to read formulas, that is, to understand what the form of a formula tells us. We also learn to find and use formulas.

For example, suppose you're paid \$8 an hour at your part-time job. If we let n stand for the number of hours you work and P stand for your pay, then your pay is modeled by the formula $P = 8n$. This formula works as long as the pay is \$8 an hour. We can find a formula that models your pay for *any* hourly wage w:

$$P = wn$$

where P = pay, w = wage, and n = number of hours worked. (Notice how we use letters that help us remember what the variables mean: P for pay, w for wage, n for number of hours worked.) We can read this formula as

"Pay equals hourly wage times the number of hours worked"

The algebraic structure of this formula tells us how the variables are related. For example, since w and n are multiplied together to give P, it follows that the larger w or n is, the larger P is. In other words, if you get a larger hourly wage or you work more hours, you'll get paid more.

Finding Formulas

In the next two examples we explore the process of finding formulas that model real-life situations. In trying to find a formula, it's often helpful to try a simple example with small numbers to see more easily how the variables in the model are related.

example 1 Finding a Formula for Gas Mileage

The "gas mileage" of a car is the number of miles it can travel on one gallon of gas.

(a) Find a formula that models gas mileage in terms of the number of miles driven and the number of gallons of gasoline used.

(b) Mike's car uses 10.5 gallons to drive 230 miles. Find its gas mileage.

Thinking About the Problem

Let's try a simple case. If a car uses 2 gallons to travel 100 miles, then it would travel just 50 miles on one gallon; that is, we can see that

$$\text{gas mileage} = \frac{100 \text{ miles}}{2 \text{ gallons}} = 50 \text{ mi/gal}$$

So "gas mileage" is the number of miles driven divided by the number of gallons used.

Solution

(a) From the simple example above we can express the formula in words as

$$\text{gas mileage} = \frac{\text{number of miles driven}}{\text{number of gallons used}}$$

To express the model as a formula, we need to assign symbols to the variables involved.

In Words	In Algebra
Number of miles driven	N
Number of gallons used	G
Gas mileage (mi/gal)	M

We can now express the formula we want as follows:

$$M = \frac{N}{G}$$

(b) To get the gas mileage, we use the formula we have just found. We substitute 230 for N and 10.5 for G in the formula.

$$M = \frac{N}{G} \qquad \text{Formula}$$

$$= \frac{230}{10.5} \qquad \text{Replace } N \text{ by 230 and } G \text{ by 10.5}$$

$$\approx 21.9 \qquad \text{Calculator}$$

The gas mileage for Mike's car is 21.9 mi/gal.

■ NOW TRY EXERCISE 21 ■

example 2 Finding a Formula for Surface Area

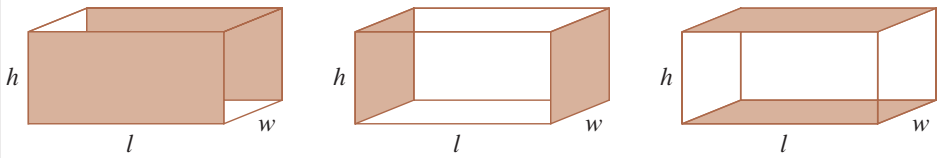
A rectangular box is to be made of plywood.

(a) Find a formula for the area of plywood needed to build a box of any size.

(b) Find the area of plywood needed to build a box 8 ft long, 5 ft wide, and 3 ft high.

Thinking About the Problem

We want a formula for the surface area of a box. A sketch of a box shows that the front and back of the box have identical areas; the same holds for the left and right sides and the top and bottom.



So we conclude that

surface area = 2 × (area of front) + 2 × (area of side) + 2 × (area of bottom)

Since the front, side, and bottom are rectangles, we can easily find their areas.

Solution

(a) To express the surface area as a formula, we need to assign symbols to the variables involved.

In Words	In Algebra
Length	l
Width	w
Height	h
Surface area	S

We need to express the following "word" formula as an algebraic formula:

surface area = 2 × (area of front) + 2 × (area of side) + 2 × (area of bottom)

Now, the area of the front is lh, the area of a side is wh, and the area of the bottom is lw. So we can now express the formula we want as

$$S = 2lh + 2wh + 2lw$$

(b) To get the surface area of the box that we want to build, we use the formula we found in part (a), replacing l by 8, w by 5, and h by 3:

$$\begin{aligned} S &= 2lh + 2wh + 2lw && \text{Formula} \\ &= 2(8\cdot 3) + 2(5\cdot 3) + 2(8\cdot 5) && \text{Replace } l \text{ by 8, } w \text{ by 5, and } h \text{ by 3} \\ &= 158 && \text{Calculator} \end{aligned}$$

The area of plywood needed to build this box is 158 ft^2.

■ NOW TRY EXERCISE 17 ■

Variables with Subscripts

In writing a formula, the goal is to express the relationship between the variables as clearly and concisely as possible. One way we do this is to use letters that clearly remind us of the quantities the letters denote—for example, A for area, V for volume, P for pressure, S for surface area, and so on. Sometimes two or more variables of the same "type" occur in a formula. In this case we use the same letter with different subscripts to denote these variables. For example, the formula for the gravitational force between two objects involves the two masses of these objects. We could use a different letter for each mass (such as M and N), but it's much clearer to denote the two masses using the same letter m but with different subscripts. So m_1 and m_2 denote two different variables. With this notation, Newton's formula for the gravitational force between two objects is

$$F = G\frac{m_1 m_2}{d^2}$$

where F is the gravitational force, m_1 and m_2 are the masses of the two objects, d is the distance between them, and G is the universal gravitational constant ($G \approx 6.673 \times 10^{-11}$ N-m^2/kg^2). We will use this formula in Example 6.

example 3 Finding a Formula for Average

In a mathematics course there are three exams during the semester.

(a) Find a formula for a student's average score for the three exams.

(b) Jim's test scores are 78, 81, and 93. Find Jim's average test score.

Thinking About the Problem

To find the average of two numbers, we add them and divide by 2. For example, the average of 10 and 30 is

$$\frac{10 + 30}{2} = 20$$

right in the middle between 10 and 30. To find the average of three numbers, we add them and divide by 3. So the average of three test scores is

$$\text{average} = \frac{\text{Score 1} + \text{Score 2} + \text{Score 3}}{3}$$

For example, the average of 10, 30, and 80 is

$$\frac{10 + 30 + 80}{3} = 40$$

Solution

(a) To express the average as a formula, we need to assign symbols to the variables involved. Let's use subscripts and write the three test scores as s_1, s_2, and s_3.

In Words	In Algebra
Score on test 1	s_1
Score on test 2	s_2
Score on test 3	s_3
Average	A

We need to express the following "word" formula as an algebraic formula:

$$\text{average} = \frac{\text{Score 1} + \text{Score 2} + \text{Score 3}}{3}$$

We write this as

$$A = \frac{s_1 + s_2 + s_3}{3}$$

(b) To get the average of Jim's test scores, we use the formula we have just found. We replace s_1 by 78, s_2 by 81, and s_3 by 93 in the formula:

$$A = \frac{s_1 + s_2 + s_3}{3} \qquad \text{Formula}$$

$$= \frac{78 + 81 + 93}{3} \qquad \text{Replace } s_1 \text{ by 78, } s_2 \text{ by 81, and } s_3 \text{ by 93}$$

$$= 84 \qquad \text{Calculator}$$

Jim's average test score is 84.

NOW TRY EXERCISE 23

Reading and Using Formulas

Since a formula is an algebraic expression, we can read what the formula is telling us by examining its algebraic form. For example, the formula

$$P = k\frac{T}{V}$$

mentioned at the beginning of this section contains the fraction T/V. Since V is in the denominator, the larger the volume V becomes, the smaller the fraction T/V becomes and hence the smaller the pressure P. On the other hand, the higher the temperature T becomes the larger the fraction T/V becomes and hence the greater the pressure P.

We can also use the rules of algebra to rewrite a formula in many different equivalent ways. For example, you can check that the above formula can be written in any of the following ways:

$$P = k\frac{T}{V} \qquad V = k\frac{T}{P} \qquad T = \frac{1}{k}PV \qquad PV = kT$$

These four formulas are equivalent—they contain exactly the same information about the relationship of the variables P, V, and T. But the first formula is most

useful for calculating pressure, the second for volume, the third for temperature, and the fourth looks the nicest. We explore these ideas in the following examples.

example 4 The Score You Need on Test 3

Martha's scores on her first two tests are 78 and 75. She wants to know what score she needs on her third test to have an average of 80 for the three tests. Martha knows the formula for average,

$$A = \frac{s_1 + s_2 + s_3}{3}$$

But she wants to know how to use this formula to answer her question.

(a) Solve the formula for the variable s_3.

(b) Find the score Martha needs on her third test.

Solution

(a) We solve for s_3 as follows:

$$A = \frac{s_1 + s_2 + s_3}{3} \qquad \text{Formula}$$

$$3A = s_1 + s_2 + s_3 \qquad \text{Multiply each side by 3}$$

$$3A - s_1 - s_2 = s_3 \qquad \text{Subtract } s_1 \text{ and } s_2 \text{ from each side}$$

Solving for one variable in terms of another is reviewed in Algebra Toolkit C.1, page T47.

So we can write the formula

$$s_3 = 3A - s_1 - s_2$$

This formula allows us to find s_3 if we know s_1, s_2, and A.

(b) We use the formula we found in part (a) with 78 for s_1, 75 for s_2, and 80 for A.

$$\begin{aligned} s_3 &= 3A - s_1 - s_2 && \text{Formula} \\ &= 3 \cdot 80 - 78 - 75 && \text{Replace } s_1 \text{ by 78, } s_2 \text{ by 75, and } A \text{ by 80} \\ &= 87 \end{aligned}$$

So Martha needs to get 87 on her third test to have an average score of 80.

■ **NOW TRY EXERCISE 27** ■

example 5 The Height of a Box

In Example 2 we found that the formula for the surface area of a rectangular box is

$$S = 2lh + 2wh + 2lw$$

(a) Solve this formula for the height h.

(b) Find the height of a box if its length is 8 in, its width is 5 in, and its surface area is 184 in^2.

Solution

(a) To solve for h, we need to put h alone on one side of the equation. We proceed as follows:

We used the Distributive Property to factor *h*. This Property is reviewed in Algebra Toolkit A.1, page T1.

$$S = 2lh + 2wh + 2lw \qquad \text{Formula}$$

$$S - 2lw = 2lh + 2wh \qquad \text{Subtract } 2lw \text{ from each side}$$

$$S - 2lw = h(2l + 2w) \qquad \text{Factor } h \text{ from the right-hand side}$$

$$\frac{S - 2lw}{2l + 2w} = h \qquad \text{Divide each side by } 2l + 2w$$

So we can write the formula as

$$h = \frac{S - 2lw}{2l + 2w}$$

This formula allows us to find h if we know l, w, and S.

(b) We use the formula we found in part (a) with l replaced by 8, w by 5, and S by 184:

$$h = \frac{S - 2lw}{2l + 2w} \qquad \text{Formula}$$

$$= \frac{184 - 2 \cdot 8 \cdot 5}{2 \cdot 8 + 2 \cdot 5} \qquad \text{Replace } l \text{ by 8, } w \text{ by 5, and } S \text{ by 184}$$

$$= 4 \qquad \text{Calculator}$$

The height of the box is 4 ft.

NOW TRY EXERCISE 25

Have you ever wondered how much the world weighs? In the next example we answer this question without having to put the world in oversized scales. We simply use Newton's formula for gravitational force. The fact that we can answer such a question using a simple formula shows the stupendous power of reasoning with formulas.

example 6 Weighing the Whole World



Sammy wants to find out how much the world weighs; more precisely, she wants to find the mass of the world. She knows two of Newton's formulas. The first gives the force F required to move an object of mass m at acceleration a:

$$F = ma$$

The second gives the gravitational force F between two objects a distance d apart with masses m and M:

$$F = G\frac{mM}{d^2}$$

where G is the gravitational constant 6.67×10^{-11} N-m^2/kg^2 (in the metric system).

(a) Let M be the mass of the earth and m be the mass of a lead ball. Use the fact that the force F is the same in each formula to find a formula for M.

(b) The distance d between the lead ball and the earth is the radius of the earth, so $d \approx 6.38 \times 10^6$ m, and the acceleration due to gravity at the surface of the earth is 9.8 m/s. Use these facts to find the mass of the earth.

Solution

(a) We equate the two different expressions for F and solve for M:

$$ma = G\frac{mM}{d^2} \qquad \text{Because } F = ma \text{ and } F = G\frac{mM}{d^2}$$

$$a = G\frac{M}{d^2} \qquad \text{Divide by } m$$

$$\frac{ad^2}{G} = M \qquad \text{Multiply by } d^2 \text{ and divide by } G$$

So we can write a formula for the mass of the earth:

$$M = \frac{ad^2}{G}$$

This formula allows us to find M if we know a, d, and G.

(b) We use the formula we found in part (a):

$$M = \frac{ad^2}{G} \qquad \text{Formula}$$

$$= \frac{(9.8)(6.38 \times 10^6)^2}{6.67 \times 10^{-11}} \qquad \text{Substitute the value of each quantity}$$

$$\approx 5.98 \times 10^{24} \qquad \text{Calculator}$$

Scientific notation is studied in Exploration 1, page 312.

So the mass of the earth is just a bit less than 6×10^{24} kg. Written out in full, this is about

$$6{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \text{ kg}$$

or 6 septillion kilograms.

■ **NOW TRY EXERCISE 33** ■

1.9 Exercises

CONCEPTS

Fundamentals

1. The model $L = nS$ gives the total number of legs that S animals have, where each animal has n legs. Using this model, we find that 12 spiders have $L =$ ____ $\times$ ____ = ____ legs, whereas 20 chickens have $L =$ ____ $\times$ ____ = ____ legs.

2. The formula $d = rt$ models the distance d in miles that you travel in t hours at a speed of r miles per hour. So the formula that models the time t it takes to go a distance d at a speed r is given by $t =$ $\frac{\square}{\square}$. We use this formula to find how long it takes to go 350 miles at a speed of 55 miles per hour: $t =$ $\frac{\square}{\square}$ = _____.

SKILLS

3–8 ■ Solve the equation to find a formula for the indicated variable.

3. $PV = nRT$; for R

4. $F = G\dfrac{m_1 m_2}{d^2}$; for d

5. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1

6. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i

7. $S = 2lw + 2wh + 2lh$; for w

8. $V = \frac{1}{3}\pi r^2 h$; for r

9–16 ■ Find a formula that models the quantity described.

9. The average A of two numbers a_1 and a_2.

10. The average A of three numbers a_1, a_2, and a_3.

11. The sum S of the squares of n and m.

12. The sum S of the square roots of n and m.

13. The product P of an integer n and two times an integer m.

14. The product P of the squares of n and m.

15. The time t it takes an airplane to travel d miles if its speed is r miles per hour.

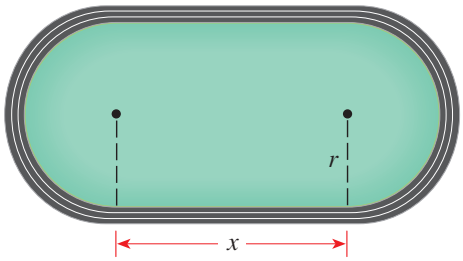
16. The speed r of a boat that travels d miles in t hours.

17–20 ■ Find a formula that models the quantity described. You may need to consult the formulas for area and volume listed on the inside back cover of this book.

17. The surface area A of a box with an open top of dimensions l, w, and h.

18. The surface area A of a cylindrical can with height h and radius r.

19. The length L of the race track shown in the figure.



20. The area A enclosed by the race track in the preceding figure.

CONTEXTS

21. CO_2 Emissions Many scientists believe that the increase of carbon dioxide (CO_2) in the atmosphere is a major contributor to global warming. The Environmental Protection Agency estimates that one gallon of gasoline produces on average about 19 pounds of CO_2 when it is combusted in a car engine.

(a) Find a formula for the amount A of CO_2 a car produces in terms of the number n of miles driven and the gas mileage G of the car.

(b) Debbie owns an SUV that has a gas mileage of 21 mi/gal. Debbie drives 15,000 miles in one year. Use the formula you found in part (a) to find how much CO_2 Debbie's car produces in one year.

(c) Debbie's friend Lisa owns a hybrid car that has a gas mileage of 55 mi/gal. Lisa also drives 15,000 miles in one year. Use the formula you found in part (a) to find how much CO_2 Lisa's car produces in one year.

22. Prehistoric Vegetation Gasoline is refined from crude oil, which was formed from prehistoric organic matter buried under layers of sediment. High pressures and temperatures transformed this material into the hydrocarbons that we call crude oil. Scientists estimate that it takes about 98 tons of prehistoric vegetation to produce one gallon of gasoline. (Today, it takes about 40 acres of farmland to produce 98 tons of vegetation in one season.)

(a) Find a formula for the amount V of prehistoric vegetation it took to produce the gasoline needed to drive a car in terms of the number n of miles driven and the gas mileage G of the car.

(b) Sonia owns an SUV that has a gas mileage of 18 mi/gal. Sonia drives 10,000 miles a year. Use the formula you found in part (a) to find the amount of prehistoric vegetation it took to produce the gas that Sonia's SUV uses in one year.

23. Investing in Stocks Some investors buy shares of individual stocks and hope to make money by selling the stock when the price increases.

(a) Find a formula for the profit P an investor makes in terms of the number of shares n she buys, the original price p_0 of a share, and the selling price p_s of a share.

(b) Silvia plans to make money by buying and selling shares of stock in her favorite retail store. She buys 1000 shares for \$21.50 and waits patiently for many months until the price finally increases to \$25.10. Use the formula found in part (a) to find the profit Silvia will make on her investment if she sells at \$25.10.

24. Growth of a CD If you invest in a 24-month CD (certificate of deposit), then the amount A at maturity is given by the formula

$$A = P\left(1 + \frac{r}{100}\right)^2$$

where P is the principal and the interest rate is $r\%$.

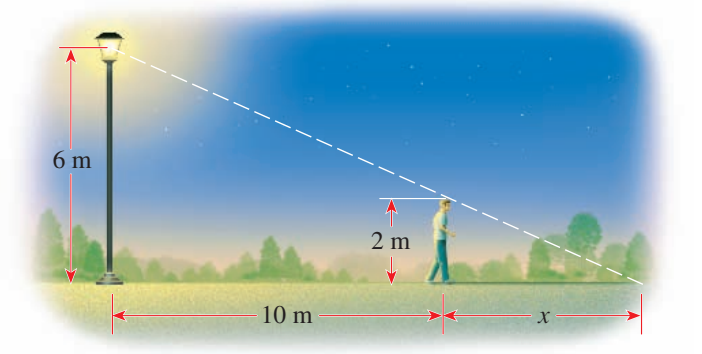
Carlos invests \$2000 in a 24-month CD that has a 2.75% interest rate. Use the formula to find how much Carlos' CD is worth at maturity.

25. Length of Shadow A man is walking away from a lamppost with a light source 6 m above the ground. If the man is h meters tall and y meters from the lamppost, then the length x of his shadow satisfies the equation

$$\frac{y + x}{6} = \frac{x}{h}$$

(a) Find a formula for x.

(b) The figure shows a 2-meter-tall man walking away from the lamppost. Use the formula to find the length of his shadow when he is 10 m from the lamppost.



26. Printing Costs The cost of printing a magazine depends on the number p of pages in the magazine and the number m of copies printed. The cost C is given by the formula

$$C = kpm$$

where k depends on per page printing price.

(a) Find a formula for k.

(b) Find the value of k using the fact that it costs \$12,000 to print 4000 copies of a 120-page magazine.

(c) Use the formula to determine how many copies of a 92-page magazine can be printed if the cost must be no more than \$40,000.

27. Electrical Resistance When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Suppose that an 8-Ω resistor is connected in parallel with a 10-Ω resistor. Use the formula to find their combined resistance R.

28. Temperature of Toaster Wire The resistance R of the heating wire for a toaster depends on temperature. The resistance R at temperature T is given by the formula

$$R = R_0[1 + 0.00045(T - T_0)]$$

where R_0 is the resistance at the initial temperature T_0 (in degrees Celsius). If the initial temperature of a toaster is 20°C and the resistance at that temperature is 147 Ω, find the resistance when the heating wire reaches a temperature of 360°C.



29. The Doppler Effect As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest. This phenomenon is called the *Doppler effect*. The observed pitch P_o is given by the formula

$$P_o = \frac{P_s}{1 - \frac{v_s}{s_o}}$$

where P_s is the actual pitch of the whistle at the source, v_s is the speed of the train, and $s_o = 332$ m/s is the speed of sound in air. Suppose the train has a whistle pitched at $P_s = 440$ Hz. Find the pitch of the whistle as perceived by an observer if the speed of the train is 44.7 m/s.

30. Boyle's Law Boyle's Law states that the pressure P in a sample of gas is related to the temperature T and the volume V by the formula

$$P = k\frac{T}{V}$$

where k is a constant.

(a) A certain sample of gas has a volume of 100 L and exerts a pressure of 33.2 kPa at a temperature of 400 K (absolute temperature measured on the Kelvin scale). Use these facts to determine the value of k for this sample.

(b) If the temperature of this sample is increased to 500 K and the volume is decreased to 80 L, use the formula to find the pressure of the gas.

(c) If the volume is quadrupled and the temperature is halved, does the pressure increase or decrease? By what factor?

31. Spread of a Disease The rate r at which a disease spreads in a population of size P is related to the number x of infected people and the number $P - x$ of those who are not infected, by the formula

$$r = kx(P - x)$$

where k is a constant that depends on the particular disease. An infection spreads in a town with a population of 5000.

(a) Compare the rate of spread of this infection when 10 people are infected to the rate of spread when 1000 people are infected. Which rate is larger?

(b) Calculate the rate of spread when the entire population is infected. Why does your answer make intuitive sense?



32. Skidding in a Curve A car is traveling on a curve that forms a circular arc. The force F needed to keep the car from skidding is related to weight w of the car and the speed s and the radius r of the curve by the formula

$$F = k\frac{ws^2}{r}$$

where k is a constant that depends on the friction between the tires and the road. A car weighing 1600 lb travels around a curve at 60 mi/h. The next car to round this curve weighs 2500 lb and requires the same force as the first car to keep from skidding. How fast is the second car traveling?

33. Flying Speeds of Migrating Birds Many birds migrate thousands of miles each year between their winter feeding grounds and their summer nesting sites. For instance, the 15-gram blackpoll warbler travels 12,000 miles from western Alaska to South America. The air speed v at which a migrating bird flies depends on its weight w and the surface area S of its wings; these quantities satisfy the equation

$$Sv^2 = 94{,}700w$$

where w is measured in pounds, S in square inches, and v in miles per hour.

(a) Find a formula for v in terms of w and S.

(b) Complete the table to find the ratio w/S and the migrating air speed v for the indicated sea birds.

(c) The ratio w/S is called the *wing loading*; the greater a bird's wing loading, the faster it must fly. A certain bird has a wing loading twice that of the sooty albatross. What is its migrating air speed?



Armin Rose/Shutterstock.com 2009

Bird	w (lb)	S (in^2)	w/S	v (mi/h)
Common tern	0.26	76		
Black-headed gull	0.52	120		
Common gull	0.82	180		
Royal tern	1.1	170		
Herring gull	2.1	280		
Great skua	3.0	330		
Sooty albatross	6.3	530		
Wandering albatross	19.6	960		

CHAPTER 1 REVIEW

CHAPTER 1 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below, section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

1.1 Making Sense of Data

A set of **one-variable data** is a list of numbers, usually obtained by recording values of a varying quantity. The **average** of a list of n numbers is their sum divided by n. If the list is written in order, then its **median** is either the middle number (if n is odd) or the average of the two middle numbers (if n is even).

A set of **two-variable data** involves *two* varying quantities that are related to each other. For example, recording the heights and weights of all the students in a class gives a set of two-variable data. We can use a table with two columns or two rows to record two-variable data.

1.2 Visualizing Relationships in Data

Two-variable data are sets of related ordered pairs of numbers. Any set of ordered pairs is a **relation**. The first element in each ordered pair is the **input**, and the second is the corresponding **output**. The **domain** of the relation is the set of all inputs, and the **range** is the set of all outputs.

To see patterns and trends in two-variable data, we can **graph** the ordered pairs in the relation given by the data, on a coordinate plane. Such a graph is called a **scatter plot**.

1.3 Equations: Describing Relationships in Data

A **mathematical model** is an equation that describes the relationship between the variables in a real-world situation. Data can often be modeled by using a **linear model**, which is an equation of the form

$$y = A + Bx$$

In this equation, A is the **initial value** of y (the value when $x = 0$), and B is the amount by which y changes for every unit increase in x. A scatter plot can often tell us whether data are best modeled with a linear model. If a set of data has equally spaced inputs, then we can use the **first differences** of the outputs to determine whether a linear model is appropriate for the data.

Using a model, we can predict output values for any input by substituting the input into the equation and solving for the output.

1.4 Functions: Describing Change

A **function** is a relation in which each input gives exactly one output. We say that y is a function of x if for every input x there is exactly one output y, and we refer to x as the **independent** (input) variable and y as the **dependent** (output) variable.

The **Vertical Line Test** states the fact that a relation is a function if and only if its graph has the property that no vertical line intersects the graph more than once.

We can represent a function in **four different ways**:

- Verbally, using words
- Numerically, using a table of two-variable data
- Symbolically, using an equation
- Graphically, using a graph in the coordinate plane

1.5 Function Notation: The Concept of Function as a Rule

A function can be thought of as a **rule** that produces exactly one output for every input. To help us describe the rule of a function, we use **function notation**. If a function with the name "f" acts on the input x to produce the output y, then we write

$$f(x) = y$$

For example, if $f(x) = x^2$ then f *squares* each input, so that, for instance, $f(3) = 9$ and $f(-2) = 4$.

The **net change in the value of a function** f from the input a to the input b (where $a \leq b$) is the difference

$$f(b) - f(a)$$

The **domain** of a function is the set of all inputs. If the domain is not explicitly given, then we assume that the domain is the set of all real numbers for which the output is defined.

A **piecewise defined function** is one that is defined by different rules on different parts of its domain. For example, the absolute value function is a piecewise defined function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

1.6 Working with Functions: Graphs and Graphing Calculators

The **graph of the function** f is the set of all ordered pairs (x, y), where $y = f(x)$, plotted in a coordinate plane. You should be familiar with the graphs of basic functions such as $f(x) = c, f(x) = x, f(x) = x^2, f(x) = x^3, f(x) = |x|$, and $f(x) = \sqrt{x}$.

In using a **graphing calculator** to graph a function, it's important to select an appropriate **viewing rectangle** ("window") to display the essential features of the graph properly.

To graph a **piecewise defined** function, graph each "piece" separately over the appropriate portion of the domain.

1.7 Working with Functions: Getting Information from the Graph

We can read the behavior or "life history" of a function from its graph, keeping in mind that the height of the graph at each point represents the value of the function at that point. From the graph we can determine:

- Values of the function at specific inputs
- Net change in the value of the function between two inputs
- The domain and range of the function
- Intervals on which the function is increasing or decreasing
- Local maximum and minimum values of the function
- Where two functions have the same value
- The intervals where the values of one function are greater than (or less than) another

1.8 Working with Functions: Modeling Real-World Relationships

Functions can be used to **model** real-world relationships. In this section we construct models from verbal descriptions of the relationships (whereas in Section 1.3 we constructed models from data).

Once we have found a function that models a real-world situation, we can use the model to answer questions and make predictions about the situation being modeled. For instance, we can determine the time when the level of water in a reservoir will drop to a given depth or the maximum height that will be achieved by a missile fired from a cannon.

1.9 Making and Using Formulas

A **formula** is an equation that involves two or more variables. Some familiar formulas are $E = mc^2$, $A = \pi r^2$, $C = 2\pi r$, and $P = 2l + 2w$. Formulas are generally used to describe relationships between real-world quantities. When using formulas, we often need to solve for one of the variables in terms of the others.

Some formulas use **subscripted** variables to denote related quantities. For instance, the average A of the three numbers n_1, n_2, and n_3 is given by the formula

$$A = \frac{n_1 + n_2 + n_3}{3}$$

CHAPTER 1 REVIEW EXERCISES

SKILLS

1–2 ■ A list of one-variable data is given.

(a) Find the average of the data.

(b) Find the median of the data.

1.

x	12	4	10	−4	4	0

2.

x	10.3	6.5	7.2	4.6	9.1	8.4	5.0

3–4 ■ Two-variable data are given.

(a) Express the data as a set of ordered pairs, where x is the input and y is the output of the relation.

(b) Find the domain and range of the relation.

(c) Find the output corresponding to the input 6.

(d) Make a scatter plot of the data.

(e) Does there appear to be a relationship between the variables? If so, describe the relationship.

3.

x	2	4	6	8	10	12
y	14	12	12	8	5	3

4.

x	2	4	4	6	10	11	12
y	9	1	4	7	3	15	6

5–6 ■ The graph of a relation is given.

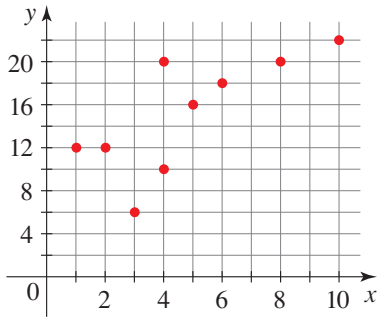
(a) List four of the ordered pairs in the relation.

(b) Find the output(s) corresponding to the input 8.

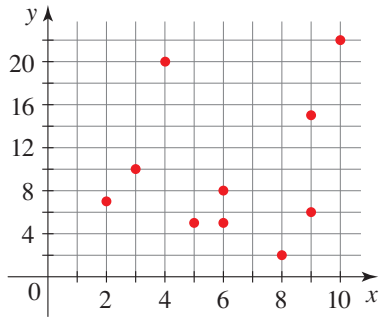
(c) Find the input(s) corresponding to the output 20.

(d) What are the maximum and minimum outputs?

5.



6.



7–10 ■ A table or a scatter plot of a data set is given.

(a) Find a linear model for the data.

(b) Use the model to predict the output for the inputs 3.5 and 30.

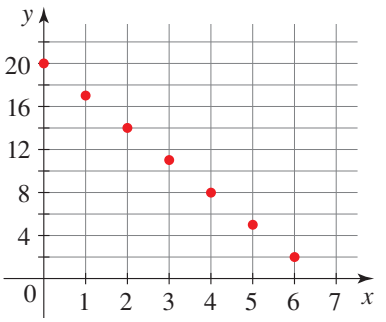
7.

x	0	2	4	6	8	10
y	6	10	14	18	22	26

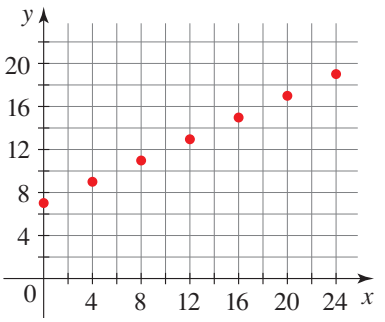
8.

x	0	1	2	3	4	5
y	8	5	2	−1	−4	−7

9.



10.



11–12 ■ A set of data is given.

(a) Find the first differences to see whether a linear model is appropriate.

(b) If a linear model is appropriate, then find the model.

(c) Use the model to complete the table.

11.

x	y	First differences
0	10.7	—
1	8.4	
2	6.1	
3	3.8	
4		
5		
6		

12.

x	y	First differences
0	56	—
2	60	
4	65	
6	71	
8		
10		
12		

13–14 ■ A set of ordered pairs defining a relation is given. Is the relation a function?

13. $\{(1, 3), (2, 3), (3, -1), (5, 2), (7, 2), (9, -1)\}$

14. $\{(1, 3), (1, -3), (2, -1), (2, 1), (3, 2), (4, -3), (5, 5)\}$

15–16 ■ Two-variable data are given in the table.

(a) Is the variable y a function of the variable x? If so, which is the independent variable and which is the dependent variable?

(b) Is the variable x a function of the variable y? If so, which is the independent variable and which is the dependent variable?

15.

x	1	2	3	2	1
y	−2	−1	0	1	2

16.

x	1	−1	2	−2	3	−3	4	−4
y	1	1	4	4	9	9	16	16

17–18 ■ An equation is given in function form.

(a) What is the independent variable and what is the dependent variable?

(b) Find the value of the dependent variable when the independent variable is 4.

(c) Find the net change in the dependent variable when the independent variable changes from 1 to 4.

17. $y = 3x + x^2$

18. $z = 5\sqrt{t} - 2t$

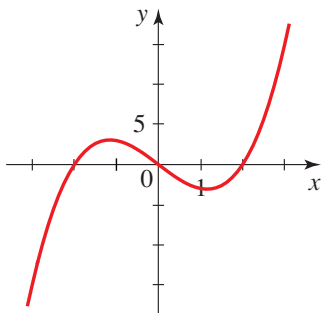
19–20 ■ An equation and its graph are given.

(a) Find the x- and y-intercepts.

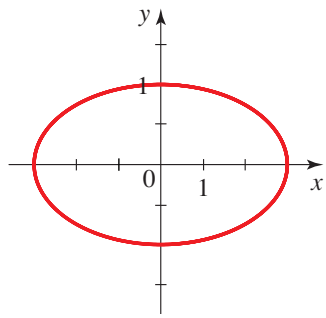
(b) Use the Vertical Line Test to see whether the equation defines y as a function of x.

(c) If y is a function of x, put the equation in function form, and find the net change in y when x changes from −3 to 1.

19. $x^3 - y - 4x = 0$



20. $x^2 + 9y^2 = 9$



21–24 ■ An equation in two variables is given.

(a) Does the equation define y as a function of x?

(b) Does the equation define x as a function of y?

21. $5x - y^2 = 10$

22. $4x^2 + y^2 = 16$

23. $2\sqrt{x} + 3y = 0$

24. $x^3 - 6y^2 - 6 = 0$

25–28 ■ A verbal rule describing a function is given.

(a) Give an algebraic representation of the function (an equation).

(b) Give a numerical representation of the function (a table of values).

(c) Give a graphical representation of the function.

(d) Express the rule in function notation.

25. "Multiply by 2, then subtract 7."

26. "Add 3, then divide by 4."

27. "Add 2, square, then divide by 3."

28. "Square, subtract 1, then multiply by $\frac{1}{2}$."

29–32 ■ A function is given.

(a) Give a verbal description of the function.

(b) Express the function as an equation. Identify the dependent and the independent variable.

(c) Find the net change in the value of the function when x changes from 3 to 5.

29. $f(x) = \dfrac{x}{3} + 5$

30. $g(x) = 7(x - 5)$

31. $h(x) = \frac{1}{2}(x - 3)^2$

32. $k(x) = 2x^3 + \frac{1}{4}$

33–36 ■ Find the indicated values of the given function.

33. $f(x) = 2x^2 - 4x + 3$

(a) $f(0)$ **(b)** $f(2)$ **(c)** $f(-1)$ **(d)** $f(a)$

34. $g(x) = \sqrt{x^2 + 4x + 4}$

(a) $g(0)$ **(b)** $g(-3)$ **(c)** $g(5)$ **(d)** $g(b)$

35. $h(x) = \begin{cases} 2 & \text{if } x < 0 \\ 3x & \text{if } x \geq 0 \end{cases}$

(a) $h(-4)$ **(b)** $h(0)$ **(c)** $h(1)$ **(d)** $h(100)$

36. $k(x) = \begin{cases} -x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$

(a) $k(-2)$ **(b)** $k(0)$ **(c)** $k(1)$ **(d)** $k(1.5)$

37–38 ■ A function is given. Complete the table of values, and then graph the function.

37. $f(x) = 5 - 2x$

x	-2	-1	0	1	2	3	4
$f(x)$							

38. $g(x) = x^2 - 4$

x	-3	-2	-1	0	1	2	3
$g(x)$							

39–44 ■ Sketch a graph of the function by first making a table of values.

39. $g(x) = 3 + \frac{1}{2}x$

40. $g(x) = -x + 2$

41. $h(x) = 4 - x^2, \quad -1 \leq x \leq 2$

42. $h(x) = x^2 - 4x, \quad -1 \leq x \leq 5$

43. $k(x) = \sqrt{x + 2}$

44. $k(x) = 2 - \sqrt{x}$

45–48 ■ Draw a graph of the function in an appropriate viewing rectangle.

45. $f(x) = 5 + 4x - x^2$

46. $f(x) = x^2 - 2x - 2$

47. $g(x) = 2 - \sqrt{x + 1}$

48. $g(x) = 4x - x^3$

49–50 ■ Sketch a graph of the piecewise-defined function.

49. $f(x) = \begin{cases} 3 & \text{if } x < -1 \\ x & \text{if } x \geq -1 \end{cases}$

50. $g(x) = \begin{cases} -x & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$

51–52 ■ The graph of a function f is given.

(a) Find $f(-1)$, $f(0)$, and $f(2)$.

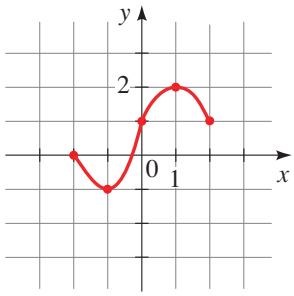
(b) Find the net change in the value of f when x changes from -1 to 1.

(c) Find the value(s) of x for which $f(x) = 1$.

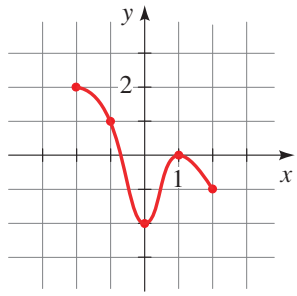
(d) Find the domain and range of f.

(e) Find the intervals on which f is increasing and on which it is decreasing.

51.



52.



53–54 ■ A function f is given.

(a) Use a graphing calculator to draw the graph of f.

(b) Find the domain and range of f from the graph.

(c) Find the intervals on which the function is increasing and on which it is decreasing.

53. $f(x) = \sqrt{9 - 4x^2}$

54. $f(x) = 1 - \sqrt{x + 2}$

55–56 ■ A function f is given.

(a) Use a graphing calculator to draw the graph of f.

(b) Find the local maximum and minimum values of f and the value of x at which each occurs.

55. $f(x) = x^3 - 3x^2 + 1$

56. $f(x) = 3 - 4x^2 + 4x^3 - x^4$

57–58 ■ Find a function that models the quantity described.

57. The number H of hours in d days.

58. The area A of a rectangle whose length is three times its width w.

59–60 ■ Find a formula that models the quantity described.

59. The sum S of the square roots of the numbers n_1, n_2, and n_3.

60. The distance d that a person walking at r feet per second can walk in t minutes.

61–62 ■ Solve the equation to find a formula for the indicated variable.

61. $K = \frac{1}{2}mv^2$; for v

62. $D = 2x + 2\pi r$; for r

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

63. Relationship Between Age and Weight The table below gives the ages and weights of the members of the Hendron family.

	John	Andrea	Helen	Louis	Ally	Victor
Age (years)	46	48	20	16	16	10
Weight (lb)	310	142	120	142	114	90

(a) Find the average age and the median age of the family members.

(b) Find the average weight and the median weight of the family members. Why is there such a large difference between the average and the median weights? Which number gives a better description of the central tendency of the Hendrons' weights?

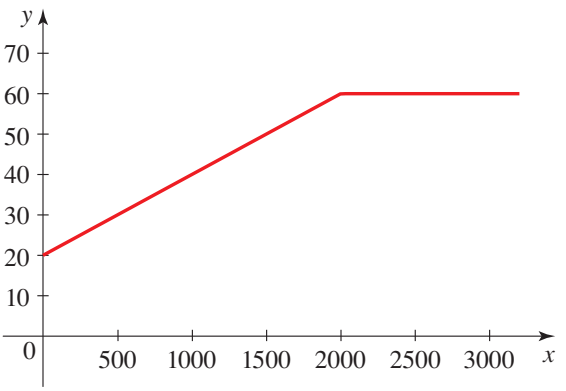
(c) List the ordered pairs in the relation obtained by treating age as the input (x) and weight as the output (y). What is the domain of this relation? What is the range?

(d) Make a scatter plot of the relation. Is the relation a function? Why or why not?

(e) Does the scatter plot indicate any general trend in the relationship between age and weight?

(f) Sketch the line $y = 5x$ on your scatter plot. Use the graph to determine how many family members have a weight (in pounds) that is more than five times their age (in years).

64. Internet Access An Internet provider serves a community action group and charges its members $f(x)$ dollars each month for x minutes of online access time, where f is the function graphed in the figure.



(a) Use the graph to complete the following table of values.

x	0	100	500	1000	1500	2000	3000
$f(x)$		$22		$40			

(b) What are the domain and range of f?

(c) What does the y-intercept of the graph represent?

(d) On what interval is the function increasing?

(e) What is the minimum monthly charge under this plan? What is the maximum monthly charge?

(f) Give a verbal description of how the monthly charge is calculated if a customer uses less than 2000 minutes and if she uses more than 2000 minutes of access time.

(g) Express f algebraically as a piecewise defined function:

$$f(x) = \begin{cases} _____ & \text{if } x < 2000 \\ _____ & \text{if } x \geq 2000 \end{cases}$$

(h) Find the net change in the value of f when x changes from 100 to 500 minutes, and when x changes from 2000 to 3000 minutes.

CONTEXTS

65. Test Scores The table gives Rianna's scores on the first five quizzes in her chemistry class.

Score	9	5	6	6	4

(a) What is her average score?

(b) What is her median score?

(c) If she receives a 9 on her sixth quiz, what are her new average and median scores?

66. Price of Gasoline The price of gasoline dropped rapidly at the Oxxo station in East Springfield during one week in November 2008. The list gives the price per gallon for regular unleaded for the first six days that week, rounded to the nearest cent.

Day	Sun.	Mon.	Tue.	Wed.	Thu.	Fri.
Price	$2.14	$2.08	$2.02	$1.98	$1.98	$1.92

(a) What was the average price per gallon over this period?

(b) What was the median price per gallon over this period?

(c) If Saturday's price remained the same as Friday's, what were the average and median price per gallon for the entire week?

67. Fishing for Salmon A group of tourists hires a boat to fish for salmon in the Strait of Georgia, British Columbia. The table gives the number of fish each person caught and the total weight of each person's catch.

(a) Find the average and the median *number* of fish caught by each person.

(b) Find the average and the median *weight* of fish caught by each person.

(c) For each person, find the average weight of the fish in that person's catch. (For example, Jack caught 4 fish weighing a total of 18 lb, which gives his fish an average weight of $18/4 = 4.5$ lb per fish.)

(d) Whose catch had the largest average weight per fish?

(e) What was the average weight per fish for all the fish caught by the entire group?

Person	Jack	Helen	Steve	Kalpana	Dieter	Magda
Number of fish	4	1	2	4	2	2
Weight of fish (lb)	18	4	11	22	13	16

68. Bungee Cord Experiment As part of a science experiment, Qiang attaches a weight to one end of a strong highly elastic cord whose other end is fixed to a high platform. He drops the weight and uses a strobe light and a camera to track its fall. The following table shows the distance between the weight and the platform, at one-second intervals.

(a) How far has the weight fallen after 2 seconds? After 6 seconds?

(b) What is the approximate time (to the nearest second) that it takes the weight to fall 90 feet?

(c) What do the data tell us happens to the path of the weight at some time near 6 seconds into its fall?

(d) How far does the weight fall between 0 seconds and 1 second? Between 3 seconds and 4 seconds? Between 5 seconds and 6 seconds?

(e) What pattern or trend do you see in these data?

Time (s)	0	1	2	3	4	5	6	7	8	9
Distance (ft)	0	6.1	23.0	46.5	70.8	90.1	99.5	96.8	82.7	60.5

69. Vending Machines A vending machine operator decides to experiment with the price she charges for candy bars in her machines, hoping that a higher price might bring more profits. After several weeks of adjusting her prices, she obtains the results in the table below. (The cost is calculated on the basis of the fact that each candy bar costs her 25 cents.)

(a) Use the fact that profit = revenue − cost to fill in the last column of the table.

(b) Make a scatter plot of the relationship between price and profit.

(c) What trend do you detect in your graph? What seems to be the best price to charge?

Price ($)	**Number sold**	**Revenue ($)**	**Cost ($)**	**Profit ($)**
0.50	300	150	75	
0.70	240	168	60	
0.85	220	187	55	
1.00	180	180	45	
1.25	100	125	25	
1.50	60	90	15	

70. Air Pressure Declines with Elevation As you rise in elevation above sea level, the air pressure and the density of the atmosphere decline. This has many consequences for human beings, including reduced endurance and even difficulty in breathing. (See Exercise 21 of Section 1.3 on page 33 to see how increasing elevation affects the boiling point of water.) The following table shows the average atmospheric pressure at various heights, measured in atmospheres. (1 atm is the air pressure at sea level.)

(a) Make a scatter plot of the data. Do the data points appear to lie on a line?

(b) Use first differences to show that a linear model is appropriate for these data.

(c) Find a linear model for the relationship between air pressure and elevation.

(d) Use your model to predict the air pressure at an elevation of 3500 ft.

(e) At what elevation will the air pressure be 0.925 atm?

(f) At an elevation of 20,000 ft, the actual air pressure is about 0.45 atm. How does this compare with the value predicted by your model? What lesson does this teach us about using linear models carefully?

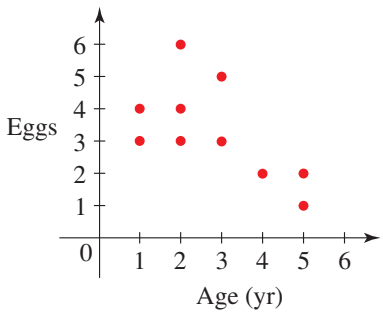
Elevation (ft)	Pressure (atm)	First differences
0	1.00	—
500	0.98	
1000	0.96	
1500	0.94	
2000	0.92	
2500	0.90	

71. Weight Loss A weight-loss clinic keeps records of its customers' weight when they first joined and their weight 12 months later. Some of these data are shown in the following table.

(a) Graph the relation given by the table.

(b) Use the Vertical Line Test to determine whether the variable y is a function of the variable x.

x = Beginning weight	212	161	165	142	170	181	165	172	312	302
y = Weight after 12 months	188	151	166	131	165	156	156	152	276	289



72. Bantam Eggs A hobby farmer maintains a small flock of bantam hens, whose eggs he sells to a local health food store. He is interested in finding how the hens' egg-laying capacity is related to their age, so he gathers the data shown in the graph in the margin. The scatter plot relates the age of each hen (in years) to the number of eggs she laid in one week.

(a) Make a table of the ordered pairs in the relation given by the graph.

(b) List the ordered pair(s) with input 2 and the ordered pair(s) with input 4.

(c) How many hens laid three eggs this week? What are their ages?

(d) Use the Vertical Line Test to determine whether the relation is a function.

73. Sales Commission André sells office furniture and earns a commission based on his sales. The piecewise defined function C gives his commission as a function of his monthly sales x (in dollars):

$$C(x) = \begin{cases} 0 & \text{if } 0 \le x < 3000 \\ 0.1x & \text{if } 3000 \le x < 10{,}000 \\ 0.15x & \text{if } x \ge 10{,}000 \end{cases}$$

(a) Find $C(1000)$, $C(4000)$, and $C(20{,}000)$. What do these values represent?

(b) Sketch a graph of C. Use your graph to determine whether it is possible for André to earn a commission of \$1250. Why or why not?

(c) What sales level will earn André a commission of \$800? Of \$2000?

(d) Find the net change in André's commission if his sales increase from \$9000 to \$14,000 per month.

74. Cooking a Roast Jason takes a roast beef out of the freezer before going to work in the morning, leaving it to defrost on the kitchen counter. He cooks it for two hours in the oven, then lets it rest for 30 minutes before serving it to his dinner guests. Draw a rough graph of the changes in the temperature of the roast over the course of the day.

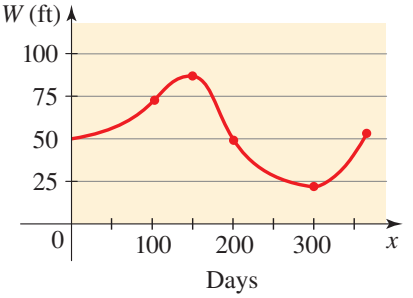
75. Depth of a Reservoir The graph shows the depth of water W in a reservoir over a 1-year period, as a function of the number of days x since the beginning of the year.

(a) What are the domain and range of W?

(b) Find the intervals on which the function W is increasing and on which it is decreasing.

(c) What is the water level in the reservoir on the 100th day?

(d) What was the highest water level in this period, and on what day was it attained?

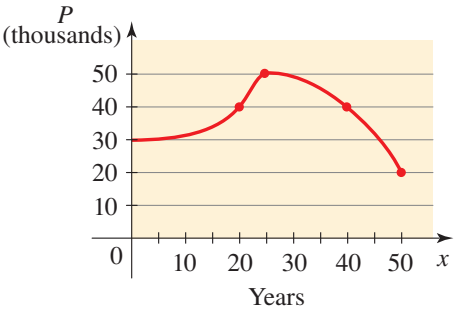


76. Population Growth and Decline The graph below shows the population P in a small industrial city from 1950 to 2000. The variable x represents the number of years since 1950.

(a) What are the domain and range of P?

(b) Find the intervals on which the function P is increasing and on which it is decreasing.

(c) What was the largest population in this time period, and in what year was it attained?

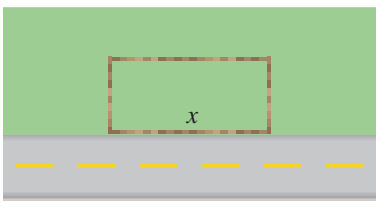


77. Fencing a Garden Plot A property owner wants to fence a garden plot adjacent to a road, as shown in the figure. The fencing next to the road must be sturdier and costs \$5 per foot, but the other fencing costs just \$3 per foot. The garden is to have an area of 1200 ft^2.

(a) Find a function C that models the cost of fencing the garden.

(b) Use a graphing calculator to find the garden dimensions that minimize the cost of fencing.

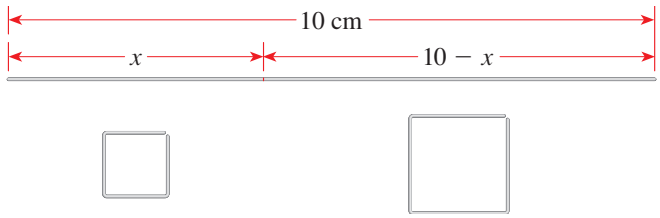
(c) If the owner has at most \$600 to spend on fencing, find the range of lengths he can fence along the road.



78. Maximizing Area A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$, as shown in the figure. Each piece is bent into the shape of a square.

(a) Find a function A that models the total area enclosed by the two squares.

(b) Use a graphing calculator to find the value of x that minimizes the total area of the two squares.



79. Mailing a Package One measure of the size of a package, used by the postal service in many countries, is "length plus girth"—that is, the length of the package plus the distance around it. So if a package has length l, width w, and height h, then its size S is given by the formula $S = l + 2w + 2h$.

(a) The maximum size parcel that the U.S. Postal Service will accept has a length plus girth of 84 in. What is the width of a package of this maximum size if its length is 64 in. and its width equals its height?

(b) Using the fact that the volume of a box is $V = lwh$, complete the following table for various packages that are all assumed to have a length plus girth equal to 84 in.

Length (in.)	Width (in.)	Height (in.)	Volume (in^3)
36	14		
40		10	
	12	12	
	16	8	
72	3		
	10	10	

CHAPTER 1 TEST

1. Find the average and the median of the values of x given in the list.

(a)

x	7	1	6	6	2	10	4	0

(b)

x	1.2	−2.4	4.4	−5.1	3.9

2. A set of two-variable data is given.

(a) Express the data as a relation (a set of ordered pairs), using x as the input and y as the output.

(b) Find the domain and range of the relation.

(c) Make a scatter plot of the data.

(d) Is the relation a function? Why or why not?

(e) Does there appear to be a relationship between the variables? If so, describe the relationship.

(f) Which input(s) correspond to the output 5?

x	0	1	2	3	4	5	6
y	1	5	7	7	5	1	−5

3. A set of two-variable data is given.

(a) Find a linear model for the data.

(b) Use the model to predict the outputs for the inputs 4.5 and 10.

(c) Sketch a graph of the model.

x	0	1	2	3	4	5	6
y	4.0	4.5	5.0	5.5	6.0	6.5	7.0

4. An equation in two variables is given. Does the equation define y as a function of x?

(a) $3x + y^2 = 9$

(b) $x^2 + 3y = 9$

5. The function f has the verbal description "Square, subtract 4, then divide by 5." Express the function algebraically using function notation.

6. A car dealership is offering discounts to buyers of new cars, with the amount of the discount based on the original price of the car, as indicated in the following table.

Original price x ($)	Discount
$x < 20{,}000$	20% of x
$20{,}000 \le x < 40{,}000$	$5000
$x \ge 40{,}000$	$8000

(a) Let $f(x)$ be the discounted price of a car whose original price was x dollars. Express the function f as a piecewise defined function.

(b) Evaluate $f(19{,}000)$, $f(22{,}000)$, $f(38{,}000)$, and $f(40{,}000)$.

(c) What two different values of x both give a discounted price of $34,000?

7. Ella is an editor of gardening books. She is planning a reception at a bookstore to promote a new book by one of her authors. Use of the facility costs $300, and refreshments for the guests cost $4 per person.

(a) Find a function C that models the total cost of the reception if x guests attend.

(b) Sketch a graph of C.

(c) Evaluate $C(50)$ and $C(200)$. What do these numbers represent? Plot the points that correspond to these values on your graph.

(d) If the total cost of the reception was $600, how many guests attended?



8. When a bullet is fired straight up with muzzle velocity 800 ft/s, its height above the ground after t seconds is given by the function $h(t) = 800t - 16t^2$.

(a) Use a graphing calculator to draw a graph of h.

(b) Use your graph to determine the maximum height that the bullet reaches and the time when it reaches this height.

(c) When does the bullet fall back to ground level?

(d) For what time interval is h decreasing? What is happening to the bullet during this time?

9. The surface area of a rectangular box with length l, width w, and height h is given by the formula $A = 2lw + 2lh + 2wh$.

(a) Find the surface area of a box that is 30 cm long, 12 cm wide, and 4 cm high.

(b) Find a formula that expresses h in terms of the other variables.

1 Bias in Presenting Data

OBJECTIVE To learn to avoid misleading ways of presenting data

In this chapter we studied how data can be used to discover hidden relationships in the real world. But collecting and analyzing data are human activities, so they are not immune from bias. When we are looking for trends in data, our goal should be to discover some true property of the thing or process we are studying and not merely to support a preconceived opinion. However, it sometimes happens that data are presented in a misleading way to support a hypothesis that is not valid. This is sometimes called *fudging the data*, that is, reporting only part of the data (the part that supports our hypothesis) or even simply making up false data. Mark Twain succinctly described these practices when he said, "People commonly use statistics like a drunk man uses a lamppost; for support rather than illumination."



www.grimmingplanet.com

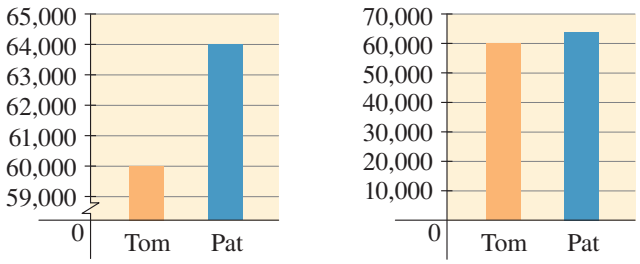
Winner: Tom; Loser: Harry

Here's a simple example of how we may choose to present data in a biased fashion. Tom and Harry compete in a two-man race, which Tom wins and Harry loses. Harry tells his friends, "I came in second, and Tom came in next-to-last." Although correct, Harry's statement gives a misleading impression of what actually happened.

But misrepresenting or misinterpreting data is not always just silly—it can be a very serious matter. For example, if experimental data on the effectiveness of a new drug are misrepresented to bolster the claim of its effectiveness, the result could be tragic to patients using the drug. So in using data, as in all scientific activities, the goal is to discover truth and to present the results of our discoveries as accurately and as fairly as possible. In this exploration we investigate different ways in which data can be misrepresented, as a warning to avoid such practices.

I. Misleading Graphs

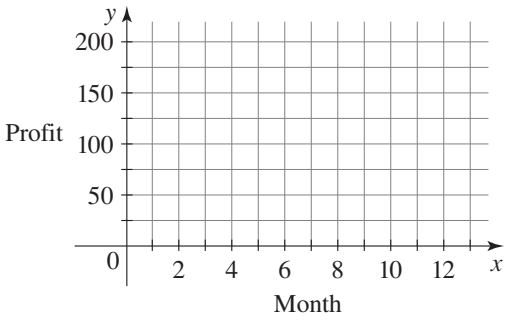
Although graphs are useful in visualizing data, they can also be misleading. One common way to mislead is to start the vertical axis well above zero. This makes small variations in data look large. The following graphs show Tom's and Pat's annual salaries. Does Pat make a lot more money than Tom, or do they make about the same? Look carefully at the scale on the y-axis before answering this question.

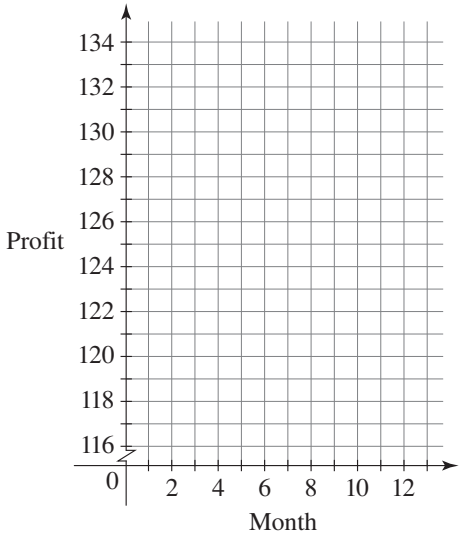


1. The business manager of a furniture company obtains the following data from the accounts department. She needs to present a report on the financial state of the company to the executive board at their annual meeting.

Month	Labor (× $1000)	Materials (× $1000)	Advertising (× $1000)	Revenue (× $1000)	Profit (× $1000)
Jan.	320	247	14	709	122
Feb.	343	330	12	810	118
Mar.	332	330	16	806	123
Apr.	367	295	7	795	120
May	405	370	33	938	124
June	430	424	18	1002	123
July	440	407	13	992	126
Aug.	427	395	12	967	128
Sept.	392	363	21	912	130
Oct.	295	284	6	722	131
Nov.	288	260	28	714	132
Dec.	315	310	8	784	134

(a) Plot the profit data on each of the axes shown below.

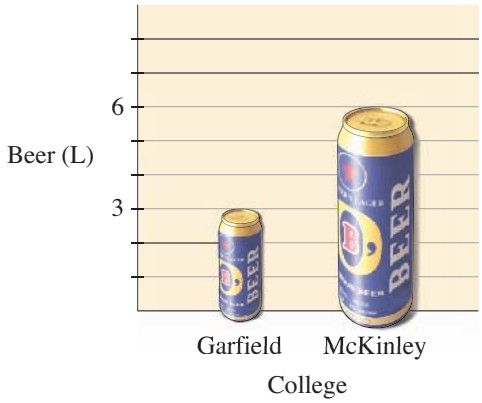




(b) The two graphs you sketched in part (a) display the same information. But which gives the impression that profit increased dramatically? What graph gives a more realistic impression of profit growth for this year?

(c) The business manager also wishes to give a graph of monthly revenue in her report. Sketch two graphs of the monthly revenue: one that gives the impression that revenues were dramatically higher in the middle of the year and one that shows that revenue remained more or less steady throughout the year.

2. The students at James Garfield College drink on average about 3 liters of beer each per month. The rival college across town, William McKinley University, has the reputation of being a "party school." Its students drink about 6 liters of beer each per month. In the following graph, each college's average beer consumption is represented by the height of a beer can.



(a) Does the graph give the correct impression about the relative amount of beer consumed at each college? Why is using images of three-dimensional cans instead of just drawing a plain bar graph deceptive? [*Hint*: By what factor does the volume of a cylindrical can increase if its height and radius are doubled?]

(b) The table gives estimates of the daily oil consumption of four large countries. Draw a bar graph of the data, using the height of a three-dimensional oil barrel to represent oil consumption. How does the graph give a misleading impression of the data?



Country	Oil consumption per day (millions of barrels)
United States	20.7
China	7.9
Russia	2.7
Mexico	2.1

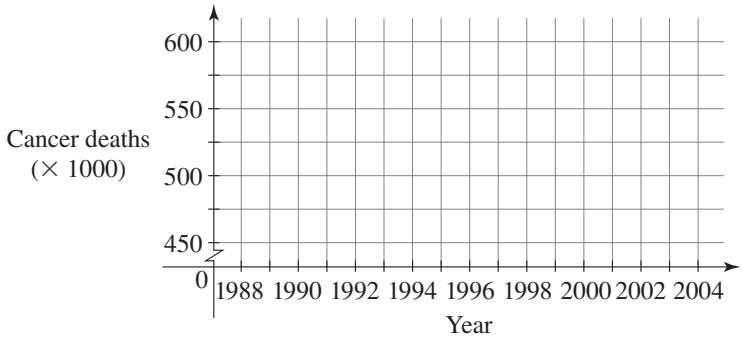
II. Are We Measuring the Right Thing?

Suppose we are told that one school district employs 50 science teachers while another district has just 5. Can we assume that the parents in the first district care more about science education than those in the second? Of course not. The relative sizes of the school districts must be taken into account before we can come to any conclusions. If the population of the first district is ten times that of the second, then both have the same relative proportion of science teachers. In many cases data have to be properly scaled to understand their significance. Let's analyze the following data about two of the leading causes of death in the United States.

1. The following table gives the number of deaths from all forms of cancer and the number of deaths from automobile accidents in the United States from 1988 to 2003, together with population data for those years.

Year	Cancer deaths	Auto fatalities	Population (× $1000)	Cancer mortality rate	Auto mortality rate
1988	485,048	49,078	244,480	198.4	20.1
1989	496,152	47,575	246,819		
1990	505,322	44,599	248,710		
1991	514,657	41,508	252,177		
1992	520,578	39,250	255,078		
1993	529,904	40,150	257,783		
1994	534,310	40,716	260,341		
1995	538,455	41,817	262,755		
1996	539,533	42,065	265,284		
1997	539,577	42,013	267,636		
1998	541,532	41,501	270,360		
1999	549,838	41,611	272,737		
2000	553,091	41,945	275,307		
2001	553,768	42,196	284,869		
2002	557,271	43,005	288,443		
2003	556,902	42,884	290,810		

(a) Graph the cancer mortality data on the given axes.

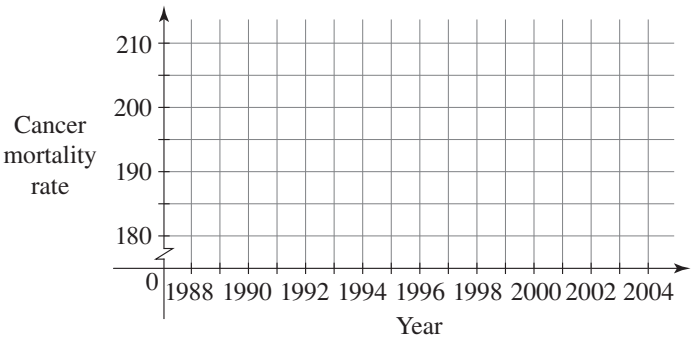


(b) Describe the trend in the number of cancer deaths over this 16-year period. Did the number of deaths per year increase, decrease, or hold steady?

(c) Complete the column in the table headed "Cancer mortality rate" by calculating the cancer mortality rates per 100,000 population. For example, the cancer death rate for 1988 is

$$\frac{485{,}048}{244{,}480{,}000} \times 100{,}000 = 198.4$$

(d) Graph the cancer mortality rate data on the given axes.



(e) Describe the general trend in the cancer mortality rate. Compare this to the trend in the actual number of cancer deaths that you found in part (b). Which do you think gives a better representation of the truth about cancer deaths: the data on number of cancer deaths per year or the data on the cancer mortality rate per year?

(f) Do the same analysis as in parts (a) to (e) for the automobile fatality data.

2. The U.S. public debt, usually referred to as the *national debt*, consists primarily of bonds issued by the Department of the Treasury. Since it was founded in 1790, the department has kept meticulous records of the national debt. The table below gives the value of the national debt at 2-year intervals between 1986 and 2006.

Year	National debt (× $ trillion)	GDP (× $ trillion)	National debt as percent of GDP
1986	2.13	4.55	46.8%
1988	2.60	5.25	
1990	3.23	5.85	
1992	4.06	6.48	
1994	4.69	7.23	
1996	5.22	8.00	
1998	5.53	8.95	
2000	5.67	9.95	
2002	6.23	10.59	
2004	7.41	11.97	
2006	8.51	13.49	

(a) Sketch a graph of the national debt data. Describe any trends you see from the graph.

(b) As the population of the United States grows, so does the economy. So it's probably not meaningful to compare the size of the national debt in 1790 (about 75 million dollars) to its current size (over 9 trillion dollars). To take into account the overall size of the economy, economists calculate the national debt as a percentage of gross domestic product (GDP). For example, for 1986 we get

$$\frac{4.55}{2.13} \times 100 = 46.8\%$$

Use similar calculations to complete the rest of the table.

(c) Sketch a graph of the percentage data that you calculated in part (b). Compare your graph to the graph you sketched in part (a). What information does your new graph provide that was not apparent from your graph in part (a)?

(d) What other factors do you think should be taken into account in considering the impact of the national debt? For example, does inflation play a part in the size of the debt?

3. The following table gives the numbers of homicides and burglaries committed in some California counties in 2005.

County	Homicides	Burglaries
Calaveras	2	308
Humboldt	3	1334
Los Angeles	1068	58,861
Monterey	14	2809
San Bernardino	174	14,548
San Luis Obispo	4	1469
Sonoma	5	2340
Stanislaus	30	4836

(a) In which county were homicides most prevalent? Least prevalent? What about burglaries?

(b) Of course, you know that Los Angeles has a lot more people than the other counties in the table. What role does the population play in determining in which county one would be least likely to be a victim of these crimes? Use the population data below to determine a better measure of the homicide rate and the burglary rate in each of these counties. Which county has the highest homicide *rate* and which has the lowest? What about burglary rates? Do you find your answers surprising?

Population data: Calaveras, 46,871; Humboldt, 128,376; Los Angeles, 9,935,475; Monterey, 412,104; San Bernardino, 1,963,535; San Luis Obispo, 255,478; Sonoma, 466,477; Stanislaus, 505,505.

2 Collecting and Analyzing Data

OBJECTIVE To experience the process of first collecting data, then getting information from the data.



French School, (16th century)/Bibliotheque de l'Institut de France, Pris, France/Lauros/Giraudon/The Bridgeman Art Library

A drawing from the 1735 La Condamine expedition to Peru

In many of the exercises in this book, data are given, and we are asked to analyze the data. Sometimes we are asked to search for data on the Internet. But the process of gathering real-life data can be quite difficult or tedious. For example, obtaining scientific data often requires complicated equipment. The U.S. Census Bureau employs hundreds of agents to survey thousands of citizens to obtain demographic data. Financial data are collected from hundreds of sources in order to measure the economic health of the country.

Sometimes data collection can be an adventure. For example, to determine how closely the earth resembles a sphere, the French Academy sent an expedition to Peru in 1735 to obtain data on the length of a degree at the equator. The expedition, headed by Charles-Marie de La Condamine, included an Atlantic crossing by boat, traveling by mule and on foot, and working at high altitudes. It took over eight years to obtain the needed data.

In this exploration we get some experience in data collection by obtaining data within the classroom and from our classmates.

I. Collecting Data

To collect data from our classmates, we make measurements and take a survey.

1. Let's make some measurements.

(a) For each student in the class, make the following measurements: height, hand span, hat size, shoe size. Make these measurements using the same unit (either inches or centimeters).



Hand span



Hat size



Shoe size

Keep a record of these data. You'll need them for several exercises in Section 2.5.

(b) Enter the information you obtained in a table as shown. Note that the information for each individual is entered in a row of the table.

Measurement Data

Name	Gender	Height	Hand span	Hat size	Shoe size
Adam					
Betty					
Cathy					
David					
⋮					

2. Let's make a survey.

(a) Make a survey that asks the following questions of your classmates. Note that the survey is anonymous; students do not need to put their names on the survey sheet.

Survey

1. Gender: Male ❑ Female ❑

2. What is the value (in cents) of the coins in your pocket or purse? _____

3. How far is your daily commute to school (in miles)? _____

4. How many siblings do you have (including yourself)? _____

5. How many hours a week do you spend on the Internet? _____

6. How many hours a week do you spend on homework? _____

7. Rate your happiness.

① not happy ② happy ③ very happy

8. Rate your satisfaction with your school work.

① not satisfied ② satisfied ③ very satisfied

(b) After the survey sheets have been collected, give each sheet an ID number—for example, 1, 2, 3, ... Now enter the information you obtained in a table like the following. For the column headings let's use short descriptions (instead of the question number) so that we can quickly recognize what each column represents.

Survey Data

ID	Gender	Coins	Commute	Siblings	Internet	Homework	Happy	Satisfied
1								
2								
3								
⋮								

3. Think of other measurements or survey questions that would help us get useful data from the class.

II. One-Variable Data Analysis

Each column in the data tables we completed in Questions 1(b) and 2(b) above is a set of one-variable data. The methods we've learned for getting information from such data are to calculate the mean or the median.

1. Consider the "Height" column in the measurement data table.

(a) Make two sets of data: one for the heights of male students and one for the heights of female students.

(b) Find the mean of each set of data. Compare the mean heights of males and females in your class. What conclusions can you make?

(c) Find the median of each set of data. Compare the median heights of males and females in your class. What conclusions can you make?

2. Consider the "Hand span" column in the measurement data table.

(a) Make two sets of data: one for the hand spans of male students and one for the hand spans of the female students in the class.

(b) Find the mean of each set of data. Compare the mean hand spans of males and females in your class. What conclusions can you make?

(c) Find the median of each set of data. Compare the median hand spans of males and females in your class. What conclusions can you make?

3. Make an analysis similar to the one in Question 1 for one other set of one-variable data from the measurement data table.

4. Consider the "Coins" column from the survey data table.

(a) Find the mean of the data.

(b) Do you suspect that there is a difference in the average value of the coins carried by males and females? If you think there might be a difference, then calculate the mean value of coins for men and women separately to find out.

5. Consider the "Happy" column in the survey data table.
   (a) How do these data differ from the measurement data or the data in the first five survey questions?
   (b) Find the mean of the data. What does a mean of 1.5 represent? What does the mean of your class tell you about the state of happiness of your class?
   (c) A measure of central tendency that is sometimes used for subjective data is the mode. The **mode** is the number that appears most often in the data. So if there are more 2's than 0's or 1's in the "Happy" column, then the mode is 2. Find the mode for your class. Do you think the mode is a better measure of central tendency than the mean in this case?
6. Make an analysis of one other set of one-variable data from the survey data table.

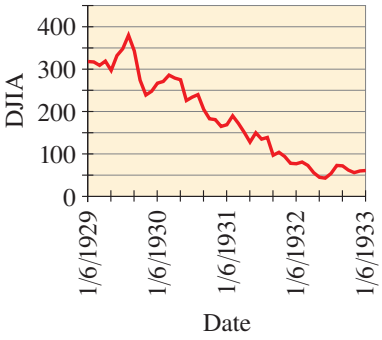
III. Two-Variable Data

Each pair of columns in the tables we completed in Questions 1(b) and 2(b) above is a set of two-variable data. For example, in the measurement data table let's choose the "Height" and "Hand span" columns. Then for each individual we form the ordered pair (a, b), where a is the height of the person and b is the hand span. In this chapter we learned to get information from two-variable data by graphing.

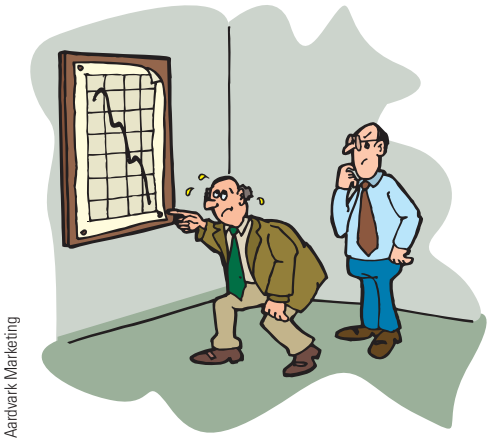
1. Consider the "Height" and "Hand span" columns in the measurement data table.
   (a) Make a set of two-variable data for the height and hand span of male students.
   (b) Make a scatter plot of the data in part (a), with the horizontal axis representing height and the vertical axis representing hand span.
   (c) Do you see any trends in the data? Do taller male students tend to have larger (or smaller) hand spans?
   (d) Repeat parts (a)–(c) for the "Height" and "Hand span" data of female students.
2. Make a graphical analysis of the "Height" and "Shoe size" data by following the steps in Question 1.
3. Consider the "Internet" and "Happy" columns in the survey data table.
   (a) Make a scatter plot of the data, with the horizontal axis representing "Internet" and the vertical axis representing "Happy."
   (b) Do you see any trends in the data? Do heavy Internet users tend to be happier?
4. Consider the survey data table.
   (a) Do you think that there may be a relationship between the "Commute" and "Coins" columns? How about the "Homework" and "Satisfied" columns? Or the "Commute" and "Homework" columns? Make a research hypothesis stating whether or not you think there is a relationship.
   (b) Test any research hypotheses you made in part (a) by making a scatter plot and observing any resulting trends.

3 Every Graph Tells a Story

OBJECTIVE To learn how to give a verbal summary of a situation described by a graph.



If a picture is worth a thousand words, then a graph is worth at least a few sentences of prose. In fact, a graph can sometimes tell a story much more quickly and effectively than many words. The **devastating** impact of the stock market crash of 1929 is immediately evident from a graph of the Dow Jones index (DJIA). Such graphs were printed in the newspapers of the time as an effective way of conveying the magnitude of the crash. No words are needed to convey the message in the cartoon below. The graph tells a simple story: Something is way down—perhaps sales, profits, or productivity—and the responsible person is very very worried.



Aardvark Marketing

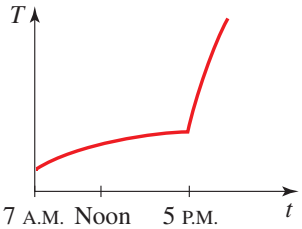
In this exploration we read the stories that graphs tell, and we make graphs that tell stories.

I. Reading a Story from a Graph

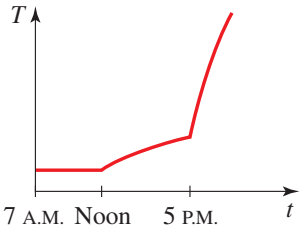
1. Four graphs of temperature versus time (starting at 7:00 A.M.) are shown below, followed by three stories.

(a) Match each of the stories with one of the graphs.

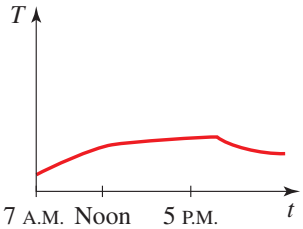
(b) Write a similar story for the graph that didn't match any story.



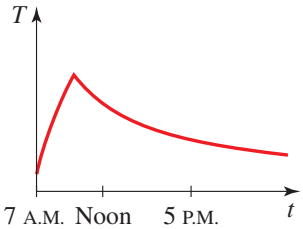
Graph A



Graph B



Graph C



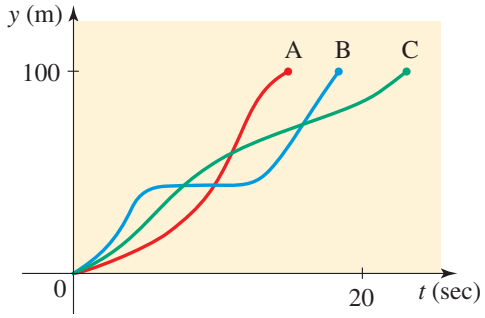
Graph D

Story 1 I took a roast out of the freezer at noon and left it on the counter to thaw. Then I cooked it in the oven when I got home.

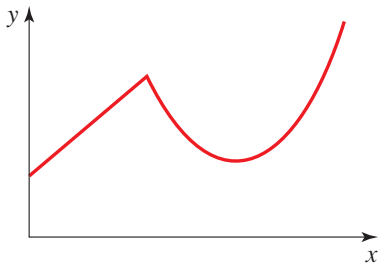
Story 2 I took a roast out of the freezer in the morning and left it on the counter to thaw. Then I cooked it in the oven when I got home.

Story 3 I took a roast out of the freezer in the morning and left it on the counter to thaw. I forgot about it and went out for Chinese food on my way home from work. I put the roast in the refrigerator when I finally got home.

2. Three runners compete in a 100-meter hurdle race. The graph shows the distance run as a function of time for each runner. Describe what the graph tells you about this race. Who won the race? Did each runner finish the race? What do you think happened to Runner B?



3. Make up a story involving any situation that would correspond to the graph shown below.



2 Linear Functions and Models



Keith Levit/Shutterstock.com 2009

2.1 Working with Functions: Average Rate of Change

2.2 Linear Functions: Constant Rate of Change

2.3 Equations of Lines: Making Linear Models

2.4 Varying the Coefficients: Direct Proportionality

2.5 Linear Regression: Fitting Lines to Data

2.6 Linear Equations: Getting Information from a Model

2.7 Linear Equations: Where Lines Meet

EXPLORATIONS

1 When Rates of Change Change

2 Linear Patterns

3 Bridge Science

4 Correlation and Causation

5 Fair Division of Assets

Global warming? Is the world getting hotter, or are we just having a temporary warm spell? To answer this question, scientists collect huge amounts of data on global temperature. A graph of the data can help to reveal long term changes in temperature, but more precise algebra methods must be used to detect any trend that is different from normal temperature fluctuations. Significant global warming could have drastic consequences for the survival of many species. For example, melting polar ice eliminates the ice paths polar bears need to reach their feeding grounds, resulting in the bears' starving or possibly drowning. (See Section 2.5, Exercise 15, page 197.) Of course, any changes in the global ecology also have implications for our own well-being.

2.1 Working with Functions: Average Rate of Change

- Average Rate of Change of a Function
- Average Speed of a Moving Object
- Functions Defined by Algebraic Expressions

IN THIS SECTION... we consider the net change of a function on an interval. This leads to the concept of *average rate of change* of a function.

GET READY... by reviewing the concept of net change from Section 1.5.

In Section 1.7 we used the graph of a function to determine where the function was increasing and where it was decreasing. In this section we take a closer look at these properties; namely, we ask, "How fast is the function increasing?" or "At what rate is the function decreasing?" To answer these questions, we give a precise meaning to the term *how fast* or *at what rate*.

Average Rate of Change of a Function

Month	Month number	Total number of books
Jan.	1	126,554
Feb.	2	126,672
Mar.	3	126,823
Apr.	4	127,003
May	5	127,187
June	6	127,523
July	7	127,634
Aug.	8	127,754
Sept.	9	127,896
Oct.	10	128,214
Nov.	11	128,415
Dec.	12	128,729

The rate at which a function increases or decreases can vary dramatically, as in the case of the speed of a runner in a marathon or the rate at which books are added to a library. However, a good approximation is the *average rate of change* of a function over an interval.

A university library receives new books each month. The head librarian needs to estimate how many books "on average" the library receives monthly. The data listed in the margin give the number of books the library has on the last day of each month in 2007.

Let's find the average monthly increase in the number of books for the summer months from the end of May to the end of August. First we find the net change (see Section 1.5) in the number of books during these months:

$$\text{net change in number of books} = 127{,}754 - 127{,}187 = 567$$

To find the number of months over which this increase in books took place, we subtract:

$$\text{number of months} = 8 - 5 = 3$$

So the average rate of change per month in the number of books is

$$\text{average rate of change} = \frac{\text{net change in number of books}}{\text{change in time}}$$

$$= \frac{567 \text{ books}}{3 \text{ months}} = 189 \text{ books/month}$$

So from May to August the library added an average of 189 books per month.

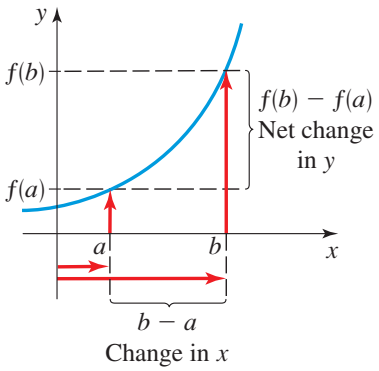


figure **1** Average rate of change of a function f

In general, we find the average rate of change of a function by calculating the net change in the function values and dividing by the net change in the x-values.

Average Rate of Change of a Function

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{net change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The graph in Figure 1 shows that $f(b) - f(a)$ is the net change in the value of f and $b - a$ is the change in the value of x.

example **1**

Average Rate of Installation

x Hours	$f(x)$ Tiles
0	0
1	21
2	69
3	126
4	189
5	216
6	245
7	347
8	403

Sima is installing new Italian ceramic flooring in her house. The table in the margin gives the total number $f(x)$ of tiles she has installed after working for x hours.

(a) Find the average rate of installation in the first hour.

(b) Find the average rate of installation for the first 4 hours.

(c) Find the average rate of installation from hour 6 to hour 8.

(d) Draw a graph of f and use the graph to find the hour in which Sima had the fastest average rate of installation.

Solution

(a) We find the average rate of change of the function f between $x = 0$ and $x = 1$:

$$\text{average rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{21 \text{ tiles} - 0 \text{ tiles}}{1 \text{ hour}} = 21 \text{ tiles/hour}$$

So the average rate of installation for the first hour is 21 tiles per hour.

(b) We find the average rate of change of the function f between $x = 0$ and $x = 4$:

$$\text{average rate of change} = \frac{f(4) - f(0)}{4 - 0} = \frac{189 - 0}{4} = 47.25 \text{ tiles/hour}$$

So the average rate of installation for the first 4 hours is about 47 tiles per hour.

(c) We find the average rate of change of the function f between $x = 6$ and $x = 8$:

$$\text{average rate of change} = \frac{f(8) - f(6)}{8 - 6} = \frac{403 - 245}{2} = 79 \text{ tiles/hour}$$

So the average rate of installation for this time period is about 79 tiles per hour.

(d) A scatter plot is shown in Figure 2. We can see from the graph that the steepest rise in the graph is during the seventh hour (from hour 6 to hour 7), so the fastest average rate of installation occurred in the seventh hour.

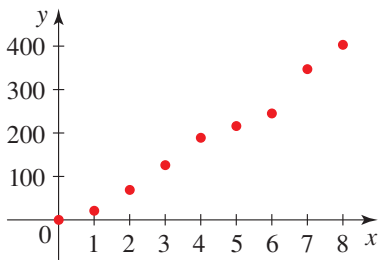


figure **2**

■ **NOW TRY EXERCISE 21** ■

IN CONTEXT ➤

Farming has always been an important part of the U.S. economy. In the 19th century the United States was very much an agrarian society; more than 75% of the labor force was engaged in some aspect of farming. Westward expansion of the U.S. population was fueled to a great extent by the search for new land to homestead and farm. As a result there was a dramatic increase in the number of farms in the United States. Most farms were family owned and operated. In the mid-20th century new automated farming methods and improved strains of crops led to an increase in farm productivity. Large corporate farming enterprises arose, and many smaller farmers found it impossible to remain profitable and sold their land to the corporate enterprises. As a result the number of farms decreased as the size of individual farms increased. Because of the efficiencies of scale, fewer workers were required to operate these larger farms, resulting in a population shift from rural to urban areas.



© Minnesota Historical Society/CORBIS

Farming in the 19th century


Stephen Mcsweeny/Shutterstock.com 2009

Farming in the 21st century

example 2 Number of Farms in the United States

The table in the margin gives the number of farms in the United States from 1850 to 2000.

Year	Farms (× 1000)
1850	1449
1860	2044
1870	2660
1880	4009
1890	4565
1900	5740
1910	6366
1920	6454
1930	6295
1940	6102
1950	5388
1960	3711
1970	2780
1980	2440
1990	2143
2000	2172

(a) Draw a scatter plot of the data.

(b) Find the average rate of change in the number of farms between the following years: (i) 1860 and 1890; (ii) 1950 and 1970.

(c) In which decade did the number of farms experience the greatest average rate of decline?

Solution

(a) A scatter plot is shown in Figure 3.

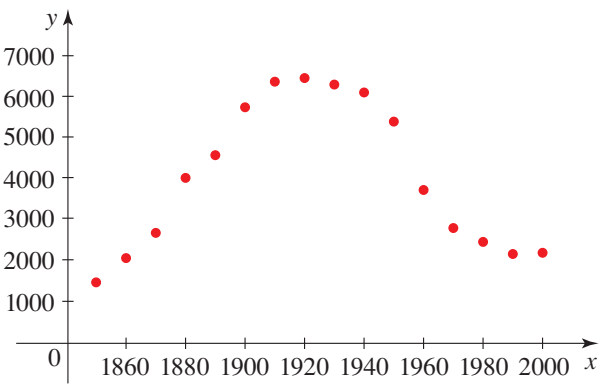


figure 3 Number of farms in the United States (in thousands)

(b) (i) The average rate of change between 1860 and 1890 is

$$\text{average rate of change} = \frac{4565 - 2044}{1890 - 1860} = 84.0 \text{ thousand farms/year}$$

So the number of farms increased at an average rate of 84,000 farms per year.

(ii) The average rate of change between 1950 and 1970 is

$$\text{average rate of change} = \frac{2780 - 5388}{1970 - 1950} = -130.4 \text{ thousand farms/year}$$

Since the average rate of change is *negative*, the number of farms *decreased* at an average rate of 130,400 farms per year.

(c) From the graph we see that the steepest drop in a single decade occurred between 1950 and 1960.

NOW TRY EXERCISE 25

Average Speed of a Moving Object

If you drive your car a distance of 60 miles in 2 hours, then your average speed is

$$\frac{60 \text{ miles}}{2 \text{ hours}} = 30 \text{ miles/hour}$$

In general, if a function represents the distance traveled, the average rate of change of the function is the average speed of the moving object.

Average Speed of a Moving Object

For a moving object, let $s(t)$ be the distance it has traveled at time t. Then the average rate of change of the function s from time t_1 to time t_2 is called the **average speed**:

$$\text{Average speed} = \frac{\text{net change in distance}}{\text{change in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

example **3** **Average Speed in a Bicycle Race**



James and Jodi compete in a bicycle race. The graphs in Figure 4 on the next page show the distance each has traveled as a function of time. We plot the time (in hours) on the t-axis and the distance (in miles) on the y-axis.

(a) Describe the bicycle race.

(b) Find James's and Jodi's average speeds for the entire race.

(c) Find Jodi's average speed between $t = 2$ hours and $t = 4$ hours.

(d) Find James's average speed between $t = 2$ hours and $t = 4$ hours.

(e) Find Jodi's average speed in the final hour of the race.

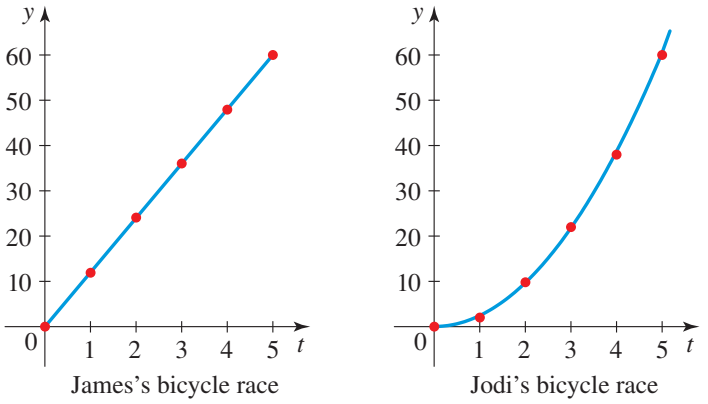


figure **4**

Solution

(a) From the graphs in Figure 4 we see that the race had a fair start with both James and Jodi starting the race at time zero. However, the differences in the graphs show that they don't always cycle at the same pace. James travels at a steady speed throughout the race, but Jodi varies her speed, traveling slowly in the beginning of the race and speeding up in the end. Even though the graphs are very different, in the end the race is a tie, since both competitors finish the race at the same time.

(b) The average speed for Jodi and James is the same, since they both travel 60 miles in 5 hours:

$$\text{average speed} = \frac{\text{net change in distance}}{\text{change in time}} = \frac{60 \text{ mi} - 0 \text{ mi}}{5 \text{ h} - 0 \text{ h}} = 12 \text{ mi/h}$$

So they each cycle at an average speed of 12 miles per hour.

(c) The total time elapsed is $4 - 2 = 2$ hours. From the graph we see that the distance Jodi traveled in this time is $38 - 10 = 28$ miles. So Jodi's average speed in this time interval is

$$\text{average speed} = \frac{\text{net change in distance}}{\text{change in time}} = \frac{38 - 10}{4 - 2} = 14 \text{ mi/h}$$

(d) Similarly, James's average speed between $t = 2$ h and $t = 4$ h is

$$\text{average speed} = \frac{\text{net change in distance}}{\text{change in time}} = \frac{48 - 24}{4 - 2} = 12 \text{ mi/h}$$

(e) Jodi's average speed in the final hour of the race is

$$\text{average speed} = \frac{\text{net change in distance}}{\text{change in time}} = \frac{60 - 38}{5 - 4} = 22 \text{ mi/h}$$

■ **NOW TRY EXERCISE 27** ■

Functions Defined by Algebraic Expressions

The concept of average rate of change applies to any function. In the next example we find average rates of change for a function defined by an algebraic expression.

example 4 Average Rate of Change of a Function

Find the average rate of change of the function $f(x) = x^2 + 4$ between the following values of x.

(a) $x = -3$ and $x = 1$

(b) $x = 2$ and $x = 5$

Solution

(a) The average rate of change of f between $x = -3$ and $x = 1$ is

$$\text{average rate of change} = \frac{f(1) - f(-3)}{1 - (-3)} = \frac{(1^2 + 4) - ((-3)^2 + 4)}{1 + 3} = -2$$

So on the interval $[-3, 1]$ the values of the function f decrease an average of 2 units for each unit change in x.

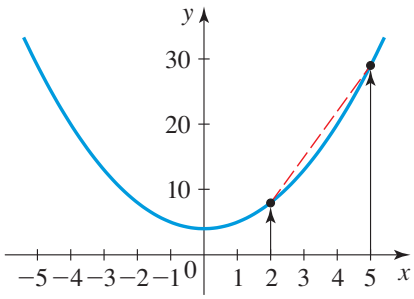
(b) The average rate of change of f between $x = 2$ and $x = 5$ is

$$\text{average rate of change} = \frac{f(5) - f(2)}{5 - 2} = \frac{(5^2 + 4) - (2^2 + 4)}{5 - 2} = 7$$

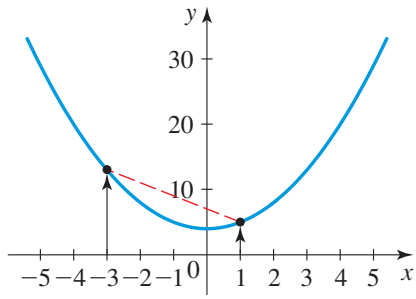
So on the interval $[2, 5]$ the values of the function f increase an average of 7 units for each unit change in x.

NOW TRY EXERCISES 13 AND 15

From the graphs in Figure 5 we can see why the average rate of change of f is positive between $x = 2$ and $x = 5$ but negative between $x = -3$ and $x = 1$.



A net increase in the value of f between $x = 2$ and $x = 5$



A net decrease in the value of f between $x = -3$ and $x = 1$

figure 5

example 5 Bungee Jumping



It is known that when an object is dropped in a vacuum, the distance the object falls in t seconds is modeled by the function

$$s(t) = 16t^2$$

where s is measured in feet (ignoring the effects of wind resistance on the speed). Use the function s to find the average speed of a bungee jumper during the following time intervals:

(a) The first second of the jump (that is, between $t = 0$ and $t = 1$)

(b) The third second of the jump (that is, between $t = 2$ and $t = 3$)

Solution

(a) The average speed of the bungee jumper in the first second of the jump is

$$\text{average speed} = \frac{s(1) - s(0)}{1 - 0} = \frac{16(1^2)\text{ ft} - 16(0^2)\text{ ft}}{1\text{ s} - 0\text{ s}} = 16\text{ ft/s}$$

(b) Similarly, the average speed of the bungee jumper in the third second of the jump is

$$\text{average speed} = \frac{s(3) - s(2)}{3 - 2} = \frac{16(3^2) - 16(2^2)}{1} = 80\text{ ft/s}$$

NOW TRY EXERCISE 29

2.1 Exercises

CONCEPTS

Fundamentals

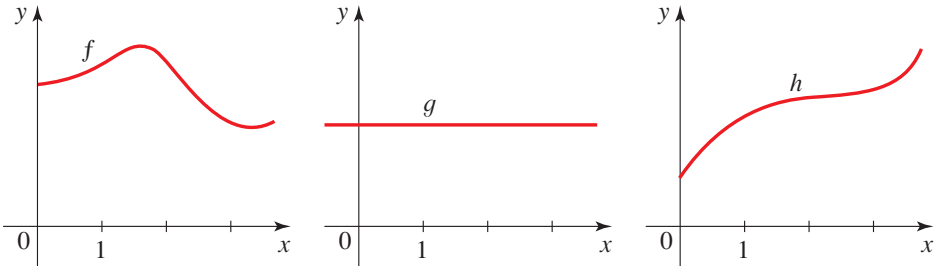
1. **(a)** The *average rate of change* of a function $y = f(x)$ between $x = a$ and $x = b$ is

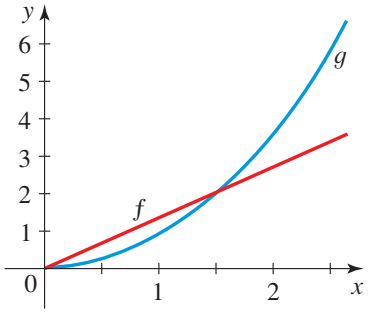
$$\frac{\text{change in } ____}{\text{change in } ____} = \frac{f(\square) - f(\square)}{\square - \square}.$$

(b) If $f(2) = 3$ and $f(5) = 10$, then the average rate of change of f between $x = 2$ and $x = 5$ is

$$\frac{\square - \square}{\square - \square} = _____.$$

2. The graphs of functions f, g, and h are shown. Between $x = 0$ and $x = 3$ the function _____ has average rate of change of 0, the function _____ has positive average rate of change, and the function _____ has negative average rate of change.





Think About It

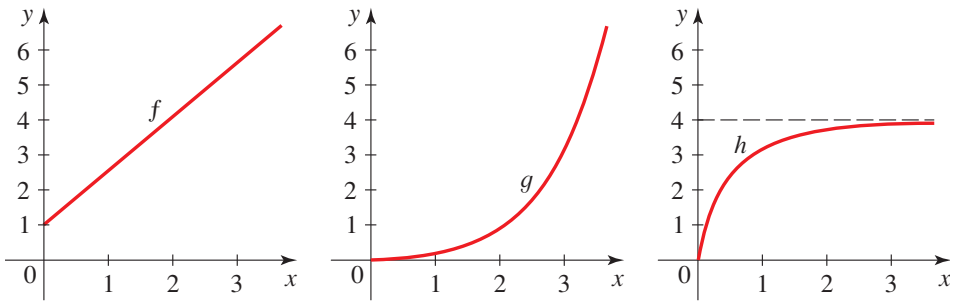
3. True or false?

(a) If a function has positive net change between $x = 0$ and $x = 1$, then the function has positive average rate of change between $x = 0$ and $x = 1$.

(b) If a function has positive average rate of change between $x = 0$ and $x = 1$, then the function has positive net change between $x = 0$ and $x = 1$.

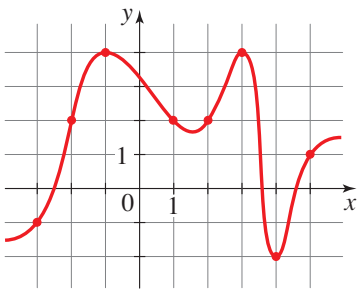
4. The graphs of the functions f and g are shown in the margin. The function _____ (f or g) has a greater average rate of change between $x = 0$ and $x = 1$. The function _____ (f or g) has a greater average rate of change between $x = 1$ and $x = 2$. The functions f and g have the same average rate of change between $x =$ _____ and $x =$ _____.

5. Graphs of the functions f, g, and h are shown below. What can you say about the average rate of change of each function on the successive intervals [0, 1], [1, 2], [2, 3], . . .?



6. The graph of a function f is shown below. Find x-values a and b so that the average rate of change of f between a and b is

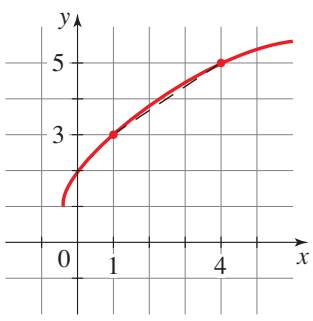
(a) 0 **(b)** 2 **(c)** -1 **(d)** -2



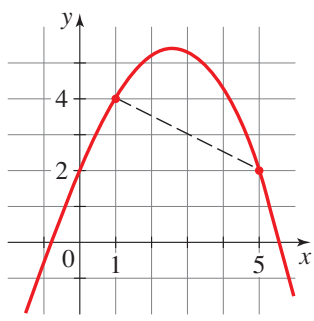
SKILLS

7–10 ■ The graph of a function is given. Determine the average rate of change of the function between the indicated points.

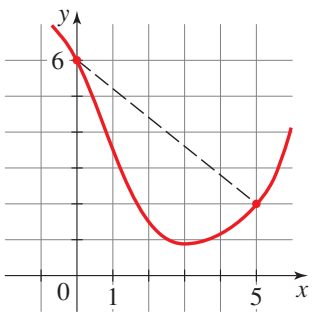
7.



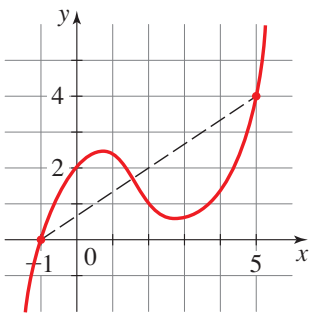
8.



9.



10.



11–12 ■ A function is given by a table.

(a) Determine the average rate of change of the function between the given values of x.

(b) Graph the function.

(c) From your graph, find the two successive points between which the average rate of change is the largest. What is this rate of change?

11. **(i)** Between $x = 2$ and $x = 4$

(ii) Between $x = 4$ and $x = 9$

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	10	20	50	70	90	80	65	60	50	60	80

12. **(i)** Between $x = 0$ and $x = 50$

(ii) Between $x = 20$ and $x = 90$

x	0	10	20	30	40	50	60	70	80	90	100
$G(x)$	3.3	3.0	2.5	1.7	1.7	0.8	2.2	4.5	5.0	5.5	6.0

13–20 ■ A function is given. Determine the average rate of change of the function between the given values of x.

13. $f(x) = 3x + 2; \quad x = 2, x = 5$

14. $f(x) = 5 - 7x; \quad x = -1, x = 3$

15. $g(x) = 1 - 2x^2; \quad x = 0, x = 1$

16. $g(x) = \frac{1}{2}x^2 + 4; \quad x = -2, x = 0$

17. $h(x) = x^2 + 3x; \quad x = -1, x = 1$

18. $h(x) = 2x - x^2; \quad x = 2, x = 4$

19. $k(x) = \dfrac{6}{x}; \quad x = 1, x = 3$

20. $k(x) = x^3; \quad x = 2, x = 4$

CONTEXTS

21. Population of Atlanta In the latter part of the 20th century the United States experienced a large population shift from the cities to the suburbs. This is true of Atlanta, for example, whose population grew steadily for its first hundred years, then began to decline. Within the last two decades Atlanta's population has started to rise again, as seen in the table at the top of the next page.

(a) Draw a scatter plot of the data.

(b) Find the average rate of change of the population of Atlanta between the following years: **(i)** 1850 and 1950 **(ii)** 1950 and 2000 **(iii)** 1950 and 1970

(c) Use the scatter plot to find the decade in which Atlanta's population experienced the greatest average rate of increase.

Population of Atlanta, Georgia

Year	Population	Year	Population
1850	2572	1930	270,688
1860	9554	1940	302,288
1870	21,789	1950	331,000
1880	37,409	1960	487,000
1890	65,533	1970	497,000
1900	89,872	1980	425,000
1910	154,839	1990	394,017
1920	200,616	2000	416,474

22. Cooling Soup When a bowl of hot soup is left in a room, the soup eventually cools down to the temperature of the room. The temperature of the soup $T(t)$ is a function of time t. The temperature changes more slowly as the soup gets closer to room temperature. The table below shows the temperature (in degrees Fahrenheit) of the soup t minutes after it was set down.

(a) What was the temperature of the soup when it was initially placed on the table?

(b) Find the average rate of change of the temperature of the soup over the first 20 minutes and over the next 20 minutes. On which interval did the temperature decline more quickly?

t (min)	T (°F)	t (min)	T (°F)
0	200	35	94
5	172	40	89
10	150	50	81
15	133	60	77
20	119	90	72
25	108	120	70
30	100	150	70

Age (yr)	Height (in.)	Age (yr)	Height (in.)
0	19.25	12	58.75
1	28.00	14	64.00
2	32.50	15	66.50
3	36.25	16	69.13
4	39.63	17	69.50
6	44.50	18	69.75
8	49.25	19	69.88
10	54.38	20	69.88

23. Growth Rate Jason's height $H(x)$ is a function of his age x (in years). At various stages in his life he grows at different rates, as shown in the table in the margin, which gives his height every year on his birthday.

(a) Find $H(0)$, $H(4)$, and $H(8)$.

(b) Find the average rate of change in Jason's height from birth to 4 years and from 4 years to 8 years. Over which period did Jason have the faster average rate of growth?

(c) Draw a graph of H, and use the graph to find the year in which Jason had the greatest average rate of growth between his 14th and 20th birthdays.

24. Rare Book Collection Between 1985 and 2007 a rare book collector purchased books for his collection at the rate of 40 books per year. Use this information to complete the table below showing the number of books in his collection between 1985 and 2007. (Note that not every year is given in the table.)

Year	1985	1986	1987	1990	1995	1997	2000	2001	2005	2006	2007
Number of books	420	460									1300

Value of the Euro in U.S. dollars

Year	Value (U.S. $)
1999	0.86
2000	1.01
2001	0.94
2002	0.89
2003	1.05
2004	1.26
2005	1.35
2006	1.18
2007	1.32
2008	1.46

25. Currency Exchange Rates The euro was introduced in 1990 as a common currency for twelve member countries of the European Union. The table in the margin shows the value of the euro in U.S. dollars on the first business day of each year from 1999 to 2008.

(a) Draw a scatter plot of the data.

(b) Find the average rate of change of the value of the euro in U.S. dollars between the following years: **(i)** 1999 and 2008 **(ii)** 2002 and 2006 **(iii)** 2005 and 2008

(c) Use the scatter plot to find the year in which the value of the Euro experienced the largest average rate of increase in terms of the U.S. dollar.

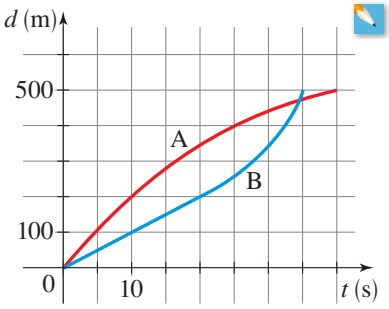
26. Rate of Increase of an Investment Julia invested $500 in a mutual fund on June 30, 2000, using a generous high school graduation gift from her aunt. Every June 30th she records the amount in the fund. In early 2001 the stock market crashed, and by June 30 that year Julia's investment had lost half its value. Over the course of the next year the fund again lost half its value, but in 2003 its value tripled. The value of the mutual fund continued to increase, and by June 30, 2006, Julia's investment was worth $600. By June 2007 the mutual fund had a 30% increase, that is, it increased by 30% of its value on June 30, 2006. The value of Julia's mutual fund is a function $A(t)$ where t is the year.

(a) Find $A(2000)$, $A(2001)$, $A(2002)$, $A(2003)$, $A(2006)$, and $A(2007)$.

(b) Find the annual average rate of change of the function A between 2000 and 2007.

(c) Draw a scatter plot of A using your data from part (a).

(d) Which is greater: the annual rate of change of A between 2002 and 2003 or between 2003 and 2006?



27. Speed Skating At the 2006 Winter Olympics in Turin, Italy, the United States won three gold medals in men's speed skating. The graph in the margin shows distance as a function of time for two speed skaters racing in a 500-meter event.

(a) Who won the race?

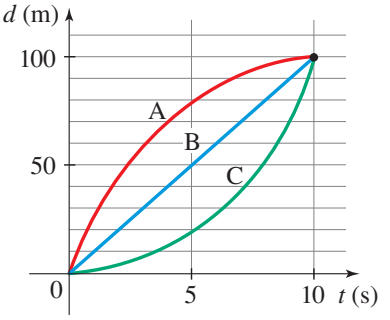
(b) Find the average speed during the first 10 seconds for each skater.

(c) Find the average speed during the last 15 seconds for each skater.

28. 100-Meter Race A 100-meter race ends in a three-way tie for first place. The graph shows distance as a function of time for each of the three winners.

(a) Find the average speed for each winner.

(b) Describe the differences in the way the three runners ran the race.



29. Phoenix Mars Lander On August 4, 2007, NASA's Phoenix Mars Lander was launched into space to search for life in the icy northern region of the planet Mars; it touched down on Mars on May 25, 2008. As the ship raced into space, its jet fuel tanks dropped off when they were used up. The distance one of the tanks falls in t seconds is modeled by the function

$$s(t) = 16t^2$$

where s is measured in feet per second and $t = 0$ is the instant the tank left the ship. Use the function s to find the average speed of a tank during the following time intervals.

(a) The first 10 seconds after separation from the ship

(b) The first 30 seconds after separation from the ship



30. A Falling Skydiver When a skydiver jumps out of an airplane from a height of 13,000 feet, her height h (in feet) above the ground after t seconds is given by the function

$$h(t) = 13{,}000 - 16t^2$$

(a) Use the function h to find the average speed of the skydiver during the first 5 seconds.

(b) The skydiver opens her parachute after 24 seconds. What is her average speed during her 24 seconds of free fall?

31. Falling Cannonballs Galileo Galilei is said to have dropped cannonballs of different sizes from the Tower of Pisa to demonstrate that their speed is independent of their mass. The function

$$h(t) = 183.27 - 16t^2$$

models the height h (in feet) of the cannonball above the ground t seconds after it is dropped. Use the function h to find the average speed of the cannonball during the following time intervals.

(a) The first 2 seconds **(b)** The first 3 seconds **(c)** Between $t = 2$ and $t = 3.25$

32. Path of Bullet A bullet is shot straight upward with an initial speed of 800 ft/s. The height of the bullet after t seconds is modeled by the function

$$h(t) = -16t^2 + 800t$$

where h is measured in feet. Use the function h to find the average speed of the bullet during the given time intervals.

(a) The first 5 seconds **(b)** The first 40 seconds **(c)** Between $t = 10$ and $t = 40$

2.2 Linear Functions: Constant Rate of Change

- **Linear Functions**
- **Linear Functions and Rate of Change**
- **Linear Functions and Slope**
- **Using Slope and Rate of Change**

IN THIS SECTION... we consider functions for which the average rate of change is constant. Such functions have the form $f(x) = b + mx$, and their graphs are straight lines. We'll see that the number m can be interpreted as the rate of change of f or the slope of the graph of f.

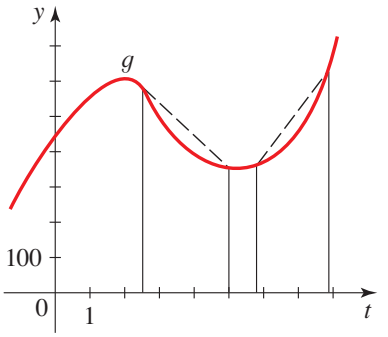


Ralph Hagen/www.CartoonStock.com

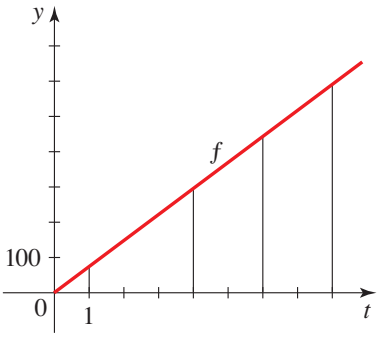
In Section 2.1 we studied the average rate of change of a function on an interval. Most of the functions we considered had different average rates of change on different intervals. But what if a function has constant average rate of change? That is, what if the function has the same average rate of change on every interval?

The graph in Figure 1(b) on page 154 shows the number of chocolates $f(t)$ produced by a chocolate-manufacturing machine in t minutes (the start of a work shift is represented by $t = 0$). We can see that chocolates are produced by the machine at a fixed rate of 100 per minute. The graph in Figure 1(a) on page 154 shows the number of chocolates $g(t)$ produced by a malfunctioning machine (which sometimes destroys chocolates it has produced). That machine's production rate varies wildly

as it attempts to produce chocolates. Notice that the graph of a function that has constant rate of change on every interval is a line.



(a) g has different average rates of change on different intervals.



(b) f has the same average rate of change on every interval.

figure **1**

Linear Functions

In Section 1.3 we encountered functions that have constant rate of change, and we've worked with such functions both numerically and graphically. In fact, the graphs of functions with constant rate of change are lines. Algebraically, functions with constant rate of change are *linear* in the following sense.

Linear Functions

- A **linear function** is a function of the form

$$f(x) = b + mx$$

  where b and m are real numbers.
- The graph of a linear function is a line.

example **1** **Identifying Linear Functions**

Determine whether the following functions are linear.

(a) $f(x) = 2 + 3x$ **(b)** $g(x) = -2x + 1$

(c) $h(x) = 3x^2 + 1$ **(d)** $k(x) = \dfrac{1 - 5x}{4}$

Solution

(a) f is a linear function, where b is 2 and m is 3.

(b) g is a linear function, where b is 1 and m is -2.

(c) h is not linear because the variable x is squared.

(d) k is a linear function where b is $\frac{1}{4}$ and m is $-\frac{5}{4}$.

■ **NOW TRY EXERCISES 9 AND 11** ■

example 2 Graphing a Linear Function

Let f be the linear function $f(x) = 2 + 3x$,

(a) Make a table of values of f.

(b) Sketch a graph of f.

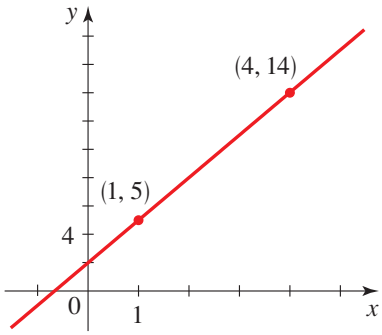


figure **2** Graph of the linear function $f(x) = 2 + 3x$

Solution

(a) A table of values is shown below.

x	-2	-1	0	1	2	3	4	5
$f(x)$	-4	-1	2	5	8	11	14	17

(b) Since f is a linear function its graph is a line. So to obtain the graph of f, we plot any two points from the table and connect them with a straight line. The graph is shown in Figure 2.

■ **NOW TRY EXERCISE 17** ■

example 3 Average Rate of Change of a Linear Function

Let f be the linear function $f(x) = 2 + 3x$. Find the average rate of change on the following intervals.

(a) Between $x = 0$ and $x = 1$

(b) Between $x = 1$ and $x = 4$

(c) Between $x = c$ and $x = d$

What conclusion can you draw from your answers?

Solution

(a) average rate of change $= \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{(2 + 3\cdot 1) - (2 + 3\cdot 0)}{1} = 3$

(b) average rate of change $= \dfrac{f(6) - f(1)}{6 - 1} = \dfrac{(2 + 3\cdot 6) - (2 + 3\cdot 1)}{5} = 3$

(c) average rate of change $= \dfrac{f(d) - f(c)}{d - c}$ Definition

$= \dfrac{(2 + 3\cdot d) - (2 + 3\cdot c)}{d - c}$ Use $f(x) = 2 + 3x$

$= \dfrac{2 + 3d - 2 - 3c}{d - c}$ Expand

$= \dfrac{3d - 3c}{d - c}$ Simplify numerator

$= \dfrac{3(d - c)}{d - c}$ Factor 3

$= 3$ Cancel common factor

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the average rate of change for this function between *any* two points $x = c$ and $x = d$ is 3.

NOW TRY EXERCISE 21

Linear Functions and Rate of Change

In Example 3 we saw that the average rate of change of a linear function is the same between any two points. In fact, for any linear function $f(x) = b + mx$ the average rate of change between any two points is the constant m. For this reason, when dealing with linear functions, we refer to the average rate of change as simply the **rate of change**. We also call the number b the **initial value** of f. Of course, $b = f(0)$; in many applications we can think of b as the "starting value" of the function (see Section 1.3).

Linear Functions and Rate of Change

Let f be the linear function $f(x) = b + mx$.

- The rate of change of f is the constant m.
- The initial value of f is the constant b.

Initial value: b; Rate of change: m

$$f(x) = b + mx$$

example 4 Linear Model of Growth

Toddlers generally grow at a constant rate. Little Jason's height between his first and second birthdays is modeled by the function

$$h(x) = 25.0 + 0.4x$$

where $h(x)$ is measured in inches and x is the number of months since his first birthday.

(a) Is h a linear function?

(b) Make a table of values for h.

(c) What is Jason's initial height?

(d) At what rate is Jason growing?

(e) Sketch a graph of h.

Month x	Height $h(x)$ (in.)
0	25.0
1	25.4
2	25.8
3	26.2
4	26.6
5	27.0
6	27.4
7	27.8
8	28.2
9	28.6
10	29.0
11	29.4
12	29.8

Solution

(a) Yes, h is a linear function for which b is 25.0 and m is 0.4.

(b) A table of values is shown in the margin.

(c) The initial value is the constant b, which is 25.0. This means that his height on his first birthday was 25.0 in.

(d) Since h is a linear function its rate of change is the constant m, which in this case is 0.4. This means that Jason grows 0.4 inch per month. Notice how the table of values agrees with this observation.

(e) Since h is a linear function, its graph is a straight line. So to graph h, we can plot any two points in the table and connect them with a straight line. See Figure 3.

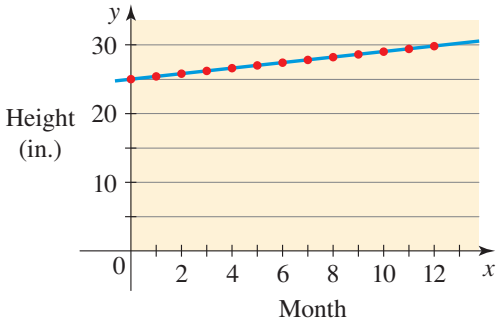
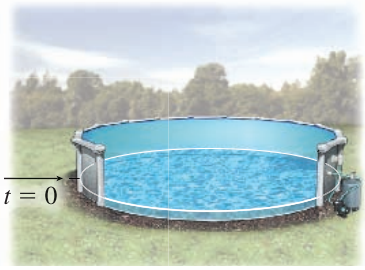


figure **3** Graph of $h(x) = 25.0 + 0.4x$

NOW TRY EXERCISE 53

example **5** **Finding a Linear Model from a Rate of Change**



There are 200 gallons of water in the pool at time $t = 0$.

Water is being pumped into a swimming pool at the rate of 5 gal/min. Initially, the pool has 200 gallons of water. Find a linear function that models the volume of water in the pool at any time.

Solution

We need to find a linear function

$$V(t) = b + mt$$

that models the volume $V(t)$ of water in the pool at time t. The rate of change of volume is 5 gal/min, so $m = 5$. Since the pool has 200 gallons to begin with, we have $V(0) = 200$. So the initial value is $b = 200$. Now that we know m and b, we can write

$$V(t) = 200 + 5t$$

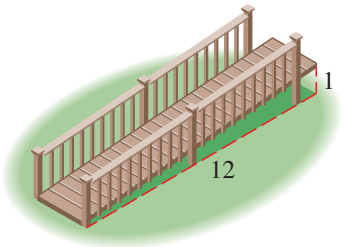
NOW TRY EXERCISE 55

Linear Functions and Slope

An important property of a line is its "steepness," or how quickly it rises or falls as we move along it from left to right. If we move between two points on a line, the *run* is the distance we move from left to right, and the *rise* is the corresponding distance that we move up (or down). The *slope* of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Figure 4 on page 158 shows some situations in which slope is important in real life and how it is measured in each case. Builders use the term *pitch* for the slope of a roof. The uphill or downhill slope of a road is called its *grade*.



Slope of a ramp
Slope $= \frac{1}{12}$



Pitch of a roof
Slope $= \frac{1}{3}$



Grade of a road
Slope $= \frac{8}{100}$

figure **4**

The graph of a linear function $f(x) = mx + b$ is a line. In this case the **run** is the change in the x-coordinate, and the **rise** is the corresponding change in the y-coordinate. The *slope* of a line is the ratio of rise to run.

Slope of a Line

If (x_1, y_1) and (x_2, y_2) are different points on a line graphed in a coordinate plane, then the **slope** of the line is defined by

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

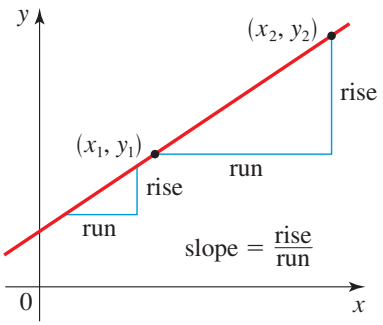


figure **5** The slope of a line

From the similar triangles property in geometry it follows that the ratio of rise to run is the same no matter which points we pick, so the slope is independent of the points we choose to measure it (see Figure 5).

example **6**

Finding the Slope of a Staircase

In Figure 6 we've placed a staircase in a coordinate plane, with the origin at the bottom left corner. The red line in the figure is the edge of the trim board of the staircase. Find the slope of this line.

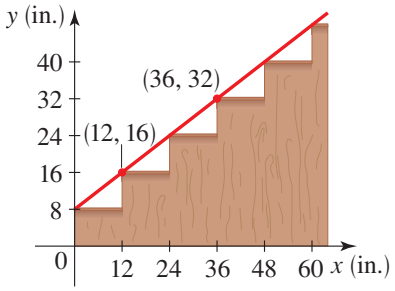


figure **6** Slope of a staircase

Solution 1

We observe that each of the steps is 8 inches high (the rise) and 12 inches deep (the run), so the slope of the line is

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{8}{12} = \frac{2}{3}$$

Solution 2

From Figure 6 we see that two points on the line are (12, 16) and (36, 32). So from the definition of slope we have

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{32 - 16}{36 - 12} = \frac{16}{24} = \frac{2}{3}$$

NOW TRY EXERCISE 57

For any linear function $f(x) = b + mx$, the **slope** of the graph of f is the constant m and the ***y*-intercept** is $f(0) = b$. So we have the following description of the graph of a linear function.

Linear Functions and Slope

y-intercept Slope
$f(x) = b + mx$

Let f be the linear function $f(x) = b + mx$.

- The graph of f is a line with slope m.
- The y-intercept of the graph of f is the constant b.

example 7

Finding the Slope of a Line

Let f be the linear function $f(x) = 2 + x$.

(a) Sketch the graph of f.

(b) Find the slope of the graph of f.

(c) Find the rate of change of f.

Solution

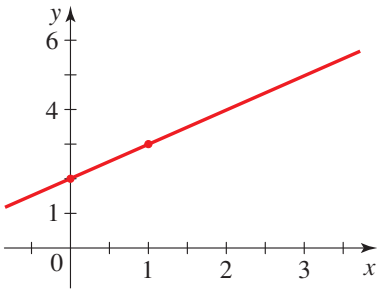


figure 7 Graph of $f(x) = 2 + x$

(a) The graph of f is a line. To sketch the graph, we need only find two points on the line. Since $f(0) = 2$ and $f(1) = 3$, two points on the graph are

$$(0, 2) \quad \text{and} \quad (1, 3)$$

Sketching these points and connecting them by a straight line, we get the graph in Figure 7.

(b) Since $f(x) = 2 + 1 \cdot x$, we see that $m = 1$, so the graph of f is a line with slope 1. We can also find the slope using the definition of slope and the two points we found in part (a):

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3 - 2}{1 - 0} = \frac{1}{1} = 1$$

So the graph of f is a line with slope 1.

(c) Since f is a linear function in which m is 1, it follows that the rate of change of f is 1. We can also calculate the rate of change directly. Using the two values of the function we found in part (a), we have

$$\text{rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{3 - 2}{1} = 1$$

NOW TRY EXERCISE 25

Using Slope and Rate of Change

For the linear function f in Example 7 we found that the rate of change of f is the same as the slope of the graph of f. This is true for any linear function. In fact if x_1 and x_2 are two different values for x, let's put $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then the points (x_1, y_1) and (x_2, y_2) are on the graph of f. From the definitions of slope and rate of change we have

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{rate of change}$$

We summarize this very useful observation.

Slope and Rate of Change

For the linear function $f(x) = b + mx$ we have

$$\text{slope} = \text{rate of change} = m$$

The difference between "slope" and "rate of change" is simply a difference in point of view.

- For the staircase in Example 6 we prefer to think that the trim board has a *slope* of 2/3, although we can also think of the rate of change in the height of the trim board as 2/3 (the trim board rises 2 inches for each 3-inch change in the run).
- For the swimming pool in Example 5 it is more natural to think that the *rate of change* of volume is 5 gal/min, although we can also view 5 as being the slope of the graph of the volume function (the graph rises 5 gallons for every 1-minute change in time).

example **8** **Finding Linear Functions from a Graph**

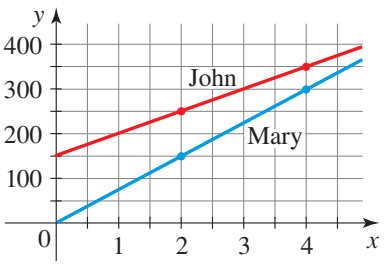


figure **8** John and Mary's trip

John and Mary are driving westward along I-76 at constant speeds. The graphs in Figure 8 show the distance y (in miles) that they have traveled from Philadelphia at time x (in hours), where $x = 0$ corresponds to noon.

(a) At what speeds are John and Mary traveling? Who is traveling faster, and how does this show up in the graph?

(b) Express the distances that John and Mary have traveled as functions of x.

(c) How far will John and Mary have traveled at 5:00 P.M.?

Solution

(a) From the graph we see that John has traveled 250 miles at 2:00 P.M. and 350 miles at 4:00 P.M. The speed is the rate of change, which is the slope of the graph, so John's speed is

$$\text{slope} = \frac{350 \text{ mi} - 250 \text{ mi}}{4 \text{ h} - 2 \text{ h}} = 50 \text{ mi/h} \qquad \text{John's speed}$$

Mary has traveled 150 miles at 2:00 P.M. and 300 miles at 4:00 P.M., so we calculate Mary's speed to be

$$\text{slope} = \frac{300 \text{ mi} - 150 \text{ mi}}{4 \text{ h} - 2 \text{ h}} = 75 \text{ mi/h} \qquad \text{Mary's speed}$$

(b) Let $f(x)$ be the distance John has traveled at time x. We know that f is a linear function, since the speed (average rate of change) is constant, so we can write f in the form

$$f(x) = b + mx$$

In part (a) we showed that m is 50, and from the graph we see that the y-intercept of John's graph is $b = 150$. So the distance John has traveled at time x is

$$f(x) = 150 + 50x \qquad \text{John's distance}$$

Similarly, Mary is traveling at $m = 75$ mi/h, and the y-intercept of her graph is $b = 0$, so the distance she has traveled at time x is

$$g(x) = 75x \qquad \text{Mary's distance}$$

(c) Replacing x by 5 in the equations we obtained in part (b), we find that at 5:00 P.M. John has traveled

$$f(5) = 150 + 50(5) = 400 \text{ miles}$$

and Mary has traveled

$$g(5) = 75(5) = 375 \text{ miles}$$

NOW TRY EXERCISE 59

2.2 Exercises

CONCEPTS

Fundamentals

1. Let f be a function with constant rate of change. Then

(a) f is a _____ function.

(b) f is of the form $f(x) =$ ☐ + ☐ x.

(c) The graph of f is a _____.

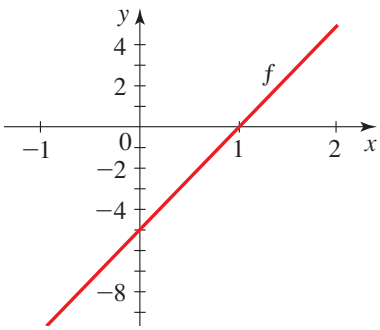
2. Let f be the linear function $f(x) = 7 - 2x$.

(a) The rate of change of f is _____, and the initial value is _____.

(b) The graph of f is a _____, with slope _____ and y-intercept _____.

3. We find the "steepness," or slope, of a line passing through two points by dividing the difference in the ____-coordinates of these points by the difference in the ____-coordinates. So the line passing through the points (0, 1) and (2, 5) has slope

$$\frac{\square - \square}{\square - \square} = _____$$



4. The graph of a linear function f is given in the margin. The y-intercept of f is _____, the slope of the graph of f is _____, and the rate of change of f is _____.

Think About It

5. Which of the following functions is not linear? Give reasons for your answer.

x	$f(x)$
0	5
2	7
4	9
6	11

x	$g(x)$
0	5
1	6
3	7
4	9

x	$h(x)$
2	7
5	16
8	25
12	37

6. If a linear function has positive slope, does its graph slope upward or downward? What if the linear function has negative slope?

7. Is $f(x) = 2$ a linear function? If so, find the slope and the y-intercept of the graph of f.

8. **(a)** Graph $f(x) = mx$ for $m = \frac{1}{2}$, $m = 1$, and $m = 2$, all on the same set of axes. How does increasing the value of m affect the graph of f?

(b) Graph $f(x) = x + b$ for $b = \frac{1}{2}$, $b = 1$, and $b = 2$, all on the same set of axes. How does increasing the value of b affect the graph of f?

SKILLS

9–16 ■ Determine whether the given function is linear.

9. $f(x) = 3 + \frac{1}{2}x$

10. $f(x) = 8 - \frac{4}{3}x$

11. $f(x) = 4 - x^2$

12. $f(x) = \sqrt{5 + 2x}$

13. $f(x) = -x + \sqrt{6}$

14. $f(x) = \dfrac{x - 3}{6}$

15. $f(x) = \frac{2}{3}(t - 2)$

16. $f(x) = (2t + 7)^3$

17–20 ■ For the given linear function, make a table of values and sketch its graph.

17. $f(x) = 6x + 5$

18. $g(x) = 4 + 2x$

19. $h(t) = 6 - 3t$

20. $s(t) = -2t - 6$

21–24 ■ For the given linear function, find the average rate of change on the following intervals.

(a) Between $x = -1$ and $x = 1$

(b) Between $x = 1$ and $x = 2$

(c) Between $x = a$ and $x = a + h$

21. $f(x) = 4 + 2x$

22. $g(x) = 5 + 15x$

23. $h(x) = -2x - 5$

24. $s(x) = -3x - 9$

25–34 ■ For the given linear function,

(a) Sketch the graph.

(b) Find the slope of the graph.

(c) Find the rate of change of the function.

25. $f(x) = 4 + 2x$

26. $f(x) = 3 - 4x$

27. $g(x) = -3 - x$

28. $g(x) = -2x + 10$

29. $f(t) = 6 + 3t$

30. $f(t) = 4 + 2t$

31. $h(t) = 2t - 6$

32. $h(t) = -3t - 9$

33. $F(x) = -0.5x - 2$

34. $F(x) = -0.3x - 6$

35–38 ■ A verbal description of a linear function is given. Find the function.

35. The linear function f has rate of change 7 and initial value -3.

36. The linear function g has rate of change $-\frac{1}{3}$ and initial value $\frac{2}{3}$.

37. The graph of the linear function h has slope -3 and y-intercept 9.

38. The graph of the linear function k has slope 2.5 and y-intercept 10.7.

39–42 ■ A table of values for a linear function f is given.

(a) Find the rate of change and the initial value of f.

(b) Express f in the form $f(x) = b + mx$.

39.

x	$f(x)$
0	3
2	4
4	5
6	6

40.

x	$f(x)$
0	8
1	5
2	2
3	−1

41.

x	$f(x)$
−2	16
0	12
1	10
3	6
7	−2

42.

x	$f(x)$
−3	5
0	6
6	8
15	11

43–48 ■ Find the slope of the line passing through the two given points.

43. (0, 0) and (4, 2)

44. (0, 0) and (2, −6)

45. (2, 2) and (−10, 0)

46. (1, 2) and (3, 3)

47. (2, 4) and (4, 3)

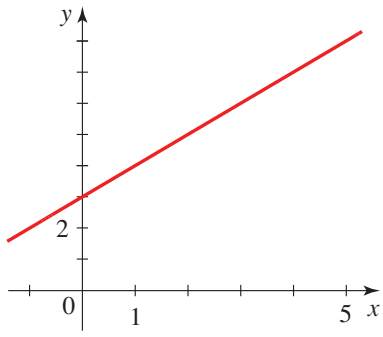
48. (2, −5) and (−4, 3)

49–52 ■ The graph of a linear function f is given.

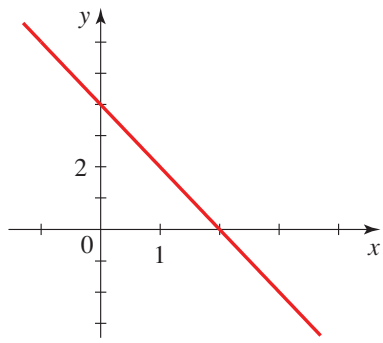
(a) Find the slope and the y-intercept of the graph.

(b) Express f in the form $f(x) = b + mx$.

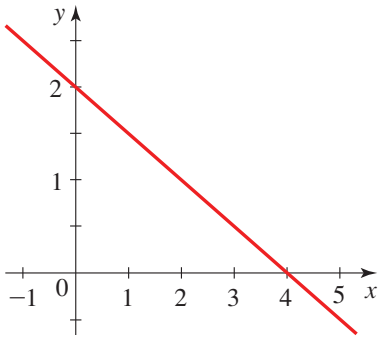
49.



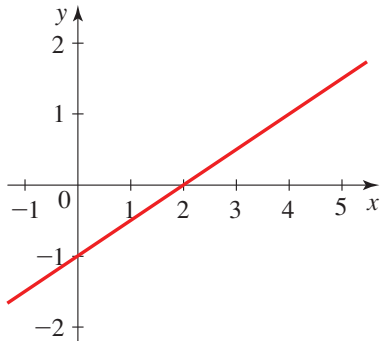
50.



51.



52.



CONTEXTS

53. Landfill The amount of trash in a county landfill is modeled by the function

$$T(x) = 32{,}400 + 4x$$

where x is the number of days since January 1, 1996, and $T(x)$ is measured in thousands of tons.

(a) Is T a linear function?

(b) What is the initial amount of trash in the landfill in 1996?

(c) At what rate is the landfill receiving trash?

(d) Sketch a graph of T.

54. Cell Phone Costs Ingrid is in the process of choosing a cell phone and a cell phone plan. Her choice of phones and plans is as follows:

Phones: \$50 (basic phone), \$100 (camera phone)

Plans: \$30, \$40, or \$60 per month

The first graph on the following page represents the cost per month of cell phone service for the \$30 plan; the second graph represents the total cost $C(x)$ of purchasing the \$50 cell phone and receiving cell phone service for x months at \$40 per month.

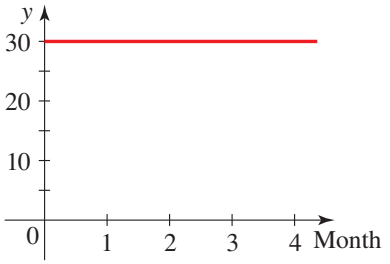
(a) Why is the line in the first graph horizontal?

(b) What do the x and y coordinates in the graph of C (second graph) represent?

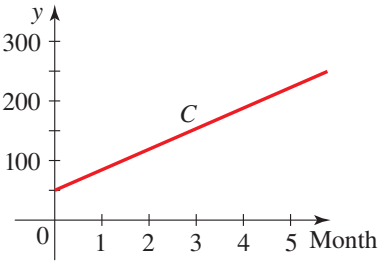
(c) If Ingrid upgrades to the \$60 per month plan, how does that change the graph of C?

(d) If Ingrid downgrades to a plan of \$30 a month, how does that change the graph of C?

(e) If Ingrid chooses the \$100 cell phone, how does that change the graph of C?



Monthly cost

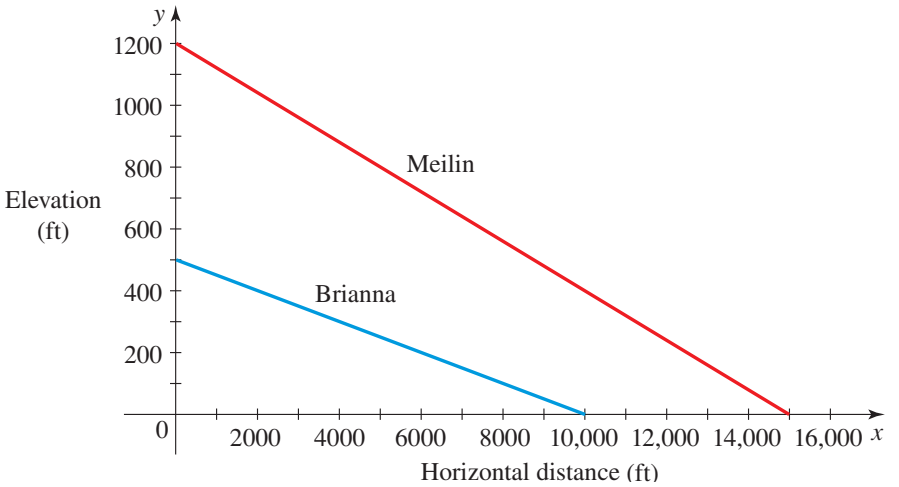


Total cost

55. Weather Balloon Weather balloons are filled with hydrogen and released at various sites to measure and transmit data such as air pressure and temperature. A weather balloon is filled with hydrogen at the rate of 0.5 ft^3/s. Initially, the balloon has 2 ft^3 of hydrogen. Find a linear function that models the volume of hydrogen in the balloon after t seconds.

56. Filling a Pond A large koi pond is filled from a garden hose at the rate of 10 gal/min. Initially, the pond contains 300 gallons of water. Find a linear function that models the volume of water in the pond (in gallons) after t minutes.

57. Mountain Biking Meilin and Brianna are avid mountain bikers. On a spring day they cycle down straight roads with steep grades. The graphs give a representation of the elevation of the path on which each of them cycles. Find the grade of each road.



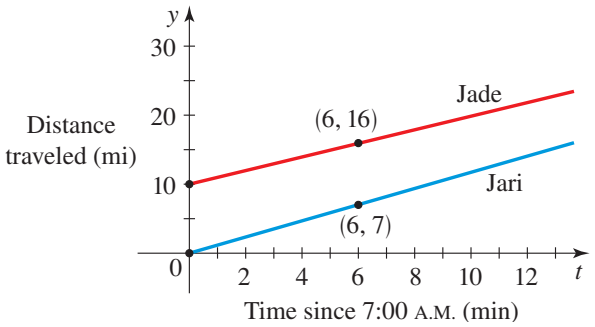
58. Wheelchair Ramp A local diner must build a wheelchair ramp to provide handicap access to their restaurant. Federal building codes require that a wheelchair ramp must have a maximum rise of 1 inch for every horizontal distance of 12 inches. What is the maximum allowable slope for a wheelchair ramp?

59. Commute to Work Jade and her roommate Jari live in a suburb of San Antonio, Texas, and both work at an elementary school in the city. Each morning they commute to work traveling west on I-10. One morning Jade left for work at 6:50 A.M., but Jari left 10 minutes later. Both drove at a constant speed. The following graphs show the distance (in miles) each of them has traveled on I-10 at time t (in minutes), where $t = 0$ is 7:00 A.M.

(a) Use the graph to decide which of them is traveling faster.

(b) Find the speed at which each of them is driving.

(c) Find linear functions that model the distances that Jade and Jari travel as functions of t.





© Charles O. Cecil/Alamy

Examining core samples

60. Sedimentation Rate Geologists date geologic events by examining sedimentary layers in the ocean floors. For instance, they can tell when an ancient volcano erupted by noting where in the sedimentary layers its ash was deposited. This is possible because the rate of sedimentation is assumed to be largely constant over geologic eras, so the depth of the sedimentation is modeled by a linear function of time. To examine sedimentary strata, core samples (often many meters long) are drilled from the ocean floor, then brought to the laboratory for study. Special features in the layers, including the chemical composition of the deposits, can indicate climate change or other geologic events. It is estimated that the rate of sedimentation in Devil's Lake, South Dakota, is 0.24 cm per year. The depth beneath the lake bottom at which a layer of sediment lies is a function of the time elapsed since it was deposited.

(a) Find a linear function that models the depth of sediment as a function of time elapsed.

(b) To what depth must we drill to reach sediment that was deposited 300 years ago?

2.3 Equations of Lines: Making Linear Models

- **Slope-Intercept Form**
- **Point-Slope Form**
- **Horizontal and Vertical Lines**
- **When Is the Graph of an Equation a Line?**

IN THIS SECTION... we find different ways of representing the equation of a line. These help us to construct linear models of real-life situations in which we know "any two points" or "a point and the rate of change" of a linear process.

GET READY... by reviewing Section 1.3, particularly how linear models are constructed using the "rate of change" and "initial value" of a linear process.

In Section 2.2 we learned that the graph of a function with constant rate of change is a line. We also saw that linear functions are useful in modeling many real-world situations. To find a linear model, we needed to know the "initial value" and the "rate of change" (or "y-intercept" and "slope"). But often this particular information is not available: We might know the rate of change and a particular value but not the initial value, or we might simply know two values of the linear function.

For example, one way to find the speed of a moving car is to use radar to find the car's distance at two different times. The two observations can be used to find a linear function that models the motion of the car (see Example 2).

Our goal in this section is to find different forms for the equation of a line that will help us in constructing linear models in different real-world situations.

Slope-Intercept Form

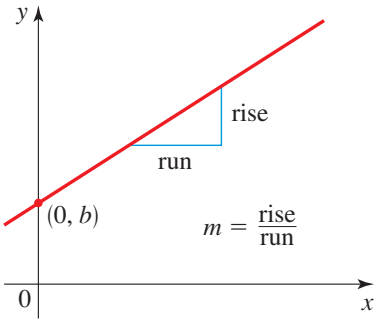


figure **1** A line with slope m and y-intercept b

In this section we express linear functions as equations. Recall from Section 1.5 that a function such as $f(x) = 5 + 3x$ can be expressed in equation form as $y = 5 + 3x$, where y is the dependent variable and x is the independent variable. In general, a **linear equation in two variables** is an equation that can be put into the form $y = b + mx$. We recognize this equation from Section 2.2 as the equation of a line (that is, an equation whose graph is a line) with slope m and y-intercept b. See Figure 1. This form of the equation of a line is called the *slope-intercept form*.

Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y-intercept b is

$$y = b + mx$$

(The slope-intercept form can also be expressed as $y = mx + b$.)

example **1** **Lines in Slope-Intercept Form**

(a) Find an equation of the line with slope 3 and y-intercept -2.

(b) Put the equation $3y - 2x = 1$ in slope-intercept form.

Solution

(a) Using the slope-intercept form with $m = 3$ and $b = -2$, we get

$y = b + mx$ Slope-intercept form

$y = -2 + 3x$ Replace m by 3 and b by -2

So the slope-intercept form of the equation of the line is $y = -2 + 3x$.

(b) To put the equation into slope-intercept form, we solve for y.

$3y - 2x = 1$ Given equation

$3y = 2x + 1$ Add $2x$

$y = \frac{2}{3}x + \frac{1}{3}$ Divide by 3

So the slope-intercept form of the given equation is $y = \frac{1}{3} + \frac{2}{3}x$.

■ **NOW TRY EXERCISES 11 AND 17** ■

example **2** **Constructing a Linear Model for Radar**

A police officer uses radar to record his distance from a car traveling on a straight stretch of road at two different times. The first measurement indicates that the car is



Marty Bucella/www.CartoonStock.com

350 feet from the officer; half a second later the car is 306 feet from the officer. Assume that the car is traveling at a constant speed.

(a) Find a linear equation that models the distance y of the car from the police officer at any time x.

(b) What is the speed of the car?

Solution

(a) Let's choose the time of the first measurement to be time 0, that is, $x = 0$. So the next measurement is at time $x = 0.5$. Then from the given data we get the following two points:

$$(0, 350) \quad \text{and} \quad (0.5, 306)$$

The first of these two points tells us that the y-intercept is 350. Using the two points together, we get the slope:

$$m = \frac{306 \text{ ft} - 350 \text{ ft}}{0.5 \text{ s} - 0 \text{ s}} = \frac{-44 \text{ ft}}{0.5 \text{ s}} = -88 \text{ ft/s}$$

So the linear equation that models the distance y that the car travels in x seconds is

$$y = 350 - 88x$$

(b) The slope of the line we found in part (a) is the same as the *rate of change* of distance with respect to time; that is, the slope is the speed of the car. So the speed of the car is

$$\text{speed} = \text{slope} = -88 \text{ ft/s}$$

The negative sign indicates that the distance between the car and the police officer is decreasing (because the car is heading in the direction of the officer). For the purposes of giving the driver a ticket, the speed of the car is 88 feet per second.

NOW TRY EXERCISE 57

We can convert the answer to part (b) of Example 2 into the more familiar miles per hour. There are 5280 feet in a mile and 3600 seconds in an hour. So

$$88\frac{\text{feet}}{\text{seconds}} = 88\frac{\cancel{\text{feet}}}{\cancel{\text{seconds}}} \times 3600\frac{\cancel{\text{seconds}}}{\text{hour}} \times \frac{1}{5280}\frac{\text{miles}}{\cancel{\text{feet}}}$$

$$= 88 \times \frac{3600}{5280}\frac{\text{miles}}{\text{hour}}$$

$$= 60 \text{ mi/h}$$

So the car is traveling at a speed of 60 miles per hour.

Point-Slope Form

We now find the equation of a line if we know the slope of the line and any point on the line (not necessarily the y-intercept). So suppose a line has slope m and

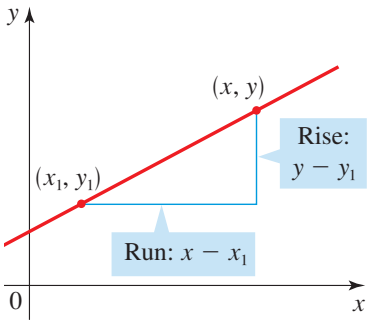


figure **2**

the point (x_1, y_1) is on the line (see Figure 2). Any other point (x, y) lies on the line if and only if the slope of the line through (x_1, y_1) and (x, y) is equal to m. This means that

$$\frac{\text{rise}}{\text{run}} = \frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$. This is the *point-slope form* for the equation of a line.

Point-Slope Form of the Equation of a Line

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

example **3**

Finding the Equation of a Line When We Know a Point and the Slope

(a) Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$.

(b) Sketch a graph of the line.

Solution

(a) We know the slope m is $-\frac{1}{2}$, and a point (x_1, y_1) on the line is $(1, -3)$. So we use the point-slope form with m replaced by $-\frac{1}{2}$, x_1 by 1, and y_1 by -3:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - (-3) &= -\tfrac{1}{2}(x - 1) && \text{Replace } m \text{ by } -\tfrac{1}{2},\ x_1 \text{ by } 1, \text{ and } y_1 \text{ by } -3 \\ y + 3 &= -\tfrac{1}{2}x + \tfrac{1}{2} && \text{Distributive Property} \\ y &= -\tfrac{1}{2}x - \tfrac{5}{2} && \text{Subtract } 3 \end{aligned}$$

So an equation for the line is $y = -\frac{5}{2} - \frac{1}{2}x$.

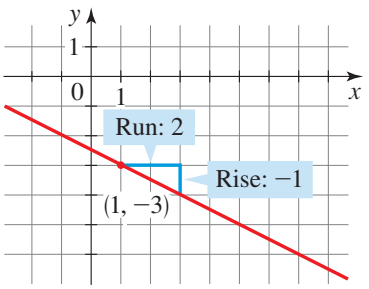


figure **3**

b) The fact that the slope is $-\frac{1}{2}$ tells us that when the run is 2, the rise is -1, so when we move 2 units to the right, the line *drops* by 1 unit. This enables us to sketch the line as in Figure 3.

■ **NOW TRY EXERCISE 23** ■

In the next example we construct a linear model for a real-world situation in which we don't know the initial value. Compare it with Example 3 in Section 2.2, in which we were given the initial value.

example **4**

Constructing a Linear Model for Volume

Water is being pumped into a swimming pool at the rate of 5 gal/min. After 20 minutes the pool has 300 gallons of water. Find a linear equation that models the volume of water in the pool at any time.

Solution

We need to find a linear equation

$$y = b + mx$$

that models the volume y of water in the pool at time x. The rate of change of volume is 5 gallons per minute, so $m = 5$. Since the pool has 300 gallons after 20 minutes, the point (20, 300) is on the desired line. So we have the following information

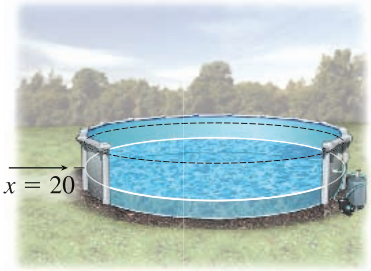
Point: $(x_1, y_1) = (20, 300)$

Slope: $m = 5$

Now, using the point-slope form for the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 300 &= 5(x - 20) && \text{Replace } m \text{ by } 5,\ x_1 \text{ by } 20, \text{ and } y_1 \text{ by } 300 \\ y - 300 &= 5x - 100 && \text{Distributive Property} \\ y &= 200 + 5x && \text{Add } 300 \end{aligned}$$

So the linear equation that models the volume of water is $y = 200 + 5x$.



There are 300 gallons of water in the pool at time $x = 20$.

NOW TRY EXERCISE 61

In Examples 3 and 4 we were given one point on a line and the slope. If instead we are given two points on the line, we can still use the point-slope form to find an equation, because we can first use the two given points to find the slope.

example **5** ## Finding an Equation for a Line Through Two Given Points

Find an equation of the line passing through the points $(-1, 2)$ and $(3, -4)$.

Solution

We first use the two given points to find the slope:

$$m = \frac{-4 - 2}{3 - (-1)} = \frac{-6}{4} = -\frac{3}{2}$$

We now have the following information about the desired line:

Point: $(x_1, y_1) = (-1, 2)$

Slope: $m = -\frac{3}{2}$

Using the point-slope form for the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 2 &= -\tfrac{3}{2}(x - (-1)) && \text{Replace } m \text{ by } -\tfrac{3}{2},\ x_1 \text{ by } -1,\ y_1 \text{ by } 2 \\ y - 2 &= -\tfrac{3}{2}(x + 1) && \text{Simplify} \\ y - 2 &= -\tfrac{3}{2}x - \tfrac{3}{2} && \text{Distributive Property} \\ y &= \tfrac{1}{2} - \tfrac{3}{2}x && \text{Add } 2 \end{aligned}$$

So the equation of the desired line is $y = \frac{1}{2} - \frac{3}{2}x$.

We can use either point, $(-1, 2)$ or $(3, -4)$, in the point-slope equation. We end up with the same linear equation. Try it!

NOW TRY EXERCISE 31

example 6 Constructing a Linear Model for Demand

A vending machine operator services the soda pop machines in a freeway rest area. He finds that the number of cans of soda that he sells each week depends linearly on the price that he charges. At a price of $1.00 per can of soda he sells 600 cans, but for every 25-cent increase in the price, he sells 75 fewer cans each week.

(a) Model the number of cans y that he sells each week at price x by a linear equation (or line), and sketch a graph of this equation.

(b) How many cans of soda will he sell if he charges 60 cents per can? If he charges $2.00 per can?

(c) Find the slope of the line. What does the slope represent?

(d) Find the y-intercept of the line. What does it represent?

(e) Find the x-intercept of the line. What does it represent?

Solution

(a) Since the number of cans y that the vendor sells depends linearly on the price x, the model we want is a linear equation:

$$y = b + mx$$

We already know two points on this line:

$$(1, 600) \quad \text{and} \quad (1.25, 525)$$

This is because when the price is $1.00, the number of cans sold is 600, and when the price is $1.25, the number of cans sold is 525. From these two points we obtain the slope of the line:

$$m = \frac{525 - 600}{1.25 - 1} = -300$$

Notice that the rate of change of y (the number of cans sold) is negative. This reflects the fact that as the price x increases, the number of cans sold *decreases*.

So we have the following information about the desired line:

Point: $(x_1, y_1) = (1, 600)$

Slope: $m = -300$

Now, using the point-slope form for the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Point-slope form} \\ y - 600 &= -300(x - 1) && \text{Replace } m \text{ by } -300,\ x_1 \text{ by } 1,\ y_1 \text{ by } 600 \\ y - 600 &= -300x + 300 && \text{Distributive Property} \\ y &= -300x + 900 && \text{Add } 600 \end{aligned}$$

So the desired equation is $y = 900 - 300x$. A graph of this equation is shown in Figure 4.

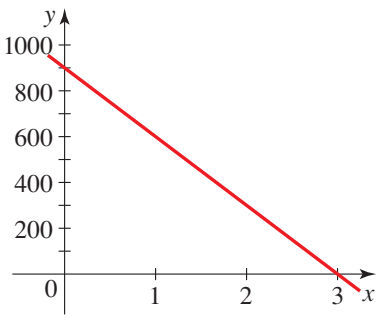


figure 4 Graph of the equation $y = 900 - 300x$

(b) At a price of 60 cents he sells $y = 900 - 300(0.60) = 720$ cans. At a price of $2.00 he sells $y = 900 - 300(2.00) = 300$ cans.

(c) The slope is -300. This means that sales drop by 300 cans for every $1.00 increase in price.

(d) The y-intercept is 900. This represents the theoretical number of cans he would "sell" at a price of $x = 0$, that is, if he were giving the soda away for free.

(e) To find the x-intercept, set $y = 0$ and solve for x.

$$0 = -300x + 900 \qquad \text{Replace } y \text{ by } 0$$

$$300x = 900 \qquad \text{Add } 300x \text{ to each side}$$

$$x = 3 \qquad \text{Divide by } 300$$

So the x-intercept is 3. This means that if he charges \$3.00 per can, he will sell no soda at all.

NOW TRY EXERCISE 63

Economists call equations like those in Example 4 *demand equations* because they express the demand for a product (the number of units sold) in terms of the price. With the aid of such equations economists can estimate the optimal price a vendor should charge to get the maximum profit from sales (see Section 2.7).

Horizontal and Vertical Lines

If a line is horizontal, then it neither rises nor falls, which means that between any two points, the rise is 0 and hence the slope m is also 0. So from the slope-intercept form of a line we see that a horizontal line has equation $y = b$. A vertical line does not have a slope, because between any two points the run is zero, and division by 0 is impossible. Nevertheless, we can write the equation of a vertical line as $x = a$, where a is its x-intercept, because the x-coordinate of every point on the line is a. (See Figure 5.)

Horizontal and Vertical Lines

- An equation of the vertical line through (a, b) is $x = a$.
- An equation of the horizontal line through (a, b) is $y = b$.

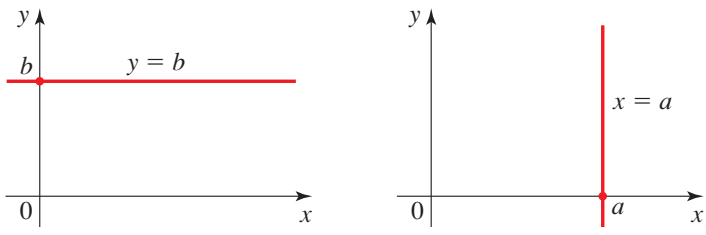


figure **5**

example 7 Horizontal and Vertical Lines

(a) The graph of the equation $x = 3$ is a vertical line with x-intercept 3.

(b) The graph of the equation $y = -2$ is a horizontal line with y-intercept -2.

The lines are graphed in Figure 6 on the next page.

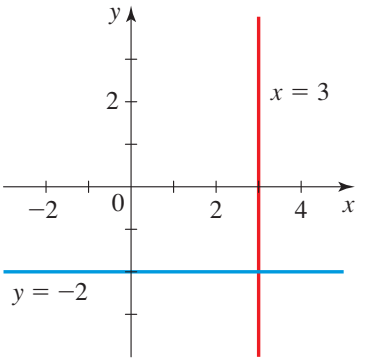


figure **6**

NOW TRY EXERCISES 41 AND 45

When Is the Graph of an Equation a Line?

Every line in the coordinate plane is either vertical, horizontal, or slanted at a nonzero slope m. Thus every line has an equation of the form $x = a$, $y = b$, or $y = b + mx$. All of these equations can be put into the form $Ax + By + C = 0$, so this is called the *general form of the equation of a line*.

General Form of the Equation of a Line

The graph of every general linear equation

$$Ax + By + C = 0$$

is a line (where not *both* A and B are 0). Conversely, every line has an equation of this form.

From the general form of a linear equation we can readily find the x- and y-intercepts and then use them to graph the equation, as illustrated in the next example.

example 8 Graphing a General Linear Equation

For the linear equation $2x - 3y - 12 = 0$,

(a) Find the slope of the line.

(b) Find the x- and y-intercept.

(c) Use the intercepts to sketch a graph.

Solution

(a) To find the slope, we put the equation into slope-intercept form:

$$\begin{aligned} 2x - 3y - 12 &= 0 && \text{General form of the equation of the line} \\ -3y &= -2x + 12 && \text{Subtract } 2x - 12 \text{ from each side} \\ y &= \tfrac{2}{3}x - 4 && \text{Divide by } -3 \end{aligned}$$

Comparing this last equation with the slope-intercept form $y = b + mx$, we see that the slope m is $\frac{2}{3}$.

(b) To find the x-intercept, we replace y by 0 and solve for x.

$$2x - 3(0) - 12 = 0 \quad \text{Replace } y \text{ by } 0$$
$$2x = 12 \quad \text{Add 12 to each side}$$
$$x = 6 \quad \text{Divide by 2}$$

So the x-intercept is 6.

To find the y-intercept, we replace x by 0 and solve for y.

$$2(0) - 3y - 12 = 0 \quad \text{Replace } x \text{ by } 0$$
$$-3y = 12 \quad \text{Add 12 to each side}$$
$$y = -4 \quad \text{Divide by } -3$$

So the y-intercept is -4.

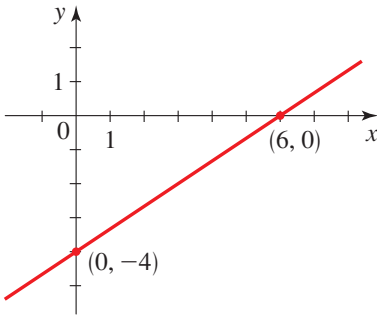


figure **7** Graph of $2x - 3y - 12 = 0$

(c) To graph a line, we need two points. It is convenient to use the two points at the intercepts: $(0, -4)$ and $(6, 0)$. We plot the intercepts and sketch the line that contains them in Figure 7.

■ **NOW TRY EXERCISE 53** ■

2.3 Exercises

CONCEPTS

Fundamentals

1. An equation of the line with slope m and y-intercept b is $y = \square + \square x$. So an equation of the line with slope 2 and y-intercept 4 is _____.

2. An equation of the line with slope m and passing through the point (x_1, y_1) is $y - \square = \square(x - \square)$. So an equation of the line with slope 3 passing through the point (1, 2) is _____.

3. **(a)** The slope of a horizontal line is _____ (zero/undefined). An equation of the horizontal line passing through (2, 3) is _____.

(b) The slope of a vertical line is _____ (zero/undefined). An equation of the vertical line passing through (2, 3) is _____.

4. To find an equation for the line passing through two points, we first find the slope determined by these two points, then use the _____ _____ form for the equation of a line. The line passing through the points (2, 5) and (3, 7) has slope _____, so the point-slope form of its equation is _____.

5. **(a)** The graph of the equation $y = 5$ is a _____ (horizontal/vertical) line. The slope of the line is _____ (zero/undefined).

(b) The graph of the equation $x = 5$ is a _____ (horizontal/vertical) line. The slope of the line is _____ (zero/undefined).

(c) An equation of the line passing through the point (0, 2) with slope 0 is _____.

6. The graph of the general linear equation $Ax + By + C = 0$ is a _____. The graph of the equation $3x + 5y - 15 = 0$ is a line with x-intercept _____ and y-intercept _____.

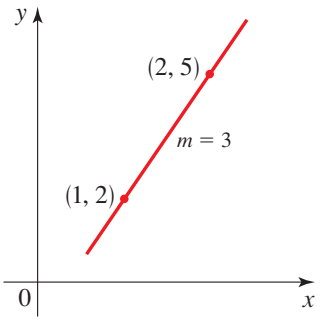
Think About It

7. Suppose that the graph of the outdoor temperature over a certain period of time is a line. How is the weather changing if the slope of the line is positive? If it is negative? If it is zero?

8. Find equations of two different lines with y-intercept 5. Find equations of two different lines with slope 5. Can two different lines with the same slope have the same y-intercept?

9. Suppose you want to find out whether three points in a coordinate plane lie on the same line. How can you do that using slopes? Can you think of another method?

10. Suppose that you know the slope and two points on a line. To find an equation for the line, it doesn't matter which of the two points we use in the point-slope form. Experiment with the line and points shown in the figure to the left by finding the point-slope equation of the line in two different ways: first using the point (1, 2), and then using the point (2, 5). Compare your answers by putting each of the equations you found into slope-intercept form. Do you get the same equation in each case?



SKILLS

11–16 ■ Find an equation of the line with the given slope and y-intercept.

11. Slope 5, y-intercept 2

12. Slope 2, y-intercept 7

13. Slope -1, y-intercept -3

14. Slope -5, y-intercept -1

15. Slope $\frac{1}{2}$, y-intercept 5

16. Slope $-\frac{1}{2}$, y-intercept $-\frac{1}{4}$

17–22 ■ Express the given equation in slope-intercept form.

17. $3x + y = 6$

18. $x + 4y = 10$

19. $9x - 3y - 4 = 0$

20. $2x - 8y + 5 = 0$

21. $4y + 5x = 10$

22. $4y + 8 = 0$

23–30 ■

(a) Find an equation of the line with the given slope that passes through the given point.
(b) Simplify the equation by putting it into slope-intercept form.
(c) Sketch a graph of the line.

23. Slope 2, through $(0, 4)$

24. Slope $\frac{3}{2}$, through $(0, -2)$

25. Slope $\frac{2}{3}$, through $(1, 7)$

26. Slope -3, through $(1, 6)$

27. Slope $-\frac{1}{3}$, through $(-6, 4)$

28. Slope $-\frac{3}{4}$, through $(-4, -3)$

29. Slope 0, through $(4, -5)$

30. Slope 0, through $(-1, 1)$

31–36 ■ Two points are given.

(a) Find an equation of the line that passes through the given points.
(b) Simplify the equation by putting it in slope-intercept form.
(c) Sketch a graph of the line.

31. $(-2, 1)$ and $(4, 7)$

32. $(-1, 6)$ and $(1, 2)$

33. $(-1, 7)$ and $(2, -2)$

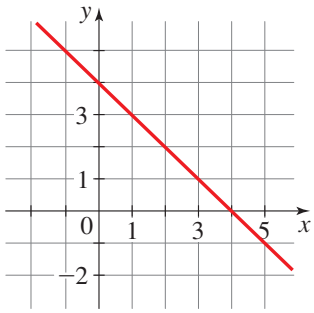
34. $(-1, -1)$ and $(3, 7)$

35. $(2, 3)$ and $(5, 7)$

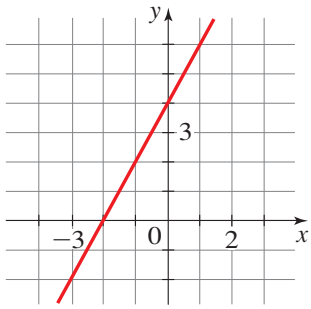
36. $(2, -1)$ and $(5, 3)$

37–40 ■ Find an equation in slope-intercept form for the line graphed in the figure.

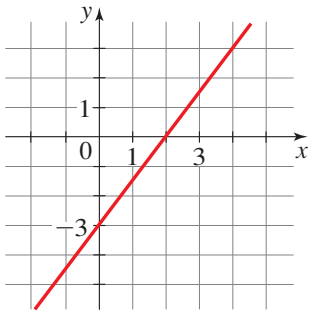
37.



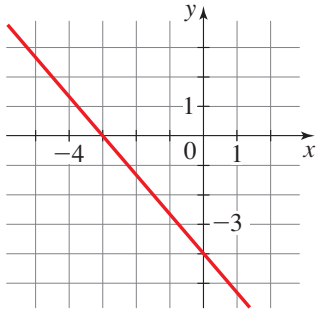
38.



39.



40.



41–44 ■ Find an equation of the horizontal line with the given property.

41. Has y-intercept -7

42. Has y-intercept 3

43. Passes through the point $(8, 10)$

44. Passes through the point $(-1, 2)$

45–48 ■ Find an equation of the vertical line with the given property.

45. Has x-intercept 8

46. Has x-intercept -3

47. Passes through the point $(8, 10)$

48. Passes through the point $(-1, 2)$

49–52 ■ An equation of a line is given. Find the slope and y-intercept of the line.

49. $3x + 4y = 24$

50. $5x = 20 + 2y$

51. $-6x = 15 - 5y$

52. $-3x - 7y = 42$

53–56 ■ A general linear equation is given.

(a) Find the slope of the line.

(b) Find the x- and y-intercepts.

(c) Use the intercepts to sketch the graph.

53. $3x + 5y - 15 = 0$

54. $2x - 7y + 14 = 0$

55. $4x - 5y - 10 = 0$

56. $-3x + 2y + 1 = 0$

CONTEXTS

57. Air Traffic Control Air traffic controllers at most airports use a radar system to identify the speed, position, and other information about approaching aircraft. Using radar, an air traffic controller identifies an approaching aircraft and determines that it is 45 miles from the radar tower. Five minutes later, she determines that the aircraft is 25 miles from the radar tower. Assume that the aircraft is approaching the radar tower directly at a constant speed.

(a) Find a linear equation that models the distance y of the aircraft from the radar tower x minutes after it was first observed.

(b) What is the speed of the approaching aircraft?

58. Depreciation A small business owner buys a truck for $25,000 to transport supplies for her business. She anticipates that she will use the truck for 5 years and that the truck will be worth $10,000 in 5 years. She plans to claim a depreciation tax credit using the *straight-line depreciation* method approved by the Internal Revenue Service. This means that if V is the value of the truck at time t, then a linear equation is used to relate V and t.

(a) Find a linear equation that models the depreciated value V of the truck t years since it was purchased.

(b) What is the rate of depreciation?

59. Depreciation A small business buys a laptop computer for $4000. After 4 years the value of the computer is expected to be $200. For accounting purposes the business uses straight-line depreciation (see Exercise 58) to assess the value of the computer at a given time.

(a) Find a linear equation that models the value V of the computer t years since its purchase.

(b) Sketch a graph of this linear equation.

(c) What do the slope and V-intercept of the graph represent?

(d) Find the depreciated value of the computer 3 years from the date of purchase.

60. Filling a Pond A large koi pond is filled with a garden hose at a rate of 10 gallons per minute. After 5 minutes the pond has 350 gallons of water. Find a linear equation that models the number of gallons y of water in the pond after t minutes.

61. Weather Balloon A weather balloon is filled with hydrogen at the rate of 0.5 ft^3/s. After 2 seconds the balloon contains 5 ft^3 of hydrogen. Find a linear equation that models the volume V of hydrogen in the balloon at any time t.

62. Manufacturing Cost The manager of a furniture factory finds that the cost of manufacturing chairs depends linearly on the number of chairs produced. It costs $2200 to make 100 chairs and $4800 to make 300 chairs.

(a) Find a linear equation that models the cost y of making x chairs, and sketch a graph of the equation.

(b) How much does it cost to make 75 chairs? 425 chairs?

(c) Find the slope of the line. What does the slope represent?

(d) Find the y-intercept of the line. What does it represent?

63. Demand for Bird Feeders A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. They sell 20 per week at a price of $10 each. They are considering raising the price, and they find that for every dollar increase, they lose two sales per week.

(a) Find a linear equation that models the number y of feeders that they sell each week at price x, and sketch a graph of the equation.

(b) How many feeders would they sell if they charged $14 per feeder? If they charged $6?

(c) Find the slope of the line. What does the slope represent?

(d) Find the y-intercept of the line. What does it represent?

(e) Find the x-intercept of the line. What does it represent?

64. Crickets and Temperature Biologists have observed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70°F and 168 chirps per minute at 80°F.

(a) Find a linear equation that models the temperature T when crickets are chirping at x chirps per minute. Sketch a graph of the equation.

(b) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

(c) Find the y-intercept of the line. What does it represent?

2.4 Varying the Coefficients: Direct Proportionality

- **Varying the Constant Coefficient: Parallel Lines**
- **Varying the Coefficient of *x*: Perpendicular Lines**
- **Modeling Direct Proportionality**

IN THIS SECTION... we explore how changing the coefficients b and m in the linear equation $y = b + mx$ affects the graph. We'll see that these coefficients tell us when two linear equations represent parallel or perpendicular lines. We'll also see that the number m plays a crucial role in modeling real-world quantities that are "directly proportional."

We know that the graph of a linear equation $y = b + mx$ is a line. How does varying the numbers b and m change this line? That is the question we seek to answer in this section.

■ Varying the Constant Coefficient: Parallel Lines

The **coefficients** of a linear equation $y = b + mx$ are the numbers b and m. The number b is called the **constant coefficient**, and m is called the **coefficient of x**.

We know that the constant coefficient b is the y-intercept of the line. So if we vary the constant b, we change only the y-intercept of the line; the slope remains the same. The next example illustrates the effect of varying the constant term.

example 1 Graphing a Family of Linear Equations

Sketch graphs of the family of linear equations $y = b + 2x$ for $b = -1, 0, 1, 2, 3, 4$. How are the graphs related?

Solution

We need to sketch graphs of the following six linear equations:

$$y = -1 + 2x \qquad y = 0 + 2x \qquad y = 1 + 2x$$
$$y = 2 + 2x \qquad y = 3 + 2x \qquad y = 4 + 2x$$

The graphs are shown in Figure 1 on page 178.

Each of these lines has the same slope (but not the same y-intercept).

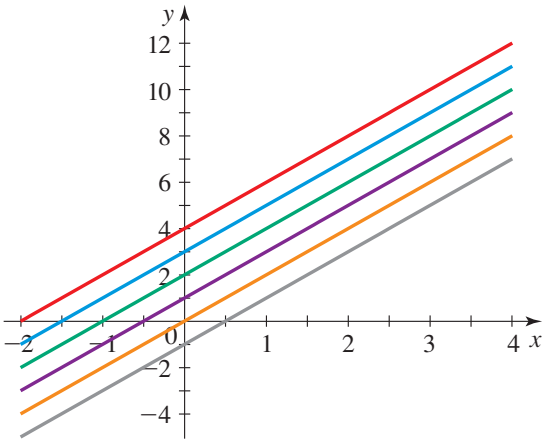


figure **1** Graph of the family of lines $y = b + 2x$, $b = -1, 0, 1, 2, 3$

NOW TRY EXERCISE 9

All the lines in Example 1 have the same slope. Since slope measures the steepness of a line, it seems reasonable that lines with the same slope are parallel. To prove this, let's consider any two parallel lines l_1 and l_2 with slopes m_1 and m_2, as shown in Figure 2. Since the triangles shown are similar, the ratios of their sides are equal:

$$m_1 = \frac{b_1}{a_1} = \frac{b_2}{a_2} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so corresponding angles are equal and the lines are parallel.

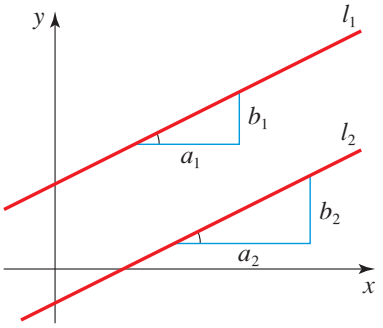


figure **2** Lines with the same slope are parallel.

Parallel Lines

Suppose that two nonvertical lines have slopes m_1 and m_2. The lines are parallel if and only if their slopes are the same:

$$m_1 = m_2$$

example **2** **Finding Equations of Parallel Lines**

Let l be the line with equation $y = 4 - 3x$.

(a) Find an equation for the line parallel to l and passing through the point $(2, 0)$.

(b) Sketch a graph of both lines.

Solution

(a) The line $y = 4 - 3x$ has slope -3. So the line we want also has slope -3. A point (x_1, y_1) on the line is $(2, 0)$. Using the point-slope form, we get

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 0 = -3(x - 2) \qquad \text{Replace } m \text{ by } -3,\ x_1 \text{ by } 2, \text{ and } y_1 \text{ by } 0$$

$$y = -3x + 6 \qquad \text{Distributive Property}$$

So an equation for the line is $y = 6 - 3x$.

(b) A graph of the lines is shown in Figure 3.

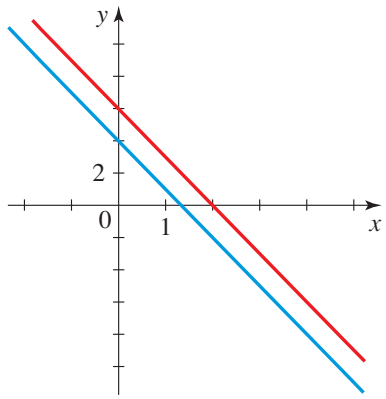


figure **3** Graphs of $y = 4 - 3x$ and $y = 6 - 3x$

NOW TRY EXERCISE 13

example **3**

Trains



Two trains are heading north along the California coast on the same track. The first train leaves Solana Beach 25 miles north of San Diego at 8:00 A.M.; the second train leaves San Clemente 58 miles north of San Diego at 9:00 A.M. Each train maintains a constant speed of 80 miles per hour.

(a) For each train, find a linear equation that relates its distance y from San Diego at time t.

(b) Sketch graphs of the linear equations you found in part (a).

(c) Will the trains collide?

Solution

(a) For each train we need to find an equation of the form

$$y = b + mt$$

Since each train travels at a speed of 80 mi/h, we have $m = 80$ for each train. Let's take time $t = 0$ to be 8:00 A.M., so 9:00 A.M. would be time $t = 1$. Let y be the distance north of San Diego.

- For the first train, $y = 25$ when $t = 0$, so the point $(0, 25)$ is on the desired line. Using the point-slope formula for the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(t - t_1) && \text{Point-slope form} \\ y - 25 &= 80(t - 0) && \text{Replace } m \text{ by } 80,\ t_1 \text{ by } 0, \text{ and } y_1 \text{ by } 25 \\ y &= 25 + 80t && \text{Add } 25 \end{aligned}$$

- For the second train, $y = 58$ when $t = 1$, so the point $(1, 58)$ is on the desired line. Using the point-slope formula for the equation of a line, we get

$$\begin{aligned} y - y_1 &= m(t - t_1) && \text{Point-slope form} \\ y - 58 &= 80(t - 1) && \text{Replace } m \text{ by } 80,\ t_1 \text{ by } 1, \text{ and } y_1 \text{ by } 58 \\ y - 58 &= 80t - 80 && \text{Distributive Property} \\ y &= -22 + 80t && \text{Add } 58 \end{aligned}$$

(b) The two equations are graphed in Figure 4. We know that the two lines are parallel because each has the same slope $m = 80$.

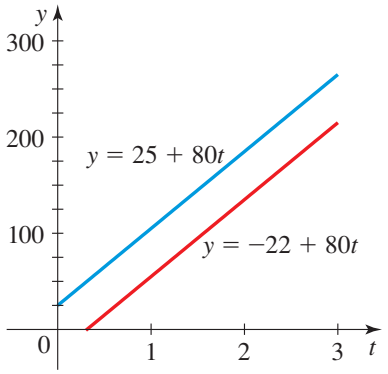


figure **4**

(c) Since the graphs in part (b) are parallel, they have no point in common. So the trains will never be at the *same place* at the *same time;* hence they won't collide.

NOW TRY EXERCISE 51

Varying the Coefficient of *x*: Perpendicular Lines

For a linear equation $y = b + mx$ we know that m, the coefficient of x, is the slope of the line. So if we vary m, we change how the line "leans." The next example illustrates this effect.

example 4 Graphing a Family of Linear Equations

Sketch graphs of the family of linear equations $y = 1 + mx$ for the following values of m.

(a) $m = 1, 2, 3$

(b) $m = -1, -2, -3$

(c) How are the graphs related?

Solution

(a) We need to sketch graphs of the following linear equations:

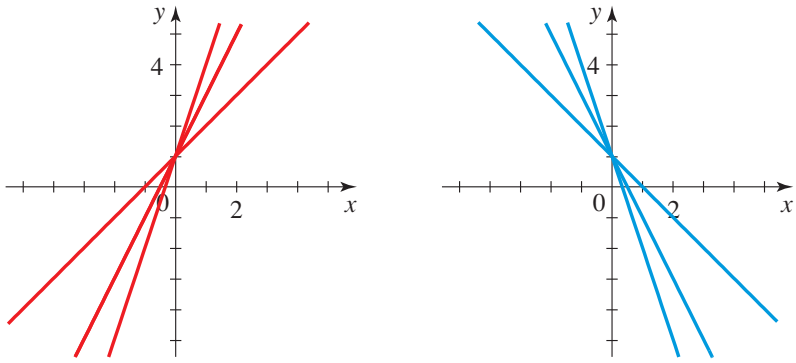
$$y = 1 + x \qquad y = 1 + 2x \qquad y = 1 + 3x$$

The graphs are shown in Figure 5(a).

(b) We need to sketch graphs of the following linear equations:

$$y = 1 - x \qquad y = 1 - 2x \qquad y = 1 - 3x$$

The graphs are shown in Figure 5(b).



(a) $m = 1, 2, 3$ (b) $m = -1, -2, -3$

figure **5** Graph of a family of lines $y = 1 + mx$

(c) All these lines have the same y-intercept 1 (but not the same slope). We see that the larger the slope, the more steeply the line rises as we move from left to right. When the slope is negative, the line falls as we move from left to right.

NOW TRY EXERCISE 11

From Example 4 we see that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. This observation is illustrated in Figure 6. The steepest lines are those for which the absolute value of the slope is the largest.

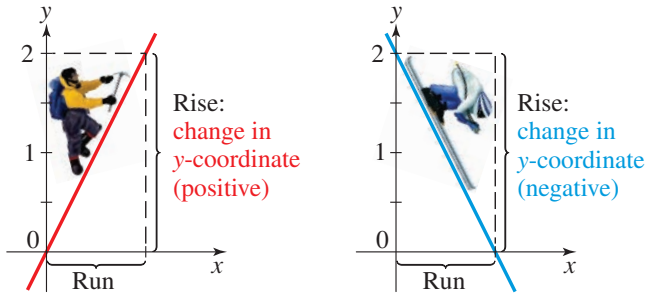


figure **6** If the rise is positive, the slope is positive. If the rise is negative, the slope is negative.

If two lines are perpendicular, how are their slopes related? Since perpendicular lines "lean" in different directions, we expect that their slopes would be quite different. In fact, if the slope of a line is positive, we can see that the slope of a perpendicular line should be negative. The exact relationship is as follows.

Perpendicular Lines

Suppose that two nonvertical lines have slopes m_1 and m_2. The lines are perpendicular if and only if their slopes are negative reciprocals:

$$m_1 = -\frac{1}{m_2}$$

Why do perpendicular lines have negative reciprocal slopes? To answer this question, let's consider two perpendicular lines l_1 and l_2 as shown in Figure 7.

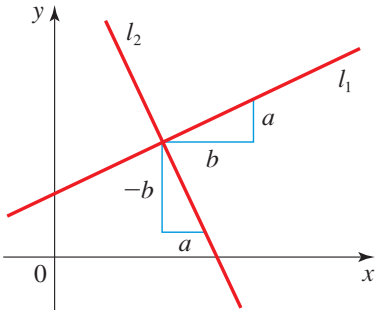


figure **7** Perpendicular lines

Since the lines are perpendicular, the two right triangles in the figure are congruent. From the figure we see that the slope of l_1 is a/b and the slope of l_2 is $-b/a$ (since the rise for l_2 is negative). This means that the slopes are negative reciprocals of each other.

example **5** **Finding Equations of Lines Parallel and Perpendicular to a Given Line**

Let l be the line with equation $y = 2x - 6$. Find equations for two lines that pass through the point (2, 0), one parallel to l and one perpendicular to l. Sketch a graph showing all three lines.

Solution

The line $y = 2x - 6$ has slope 2, so the parallel line also has slope 2, and the perpendicular line has slope $-\frac{1}{2}$, the negative reciprocal of 2. We use the point-slope form in each case to find the equation of the line we want.

- **Parallel line:** Using slope $m = 2$ and the point (2, 0), we get the equation

$$y - 0 = 2(x - 2) \quad \text{Point-slope form}$$
$$y = 2x - 4 \quad \text{Simplify}$$

- **Perpendicular line:** Using slope $m = -\frac{1}{2}$ and the point (2, 0), we get the equation

$$y - 0 = -\tfrac{1}{2}(x - 2) \quad \text{Point-slope form}$$
$$y = -\tfrac{1}{2}x + 1 \quad \text{Simplify}$$

All three lines are graphed in Figure 8.

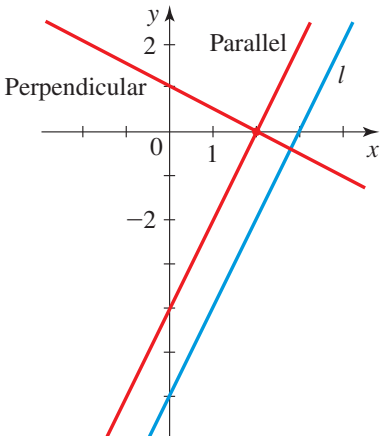


figure **8**

NOW TRY EXERCISE 19

Modeling Direct Proportionality

The electrical capacity of solar panels is directly proportional to the surface area of the panels. This means that if we double the area of the panels, we double the electrical capacity; if we triple the area of the panels, we triple the electrical capacity;

and so on (see Figure 9). This type of relationship between two variables is called *direct proportionality* and is described by a linear equation.

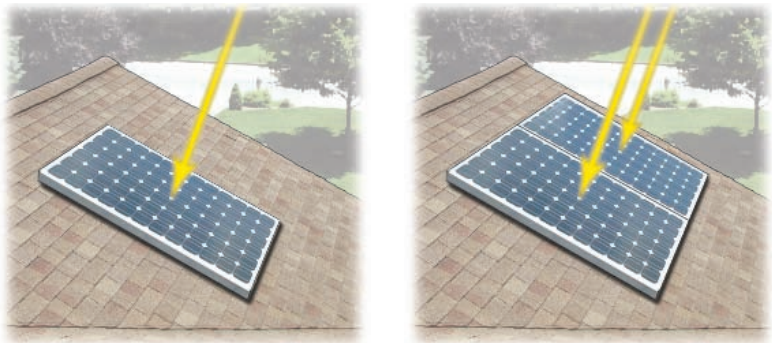


figure **9** Doubling the area of a solar panel doubles the electrical capacity

Direct Proportionality

We say that the variable y **is directly proportional to** the variable x (or y **varies directly as** x) if x and y are related by an equation of the form

$$y = kx$$

The constant k is called the **constant of proportionality**.

Notice that the equation that defines direct proportionality is a linear equation with y-intercept 0. The graph of this linear equation is a line passing through the origin with slope k.

example **6** **Direct Proportionality**

A solar electric company installs solar panels on the roofs of houses. A customer is informed that when 12 solar panels are installed, they produce 2.4 kilowatts of electricity.

(a) Find the equation of proportionality that relates the number of panels installed to the number of kilowatts produced.

(b) How many kilowatts of electricity are produced by 16 panels?

(c) Sketch a graph of the equation you found in part (a).

Solution

(a) The equation of proportionality is an equation of the form $y = kx$, where x represents the number of solar panels and y represents the number of kilowatts produced. To find the constant k of proportionality, we replace x by 12 and y by 2.4 in the equation.

$$y = kx \qquad \text{Equation of proportionality}$$

$$2.4 = k \cdot 12 \qquad \text{Replace } x \text{ by 12 and } y \text{ by 2.4}$$

$$k = \frac{2.4}{12} \qquad \text{Divide by 12, and switch sides}$$

$$k = 0.2 \qquad \text{Calculate}$$

Now that we know k is 0.2, we can write the equation of proportionality:

$$y = 0.2x$$

(b) To find how much electricity is produced by 16 panels, we replace x by 16 in the equation

$$y = 0.2x \qquad \text{Equation of proportionality}$$
$$y = 0.2 \cdot 16 \qquad \text{Replace } x \text{ by } 16$$
$$y = 3.2 \qquad \text{Calculate}$$

So 16 panels produce 3.2 kilowatts.

(c) A graph of the equation $y = 0.2x$ is shown in Figure 10. Notice that this is the equation of a line with slope 0.2 and y-intercept 0.

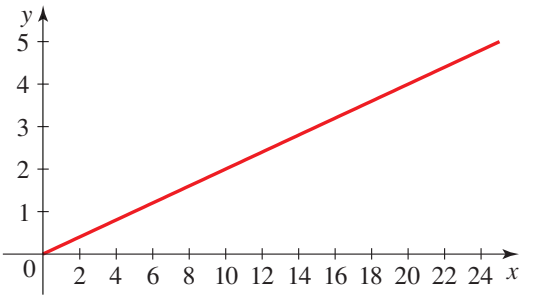


figure **10** Graph of $y = 0.2x$

NOW TRY EXERCISE 53

2.4 Exercises

CONCEPTS

Fundamentals

1. The graph of the line $y = 5 + 3x$ and the graph of the line $y = 6 + 3x$ have the same _____, so they are _____ lines.

2. The graph of the line $y = 4 + 2x$ and the graph of the line $y = 4 + 3x$ have the same _____. Are these two lines parallel, perpendicular, or neither?

3. A line has the equation $y = 3x + 2$.

(a) This line has slope _____.

(b) Any line parallel to this line has slope _____.

(c) Any line perpendicular to this line has slope _____.

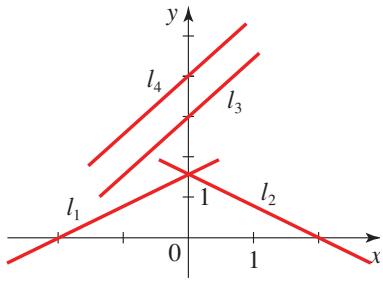
4. **(a)** If the quantities x and y are related by the equation $y = 3x$, then we say that y is _____ _____ to x and the constant of proportionality is _____.

(b) If y is directly proportional to x and if the constant of proportionality is 3, then x and y are related by the equation $y =$ ☐ x.

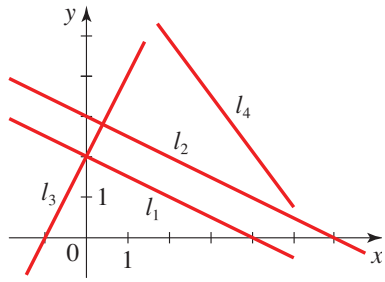
Think About It

5. Compare the slopes and y-intercepts of the lines l_1, l_2, l_3, and l_4. Which lines have the greatest slope? Which lines are parallel to each other? Which lines are perpendicular to each other?

(a)



(b)



6. Suppose you are given the coordinates of three points in the plane and you want to find out whether they form a right angle. How can you do this using slopes? Can you think of another method?

7. True or false?

(a) If electricity costs \$0.20 per kilowatt-hour, then the cost of electricity is directly proportional to the number of kilowatt-hours used.

(b) If a phone plan costs \$5.95 per month plus \$0.22 per minute, then the monthly phone cost is directly proportional to the number of minutes used.

(c) If the price of apples is \$0.79 per pound, then the cost of a bag of apples is directly proportional to the weight.

(d) If the price of tomatoes is \$2.99 per pound, then the cost of a box of tomatoes is directly proportional to the number of tomatoes in the box.

8. Use the following tables to determine which variable(s) (r, w, or t) could be directly proportional to x.

x	r
0	3.200
1	3.400
2	3.440
3	3.448
4	3.450

x	w
0	0.0
1	5.7
2	11.4
3	17.1
4	22.8

x	t
0	0
1	1/3
2	1/9
3	1/27
4	1/81

SKILLS

9–12 ■ Use a graphing device to graph the given family of linear equations in the same viewing rectangle. What do the lines have in common?

9. **(a)** $y = 2(x + b)$ for $b = 0, 1, 2, 3$

(b) $y = 2(x + b)$ for $b = 0, -1, -2, -3$

10. **(a)** $y = mx - 3$ for $m = 0, 0.25, 0.75, 1.5$

(b) $y = mx - 3$ for $m = 0, -0.25, -0.75, -1.5$

11. **(a)** $y = m(x - 3)$ for $m = 0, 0.25, 0.75, 1.5$

(b) $y = m(x - 3)$ for $m = 0, -0.25, -0.75, -1.5$

12. **(a)** $y = 2 + m(x + 3)$ for $m = 0, 0.5, 1, 2$

(b) $y = 2 + m(x + 3)$ for $m = 0, -0.5, -1, -2$

13–16 ■ An equation of a line l and the coordinates of a point P are given.

(a) Find an equation for the line that is parallel to l and passes through P.

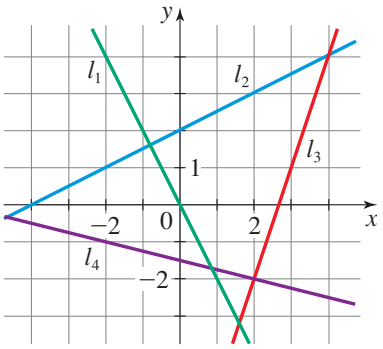
(b) Sketch a graph of both lines on the same coordinate axes.

13. $y = 3x - 4$; $(2, 4)$

14. $y - 3 = 5(x - 2)$; $(-1, 2)$

15. $2x + 3y + 6 = 0$; $(4, 0)$

16. $x - 2y = 3$; $(3, -2)$



17–18 ■ Graphs of lines l_1, l_2, l_3, and l_4 are shown in the figure in the margin. Are the given pairs of lines perpendicular?

17. l_1 and l_2

18. l_3 and l_4

19–22 ■ An equation of a line l and the coordinates of a point P are given.

(a) Find an equation for the line that is parallel to l and passes through P.

(b) Find an equation for the line that is perpendicular to l and passes through P.

(c) Sketch a graph of all three lines on the same coordinate axes.

19. $y = -\frac{1}{2}x + 1$; $(-1, 4)$

20. $y + 1 = 2(x + 3)$; $(-6, 12)$

21. $4x - 3y + 1 = 0$; $(0, 3)$

22. $5x + 4y = 2$; $(2, 1)$

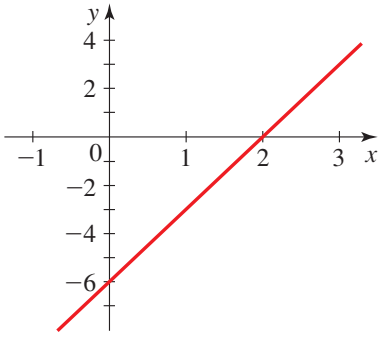
23–26 ■ The graph of a line l is sketched.

(a) Find an equation for l.

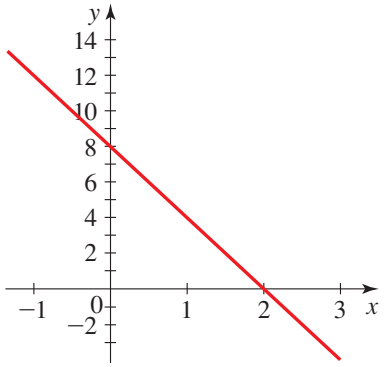
(b) Find an equation for the line parallel to l and with x-intercept 3.

(c) Find an equation for the line perpendicular to l and passing through the y-intercept of l.

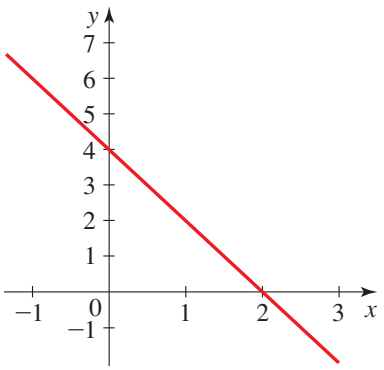
23.



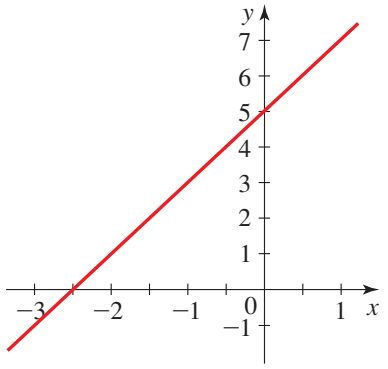
24.



25.



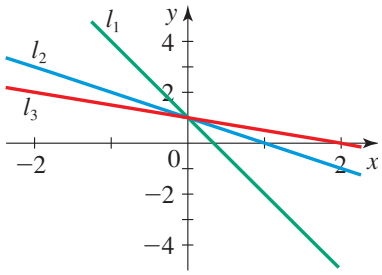
26.

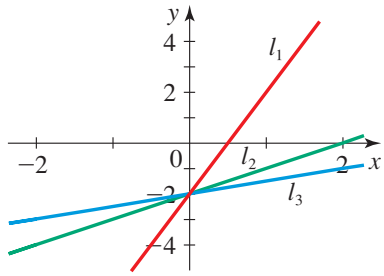


27–28 ■ Match the equation of the line with one of the lines l_1, l_2, or l_3 in the graph.

27. **(a)** $y = -\frac{1}{2}x + 1$
(b) $y = -x + 1$
(c) $y = -3x + 1$

28. **(a)** $y = 4x - 2$
(b) $y = \frac{1}{2}x - 2$
(c) $y = x - 2$





29–40 ■ Find an equation of the line that satisfies the given conditions.

29. Through $(-1, 2)$; parallel to the line $y = 4x + 7$

30. Through $(2, 3)$; parallel to the line $y = -2x - 5$

31. Through $(2, 2)$; parallel to the line $y = 5$

32. Through $(4, 5)$; parallel to the y-axis

33. Through $(3, -3)$; parallel to the line $x = -1$

34. Through $(4, 5)$; parallel to the x-axis

35. Through $(2, 6)$; perpendicular to the line $y = \frac{2}{3}x + 5$

36. Through $(-6, 6)$; perpendicular to the line $y = \frac{3}{4}x - 2$

37. Through $(5, -\frac{1}{5})$; perpendicular to the line $4y = 5x - 3$

38. Through $(\frac{1}{2}, -\frac{1}{4})$; perpendicular to the line $4x - 8y = 1$

39. Through $(1, 7)$; parallel to the line passing through $(2, 5)$ and $(-2, 1)$

40. Through $(-2, -11)$; perpendicular to the line passing through $(1, 1)$ and $(5, -1)$

41–42 ■ Write an equation that expresses the statement.

41. T is directly proportional to x.

42. P is directly proportional to w.

43–44 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

43. y is directly proportional to x. If $x = 6$, then $y = 42$.

44. z is directly proportional to t. If $t = 3$, then $z = 5$.

45–46 ■ Use the given information to solve the problem.

45. y is directly proportional to x, with constant of proportionality $k = 2.4$. Find y when $x = 5$.

46. z is directly proportional to t, with constant of proportionality $k = 0.25$. Find z when $t = 25$.

47–50 ■ Find the equation of proportionality that relates y to x.

47. The number of feet y in x miles. (One mile is equal to 5280 feet.)

48. The number of seconds y in x hours.

49. The number of fluid ounces y in x barrels of crude oil. (One barrel of crude oil contains 42 gallons of oil, and each gallon contains 128 fluid ounces.)

50. The number of ounces y in x tons. (One ton is equal to 2000 pounds and one pound is equal to 16 ounces.)

CONTEXTS

51. Kayaking Mauricio and Thanh are kayaking south down a river heading toward some rapids. Mauricio leaves 45 miles north of the rapids at 6:00 A.M., and Thanh leave 24 miles north of the rapids at 8:00 A.M. Both boys maintain a constant speed of 5 mi/h.

(a) For each boy, find a linear equation that relates their distance y from the rapids at time x. (Take time $x = 0$ to be 6:00 A.M., so 8:00 A.M. would be time $x = 2$.)

(b) Sketch a graph of the linear equations you found in part (a).

(c) Will Mauricio ever pass Thanh?

52. Catching Up Kathie and her friend Tia are on their motorcycles heading north to Springfield on the same straight highway. Kathie leaves from a point 120 miles south of Springfield at 10:00 A.M., and Tia leaves from a point 35 miles south of Springfield at 11:00 A.M. Both girls maintain a constant speed of 75 mi/h.

(a) For each girl, find a linear equation that relates her distance y from Springfield at time x. (Take time $x = 0$ to be 10:00 A.M., so 11:00 A.M. would be time $x = 1$.)

(b) Sketch a graph of the linear equations you found in part (a).

(c) Will Kathie ever pass Tia?

53. Crude Oil in Plastic A small amount of crude oil is used for manufacturing plastic. Scientists estimate that about 3 fluid ounces of crude oil is used to manufacture 1 million plastic bottles. So the number of fluid ounces of crude oil y used to manufacture plastic bottles is directly proportional to the number of bottles x that are manufactured.

(a) Find the equation of proportionality that relates y to x.

(b) In 2006, about 29 billion plastic bottles were used in the United States. Use the equation found in part (a) to determine the number of barrels of crude oil that were used to manufacture these plastic bottles. (Use the equation found in Exercise 49 to convert fluid ounces to barrels of oil.)

(c) Search the Internet to confirm your answer to part (b).



54. Power from Windmills A power company installs 100 wind turbines in a large field. When all the windmills are operating the electrical capacity is 55,000 kilowatts (kW) of electricity.

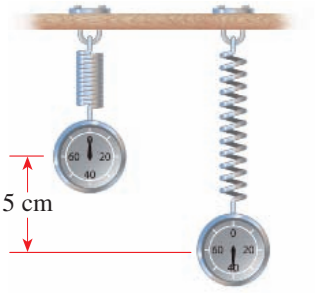
(a) Find the equation of proportionality that relates the number of operating windmills to the total electrical capacity (in kW).

(b) What is the electrical capacity (in kW) when 56 windmills are operating?

55. Interest The amount of interest i earned from a CD is directly proportional to the amount of money P invested in the CD. Hiam invests $1500 in a 12-month CD and earns $90.00 in interest at maturity. Find the equation of proportionality that relates i to P. What does the constant of proportionality represent?

56. Interest Perry invests in a high-yield money market account that has an APY of 6.17%. This means that when the effects of compounding are included, Perry's investment yields 6.17% each year. Find the equation of proportionality that relates the amount of interest i earned in one year to the amount of the investment P. If Perry invests $2500, what is the amount of interest that the investment earns after one year?

57. Hooke's Law Hooke's Law states that the force F needed to keep a spring stretched x units beyond its natural length is directly proportional to x. The constant of proportionality is called the **spring constant**.



(a) Write Hooke's Law as an equation.

(b) If a spring has a natural length of 10 cm and a force of 40 N is required to maintain the spring stretched to a length of 15 cm, find the spring constant.

(c) What force is needed to keep the spring stretched to a length of 14 cm?

58. Weighing Fish An Alaskan fisherman uses a spring scale to weigh the fish he catches. He observes that a salmon weighing 10 pounds stretches the spring of the scale 1.5 inches beyond its natural length.

(a) Find the spring constant for the scale.

(b) If a halibut stretches the scale by 4 in. what is its weight?

R **59. Calorie Calculator** The number of calories a person's body burns is directly proportional to the amount of energy the body uses. Search the Internet to find a site that calculates the number of calories burned for a given level of activity. Choose a weight and activity, and then calculate the projected number of calories burned for one hour spent on that activity. Use the calculation to find the equation of proportionality that relates the number of calories burned y to the duration of time x spent performing the chosen activity for the chosen weight.

R **60. Blood Alcohol Level** The blood alcohol level in your body is directly proportional to the number of alcoholic drinks you have had. Search the Internet to find a site that calculates blood alcohol level (there are many of them). Choose a weight, type of drink, and time elapsed since drinking. Then calculate the projected blood alcohol level for one drink. Use the calculation to find the equation of proportionality that relates the blood alcohol level y to the number of alcoholic drinks x for your chosen parameters.

2.5 Linear Regression: Fitting Lines to Data

- The Line That Best Fits the Data
- Using the Line of Best Fit for Prediction
- How Good Is the Fit? The Correlation Coefficient

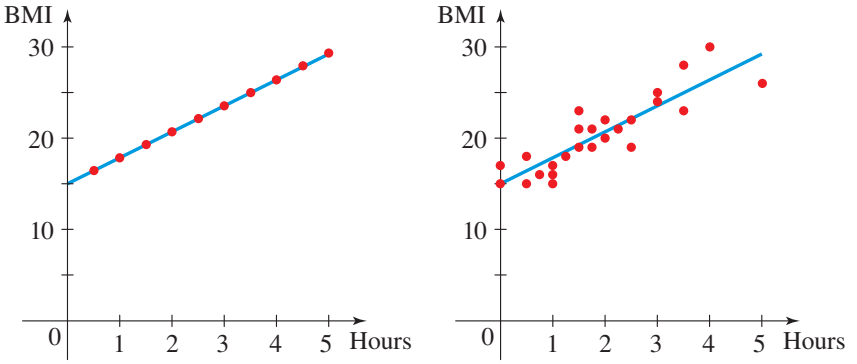
IN THIS SECTION... we model real-life data whose scatter plot doesn't lie exactly on a line but only appears to exhibit a "linear trend." We can model such data by a line that best fits the data—called the *line of best fit* or the *regression line.*

GET READY... by reviewing Section 1.6 on graphing with a graphing calculator. We need a graphing calculator or computer spreadsheet program for this section.



So far in this chapter we have used linear functions to model data whose scatter plots lie exactly on a line. But in most real-life situations data seldom fall into a precise line. Because of measurement errors or other random factors, a scatter plot of real-world data may appear to lie more or less on a line, but not exactly. For example, the scatter plot in Figure 1(b) on page 190 shows the results of a study on childhood obesity; the graph plots the body mass index (BMI) versus the number of hours of television watched per day for 25 adolescent subjects. Of course, we would not expect an exact relationship between these variables, as in Figure 1(a), but the scatter plot clearly indicates a linear trend: The more hours a subject spends watching TV, the

higher the BMI tends to be. So although we cannot fit a line exactly through the data points, a line like the one in Figure 1(b) shows the general trend of the data.



(a) Line fits data exactly (b) Line of best fit

figure **1**

Fitting lines to data is one of the most important tools available to researchers who need to analyze numerical data. In this section we learn how to find and use lines that best fit almost-linear data.

The Line That Best Fits the Data

table **1**
U.S. infant mortality

Year	Rate
1950	29.2
1960	26.0
1970	20.0
1980	12.6
1990	9.2
2000	6.9

Until recently, infant mortality in the United States was declining steadily. Table 1 gives the nationwide infant mortality rate for the period from 1950 to 2000; the rate is the number of infants who died before reaching their first birthday out of every 1000 live births. Over this half century the mortality rate was reduced by over 75%, a remarkable achievement in neonatal care.

The scatter plot in Figure 2 shows that the data lie roughly on a straight line. We can try to fit a line visually to approximate the data points, but since the data aren't exactly linear, there are many lines that might seem to work. Figure 3 shows two attempts at "eyeballing" a line to fit the data.

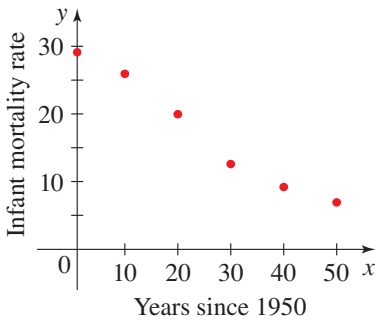


figure **2** U.S. infant mortality rate

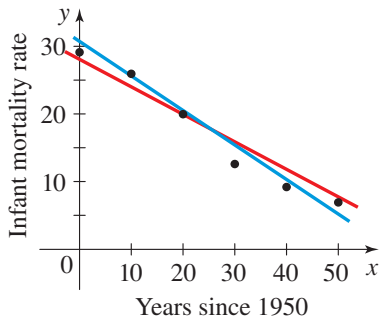


figure **3** Attempts to visually fit line to data

Of all the lines that run through these data points, there is one that "best" fits the data, in the sense that it provides the most accurate linear model for the data. We now describe how to find this line.

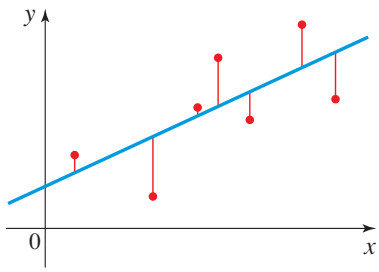


figure **4** The line of best fit

It seems reasonable that the line of best fit is the line that is as close as possible to all the data points. This is the line for which the sum of the vertical distances from the data points to the line is as small as possible (see Figure 4).

For technical reasons it is best to use the line for which the sum of the squares of these distances is smallest. This is called the *regression line*. The formula for the regression line is found by using calculus, but fortunately, the formula is programmed into most graphing calculators. In Example 1 we see that we can use a TI-83 calculator to find the regression line for the infant mortality data just described. (The process for other calculator models is similar.)

example **1**

Finding the Regression Line for the U.S. Infant Mortality Data

Find the regression line for the infant mortality data in Table 1. Graph the regression line on a scatter plot of the data.

Solution

To find the regression line using a TI-83 calculator, we must first enter the data into the lists L1 and L2, which are accessed by pressing the STAT key and selecting `Edit`. Figure 5 shows the calculator screen after the data have been entered. (Note that we are letting $x = 0$ correspond to the year 1950 so that $x = 50$ corresponds to 2000. This makes the equations easier to work with.)

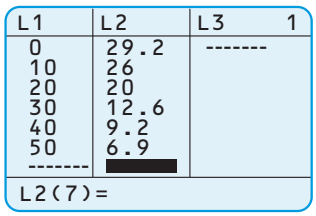
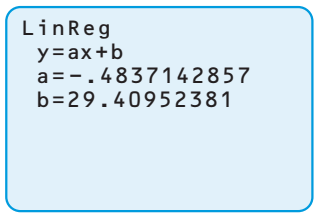


figure **5** Entering the data

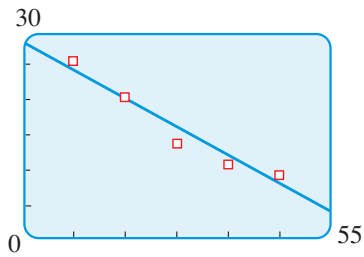
We then press the STAT key again and select `CALC`, then `4:LinReg(ax+b)`, which provides the output shown in Figure 6(a). This tells us that the regression line is

$$y = -0.48x + 29.4$$

Here, x represents the number of years since 1950, and y represents the corresponding infant mortality rate.



(a) Output of the `LinReg` command



(b) Scatter plot and regression line

figure **6**

The scatter plot and the regression line are plotted on a graphing calculator screen in Figure 6(b).

NOW TRY EXERCISE 13

Using the Line of Best Fit for Prediction

The main use of the regression line is to help us make predictions about new data points that are outside the domain of the collected data. In general, **extrapolation** means using a model to make predictions about data points that are *beyond* the

domain of the given data, and **interpolation** means using a model to make prediction about new data points that are *between* the given data points. In the next example we use the regression line of Example 1 to predict the infant mortality rate for a future year (extrapolation) and for a year between those given in the data points (interpolation).

example 2 Using the Regression Line for Prediction

Use the regression line from Example 1 to do the following:

(a) Estimate the infant mortality rate in 1995.

(b) Predict the infant mortality rate in 2006. Search the Internet for the actual rate for 2006. Compare it with your prediction.

Solution

(a) The year 1995 is 45 years after 1950, so substituting 45 for x, we find that

$$y = -0.48(45) + 29.4 = 7.8$$

So the infant mortality rate in 1995 was about 7.8.

(b) To predict the infant mortality rate in 2006, we replace x by 56 in the regression equation to get

$$y = -0.48(56) + 29.4 = 2.5$$

So we predict that the infant mortality rate in 2006 would be about 2.5.

■ **NOW TRY EXERCISE 15** ■

An Internet search shows that the actual infant mortality rate was 7.6 in 1995 and 6.4 in 2006. So the regression line is fairly accurate for 1995 (the actual rate was slightly lower than the predicted rate), but it is considerably off for 2006 (the actual rate was more than twice the predicted rate). The reason is that infant mortality in the United States stopped declining and actually started rising in 2002, for the first time in more than a century. This shows that we have to be very careful about extrapolating linear models outside the domain over which the data are spread.

IN CONTEXT ➤



AP Images

Tim Mack, 2004 Olympic gold medal winner

Since the modern Olympic Games began in 1896, achievements in track and field events have been improving steadily. One example in which the winning records have shown an upward linear trend is the pole vault. Pole vaulting began in the northern Netherlands as a practical activity; when traveling from village to village, people would vault across the many canals that crisscrossed the area to avoid having to go out of their way to find a bridge. Households maintained a supply of wooden poles of lengths appropriate for each member of the family. Pole vaulting for height rather than distance became a collegiate track and field event in the mid-1800s and was one of the events in the first modern Olympics. In the next example we find a linear model for the gold-medal-winning records in the men's Olympic pole vault.

example 3 Regression Line for Olympic Pole Vault Records

Table 2 gives the men's Olympic pole vault records up to 2004.

(a) Find the regression line for the data.

(b) Make a scatter plot of the data and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the slope of the regression line represent?

(d) Use the model to predict the winning pole vault height for the 2008 Olympics.

table 2
Men's Olympic pole vault records

Year	Gold medalist	Height (m)	Year	Gold medalist	Height (m)
1896	William Hoyt, USA	3.30	1956	Robert Richards, USA	4.56
1900	Irving Baxter, USA	3.30	1960	Don Bragg, USA	4.70
1904	Charles Dvorak, USA	3.50	1964	Fred Hansen, USA	5.10
1906	Fernand Gonder, France	3.50	1968	Bob Seagren, USA	5.40
1908	A. Gilbert, E. Cook, USA	3.71	1972	W. Nordwig, E. Germany	5.64
1912	Harry Babcock, USA	3.95	1976	T. Slusarski, Poland	5.64
1920	Frank Foss, USA	4.09	1980	W. Kozakiewicz, Poland	5.78
1924	Lee Barnes, USA	3.95	1984	Pierre Quinon, France	5.75
1928	Sabin Can, USA	4.20	1988	Sergei Bubka, USSR	5.90
1932	William Miller, USA	4.31	1992	M. Tarassob, Unified Team	5.87
1936	Earle Meadows, USA	4.35	1996	Jean Jaffione, France	5.92
1948	Guinn Smith, USA	4.30	2000	Nick Hysong, USA	5.90
1952	Robert Richards, USA	4.55	2004	Timothy Mack, USA	5.95

```
LinReg
 y=ax+b
 a=.0265652857
 b=3.400989881
```

Output of the `LinReg` function on the TI-83

Solution

(a) Let x = year − 1900 so that 1896 corresponds to $x = -4$, 1900 to $x = 0$, and so on. Using a calculator, we find the regression line:

$$y = 0.0266x + 3.40$$

(b) The scatter plot and the regression line are shown in Figure 7. The regression line appears to be a good model for the data.

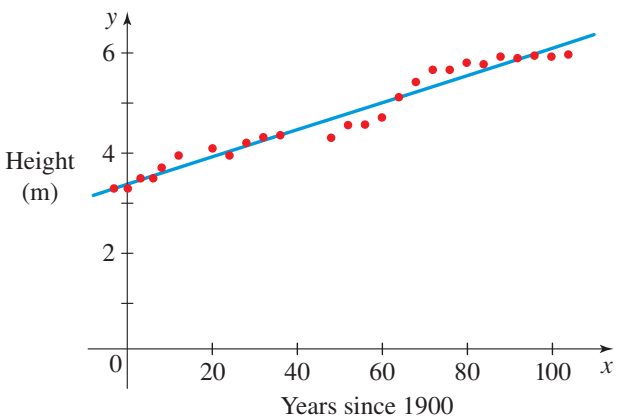


figure 7 Scatter plot and regression line for pole vault data

(c) In general, the slope is the average rate of change of the function. In this case the slope is the average rate of increase in the pole vault record per year. So on average, the pole vault record increased by 0.0266 meters per year.

(d) The year 2008 corresponds to $x = 108$ in our model. The model gives

$$y = 0.0266(108) + 3.40 \approx 6.27$$

So the model predicts that in 2008 the winning pole vault record will be 6.27 meters.

We can make a more accurate prediction by using only recent data. See Exercise 18.

NOW TRY EXERCISE 17

In the 2008 Olympic Games, Steven Hooker of Australia set an Olympic pole vault record of 5.96 meters. This is considerably less than the prediction of the model of Example 3. In Exercise 18 we explore how using only recent data can help to improve our prediction.

example 4 Regression Line for Links Between Asbestos and Cancer



Eric & David Hosking/Corbis

When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. Table 3 lists the results of several experiments by different scientists.

(a) Find the regression line for the data. (Let x be the asbestos exposure and y the percent of rats that develop tumors.)

(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the y-intercept of the regression line represent?

table 3
Asbestos-tumor data

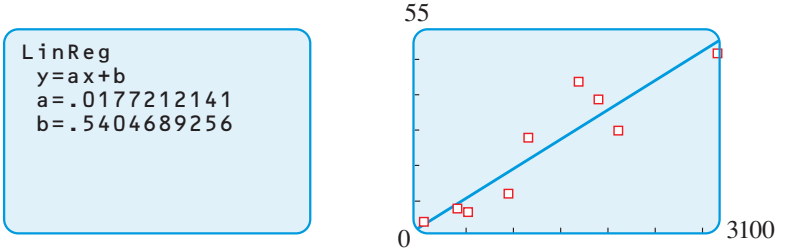
Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50

Solution

(a) Using a calculator, we find the regression line (see Figure 8(a)):

$$y = 0.0177x + 0.5405$$

(b) The scatter plot and regression line are shown in Figure 8(b). The regression line appears to be a reasonable model for the data.



(a) Output of the `LinReg` command

(b) Scatter plot and regression line

figure 8 Linear regression for the asbestos-tumor data

(c) The y-intercept is the percentage of mice that develop tumors when no asbestos fibers are present. In other words, this is the percentage that normally develop lung tumors (for reasons other than asbestos).

NOW TRY EXERCISE 19

How Good Is the Fit? The Correlation Coefficient

For any given set of two-variable data it is always possible to find a regression line, even if the data points don't tend to lie on a line and even if the variables don't seem to be related at all. Look at the three scatter plots in Figure 9. In the first scatter plot, the data points lie close to a line. In the second plot, there is still a linear trend, but the points are more scattered. In the third plot, there doesn't seem to be any trend at all, linear or otherwise.

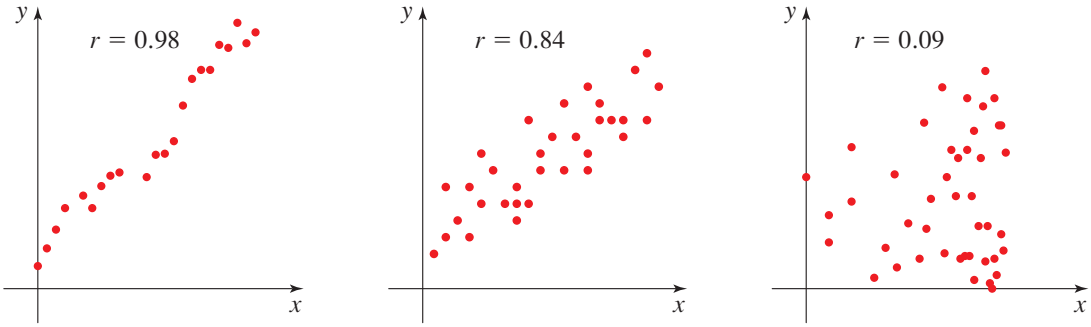


figure **9**

To ensure that your TI-83 calculator returns the value of r when you perform the `LinReg` command, open the `Catalog` menu and start `DiagnosticOn`.

A graphing calculator can give us a regression line for each of these scatter plots. But how well do these lines represent or "fit" the data? To answer this question, statisticians have invented the *correlation coefficient,* usually denoted r. The correlation coefficient is a number between -1 and 1 that measures how closely the data follow the regression line—or, in other words, how strongly the variables are *correlated*. Many graphing calculators give the value of r when they compute a regression line. If r is close to -1 or 1, then the variables are strongly correlated—that is, the scatter plot follows the regression line closely. If r is close to 0, then there is little correlation. (The sign of r depends on the slope of the regression line.) The correlation coefficients of the scatter plots in Figure 9 are indicated on the graphs. For the first plot, r is close to 1 because the data are very close to linear. The second plot also has a relatively large r, but not as large as the first, because the data, while fairly linear, are more diffuse. The third plot has an r close to 0, since there is virtually no linear trend in the data.

The correlation coefficient is a guide in helping us decide how closely data points lie along a line. In Example 1 the correlation coefficient is -0.99, indicating a very high level of correlation, so we can safely say that the drop in infant mortality rates from 1950 to 2000 was strongly linear. (The value of r is negative, since infant mortality *declined* over this period.) In Example 4 the correlation coefficient is 0.92, which also indicates a strong correlation between the variables. So exposure to asbestos is clearly associated with the growth of lung tumors in rats. Does this mean that asbestos *causes* lung cancer? See Exploration 4 for a closer look at this question.

2.5 Exercises

CONCEPTS

Fundamentals

1. For a set of data points, the line that best fits the scatter plot of the data is called the _____ line.

2. **(a)** Using a regression line to predict trends outside the domain of our existing data is called _____.

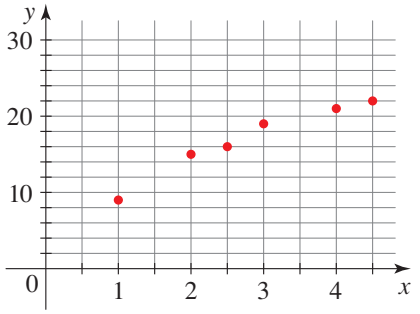
(b) Using a regression line to predict trends within the domain of our existing data is called _____.

Think About It

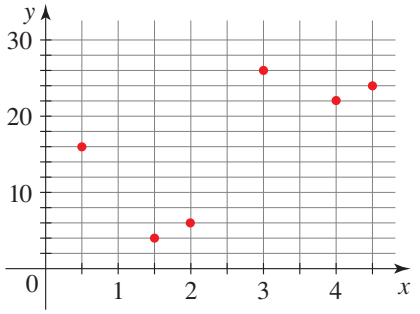
3. Can you find the regression line for any set of two-variable data, even if these data don't exhibit a linear trend?

4. Try to visually draw the line of best fit to the scatter plot shown, then find the regression line and compare to your "visual line." Which of your lines is closer to the regression line?

(a)



(b)



SKILLS

5–12 ■ A table of values is given.

(a) Make a scatter plot of the data.

(b) Find and graph the linear regression line that models the data.

5.

x	1	2	3	4	5	6
y	3.1	8.2	13.5	18.2	23.4	28.7

6.

x	10	20	30	40	50	60
y	17.2	27.0	36.8	45.1	57.1	67.2

7.

x	71	90	102	103	105	121
y	21.1	20.9	19.0	19.5	19.2	17.3

8.

x	2	3	7	8	9	11
y	5.7	9.2	21.3	2.2	13.1	12.2

9.

x	13	14	20	24	27	33
y	1.32	11.33	9.30	5.30	21.27	22.22

10.

x	21	29	35	42	57	103
y	51	35	47	90	26	11

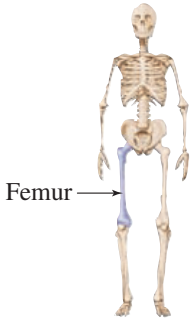
11.

x	1	5	7	11	15	27
y	3.9	11.4	16.6	24.1	31.6	56.8

12.

x	5	6	10	15	17	21
y	173	150	148	121	110	94

CONTEXTS



13. Height and Femur Length Anthropologists use a linear model that relates femur (thigh bone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analyzing the data on femur length and height for the eight males given in the table below.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) An anthropologist finds a femur of length 58 cm. How tall was the person?

Femur length (cm)	Height (cm)
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

14. Carbon Dioxide Levels The Mauna Loa Observatory, located on the island of Hawaii, has been monitoring carbon dioxide (CO_2) levels in the atmosphere since 1958. The following table lists the average annual CO_2 levels measured in parts per million (ppm) from 1984 to 2006.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) Use the linear model in part (b) to estimate the CO_2 level in the atmosphere in 2005.

(d) Search the Internet to find the actual reported CO_2 level in 2005, and compare this to your answer in part (c).

Year	CO_2 level (ppm)	Year	CO_2 level (ppm)
1984	344.3	1996	362.7
1986	347.0	1998	366.5
1988	351.3	2000	369.4
1990	354.0	2002	372.0
1992	356.3	2004	377.5
1994	358.9	2006	380.9

15. Extent of Arctic Sea Ice The National Snow and Ice Data Center monitors the amount of ice in the Arctic year round. The table on the next page gives approximate

values for the arctic ice extent in millions of square kilometers from 1980 to 2006, in 2-year intervals.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) Use the linear model in part (b) to estimate the arctic ice extent in 2001.

(d) Predict the arctic ice extent in 2008. Search the Internet to find the actual reported arctic ice extent in 2008, and compare this to your prediction.

Year	Ice extent (million km^2)	Year	Ice extent (million km^2)
1980	7.9	1994	7.1
1982	7.4	1996	7.9
1984	7.2	1998	6.6
1986	7.6	2000	6.3
1988	7.5	2002	6.0
1990	6.2	2004	6.1
1992	7.6	2006	5.7

16. Temperature and Chirping Crickets Biologists have observed that the chirping rate of crickets of a certain species appear to be related to temperature. The following table shows the chirping rate for various temperatures.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping rate (chirps/min)
50	20
55	46
60	79
65	91
70	113
75	140
80	173
85	198
90	211

17. Life Expectancy The average life expectancy in the United States has been rising steadily over the past few decades, as shown in the table.

Year	Life expectancy
1920	54.1
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	76.9

(a) Find the regression line for the data.

(b) Make a scatter plot of the data, and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the slope of the regression line represent?

(d) Use the linear model you found in part (b) to predict the life expectancy in the year 2006.

(e) Search the Internet to find the actual 2006 average life expectancy. Compare it to your answer in part (c).

18. Olympic Pole Vault The graph in Figure 7 (page 193) indicates that in recent years the winning Olympic men's pole vault height has fallen below the value predicted by the regression line in Example 3. This might have occurred because when the pole vault

Year	x	Height (m)
1972	0	5.64
1976	4	
1980	8	
1984		
1988		
1992		
1996		
2000		
2004		

was a new event, there was much room for improvement in vaulters' performances, whereas now even the best training can produce only incremental advances. Let's see whether concentrating on more recent results gives a better predictor of future records.

(a) Use the data in Table 2 (page 193) to complete the table of winning pole vault heights in the margin. (Note that we are using $x = 0$ to correspond to the year 1972, where this restricted data set begins.)

(b) Find the regression line for the data in part (a).

(c) Plot the data and the regression line on the same axes. Does the regression line seem to provide a good model for the data?

(d) What does the regression line predict as the winning pole vault height for the 2008 Olympics? Has this new regression line provided a better prediction than the line in Example 2? Compare to the actual result given on page 194.

19. Mosquito Prevalence The following table lists the relative abundance of mosquitoes (as measured by the "mosquito positive rate") versus the flow rate (measured as a percentage of maximum flow) of canal networks in Saga City, Japan.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) What does the y-intercept of the regression line represent?

(d) Use the linear model in part (b) to estimate the mosquito positive rate if the canal flow is 70% of maximum.

Flow rate (%)	Mosquito positive rate (%)
0	22
10	16
20	12
60	11
90	6
100	2

20. Noise and Intelligibility Audiologists study the intelligibility of spoken sentences under different noise levels. Intelligibility, the MRT score, is measured as the percentage of a spoken sentence that the listener can decipher at a certain noise level in decibels (dB). The following table shows the results of one such test.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Find the correlation coefficient. Is a linear model appropriate?

(d) Use the linear model in part (b) to estimate the intelligibility of a sentence at a 94-dB noise level.

Noise level (dB)	MRT score (%)
80	99
84	91
88	84
92	70
96	47
100	23
104	11

21. Olympic Swimming Records The tables give the gold medal times in the men's and women's 100-m freestyle Olympic swimming event.

(a) Find the regression lines for the men's data and the women's data.

(b) Sketch both regression lines on the same graph. When do these lines predict that the women will overtake the men in the event?

(c) Does this conclusion seem reasonable? Search the Internet for the 2008 Olympic results to check your answer.

MEN

Year	Gold medalist	Time (s)
1908	C. Daniels, USA	65.6
1912	D. Kahanamoku, USA	63.4
1920	D. Kahanamoku, USA	61.4
1924	J. Weissmuller, USA	59.0
1928	J. Weissmuller, USA	58.6
1932	Y. Miyazaki, Japan	58.2
1936	F. Csik, Hungary	57.6
1948	W. Ris, USA	57.3
1952	C. Scholes, USA	57.4
1956	J. Henricks, Australia	55.4
1960	J. Devitt, Australia	55.2
1964	D. Schollander, USA	53.4
1968	M. Wenden, Australia	52.2
1972	M. Spitz, USA	51.22
1976	J. Montgomery, USA	49.99
1980	J. Woithe, E. Germany	50.40
1984	R. Gaines, USA	49.80
1988	M. Biondi, USA	48.63
1992	A. Popov, Russia	49.02
1996	A. Popov, Russia	48.74
2000	P. van den Hoogenband, Netherlands	48.30
2004	P. van den Hoogenband, Netherlands	48.17

WOMEN

Year	Gold medalist	Time (s)
1912	F. Durack, Australia	82.2
1920	E. Bleibtrey, USA	73.6
1924	E. Lackie, USA	72.4
1928	A. Osipowich, USA	71.0
1932	H. Madison, USA	66.8
1936	H. Mastenbroek, Holland	65.9
1948	G. Andersen, Denmark	66.3
1952	K. Szoke, Hungary	66.8
1956	D. Fraser, Australia	62.0
1960	D. Fraser, Australia	61.2
1964	D. Fraser, Australia	59.5
1968	J. Henne, USA	60.0
1972	S. Nielson, USA	58.59
1976	K. Ender, E. Germany	55.65
1980	B. Krause, E. Germany	54.79
1984	(Tie) C. Steinseifer, USA	55.92
	N. Hogshead, USA	55.92
1988	K. Otto, E. Germany	54.93
1992	Z. Yong, China	54.64
1996	L. Jingyi, China	54.50
2000	I. DeBruijn, Netherlands	53.83
2004	J. Henry, Australia	53.84

22. Living Alone According to the latest U.S. Census Bureau survey, the number of Americans living alone now exceeds the number of households composed of the classic nuclear family: a married couple and their natural children. The table below shows a detailed account of this trend from 1970 to 2000.

(a) Find the regression line for the percentage of single households as related to the number of years since 1970.

(b) Find the regression line for the percentage of married households as related to the number of years since 1970.

(c) Graph the regression lines found in parts (a) and (b), along with a scatter plot of the data.

(d) Use the models found in parts (a) and (b) to predict the percentage of single households and percentage of married households in 2010. Do you think this is a reasonable prediction?

Year	Percentage of single households	Percentage of married households with children under age 18
1970	17.6	40.3
1980	22.7	30.9
1990	24.6	26.3
2000	25.5	24.1

Science and engineering doctorates

Year	All S&E doctorates	Women S&E doctorates
2002	24,609	9172
2003	25,282	9519
2004	26,275	9856
2005	27,989	10,539
2006	29,854	11,469

23. Women in Science and Engineering The number of women receiving doctoral degrees in science and engineering has been rising steadily. The table gives a detailed account of these doctoral degrees from 2002 to 2006.

(a) Find the regression line for the number of science and engineering doctorates as a function of the number of years since 2002.

(b) Find the regression line for the number of women receiving doctoral degrees in science and engineering as a function of the number of years since 2002.

(c) Graph the regression lines along with the scatter plots of the data.

R **(d)** Use the model found in part (b) to predict the number of women who receive doctorates in science and engineering in 2007. Search the Internet to find the actual number of women who received doctorates in science and engineering in 2007, and compare it to your answer.

2.6 Linear Equations: Getting Information from a Model

- Getting Information from a Linear Model
- Models That Lead to Linear Equations

IN THIS SECTION... we learn how to get information from a model about the situation being modeled. Getting this information requires us to solve one-variable linear equations.

GET READY... by reviewing how to solve linear equations in **Algebra Toolkit C.1**. Test your skill by doing the **Algebra Checkpoint** at the end of this section.

Getting Information from a Linear Model

In Example 4 in Section 2.3 we modeled the volume y of water in a swimming pool at time x by the *two-variable* equation

$$y = 200 + 5x \qquad \text{Two-variable equation}$$

If the pool has a capacity of 10,000 gallons, when will the pool be filled? To answer this question, we replace y by 10,000 in the equation. This gives us the *one-variable* equation

$$10{,}000 = 200 + 5x \qquad \text{One-variable equation}$$

This equation says that at time x the volume y of water in the pool is 10,000. So the value of x that makes the last equation true is the answer to our question. We'll solve this equation graphically and algebraically in Example 1.

example 1 Getting Information from a Linear Model

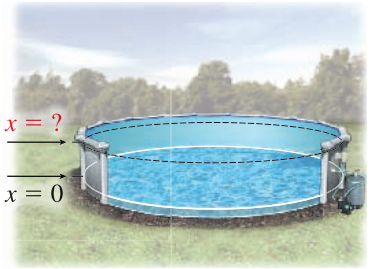
A swimming pool is being filled with water. The linear equation

$$y = 200 + 5x$$

models the volume y of water in the pool at time x (see Example 5, p. 157). If the pool has a capacity of 10,000 gallons, when will the pool be filled?

Solution 1 Graphical

A graph of $y = 200 + 5x$ is shown in Figure 1 on page 202. Since the pool is filled when the volume y is 10,000, we also graph the line $y = 10{,}000$. The pool is filled at



We want to find the time x when the pool is filled.

the time x where these two graphs intersect. From the figure we see the height of the graph of $y = 200 + 5x$ reaches 10,000 when x is about 1950. So the pool is filled in about 1950 minutes.

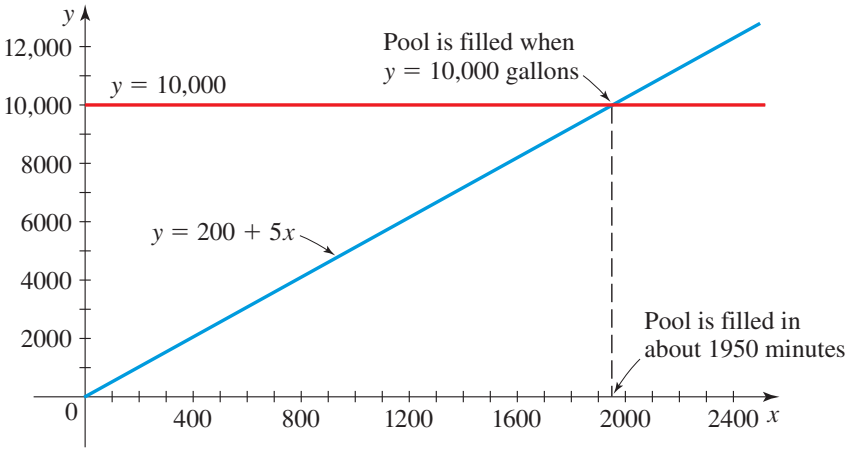


figure **1** Graphs of $y = 200 + 5x$ and $y = 10{,}000$

Solution 2 Algebraic

The pool is filled when the volume y is 10,000 gallons. Replacing y by 10,000 in the equation $y = 200 + 5x$ gives us a one-variable equation in the variable x. We solve this equation for x, the time when the pool is filled.

$$y = 200 + 5x \qquad \text{Equation}$$

$$10{,}000 = 200 + 5x \qquad \text{Replace } y \text{ by } 10{,}000$$

$$9800 = 5x \qquad \text{Subtract 200 from each side}$$

$$x = \frac{9800}{5} \qquad \text{Divide by 5, and switch sides}$$

$$x = 1960 \qquad \text{Calculator}$$

So 1960 is a solution to the equation. This means that it takes 1960 minutes to fill the pool. Notice that the algebraic solution gives us an exact answer, whereas the graphical solution is approximate.

NOW TRY EXERCISE 21

In Example 1 the solution is 1960 minutes. Such a large number of minutes is better understood if it is stated in hours:

$$1960 \text{ minutes} = \frac{1960 \text{ minutes}}{60 \dfrac{\text{minutes}}{\text{hour}}} = \frac{1960 \cancel{\text{minutes}}}{60} \frac{\text{hour}}{\cancel{\text{minutes}}} = 32\frac{2}{3} \text{ hours}$$

So it takes 32 hours and 40 minutes to fill the pool.

Models That Lead to Linear Equations

Recall that a model is a function that represents a real-world situation. Here we construct models that lead to linear equations. (Getting information from these models requires us to solve linear equations.) In constructing models, we use the following guidelines.

How to Construct a Model

1. **Choose the variable.** Identify the varying quantity in the problem (the independent variable) and give it a name, such as x.
2. **Translate words to algebra.** Express all the quantities given in the problem in terms of the variable x.
3. **Set up the model.** Express the model algebraically as a function of the variable x.

In the next example we construct a model involving simple interest. We use the following **simple interest formula**, which gives the amount of interest I earned when a principal P is deposited for t years at an interest rate r:

$$I = Prt$$

When using this formula, remember to convert r from a percentage to a decimal. For example, in decimal form, 5% is 0.05. So at an interest rate of 5% the interest paid on a \$1000 deposit over a 3-year period is $I = Prt = 1000(0.05)(3) = \$150$.

example 2 Constructing and Using a Model (Interest)

Mary inherits \$100,000 and invests it in two one-year certificates of deposit. One certificate pays 6%, and the other pays $4\frac{1}{2}\%$ simple interest annually.

(a) Construct a model for the total interest Mary earns in one year on her investments.

(b) If Mary's total interest is \$5025, how much money did she invest in each certificate?

Solution

(a) **Choose the variable.** The variable in this problem is the amount that Mary invests in each certificate. So let

$$x = \text{amount invested at } 6\%$$

Translate words to algebra. Since Mary's total inheritance is \$100,000 and she invests x dollars at 6%, it follows that she invested $100{,}000 - x$ at $4\frac{1}{2}\%$ in the second certificate. Let's translate all the information given in the problem into the language of algebra.

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100{,}000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100{,}000 - x)$

Set up the model. We are now ready to set up the model. The function we want gives the total interest Mary earns (the interest she earns at 6% plus the interest she earns at $4\frac{1}{2}\%$).

Interest earned at 6% ↓ Interest earned at $4\frac{1}{2}\%$ ↓

$$y = 0.06x + 0.045(100{,}000 - x)$$

$$= 0.06x + 4500 - 0.045x \quad \text{Distributive property}$$

$$= 4500 + 0.015x \quad \text{Simplify}$$

So the model we want is the linear equation $y = 4500 + 0.015x$.

(b) Since Mary's total interest is \$5025, we replace y by 5025 in the model and solve the resulting one-variable linear equation.

$$y = 4500 + 0.015x \quad \text{Model}$$

$$5025 = 4500 + 0.015x \quad \text{Replace } y \text{ by } 5025$$

$$525 = 0.015x \quad \text{Subtract } 4500$$

$$x = \frac{525}{0.015} \quad \text{Divide by } 0.015 \text{ and switch sides}$$

$$x = 35{,}000 \quad \text{Calculator}$$

So Mary invested \$35,000 at 6% and the remaining \$65,000 at $4\frac{1}{2}\%$.

NOW TRY EXERCISE 27

Many real-world problems involve mixing different types of substances. For example, construction workers may mix cement, gravel, and sand; fruit juice from concentrate may involve mixing different types of juices. Problems involving mixtures and concentrations make use of the fact that if an amount x of a substance is dissolved in a solution with volume V, then the concentration C of the substance is given by

$$C = \frac{x}{V}$$

So if 10 grams of sugar are dissolved in 5 liters of water, then the sugar concentration is $C = 10/5 = 2$ g/L.

example 3 Constructing and Using a Model (Mixtures and Concentration)

A manufacturer of soft drinks advertises its orange soda as "naturally flavored," although the soda contains only 5% orange juice. A new federal regulation stipulates that to be called "natural," a drink must contain at least 10% fruit juice. The manufacturer has a 900-gallon vat of soda and decides to add pure orange juice to the vat.

(a) Construct a model that gives the fraction of the mixture that is pure orange juice.

(b) How much pure orange juice must be added for the mixture to satisfy the 10% rule?

Solution

(a) **Choose the variable.** The variable in this problem is the amount of pure orange juice added to the vat. So let

x = the amount (in gallons) of pure orange juice added

Translate words to algebra. Let's translate all the information given in the problem into the language of algebra. First note that to begin with, 5% of the 900 gallons in the vat is orange juice, so the amount of orange juice in the vat is $(0.05)900 = 45$ gallons.

In Words	In Algebra
Amount of orange juice added	x
Amount of the mixture	$900 + x$
Amount of orange juice in the mixture	$45 + x$

Set up the model. We are now ready to set up the model. To find the fraction of the mixture that is orange juice, we divide the amount of orange juice in the mixture by the amount of the mixture.

$$y = \frac{45 + x}{900 + x}$$ ← Amount of orange juice in the mixture ← Amount of the mixture

So the model we want is the equation $y = (45 + x)/(900 + x)$.

(b) We want the mixture to have 10% orange juice. This means that we want the fraction of the mixture that is orange juice to be 0.10. So we replace y by 0.10 in our model and solve the resulting one-variable linear equation for x.

$$0.10 = \frac{45 + x}{900 + x}$$ Replace y by 0.10

$$0.10(900 + x) = 45 + x$$ Cross multiply

$$90 + 0.10x = 45 + x$$ Distributive Property

$$45 = 0.90x$$ Subtract 45, subtract $0.10x$

$$x = \frac{45}{0.90}$$ Divide by 0.90 and switch sides

$$x = 50$$ Calculator

So the manufacturer should add 50 gallons of pure orange juice to the soda.

NOW TRY EXERCISE 29

example 4 Constructing and Using a Model (Geometry)

Al paints with water colors on a sheet of paper that is 20 inches wide by 15 inches high. He then places this sheet on a mat so that a uniformly wide strip of the mat shows all around the picture.

(a) Construct a model that gives the perimeter of the mat.

(b) If the perimeter of the mat is 102 inches, how wide is the strip showing around the picture?

Solution

(a) Choose the variable. The variable in this problem is the width of the strip. So let

x = the width of the strip

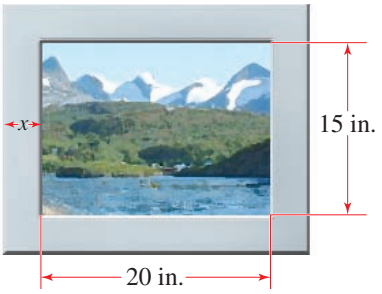


figure **2**

Translate words to algebra. Figure 2 helps us to translate information given in the problem into the language of algebra.

In Words	In Algebra
Width of mat	$20 + 2x$
Length of mat	$15 + 2x$

Set up the model. We are now ready to set up the model. To find the perimeter of the mat, we add twice the width and twice the length.

$$\begin{aligned} y &= 2(\overbrace{20 + 2x}^{\text{Width}}) + 2(\overbrace{15 + 2x}^{\text{Length}}) \\ &= 40 + 4x + 30 + 4x && \text{Distributive Property} \\ &= 70 + 8x && \text{Simplify} \end{aligned}$$

So the model we want is the equation $y = 70 + 8x$.

(b) Since the perimeter is 102 inches, we replace y by 102 in our model and solve the resulting one-variable equation for x.

$$\begin{aligned} y &= 70 + 8x && \text{Model} \\ 102 &= 70 + 8x && \text{Replace } y \text{ by } 102 \\ 8x &= 32 && \text{Subtract 70, switch sides} \\ x &= \frac{32}{8} && \text{Divide by 8} \\ x &= 4 && \text{Calculate} \end{aligned}$$

So if the perimeter is 102 inches, the width of the strip is 4 inches.

■ **NOW TRY EXERCISE 31** ■

Algebra Checkpoint ✓

Check your knowledge of solving linear equations by doing the following problems. You can review these topics in **Algebra Toolkit C.1** on page T47.

1. Determine whether each given value is a solution of the equation.

(a) $4x + 7 = 9x - 3$; $2, 3$ **(b)** $\dfrac{4x - 6}{3} = 2x - 6$; $0, 6$

(c) $\dfrac{x + 6}{x + 2} = 5$; $1, -1$

2. Solve the given equation.

(a) $2x + 7 = 31$ **(b)** $\frac{1}{2}(x - 8) = 1$ **(c)** $3(t - 8) = 2(1 - 5t)$

(d) $\dfrac{z}{5} + 3 = \dfrac{3}{10}z + 7$ **(e)** $3.2x + 1.4 = 10.9$

(f) $-2(x - 1) = 5(2x + 3) - 5x$

3. Solve the given equation. Check that the solution satisfies the equation.

(a) $\dfrac{x}{3x - 8} = 3$ **(b)** $\dfrac{2}{t + 6} = \dfrac{3}{t - 1}$ **(c)** $\dfrac{2x - 2}{x + 2} = \dfrac{4}{5}$

(d) $\dfrac{1}{x + 1} + \dfrac{1}{x - 2} = \dfrac{x + 3}{x^2 - x - 2}$ **(e)** $2 + \dfrac{5}{x - 4} = \dfrac{x + 1}{x - 4}$

2.6 Exercises

CONCEPTS

Fundamentals

1. **(a)** The equation $y = 8 + 5x$ is a _____-variable equation.

(b) The equation $3 = 8 + 5x$ is a _____-variable equation.

2. **(a)** To find the x-value when y is 6 in the equation $y - 2 = 3x + 1$, we solve the one-variable equation _____.

(b) The solution of the one-variable equation in part (a) is _____.

3. Suppose the two-variable equation $y = 500 + 10x$ models the cost y of manufacturing x flash drives. If the company spends \$4000 on manufacturing flash drives, then to find how many drives were manufactured, we solve the equation _____.

4. Alice invests \$1000 in two certificates of deposit; the interest on the first certificate is 4%, and the interest on the second is 5%. Let's denote the amount she invests in the first certificate by x. Then the amount she invests in the second certificate is _____. The interest she receives on the first certificate is _____, and the interest she receives on the second certificate is _____. So a function that models the total interest she receives from both certificates is $y =$ _____.

Think About It

5–6 ■ True or false?

5. Bill is draining his aquarium. The equation $y = 50 - 2x$ models the volume y of water remaining in the tank after x minutes. To find how long it takes for the tank to empty, we replace y by 0 in the model and solve for x.

6. The equation $y = -15 + 3x$ models the profit y a street vendor makes from selling x sandwiches. To find the number of sandwiches he must sell to make a profit of \$45, we replace x by 45 in the model and solve for y.

SKILLS

7–10 ■ Find the value of x that satisfies the given equation when y is 4.

7. $y = 24 + 5x$

8. $y = 6 - 4x$

9. $2x - 3y = 8$

10. $-3y = 8 + 2x$

11–16 ■ The given equation models the relationship between the quantities x and y. Find the value of x for the given value of y.

11. $y = 5 + 2x$; 25

12. $y = 500 - 0.25x$; 100

13. $y = 0.05x + 0.06(1000 - x)$; 57.50

14. $y = 0.025x + 0.035(50{,}000 - x)$; 1350

15. $y = \dfrac{10 + x}{50 + x}$; 0.5

16. $y = \dfrac{5000 + 2x}{250 + x}$; 8

17–20 ■ A quantity Q is related to a quantity R by the given equation. Find the value of Q for the given value of R.

17. $\dfrac{12}{R + 1} = 2(Q - 3)$; 5

18. $\dfrac{5}{R - 2} = 4(2Q + 1)$; 3

19. $\dfrac{2}{R} = \dfrac{Q - 5}{2Q - 1}$; -2

20. $\dfrac{6}{R - 1} = \dfrac{5 - 2Q}{3Q - 4}$; -5

CONTEXTS

21. Weather Balloon A weather balloon is being filled. The linear equation

$$V = 2 + 0.05t$$

models the volume V (in cubic feet) of hydrogen in the balloon at any time t (in seconds). (See Exercise 55 in Section 2.2.) How many minutes will it take until the balloon contains 55 ft^3 of hydrogen?

22. Filling a Pond A large koi pond is being filled. The linear equation

$$y = 300 + 10x$$

models the number y of gallons of water in the pool after x minutes. (See Exercise 56 in Section 2.2.) If the pond has a capacity of 4000 gallons, how many minutes will it take for the pond to be filled? Answer this question in two ways:

(a) Graphically (by graphing the equation and estimating the time from the graph)

(b) Algebraically (by solving an appropriate equation)

23. Landfill A county landfill has a maximum capacity of about 131,000,000 tons of trash. The amount in the landfill on a given day since 1996 is modeled by the function

$$T(x) = 32{,}400 + 4x$$

Here x is the number of days since January 1, 1996, and $T(x)$ is measured in thousands of tons. (See Exercise 53 in Section 2.2.) After how many days will the landfill reach maximum capacity? Answer this question in two ways:

(a) Graphically (by graphing the equation and estimating the time from the graph)

(b) Algebraically (by solving an appropriate equation)

24. Air Traffic Controller An aircraft is approaching an international airport. Using radar, an air traffic controller determines that the linear equation

$$y = -4x + 45$$

models the distance (measured in miles) of the approaching aircraft from the radar tower x minutes since the radar identified the aircraft. (See Exercise 57 in Section 2.3.) How many minutes will it take for the aircraft to reach the radar tower?

25. Crickets and Temperature Biologists have observed that the chirping rate of a certain species of cricket is modeled by the linear equation

$$t = \tfrac{5}{24}n + 45$$

where t is the temperature (in degrees Fahrenheit) and n is the number of chirps per minute. (See Exercise 64 in Section 2.3.) If the temperature is 80°F, estimate the cricket's chirping rate (in chirps per minute).

26. Manufacturing Cost The manager of a furniture factory finds that the cost C (in dollars) to manufacture x chairs is modeled by the linear equation

$$C = 13x + 900$$

How many chairs are manufactured if the cost is \$3500?

27. Investment Lili invests \$12,000 in two one-year certificates of deposit. One certificate pays 4%, and the other pays $4\frac{1}{2}$% simple interest annually.

(a) Construct a model for the total interest Lili earns in one year on her investments. (Let x represent the amount invested at 4%.)

(b) If Lili's total interest is \$526.00, how much money did she invest in each certificate?

28. Investment Ronelio invests \$20,000 in two one-year certificates of deposit. One certificate pays 3%, and the other pays $3\frac{3}{4}$% simple interest annually.

(a) Construct a model for the total interest Ronelio earns in one year on his investments. (Let x represent the amount invested at 3%.)

(b) If Ronelio's total interest is \$697.50, how much money did he invest in each certificate?

29. Mixture A jeweler has five rings, each weighing 18 grams, made of an alloy of 10% silver and 90% gold. He decides to melt down the rings and add enough silver to reduce the gold content to 75%.

(a) Construct a model that gives the fraction of the new alloy that is pure gold. (Let x represent the number of grams of silver added.)

(b) How much pure silver must be added for the mixture to have a gold content of 75%?

30. Mixture Monique has a pot that contains 6 liters of brine (salt water) at a concentration of 120 g/L. She needs to boil some of the water off to increase the concentration of salt.

(a) Construct a model that gives the concentration of the brine after boiling it. (Let x represent the number of liters of water that is boiled off.)

(b) How much water must be boiled off to increase the concentration of the brine to 200 g/L?

31. Geometry A graphic artist needs to construct a design that uses a rectangle whose length is 5 cm longer than its width x.

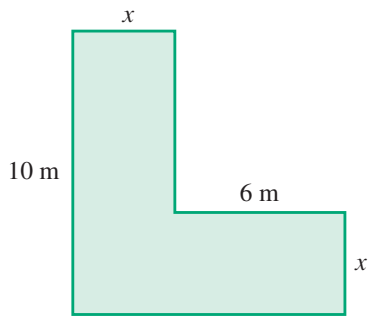
(a) Construct a model that gives the perimeter of the rectangle.

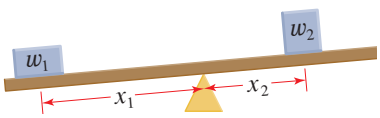
(b) If the perimeter of the rectangle is 26 cm, what are the dimensions of the rectangle?

32. Geometry An architect is designing a building whose footprint has the shape shown below.

(a) Construct a model that gives the total area of the footprint of the building.

(b) Find x such that the area of the building is 144 square meters.



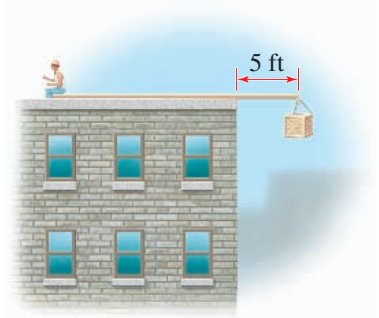


33. Law of the Lever The figure at left shows a lever system, similar to a seesaw that you might find in a children's playground. For the system to balance the product of the weight and its distance from the fulcrum must be the same on each side; that is

$$w_1x_1 = w_2x_2$$

This equation is called the Law of the Lever and was first discovered by Archimedes (see page 568).

A mother and her son are playing on a seesaw. The boy is at one end, 8 ft from the fulcrum. If the boy weights 100 lb and the mother weigh 125 lb, at what distance from the fulcrum should the woman sit so that the seesaw is balanced?



34. Law of the Lever A 30 ft plank rests on top of a flat roofed building, with 5 ft of the plank projecting over the edge, as shown in the figure. A 240 lb worker sits on one end of the plank. What is the largest weight that can be hung on the projecting end of the plank if it is to remain in balance? (Use the Law of the Lever as stated in Exercise 33.)

2.7 Linear Equations: Where Lines Meet

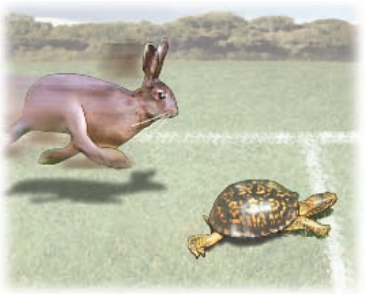
■ Where Lines Meet
■ Modeling Supply and Demand

IN THIS SECTION... we learn how to find where the graphs of two linear functions intersect. To find intersection points algebraically, we need to solve equations like the ones we solved in the preceding section.

GET READY... by reviewing Section 1.7 on how to find graphically where the graphs of two functions meet.

In many situations involving two linear functions we need to find the point where the graphs of these functions intersect. In Section 1.7 we used the graphs of functions to find their intersection points. In this section we will learn to use algebraic techniques to find the intersection point of two linear functions.

■ Where Lines Meet



The hare is much faster than the tortoise, but will he win the race?

A well-known story from Aesop's fables is about a race between a tortoise and a hare. The hare is so confident of winning that he decides to take a long nap at the beginning of the race. When he wakes up, he sees that the tortoise is very close to the finish line; he realizes that he can't catch up.

The hare's confidence is justified; hares can sprint at speeds of up to 30 mi/h, whereas tortoises can barely manage 0.20 mi/h. But let's say that in this 2-mile race, the tortoise tries really hard and keeps up a 0.8 mi/h pace; the complacent hare runs at a leisurely 4 mi/h after a two and a quarter hour nap. Under these conditions, who will win the race? Let's see.

The tortoise starts the race at time zero, so the distance y he reaches in the race course at time t is modeled by the equation $y = 0 + 0.8t$ or

$$y = 0.8t \qquad \text{Tortoise equation}$$

Let's find the linear equation that models the distance the hare reaches at time t. The hare's distance y is 0 when t is 2.25, so the point (2.25, 0) is on the graph of the desired equation. Using the point-slope formula, we get $y - 0 = 4(t - 2.25)$, or

$$y = -9 + 4t \qquad \text{Hare equation}$$

for $t \geq 2.25$. The graph in Figure 1 shows that the tortoise reaches the two-mile finish line before the hare does.

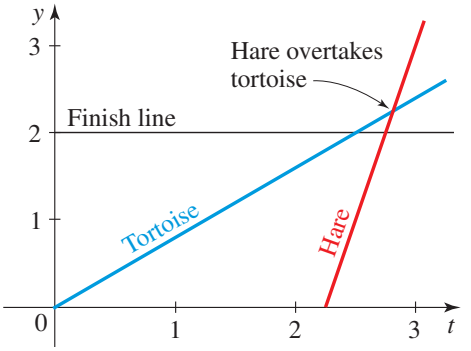


figure **1** The hare and tortoise race

If the race is to continue beyond the finish line, when does the hare overtake the tortoise? We can answer this question graphically (as in Section 1.7) or algebraically:

The accuracy of the graphical method depends on how precisely we can draw the graph. The algebraic method always gives us the exact answer.

Graphically: From Figure 1 we estimate that the intersection point of the two lines is approximately (2.8, 2.2). This point tells us that the hare overtakes the tortoise about 2.8 hours into the race.

Algebraically: The hare and the tortoise have reached the same point in the race course when the y-values (the distance traveled) in each equation are equal. So we equate the y-values from each equation:

$$0.8t = -9 + 4t$$

Solving for t, we get $t = 2.8125$. So the rabbit overtakes the hare 2.8125 hours into the race.

In general, the graphs of the functions f and g intersect at those values of x for which $f(x) = g(x)$. In the special case in which the functions are linear, we have the following.

Intersection Points of Linear Functions

To find where the graphs of the linear functions

$$y = b_1 + m_1x \qquad \text{and} \qquad y = b_2 + m_2x$$

intersect, we solve for x in the equation

$$b_1 + m_1x = b_2 + m_2x$$

example **1** **Finding Where Two Lines Meet**

Let $f(x) = -8 + 5x$ and $g(x) = 2 + 3x$. Find the value of x where the graphs of f and g intersect, and find the point of intersection.

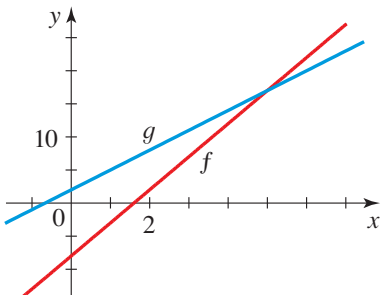


figure **2** Graphs of f and g

Solution 1 Graphical

We graph f and g in Figure 2. From the graph we see that the x-value of the intersection point is about $x = 5$. Also from the graph we see that the y-value of the intersection point is more than 15, but it is difficult to see its exact value from the graph. So from the graph we can estimate that the intersection point is (5, 15). (Note that this answer is only an approximation and depends on how precisely we can draw the graph.)

Solution 2 Algebraic

We solve the equation

$$\begin{aligned} -8 + 5x &= 2 + 3x && \text{Set functions equal} \\ 5x &= 10 + 3x && \text{Add 8} \\ 2x &= 10 && \text{Subtract } 3x \\ x &= 5 && \text{Divide by 2} \end{aligned}$$

So $f(x) = g(x)$ when $x = 5$. This means that the value of x where the graphs intersect is 5.

Let's find the common value of f and g when $x = 5$:

$$f(5) = -8 + 5(5) = -8 + 25 = 17$$

$$g(x) = 2 + 5(3) = 17$$

So the point (5, 17) is on both graphs, thus the graphs intersect at (5, 17).

NOW TRY EXERCISE 5

IN CONTEXT ➤

With rising fuel costs, there is an increasing interest in hybrid-electric vehicles, which consume a lot less fuel than gas-powered vehicles do. Also, several manufacturers are now introducing fully electric cars, including some sporty models. The Tesla fully electric sports car outperforms many gas-powered sports cars. The Tesla can accelerate from 0 to 60 mi/h in 3.9 seconds. That matches the Lamborghini Diablo's acceleration. So now it is possible to have a fully electric car of your dreams! Moreover, electric motors have few moving parts, so they require far less maintenance (for example, the Tesla's electric motor has only twelve moving parts).



Jonathan Larsen/Shutterstock.com 2009

Plug-in hybrid-electric car



Image by © Car Culture/Corbis

Fully electric sports car

example 2 Cost Comparison of Gas-Powered and Hybrid-Electric Cars

Kevin needs to buy a new car. He compares the cost of owning two cars he likes:

Car A: An $18,000 gas-powered car that gets 20 mi/gal

Car B: A $25,000 hybrid-electric car that gets 48 mi/gal

For this comparison he assumes that the price of gas is $4.50 per gallon.

(a) Find a linear equation that models the cost of purchasing Car A and driving it x miles.

(b) Find a linear equation that models the cost of purchasing Car B and driving it x miles.

(c) Find the break-even point for Kevin's cost comparison. That is, find the number of miles he needs to drive so that the cost of owning Car A is the same as the cost of owning Car B.

Solution

(a) Since the gas-powered car gets 20 miles per gallon and the price of gas is assumed to be $4.50, the cost of driving x miles is $4.50(x/20) = 0.225x$. Since

the initial cost of this car is \$18,000, the model for the cost of purchasing this car and driving it x miles is

$$g(x) = 18{,}000 + 0.225x$$

(b) Similarly, the model for the cost of purchasing the hybrid-electric car and driving it x miles is

$$h(x) = 25{,}000 + 0.09375x$$

(c) To find the break-even point, we find the value of x where $g(x) = h(x)$.

$$\begin{aligned} 18{,}000 + 0.225x &= 25{,}000 + 0.09375x && \text{Set } g(x) = h(x) \\ 0.225x &= 7{,}000 + 0.09375x && \text{Subtract } 18{,}000 \\ 0.13125x &= 7{,}000 && \text{Subtract } 0.09375x \\ x &\approx 53{,}333 && \text{Divide by } 0.13125 \end{aligned}$$

So the hybrid-electric car needs to be driven about 53,333 miles before it becomes cost-effective.

NOW TRY EXERCISE 27

example 3 Catching up with the Leader

Petra and Jordanna go on a bike ride from their home to the beach. Petra was slow in getting ready to go, and she left the house 5 minutes later than Jordanna. On this trip, Jordanna rides at an average speed of 5 m/s and Petra rides at an average speed of 8 m/s. Let $t = 0$ be the time Jordanna leaves home.

(a) Find a linear equation that models Jordanna's distance y from home at time t.

(b) Find a linear equation that models Petra's distance y from home at time t.

(c) How long does it take Petra to catch up with Jordanna? How far have they cycled when they meet?

Solution

(a) Since Jordanna cycles at an average speed of 5 m/s, the rate of change is $m = 5$. For Jordanna, y is 0 when t is 0, so the initial value b is 0. So an equation of the form $y = mx + b$ that models the distance Jordanna has traveled after t seconds is $y = 5t + 0$, or

$$y = 5t \qquad \text{Jordanna's equation}$$

(b) Petra's average speed is 8 m/s, so the rate of change is $m = 8$. Petra leaves 5 minutes $= 5 \times 60 = 300$ seconds later than Jordanna. So for Petra, y is 0 when t is 300. Thus the point (300, 0) is on the graph of the desired linear equation. By the point-slope formula, the equation that models the distance Petra has traveled after t seconds is $y - 0 = 8(t - 300)$, or

$$y = 8t - 2400 \qquad \text{Petra's equation}$$

(c) We need to find the time t when the y-value in Jordanna's equation equals the y-value in Petra's equation. We set these two values equal to each other and solve for t.

$$8t - 2400 = 5t \qquad \text{Set } y\text{-values equal}$$
$$3t = 2400 \qquad \text{Subtract } 5t\text{, add } 2400$$
$$t = 800 \qquad \text{Divide by } 3$$

So the cyclists meet when Jordanna has cycled for 800 seconds, or $800/60 \approx 13.3$ minutes. Since Petra left 5 minutes later, she travels $13.3 - 5 = 8.3$ minutes to catch up with Jordanna.

To find how far the cyclists have traveled when they meet, we replace t by 800 in either one of the equations.

$$y = 5(800) = 4000 \qquad \text{Replace } t \text{ by 800 in Jordanna's equation}$$

So they have cycled 4000 meters when they meet.

NOW TRY EXERCISE 33

Modeling Supply and Demand

Economists model supply and demand for a commodity using linear functions. For example, for a certain commodity we might have

$$\textbf{Supply equation:} \quad y = 8p - 10$$
$$\textbf{Demand equation:} \quad y = -3p + 15$$

where p is the price of the commodity. In the supply equation, y (the *amount produced*) increases as the price increases because if the price is high, more suppliers will manufacture the commodity. The demand equation indicates that y (the *amount sold*) decreases as the price increases. The **equilibrium point** is the point of intersection of the graphs of the supply and demand equations; at that point, the amount produced equals the amount sold.

example 4 Supply and Demand for Wheat

An economist models the market for wheat by the following equations.

$$\text{Supply:} \quad y = 3.82p - 10.51$$
$$\text{Demand:} \quad y = -0.99p + 25.34$$

Here, p is the price per bushel (in dollars), and y is the number of bushels produced and sold (in millions).

(a) Use the model for supply to determine at what point the price is so low that no wheat is produced.

(b) Use the model for demand to determine at what point the price is so high that no wheat is sold.

(c) Find the equilibrium price and the quantities produced and sold at equilibrium.

Solution

(a) If no wheat is produced, then $y = 0$ in the supply equation.

$$0 = 3.82p - 10.51 \qquad \text{Set } y = 0 \text{ in the supply equation}$$
$$p \approx 2.75 \qquad \text{Solve for } p$$

So at the low price of \$2.75 per bushel, the production of the wheat halts completely.

(b) If no wheat is sold, then $y = 0$ in the demand equation.

$$0 = -0.99p + 25.34 \qquad \text{Set } y = 0 \text{ in the demand equation}$$

$$p \approx 25.60 \qquad \text{Solve for } p$$

So at the high price of \$25.60 per bushel, no wheat is sold.

(c) To find the equilibrium point, we set the supply and demand equations equal to each other and solve.

$$3.82p - 10.51 = -0.99p + 25.34 \qquad \text{Set functions equal}$$

$$3.82p = -0.99p + 35.85 \qquad \text{Add } 10.51$$

$$4.81p = 35.85 \qquad \text{Add } 0.99p$$

$$p \approx 7.45 \qquad \text{Divide by } 4.81$$

So the equilibrium price is \$7.45. Evaluating the supply equation for $p = 7.45$, we get

$$y = 3.82(7.45) - 10.51 \approx 17.95$$

So for the equilibrium price of \$7.45 per bushel, about 18 million bushels of wheat are produced and sold.

NOW TRY EXERCISE 35

2.7 Exercises

CONCEPTS

Fundamentals

1. **(a)** To find where the graphs of the functions $y = m_1x + b_1$ and $y = m_2x + b_2$ intersect, we solve for x in the equation _____ = _____

(b) To find where the graphs of the functions $y = 2x + 7$ and $y = 3x - 10$ intersect, we solve for x in the equation _____ = _____. So the graphs of these functions intersect when x = _____. Therefore the graphs intersect at the point (____, ____).

2. **(a)** The point where the graphs of supply and demand equations intersect is called the _____ point.

(b) The supply of a product is given by the equation $y = m_1p + b_1$, and the demand is given by the equation $y = m_2p + b_2$, where p is the price of the product. To find the price for which supply is equal to demand, we solve for p in the equation _____ = _____.

Think About It

3. Muna's and Michael's distances from home are modeled by the following equations.

Muna's equation: $y = 4 + 3t$

Michael's equation: $y = 2 + 3t$

Explain why Michael will never catch up with Muna.

4. Udit is walking from school to his home 5 miles away, starting at time $t = 0$. Udit walks slowly at a speed of 2 miles per hour. If the time t is measured in hours, then

Udit's distance from school at time t is $y = \square + \square\, t$.

Udit's distance from home at time t is $y = \square + \square\, t$.

SKILLS

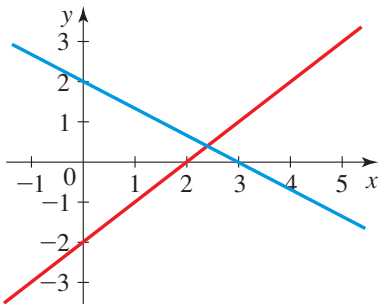
5–6 ■ A graph of two lines is given.

(a) Use the graph to estimate the coordinates of the point of intersection.

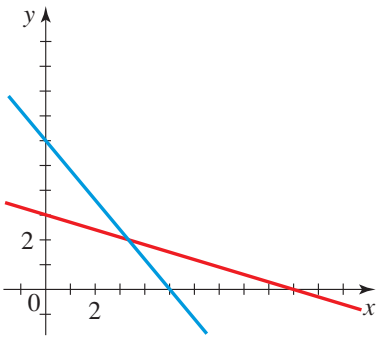
(b) Find an equation for each line.

(c) Use the equations from part (b) to find the coordinates of the point of intersection. Compare with your answer to part (a).

5.



6.



7–10 ■ Linear functions f and g are given.

(a) Graph f and g, and use the graphs to estimate the value of x where the graphs intersect.

(b) Find algebraically the value of x where the graphs of f and g intersect.

7. $f(x) = 2x - 5$; $g(x) = -2x + 3$

8. $f(x) = x + 3$; $g(x) = -x - 1$

9. $f(x) = x - 4$; $g(x) = -2x + 2$

10. $f(x) = 2x - 4$; $g(x) = -3x + 6$

11–16 ■ Find the value of x for which the graphs of the two linear equations intersect.

11. $y = 4 - x$; $y = x$

12. $y = x - 3$; $y = 7 - x$

13. $y = 3 - \frac{4}{3}x$; $y = \frac{2}{3}x - 3$

14. $y = -\frac{3}{2}x$; $y = -4 - \frac{1}{2}x$

15. $y = \frac{5}{3} - \frac{1}{3}x$; $y = 2x - 3$

16. $y = 7 - x$; $y = \frac{2}{3}x + \frac{1}{3}$

17–22 ■ Find the point at which the graphs of the two linear equations intersect.

17. $y = 2 + x$; $y = \frac{4}{3}x + 3$

18. $y = \frac{4}{3}x - \frac{28}{3}$; $y = 9x + 6$

19. $y = \frac{7}{2} - \frac{1}{2}x$; $y = 5x - 2$

20. $y = \frac{1}{3}x$; $y = 40 - 3x$

21. $y = 6 - \frac{3}{2}x$; $y = \frac{3}{10}x - 12$

22. $y = 9 + x$; $y = 6.6 + 0.6x$

23–24 ■ Two linear functions are described verbally. Find equations for the functions, and find the point at which the graphs of the two functions intersect.

23. The graph of the linear function f has slope $\frac{2}{3}$ and y-intercept 5; the graph of the linear function g has slope 5 and y-intercept 2.

24. The graph of the linear function f has x-intercept 4 and y-intercept -1; the graph of the linear function g has x-intercept -2 and y-intercept 10.

25–26 ■ Equations for supply and demand are given. Find the price (in dollars) and the amount of the commodity produced and sold at equilibrium.

25. Supply: $y = 0.45p + 4$
Demand: $y = -0.65p + 28$

26. Supply: $y = 8.5p + 45$
Demand: $y = -0.6p + 300$

CONTEXTS

27. Cell Phone Plan Comparison Dietmar is in the process of choosing a cell phone and a cell phone plan. The first plan charges 20¢ per minute plus a monthly fee of $10, and the second plan offers unlimited minutes for a monthly fee of $100.

(a) Find a linear function f that models the monthly cost $f(x)$ of the first plan in terms of the number x of minutes used.

(b) Find a linear function g that models the monthly cost $g(x)$ of the second plan in terms of the number x of minutes used.

(c) Determine the number of minutes for which the two plans have the same monthly cost.

28. Solar Power Lina is considering installing solar panels on her house. Solar Advantage offers to install solar panels that generate 320 kWh of electricity per month for an installation fee of $15,000. She uses 350 kWh of electricity per month, and her local utility company charges 20¢ per kWh.

(a) If Lina gets all her electrical power from the local utility company, find a linear function U that models the cost $U(x)$ of electricity for x months of service.

(b) If Lina has Solar Advantage install solar panels on her roof that generate 320 kWh of power per month, find a linear function S that models the cost S(x) of electricity for x months of service.

(c) Determine the number of months it would take to reach the break-even point for installation of Solar Advantage's solar panels, that is, determine when $S(x) = U(x)$.

29. Renting Versus Buying a Photocopier A certain office can purchase a photocopier for $5800 with a maintenance fee of $25 a month. On the other hand, they can rent the photocopier for $95 a month (including maintenance). If they purchase the photocopier, each copy would cost 3¢; if they rent, the cost is 6¢ per copy. The office manager estimates that they make 8000 copies a month.

(a) Find a linear function C that models the cost $C(x)$ of purchasing and using the copier for x months.

(b) Find a linear function S that models the cost $S(x)$ of renting and using the copier for x months.

(c) Make a table of the cost of each method for 1 year to 3 years of use, in 6-month increments.

(d) For how many months of use would the cost be the same for each method?

30. Cost and Revenue A tire company determines that to manufacture a certain type of tire, it costs $8000 to set up the production process. Each tire that is produced costs $22 in material and labor. The company sells this tire to wholesale distributors for $49 each.

(a) Find a linear function C that models the total cost $C(x)$ of producing x tires.

(b) Find a linear function R that models the revenue $R(x)$ from selling x tires.

(c) Find a linear function P that models the profit $P(x)$ from selling x tires. [*Note:* profit = revenue − cost.]

(d) How many tires must the company sell to break even (that is, when does revenue equal cost)?

31. Car Rental A businessman intends to rent a car for a 3-day business trip. The rental is $35 a day and 15¢ per mile (Plan 1) or $90 a day with unlimited mileage (Plan 2). He is not sure how many miles he will drive but estimates that it will be between 1000 and 1200 miles.

(a) For each plan, find a linear function C that models the cost $C(x)$ in terms of the number x of miles driven.

(b) Which rental plan is cheaper if the businessman drives 1000 miles? 1200 miles? At what mileage do the two plans cost the same?

32. Buying a Car Kofi wants to buy a new car, and he has narrowed his choices to two models.

Model A sells for \$15,500, gets 25 mi/gal, and costs \$350 a year for insurance

Model B sells for \$19,100, gets 48 mi/gal, and costs \$425 a year for insurance

Kofi drives about 36,000 miles a year, and gas costs about \$4.50 a gallon.

(a) Find a linear function A that models the total cost $A(x)$ of owning Model A for x years.

(b) Find a linear function B that models the total cost $B(x)$ of owning Model B for x years.

(c) Find the number of years of ownership for which the cost to Kofi of owning Model A equals the cost of owning Model B.

33. Commute to Work (See Exercise 59 in Section 2.2.) Jade and her roommate Jari live in a suburb of San Antonio, Texas, and both work at an elementary school in the city. Each morning they commute to work traveling west on I-10. One morning Jade leaves for work at 6:50 A.M., but Jari leaves 10 minutes later. On this trip Jade drives at an average speed of 65 mi/h, and Jari drives at an average speed of 72 mi/h.

(a) Find a linear equation that models the distance y Jari has traveled x hours after she leaves home.

(b) Find a linear equation that models the distance y Jade has traveled x hours after Jari leaves home.

(c) Determine how long it takes Jari to catch up with Jade. How far have they traveled at the time they meet?

34. Catching Up Kumar leaves his house at 7:30 A.M. and cycles to school. Kumar's mother notices that he has left his lunch at home. She leaves the house by car 5 minutes after Kumar left to give him his lunch. Kumar cycles at an average speed of 8 mi/h, and his mother drives at an average speed of 24 mi/h.

(a) Find a linear equation that models the distance y Kumar's mother has traveled x hours after she left home.

(b) Find a linear equation that models the distance y Kumar has traveled x hours after his mother has left home.

(d) Determine how long it takes Kumar's mother to catch up with Kumar. How far have they traveled at the time they meet?

35. Supply and Demand for Corn An economist models the market for corn by the following equations:

$$\text{Supply:} \quad y = 4.18p - 11.5$$

$$\text{Demand:} \quad y = -1.06p + 19.3$$

Here, p is the price per bushel (in dollars), and y is the number of bushels produced and sold (in billions).

(a) Use the model for supply to determine at what point the price is so low that no corn is produced.

(b) Use the model for demand to determine at what point the price is so high that no corn is sold.

(c) Find the equilibrium price and the quantities that are produced and sold at equilibrium.

36. Supply and Demand for Soybeans An economist models the market for soybeans by the following equations:

$$\text{Supply:} \quad y = 0.37p - 1.59$$

$$\text{Demand:} \quad y = -0.23p + 5.22$$

Here p is the price per bushel (in dollars), and y is the number of bushels produced and sold (in billions).

(a) Use the model for supply to determine at what point the price is so low that no soybeans are produced.

(b) Use the model for demand to determine at what point the price is so high that no soybeans are sold.

(c) Find the equilibrium price and the quantities that are produced and sold at equilibrium.

R **37. Median Incomes of Men and Women** The gap between the median income of men and that of women has been slowly shrinking over the past 30 years. Search the Internet to find the median incomes of men and women for 1965 to 2005.

(a) Find the regression line for the data found on the Internet for the median income of men.

(b) Find the regression line for the data found on the Internet for the median income of women.

(c) Use the regression lines found in parts (a) and (b) to predict the year when the median incomes of men and women will be the same.

R **38. Population** The populations in many large metropolitan districts have recently been decreasing as residents move to the suburbs. Search the Internet for a city of your choice where there has been a decrease in the metropolitan population and an increase in the suburban population.

(a) Find the regression line for the metropolitan population data for the past 40 years.

(b) Find the regression line for the suburban population data for the past 40 years.

(c) Use the regression lines found in parts (a) and (b) to estimate the year in which the suburban population will equal the metropolitan population.

R **39. Teacher Salaries** The gap between the median salaries for men teachers and women teachers has recently been shrinking. Search the Internet to find a history for the past 30 years of men and women teachers' median salaries in your state.

(a) Find the regression line for the data on the men teachers' median salary.

(b) Find the regression line for the data on the women teachers' median salary.

(c) Use the regression lines found in parts (a) and (b) to predict the year when the median teacher's salary will be equal for men and women.

CHAPTER 2 REVIEW

CHAPTER 2 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

2.1 Working with Functions: Average Rate of Change

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{net change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change measures how quickly the dependent variable changes with respect to the independent variable. A familiar example of a rate of change is the average speed of a moving object such as a car.

2.2 Linear Functions: Constant Rate of Change

A **linear function** is a function of the form $f(x) = b + mx$. Such a function is called *linear* because its graph is a straight line. The average rate of change of a linear function between any two values of x is always m, so we refer to m simply as the **rate of change** of f. The number b is the **starting value** of the function (its value when x is 0).

To find the **slope** of a line graphed in a coordinate plane, we choose two different points on the line, (x_1, y_1) and (x_2, y_2), and calculate

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The **y-intercept** of the graph of any function f is $f(0)$.

If $f(x) = b + mx$ is a linear function, then

- The graph of f is a line with slope m.
- The y-intercept of the graph of f is b.

Thus the rate of change of f is the slope of its graph, and the initial value of f is the y-intercept of its graph.

2.3 Equations of Lines: Constructing Linear Models

A **linear model** is a linear function that models a real-life situation. To construct a linear model from data or from a verbal description of the situation, we use one of the following equivalent forms of the equation of a line:

- **Slope-intercept form:** $y = b + mx$
  when we know the slope m and the y-intercept b.
- **Point-slope form:** $y - y_1 = m(x - x_1)$
  when we know the slope m and a point (x_1, y_1) that lies on the line.

If we know two points that lie on a line, we first use the points to find the slope m and then use this slope and one of the given points in the point-slope form to find the equation of the line.

Horizontal and vertical lines have simple equations:

- The vertical line through the point (a, b) has equation $x = a$.
- The horizontal line through the point (a, b) has equation $y = b$.

The **general form of the equation of a line** is

$$Ax + By + C = 0$$

Every line has an equation of this form, and the graph of every equation of this form is a line.

2.4 Varying the Coefficients: Direct Proportionality

The numbers b and m in the linear equation $y = b + mx$ are called the **coefficients** of the equation: b is the **constant coefficient** and m is the **coefficient of x**.

If two nonvertical lines have slopes m_1 and m_2, then they are

- **parallel** if they have the same slope: $m_1 = m_2$.
- **perpendicular** if they have negative reciprocal slopes: $m_1 = -\dfrac{1}{m_2}$.

The variable y is **directly proportional to** the variable x (or y **varies directly as** x) if there is a constant k such that x and y are related by the equation $y = kx$. The constant k is called the **constant of proportionality**. The graph of a proportionality equation is a straight line with slope k passing through the origin.

2.5 Linear Regression: Fitting Lines to Data

In Section 2.3 we found linear models for data whose scatter plots lie exactly on a line. Most real-life data are not exactly linear but may appear to lie approximately on a line. In this case a linear model can still be useful for revealing trends and patterns. The linear model that we use is called the **regression line**; it is the "line of best fit" for the data.

A simple way to find the regression line for a set of data is to use a graphing calculator. (The command for this is `LinReg` on many calculators.)

A regression line can be found for any set of two-variable data, but this line isn't meaningful if too many of the data points lie far away from the line in a scatter plot. The **correlation coefficient** r is a measure of how closely data fit their regression line (that is, how closely the variables *correlate*). For any regression line we have $-1 \le r \le 1$; values of r close to 1 or -1 indicate a high degree of correlation, and values of r close to 0 indicate little or no correlation.

2.6 Linear Equations: Getting Information from a Model

To construct a model from a verbal description of a real-life situation, we follow three general steps:

1. Choose the variable in terms of which the model will be expressed. Assign a symbol (such as x) to the variable.
2. Translate words into algebra by expressing the other quantities in the problem in terms of the variable x.
3. Set up the model by expressing the fact(s) given about the quantity modeled as a function of x.

Once we have found a model, we can use it to answer questions about the quantity being modeled. This will often require solving equations.

2.7 Linear Equations: Where Lines Meet

Some models involve two linear functions that model two different but related quantities. We are often interested in determining when the two functions have the same value. In this case we need to find the point at which the graphs of the two lines meet. This can be accomplished in one of two ways:

- Graphically, by graphing both lines on the same coordinate plane and estimating the coordinates of the point of intersection
- Algebraically, by setting the two functions equal to each other and solving the resulting equation

The graphical method usually gives just an estimate, but the algebraic method always gives the exact answer.

CHAPTER 2 REVIEW EXERCISES

SKILLS

1–6 ■ A function is given (either numerically, graphically, or algebraically). Find the average rate of change of the function between the indicated points.

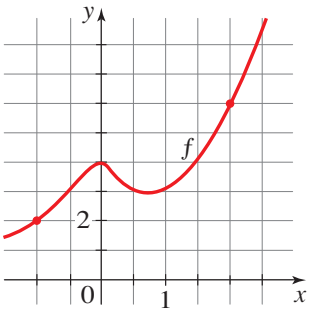
1. Between $x = 4$ and $x = 10$

x	$F(x)$
2	14
4	12
6	12
8	8
10	6
12	3

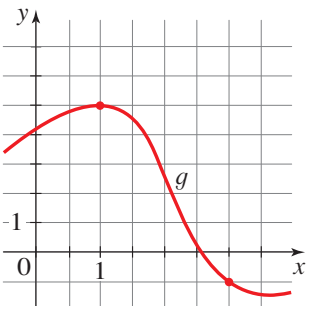
2. Between $x = 10$ and $x = 30$

x	$G(x)$
0	25
10	−5
20	10
30	30
40	0
50	15

3. Between $x = -1$ and $x = 2$



4. Between $x = 1$ and $x = 3$



5. $f(x) = x^2 + 2x$, between $x = 1$ and $x = 4$

6. $g(t) = 6t^2 - t^3$, between $t = -1$ and $t = 3$

7–8 ■ Determine whether the given function is linear.

7. $f(x) = (2 + 3x)^2$

8. $g(x) = \dfrac{x + 5}{2}$

9–12 ■ A linear function is given.

(a) Sketch a graph of the function.

(b) What is the slope of the graph?

(c) What is the rate of change of the function?

9. $f(x) = 3 + 2x$ **10.** $f(x) = 6 - 2x$ **11.** $g(x) = 3 - \frac{1}{2}x$ **12.** $g(x) = -3 + x$

13–18 ■ A linear function f is described, either verbally, numerically, or graphically. Express f in the form $f(x) = b + mx$.

13. The function has rate of change −5 and initial value 10.

14. The graph of the function has slope $\frac{1}{3}$ and initial value 2.

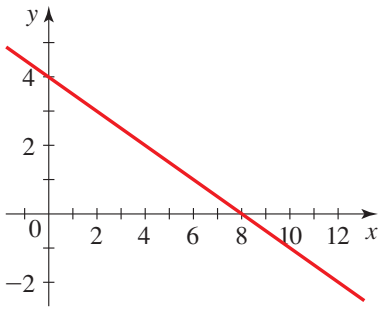
15.

x	$f(x)$
0	3
1	5
2	7
3	9

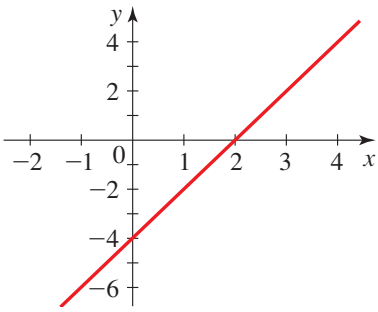
16.

x	$f(x)$
0	6
2	5.5
4	5
6	4.5

17.



18.



19–28 ■ Find the equation, in slope-intercept form, of the line described.

19. The line has slope 5 and y-intercept 0.

20. The line has slope $-\frac{1}{2}$ and y-intercept 4.

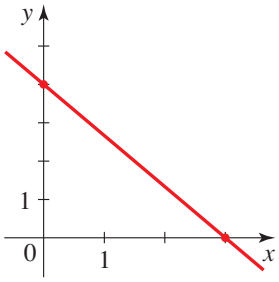
21. The line has slope -6 and passes through the point $(1, 3)$.

22. The line has slope 0.25 and passes through the point $(-6, -3)$.

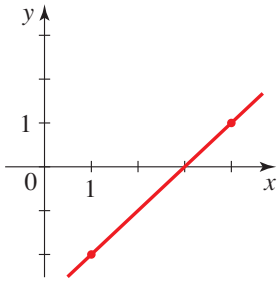
23. The line passes through the points $(2, 4)$ and $(4, 0)$.

24. The line passes through the points $(-3, 0)$ and $(6, 2)$.

25.



26.



27. The line is horizontal and has y-intercept 26.

28. The line is vertical and passes through the point $(-3, 26)$.

29–32 ■ An equation of a line is given.

(a) Find the slope, the y-intercept, and the x-intercept of the line.

(b) Sketch a graph of the line.

(c) Express the equation of the line in slope-intercept form.

29. $1 - y = \dfrac{x}{2}$

30. $3 + x = 6 + 2y$

31. $3x - 4y - 12 = 0$

32. $5x + 4y + 20 = 0$

33–34 ■ An equation of a line l and the coordinates of a point P are given.

(a) Find an equation in general form of the line parallel to l passing through P.

(b) Find an equation in general form of the line perpendicular to l passing through P.

(c) Graph all three lines on the same coordinate axes.

33. $y = -2 + \frac{1}{2}x; \quad P(2, -3)$

34. $2x + 3y = 6; \quad P(-1, 2)$

35–38 ■ Find an equation for the line that satisfies the given conditions.

35. Passes through the origin, parallel to the line $2x - 4y = 8$

36. Passes through $(4, -3)$, perpendicular to the y-axis

37. Passes through $(3, -2)$, perpendicular to the line $x + 4y + 8 = 0$

38. Passes through $(-2, 4)$, parallel to the line that contains $(1, 2)$ and $(3, 8)$

39–40 ■ Suppose that y is directly proportional to x. Find the constant of proportionality k under the given condition, and use it to solve the rest of the problem.

39. If $x = 3$, then $y = 15$. Find y when $x = 7$.

40. If $x = 6$, then $y = 14$. Find x when $y = 35$.

41–42 ■ Find a proportionality equation of the form $y = kx$ that relates y to x.

41. The number y of millimeters in x inches. (1 inch equals 25.4 millimeters.)

42. The number y of legs on x horses (assuming that none of the horses has lost a leg).

43–44 ■ A table of values is given.

(a) Make a scatter plot of the data.

(b) Find the regression line for the data, and graph it on your scatter plot.

(c) Does the regression line appear to fit the data well?

43.

x	1	2	2	3	4	6	8
y	5	7	8	7	10	12	14

44.

x	0	2	3	3	5	6	8
y	10	3	6	12	14	5	4

45–50 ■ Linear functions f and g are given.

(a) Sketch graphs of f and g.

(b) From your graph, estimate the coordinates of the points where the graphs of f and g intersect.

(c) Find the intersection point of the graphs of f and g algebraically.

45. $f(x) = 2x + 4$; $g(x) = 5x - 8$

46. $f(x) = 10 - 3x$; $g(x) = 5x$

47. $f(x) = 5 - 3x$; $g(x) = 12 + 4x$

48. $f(x) = 13$; $g(x) = 7x - 8$

49. $f(x) = 6 + 2x$; $g(x) = 15 - 3x$

50. $f(x) = 1 + 5x$; $g(x) = 2x - 7$

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

51. Points and Lines The table lists the coordinates of six points, P_1 through P_6. Answer the following questions about these points and the lines that they determine.

Point	P_1	P_2	P_3	P_4	P_5	P_6
Coordinates	$(-1, 1)$	$(-1, -2)$	$(1, 0)$	$(2, 1)$	$(0, 3)$	$(3, 4)$

(a) Plot the points on a coordinate plane.

(b) Which two points lie on the same vertical line? What is the equation of that line?

(c) Which two points lie on the same horizontal line? What is the equation of that line?

(d) Find an equation in slope-intercept form for the line containing P_1 and P_3.

(e) Which two points lie on a line perpendicular to the line you found in part (c)? What is the equation of the line containing these two points?

(f) Let f be the function whose graph contains P_5 and P_6. What is the rate of change of f?

(g) Find equations in slope-intercept form for the line that contains P_2 and P_5, and the line that contains P_4 and P_6. Find the coordinates of the point of intersection of these two lines.

(h) Find the regression line for the six points in the table, and graph it on your scatter plot from part (a). Do most of the points lie close to the line, or are they scattered far away from it?

52. Supply and Demand for Milk The price of fresh milk varies widely from state to state. The Ackamee grocery store in Dalewood, Maryland, determines from sales data that when milk is priced at \$1.10 per quart, they sell 4000 quarts every week, but when it is priced at \$1.25 per quart, they sell only 3700 quarts in a week.

(a) Use the given price and quantity data to find a linear demand equation of the form $y = b + mp$ that gives the weekly demand y (in quarts) when the price of milk is p dollars per quart. Sketch a graph of the equation.

(b) What does the slope of the graph of the demand equation represent? Why is the slope negative?

(c) What do the p- and y-intercepts of the demand equation represent?

(d) At what price will the demand for milk drop to 3000 quarts?

(e) A survey of local dairies indicates that the supply equation for milk sold at Ackamee is $y = 7800p - 5070$. Sketch this line on your graph from part (a).

(f) What does the p-intercept of the supply equation represent?

(g) From your graph, estimate the coordinates of the equilibrium point (the point at which the graphs of the supply and demand equations meet).

(h) Find the coordinates of the equilibrium point algebraically. What is the equilibrium price, and how many quarts of milk are sold at this price?

CONTEXTS

53. U.S. Population The Constitution of the United States requires a national census every 10 years. The census data for 1800–2000 are given in the table. (Express all rates in millions/yr.)

(a) Draw a partial scatter plot of the data, using only the data for years divisible by 20 (that is, 1800, 1820, 1840, and so on).

(b) From your scatter plot in part (a), do you detect a general trend in the rate of change of population over the past two centuries?

(c) Find the average rate of change of population for the years 1900–1950.

(d) Find the average rate of change of population in the 19th century and in the 20th century.

(e) Find the average rate of change of population for the years 1800–2000. Compare this to the rates you found in part (d).

(f) In which decade of the 20th century was the rate of change of population the greatest? What was that rate of change?

Year	Population (millions)	Year	Population (millions)	Year	Population (millions)
1800	5.3	1870	38.6	1940	132.2
1810	7.2	1880	50.2	1950	151.3
1820	9.6	1890	63.0	1960	179.3
1830	12.9	1900	76.2	1970	203.3
1840	17.1	1910	92.2	1980	226.5
1850	23.2	1920	106.0	1990	248.7
1860	31.4	1930	123.2	2000	281.4



54. Price of Gasoline The price of gasoline in the United States has fluctuated considerably in recent years. A function that models the average price per gallon of gas in Montgomery County, Pennsylvania, for the period from July 1, 2008, to March 31, 2009, is

$$f(x) = 4 - 0.08x - 0.002x^2 + 0.00008x^3$$

where x is the number of weeks since July 1, 2008, and $f(x)$ is the average price in dollars of a gallon of gas in week x.

(a) Draw a graph of f with a graphing calculator, using the viewing rectangle $[0, 40]$ by $[1, 4.5]$.

(b) What does the graph indicate about the general trend of the rate of change of the price of gasoline over this period?

(c) Find the average rate of change (in dollars per week) in the price of gasoline from July 1 to December 31, 2008 (i.e., from $x = 0$ to $x = 26$).

(d) Find the average rate of change in the price of gasoline from January 1 to March 31, 2009 (i.e., from $x = 26$ to $x = 39$).

55. Growing Corn During its initial growth phase, a corn plant's height increases linearly. It then stops growing as the ears of corn mature. The official start of corn season in a certain Pennsylvania farming community is May 15, a couple of weeks after most of the farmers have planted their corn. Let $h(x)$ be the height (in inches) of a particular cornstalk x days after May 15. The table in the margin shows the height of this cornstalk on various days of the corn season.

Day x	Height $h(x)$
5	21
10	28
20	42
35	63

(a) Find the rate at which the cornstalk is growing.

(b) Find the linear function h that models the growth of the cornstalk.

(c) Sketch a graph of h. What is the slope of the graph?

(d) The cornstalk stops growing on day 40 of the season. What height does it reach on this day?

(e) What was the height of the cornstalk on May 15 (that is, when x is 0)?

(f) How many days before May 15 did the seedling break the surface of the soil (that is, start at height 0)?

(g) When did the cornstalk reach a height of 35 inches?

56. Length of a Coho Salmon Coho salmon inhabit the coastal waters of the Pacific Northwest, living a fixed lifespan of four years before they die as they spawn in the same riverbed in which they were born. They grow steadily throughout their lives, and their length is proportional to their age. A coho two years old is 16 inches long.

(a) Find the equation of proportionality that relates a coho's length in inches to its age in months.

(b) How long is a coho 30 months old?

(c) How old is a coho 24 inches long?

(d) What length are cohoes when they die?

(e) A few cohoes do survive spawning, returning to the ocean for another four-year cycle. A fisherman catches such a coho that is 40 inches long. How old is this coho?

57. Super-Size Portions A study published in the *American Journal of Public Health* found that the portion sizes in many fast-food restaurants far exceed the sizes of those offered in the past. For instance, in the mid-1950s, McDonald's had only one size of french fries, and now that size is labeled "Small" and is one-third the weight of the largest size available. The problem with larger portions is that we tend to eat more when we are offered more food, contributing to the national obesity epidemic. The table at left shows the number of ounces in a "large" drink at a particular convenience store chain since 1975.

Year	Large size (oz)
1975	12
1980	15
1985	16
1990	20
1995	24
2000	32
2005	36

(a) Find the regression line for the number of ounces in the "large" drink as a function of the number of years since 1975.

(b) Graph the regression line along with a scatter plot of the data.

(c) Use the model found in part (a) to predict the number of ounces in a "large" drink in 2010. Does this prediction make sense?

(d) If the model is accurate, in what year will a "large" drink contain 60 oz?

58. Weight Loss A weight-loss clinic keeps records of its clients' weights when they first joined and their weight 12 months later. The data for ten clients are shown in the table. (See Chapter 1 Review, Exercise 71, page 123.)

(a) Find the regression line for the weight after 12 months in terms of the beginning weight.

(b) Make a scatter plot of the weights, together with the regression line you found in part (b). Does the line seem to fit the data well?

(c) What does the regression line predict that the weight of a 200-lb person will be 12 months after entering the program?

x = Beginning weight	212	161	165	142	170	181	165	172	312	302
y = Weight after 12 months	188	151	166	131	165	156	156	152	276	289

59. Distance, Time, and Speed Annette leaves her home at 8:00 A.M. and cycles at a constant speed of 7 mi/h to the gardening supply store where she works. Fifteen minutes later, her husband Theo notices that she has forgotten her keys, so he leaves the house and follows her in his car, driving at 24 mi/h, so that he can give her the keys.

(a) Find linear functions f and g that model the distance Annette and Theo have traveled, as a function of the time t (in hours) that has elapsed since 8:00 A.M.

(b) At what time will Theo catch up to Annette? How far will he have driven when he catches up?

60. Cost and Revenue Amanda silkscreens artistic designs on tee shirts, which she sells to souvenir shops in her beachside town. She has fixed costs of \$60 every week, and each tee shirt costs her \$3.50, while dyes and other supplies cost 50 cents per shirt. She sells each completed tee shirt for \$15.

(a) Find a linear function C that models Amanda's total cost $C(x)$ when she produces x tee shirts in a week.

(b) Find a linear function R that models Amanda's total revenue $R(x)$ when she sells x tee shirts.

(c) What is Amanda's profit $P(x)$ when she sells x tee shirts in a week?

(d) How many shirts must Amanda sell to break even?

CHAPTER 2 TEST

1. Find the average rate of change of the function f between the given values of x.

(a) $f(x) = x^2 - 3x$ between $x = 1$ and $x = 4$

(b) $f(x) = 3 + 5x$ between $x = 2$ and $x = 2 + h$

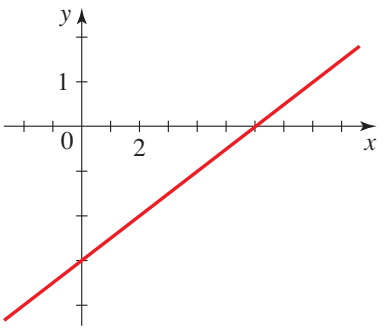
2. Let $f(x) = x^2 - 5$, and let $g(x) = 6 - 3x$.

(a) Only one of the two functions f and g is linear. Which one is linear, and why is the other one not linear?

(b) Sketch a graph of the linear function.

(c) What is the rate of change of the linear function?

3. The figure shows the graph of a linear function. Express the function in the form $f(x) = b + mx$.



4. A description of a line is given. Find the equation of the line in slope-intercept form.

(a) The line contains the points (2, 5) and (4, 13).

(b) The line contains the point $(-3, 3)$ and is parallel to the line $2x + 5y = 20$.

(c) The line contains the origin and is perpendicular to the line $2x + 5y = 20$.

5. Find equations for the vertical line and the horizontal line that pass through the point $(6, -8)$.

6. A linear equation is given. Put the equation in general form and in slope-intercept form. Then graph the equation.

(a) $x = \dfrac{y - 1}{3}$ **(b)** $2x + 3y = 6 + y$

7. The distance by which a rubber band stretches when it is pulled is proportional to the force used in pulling it. If a rubber band stretches by 6 inches when a 2-pound weight is hung from it, by how much does it stretch when a weight of 3.5 pounds is hung from it?

Weight of guinea pig (oz)	Food per week (oz)
18	15
22	12
23	14
23	15
25	17
27	18

8. The data in the table give the weight of six guinea pigs and the amount of food each one eats in one week (both measured in ounces).

(a) Make a scatter plot of the data.

(b) Find the regression line for the data, and graph it on your scatter plot.

(c) Use the regression line to predict how much food a guinea pig weighing 30 ounces would eat.

9. A pair of supply and demand equations is given.

(a) Find the coordinates of the point at which the graphs of the lines intersect (the equilibrium point).

(b) What is the equilibrium price of the commodity being modeled?

$$\text{Supply:} \quad y = 4p - 100$$

$$\text{Demand:} \quad y = -2p + 350$$

1 When Rates of Change Change

OBJECTIVE To recognize changes in the average rate of change of a function and to see how such changes affect the graph of the function.



In an episode of the popular television show *The Simpsons*, Homer reads from his *US of A Today* newspaper and says, "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate."

In this statement Homer is talking about the rate of change of a rate of change: SAT scores are changing (declining), but the rate at which they are declining is itself changing (slowing down).* In the real world, rates of change are changing all the time. When you drive, your speed (rate of change of distance) increases when you accelerate and decreases when you decelerate. As teenagers grow, the rate at which their height increases slows down and eventually stops as they become adults. In this Exploration we investigate how a changing rate of change affects the graph of a function.

I. Changes in Tuition Fees

1. The table gives the average annual cost of tuition at public 4-year colleges in the United States. The first section in the table gives the tuition in actual current dollars; the second section adjusts these numbers for inflation to constant 2006 dollars.

(a) Fill in the "Rate of change" columns by finding the change in tuition and fees in dollars per year, over the preceding year.

(b) Fill in the "Annual percentage change" columns by expressing the rate of change as a percentage of the preceding year's tuition (to the nearest percent).

Current dollars

Academic year	Tuition	Rate of change	Annual % change
99–00	$3362	—	—
00–01	$3508	$146	4.3%
01–02	$3766		
02–03	$4098		
03–04	$4645		
04–05	$5126		
05–06	$5492		
06–07	$5836		

Inflation-adjusted dollars

Tuition	Rate of change	Annual % change
$4102	—	—
$4139	$37	0.9%
$4326		
$4624		
$5131		
$5516		
$5702		
$5836		

Source: The College Board, New York, NY.

*References to mathematical ideas are frequent on *The Simpsons.* Professor Sarah Greenwald of Appalachian State University and Professor Andrew Nestler of Santa Monica College maintain a website devoted to "the mathematics of *The Simpsons*": www.mathsci.appstate.edu/~sjg/simpsonsmath/.

2. From the table we see that the cost of tuition has changed from year to year over this ten-year period, in both actual and inflation-adjusted dollars.

(a) Did tuition increase every year over this period?

(b) Did the rate of change increase every year over this period? If not, describe how the rate of change of tuition changed over this period.

3. Consider the actual current dollar data shown in the table.

(a) Over what period was the rate of change of tuition increasing?

(b) Over what period was the rate of change of tuition decreasing?

4. Repeat Question 3 for the inflation-adjusted data.

5. **(a)** Which do you think is a better way of measuring the change in tuition, actual dollars or inflation-adjusted dollars?

(b) Which do you think is a better way of expressing the rate of change of tuition, dollars per year or the percentage change per year?

II. Rates of Change and the Shapes of Graphs

The graphs in Figure 1 show the temperatures in Springfield over a 12-hour period on two different days, starting at midnight. From the graphs we see that a warm front was moving in overnight, causing the temperature to rise.

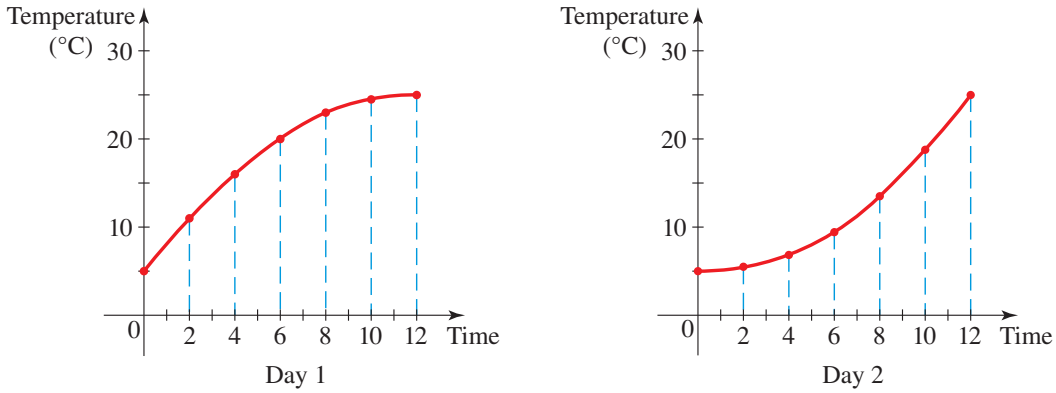


figure **1**

1. Complete the following tables by finding the average rates of change of temperature over consecutive 2-hour intervals. Read the temperature from each graph as accurately as you can.

Temperature on Day 1

Time interval (h)	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
Average rate of change on interval	$\frac{11 - 5}{2 - 0} = 3.0$					

Temperature on Day 2

Time interval (h)	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
Average rate of change on interval	$\frac{5.5 - 5}{2 - 0} = 0.25$					

2. It is obvious from the graphs that the temperature increases on both days. Is the average rate of change of temperature increasing or decreasing on Day 1? On Day 2?

3. Which of the basic shapes in Figure 2 below best describe each of the graphs in Figure 1?



figure **2** Basic shapes

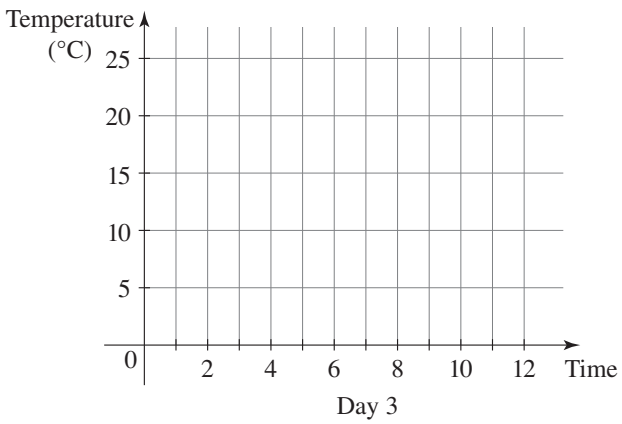
4. Use your answers to Questions 2 and 3 to complete the following table.

	Day 1	**Day 2**
Temperature	Increasing	
Average rate of change	Decreasing	
Shape of graph		

5. On Day 3 a cold front moves into Springfield. The following table shows the temperatures on that day.

(a) Plot the data and connect the points with a smooth curve.

Time	**Temperature (°C)**
0	24
2	23
4	21
6	18
8	14
10	9
12	2



(b) Complete the following table by finding the average rates of change of temperature over consecutive 2-hour intervals.

Temperature on Day 3

Interval time (h)	[0, 2]	[2, 4]	[4, 6]	[6, 8]	[8, 10]	[10, 12]
Average rate of change on interval	$\frac{23 - 24}{2 - 0} = -0.5$					

(c) The temperature is decreasing in this 12-hour period. Is the average rate of change of temperature increasing or decreasing?

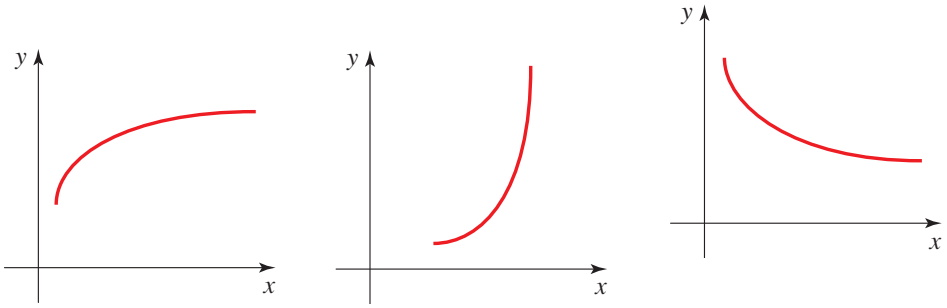
(d) Which of the basic shapes in Figure 2 best describes the graph for Day 3?

6. On Day 4 the temperature also decreases from 24°C to 2°C between midnight and noon. But this time, the rate of change of temperature is increasing. Sketch a rough graph that describes this situation. (You may find it helpful to consider the basic shapes in Figure 2.)

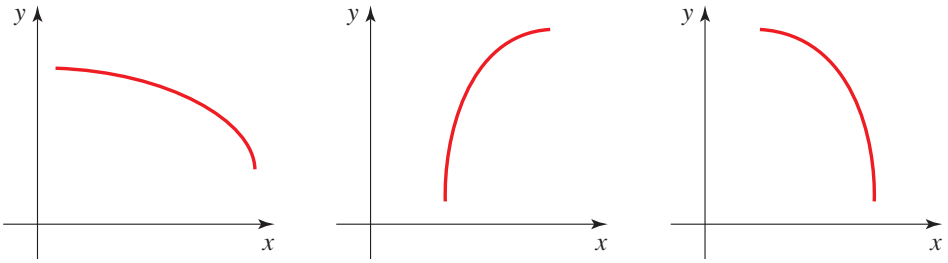
7. Use your answers to Questions 5 and 6 to complete the following table.

	Day 3	**Day 4**
Temperature		
Average rate of change		
Shape of graph		

8. Six different functions are graphed. For each function, determine whether the function is increasing or decreasing and whether the average rate of change is increasing or decreasing.



Function:	Increasing		
Rate of change:	Decreasing		



Function: _____ _____ _____

Rate of change: _____ _____ _____

III. SAT Scores

We began this exploration with a discussion of the change in the rate of change of SAT scores. Let's examine what really happened to SAT scores. The table shows the combined verbal and mathematical SAT scores between 1988 and 2002.

Year	1988	1990	1992	1994	1996	1998	2000	2002
SAT score	1006	1001	1001	1003	1013	1017	1019	1020

Let's examine SAT scores from 1994 to 2002.

1. Did SAT scores decrease or increase in this period?
2. Did the rate of change in SAT scores increase or decrease in this period?
3. Did Homer Simpson's *US of A Today* newspaper report the facts accurately?

2 Linear Patterns

OBJECTIVE To recognize linear data and find linear functions that fit the data exactly.



Data mining

Finding patterns is a basic human urge. Recognizing patterns in our environment helps us to avoid the unexpected and so feel more secure in our daily lives. In fact, psychologists tell us that our need to find patterns is so strong that sometimes we make up patterns that aren't really there!

But finding real patterns in large data sets can be very profitable for those who know how to do it. For example, *Business Week* reported that the Sunnyvale campus of Yahoo employs a team of more than 100 very highly paid mathematicians and computer scientists to sift through Yahoo's immense pool of data. Their goal is to find patterns in the online activity of the over 200 million registered Yahoo users. Yahoo's employees "mine" the data for useful information using methods from a new branch of mathematics called data mining. The patterns they find help the company to be more responsive to its customers' needs.

In this exploration we consider the simplest type of pattern: linear patterns. In later explorations (in Chapters 3–5) we build on what we learned here to find more complex patterns.

I. Properties of Linear Data

One of the main ways of finding patterns in data is to look at the differences between consecutive data points. In this exploration we assume that the inputs are the equally spaced numbers

$$0, 1, 2, 3, \ldots$$

We want to find properties of the outputs which guarantee that there is a linear function that exactly models the data. So let's consider the linear function

$$f(x) = b + mx$$

1. Find the outputs of the linear function $f(x) = b + mx$ corresponding to the inputs 0, 1, 2, 3, ...

$f(0) =$ ___b___, $f(1) =$ _____, $f(2) =$ _____, $f(3) =$ _____

2. Use your answers to Question 1 to find the first differences of the outputs.

$f(1) - f(0) =$ ___m___, $f(2) - f(1) =$ _____, $f(3) - f(2) =$ _____

3. Use your answers to Questions 1 and 2 to confirm that the entries in the first difference table below are correct. Use the pattern you see in the table to fill in the columns for the inputs 4, 5, and 6.

table **1**
First differences for $f(x) = b + mx$

x	0	1	2	3	4	5	6
y	b	$b+m$	$b+2m$	$b+3m$			
First difference	—	m	m	m			

4. A data set is given in the following table.

x	0	1	2	3	4	5	6	7
y	5	9	13	17	21	25	29	33
First difference	—	4						

(a) Fill in the entries for the first differences.

(b) Observe that the first differences are constant, so there is a linear function

$$f(x) = b + mx$$

that models the data. To find b and m, let's compare this data table to Table 1. Comparing the output corresponding to the input 0 in each of these tables, we conclude that

$b =$ _____.

Comparing the first differences in each of these tables, we conclude that

$$m = _____.$$

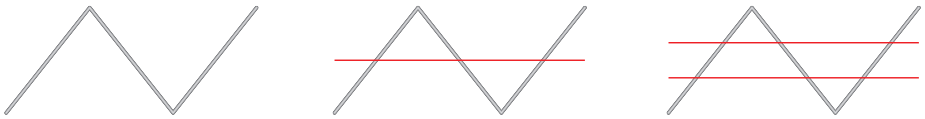
So a linear function that models the data is

$$f(x) = ____ + ____\, x$$

(c) Check that $f(0), f(1), f(2), \ldots$ match the y-values in the data.

II. Linear Patterns

1. A piece of wire is bent as shown in the figure. You can see that one cut through the wire produces four pieces and two parallel cuts produce seven pieces.



(a) Complete the table for the number of pieces produced by parallel cuts, and then calculate the first differences.

Number of cuts x	0	1	2	3	4
Number of pieces y	1				
First difference	—				

(b) Are the first differences constant? Is there a linear pattern relating the number of cuts to the number of pieces? If so, find b and m by comparing the entries in this table with the corresponding ones in Table 1.

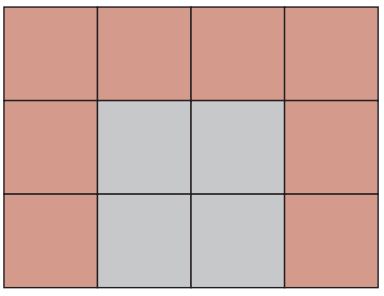
$$b = _____ \qquad m = _____.$$

Find a linear function that models the pattern in the data:

$$f(x) = ____ + ____\, x$$

(c) How many pieces are produced by 142 parallel cuts?

2. Mia wants to make a square patio made of gray tile with a red tile border on three sides. The figure shows a possible patio for which the gray part is a square of side 2.



(a) Complete the table for the number of red border tiles needed when the gray part is a square of side x, and then calculate the first differences.

Square of side x	1	2	3	4
Number of border tiles y				
First difference				

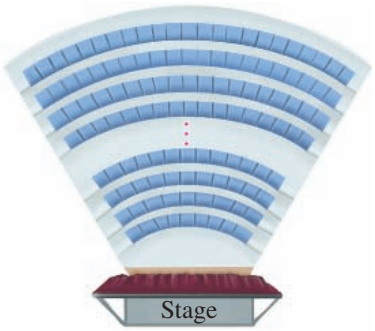
(b) Are the first differences constant? If so, find b and m by comparing the entries in this table with the corresponding ones in Table 1.

$b + m$ _____ $m =$ _____.

Use these equations to find b. Now find a linear function that models the pattern in the data:

$f(x) =$ ____ + ____ x

(c) How many border tiles are needed if the square part is 25 tiles wide?



3. An amphitheater has several rows of seats, with 30 seats in the first row, 32 in the second row, and so on.

(a) Complete the table for the number of seats in each row, and then calculate the first differences.

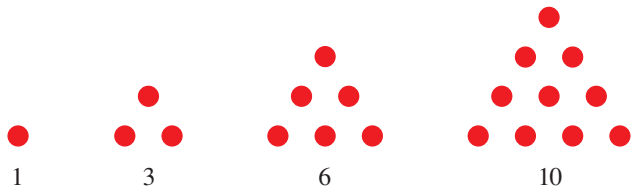
Row number x	1	2	3	4	5
Number of seats y	30				
First difference	—				

(b) Are the first differences constant? If so, find b and m by comparing the entries in this table with the corresponding ones in Table 1; then find a linear function that models the pattern in the data:

$f(x) =$ _____

(c) If the amphitheater has 120 rows, how many seats are in the last row?

4. Let's make triangles out of dots as shown in the figure.



(a) Complete the table for the number of dots in each triangle, and then calculate the first differences.

Triangle number x	1	2	3	4	5	6
Number of dots y	1					
First difference	—					

(b) Are the first differences constant? Is there a linear pattern relating the number of dots in a triangle and the number of dots in its base?

3 Bridge Science

OBJECTIVE To experience the process of collecting data and then analyzing the data using linear regression.

If you want to build a bridge, how can you be sure that it will be strong enough to support the cars that will drive over it? You wouldn't want to just build one and hope for the best. Mathematical models of the forces the bridge is expected to experience can help us determine the strength of the bridge before we actually build it.

The most famous bridge collapse in modern history is that of the first Tacoma Narrows bridge in Washington State. From the day of its opening on July 1, 1940, the bridge exhibited strange behavior. On windy days it would sway up and down; drivers would see approaching cars disappear and reappear as moving dips and humps formed in the roadway. The locals nicknamed the bridge "Galloping Gertie." On the day of its collapse, November 7, 1940, there was a particularly strong wind, and the bridge began to sway violently. Leonard Coatsworth, a driver stranded on the bridge described it this way:

> Just as I drove past the towers, the bridge began to sway violently from side to side. Before I realized it, the tilt became so violent that I lost control of the car. . . . On hands and knees most of the time, I crawled 500 yards or more to the towers. . . . Safely back at the toll plaza, I saw the bridge in its final collapse and saw my car plunge into the Narrows.

To see a video of the bridge collapse, search YouTube for *Tacoma Narrows bridge collapse*.



AP Images

Bridge swaying



AP Images

Bridge collapsing

In this exploration we perform an experiment on simple paper bridges, generating data on the strength of the bridges. We then use linear regression to analyze the data.

I. Setting up the Experiment

In this experiment you will construct bridges out of paper and use pennies as weights to determine how strong each bridge is.

You will need:

- 21 pieces of paper, each 11 by 4.25 inches (half of a regular sheet of paper cut lengthwise)
- 100 pennies (two rolls)
- 2 books that have the same thickness
- 1 small paper cup

Procedure:

1. Fold up a 1-inch strip on both long sides of each piece of paper. These paper "U-beams" will be used to construct the bridges.
2. Suspend one or more U-beams (nested inside each other if you are using more than one) across the space between the books, as shown in the figure, and center the paper cup on the bridge. There should be about 9 inches of space between the books.



3. One by one, put pennies into the cup until the bridge collapses. We'll call the number of pennies that it takes to make the bridge collapse the *load strength* of the bridge.

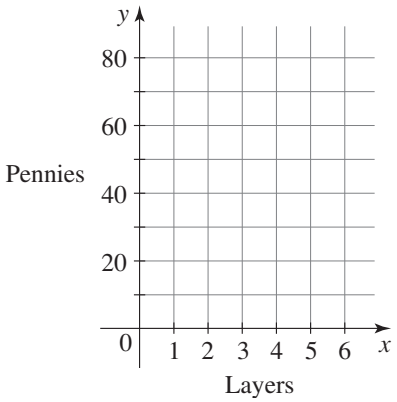
II. Collecting the Data

1. Determine the load strength for bridges that are 1, 2, 3, 4, and 5 layers thick. Make sure you use new U-beams to build each bridge, since U-beams from collapsed bridges will be weaker than new ones.
2. Complete the following table with the load strength data for your bridges.

Layers in bridge	0	1	2	3	4	5	6
Load strength	0						

III. Analyzing the Data

1. Make a scatter plot of your data. Do the data follow an approximately linear trend?



2. Use a graphing calculator to find the regression line for your data.

   Regression line: $y =$ _____

3. Graph the regression line on your scatter plot above.
4. What is the slope of the regression line, and what does it tell us?
5. Use the regression line to predict the load strength of a bridge with six layers.
6. Build a bridge with six layers, and find its load strength. How does this compare to the prediction from the regression line?

4 Correlation and Causation

OBJECTIVE To understand real-life examples in which two variables are mathematically correlated but changes in one variable do not necessarily *cause* changes in the other.

If two variables are correlated, it does not necessarily mean that a change in one variable causes a change in the other. For example, the mathematician John Allen Paulos points out that shoe size is strongly correlated to mathematics scores among school children. Does this mean that we should stretch children's feet to improve their mathematical ability? Certainly not—both shoe size and math skills increase independently with age. We refer to "age" here as the *hidden variable*; increasing age is the real reason that shoe size and mathematical ability increase simultaneously. Here are some more examples of the hidden variable phenomenon.

Do churches cause murder? In the 1980s several studies used census data to show that the more churches a city has, the more murders occur in the city each year. With tongue in cheek, the authors claimed to have proved that the presence of churches increases the prevalence of murders. While the data were correct, the conclusion was obviously nonsense. What is the hidden variable here? Larger cities tend

to have more of everything, including both churches and murders, so the hidden variable is population size.



Vladislav Gurfinkel/Shutterstock.com 2009

More churches . . .



Jack Dagley/Shutterstock.com 2009

More crime?

Do video games cause violent behavior? Some studies have shown that playing violent video games and aggressive behavior are strongly correlated. While it is certainly possible that violent video games might cause desensitization to actual aggressive behavior, could there be a hidden variable here as well? Some people are innately more violent than others, quicker to lose their temper or lash out. Perhaps these inborn tendencies cause both an interest in violent video games and a tendency toward real-life violence. More research is needed to settle these questions.

It is important not to jump to conclusions. Correlation and causation are not the same thing. Correlation is a useful tool for bringing important cause-and-effect relationships to light, but to prove causation, we must explain the mechanism by which one variable affects the other. For example, the link between smoking and lung cancer was observed as a correlation long before medical science determined how the toxins in tobacco smoke actually cause lung cancer.

I. Hidden Variables

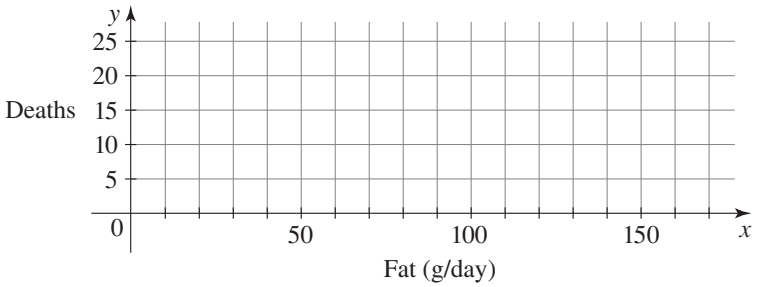
1. A public health student gathers data on bottled water use in her state. Her data indicate that households that use bottled water have healthier children than households that don't. She concludes that drinking bottled water instead of tap water helps to prevent childhood diseases. Do you think her conclusion is valid, or is there likely to be a hidden variable that accounts for this correlation? Write a short paragraph to explain your reasoning.

2. The residents of a seaside town have noticed that on days when the local ice cream parlor is busy, a lot of people go swimming in the ocean. They wonder whether going swimming causes people to crave ice cream or whether eating ice cream makes people want to go swimming. Which alternative is correct? Or does a hidden variable cause both phenomena? Explain your answer.

3. A study investigating the connection between dietary fat and cancer in various countries came up with the data shown in the table, relating daily fat intake (in grams) and annual cancer death rates (in deaths per 100,000 population).

Country	Fat (g/day)	Cancer deaths
Austria	120	18
Denmark	160	23
El Salvador	40	1
Greece	95	7
Hungary	105	14
Norway	130	17
Poland	92	10
Portugal	74	13
United States	150	20

(a) Make a scatter plot of the data. Do they appear to be linearly correlated?



(b) Does it seem reasonable to conclude that a high-fat diet causes cancer? What else would you need to know to settle the question?

(c) In the 35 years since the study was conducted, no widely accepted mechanism has been discovered to explain how eating fat would cause cancer. Can you think of any hidden variable that would account for this correlation?

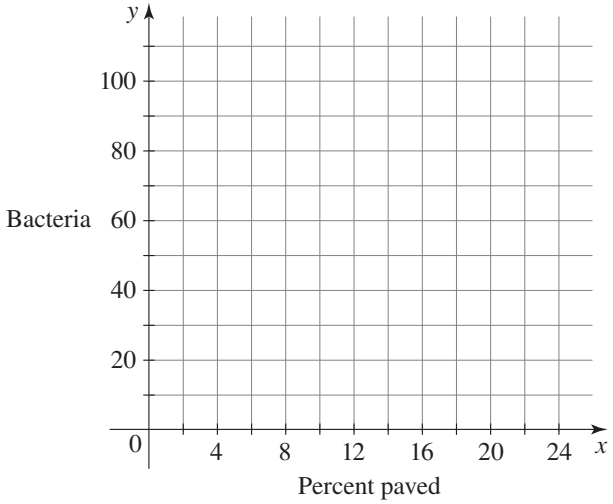
4. A researcher wants to test the hypothesis that paving over ground surfaces causes an increase in fecal bacteria in groundwater. He studies six creeks in Maryland and gathers the following data relating the percentage of each creek's watershed that is covered with pavement and the average fecal coliform bacteria count (in units/100 mL).

Paved	Bacteria
7%	18
9%	22
14%	40
18%	65
21%	80
22%	92

(a) Make a scatter plot of the data on the coordinate plane provided on the next page. Do the data appear to be linearly correlated?

(b) Can you think of any reasons why the presence of pavement might increase the bacteria count?

(c) Some researchers don't believe that increasing paved surface area increases fecal bacteria counts. Can you think of a hidden variable that might cause both increases?



5. From your own experience or your reading, think of two variables that are correlated but for which the correlation is caused by some third hidden variable. Write a short paragraph describing the correlation and explaining how the hidden variable is the true reason for the correlation.

II. Are We Measuring the Right Thing?

Just because not every correlation is evidence of causation doesn't mean that we should be too skeptical about studies that use correlation to link two variables. But sometimes people get the direction of causation wrong; instead of concluding correctly that A causes B, they conclude wrongly that B causes A. Here are some examples of this type of thinking.

1. Some diet books recommend that people who want to lose weight should not drink diet soda, because studies have shown that diet soda drinkers tend to be heavier than regular soda drinkers. Does this really mean that drinking diet soda causes weight gain? How do you explain these studies?

2. Inhabitants of the New Hebrides used to believe that body lice improved their health. They correctly observed that healthy people had lice and sick people didn't. Do you think their belief was reasonable? What do you think might account for their observation? (Note that lice don't like high body temperatures.)

5 Fair Division of Assets

OBJECTIVE To understand the real-life problem of distributing assets fairly among several claimants and how linear equations can help solve such problems.

When companies or individuals cannot meet their financial obligations they can sometimes file for bankruptcy. Under the Chapter 7 bankruptcy rules, all debts are terminated, and the company's assets are divided among its creditors. The primary goal of the legal process of dividing the assets among the claimants is fairness. This



process can be quite complicated because there are different ways of interpreting "fairness." One of the largest bankruptcies in U.S. history was that of Worldcom in 2002. The telecommunications giant, which operated the world's largest Internet network, had $107 billion in assets and $41 billion in debt, owed to over 1000 creditors. As you can imagine, restructuring such a huge debt among so many creditors is a complicated process.

In this Exploration we see how algebra can be used to analyze different possible ways of dividing assets fairly among several claimants.

I. Dividing Assets Fairly

When high-tech Company C goes bankrupt, it owes $120 million to Company A and $480 million to Company B. Unfortunately, Company C has only $300 million in assets. How should the court divide these assets between Companies A and B? Here are some possible ways:

- Companies A and B divide the assets equally.
- Companies A and B share the losses equally.
- Companies A and B each get the same fixed percentage of what they are owed.
- Companies A and B get an amount proportional to what they are owed.

Let's explore these alternatives to see which is fairest.

1. Suppose Companies A and B divide the $300 million equally.

(a) How much does each company receive?

A gets: _____ B gets: _____

(b) After the assets have been distributed, what is the net loss for each company?

A's net loss: _____ B's net loss: _____

(c) Do you think this is a fair method? Explain.

2. Suppose Companies A and B share the losses equally; that is, each company loses the same amount.

(a) Let x be the amount Company A receives and y be the amount Company B receives when the \$300 million are distributed. What is the net loss for each company?

A's net loss: _____ B's net loss: _____

(b) To find x and y, we need to solve two equations. What does the fact that the assets total \$300 million tell us about x and y?

First equation: $x + y =$ _____

(c) What does the fact that the net losses in part (a) are equal tell us about x and y?

Second equation: _____ = _____

(d) Write the equation from parts (b) and (c) in the form $y = b + mx$.

First equation: $y =$ _____

Second equation: $y =$ _____

(e) Find where the lines in part (d) meet. How much does each company receive?

A gets: _____ B gets: _____

(f) One of the values in part (e) is negative. What does this mean? What would the company getting the negative amount have to do?

(g) Do you think this is a fair method? Explain.

3. Suppose Companies A and B get the same fixed percentage of what they are owed.

(a) Let x be the percentage of what they are owed. How much does each company receive?

A gets: $\frac{x}{100} \times$ _____ B gets: $\frac{x}{100} \times$ _____

(b) The total amount to be distributed to Companies A and B is \$300 million. Use this fact together with the amounts in part (a) to write an equation involving x.

_____ + _____ = 300

(c) Solve for the percentage x.

$x =$ _____

(d) Use the percentage x to calculate how much each company receives.

A gets: $\frac{\square}{100} \times$ _____ = _____ B gets: $\frac{\square}{100} \times$ _____ = _____

(e) Do you think this is a fair method? Explain.

4. Suppose Companies A and B get an amount proportional to what they are owed.

(a) What is the total amount T owed to Companies A and B together?

$$T = _____ + _____ = _____$$

(b) What proportion of the total claim T is owed to Company A? To Company B?

$$\text{A's proportion: } \frac{\square}{T} = _____$$

$$\text{B's proportion: } \frac{\square}{T} = _____$$

(c) Let's divide the \$300 million between Companies A and B according to the proportions in part (b).

$$\text{A gets: } _____ \times 300 = _____$$

$$\text{B gets: } _____ \times 300 = _____$$

(d) Do you think this is a fair method? Explain.

5. Notice that the amounts distributed to Companies A and B in Questions 3 and 4 are the same. Is this a coincidence, or are the methods really the same? Give reasons for your answer.

6. Can you think of another fair way to divide the assets between Companies A and B? Is your method fairer than the methods already described? Explain.

II. Profit Sharing

Anita and Karim pool their savings to start a business that imports wicker furniture. Anita invests \$2.6 million, and Karim invests \$1.4 million. After three years of successful operation, the business is sold to a national chain of furniture stores for \$6.4 million. How should Anita and Karim divide the \$6.4 million? Here are several possibilities:

- They divide the \$6.4 million equally.
- Each gets their original investment back, and they share the profit equally.
- Each gets a fraction of the \$6.4 million proportional to the amount they invested.
- Each gets their original investment back plus a fraction of the profit proportional to the amount they invested.

1. Investigate each of these methods to see which alternative is most fair.

2. You should have found that the last two methods result in Anita and Karim receiving the same amount. Explain why the two methods are really the same.

3. How do you think partnerships like the one described here normally divide their profits in the real world?

3 Exponential Functions and Models



© David Iliff; (inset) MPI/Getty Images

3.1 Exponential Growth and Decay
3.2 Exponential Models: Comparing Rates
3.3 Comparing Linear and Exponential Growth
3.4 Graphs of Exponential Functions
3.5 Fitting Exponential Curves to Data

EXPLORATIONS
1 Extreme Numbers: Scientific Notation
2 So You Want to Be a Millionaire?
3 Exponential Patterns
4 Modeling Radioactivity with Coins and Dice

Population explosion? Cities and towns all over the world have experienced huge increases in population over the past century. The above photos of Hollywood in the 1920s and in 2000 tell the population story quite dramatically. But to find out where population is really headed, we need a mathematical model. Population grows in much the same way as money grows in a bank account: The more money in the account, the more interest is paid. In the same way, the more people there are in the world, the more babies are born. This type of growth is modeled by exponential functions, the topic of this chapter. According to some exponential models, world population will double in successive 40-year periods, ending in a population explosion. However, a different exponential model, which takes into account the limited resources available for growth, predicts that world population will eventually stabilize at a supportable level (see Exercise 33 on page 285).

3.1 Exponential Growth and Decay

- An Example of Exponential Growth
- Modeling Exponential Growth: The Growth Factor
- Modeling Exponential Growth: The Growth Rate
- Modeling Exponential Decay

IN THIS SECTION... we find functions that model growth and decay processes (such as population growth or radioactive decay). We use these models to predict future trends. The functions we use are called *exponential* functions because the independent variable is in the exponent.

GET READY... by reviewing the properties of exponents in **Algebra Toolkits A.3** and **A.4**. Test your understanding by doing the **Algebra Checkpoint** on page 255.

In Chapter 2 we studied linear functions, that is, functions that have constant rates of change. We learned that linear functions can be used to model many real-world situations. But there are important real-world phenomena in which the rate of change is not constant. For example, the rate at which population grows is not constant; we can easily see that the larger the population, the greater the number of offspring and hence the more quickly the population grows. In the same way the larger your bank account, the more interest you earn. Both these types of growth exhibit a phenomenon called *exponential growth*. In this section we develop the kinds of functions, called *exponential functions*, that model this type of growth.

An Example of Exponential Growth



Sebastian Kaulitzki/Shutterstock.com 2009

Streptococcus A bacterium (12,000×)

Bacteria are the most common organisms on the earth. Many bacterial species are beneficial to humans; for example, they play an essential role in the digestive process and in breaking down waste products. But some types of bacteria can be deadly. For example, the *Streptococcus A* bacterium can cause a variety of diseases, including strep throat, pneumonia, and other respiratory illnesses as well as *necrotic fasciitis* ("flesh-eating" disease). Although bacteria are invisible to the naked eye, their huge impact in our world is due to their ability to multiply rapidly. Under ideal conditions the *Streptococcus A* bacterium can divide in as little as 20 minutes. So an infection of just a few bacteria can soon grow to such large numbers as to overwhelm the body's natural defenses. How does such a massive infection happen?

Suppose that a person becomes infected with 10 streptococcus bacteria from a sneeze in a crowded room. Let's monitor the progress of the infection. If each bacterium splits into two bacteria every hour, then the population doubles every hour. So after one hour the person will have 10×2 or 20 bacteria in his body. After another hour the bacteria count doubles again, to 40, and so on. Of course, doubling the number of bacteria means multiplying the number by 2. So if we start

table **1**
Bacterial growth

Time (hours)	Number of bacteria	Number of bacteria
0	10	10
1	$10 \cdot 2^1$	20
2	$10 \cdot 2^2$	40
3	$10 \cdot 2^3$	80
4	$10 \cdot 2^4$	160
5	$10 \cdot 2^5$	320
6	$10 \cdot 2^6$	640
7	$10 \cdot 2^7$	1280
8	$10 \cdot 2^8$	2560
9	$10 \cdot 2^9$	5120
10	$10 \cdot 2^{10}$	10,240
11	$10 \cdot 2^{11}$	20,480
12	$10 \cdot 2^{12}$	40,960
13	$10 \cdot 2^{13}$	81,920
14	$10 \cdot 2^{14}$	163,840
15	$10 \cdot 2^{15}$	327,680
16	$10 \cdot 2^{16}$	655,360
17	$10 \cdot 2^{17}$	1,310,720
18	$10 \cdot 2^{18}$	2,621,440
19	$10 \cdot 2^{19}$	5,242,880
20	$10 \cdot 2^{20}$	10,485,760
21	$10 \cdot 2^{21}$	20,971,520
22	$10 \cdot 2^{22}$	41,943,040
23	$10 \cdot 2^{23}$	83,886,080
24	$10 \cdot 2^{24}$	167,772,160

with 10 bacteria, the number of bacteria after the first, second, and third hours are as follows.

Time	Number of bacteria
1 hour	$10 \times 2 = 10 \cdot 2^1$
2 hours	$10 \times 2 \times 2 = 10 \cdot 2^2$
3 hours	$10 \times 2 \times 2 \times 2 = 10 \cdot 2^3$

Table 1 shows the number of bacteria at one-hour intervals for a 24-hour period. We can see from the table that the bacterial population increases more and more rapidly over the course of the day. What type of a function can we use to model such growth? From the second column in the table we see that the population P after x hours is given by

$$P = 10 \cdot 2^x \qquad \text{Model}$$

This is called an *exponential function*, because the independent variable x is the exponent. Let's use this function to estimate the number of bacteria after another half day has passed (36 hours in all). Replacing x by 36 in the model, we get $P = 10 \cdot 2^{36} \approx 6.87 \times 10^{11}$. This is almost a trillion bacteria. It's no wonder that unchecked infections can quickly cause serious damage to the human body.

In Figure 1 we sketch a graph of the bacteria population P for x between 0 and 5; it would be difficult to draw a graph (to scale) for x between 0 and 24. Do you see why?

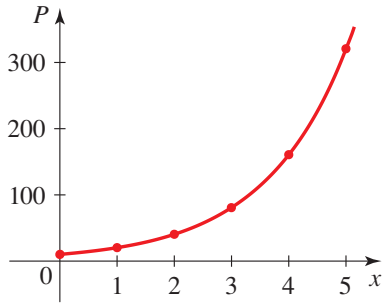


figure **1** Graph of bacteria population

Modeling Exponential Growth: The Growth Factor

In the preceding discussion we modeled a bacteria population by the function $f(x) = 10 \cdot 2^x$, where 10 is the *initial population*. The base 2 is called the *growth factor* because the population is multiplied by the factor 2 in every time period. (In this example the time period is one hour.)

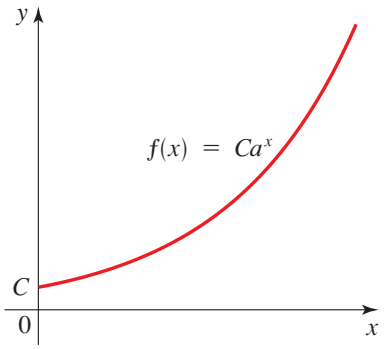
Exponential Growth Models: The Growth Factor

Exponential growth is modeled by a function of the form

$$f(x) = Ca^x \qquad a > 1$$

- The variable x is the number of **time periods**.
- The base a is the **growth factor**, the factor by which $f(x)$ is multiplied when x increases by one time period.
- The constant C is the **initial value** of f (the value when x is 0).
- The graph of f has the general shape shown.

$f(0) = C \cdot a^0 = C$



example 1 Finding Models for Exponential Growth

Find a model for the following exponential growth situations.

(a) A bacterial infection starts with 100 bacteria and triples every hour.

(b) A pond is stocked with 5800 fish, and each year the fish population is multiplied by a factor of 1.2.

Solution

Since these populations grow exponentially, we are looking for a model of the form $f(x) = Ca^x$.

(a) The initial population is 100, so C is 100. The population triples every hour, so the one-hour growth factor is 3; that is, a is 3. Thus the model we seek is

$$f(x) = 100 \cdot 3^x$$

where x is the number of time periods (hours) since infection.

(b) The initial population is 5800, so C is 5800. The population is multiplied by 1.2 every year, so the one-year growth factor is 1.2; that is, a is 1.2. Thus the model we seek is

$$f(x) = 5800(1.2)^x$$

where x is the number of years (time periods) since the pond was stocked.

NOW TRY EXERCISES 25 AND 33

Suppose a population is modeled by $f(x) = Ca^x$. The growth factor a is the number by which the population is multiplied in every time period x. So increasing

x by one time period, we get $f(x + 1) = a \cdot f(x)$, or

$$a = \frac{f(x + 1)}{f(x)}$$

Let's write this last formula in words:

$$\text{growth factor} = \frac{\text{population after } x + 1 \text{ time periods}}{\text{population after } x \text{ time periods}}$$

If we know the population at two different times, we can use this formula to find the growth factor, as the next example illustrates.

example 2 Finding the Growth Factor

A chinchilla farm starts with 20 chinchillas, and after 3 years there are 128 chinchillas. Assume that the number of chinchillas grows exponentially. Find the 3-year growth factor.

Solution

From the discussion preceding this example we have

$$\text{3-year growth factor} = \frac{\text{population in year 3}}{\text{population in year 0}} = \frac{128}{20} = 6.4$$

■ NOW TRY EXERCISES 21 AND 39 ■

■ Modeling Exponential Growth: The Growth Rate

The **growth rate** of a population is the proportion of the population by which it increases during one time period. So if the population after x time periods is $f(x)$, then the growth rate is

$$r = \frac{f(x + 1) - f(x)}{f(x)}$$

The growth rate r is expressed as a decimal. If the population increases by 40% per time period, then the growth rate is $r = 0.40$.

For instance, let's suppose that a certain population increases by 40% each 1-year time period and that $f(x)$ is the population after x years (time periods). Then after one more time period the population is

$$\begin{aligned} f(x + 1) &= (\text{population at } x \text{ years}) + (40\% \text{ of the population}) \\ &= f(x) + 0.40\, f(x) && 40\% \text{ of } f(x) \text{ is } 0.40\, f(x) \\ &= (1 + 0.40)\, f(x) && \text{Distributive Property} \\ &= 1.40\, f(x) && \text{Simplify} \end{aligned}$$

So $f(x + 1) = 1.40\, f(x)$, and thus the growth factor is $1 + 0.40 = 1.40$. In general, for an exponential growth model $f(x) = Ca^x$ we have

$$r = \frac{f(x + 1) - f(x)}{f(x)} = \frac{f(x + 1)}{f(x)} - 1 = \frac{Ca^{x+1}}{Ca^x} - 1 = a - 1$$

So $r = a - 1$ and $a = 1 + r$.

The Growth Factor and the Growth Rate

For an exponential growth model, the growth factor a is greater than 1. The growth rate r is *positive* and satisfies

$$a = 1 + r \qquad r > 0$$

example 3

An Exponential Growth Model for a Rabbit Population

Fifty rabbits are introduced onto a small uninhabited island. They have no predators, and food is plentiful on the island, so the population grows exponentially, increasing by 60% each year.

(a) Find a function P that models the rabbit population after x years.

(b) How many rabbits are there after 8 years?

Solution

(a) The population grows by 60% each year, so the one-year growth rate r is 0.60. This means that the one-year growth factor is

$$a = 1 + r = 1 + .60 = 1.60$$

Since the initial population C is 50, the exponential growth model $P(x) = Ca^x$ that we seek is

$$P(x) = 50 \cdot (1.60)^x \qquad \text{Model}$$

where x is measured in years.

(b) Replacing x by 8 in our model, we get

$$P(8) = 50 \cdot (1.60)^8 \qquad \text{Replace } x \text{ by } 8$$

$$\approx 2147.48 \qquad \text{Calculator}$$

So after 8 years there are about 2147 rabbits on the island.

NOW TRY EXERCISE 35

Modeling Exponential Decay

In Exploration 4 of Chapter 6, page 563, we use a "surge function" to model the amount of a drug in the body from the moment it is ingested. Here, we model the decay part of the surge function.

When a patient is injected with a drug, the concentration of the drug in the blood typically reaches a maximum quickly, but then as the liver metabolizes the drug, the amount remaining in the bloodstream decreases exponentially. For example, suppose a patient is injected with 10 mg of a therapeutic drug. It is known that 20% of the drug is expelled by the body each hour, so after one hour 80% of the drug remains in the body. The table below shows the amount of the drug remaining at the end of 1, 2, and 3 hours.

Time	Amount remaining
1 hour	$10 \times 0.80 = 10 \cdot (0.80)^1$
2 hours	$10 \times 0.80 \times 0.80 = 10 \cdot (0.80)^2$
3 hours	$10 \times 0.80 \times 0.80 \times 0.80 = 10 \cdot (0.80)^3$

From the pattern in the table we see that the amount remaining after x hours is modeled by

$$m(x) = 10 \cdot (0.80)^x$$

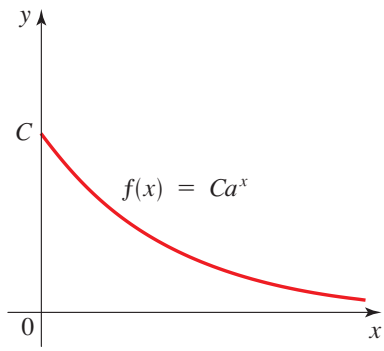
This model has the same form as the model for exponential growth, except that the "growth factor" (0.80) is less than 1, so the values of the function get *smaller* (instead of larger) as time increases. Also, the "growth rate" is $r = a - 1 = 0.80 - 1 = -0.20$. The negative growth rate -0.20 indicates that 20% of the drug is *subtracted* from (instead of added to) the body. We use the terms *exponential decay*, *decay factor*, and *decay rate* to describe this situation.

Exponential Decay Models

Exponential decay is modeled by a function of the form

$$f(x) = Ca^x \qquad 0 < a < 1$$

- The variable x is the number of **time periods**.
- The **decay factor** is a, where a is a positive number less than 1.
- The **decay rate** r satisfies $a = 1 + r$, so the decay rate is a *negative* number.
- The graph of f has the general shape shown.



example 4

An Exponential Decay Model for a Medication

A patient is administered 75 mg of a therapeutic drug. It is known that 30% of the drug is expelled from the body each hour.

(a) Find an exponential decay model for the amount of drug remaining in the patient's body after x hours.

(b) Use the model to predict the amount of the drug that remains in the patient's body after 4 hours.

Solution

(a) The amount of the drug decreases by 30% each hour, so the decay rate r is -0.30. This means that the decay factor is

$$a = 1 + r = 1 + (-0.30) = 0.70$$

Since the initial amount C is 75, the exponential decay model $m(x) = Ca^x$ that we seek is

$$m(x) = 75 \cdot (0.70)^x \qquad \text{Model}$$

where x is the number of hours since the drug was administered.

(b) Replacing x by 4 in our model, we get

$$m(4) = 75 \cdot (0.70)^4 \qquad \text{Replace } x \text{ by } 4$$
$$\approx 18.008 \qquad \text{Calculator}$$

So after 4 hours approximately 18 mg of the drug remains.

NOW TRY EXERCISE 41

The half-lives of **radioactive elements** vary widely.

Element	Half-life
Thorium-232	14.5 billion years
Uranium-235	4.5 billion years
Thorium-230	80,000 years
Plutonium-239	24,360 years
Carbon-14	5,730 years
Radium-226	1,600 years
Cesium-137	30 years
Strontium-90	28 years
Polonium-210	140 days
Thorium-234	25 days
Iodine-135	8 days
Radon-222	3.8 days
Lead-211	3.6 minutes
Krypton-91	10 seconds

Radioactive elements decay when their atoms spontaneously emit radiation and become smaller, stable atoms. For instance, uranium-238 emits alpha and beta particles (two common types of radiation) and decays into nonradioactive lead. Physicists express the rate of decay in terms of **half-life**, the time required for half the mass of the radioactive substance to decay. The half-life of radium-226 is 1600 years, so a 100-g sample decays to 50 g (or $\frac{1}{2} \times 100$ g) in 1600 years. This means that the 1600-year decay factor of radium-226 is $\frac{1}{2}$.

example 5 An Exponential Decay Model for Radium

The half-life of radium-226 is 1600 years. A 50-gram sample of radium-226 is placed in an underground disposal facility and monitored.

(a) Find a function that models the mass $m(x)$ of radium-226 remaining after x half-lives.

(b) Use the model to predict the amount of radium-226 remaining after 4000 years.

(c) Make a table of values for $m(x)$ with x varying between 0 and 5.

(d) Graph the function $m(x)$ and the entries in the table. What does the graph tell us about how radium-226 decays.

Solution

(a) The initial mass is 50 g, and the decay factor is

$$a = \tfrac{1}{2} \qquad \text{1600-year decay factor}$$

So the model $m(x) = Ca^x$ that we seek is

$$m(x) = 50 \cdot \left(\tfrac{1}{2}\right)^x$$

where x is the number of 1600-year time periods.

(b) To find the number of time periods that 4000 years represents, we divide:

$$\frac{4000}{1600} = 2.5$$

So 4000 years represents 2.5 time periods. Replacing x by 2.5 in the model, we get

$$\begin{aligned} m(x) &= 50\left(\tfrac{1}{2}\right)^x && \text{Model} \\ &= 50\left(\tfrac{1}{2}\right)^{2.5} && \text{Replace } x \text{ by } 2.5 \\ &\approx 8.84 && \text{Calculator} \end{aligned}$$

So the mass remaining after 4000 years is about 8.84 g.

(c) We use a calculator to evaluate the entries in the table in the margin.

x	$m(x)$
0	50
1	25
2	12.5
3	6.25
4	3.125
5	1.562

(d) A graph of the function m and the data points in the table are shown in Figure 2. From the graph we see that radium-226 decays rapidly at first but that the rate of decay slows down as time goes on.

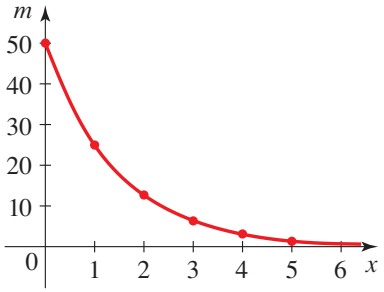


figure **2**
Graph of $m(x) = 50 \cdot \left(\frac{1}{2}\right)^x$

NOW TRY EXERCISE 45

We see from Example 5 that for an initial amount C of a radioactive substance, the model

$$m(x) = C\left(\tfrac{1}{2}\right)^x$$

gives the amount remaining after x half-lives.

Algebra Checkpoint ✓

Check your knowledge of the rules of exponents by doing the following problems. You can review the rules of exponents in **Algebra Toolkits A.3** and **A.4** on pages T14 and T20.

1. Evaluate each expression.

(a) $5^3 \cdot 5^7$ **(b)** $4^3 \cdot 4^{-5}$ **(c)** $\dfrac{10^8}{10^3}$ **(d)** $(5^6)^3$

2. Simplify each expression.

(a) x^4x^3 **(b)** $(5x^3)^2$ **(c)** $\dfrac{z^5}{z^3}$ **(d)** $\dfrac{y^6}{y^{-2}}$

3. Write each expression using exponents.

(a) $\sqrt{3}$ **(b)** $\dfrac{1}{\sqrt{3}}$ **(c)** $\sqrt[3]{5}$ **(d)** $\sqrt[3]{5^4}$

4. Write each expression using radicals.

(a) $5^{1/2}$ **(b)** $7^{-1/2}$ **(c)** $6^{1/3}$ **(d)** $6^{2/3}$

5. Evaluate each expression without using a calculator.

(a) $\sqrt[4]{16}$ **(b)** $\sqrt{2}\sqrt{18}$ **(c)** $8^{2/3}$ **(d)** $\left(\frac{25}{64}\right)^{1/2}$

6. Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

(a) $x^{1/3}x^{2/3}$ **(b)** $\dfrac{x}{x^{1/2}}$ **(c)** $(27x^3)^{2/3}$ **(d)** $(x^6y^9)^{-1/3}$

3.1 Exercises

CONCEPTS

Fundamentals

1. The number of bacteria in a culture at time t (in hours) is modeled by $N(t) = 200(1.8)^t$.

(a) The initial bacteria population is _____.

(b) The growth factor is _____.

(c) The growth rate is _____ (in decimal form).

(d) The growth rate is _____ (in percentage form).

2. The amount (in grams) of a radioactive substance remaining at time t (years) is modeled by $f(t) = 100(0.4)^t$.

(a) The initial amount is _____.

(b) The decay factor is _____.

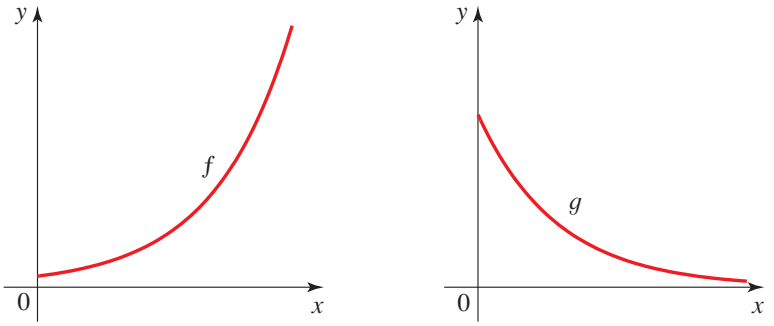
(c) The decay rate is _____ (in decimal form).

(d) The decay rate is _____ (in percentage form).

3. Graphs of exponential functions f and g are shown.

(a) The function f models exponential _____ (growth/decay).

(b) The function g models exponential _____ (growth/decay).



4. **(a)** The function $f(x) = 50 \cdot (0.25)^x$ models exponential _____ (growth/decay).

(b) The function $f(x) = 200 \cdot (1.1)^x$ models exponential _____ (growth/decay).

Think About It

5. Population A increases by 10% each year, and Population B is multiplied by 1.10 each year. Compare the growth factors and growth rates of each population. Are they the same? Explain why or why not.

6. Population A is modeled by $P(x) = 200 \cdot \left(\frac{1}{3}\right)^x$, and Population B is modeled by $Q(x) = 200 \cdot 3^{-x}$. Explain why both populations decay exponentially. What are the decay factor and the decay rate for each population?

SKILLS

7. Which of the following are not models of exponential growth?

(a) $f(x) = 100 \cdot 7^x$ **(b)** $P(x) = x^{10}$ **(c)** $h(t) = 100x$ **(d)** $Q(x) = 4(0.5)^x$

8. Which of the following are not models of exponential decay?

(a) $f(x) = 100 \cdot \left(\frac{1}{5}\right)^x$ **(b)** $h(x) = 100 - x$ **(c)** $Q(x) = 10(0.3)^x$ **(d)** $P(x) = (0.5x)^4$

9–16 ■ An exponential growth or decay model is given.

(a) Determine whether the model represents growth or decay.

(b) Find the growth or decay factor.

(c) Find the growth or decay rate.

9. $f(x) = 50 \cdot (1.2)^x$

10. $g(t) = 20 \cdot (1.9)^t$

11. $y = \left(\frac{1}{3}\right)^x$

12. $h(T) = (0.15)^T$

13. $R(x) = 11(0.25)^x$

14. $Q(x) = 8 \cdot (2.5)^x$

15. $n(x) = 39 \cdot (2.8)^x$

16. $m(t) = 3.5(0.7)^t$

17–20 ■ Match the exponential function with its graph.

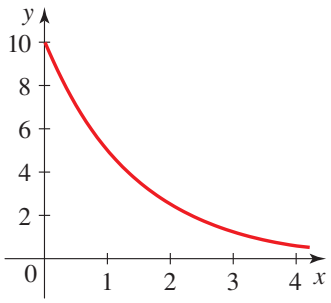
17. $f(x) = 10 \cdot 2^x$

18. $f(x) = 10 \cdot (0.5)^x$

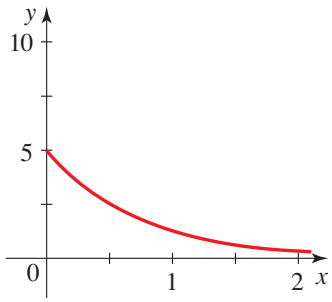
19. $f(x) = 5 \cdot \left(\frac{1}{4}\right)^x$

20. $f(x) = 5(1.4)^x$

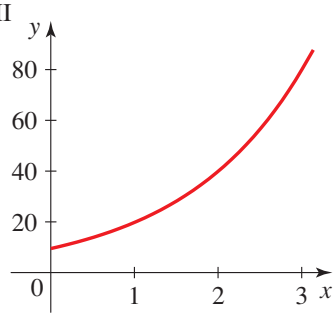
I



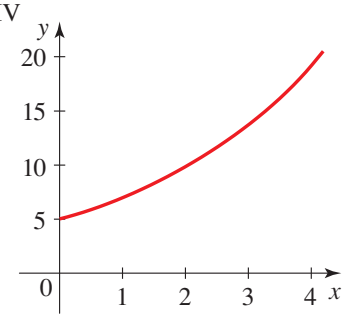
II



III



IV



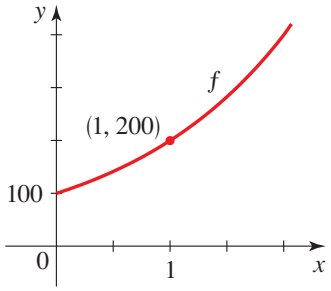
21–22 ■ A population P is initially 3750 and six hours later reaches the given number. Find the six-hour growth or decay factor.

21. **(a)** 7250 **(b)** 5000 **(c)** 2500 **(d)** 750

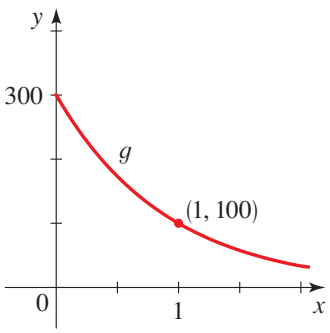
22. **(a)** 5800 **(b)** 9600 **(c)** 1875 **(d)** 1250

23–24 ■ The graph of a function that models exponential growth or decay is shown, where x represents one-year time periods. Find the initial population and the one-year growth factor.

23.



24.



25–34 ■ A population P is initially 350. Find an exponential growth model in terms of the number of time periods x if in each time period the population P

25. quadruples.

26. doubles.

27. decreases by $\frac{1}{2}$.

28. decreases by $\frac{1}{3}$.

29. decreases by 35%.

30. increases by 100%.

31. increases by 300%.

32. decreases by 2%.

33. is multiplied by 1.7.

34. is multiplied by 2.3.

CONTEXTS



Laurence Gough/Shutterstock.com 2009

35. Bacterial Infection The bacteria *Streptococcus pneumoniae* (*S. pneumoniae*) is the cause of many human diseases, the most common being pneumonia. A culture of these bacteria initially has 10 bacteria and increases at a rate of 26% per hour.

(a) Find the hourly growth factor a, and find an exponential model $f(t) = Ca^t$ for the bacteria count in the sample, where t is measured in hours.

(b) Use the model you found to predict the number of bacteria in the sample after 5 hours.

36. Bacterial Infection The bacteria *R. sphaeroides* is the cause of the disease chlamydia. A culture of these bacteria initially has 25 bacteria and increases at a rate of 28% per hour.

(a) Find the hourly growth factor a, and find an exponential model $f(t) = Ca^t$ for the bacteria count in the sample, where t is measured in hours.

(b) Use the model you found to predict the number of bacteria in the sample after 7 hours.

37. Cost-of-Living Increase Mariel teaches science at a community college in Arizona. Her starting salary in 2003 was $38,900, and each year her salary increases by 2%.

(a) Complete the table of Mariel's salary.

Year	Years since 2003	Salary ($)
2003	0	38,900
2004		
2005		
2006		
2007		

(b) Find an exponential growth model for Mariel's salary t years after 2003. What is the growth factor?

(c) Use the model found in part (b) to predict Mariel's salary in 2010, assuming that her salary continues to increase at the same rate.

(d) Use a graphing calculator to graph the model you found in part (b) together with a scatter plot of the data in the table.

38. Investment Value Kerri invested $2000 on January 1, 2005, in a mutual fund that for the past 5 years had a yearly increase of 5%. Assume that the mutual fund continues to grow at 5% each year.

(a) Complete the following table for the investment value at the beginning of each year.

Year	Years since 2005	Investment value ($)
2005	0	2000
2006		
2007		
2008		

(b) Find an exponential growth model for Kerri's investment t years since 2005. What is the growth factor?

(c) Use the model found in part (b) to predict Kerri's investment value in 2010.



(d) Use a graphing calculator to graph the model you found in part (b) together with a scatter plot of the data in the table.



39. Health-Care Spending The Centers for Medicare and Medicaid Services report that health-care expenditures per capita were $2813 in 1990 and $3329 in 1993. Assume that this rate of growth continues.

(a) Find the three-year growth factor a, and find an exponential growth model $E(x) = Ca^x$ for the annual health-care expenditures per capita, where x is the number of three-year time periods since 1990.

(b) Use the model found in part (a) to predict health-care expenditures per capita in 1996 and in 2005.



(c) Search the Internet to find the actual health-care expenditures for 1996 and 2005. Were your predictions reasonable?

40. Profit Increase In 2007, Mieko's software company made a profit of $5,165,000. Mieko projects that her company will have a profit of 6 million dollars in 2011. Assume that her predictions are correct and that this rate of growth continues.

(a) Find the four-year growth factor a, and find an exponential growth model $P(x) = Ca^x$ for the annual profit, where x is the number of four-year time periods since 2007.

(b) Use the model found in part (a) to predict the profit for Mieko's company in 2015.



41. Drug Absorption A patient is administered 100 mg of a therapeutic drug. It is known that 25% of the drug is expelled from the body each hour.

(a) Find an exponential decay model for the amount of drug remaining in the patient's body after t hours.

(b) Use the model to predict the amount of the drug that remains in the patient's body after 6 hours.

42. Drug Absorption A patient is administered 250 mg of a therapeutic drug. It is known that 40% of the drug is expelled from the body each hour.

(a) Find an exponential decay model for the amount of drug remaining in the patient's body after t hours.

(b) Use the model to predict the amount of the drug that remains in the patient's body after 10 hours.

43. Declining Housing Prices In 2006 the U.S. "housing bubble" was beginning to burst. One news article at the time stated: "The median price of a home sold in the United States in the third quarter of 2006 was $232,300, and forecasters predict that the median price of houses will fall by 8.9% each year." Assume that the median price decreases at the forecasted decay rate of -8.9%.

(a) Find the decay factor a, and find an exponential growth model $E(x) = Ca^x$ for the median price of a home sold in the United States x years since the prediction.

(b) Use the model found in part (a) to predict the median price of homes sold in the third quarter of 2009.

44. Falling Gas Prices A California newspaper published a story on October 17, 2008, stating: "The average price of a gallon of gas has dropped to $3.60, which is a drop of 10% in just one week. Forecasters predict that this trend will continue and the price of gas will be under $2.00 a gallon by the end of November." Assume that the price of gas decreases at the forecasted decay rate of -10% per week.

(a) Find the decay factor a, and find an exponential growth model $G(x) = Ca^x$ for the price of gas x weeks since October 17, 2008, when the price was $3.60 per gallon.

(b) Use the model to predict the price of gas 6 weeks later. Is this what the forecasters predicted?

45. Radioactive Cesium The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample.

(a) Find a function that models the mass $m(x)$ of cesium-137 remaining after x half-lives.

(b) Use the model to predict the amount of cesium-137 remaining after 270 years.

(c) Make a table of values for $m(x)$ with x varying between 0 and 5.

(d) Graph the function $m(x)$ and the entries in the table. What does the graph tell us about how cesium-137 decays?

46. Radioactive Iodine The half-life of iodine-135 is 8 days. Suppose we have a 22-mg sample.

(a) Find a function that models the mass $m(x)$ of iodine-135 remaining after x half-lives.

(b) Use the model to predict the amount of iodine-135 remaining after 32 days.

(c) Make a table of values for $m(x)$ with x varying between 0 and 5.

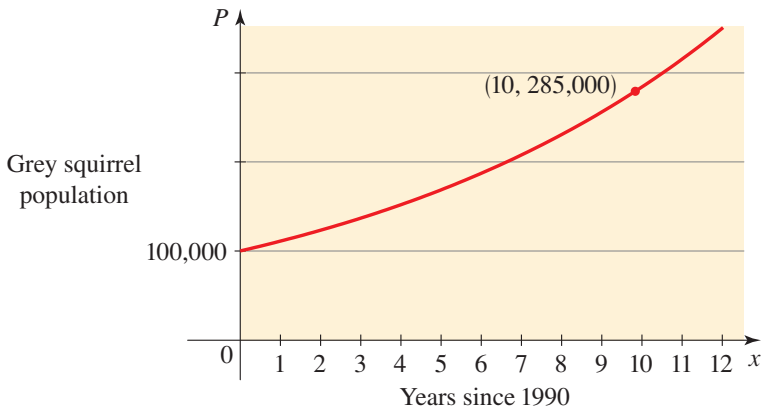
(d) Graph the function $m(x)$ and the entries in the table. What does the graph tell us about how iodine-135 decays?

47. Grey Squirrel Population The American grey squirrel is a species not native to Great Britain that was introduced there in the early twentieth century and has been increasing in numbers ever since. The graph shows the grey squirrel population in a British county between 1990 and 2000. Assume that the population continues to grow exponentially.

(a) What was the grey squirrel population in 1990?

(b) Find the ten-year growth factor a, and find an exponential growth model $P(x) = Ca^x$, where x is the number of ten-year time periods since 1990.

(c) Use the model found in part (b) to predict the grey squirrel population in 2010.

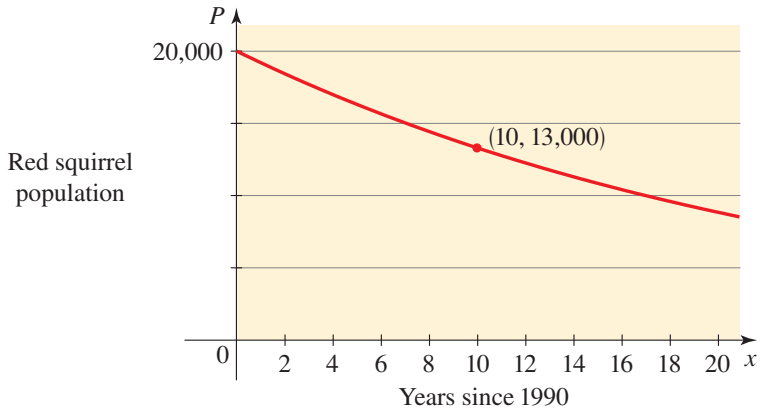


48. Red Squirrel Population The red squirrel is the native squirrel of Great Britain, but its numbers have been declining ever since the American grey squirrel was introduced to Great Britain. The graph shows the red squirrel population in a British county between 1990 and 2000. Assume that the population continues to decrease exponentially.

(a) What was the red squirrel population in 1990?

(b) Find the ten-year decay factor a, and find an exponential decay model $P(x) = Ca^x$ for the red squirrel population, where x is the number of ten-year time periods since 1990.

(c) Use the model found in part (b) to predict the red squirrel population in 2010.



49. Algebra and Alcohol After alcohol is fully absorbed into the body, it is metabolized with a half-life of 1.5 hours. Suppose Thad consumes 45 mL of alcohol (ethanol).

(a) Find an exponential decay model for the amount of alcohol remaining after x 1.5-hour time intervals.

(b) Graph the function you found in part (a) for x between 0 and 4.

3.2 Exponential Models: Comparing Rates

- Changing the Time Period
- Growth of an Investment: Compound Interest

IN THIS SECTION... we find exponential models of growth and decay for different time periods. We also study compound interest as an example of exponential growth.

GET READY... by reviewing how to solve power equations in **Algebra Toolkit C.1.** Test your understanding by doing the **Algebra Checkpoint** on page 267.



To determine the growth rate of a certain type of bacteria, a biologist prepares a nutrient solution in which the bacteria can grow. The biologist introduces a number of bacteria into the solution and then estimates the bacteria count 40 minutes later. From this information the biologist can determine the 40-minute growth factor. But the biologist wants to report the hourly growth factor, not the 40-minute growth factor. This conforms to the standards for reporting bacteria growth rates

and allows scientists to directly compare the growth rates for different types of bacteria. So the situation is as follows:

- We know the **40-minute growth factor**.
- We want to find the **one-hour growth factor**.

In this section we study how to change growth factors and growth rates from one time period to a different time period.

Changing the Time Period

A researcher may measure the growth of an insect population daily but may want to know the weekly growth factor. To do this, suppose that the daily growth factor is a. In a week the population undergoes seven daily time periods, so the weekly growth factor is a^7. Similarly, if the weekly growth factor is b, then the daily growth factor a must satisfy $a^7 = b$, so $a = b^{1/7}$.

Growth factor	
Daily: a	Weekly: a^7
Weekly: b	Daily: $b^{1/7}$
Monthly: c	Yearly: c^{12}
Yearly: d	Monthly: $d^{1/12}$

example 1 Changing the Time Period

A biologist finds that the 30-minute growth rate for a certain type of bacteria is 0.85. Find the one-hour growth rate.

Solution

The 30-minute growth factor is $1 + 0.85 = 1.85$. Let a be the one-hour growth factor. Since there are two 30-minute time periods in one hour, we have

$$a = 1.85^2 \qquad \text{30-minute growth factor is 1.85}$$

$$a \approx 3.4 \qquad \text{Calculator}$$

So the one-hour growth factor is 3.4. It follows that the one-hour growth rate is $r = a - 1 = 3.4 - 1 = 2.4$.

■ NOW TRY EXERCISE 9 ■

example 2 Changing the Time Period

A chinchilla farm starts with 20 chinchillas, and after 3 years there are 128 chinchillas. Assume that the number of chinchillas grows exponentially.

(a) Find the 3-year growth factor. **(b)** Find the one-year growth factor.

Solution

(a) The 3-year growth factor is 6.4 as shown in Example 2 of Section 3.1 (page 251).

(b) To find the one-year growth factor, we first let a be the one-year growth factor. Then after 3 years the population is multiplied by a three times. So the 3-year growth factor is a^3. Since we know that the 3-year growth factor is 6.4, we can find a as follows:

In Example 3 we solve the power equation $a^3 = 6.4$. Solving such equations is reviewed in Algebra Toolkit C.1, page T47.

$$a^3 = 6.4 \qquad \text{3-year growth factor is 6.4}$$
$$a = (6.4)^{1/3} \qquad \text{Take cube root}$$
$$a \approx 1.86 \qquad \text{Calculator}$$

So the one-year growth factor is 1.86.

NOW TRY EXERCISES 5 AND 45

example **3** **Models for Different Time Periods**

A bacterial infection starts with 100 bacteria, and the bacteria count doubles every five-hour time period. Find an exponential growth model for the number of bacteria

(a) x time periods after infection. **(b)** t hours after infection.

Solution

(a) The initial number of bacteria is 100, and the growth factor for a five-hour time period is 2. So a model for the number of bacteria is

$$P(x) = 100 \cdot 2^x$$

where x is the number of five-hour time periods since infection.

(b) We first let a be the one-hour growth factor. Then a^5 is the five-hour growth factor. From part (a) we know that the five-hour growth factor is 2. So

$$a^5 = 2 \qquad \text{The five-hour growth factor is 2}$$
$$a = 2^{1/5} \qquad \text{Solve for } a$$
$$\approx 1.15 \qquad \text{Calculator}$$

So the one-hour growth factor a is 1.15. So an exponential model is obtained as follows:

$$P(t) = Ca^t \qquad \text{Model}$$
$$P(t) = 100(1.15)^t \qquad \text{Replace } C \text{ by 100, } a \text{ by 1.15}$$

The model is $P(t) = 100(1.15)^t$, where t is measured in hours.

Another solution Let t be the number of hours since infection. Since each time periods consists of 5 hours, after x time periods we have $t = 5x$. We solve this equation for x:

$$t = 5x \qquad \text{Each time period } x \text{ is 5 h}$$
$$x = \frac{t}{5} \qquad \text{Divide by 5 and switch sides}$$

To obtain an exponential model in terms of one-hour time periods, we replace x by $t/5$ in the model in part (a):

$$P(t) = 100 \cdot 2^{t/5} \qquad \text{Replace } x \text{ by } t/5$$
$$= 100 \cdot (2^{1/5})^t \qquad \text{Property of exponents}$$
$$\approx 100 \cdot (1.15)^t \qquad \text{Calculator}$$

where t is measured in hours.

NOW TRY EXERCISE 13

example 4 Comparing Growth Rates

The growth rates of two types of bacteria, A and B, are tested.

Type A doubles every 5 hours

Type B triples every 7 hours

(a) Find the one-hour growth rate for each type of bacterium.

(b) Which type has the larger growth rate?

Solution

(a) From Example 3 we know that Type A bacteria have a one-hour growth factor of 1.15. So the one-hour growth rate is $1.15 - 1 = 0.15$, or 15% per hour.

If we let a be the one-hour growth factor for Type B bacteria, then $a^7 = 3$, so $a = 3^{1/7} \approx 1.17$. So the growth rate is $1.17 - 1 = 0.17$ or 17% per hour.

(b) From part (a) we see that Type B has a slightly larger growth rate.

NOW TRY EXERCISE 43

In Section 3.1 we learned that radioactive decay is expressed in terms of half-life, the time required for half the mass of the radioactive substance to decay. We noted that the half-life of radium-226 is 1600 years, so a 100-g sample decays to 50 g (or $\frac{1}{2} \times 100$ g) in 1600 years. In the next example we find the one-year decay factor for radium-226.

example 5 An Exponential Decay Model for Radium

The half-life of radium-226 is 1600 years. A 50-gram sample of radium-226 is placed in an underground disposal facility and monitored.

(a) Find a function that models the mass $m(t)$ of radium-226 remaining after t years.

(b) Use the model to predict the amount of radium-226 remaining after 100 years.

Solution

(a) The initial mass is 50 g, and the decay factor is

$$a = \tfrac{1}{2} \qquad \text{1600-year decay factor}$$

Let t be the number of years. Since each time period consists of 1600 years, after x time periods we have $t = 1600x$. Solving for x, we get $x = t/1600$. So an exponential growth model is

$$\begin{aligned} m(t) &= 50 \cdot \left(\tfrac{1}{2}\right)^{t/1600} && \text{Replace } x \text{ by } t/1600 \\ &= 50 \cdot (0.5^{1/1600})^t && \text{Property of exponents} \\ &\approx 50 \cdot (0.999567)^t && \text{Calculator} \end{aligned}$$

where t is measured in years.

(b) Replacing t by 100 in the model, we get

$$\begin{aligned} m(t) &= 50 \cdot (0.999567)^t && \text{Model} \\ m(100) &= 50(0.999567)^{100} && \text{Replace } t \text{ by } 100 \\ &\approx 47.88 && \text{Calculator} \end{aligned}$$

So the mass remaining after 100 years is about 47.88 g.

NOW TRY EXERCISE 47

From Example 5 we see that in general, if the half-life of a radioactive substance is h years, then the one-year growth factor is $\left(\frac{1}{2}\right)^{1/h}$, and the amount remaining after t years is

$$m(t) = C \cdot \left(\frac{1}{2}\right)^{t/h}$$

A similar statement holds if the half-life is given in other time units, such as hours, minutes, or seconds.

Growth of an Investment: Compound Interest

Money deposited in a savings account increases exponentially, because the interest on the account is calculated by multiplying the amount in the account by a fixed factor—the interest rate. Let's suppose that \$1000 is deposited in a 10-year certificate of deposit (CD) paying 6% interest annually, compounded monthly. This means that the interest rate each month is

$$\frac{6\%}{12} = 0.5\%$$

At the end of every month, the bank adds 0.5% of the amount on deposit to the CD. So the amount of the CD grows exponentially as follows:

Initial value: \$1000

Time period: 1 month

Monthly growth rate: $0.06/12 = 0.005$

Monthly growth factor: $1 + 0.005 = 1.005$

So the amount in the CD after x months is modeled by the exponential function

$$A(x) = 1000(1.005)^x \qquad \text{Model}$$

The table in the margin gives the amount after each month (time period).

Month	Amount ($)	Amount ($)
0	1000	1000.00
1	1000(1.005)	1005.00
2	$1000(1.005)^2$	1010.03
3	$1000(1.005)^3$	1015.08
4	$1000(1.005)^4$	1020.15

In general, if the annual interest rate is r (expressed as a decimal, not a percentage) and if interest is compounded n times each year, then in each time period the interest rate is r/n, and in t years there are nt time periods. This leads to the following formula for compound interest.

Compound Interest

If an amount P is invested at an annual interest rate r compounded n times each year, then the amount $A(t)$ of the investment after t years is given by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

r is often referred to as the *nominal annual interest rate.*

example 6 Comparing Yields for Different Compounding Periods

Ravi wishes to invest $5000 in a 3-year CD. He can choose either of two CDs:

Certificate A: 5.50% each year, compounded twice a year

Certificate B: 5.50% each year, compounded daily

Which certificate is the better investment?

Solution

The principal P is $5000, and the term t of the investment is 3 years. For Certificate A the rate r is 0.055, and the number n of compounding periods is 2. Using the formula for compound interest, replacing t by 3, r by 0.055, and n by 2, we get

$$A(3) = 5000\left(1 + \frac{0.055}{2}\right)^{2\cdot 3} = \$5883.84 \qquad \text{Certificate A}$$

For Certificate B the rate r is 0.055, and the number n of compounding periods is 365. Using the formula for compound interest replacing t by 3, r by 0.055, and n by 365, we get

$$A(3) = 5000\left(1 + \frac{0.055}{365}\right)^{365\cdot 3} = \$5896.89 \qquad \text{Certificate B}$$

We conclude that Certificate B is a slightly better investment.

NOW TRY EXERCISE 41

Banks advertise their interest rates and compounding periods; they are also required by law to report the **annual percentage yield (APY)**, which is the annual growth rate for money deposited in their bank.

example 7 Calculating Annual Percentage Yield

Find the annual percentage yield for Certificate B in Example 6.

Solution

After 1 year any principal P will grow to the amount

$$A = P\left(1 + \frac{0.055}{365}\right)^{365\cdot 1} \qquad \text{Amount after 1 year}$$

$$= P \cdot 1.0565 \qquad \text{Calculator}$$

So the annual growth factor a is 1.0565. Therefore the annual growth rate is $1.0565 - 1 = 0.0565$. So the annual percentage yield is 5.65%.

NOW TRY EXERCISE 37

Algebra Checkpoint ✓

Check your knowledge of solving power equations by doing the following problems. You can review the rules of exponents in **Algebra Toolkit C.1** on page T47.

1–8 Solve the equation for x.

1. $x^4 = 16$ **2.** $x^3 = \frac{1}{27}$ **3.** $5x^3 = 40$ **4.** $2x^5 = 486$

5. $x^{1/2} = 5$ **6.** $x^{1/3} = 2$ **7.** $5x^{1/6} = 10$ **8.** $3x^{1/2} = 5$

9–12 Solve the equation for x; express your answer correct to two decimal places.

9. $x^4 = 5$ **10.** $2x^3 = 9$ **11.** $x^{1/5} = 7.35$ **12.** $3x^{1/4} = 34.12$

3.2 Exercises

CONCEPTS

Fundamentals

1. The bacteria population in a certain culture grows exponentially.

(a) If the 20-minute growth factor is a, then the one-hour growth factor is a^3.

(b) If the five-hour growth factor is b, then the one-hour growth factor is _____.

(c) If the half-life of a radioactive element is h years, then the yearly decay factor is _____.

2. In the compound interest formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, the letters P, r, n, and t stand for _____, _____, _____, and _____, respectively, and $A(t)$ stands for _____.

Think About It

3. Biologists often report population growth rates in fixed time periods such as one hour, one day, one year, and so on. What are some of the advantages of this type of reporting (as opposed to reporting 38-minute growth rates or nine-day growth rates)?

4. The rate of decay of a radioactive element is usually expressed in terms of its half-life. Why do you think this is? How is this more useful than reporting yearly decay rates?

SKILLS

5–6 ■ A population P starts at 2000, and four years later the population reaches the given number.

(i) Find the four-year growth or decay factor.

(ii) Find the annual growth or decay factor.

5. **(a)** 8000 **(b)** 20,000 **(c)** 1500 **(d)** 1200

6. **(a)** 6000 **(b)** 3000 **(c)** 800 **(d)** 400

7–10 ■ The bacteria population in a certain culture grows exponentially. Find the one-hour growth rate if

7. the 10-minute growth rate is 0.02.

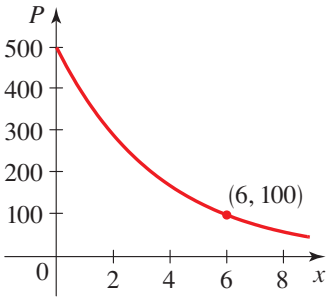
8. the 15-minute growth rate is 0.09.

9. the three-hour growth rate is 57%.

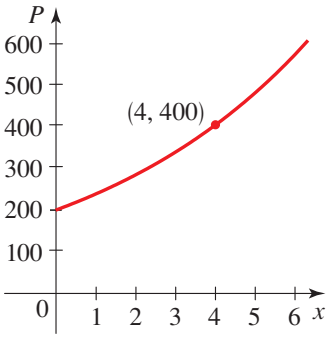
10. the four-hour growth rate is 72%.

11–12 ■ The graph of a function that models exponential growth or decay is shown. Find the initial population and the one-year growth factor.

11.



12.



13–16 ■ A bacterial infection starts with 2000 bacteria, and the bacteria count doubles in the given time period. Find an exponential growth model for the number of bacteria

(a) x time periods after infection.

(b) t hours after infection.

13. 20 minutes

14. 15 minutes

15. 90 minutes

16. 2 hours

17–24 ■ A population P is initially 1000. Find an exponential model (growth or decay) for the population after t years if the population P

17. doubles every year.

18. is multiplied by 2 every year.

19. increases by 35% every 5 years.

20. increases by 200% every 6 years.

21. decreases by $\frac{1}{2}$ every 6 months.

22. is multiplied by 0.75 every month.

23. decreases by 40% every 2 years.

24. decreases by 37% every 9 years.

25–28 ■ Information on two different bacteria populations is given.

(a) Find the one-hour growth rate for each type of bacteria.

(b) Which type of bacteria has the greater growth rate?

25. Type A: 20-minute growth factor is 1.19
Type B: 30-minute growth factor is 1.23

26. Type A: doubles every 20 minutes
Type B: triples every 30 minutes

27. Type A: triples every 5 hours
Type B: quadruples every 7 hours

28. Type A: increases by 30% every hour
Type B: increases by 60% every 2 hours

CONTEXTS

29–30 ■ **Compound Interest** An investment of \$5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amount to which the investment grows at the indicated times or interest rates.

29. $r = 4\%$

Time (years)	Amount ($)
0	5000
1	
2	
3	
4	
5	

30. $t = 5$ years

Annual rate	Amount ($)
1%	
2%	
3%	
4%	
5%	
6%	

31. Compound Interest If \$500 is invested at an interest rate of 3% per year, compounded quarterly, find the value of the investment after the given number of years.
(a) 1 year **(b)** 2 years **(c)** 5 years

32. Compound Interest If \$2500 is invested at an interest rate of 2.5% per year, compounded daily, find the value of the investment after the given number of years.
(a) 2 years **(b)** 3 years **(c)** 6 years

33. Compound Interest If \$4000 is invested at an interest rate of 1.6% per year, compounded quarterly, find the value of the investment after the given number of years.
(a) 4 years **(b)** 6 years **(c)** 8 years

34. Compound Interest If \$10,000 is invested at an interest rate of 10% per year, compounded semiannually, find the value of the investment after the given number of years.
(a) 5 years **(b)** 10 years **(c)** 15 years

35. Compound Interest If \$3000 is invested at an interest rate of 4% each year, find the amount of the investment at the end of 5 years for the following compounding methods.
(a) Annual **(b)** Semiannual **(c)** Monthly **(d)** Daily

36. Compound Interest If \$4000 is invested in an account for which interest is compounded quarterly, find the amount of the investment at the end of 5 years for the following interest rates.
(a) 6% **(b)** $6\frac{1}{2}\%$ **(c)** 7% **(d)** 8%

37. Annual Percentage Yield Find the annual percentage yield for an investment that earns 2.5% each year, compounded daily.

38. Annual Percentage Yield Find the annual percentage yield for an investment that earns 4% each year, compounded monthly.

39. Annual Percentage Yield Find the annual percentage yield for an investment that earns 3.25% each year, compounded quarterly.

40. Annual Percentage Yield Find the annual percentage yield for an investment that earns 5.75% each year, compounded semiannually.

41. Compound Interest Kai wants to invest \$5000, and he is comparing two different investment options:
(i) $3\frac{1}{4}\%$ interest each year, compounded semiannually
(ii) 3% interest each year, compounded daily
Which of the two options would provide the better investment?

42. Compound Interest Sonya wants to invest $3000, and she is comparing three different investment options:

(i) $4\frac{1}{2}\%$ interest each year, compounded semiannually

(ii) $4\frac{1}{4}\%$ interest each year, compounded quarterly

(iii) 4% interest each year, compounded daily

Which of the given interest rates and compounding periods would provide the best investment?

43. Bacteria Bacterioplankton that occur in large bodies of water are a powerful indicator of the status of aquatic life in the water. One team of biologists tests a certain location A, and their data indicate that bacterioplankton are increasing by 19% every 20 minutes. Another team of biologists test a different location B, and their data indicate that bacterioplankton are increasing by 40% every 3 hours.

(a) Find the one-hour growth factor for each location.

(b) Which location has the larger growth rate?



Tischenko Irina/Shutterstock.com 2009

B. melatensis

44. Bacteria The bacterium *Brucella melatensis* (*B. melatensis*) is one of the causes of mastitis infections in milking cows. A lab tests the growth rates of two strains of this bacterium:

Strain A increases by 30% every 4 hours

Strain B increases by 40% every 2 hours

(a) Find the one-hour growth factor for each strain.

(b) Which strain has the larger growth rate?

45. Population of India Although India occupies only a small portion of the world's land area, it is the second most populous country in the world, and its population is growing rapidly. The population was 846 million in 1990 and 1148 million in 2000. Assume that India's population grows exponentially.

(a) Find the 10-year growth factor and the annual growth factor for India's population.

(b) Find an exponential growth model P for the population t years after 1990.

(c) Use the model found in part (b) to predict the population of India in 2010.

(d) Graph the function P for t between 0 and 25.

46. U.S. National Debt The U.S. national debt was about $5776 billion on January 1, 2000, and increased to about $9229 billion on January 1, 2007. Assume that the U.S. national debt grows exponentially.

(a) Find the 7-year growth factor and the annual growth factor for national debt.

(b) Find an exponential growth model P for the national debt t years after January 1, 2000.

(c) Use the model found in part (b) to predict the national debt on January 1, 2009.

(d) Graph the function P for t between 0 and 15.

47. Radioactive Plutonium Nuclear power plants produce radioactive plutonium-239, which has a half-life of 24,360 years. A 700-gram sample of plutonium-239 is placed in an underground waste disposal facility.

(a) Find a function that models the mass $m(t)$ of plutonium-239 remaining in the sample after t years. What is the decay factor?

(b) Use the model to predict the amount of plutonium-239 remaining in the sample after 500 years.

(c) Make a table of values for $m(t)$, with t varying between 0 and 5000 years in 1000-year increments. Graph the entries in the table. What does the graph tell us about how plutonium-239 decays?

48. Radioactive Strontium One radioactive material that is produced in atomic bombs is the isotope strontium-90, with a half-life of 28 years. In 1986, a 20-gram sample of strontium-90 is taken from a site near Chernobyl.

(a) Find a function that models the mass $m(t)$ of strontium-90 remaining in the sample after t years. What is the decay factor?

(b) Use the model to predict the amount of strontium-90 remaining in the sample after 40 years.

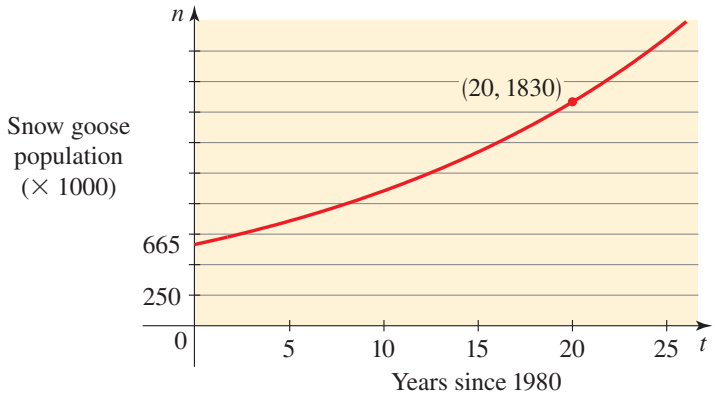
(c) Make a table of values for $m(t)$ with t varying between 0 and 100 years in 20-year increments. Make a plot of the entries in the table. What does the graph tell us about how strontium-90 decays?

49. Bird Population The snow goose population in the United States has become so large that food is becoming scarce for these birds. The graph shows the number of snow geese counted between 1980 and 2000, according to the Audubon Christmas Bird Count. Assume that the population grows exponentially.

(a) What was the snow goose population in 1980?

(b) Find the one-year growth factor a, and find an exponential growth model $n(t) = Ca^t$ for the snow goose population t years since 1980.

(c) Use the model to predict the number of snow geese in 2005.

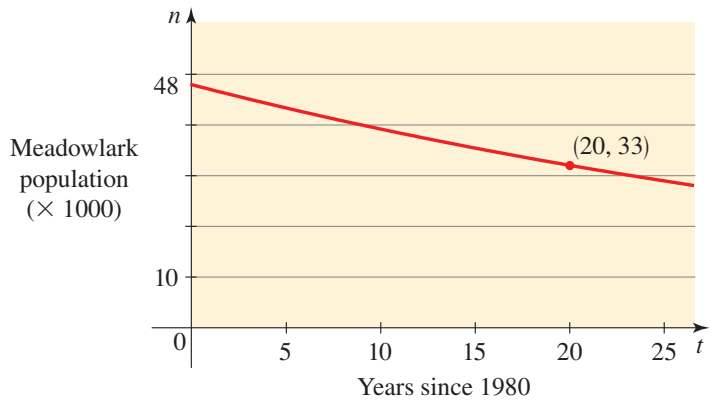


50. Bird Population Some common bird species are in decline as their habitat is lost to development. One of the most common such species is the eastern meadowlark. The graph shows the number of eastern meadowlarks in the United States between 1980 and 2005, according to the Audubon Christmas Bird Count. Assume that the population continues to decrease exponentially.

(a) What was the meadowlark population in 1980?

(b) Find the one-year decay factor a, and find an exponential decay model $n(t) = Ca^t$ for the meadowlark population t years since 1980.

(c) Use the model to predict the number of meadowlarks in 2010.



51. Algebra and Alcohol After alcohol is fully absorbed into the body, it is metabolized with a half-life of 1.5 hours. Suppose Thad consumes 45 mL of alcohol (ethanol). (See Exercise 49 in Section 3.1.)

(a) Find an exponential decay model for the amount of alcohol remaining after t hours.

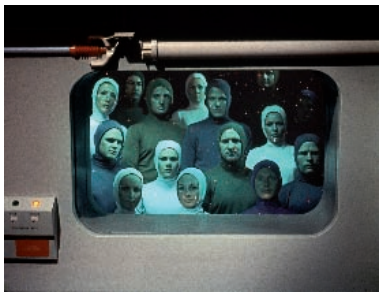
(b) Graph the function you found in part (a) for t between 0 and 6 hours.

3.3 Comparing Linear and Exponential Growth

- **Average Rate of Change and Percentage Rate of Change**
- **Comparing Linear and Exponential Growth**
- **Logistic Growth: Growth with Limited Resources**

IN THIS SECTION... we study the concept of "percentage rate of change" for exponential models. We use this concept to compare exponential growth with linear growth as well as to find growth models in the presence of limited resources (logistic growth).

CBS Entertainment/CBS Paramount Network Television



The crowded planet Gideon as seen from a view port of the starship *Enterprise*

One way to better understand exponential growth is to compare it to linear growth. Comparing the rate of change of each type of growth will help us to see why they are so different.

Exponential growth is highlighted in the *Star Trek* episode "The Mark of Gideon," in which Captain Kirk is sent to a planet where no one seems ever to get sick or die. As a result, the planet is severely overpopulated—there is barely any room for the people on the planet to move around. Captain Kirk is abducted by the planet's leader in the hope that he could introduce disease (and consequently death) to the planet. This episode tries to explain the dilemma of exponential growth: It leads to excessively large populations that continue to grow ever faster as time goes on. Realistically, however, population growth is often limited by available resources. We'll see that this type of growth, called *logistic growth*, can also be modeled by using exponential functions.

Average Rate of Change and Percentage Rate of Change

Average rate of change is studied in Section 2.1.

Let $f(x) = Ca^x$ be an exponential growth (or decay) model. The average rate of change of f over one time period is

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x)$$

Recall from Section 3.1 that the change in f over one time period as a *proportion* of the value of f at x is the growth rate r:

$$r = \frac{f(x+1) - f(x)}{f(x)}$$

The *percentage rate of change* of f over one time period is the growth rate r expressed as a percentage.

Percentage Rate of Change

For the growth or decay model $f(x) = Ca^x$ the growth or decay rate r is

$$r = \frac{f(x+1) - f(x)}{f(x)}$$

The **percentage rate of change** is this growth rate r expressed as a percentage.

Let's compare average rate of change and percentage rate of change for an exponential model.

example **1** **Rates of Change of an Exponential Function**

Find the average rate of change and the percentage rate of change for the function $f(x) = 10 \cdot 3^x$ on intervals of length 1, starting at 0. What do you observe about these rates of change?

Solution

We make a table of these rates. To show how the entries in the table are calculated, we show how the second row of the table is obtained.

- Values of f: $f(0) = 10 \cdot 3^0 = 10$, $f(1) = 10 \cdot 3^1 = 30$
- Average rate of change of f from 0 to 1:

$$\frac{f(1) - f(0)}{1 - 0} = \frac{20}{1} = 20$$

- The change in f from 0 to 1 as a proportion of the value of f at 0 is

$$r = \frac{f(1) - f(0)}{f(0)} = \frac{20}{10} = 2.00$$

So the percentage rate of change is 200%. The remaining rows of the table are calculated in the same way.

x	$f(x)$	Average rate of change	Percentage rate of change
0	10	—	—
1	**30**	**20**	**200%**
2	90	60	200%
3	270	180	200%
4	810	540	200%

The average rate of change of f appears to increase with increasing values of x (also see Figure 1). The percentage rate change is constant for all intervals of length 1.

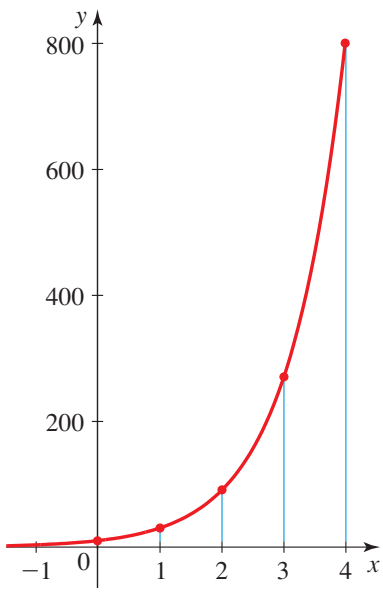


figure **1** Rate of change of f

NOW TRY EXERCISE 9

We see that the average rate of change of an exponential function varies dramatically over different intervals; this is very different from the case of a linear function, in which the average rate of change is the same on any interval. On the other hand, the percentage rate of change of an exponential function is constant on time periods of the same length. So if we have data in which the inputs are equally spaced we can analyze the data as follows:

- If the *average rate of change* for consecutive outputs is constant, then there is a *linear model* that fits the data exactly. (See Section 1.3.)
- If the *percentage rate of change* for consecutive outputs is constant, then there is an exponential model that fits the data exactly. (See Example 2.)

example 2 Fitting an Exponential Model to Data

x	y
0	10,000
1	7000
2	4900
3	3430
4	2401

A data set with equally spaced inputs is given.

(a) Find the average rate of change and the percentage rate of change for consecutive outputs.

(b) Is an exponential model appropriate? If so, find the model and sketch a graph of the model together with a scatter plot of the data.

Solution

(a) We complete the table for the average rate of change and percentage rate of change as in Example 1. Here's how we calculate the entries in the second row.

- The average rate of change in f from 0 to 1:

$$\frac{f(1) - f(0)}{1 - 0} = 7000 - 10{,}000 = -3000$$

- The change in f from 0 to 1 as a proportion of the value of f at 0 is

$$r = \frac{f(1) - f(0)}{f(0)} = \frac{-3000}{10{,}000} = -0.30$$

So the percentage rate of change is -30%. The remaining rows in the table are calculated in the same way.

x	$f(x)$	Average rate of change	Percentage rate of change
0	10,000	—	—
1	7000	−3000	−30%
2	4900	−2100	−30%
3	3430	−1470	−30%
4	2401	−1029	−30%

(b) Since the input values are equally spaced and the percentage rate of change is constant, an exponential model is appropriate. To find a model of the form $f(x) = C \cdot a^x$, we first observe that $f(0) = 10{,}000$, so C is 10,000. Since the percentage rate of change is -30%, the decay rate r is -0.30, so the decay

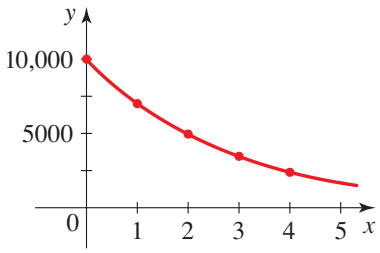


figure **2** $f(x) = 10{,}000 \cdot (0.70)^x$

factor is $a = 1 + r = 1 + (-0.30) = 0.70$. Thus an exponential model for the data is

$$f(x) = 10{,}000 \cdot (0.70)^x$$

A graph is shown in Figure 2.

NOW TRY EXERCISE 13

Comparing Linear and Exponential Growth

Let's compare linear functions and exponential functions. As we learned in Section 1.3, for a linear model $f(x) = b + mx$, increasing x by one unit has the effect of *adding* m to $f(x)$. From Section 3.1 we know that for an exponential model $f(x) = Ca^x$, increasing x by one unit has the effect of *multiplying* $f(x)$ by the growth factor a.

	Linear	Exponential
Model	$f(x) = b + mx$	$f(x) = Ca^x$
Initial value $f(0)$	b	C
Increasing x by 1 has the effect of . . .	adding m to $f(x)$.	multiplying $f(x)$ by a.

Let's test these ideas on a specific example.

example 3 Linear Growth Versus Exponential Growth

City officials in Newburgh, population 10,000, wish to make long-range plans to maintain and expand the infrastructure and services of their city. They hire two city planners to construct models for Newburgh's growth over the next 5 years.

Planner A estimates that the city's population increases by 500 people each year.

Planner B estimates that the city's population increases by 5% per year.

(a) Use Planner A's assumption to find a function P_A that models Newburgh's population t years from now. Make a table of values of the function P_A, including the average rate of change of the population increase for each year.

(b) Use Planner B's assumption to find a function P_B that models Newburgh's population t years from now. Make a table of values of the function P_B, including the percentage rate of change of population each year.

(c) Graph the functions P_A and P_B for the next 25 years. How do these functions differ?

Solution

(a) Planner A assumes that the population grows by 500 people per year, so after t years, $500t$ people will be added to the initial population. This leads to the *linear model*

$$P_A(t) = 10{,}000 + 500t$$

with initial value 10,000 and rate of change 500. See the following table.

(b) Planner B assumes that the population grows by 5% per year. So the growth rate is $r = 0.05$, and the growth factor is $a = 1 + 0.05 = 1.05$. This leads to the *exponential model*

$$P_B(t) = 10{,}000 \cdot (1.05)^t$$

See the table below.

Planner A

Year	Population	Average rate of change
0	10,000	—
1	10,500	500
2	11,000	500
3	11,500	500
4	12,000	500
5	12,500	500

Planner B

Year	Population	Percentage rate of change
0	10,000	—
1	10,500	5%
2	11,025	5%
3	11,576	5%
4	12,155	5%
5	12,762	5%

(c) The graphs are shown in Figure 3. The exponential growth function P_B eventually grows much faster than the linear function P_A.

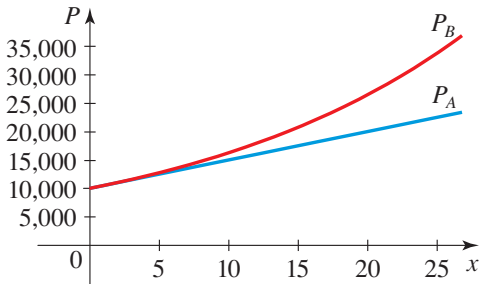


figure **3** Graphs of P_A and P_B

NOW TRY EXERCISES 19 AND 29

Logistic Growth: Growth with Limited Resources

A cell phone company has sold 3 million phones this month, but the company president would like his sales force to double their efforts and double sales every month. This means that sales would grow exponentially. The number of phones sold (in millions) in month x would be modeled by

$$S(x) = 3 \cdot 2^x$$

If the sales force could actually accomplish this feat, then in the twelfth month they would sell $S(12) = 3 \cdot 2^{12} = 12{,}288$ million phones, or

$$12{,}288{,}000{,}000 \approx 12 \text{ billion phones}$$

This number far exceeds the population of the entire world. Moreover, the sales staff would have to sell twice as many phones the next month to satisfy the quota. Clearly, sales growth must be limited by the available resources. In this case, the number of potential customers is limited to about 6 billion, say.

Nevertheless, the company can be considered very successful if it can gain an increasing *percentage* of the *remaining market share* (the people who haven't yet bought a cell phone). If the maximum number of potential customers is C and the company has sold phones to N of them, then the remaining potential market is $C - N$. The number of phones sold as a fraction of this potential market is $N/(C - N)$. As we've seen, the company can't hope for exponential growth, but maybe their sales as a fraction of the *remaining* market can grow exponentially with growth factor a, that is,

$$\frac{N}{C - N} = ka^x$$

where N is the number of cell phones sold by time period x. Solving this equation for N (the total number of cell phones sold by time period x), we get an equation of the form

$$N = \frac{C}{1 + b \cdot a^{-x}} \qquad a > 1$$

where $b = 1/k$. This type of equation, called a *logistic growth* equation, models growth under limited resources.

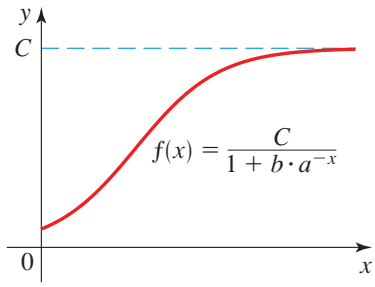
Logistic Growth

A **logistic growth** model is a function of the form

$$f(x) = \frac{C}{1 + b \cdot a^{-x}} \qquad a > 1, b > 0$$

and models growth under limited resources.

- The variable x is the number of **time periods**.
- The constant C is the **carrying capacity**, the maximum population the resources can support.
- The graph of the function f has the general shape shown. The horizontal line $y = C$ is an asymptote of the graph.



example **4** **Limited Fish Population**

A biologist models the number of fish in a small pond by the logistic growth function

$$f(x) = \frac{100}{1 + 9 \cdot 1.5^{-x}}$$

where x is the number of years since the pond was first observed.

(a) What is the initial number of fish in the pond?

(b) Make a table of values of f for x between 0 and 21. From the table, what can you conclude happens to the fish population as x increases?

(c) Use a graphing calculator to draw a graph of the function f and the line $y = 100$. From the graph, what can you conclude about the fish population as x increases?

(d) What is the carrying capacity of the pond? Does this answer agree with the table in part (b) and the graph in part (c)?

Solution

(a) When x is 0 the population is

$$f(0) = \frac{100}{1 + 9 \cdot 1.5^0} = \frac{100}{1 + 9 \cdot 1} = 10$$

Initially, there were 10 fish in the pond.

(b) We use a calculator to find the values of the function f shown in the table below. It appears that the population stabilizes at 100 fish.

Year	Population	Year	Population
0	10	11	91
1	14	12	94
2	20	13	96
3	27	14	97
4	36	15	98
5	46	16	99
6	56	17	99
7	65	18	99
8	74	19	100
9	81	20	100
10	86	21	100

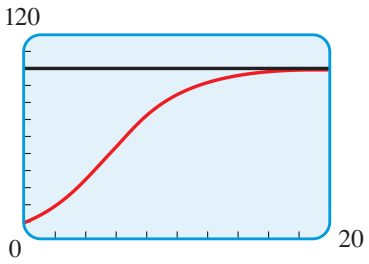


figure **4**

Graph of $f(x) = \dfrac{100}{1 + 9 \cdot 1.5^{-x}}$

(c) A graph of the function f and the line $y = 100$ are shown in Figure 4. It appears that the population gets closer and closer to 100 fish but will never exceed that number. The line $y = 100$ is a horizontal asymptote of f.

(d) Comparing the general form of the logistic equation with the model, we see that the carrying capacity C is 100. This means that, according to the model, the maximum number of fish the pond can support is 100, so the population never exceeds this capacity. The table of values and the graph of f confirm this property of the model.

■ **NOW TRY EXERCISE 31** ■

3.3 Exercises

CONCEPTS

Fundamentals

1. A population is modeled by $f(x) = 8 \cdot 5^x$.

(a) The *average rate of change* of f _____ (is constant/varies) on different intervals.

(b) The *percentage rate of change* of f _____ (is constant/varies) on all intervals of length 1.

(c) The percentage rate of change of f between $x = 3$ and $x = 4$ is _____

x	y	Percentage rate of change
0	6	—
1	18	300%
2	54	
3	162	
4	486	

2. A set of data with equally spaced inputs is given in the table in the margin. Complete the table for the percentage rate of change of the outputs y.

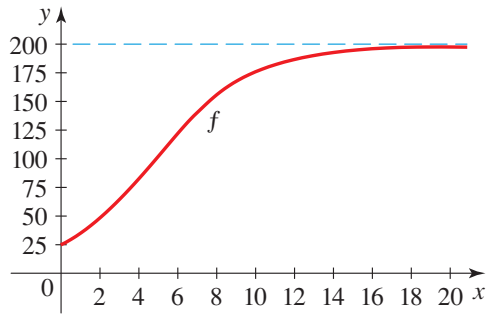
(a) Since the percentage rate of change _____ (is constant/varies), there is an exponential model $f(x) = C \cdot a^x$ that fits the data.

(b) The percentage rate of change is the constant _____%, so the growth rate r is _____, and the growth factor a is _____.

(c) An exponential model that fits the data is $f(x) = \square \cdot \square^x$.

3. A population with limited resources is modeled by the logistic growth function $f(x) = \dfrac{500}{1 + 24 \cdot a^{-x}}$. The initial population is _____, and the *carrying capacity* is _____.

4. The population of a species with limited resources is modeled by a logistic growth function f. Use the graph of f to estimate the initial population and the *carrying capacity*.



Think About It

5. True or false?

(a) The function $f(x) = 2x + 5$ has constant average rate of change on all intervals.

(b) The function $f(x) = 6^x$ has constant percentage rate of change on all intervals of length one.

6. Give some real-world examples of population growth with limited resources. For each example, explain how the population would grow if there were unlimited resources.

SKILLS

7. Identify each type of growth as linear or exponential.

(a) Doubling every 5 years **(b)** Adding 1000 units each year

(c) Increasing by 6% each year **(d)** Multiplying by 2 each year

8. Identify each type of decay as linear or exponential.

(a) Decreasing by 10% each year **(b)** Decreasing by 100 units each year

(c) Half remains each year **(d)** Multiplying by $\frac{1}{3}$ each year

9–12 ■ A population is modeled by the given function f in terms of time t, where t is measured in hours. Make a table of values for f, the average rate of change of f, and the percentage rate of change of f in each time period.

9. $f(t) = 5000(1.3)^t$

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	5000	—	—
1	6500	1500	30%
2			
3			
4			

10. $f(t) = 1000(0.95)^t$

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	1000	—	—
1	950	−50	−5%
2			
3			
4			

11. $f(t) = 30{,}000(0.80)^t$

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	30,000	—	—
1	24,000	−6000	−20%
2			
3			
4			

12. $f(t) = 20{,}000(1.7)^t$

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	2000	—	—
1	3400	1400	70%
2			
3			
4			

13–18 ■ A data set with equally spaced inputs is given.

(a) Find the average rate of change and the percentage rate of change for consecutive outputs.

(b) Is an exponential model appropriate? If so, find the model and sketch a graph of the model together with a scatter plot of the data.

13.

x	y
0	30,000
1	57,000
2	108,300
3	205,770
4	390,963

14.

x	y
0	500
1	450
2	405
3	364.5
4	328.05

15.

x	y
0	20,000
1	12,000
2	7200
3	4320
4	2592

16.

x	y
0	7000
1	8250
2	9500
3	10,750
4	12,000

17.

x	y
0	3000
1	2440
2	1880
3	1320
4	760

18.

x	y
0	40,000
1	84,000
2	176,400
3	370,440
4	777,924

19–22 ■ Suppose that the function f is a model for linear growth and that g is a model for exponential growth, with both f and g functions of time t.

(a) Fill in the table by evaluating the functions f and g at the given values of t.

(b) Construct equations for the functions f and g in terms t.

(c) Use a graphing calculator to graph the functions f and g on the same screen for $0 \le t \le 10$. What does the graph tell us about the differences between the two functions' behaviors?

19.

t	$f(t)$
0	200
1	350
2	
3	
4	

t	$g(t)$
0	200
1	350
2	
3	
4	

20.

t	$f(t)$
0	90
1	10
2	
3	
4	

t	$g(t)$
0	90
1	10
2	
3	
4	

21.

t	$f(t)$
0	500
1	300
2	
3	
4	

t	$g(t)$
0	500
1	300
2	
3	
4	

22.

t	$f(t)$
0	40
1	70
2	
3	
4	

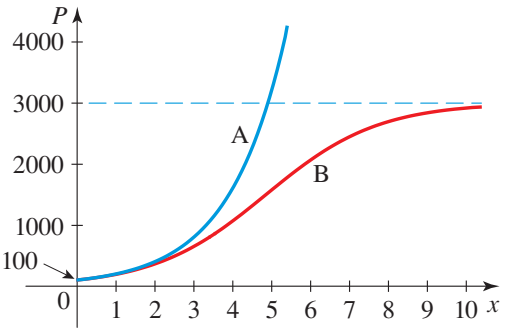
t	$g(t)$
0	40
1	70
2	
3	
4	

23. The graphs of two population models are shown. One grows exponentially and the other grows logistically.

(a) Which population grows exponentially?

(b) Which population grows logistically? What is the carrying capacity?

(c) What is the initial population for A and for B?



24. The function $f(t) = 100 \cdot 2^t$ is a model for a Population A, where t is measured in years. The function

$$g(t) = \frac{3000}{1 + 29 \cdot 2^{-t}}$$

is a model for a Population B, where t is measured in years.

(a) Fill in the tables in the margin for the functions f and g for the given values of t.

(b) Are the initial populations for A and B the same?

(c) Which population grows exponentially?

(d) Which population grows logistically? What is the carrying capacity?

t	$f(t)$
0	
2	
4	
6	
8	
10	

t	$g(t)$
0	
2	
4	
6	
8	
10	

CONTEXTS

25. World Population The population of the world was 6.454 billion in 2005 and 6.555 billion in 2006. Assume that the population grows exponentially.

(a) Find an exponential growth model $f(t) = Ca^t$ for the population t years since 2005. What is the growth rate?

(b) Sketch a graph of the function found in part (a), and plot the points $(0, f(0))$, $(1, f(1))$, $(3, f(3))$, and $(4, f(4))$.

(c) Use the model found in part (a) to find the average rate of change from 2005 to 2006 and from 2008 to 2009. Is the average rate of change the same on each of these time intervals?

(d) Use the model to find the percentage rate of change from 2005 to 2006 and from 2008 to 2009. Is the percentage change the same on each of these time intervals?

26. Population of California The population of California was 29.76 million in 1990 and 33.87 million in 2000. Assume that the population grows exponentially.

(a) Find an exponential growth model $f(t) = Ca^t$ for the population t years since 1990. What is the annual growth rate?

(b) Sketch a graph of the function found in part (a), and plot the points $(0, f(0))$, $(1, f(1))$, $(10, f(10))$, and $(11, f(11))$.

(c) Use the model found in part (a) to find the average rate of change from 1990 to 1991 and from 2000 to 2001. Is the average rate of change the same on each of these time intervals?

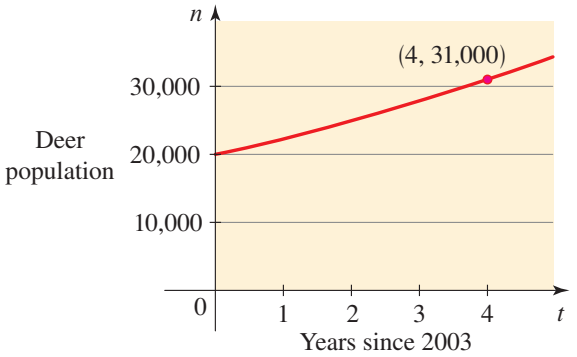
(d) Use the model to find the percentage rate of change from 1990 to 1991 and from 2000 to 2001. Is the percentage change the same on each of these time intervals?

27. Deer Population The graph shows the deer population in a Pennsylvania county between 2003 and 2007. Assume that the population grows exponentially.

(a) What was the deer population in 2003?

(b) Find an exponential growth model $n(t) = Ca^t$ for the population t years since 2003. What is the annual growth rate?

(c) Use the model to find the percentage rate of change from 2005 to 2006. Compare your answer to the growth rate from part (b).

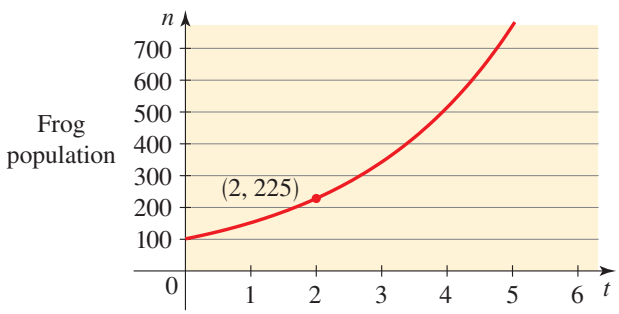


28. Frog Population Some bullfrogs were introduced into a small pond. The graph shows the bullfrog population for the next few years. Assume that the population grows exponentially.

(a) What was the initial bullfrog population?

(b) Find an exponential growth model $n(t) = Ca^t$ for the population t years since the bullfrogs were put into the pond. What is the annual growth rate?

(c) Use the model to find the percentage rate of change from $t = 2$ to $t = 3$. Compare your answer to the growth rate from part (b).



29. Salary Comparison Suppose you are offered a job that lasts three years and you are to be very well paid. Which of the following methods of payment is more profitable for you?

Offer A: You are paid \$10,000 in the starting month (month 0) and get a \$1000 raise each month.

Offer B: You are paid 2 cents in month 0, 4 cents in month 1, 8 cents in month 2, and so on, doubling your pay each month.

(a) Complete the salary tables for Offer A and Offer B.

Offer A

Month	Monthly salary ($)	Average rate of change
0	10,000	—
1	11,000	1000
2		
3		
4		
5		

Offer B

Month	Monthly salary ($)	Percentage rate of change
0	0.02	—
1	0.04	100%
2		
3		
4		
5		

(b) Find a function f_A that models your salary in month t if you accept Offer A. What kind of function is f_A?

(c) Find a function f_B that models your salary in month t if you accept Offer B. What kind of function is f_B?

(d) Find the salary for the last month (month 35) for each offer. Which offer gives the highest salary in the last month?

30. Pesticide Cleanup Many pesticides for citrus trees are toxic to fish. While the citrus groves on a large farm were treated with pesticides, high winds deposited 1000 gallons of pesticides into a lake stocked with fish. To ensure survival of the fish, 95% of the pesticides must be removed from the lake within 96 hours of contamination. There are two methods of removal:

Method A: A chemical reagent is added to the pond to neutralize the pesticide. This chemical reagent removes the pesticide at a rate of 5 gal/h.

Method B: The pond is fitted with a filtration system that removes 9% of the remaining pesticides each hour.

(a) Complete the tables for Method A and Method B.

(b) Find a function f_A that models the amount of pesticide remaining after t hours if Method A is used. What kind of function is f_A?

(c) Find a function f_B that models the amount of pesticide remaining after t hours if Method B is used. What kind of function is f_B?

Method A

Hours	Total amount of pesticide remaining (gal)	Average rate of change
0	1000	—
1	995	−5
2		
3		
4		
5		

Method B

Hours	Total amount of pesticide remaining (gal)	Percentage rate of change
0	1000	—
1	910	−9%
2		
3		
4		
5		

(d) Determine the amount of pesticide remaining after 96 hours for each method of removal. Which method would save the fish?

31. Limited Fox Population The fox population on a small island behaves according to the logistic growth model

$$n(t) = \frac{1200}{1 + 11 \cdot (1.7)^{-t}}$$



where t is the number of years since the fox population was first observed.

(a) Find the initial fox population.

(b) Make a table of values of f for t between 0 and 20. From the table, what can you conclude happens to the fox population as t increases?

(c) Use a graphing calculator to draw a graph of the function $n(t)$. From the graph, what can you conclude happens to the fox population as t increases?

(d) What is the carrying capacity? Does this answer agree with the table in part (b) and the graph in part (c)?

32. Limited Bird Population The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the logistic growth model

$$n(t) = \frac{11{,}200}{1 + 27 \cdot (1.85)^{-t}}$$

where t is the number of years since the bird population was first observed.

(a) Find the initial bird population.

(b) Make a table of values of f for t between 0 and 20. From the table, what can you conclude happens to the bird population as t increases?

(c) Use a graphing calculator to draw a graph of the function $n(t)$. From the graph, what can you conclude happens to the bird population as t increases?

(d) What is the carrying capacity? Does this answer agree with the table in part (b) and the graph in part (c)?

33. Limited World Population The growth rate of world population has been decreasing steadily in recent years. On the basis of this, some population models predict that world population will eventually stabilize at a level that the planet can support. One such logistic model is

$$P(t) = \frac{12}{1 + 0.97 \cdot (1.020214)^{-t}}$$

where $t = 0$ represents the year 2000 and population is measured in billions.

(a) What world population does this model predict for the year 2200? For 2300?

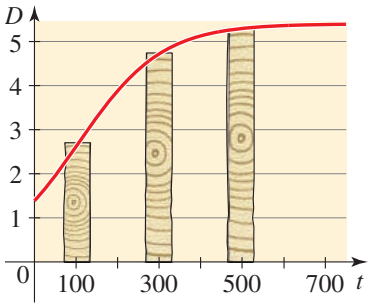
(b) Use a graphing calculator to draw a graph of the function P for the years 2000 to 2500.

(c) From the graph, what can you conclude happens to the world population as t increases?

(d) What is the carrying capacity? Does this answer agree with the graph in part (c)?

34. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years) according to the logistic growth model

$$D(t) = \frac{5.4}{1 + 2.9 \cdot (1.01009)^{-t}}$$



(a) What tree diameter does this model predict for a 150-year-old tree?

(b) Use the graph of D shown in the margin to determine what happens to the tree diameter as t increases.

(c) What is the carrying capacity? Does this answer agree with the graph?

3.4 Graphs of Exponential Functions

- Graphs of Exponential Functions
- The Effect of Varying *a* or *C*
- Finding an Exponential Function from a Graph

IN THIS SECTION... we study exponential functions on their entire domain (not only for positive numbers as in the preceding section), their graphs, and their rates of change.

GET READY... by reviewing the properties of exponents in **Algebra Toolkits A.3** and **A.4**.

Graphs of exponential growth or decay models allow us to visually compare the effects of different rates of growth or different initial values. In this section we study graphs of exponential functions in more detail.

Graphs of Exponential Functions

In Section 3.1 we modeled exponential growth and decay using functions of the form $f(x) = Ca^x$, for positive values of x. Here we study such functions for all values of x.

Exponential Functions

An **exponential function with base** a is a function of the form

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$. The domain of f is the set of all real numbers.

Rules of Exponents are reviewed in Algebra Toolkits A.3 and A.4, pages T14 and T20.

In evaluating exponential functions, we use the Rules of Exponents to get

$$a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$$

Notice that if $a > 1$, then $1/a < 1$.

example 1 **Graphing Exponential Functions**

Graph the exponential function $f(x) = 2^x$.

Solution

We calculate values of $f(x)$ and plot points to get the graph in Figure 1. For instance, to calculate $f(-3)$, we use the Rules of Exponents to get

$$f(-3) = 2^{-3} = \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}$$

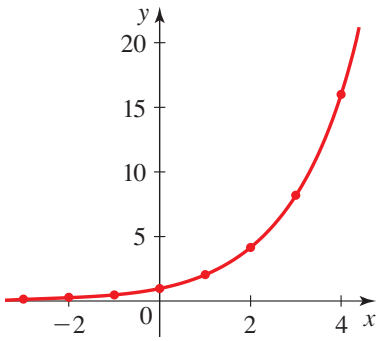


figure **1** Graph of $f(x) = 2^x$

The other entries in the table are calculated similarly.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

NOW TRY EXERCISE 5

The graphs of exponential functions have two different shapes, depending on whether the base a is greater than or less than 1. We graph both types in the next example.

example **2**

Graphing Exponential Functions

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\frac{1}{3}\right)^x$

Solution

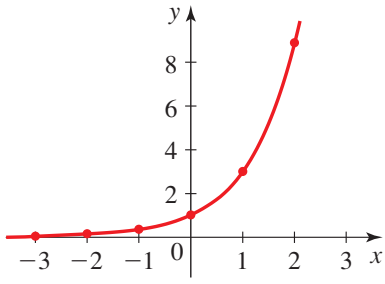
We calculate values of $f(x)$ and $g(x)$ and plot points to get the graph in Figure 2. For instance, to calculate $g(-2)$ we use the Rules of Exponents to get

$$g(-2) = \left(\tfrac{1}{3}\right)^{-2} = 3^2 = 9$$

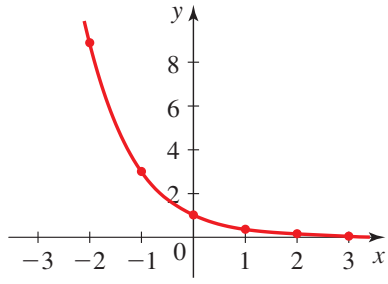
The other entries in the tables are calculated similarly.

x	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

x	-2	-1	0	1	2	3
$g(x)$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



(a) $f(x) = 3^x$



(b) $g(x) = \left(\frac{1}{3}\right)^x$

figure **2** Graphs of exponential functions

NOW TRY EXERCISE 23

From Example 2 we see that the graph of an exponential function always lies entirely above the x-axis. The graph gets close to the x-axis, but never crosses it. This means the x-axis is a *horizontal asymptote* of the graph of the function.

Graphs of Exponential Functions

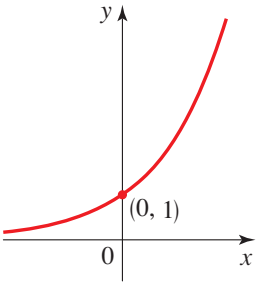
For the exponential function

$$f(x) = a^x \qquad a > 0,\ a \neq 1$$

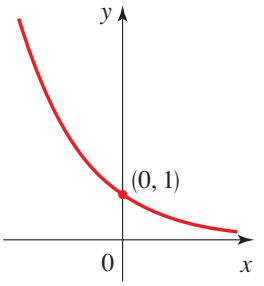
- The domain is all real numbers, and the range is all positive real numbers.
- The line $y = 0$ (the x-axis) is a *horizontal asymptote* of f.
- The graph of f has one of the shapes shown below.

  If $a > 1$, then f is an increasing function.
  If $0 < a < 1$, then f is a decreasing function.

Informally, an **asymptote** of a function is a line to which the graph of the function gets closer and closer as one travels along the line.



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

The Effect of Varying *a* or *C*

In the next two examples we take a closer look at how changing the base a and multiplying by a constant C affect the shape of the graph of an exponential function.

example 3 The Effect of Varying the Base *a*

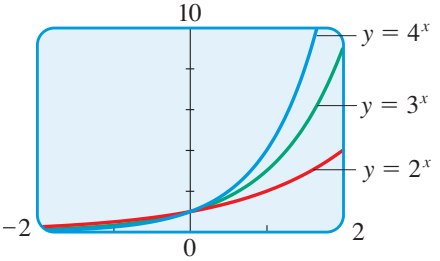
Graph the family of exponential functions for the given values of a. Explain how changing the value of a affects the graph.

(a) $f(x) = a^x$; $a = 2, 3, 4$ **(b)** $g(x) = a^x$; $a = 0.2, 0.3, 0.4$

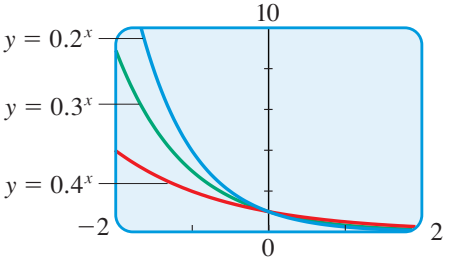
Solution

The graphs are shown in Figure 3.

(a) From Figure 3(a) we see that if the base a is greater than 1, then the larger the value of a, the more rapidly the function increases.



(a) $f(x) = a^x$; $a = 2, 3, 4$



(b) $g(x) = a^x$; $a = 0.2, 0.3, 0.4$

figure 3

(b) From Figure 3(b) we see that if the base a is less than 1, then the smaller the value of a, the more slowly the function decreases.

NOW TRY EXERCISES 15 AND 17

example **4**

Different Models for World Population

In 2000 the population of the world was 6.1 billion and was growing exponentially with a growth rate of 0.014. It is claimed that reducing the rate to 0.01 would make a significant difference in the total population in just a few decades. Test this claim graphically and algebraically as follows.

(a) Find exponential models for world population for each growth rate.

(b) Graph the models for the years 2000 to 2100 in the same viewing rectangle. What do the graphs tell us about world population?

(c) Find the predicted world population in 2100 for each model.

Solution

(a) The growth factors are $1 + 0.014 = 1.014$ and $1 + 0.01 = 1.01$. So the models we seek are

$$n(t) = 6.1(1.014)^t \qquad \text{and} \qquad m(t) = 6.1(1.01)^t$$

where t is measured in years since 2000 and $n(t)$ and $m(t)$ are measured in billions.

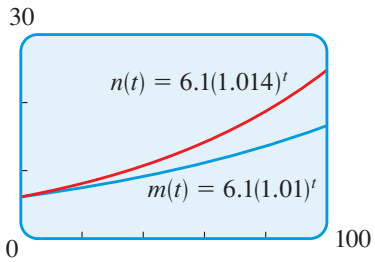


figure **4**

(b) The graph in Figure 4 shows that a small change in the relative rate of growth will, over time, make a big difference in population size.

(c) The year 2050 is 50 years after the year 2000. So replacing t by 50 in each model, we get

$$n(50) = 6.1(1.014)^{50} \approx 12.2 \qquad \text{and} \qquad m(50) = 6.1(1.01)^{50} \approx 10.0$$

So the models predict populations of about 12.2 billion and 10.0 billion, a difference of over 2 billion people.

NOW TRY EXERCISE 35

example **5**

The Effect of Varying C

Use a graphing calculator to graph the family of exponential functions

$$f(x) = C \cdot 3^x \qquad C = 10, 20, 30$$

Explain how changing the value of C affects the graph.

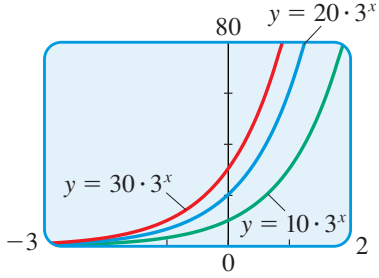


figure **5**
$f(x) = C \cdot 3^x, \quad C = 10, 20, 30$

Solution

Using a graphing calculator, we graph the family of exponential functions $f(x) = C \cdot 3^x$ for $C = 10, 20, 30$. From the graphs in Figure 5 we see that the value C is the y-intercept of the exponential function $f(x) = Ca^x$. Increasing the value of C has the effect of "stretching" the graph vertically.

NOW TRY EXERCISE 19

example 6 Different Amounts of a Radioactive Substance

A radioactive substance has a decay rate of −0.02 per year. A 100-g sample and a 200-g sample of this substance are kept in a safe storage facility. Compare the amounts remaining in each sample after t years graphically and algebraically as follows.

(a) For each sample, find an exponential model for the amount remaining after t years.

(b) Graph the models for t between 0 and 50 in the same viewing rectangle.

(c) Find the predicted amounts remaining in each sample after 100 years.

Solution

(a) Since the decay rate is −0.02, the decay factor is $1 + (-0.02) = 0.98$. So the models we seek are

$$n(t) = 100(0.98)^t \qquad \text{and} \qquad m(t) = 200(0.98)^t$$

where t is measured in years and $n(t)$ and $m(t)$ are measured in grams.

(b) The graphs are shown in Figure 6.

(c) To find the amounts remaining after 100 years, we replace t by 100 in each model.

$$n(100) = 100(0.98)^{100} \approx 13.3 \qquad \text{and} \qquad m(100) = 200(0.98)^{100} \approx 26.5$$

So the models predict amounts of 13.3 g and 26.5 g.

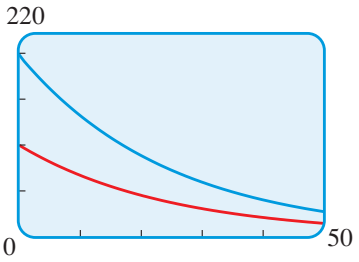


figure 6

■ NOW TRY EXERCISE 33 ■

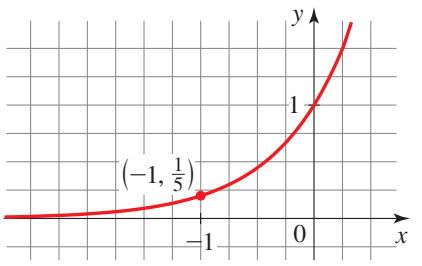
Finding an Exponential Function from a Graph

In the preceding examples we graphed several exponential functions. Now we see how to find an equation for an exponential function if we are given its graph.

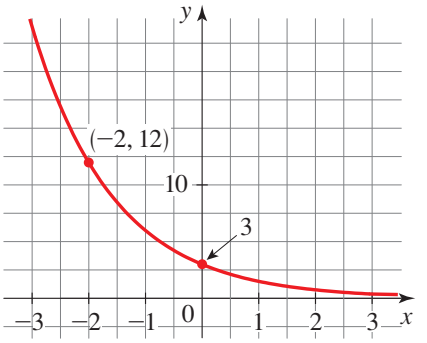
example 7 Finding an Exponential Function from a Graph

Find the function $f(x) = Ca^x$ whose graph is given.

(a)



(b)



Solution

(a) From the graph, $f(0) = Ca^0 = 1$, so C is 1. Also, $f(-1) = Ca^{-1} = \frac{1}{5}$. We solve for a:

$$Ca^{-1} = \tfrac{1}{5} \quad \text{Given}$$
$$a^{-1} = \tfrac{1}{5} \quad \text{Replace } C \text{ by 1}$$
$$a = 5 \quad \text{Rules of Exponents}$$

So $f(x) = 5^x$.

(b) From the graph, $f(0) = 3$, so C is 3. Also $f(-2) = Ca^{-2} = 12$. We solve for a:

$$Ca^{-2} = 12 \quad \text{From the graph}$$
$$3 \cdot a^{-2} = 12 \quad \text{Replace } C \text{ by 3}$$
$$a^{-2} = 4 \quad \text{Divide by 3}$$
$$(a^{-2})^{-1/2} = (4)^{-1/2} \quad \text{Raise both sides to the } -1/2 \text{ power}$$
$$a = \tfrac{1}{2} \quad \text{Rules of Exponents}$$

So $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$.

In Example 5 we solve power equations like $a^{-2} = 4$. Solving such equations is reviewed in Algebra Toolkit C.1, page T47.

■ **NOW TRY EXERCISES 29 AND 31** ■

3.4 Exercises

CONCEPTS

Fundamentals

1. Let $f(x) = 5^x$.

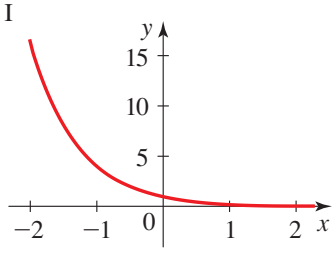
(a) The function f is an exponential function with base _____.

(b) $f(2) =$ _____, $f(0) =$ _____, and $f(-2) =$ _____.

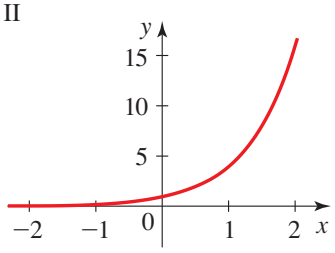
2. Match the exponential function with its graph.

(a) $f(x) = 4^x$ **(b)** $f(x) = 4^{-x}$

I



II



Think About It

3–4 ■ True or false?

3. The exponential function $f(x) = (0.25)^x$ is increasing.

4. The exponential function $f(x) = (2.5)^x$ is increasing.

SKILLS

5–6 ■ Fill in the table and sketch a graph of the given exponential function.

5. $f(x) = 6^x$

x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	

6. $g(x) = 5^{-x}$

x	$g(x)$
-3	
-2	
-1	
0	
1	
2	
3	

7–14 ■ Sketch a graph of the exponential function by making a table of values.

7. $f(x) = 4^x$

8. $g(x) = 2^{-x}$

9. $h(s) = 3^{-s}$

10. $s(t) = (1.1)^t$

11. $h(t) = \left(\frac{1}{7}\right)^t$

12. $I(t) = (0.1)^t$

13. $R(x) = (1.7)^x$

14. $n(T) = \left(\frac{4}{3}\right)^{-T}$

15–18 ■ Use a graphing calculator to graph the family of exponential functions for the given values of a. Explain how changing the value of a affects the graph.

15. $f(x) = a^x; \quad a = 4, 6, 8$

16. $f(x) = a^x; \quad a = 2, 3, 10$

17. $f(x) = a^x; \quad a = 0.5, 0.7, 0.9$

18. $f(x) = a^x; \quad a = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$

19–22 ■ Use a graphing calculator to graph the family of exponential functions for the given values of C. Explain how changing the value of C affects the graph.

19. $f(x) = C \cdot 5^x; \quad C = 10, 20, 100$

20. $f(x) = C \cdot 5^x; \quad C = -1.0, -1.5, -2.0$

21. $f(x) = C \cdot 3^x; \quad C = -2, -3, -5$

22. $f(x) = C \cdot 3^x; \quad C = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

23–26 ■ The functions f and g are given.

(a) Using a graphing calculator, graph both functions in the same screen.

(b) Do the two graphs intersect? If so, find the intersection point.

23. $f(x) = 4^x$ and $g(x) = 4^{-x}$

24. $f(x) = 5^{-x}$ and $g(x) = \left(\frac{1}{5}\right)^{-x}$

25. $f(x) = \left(\frac{2}{3}\right)^x$ and $g(x) = \left(\frac{4}{3}\right)^x$

26. $f(x) = 4^x$ and $g(x) = 7^x$

27–28 ■ The functions f and g are given.

(a) Graph both functions in the given viewing rectangle.

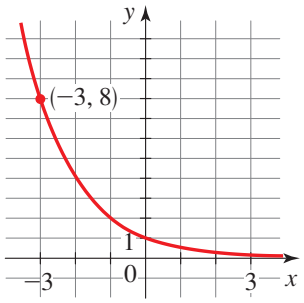
(b) Comment on how the graphs are related. Where do they intersect?

27. $f(x) = 3 \cdot 2^x$ and $g(x) = 2^{3x}; \quad [-1, 3]$ by $[0, 60]$

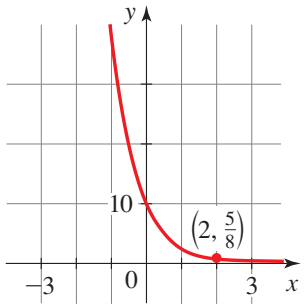
28. $f(x) = 2 \cdot 5^x$ and $g(x) = 5^{2x}$; $[-0.5, 2]$ by $[0, 40]$

29–32 ■ Find the function $f(x) = Ca^x$ whose graph is given.

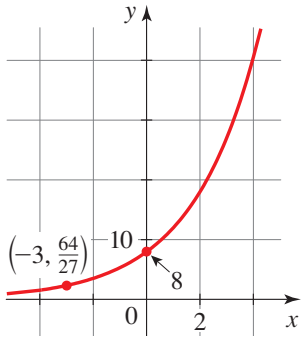
29.



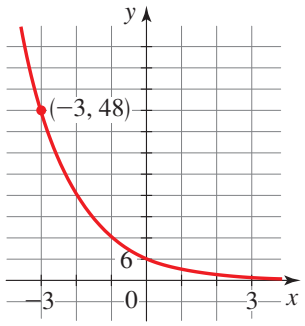
30.



31.



32.



CONTEXTS

33. Bacteria To prevent bacterial infections, it is recommended that you wash your hands and cooking utensils as often as possible. At a family barbeque, Jim prepares his meat without using proper sanitary precautions. There are about 100 colony-forming units (CFU/mL) of a certain type of bacteria on the meat. Harold prepares his meat using proper sanitary precautions, so there are only 2 CFU/mL of the bacteria on his meat. It is known that the doubling time for this type of bacteria is 20 minutes.

(a) For each situation, find an exponential model for the number of bacteria on the meat t hours since it was prepared.

(b) Graph the models for t between 0 and 6. What do the graphs tell us about the number of bacteria present on the meat?

(c) For each model, find the predicted number of bacteria after 6 hours.

34. Bacteria The bacterium *Streptococcus pyogenes* (*S. pyogenes*) is the cause of many human diseases, the most common being strep throat. The doubling time of these bacteria is 30 minutes, but in the presence of a certain antibiotic the doubling time is 6 hours. Two cultures are prepared, and the antibiotic is added to one of the cultures. Initially, each culture has 200 bacteria.

(a) Find an exponential model for the number of *S. pyogenes* bacteria in each culture after t hours.

(b) Graph the models for t between 0 and 8. What do the graphs tell us about the number of bacteria?

(c) For each model, find the predicted number of bacteria after 8 hours.

35. Investment Value On January 1, 2005, Marina invested $2000 in each of three mutual funds: a growth fund with an estimated yearly return of 10%, a bond fund with

an estimated yearly return of 6%, and a real estate fund with an estimated yearly return of 17%.

(a) Complete the table for the yearly value of each investment at the beginning of the given year.

Year	Growth fund value ($)	Bond fund value ($)	Real estate fund value ($)
2005	2000	2000	2000
2006			
2007			
2008			

(b) Find exponential growth models for each of Marina's investments t years since 2005.

(c) Graph the functions found in part (b) for t between 0 and 10. Explain how the expected yearly returns affect the graphs.

(d) Predict the value of each investment in 2011.

36. Investment Value On January 1, 2005, Isabella invested a total of $15,000 in three different bond funds, each with an expected yearly return of 5.5%. She invested $2000 in Bond Fund A, $4000 in Bond Fund B, and $9000 in Bond Fund C.

(a) Complete the table for the value of each investment at the beginning of the given year.

Year	Bond Fund A value ($)	Bond Fund B value ($)	Bond Fund C value ($)
2005	2000	4000	9000
2006			
2007			
2008			

(b) Find exponential growth models for Isabella's investments t years since 2005.

(c) Graph the functions found in part (b) for t between 0 and 10. Compare how the initial values affect the graphs.

(d) Predict the value of each investment in 2011.

37. Health-Care Expenditures U.S. health-care expenditures have been growing exponentially during the past two decades. In 2008, expenditures were 2.4 trillion dollars with a growth rate of 9%. Lawmakers hope to decrease the annual growth rate to 2%. Compare the two rates graphically and algebraically as follows.

(a) For each growth rate, find an exponential model for the health-care expenditures t years since 2008.

(b) Graph the models for the years 2008 to 2020. What do the graphs tell us about the health-care expenditures?

(c) Find the predicted health-care expenditures in 2020 for each model.

38. Algebra and Alcohol After alcohol is fully absorbed into the body, it is metabolized with a half-life of 1.5 hours. Suppose Tom consumes 30 mL of alcohol (ethanol) and Ted consumes 15 mL.

(a) Find exponential decay models for the amount of alcohol remaining in each person's body after t hours.

(b) Graph the functions in part (a) for t between 0 and 4. What do the graphs tell us about the amount of alcohol remaining?

(c) Find the predicted amount of alcohol remaining in each person's body after 2 hours.

3.5 Fitting Exponential Curves to Data

- Finding Exponential Models for Data
- Is an Exponential Model Appropriate?
- Modeling Logistic Growth

IN THIS SECTION... we learn to fit exponential curves to data and to recognize when these curves are appropriate for modeling data. We also fit logistic curves to data.

GET READY... by reviewing Section 2.5 on fitting lines to data.

In Section 2.5 we learned how to find the line that best fits data—the line models the increasing or decreasing trend of the data. But what if a scatter plot of the data does not reveal any linear trend? In general, the shape of a scatter plot can help us choose the type of curve to use in modeling the data.

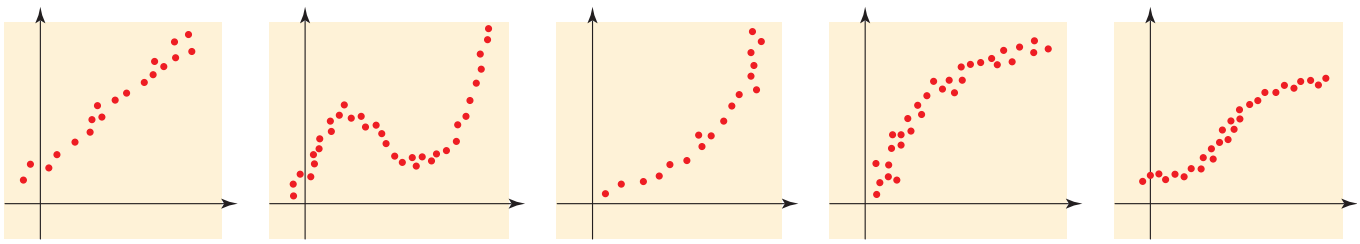


figure **1** Scatter plots of data

For example, the first plot in Figure 1 fairly begs for a line to be fitted through it. For the third plot, it seems that an exponential curve might fit better than a line. How do we decide which curve is the more appropriate model? We'll see that the properties of exponential curves that we studied in the preceding sections will help us to answer this question.

Finding Exponential Models for Data

If a scatter plot shows that the data increase rapidly, we might want to model the data using an exponential model, that is, a function of the form

$$f(x) = Ca^x$$

where C and a are constants. In the first example we model world population by an exponential model. Recall from Section 3.1 that population tends to increase exponentially.

example 1 An Exponential Model for World Population

table 1
World population

Year	x	Population P (millions)
1900	0	1650
1910	10	1750
1920	20	1860
1930	30	2070
1940	40	2300
1950	50	2520
1960	60	3020
1970	70	3700
1980	80	4450
1990	90	5300
2000	100	6060

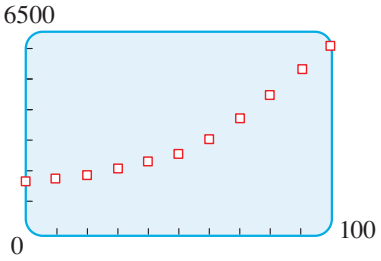


figure 2
Scatter plot of world population

Table 1 shows how the world population has changed in the 20th century.

(a) Draw a scatter plot and note that a linear model is not appropriate.

(b) Find an exponential function that models the data.

(c) Draw a graph of the function you found together with the scatter plot. How well does the model fit the data?

(d) Use the model you found to predict world population in 2020.

Solution

(a) The scatter plot is shown in Figure 2. The plotted points do not appear to lie along a straight line, so a linear model is not appropriate.

(b) Using a graphing calculator and the `ExpReg` command (see Figure 3(a)), we get the exponential model

$$P(t) = 1445(1.0137)^t$$

(c) From the graph in Figure 3(b) we see that the model appears to fit the data fairly well. The period of relatively slow population growth is explained by the depression of the 1930s and the two world wars.

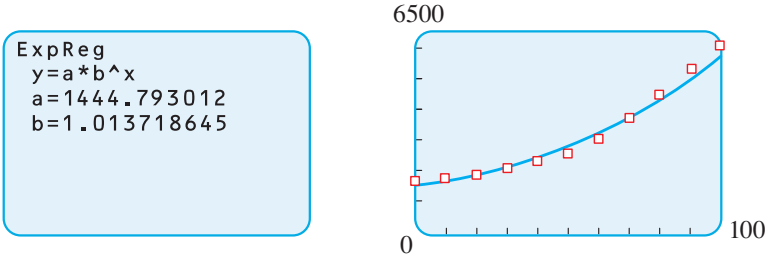


figure 3 Exponential model for world population

(d) The model predicts that the world population in 2020 (when x is 120) will be

$$P(120) = 1445(1.0137)^{120}$$
$$\approx 7396$$

So the model predicts a population of about 7400 million in 2020.

NOW TRY EXERCISE 13

Is an Exponential Model Appropriate?

Here is a way to tell whether an exponential model is appropriate. Recall from Section 3.4 that for equally spaced data the percent rate of change between consecutive points is constant. So if the inputs of our data are equally spaced, we can calculate the percent rate of change and see whether they are approximately constant. If they are, this would indicate that an exponential model is appropriate.

example 2 Using Percentage Rate of Change

table 2
Data

x	y
0	1500
1	1921
2	2366
3	2798
4	3237
5	3688

A set of data is given in Table 2.

(a) Find the average rate of change and percentage rate of change between consecutive data points.

(b) Determine whether a linear model or an exponential model is more appropriate.

Solution

(a) We first note that the inputs x are equally spaced, so we calculate the net change and the percent rate of change between consecutive data points (see the table below). Here is how the entries in the second row are calculated:

- The average rate of change in y from 0 to 1 is

$$\frac{1921 - 1500}{1 - 0} = 421$$

- The percentage rate of change in y from 0 to 1 is

$$\frac{1921 - 1500}{1500} = \frac{421}{1500} \approx 0.28$$

Expressed in percentage form, this last fraction is 28%. The remaining rows of the table are calculated in the same way.

x	$f(x)$	Average rate of change	Percentage rate of change
0	1500	—	—
1	1921	421	28%
2	2356	435	23%
3	2784	428	18%
4	3215	431	15%
5	3638	423	13%

(b) Since the average rate of change is approximately constant (but the percentage rate of change is decreasing), a linear model is more appropriate.

NOW TRY EXERCISE 9

Modeling Logistic Growth

In Section 3.3 we learned that a logistic growth model is a function of the form

$$N = \frac{C}{1 + b \cdot a^{-x}} \qquad a > 1$$

where a, b, and C are positive constants. Logistic functions are used to model populations in which the growth is constrained by available resources. When a graphing calculator is used to find the logistic function that best fits a given set of data,

the calculator uses the base e, which is approximately 2.718. We'll study the number e in more detail in Section 4.5. For now, we'll use the calculator output to graph models of logistic growth.

example 3 Stocking a Pond with Catfish

table 3

Week	Catfish
0	1000
15	1500
30	3300
45	4400
60	6100
75	6900
90	7100
105	7800
120	7900

Much of the fish sold in supermarkets today is raised on commercial fish farms, not caught in the wild. A pond on one such farm is initially stocked with 1000 catfish, and the fish population is then sampled at 15-week intervals to estimate its size. The population data are given in Table 3.

(a) Find an appropriate model for the data.

(b) Make a scatter plot of the data and graph the model you found in part (a) on the scatter plot.

(c) What does the model predict about how the fish population will change with time?

Solution

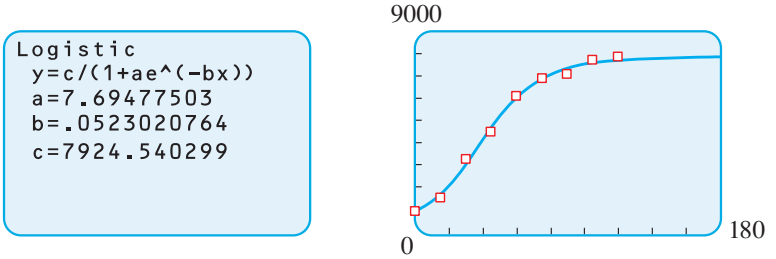
(a) Since the catfish population is restricted by its habitat (the pond), a logistic model is appropriate. Using the `Logistic` command on a calculator (see Figure 4(a)), we find the following model for the catfish population $P(t)$:

$$P(t) = \frac{7925}{1 + 7.7e^{-0.052t}}$$

where e is about 2.718.

(b) The scatter plot and the logistic curve are shown in Figure 4(b).

(c) From the graph of P in Figure 4(b) we see that the catfish population increases rapidly until about 80 weeks. Then growth slows down, and at about 120 weeks the population levels off and remains more or less constant at slightly over 7900.



(a) (b) Catfish population $y = P(t)$

figure 4

NOW TRY EXERCISE 19

The behavior exhibited by the catfish population in Example 3 is typical of logistic growth. After a rapid growth phase, the population approaches a constant level called the *carrying capacity* of the environment (see Section 3.2).

3.5 Exercises

CONCEPTS

Fundamentals

1–4 ■ Two data sets, each with equally spaced inputs, are given below.

1. Complete each table by finding the average rate of change and percentage rate of change for successive data points.

2. If the average rate of change is constant between successive data points, then _____ (an exponential/a linear) model is appropriate.

3. If the percentage rate of change is constant between successive data points, then _____ (an exponential/a linear) model is appropriate.

4. Find an appropriate model for each of the data sets.

Model: _____

x	y	Average rate of change	Percentage rate of change
0	10		
1	30		
2	50		
3	70		
4	90		
5	110		

Model: _____

x	y	Average rate of change	Percentage rate of change
0	10		
1	30		
2	90		
3	270		
4	810		
5	2430		

Think About It

5. Suppose we make a scatter plot of some given data. Can we tell from the scatter plot whether a linear or exponential model is appropriate?

6. Suppose we have data with equally spaced inputs and we calculate the growth factor between successive data points. How can we use our calculations to decide whether an exponential model is appropriate?

SKILLS

7–8 ■ A data set is given in the table.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential model for the data.

(c) Graph the model you found in part (b) together with the scatter plot. Does your model appear to be appropriate?

7.

x	y
0	12
2	34
4	99
6	275
8	783
10	2238

8.

x	y
0	500
3	571
6	647
9	742
12	851
15	978

9–12 ■ A data set is given, with equally spaced inputs.

(a) Fill in the table to find the average rate of change and the percentage rate of change between successive data points.

(b) Use your results to determine whether a linear model or an exponential model is appropriate, and use a calculator to find the appropriate model.

(c) Make a scatter plot of the data and graph the model you found in part (b). Does your model appear to be appropriate?

9.

x	$f(x)$	Average rate of change	Percentage rate of change
0	210	—	—
1	281	71	34%
2	379		
3	512		
4	689		
5	932		

10.

x	$f(x)$	Average rate of change	Percentage rate of change
0	368	—	—
1	333	−35	−10%
2	299		
3	265		
4	231		
5	196		

11.

x	$f(x)$	Average rate of change	Percentage rate of change
0	441	—	—
1	392	−49	−11%
2	345		
3	296		
4	248		
5	200		

12.

x	$f(x)$	Average rate of change	Percentage rate of change
0	120	—	—
1	202	82	68%
2	337		
3	569		
4	960		
5	1612		

CONTEXTS

13. U.S. Population The U.S. Constitution requires a census every 10 years. The census data for 1790–2000 are given in the table.

(a) Make a scatter plot of the data. Is a linear model appropriate?

(b) Use a calculator to find an exponential curve $f(x) = b \cdot a^x$ that models the population x years since 1790.

(c) Draw a graph of the function that you found together with the scatter plot. How well does the model fit the data?

(d) Use your model to predict the population at the 2010 census.

Year	x	Population (millions)	Year	x	Population (millions)
1790	0	3.9	1900	110	76.2
1800	10	5.3	1910	120	92.2
1810	20	7.2	1920	130	106.0
1820	30	9.6	1930	140	123.2
1830	40	12.9	1940	150	132.2
1840	50	17.1	1950	160	151.3
1850	60	23.2	1960	170	179.3
1860	70	31.4	1970	180	203.3
1870	80	38.6	1980	190	226.5
1880	90	50.2	1990	200	248.7
1890	100	63.0	2000	210	281.4

14. Health-Care Expenditures U.S. health-care expenditures for 1970–2008 (reported by the Centers for Medicare and Medicaid Services) are given in the table below.

(a) Make a scatter plot of the data. Is a linear model appropriate?

(b) Use a calculator to find an exponential curve $f(x) = b \cdot a^x$ that models U.S. health-care expenditures x years since 1970.

(c) Use your model to estimate U.S. health-care expenditures in 2001.

(d) Use your model to predict U.S. health-care expenditures in 2009.

(e) Search the Internet to find reported U.S. health-care expenditures for 2009. How does it compare with your prediction?

Year	x	Health-care expenditures (billions of $)	Year	x	Health-care expenditures (billions of $)
1970	0	75	1996	26	1069
1975	5	133	1998	28	1190
1980	10	253	2000	30	1353
1985	15	439	2002	32	1602
1990	20	714	2004	34	1855
1992	22	849	2006	36	2113
1994	24	962	2008	38	2400

15. Doubling Time of Bacteria A student is trying to determine the doubling time for a population of the bacterium *Giardia lamblia* (*G. lamblia*). He starts a culture in a nutrient solution and counts the bacteria every 4 hours. His data are shown in the table.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential curve $f(x) = b \cdot a^x$ that models the bacteria population x hours later.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the TRACE feature to determine how long it takes for the bacteria count to double.



Sebastian Kaulitzki/Shutterstock.com 2009

G. lamblia

Time (h)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	176

16. Half-Life of Radioactive Iodine A student is trying to determine the half-life of radioactive iodine-131. She measures the amount of iodine-131 in a sample solution every 8 hours. Her data are shown in the table.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential curve $f(x) = b \cdot a^x$ that models the amount of iodine-131 remaining after x hours.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the TRACE feature to determine the half-life of iodine-131.

Time (h)	0	8	16	24	32	40	48
Amount of ^{131}I (g)	4.80	4.66	4.51	4.39	4.29	4.14	4.04

17. Hybrid Car Sales The table shows the number of hybrid cars sold in the United States for the period 2000–2007.

(a) Complete the table to find the one-year growth factor for each one-year time period. For instance, the one-year growth factor for the time period 2000–2001 is the ratio of the sales in 2001 to the sales in 2000, as calculated in the last two columns in the table.

(b) Find the "average" one-year growth factor A. Do this by finding the average of all the growth factors you found in the last column of the table.

(c) It seems reasonable to use the average growth factor in part (b) to construct an exponential model of the form $f(x) = ba^x$ for the number of hybrid cars sold in year x. Use your answer from part (b) to construct such a model.

(d) Use a graphing calculator to find the exponential model of best fit for the number of hybrid cars sold in year x, where $x = 0$ represents the year 2000.

(e) Graph the models from (c) and (d) together with a scatter plot of the sales data.

Hybrid car sales

Year	x	**Number of hybrid cars sold (thousands)**	$\frac{\text{sales in year } x+1}{\text{sales in year } x}$	**Growth factor**
2000	0	9.5	—	—
2001	1	20.3	20.3/9.5	2.14
2002	2	35.0		
2003	3	43.4		
2004	4	85.0		
2005	5	205.8		
2006	6	254.5		
2007	7	347.1		

18. Population of Belgium Many highly developed countries, particularly in Europe, are finding that their population growth rates are declining and that a logistic function provides a much more accurate model than an exponential one for their population. The table gives $B(t)$, the midyear population of Belgium (in millions), for years between 1980 and 2000, where t represents the number of years since 1980.

(a) Use a graphing calculator (TI-89 or better) to find a logistic model for Belgium's population.

(b) Make a scatter plot of the data, using a range of 9.8 to 10.3 on the y-axis. Graph the logistic function you found in part (a) on the scatter plot. Does it seem to fit the data well?

(c) What does the model predict about Belgium's long-term population trend? At what value will the population level off?

Year	1980	1982	1984	1986	1988	1990	1992	1994	1996	1998	2000
t	0	2	4	6	8	10	12	14	16	18	20
$B(t)$	9.85	9.86	9.86	9.86	9.88	9.96	10.04	10.11	10.15	10.18	10.19



19. Logistic Population Growth The table gives the population of black flies in a closed laboratory container over an 18-day period.

(a) Use the `Logistic` command on your calculator to find a logistic model for these data.

(b) Graph the model you found in part (a) together with a scatter plot of the data.

(c) According to the model, how does the fly population change with time? At what value will the population level off?

Time (days)	0	2	4	6	8	10	12	16	18
Number of flies	10	25	66	144	262	374	446	494	498

CHAPTER 3 REVIEW

CHAPTER 3 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

3.1 Exponential Growth and Decay

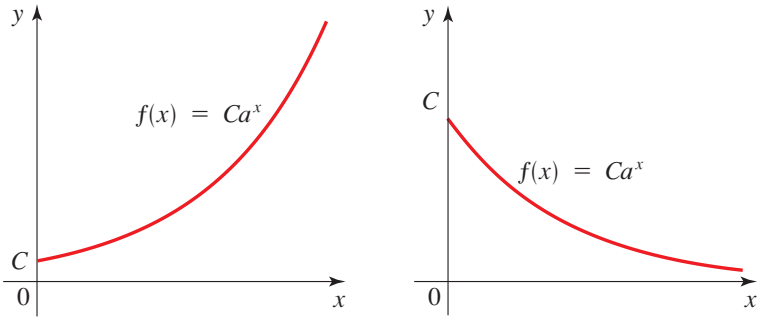
Exponential growth and **decay** are modeled by functions of the form

$$f(x) = Ca^x \qquad a > 0, a \neq 1$$

where f models *growth* if $a > 1$ and *decay* if $0 < a < 1$. In this model,

- the variable x is the number of **time periods**;
- the base a is the **growth** or the **decay factor**; and
- the constant C is the **initial value** of f, that is, $C = f(0)$.

The graph of f has one of the following shapes, depending on whether it is a model for growth or decay.



Exponential growth, $a > 1$ Exponential decay, $0 < a < 1$

In a growth (or decay) model, the **growth** (or **decay**) **rate** is the proportion by which the quantity being modeled grows (or decays) in each time period, expressed as a fraction, decimal, or percentage. The growth (or decay) rate r is related to the growth (or decay) factor a by the formula

$$a = 1 + r$$

For a growth model we have $r > 0$, and for a decay model we have $r < 0$, because in a *growth* model, the amount $f(x)$ is *increasing*, whereas in a *decay* model, the amount $f(x)$ is *decreasing*.

3.2 Exponential Models: Comparing Rates

We may wish to **change the length of the time period** that we use to measure the time x in a growth or decay model. (For instance, we may prefer to measure time in days instead of weeks.) Let t represent the number of new time intervals that correspond to x old time intervals. If x and t are related by $x = kt$, then the exponential model $f(x) = Ca^x$ can be rewritten as the model

$$f(t) = Ca^{kt} = Cb^t$$

where $b = a^k$.

An example of exponential growth is the growth of money invested in an account that pays **compound interest**. This is modeled by the exponential function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

- $A(t)$ is the amount in the account after t years.
- P is the original principal that was invested.
- r is the annual interest rate (expressed as a decimal), compounded n times a year.

The rate of **radioactive decay** is usually expressed in terms of the **half-life**, the time required for a sample to decay to half of its initial mass. The decay model can be expressed as

$$m(t) = C\left(\tfrac{1}{2}\right)^{t/h}$$

where $m(t)$ is the mass remaining of an initial mass C after time t, expressed in the same units as the half-life h.

3.3 Comparing Linear and Exponential Growth

Whereas linear models have constant average rate of change, **exponential** growth or decay models have **constant percentage rate of change**. The percentage rate of change of an exponential function is the average rate of change between any two inputs that are one unit apart (x and $x + 1$), expressed as a percentage of the value at x. This difference is the main feature that sets the two types of models apart.

When population growth experiences limited resources, a **logistic growth model** is often appropriate. A logistic model is a function of the form

$$f(x) = \frac{C}{1 + b \cdot a^{-x}} \qquad a > 1$$

Here, x is the number of time periods, and C is the **carrying capacity**, the maximum population the resources can support.

3.4 Graphs of Exponential Functions

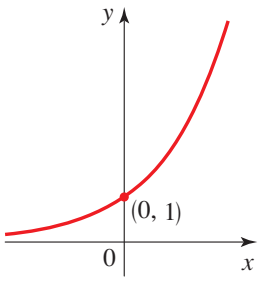
An **exponential function with base** a is a function of the form

$$f(x) = a^x \qquad a > 0, a \neq 1$$

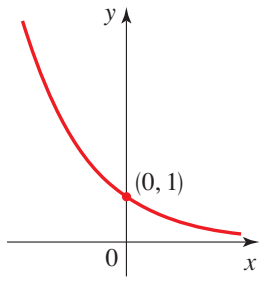
Exponential functions have the following properties:

- The domain is all real numbers, and the range is all positive real numbers.
- The line $y = 0$ (the x-axis) is a *horizontal asymptote* of f.
- The graph of f has one of the following shapes:

  If $a > 1$, then f is an increasing function.
  If $a < 1$, then f is a decreasing function.



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

3.5 Fitting Exponential Curves to Data

Suppose that the scatter plot of a data set indicates that the data seem to be best modeled by an exponential function. Then we can use the `ExpReg` feature on a graphing calculator to find the **exponential curve**

$$y = a \cdot b^x$$

that best fits the data.

To determine whether an exponential model is appropriate, we can check to see whether the **percentage rate of change** between equally spaced inputs is approximately **the same**. If so, then an **exponential model is appropriate**.

When modeling population growth that is limited by the availability of essential resources, we can use the `Logistic` feature on a graphing calculator to find the **logistic growth model** of the following form to best fit the data (where a, b, and C are positive constants):

$$N = \frac{C}{1 + b \cdot a^{-x}}$$

CHAPTER 3 REVIEW EXERCISES

SKILLS

1–4 ■ The given function is a model for exponential growth or decay.

(a) Is it a growth model or a decay model?

(b) What is the growth or decay factor?

(c) What is the growth or decay rate?

(d) What is the initial amount?

1. $f(x) = 1000(1.25)^x$

2. $g(x) = 1250(0.97)^x$

3. $h(x) = 8\left(\frac{3}{5}\right)^x$

4. $k(x) = \left(\frac{17}{5}\right)^x$

5–8 ■ A bacteria population P initially has the given size C, and in each 3-hour time period it experiences the growth (or decay) described.

(a) Find an exponential model for the population as a function of x, the number of 3-hour time periods that have passed.

(b) Find a model for the population as a function of t, the number of hours that have passed.

(c) Use both models to predict the population after 12 hours. Do you get the same answer from each model?

5. $C = 200$, P doubles

6. $C = 14{,}000$, P is cut in half

7. $C = 1320$, P decreases by 30%

8. $C = 70$, P increases by 25%

9–10 ■ An investment of \$24,000 is invested in a CD paying the given annual interest rate, compounded as indicated. Find the amount in the account after each given time period.

9. 3.6% interest, compounded monthly

(a) 1 year **(b)** 3 years **(c)** 4 years and 6 months

10. 1.2% interest, compounded daily

(a) 2 years **(b)** 10 years **(c)** 50 years

11–12 ■ A radioactive element, its initial mass, and its half-life h (in years) are given.

(a) Find a function of the form $m(t) = C\left(\frac{1}{2}\right)^{t/h}$ that models the mass of the element remaining after t years.

(b) Find the mass remaining after 25 years.

(c) Express the model in the form $m(t) = Ca^t$, where a is the annual decay factor.

(d) What is the annual decay rate?

11. Radium-226, 300 kg, 1600 years

12. Strontium-90, 35 g, 28 years

13–14 ■ Complete the table by calculating the average rate of change and the percentage rate of change for f. Then find an exponential model for the given data.

13.

t	$f(t)$	Average rate of change	Percentage rate of change
0	5000	—	—
1	6000		
2	7200		
3	8640		

14.

t	$f(t)$	Average rate of change	Percentage rate of change
0	270	—	—
1	180		
2	120		
3	80		

15–18 ■ Sketch a graph of the exponential function by first making a table of values. Is the function increasing or decreasing?

15. $f(x) = \left(\frac{3}{2}\right)^x$

16. $g(x) = (0.6)^x$

17. $h(t) = \frac{1}{2}\left(\frac{4}{5}\right)^t$

18. $k(t) = \frac{2}{3} \cdot 6^{-t}$

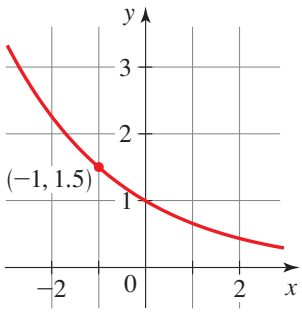
19–20 ■ Use a graphing calculator to draw graphs of both f and g on the same screen. Find the intersection point of the graphs.

19. $f(x) = 3^x$ and $g(x) = 2 \cdot 4^{-x}$

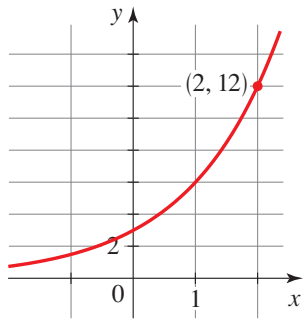
20. $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = 2(1.5)^x$

21–22 ■ Find the exponential function $f(x) = Ca^x$ whose graph is given.

21.



22.



23–24 ■ A data set is given in the table.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential model for the data.

(c) Graph the model you found in part (b) on your scatter plot from part (a). Does the model appear to fit the data well?

23.

t	$f(t)$
0	400
2	424
4	462
6	491
8	519
10	553

24.

t	$f(t)$
0	1225
5	1092
10	985
15	880
20	792
25	718

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

25. Linear, Exponential, or Logistic? Three coastal communities were established in the year 2000. The tables show the growth in the number of housing units in each community in the period from 2000 to 2006 (with year 0 representing 2000).

Avalon Acres

Year	Dwellings
0	250
1	296
2	359
3	434
4	520
5	620
6	751

Buccaneer Beach

Year	Dwellings
0	200
1	218
2	241
3	255
4	284
5	296
6	322

Coral Cay

Year	Dwellings
0	190
1	210
2	245
3	281
4	302
5	308
6	312

(a) Make scatter plots for each of the data sets.

(b) On the basis of your scatter plots, what kind of model best represents the growth of housing units in each town: linear, exponential, or logistic?

(c) Use a calculator to find a growth model of the appropriate type for each community.

(d) Use your models from part (c) to determine: **(i)** The growth rate in the town with the linear model. **(ii)** The percentage growth rate in the town with the exponential model. **(iii)** The carrying capacity in the town with the logistic model.

(e) What planning policies or other factors might account for the differences in the type of growth that each community experiences?

Time (min)	Bacteria (millions)
0	0.5
20	1.5
40	4.5
60	13.5
80	40.5

26. Bacterial Infection A horse is infected with a bacterium that may cause death if the bacteria count becomes sufficiently large. The table gives the number of bacteria in the horse at 20-minute intervals since infection.

(a) Find the growth factor (per 20-minute time period) for the bacteria population.

(b) Find a model of the form $P(x) = Ca^x$ for the bacteria population after x time periods.

(c) Express your model as an exponential function of t, the number of hours since infection.

(d) Graph your model from part (c) on a graphing calculator. Use the TRACE feature to determine how many hours it will take for the bacteria count to reach 500 million, the lethal level.

(e) Use a graphing calculator (TI-89 or better) to find a logistic model for the data. Graph the model on a scatter plot of the data. Does it seem to fit the data well?

(f) If the logistic model is in fact the correct one, will the bacteria count ever reach the lethal level of 500 million, or will the horse survive?

CONTEXTS

27. Aspirin Metabolism When a standard dose of 650 mg of aspirin is ingested, it is metabolized at an exponential rate. For the average person the half-life of the aspirin remaining in the body is 50 minutes.

(a) Find an exponential function of the form $A(x) = Ca^x$ that models the amount of aspirin remaining in the body after x 50-minute time periods.

(b) Change the time interval to find a function $A(t) = Cb^t$ that models the amount of aspirin in the body after t hours.

(c) How much aspirin is left in the body 24 hours after ingestion?

(d) Use a table of values to plot a graph of the amount of aspirin in the body over the first 10-hour period after ingesting it.

28. Fruit Fly Population A messy student leaves discarded food scattered around his dorm room, attracting fruit flies. From an initial population of 10 flies, the population grows exponentially, tripling every four hours.

(a) Find an exponential function of the form $P(x) = Ca^x$ that models the number of fruit flies in the room after x four-hour time periods.

(b) Change the time interval to find a function $P(t) = Cb^t$ that models the number of fruit flies in the room after t hours.

(c) What is the one-hour growth factor for the fruit fly population model? What is the growth rate?

(d) The student goes home for Thanksgiving without cleaning his room, allowing the flies to flourish. How many flies will there be in the room when he returns, 2 days after they made their initial appearance?

29. Investments Marie-Claire has received an unexpected bonus of $7500 from her employer and is trying to decide which of two 3-year CD offers from her bank would be a better way to invest the money:

- Plan A offers a 4% annual interest rate, compounded monthly.
- Plan B offers a 3% annual interest rate, compounded quarterly, but then adds an extra 2% of the principal at the end if the CD is held to maturity.

Calculate the amount to which her money would grow under each plan at the end of the 3 years. Which plan is better?

30. Bond Interest Some savings bonds pay a fixed rate of interest, which is issued to the owner at the end of every month. (These are known as *coupon bonds*.) Others are like CDs, for which the interest is compounded and paid out with the principal when the bond matures. Joshua purchases a one-year $5000 bond of each type, each paying 6% annual interest.

- Bond A provides Joshua a check for his interest on just the principal every month. He gets his principal back with his last interest payment at the end of the year.
- Bond B credits Joshua's bond with the interest, compounding it monthly, and then gives him his principal plus accumulated interest at the end of the year.

(a) Complete the following table giving the value of Joshua's investment (principal plus interest) at the end of each of the first six months that he owns the bonds.

Month	0	1	2	3	4	5	6
Bond A	$5000	$5025	$5050				
Bond B	$5000	$5025	$5050.13				

(b) For which bond is the growth of the investment value linear? What is the monthly growth rate?

(c) For which bond is the growth of the investment exponential? What is the monthly growth factor? What is the monthly percentage growth rate?

(d) What is the value of each of the investments when the bonds mature after one year?

(e) Which bond would you choose to buy yourself? Why?



31. Turtle Population Fifteen pet turtles escape from their owner's home on a private Caribbean island. Over the years their population increases, as indicated in the table (where "year" indicates years since escape).

(a) Make a scatter plot of the data. Which would seem to be a better model for the population: an exponential model or a logistic model?

Year	0	1	2	3	4	5	6
Turtles	15	23	34	41	48	50	51

(b) Use a graphing calculator to find both an exponential and a logistic model for the data. Graph both functions on your scatter plot, and decide which one you feel fits the data better.

(c) Determine the number of turtles the island is predicted to have with each model after 100 years. Which model seems more reasonable?



32. Diminishing Returns An athlete wishes to improve her performance in the 100-meter sprint. The data in the table below give her time (in seconds) for this sprint on each Monday of an intensive six-week training program.

(a) Use a graphing calculator (TI-89 or better) to find a logistic model for her sprint time.

(b) Graph both the data and the model on the same viewing rectangle. What does her best possible time appear to be, on the basis of the carrying capacity of the model?

(c) If she trains for six more weeks, to what level can she expect her time to improve (to two decimal places)?

Week number	0	1	2	3	4	5	6
Sprint time (s)	14.9	14.0	13.4	13.0	12.7	12.6	12.5

CHAPTER 3 TEST

1. The growth of a population is modeled by the exponential function

$$P(x) = 640(1.5)^x$$

where x represents time measured in days.

(a) What is the initial population?

(b) What is the growth factor? What is the growth rate?

(c) What will the population be after 6 days?

(d) Make a table of values for $x = 0, 1, 2, 3$, and use it to sketch a graph of P.

(e) What is the one-week growth factor for the population? Find a model of the form $P(t) = Cb^t$ for the population, where t is measured in weeks.

2. Marla opens a savings account that is guaranteed to pay an annual interest rate of 3.6%, compounded monthly. She deposits $2000 into the account.

(a) Find the amount in Marla's account after 4 months, after 1 year, and after 3 years.

(b) Marla had the option of an account that pays 3.45% annual interest, compounded daily. Would she have seen better returns with that account?

3. For the function $F(x) = \frac{1}{2} \cdot 3^x$, find the average rate of change and the percentage rate of change between the indicated values of x.

(a) Between 0 and 2 **(b)** Between 2 and 4

4. A population is modeled by the function

$$P(t) = \frac{500}{1 + 3 \cdot 2^{-t}}$$

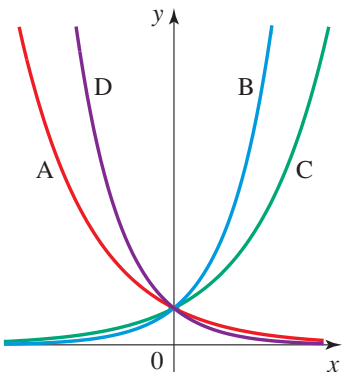
where t is time measured in years.

(a) What is this type of growth model called?

(b) What is the initial population?

(c) What is the population after 5 years?

(d) Will the population ever reach 600? Why or why not? [*Hint:* Think about the carrying capacity.]



5. The figure in the margin shows the graphs of the four exponential functions

$$f(x) = 2^x, \quad g(x) = 3^x, \quad h(x) = \left(\tfrac{1}{2}\right)^x, \quad k(x) = \left(\tfrac{1}{3}\right)^x$$

For each graph determine which function it is the graph of. Explain the reasons for your choices.

6. Some species of fungi form colonies of cells that grow without bound (assuming that sufficient resources are present). A scientist records the mass of such a colony every week; his results are given in the table below.

Week	0	1	2	3	4	5
Mass (kg)	1.2	1.7	2.8	3.9	6.1	9.0

(a) Use a graphing calculator to find (i) the linear model and (ii) the exponential model that best fits the data.

(b) Make a scatter plot of the data, and graph both the models that you found in part (a) on the scatter plot. Which model fits the data better?

(c) Use the better model to predict the mass of the fungal colony in week 6.

1 Extreme Numbers: Scientific Notation

OBJECTIVE To use scientific notation to express enormously huge and incredibly tiny numbers.



Magdalena Bujak/Shutterstock.com 2009

Have you ever wondered how many grains of sand there are on the world's beaches? In his book *The Sand Reckoner* Archimedes (ca. 200 B.C.) tries to estimate the number of grains of sand it would take to fill the then known universe. Archimedes didn't know about our decimal system or our notation for exponents; that's why it takes a whole book to explain his estimate. Using exponents, however, it takes just a few symbols to express gigantic numbers. For example, the number of grains of sand on the world's beaches is estimated at about 10^{19} grains (see `www.hawaii.edu/swemath/jsand.htm`).

Scientists routinely encounter extremely large numbers and extremely small numbers. Here are some examples:

World population: 6.5 billion = 6,500,000,000

Mass of a hydrogen atom: 0.000000000000000000000000166 g

The world's ant population is estimated at one quadrillion, and the insect population is estimated at one quintillion. There are about 6 sextillion cups of water in the world's oceans, and each cup of water contains about 24 septillion atoms.

Inventing new names for ever larger numbers is an endless task, and the many zeros needed to write extreme numbers in decimal notation can be difficult to read. So scientists express such numbers as multiples of powers of 10.

To write a number in **scientific notation**, express it in the form

$$a \times 10^n$$

where $1 \le a < 10$ and n is an integer.

Using scientific notation, we express the world population as 6.5×10^9. The positive exponent 9 indicates that the decimal point should be moved nine places to the *right*:

$$6.5 \times 10^9 = 6{,}500{,}000{,}000$$

Move decimal point 9 places to the right

Similarly, we can express the mass of a hydrogen atom as 1.66×10^{-24} g. The negative exponent -24 indicates that the decimal point should be moved 24 places to the left:

$$1.66 \times 10^{-24} = 0.\ 000000000000000000000001\ 66$$

Move decimal point 24 places to the left

I. Writing Numbers in Scientific Notation

First, let's get some practice using scientific notation.

1. A number is given in decimal notation. Write the number in scientific notation.

Distance to the star nearest to our sun (Proxima Centauri):

40,000,000,000,000 km or 4.0×10^{13} km

Mass of the earth:

5,970,000,000,000,000,000,000,000 kg or _____

Population of India in 2006:

1,100,000,000 or _____

Diameter of a red blood cell:

0.0000007 m or _____

Mass of an electron:

0.00000000000000000000000000091 g or _____

2. A number is given in scientific notation. Write the number in decimal notation.

U.S. federal budget for 2007:

$\$2.8 \times 10^{12}$ or $2,800,000,000,000

Weight of a hummingbird:

3.5×10^{-6} tons or _____

Population of China in 2006:

1.3×10^{9} or _____

Number of cells in the average human body:

4.9×10^{13} or _____

Radius of a gold atom:

1.4×10^{-10} m or _____

II. Calculating with Scientific Notation

We often need to multiply or divide numbers expressed in scientific notation. For example, to find the time it takes light to reach us from the sun, we need to divide the distance to the sun, 9.3×10^7 mi, by the speed of light, 1.86×10^5 mi/s. Scientific notation allows us to multiply or divide large numbers easily.

1. Let's see how to multiply two numbers given in scientific notation.

(a) Multiply the following two numbers. To keep your answer in scientific notation, combine the powers of 10 separately.

$$(1.6 \times 10^5) \times (2.5 \times 10^8) = (1.6 \times 2.5)(10^5 \times 10^8) = _____ \times 10^{\square}$$

(b) Multiply the following two numbers and write your answer in scientific notation. Notice that the number $a \times 10^n$ is in scientific notation only if $1 \le a < 10$.

$$(8.4 \times 10^9) \times (9.5 \times 10^{-3}) = (8.4 \times 9.5)(10^9 \times 10^{-3})$$
$$= _____ \times 10^{\square}$$
$$= _____ \times 10^{\square}$$

2. In the introduction to this Exploration we learned that the world's oceans contain 6 sextillion (6×10^{21}) cups of water and each cup of water contains about 24 septillion (2.4×10^{25}) atoms. How many atoms do the oceans contain? Express your answer in scientific notation.

3. In 2006 there were approximately 120,000,000 homes in the United States, and the average home price was $219,000.

(a) Write these numbers in scientific notation.

$$\text{number of homes} = _____$$

$$\text{average home price} = _____$$

(b) Multiply these numbers to find the total value of U.S. residential real estate in 2006.

$$\text{number} \times \text{average price} = _____ \text{ (scientific notation)}$$

$$= _____ \text{ (decimal notation)}$$

4. Let's see how to divide two numbers given in scientific notation.

(a) Divide the following two numbers. To keep your answer in scientific notation, combine the powers of 10 separately.

$$\frac{8.4 \times 10^{24}}{2.1 \times 10^{19}} = \frac{8.4}{2.1} \times \frac{10^{24}}{10^{19}} = _____ \times 10^{\square}$$

(b) Divide the following two numbers and write your answer in scientific notation. Notice that the number $a \times 10^n$ is in scientific notation only if $1 \le a < 10$.

$$\frac{2.7 \times 10^{16}}{6.5 \times 10^{-5}} = \frac{2.7}{6.5} \times \frac{10^{16}}{10^{-5}} = _____ \times 10^{\square} = _____ \times 10^{\square}$$

5. The distance from the earth to the sun is 9.3×10^7 mi and the speed of light is 1.86×10^5 mi/s. How long does it take light to reach us from the sun? Convert your answer from seconds to minutes.

6. On January 1, 2007, the U.S. national debt was $8,678,000,000,000, and the U.S. population was 300,900,000.

(a) Write these numbers in scientific notation.

$$\text{national debt} = _____$$

$$\text{population} = _____$$

(b) If the national debt is distributed over the entire population, how much does each person owe?

$$\frac{\text{national debt}}{\text{population}} = _____ \text{ (scientific notation)}$$

$$= _____ \text{ (decimal notation)}$$

2 So You Want to Be a Millionaire?

OBJECTIVE To become familiar with the rapid rate of growth of exponential functions.

If you want to be a millionaire, you can start by taking Benjamin Franklin's wise advice:

"A penny saved is a penny earned."



But you're going to have to save more and more pennies each day. So suppose you put a penny in your piggy bank today, two pennies tomorrow, four pennies the next day, and so on, doubling the number of pennies you add to the bank each day. Let's call the first day we put a penny in the bank Day 0, the next Day 1, and so on. So on Day 3 you put 8 pennies in the bank, on Day 5 you put in 32 pennies, and on Day 10 you put in 1024 pennies, or about 10 dollars worth of pennies.

This doesn't seem like an effective way of becoming a millionare; after all, we're just saving pennies. How many years does it take to save a million dollars this way? We'll work it out in this exploration, and in the process we hope to improve our intuition for exponential growth.

I. How to Save a Million Dollars

1. **(a)** Complete the table for the number of pennies saved each day.

Day x	0	1	2	3	4	5
Pennies saved $f(x)$	1					

(b) The number of pennies saved each day grows exponentially. What is the growth factor? What is the initial value? Find an exponential function $f(x) = Ca^x$ that models the number of pennies saved on day x.

$$f(x) = \square \cdot \square^{x}$$

2. Let's complete the table for the number of pennies saved on each day for 30 days, without using a calculator—but we'll make the calculations easier by making some approximations. On Day 10 we must save 2^{10}, or 1024 pennies. Since this is about \$10, we'll write \$10 instead of the 1024 pennies. On Day 20 we must save 2^{20} pennies; check that this is approximately \$10,000.

Day x	0	1	2	3	4	5	6	7	8	9
Pennies saved $f(x)$	1									

Day x	10	11	12	13	14	15	16	17	18	19
Dollars saved	\$10									

Day x	20	21	22	23	24	25	26	27	28	29
Dollars saved	\$10,000									

3. On which day does the amount we put in first exceed one million dollars? Is your answer surprising? What does this experiment tell us about exponential growth?

II. How to Manage Population Growth

We know that population grows exponentially. Let's see what this means for a type of bacteria that splits every minute. Suppose that at 12:00 noon a single bacterium colonizes a discarded food can. The bacterium and its descendents are all happy, but they fear the time when the can is completely full of bacteria—doomsday.

1. How many bacteria are in the can at 12:05? At 12:10?
2. The can is completely full of bacteria at 1:00 P.M. At what time was the can only half full of bacteria?
3. When the can is exactly half full, the president of the bacteria colony reassures his constituents that doomsday is far away—after all, there is as much room left in the can as has been used in the entire previous history of the colony. Is the president correct? How much time is left before doomsday?
4. When the can is a quarter full, how much time is left till doomsday?

3 Exponential Patterns

OBJECTIVE To recognize exponential data and find exponential functions that fit the data exactly.

In Exploration 2 of Chapter 2 (page 233) we discussed the importance of finding patterns, and we found many linear patterns. In this exploration we find exponential patterns. We've already learned how exponential functions model interest, population growth, and other real-world phenomena. But exponential patterns also occur in unexpected places, as we'll see in this exploration.

I. Recognizing Exponential Data

To find exponential patterns, we look at the ratio of consecutive outputs of the data. In this exploration we assume that the inputs are the equally spaced numbers

$$0, 1, 2, 3, \ldots$$

We want to find properties of the outputs that guarantee that there is an exponential function that exactly models the data. So let's consider the exponential function

$$f(x) = Ca^x$$

In the next question we confirm that the entries in Table 1 at the top of the next page are correct. In particular, the ratio of consecutive terms is constant.

table **1**
Ratios of consecutive outputs for $f(x) = Ca^x$

x	0	1	2	3	4	5	6
$f(x)$	C	Ca	Ca^2	Ca^3	Ca^4	Ca^3	Ca^3
Ratio	—	a	a	a	a	a	a

1. **(a)** Find the outputs of the exponential function $f(x) = Ca^x$ corresponding to the inputs 0, 1, 2, 3, ...

$f(0) =$ ___C___, $f(1) =$ _____, $f(2) =$ _____, $f(3) =$ _____

(b) Use your answers to part (a) to find the ratio between consecutive terms.

$\frac{f(1)}{f(0)} =$ _____, $\frac{f(2)}{f(1)} =$ _____, $\frac{f(3)}{f(2)} =$ _____

(c) Do your answers to parts (a) and (b) match the entries in Table 1?

2. A data set is given in the table.

x	0	1	2	3	4	5	6	7
$f(x)$	3	6	12	24	48	96	192	384
Ratios	—							

(a) Fill in the entries for the ratios of consecutive outputs.

(b) Observe that the ratios of consecutive terms are constant, so there is an exponential function $f(x) = Ca^x$ that models the data. To find C and a, let's compare the entries in this table with those of Table 1. Comparing the output corresponding to the input 0 in each of these tables, we conclude that

$C =$ _____

Comparing the ratios in each of these tables, we conclude that

$a =$ _____

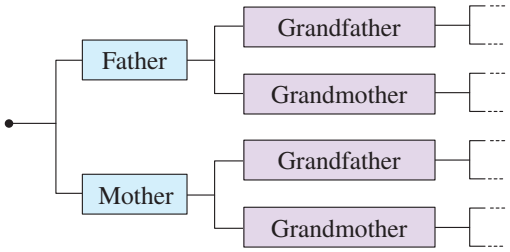
So an exponential function that models the data is

$f(x) = \square \cdot \square^x$

(c) Check that the function f you found in part (b) matches the data. That is, $f(0), f(1), f(2), \ldots$ match the values in the table.

II. Exponential Patterns

1. A person has two parents, four grandparents, eight great-grandparents, and so on.



(a) Complete the table below for the number of ancestors a person has x generations back, and then calculate the ratio of consecutive outputs.

Generation x	0	1	2	3	4
Number of ancestors $f(x)$	1	2			
Ratio	—				

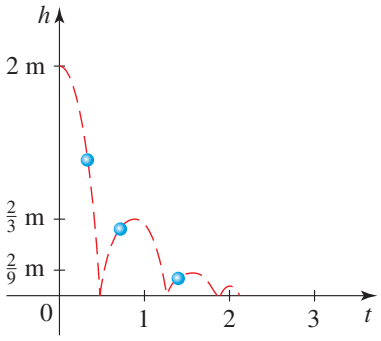
(b) Are the ratios of consecutive outputs constant? Is there an exponential pattern relating the generation and the number of ancestors? If so, find C and a by comparing the entries in this table with the corresponding entries in Table 1.

$C =$ _____ $a =$ _____

An exponential function that models the pattern in the data is:

$f(x) =$ _____

(c) How many ancestors does the model tell us a person has 15 generations back? Why might there actually be fewer ancestors than the number the model predicts 15 generations back?



2. When a particular ball is dropped, it bounces back up to one-third of the distance it has fallen. The ball is dropped from a height of 2 meters.

(a) Complete the table for the height the ball reaches on bounce x, and then calculate the ratio of consecutive outputs.

Bounce x	1	2	3	4	5
Height $f(x)$	2/3				
Ratio	—				

(b) Are the ratios of consecutive outputs constant? If so, find C and a by comparing the entries in this table with the corresponding entries in Table 1.

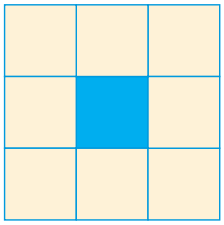
$C =$ _____ $a =$ _____

Find an exponential function that models the pattern in the data:

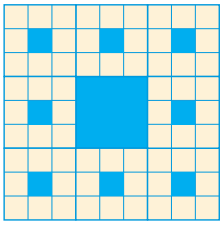
$f(x) = \square \cdot \square^{x}$

(c) How high does the ball bounce on the sixth bounce?

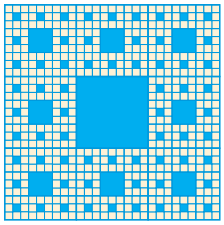
3. A yellow square of side 1 is divided into nine smaller squares, and the middle square is colored blue as shown in the figure. Each of the smaller yellow squares is in turn divided into nine squares, and each middle square is colored blue.



Stage 1



Stage 2



Stage 3

(a) Complete the table for the number of blue squares added at stage x, and then calculate the ratio of consecutive outputs.

Stage x	1	2	3	4
Number of blue squares added, $f(x)$	1			
Ratio	—			

(b) Are the ratios of consecutive outputs constant? If so, find C and a by comparing the entries in this table with the corresponding entries in Table 1.

$C =$ _____ $a =$ _____

Find an exponential function that models the pattern in the data:

$f(x) =$ _____

(c) How many blue squares are added at the fifth stage?

(d) Complete the table for the area of each of the blue squares added at stage x, then calculate the ratio of consecutive outputs.

Stage x	1	2	3	4
Area of blue squares added, $A(x)$	1/9			
Ratio	—			

(e) Find an exponential function that models the area of each blue square added at stage x.

$A(x) =$ _____

4 Modeling Radioactivity with Coins and Dice

OBJECTIVE To experience how random events can lead to exponential decay models.

Radioactive elements decay when their atoms spontaneously emit radiation and change into smaller, stable atoms. But if atoms decay randomly, how is it possible to find a function that models their behavior? We'll try to answer this question by experimenting with coins and dice.

Imagine tossing a coin 100 times (or, equivalently, tossing 100 coins all at once). How often would you expect to get 100 heads? Or 99 heads? A much more likely outcome is that the number of heads and tails will be about the same (because the "rate" at which heads appears is 50%). So although the outcome of a single toss of a coin is totally unpredictable, when we toss many coins, the number of heads is fairly predictable. Of course, there are a lot more than 100 atoms in even the tiniest sample of a radioactive substance, so the outcome of the random decay of atoms in the sample is quite predictable. Let's simulate the random decay of radioactive atoms using coin tosses and rolls of dice.

I. Modeling Radioactive Decay with Coins

In this first experiment we toss pennies to simulate the decay of atoms in a radioactive substance.



You will need:

- 40 pennies
- A cup or jar
- A table

Procedure:

1. Put the pennies in a cup and shake them well, then toss them on a table. The pennies that show tails are considered "decayed," and those that show heads are still "radioactive."
2. Discard the decayed pennies and collect the radioactive ones. Record the number of radioactive pennies remaining (in the table below).
3. Repeat Steps 1 and 2 with the remaining radioactive pennies until all the pennies have decayed.

Toss number x	Pennies remaining $f(x)$	Ratio $\frac{f(x+1)}{f(x)}$
0	40	—
1		
2		
3		
4		
5		
⋮		

Analysis:

4. Is an exponential model appropriate for the data you obtained? To decide, complete the "Ratio" column in the table to determine whether there is a reasonably constant "decay factor."

5. **(a)** Use the `ExpReg` command on a graphing calculator to find the exponential curve $Y = a \cdot b^x$ that best fits the data:

$$Y = \square \cdot \square^x$$

(b) What is the decay factor for the model you found in part (a)? Is the decay factor what we should expect when tossing pennies?

II. Modeling Radioactive Decay with Dice

In this experiment we roll dice to simulate the decay of atoms in a radioactive substance.



You will need:

- 24 dice
- A cup or jar
- A table

Procedure:

This is basically the same experiment as in Part I, except with dice.

1. Put the dice in a cup and shake them well, then roll them on a table. The dice that show a one or a six are considered "decayed," and the others are still "radioactive."

2. Discard the decayed dice and collect the radioactive ones. Record the number of radioactive dice remaining (in the table below).

3. Repeat Steps 1 and 2 with the remaining radioactive dice until all the dice have decayed.

Toss number x	Pennies remaining $f(x)$	Ratio $\frac{f(x+1)}{f(x)}$
0	24	—
1		
2		
3		
4		
5		
⋮		

Analysis:

4. Is an exponential model appropriate for the data you obtained? To decide, complete the "Ratio" column in the table to determine whether there is a reasonably constant "decay factor."

5. **(a)** Use the `ExpReg` command on a graphing calculator to find the exponential curve $Y = a \cdot b^x$ that best fits the data:

$$Y = \square \cdot \square^{x}$$

(b) What is the decay factor for the model you found in part (a)? Is the decay factor what we should expect when rolling dice?

4 Logarithmic Functions and Exponential Models



George Marks/Retrofile/Getty Images

4.1 Logarithmic Functions

4.2 Laws of Logarithms

4.3 Logarithmic Scales

4.4 The Natural Exponential and Logarithm Functions

4.5 Exponential Equations: Getting Information from a Model

4.6 Working with Functions: Composition and Inverse

EXPLORATIONS

1 Super Origami
2 Orders of Magnitude
3 Semi-Log Graphs
4 The Even-Tempered Clavier

Are we there yet? It's a basic question: *When* will we reach our destination? In Chapter 3 we modeled the growth of an investment and the growth of a population using exponential functions. The growth of an investment may seem frustratingly slow, whereas population growth can be shockingly fast. So we want to answer questions such as these: *When* will my bank account have a million dollars? *When* will the population reach a given level? If population continues to grow exponentially (or logistically), when will every street be as crowded as the New York City street shown in the photo above? These are important questions that can help us to plan for the future. To answer such questions, we need to solve exponential equations (equations in which the unknown is in the exponent), and to do this, we need logarithms. We'll see how logarithms allow us to reverse the rule of an exponential function, which in turn helps us to answer the questions posed here.

4.1 Logarithmic Functions

- Logarithms Base 10
- Logarithms Base *a*
- Basic Properties of Logarithms
- Logarithmic Functions and Their Graphs

IN THIS SECTION... we define logarithms and study their basic properties.

GET READY... by reviewing the rules of exponents in **Algebra Toolkits A.3** and **A.4**. Test your understanding by doing the **Algebra Checkpoint** at the end of this section.



Lothar Redlin

The world bird population is about 10^{11}; the logarithm of 10^{11} is just 11.

In Section 3.1 we used exponential functions to model exponential growth and decay. When working with exponential models, we often need to answer questions such as: How long will it take for a population to reach a given size? How long does it take for a radioactive sample to decay to 1% of its original size? To answer these questions, we need to solve exponential equations, and solving such equations requires the use of logarithms.

In this section we'll see how logarithms allow us to "tame" very large or very small numbers. For example, the world bird population is estimated at 10^{11} birds, but the logarithm of 10^{11} is just 11; the mass of an electron is about 10^{-29} grams, but the logarithm of 10^{-29} is just -29. Do you see the pattern?

Logarithms Base 10

The *logarithm base 10* (or *common logarithm*) of a number x is the power to which we must raise 10 to get x.

Logarithms Base 10

The **logarithm base 10** of x is defined by

$$\log_{10} x = y \quad \text{if and only if} \quad 10^y = x$$

We read $\log_{10} x$ as "log base 10 of x." Logarithms base 10 are called **common logarithms**, and $\log_{10} x$ is often written simply as $\log x$ (omitting the subscript 10).

From this definition we see that the logarithm of a positive number x is an exponent: "$\log_{10} x$ is the exponent to which we must raise 10 to get x." So if x is a power of 10, it's easy to find $\log_{10} x$. In fact, $\log_{10} 10^x = x$. The following table shows the pattern for finding logarithms.

x	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3	10^4
$\log_{10} x$	-4	-3	-2	-1	0	1	2	3	4

The next example illustrates how to find logarithms of powers of 10.

example 1 Finding Logarithms of Powers of 10

Find the following logarithms.

(a) $\log_{10} 1000$

(b) $\log_{10} 1{,}000{,}000$

(c) $\log_{10} \frac{1}{100}$

(d) $\log_{10} \sqrt{10}$

Solution

The Rules of Exponents are reviewed in Algebra Toolkits A.3 and A.5, pages T14 and T20.

To find the logarithm of a number, it's helpful to express the number as a power of 10.

(a) $\log_{10} 1000 = \log_{10} 10^3 = 3$

(b) $\log_{10} 1{,}000{,}000 = \log_{10} 10^6 = 6$

(c) $\log_{10} \frac{1}{100} = \log_{10} 10^{-2} = -2$

(d) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$

■ **NOW TRY EXERCISE 7** ■

We need to do some additional work to find the logarithms of numbers that are not easily expressed as powers of 10. To find the logarithm of 735, we need to express it as a power of 10. In other words, we need to find a number y such that $735 = 10^y$.

x (decimal form)	100	735	1000
x (exponential form)	10^2	10^y	10^3
$\log_{10} x$	2	y	3

By the definition of the logarithm, the exponent y is precisely $\log_{10} 735$. Let's estimate the value of y. Since $100 < 735 < 1000$, the power to which we need to raise 10 to get 735 is between 2 and 3. That is,

$$2 < \log_{10} 735 < 3$$

To get a better approximation, we can experiment to find a power of 10 that is closer to 735. For instance, using a calculator, we find

$$10^{2.7} \approx 501.2 \qquad 10^{2.8} \approx 631.0 \qquad 10^{2.9} \approx 794.3$$

So $\log_{10} 735$ is between 2.8 and 2.9. We can continue this process to get better estimates, but fortunately, scientific calculators have a LOG key that gives the values of base 10 logarithms.

example 2 Finding Logarithms Using a Calculator

Find the following logarithms using a calculator.

(a) $\log_{10} 735$

(b) $\log_{10} 50$

(c) $\log_{10} 0.002$

(d) $\log_{10} \sqrt{10}$

Solution

	Calculator Keystrokes	Output
(a) $\log_{10} 735$	LOG 7 3 5 ENTER	2.866287339
(b) $\log_{10} 50$	LOG 5 0 ENTER	1.698970004
(c) $\log_{10} 0.002$	LOG . 0 0 2 ENTER	−2.698970004
(d) $\log_{10} \sqrt{10}$	LOG √ 1 0 ENTER	0.5

NOW TRY EXERCISE 9

An ant is a lot smaller than an elephant, which in turn is a lot smaller than a whale. Let's find the logarithms of the weights of these animals.

example 3 Finding Logarithms

The weights of a particular ant, elephant, and whale are given. Find the logarithms of these weights.

Ant	0.000003 kilograms
Elephant	4000 kilograms
Whale	170,000 kilograms

Solution

Recall that $\log x$ is the common logarithm, which is the same as $\log_{10} x$.

We use a calculator to find the logarithms.

Ant	$\log 0.000003 \approx -5.5$
Elephant	$\log 4000 \approx 3.6$
Whale	$\log 170{,}000 \approx 5.2$

NOW TRY EXERCISE 11

Logarithms Base *a*

To analyze exponential growth models with growth factor a, we need logarithms base a. These are defined in exactly the same way that we've defined logarithms base 10.

Logarithms Base *a*

If a is a positive number then the **logarithm base *a*** of x is defined by

$$\log_a x = y \quad \text{if and only if} \quad a^y = x$$

We read $\log_a x$ as "log base a of x."

From this definition we see that the logarithm of a positive number x is an exponent: "$\log_a x$ is the exponent to which we must raise a to get x." So if x is a power of a, it's easy to find $\log_a x$. In fact, $\log_a a^x = x$. The following table shows the pattern for finding logarithms base 2.

x	2^{-4}	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3	2^4
$\log_2 x$	-4	-3	-2	-1	0	1	2	3	4

The next example illustrates how to find logarithms for different bases.

example 4 Finding Logarithms for Different Bases

Find the following logarithms.

(a) $\log_2 32$ **(b)** $\log_5 25$ **(c)** $\log_3 \frac{1}{9}$ **(d)** $\log_{16} 4$

Solution

To find the logarithm of a number, it's helpful to express the number as a power of the base.

(a) $\log_2 32 = \log_2 2^5 = 5$

(b) $\log_5 25 = \log_5 5^2 = 2$

(c) $\log_3 \frac{1}{9} = \log_3 3^{-2} = -2$

(d) $\log_{16} 4 = \log_{16} \sqrt{16} = \log_{16} 16^{1/2} = \frac{1}{2}$

■ NOW TRY EXERCISE 17 ■

From the definition of logarithms we see that there are two equivalent ways of expressing the relationship between a number and its logarithm: the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$. In switching back and forth between the logarithmic form and exponential form, it is helpful to notice that in each form, the base is the same:

Logarithmic form	Exponential form
Exponent ↓	Exponent ↓
$\log_a x = y$	$a^y = x$
↑ Base	↑ Base

example 5 Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations; if one is true, so is the other. So we can switch from one form to the other.

Logarithmic form	Exponential form
$\log 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2 \frac{1}{8} = -3$	$2^{-3} = \frac{1}{8}$
$\log_9 3 = \frac{1}{2}$	$9^{1/2} = 3$
$\log_5 \frac{1}{5} = -1$	$5^{-1} = \frac{1}{5}$
$\log_5 5 = 1$	$5^1 = 5$

■ NOW TRY EXERCISE 27 ■

Basic Properties of Logarithms

From the definition of logarithms we can establish the following basic properties.

Basic Properties of Logarithms

Property	Reason
1. $\log_a 1 = 0$	We raise a to the exponent 0 to get 1.
2. $\log_a a = 1$	We raise a to the exponent 1 to get a.
3. $\log_a a^x = x$	We raise a to the exponent x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the exponent to which a must be raised to get x.

example 6 Applying the Basic Properties of Logarithms

Evaluate the expression.

(a) $\log_8 1$ **(b)** $\log_3 3$ **(c)** $\log_2 2^{15}$ **(d)** $5^{\log_5 12}$

Solution

We use the basic properties of logarithms to evaluate the expressions.

(a) $\log_8 1 = 0$ Property 1

(b) $\log_3 3 = 1$ Property 2

(c) $\log_2 2^{15} = 15$ Property 3

(d) $5^{\log_5 12} = 12$ Property 4

■ NOW TRY EXERCISES 15 AND 21 ■

Logarithmic Functions and Their Graphs

Recall that a function is a rule f that assigns a number $f(x)$ to each number x in the domain of f. The logarithmic function with base a is defined as follows.

Logarithmic Functions

The **logarithmic function with base *a*** is the function

$$f(x) = \log_a x$$

where $a > 0$ and $a \neq 1$. The domain of f is $(0, \infty)$.

example 7 Graphing Logarithmic Functions

Graph the logarithmic function $f(x) = \log_2 x$.

Solution

To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. Then we plot the points in the table and connect them with a smooth curve as in Figure 1.

x	$f(x)$
$\frac{1}{16}$	-4
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

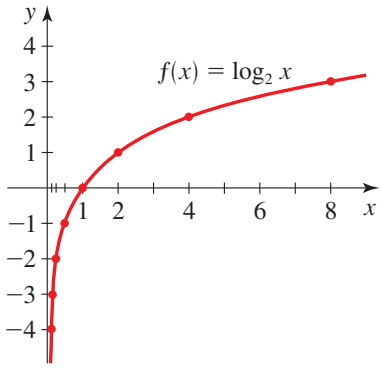


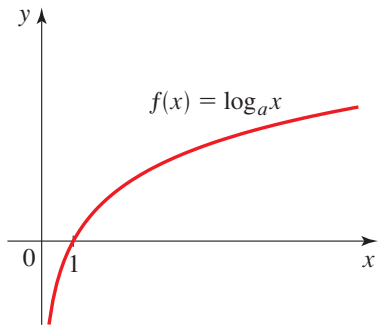
figure **1** Graph of $f(x) = \log_2 x$

NOW TRY EXERCISE 41

Logarithmic functions with base greater than 1 all have the same basic shape as the one in Example 7.

Graphs of Logarithmic Functions

The graph of the logarithmic function $f(x) = \log_a x$ for $a > 1$ has the following general shape. The line $x = 0$ (the y-axis) is a *vertical asymptote* of f.



example **8** **Graphing a Family of Logarithmic Functions**

Graph the family of logarithmic functions $f(x) = \log_a x$ for $a = 2, 3, 4$. Explain how changing the value of a affects the graph.

Solution

We plot points as an aid to sketching these graphs. In each case we choose the x-values to be powers of the base so that we can easily find their logarithms.

x	$\log_2 x$
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

x	$\log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3

x	$\log_4 x$
$\frac{1}{4}$	-1
1	0
4	1
16	2
64	3

The graphs are shown in Figure 2 for x-values between 0 and 9.

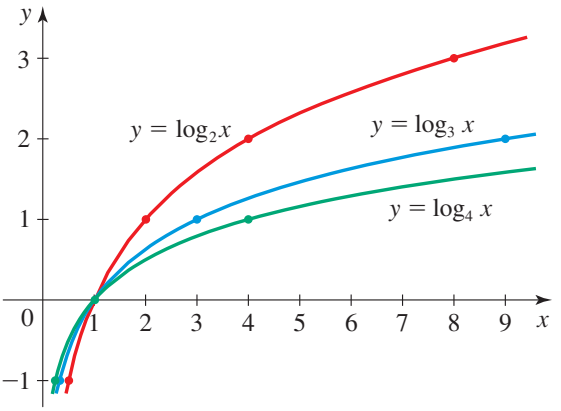


figure **2** A family of logarithmic functions

We notice from these graphs that the logarithmic function increases more slowly for larger values of the base a.

NOW TRY EXERCISE 51

Algebra Checkpoint ✓

Check your knowledge of the rules of exponents by doing the following problems. You can review the rules of exponents in **Algebra Toolkits A.3** and **A.4** on pages T14 and T20.

1. Evaluate each expression without using a calculator.

(a) 4^3 **(b)** 5^{-1} **(c)** 5^{-2} **(d)** $4^{1/2}$ **(e)** $27^{-1/3}$

2. Express each number as a power of 10.

(a) 100 **(b)** 10,000 **(c)** $\frac{1}{10}$ **(d)** $\frac{1}{100}$ **(e)** $\frac{1}{\sqrt{10}}$

3. Express each number as a power of 2.

(a) 16 **(b)** 64 **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{16}$ **(e)** $\frac{1}{\sqrt[3]{2}}$

4. Express each number as a power of 9.

(a) 81 **(b)** 729 **(c)** $\frac{1}{9}$ **(d)** 3 **(e)** $\frac{1}{3}$

4.1 Exercises

CONCEPTS

Fundamentals

1. $\log x$ is the exponent to which the base 10 must be raised to get _____. Use this fact to complete the following table for $\log x$.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$	$10^{3.4}$
$\log x$									

2. **(a)** $5^3 = 125$, so $\log_{\square} \square = \square$

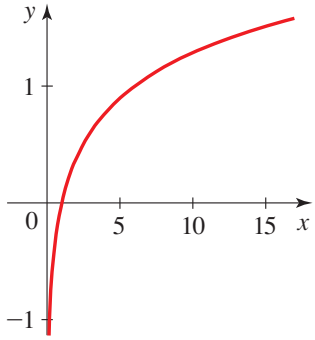
(b) $\log_5 25 = 2$, so $\square^{\square} = \square$

3. The function $f(x) = \log_9 x$ is the logarithm function with base _____. So $f(9) =$ _____, $f(1) =$ _____, $f(\frac{1}{9}) =$ _____, $f(81) =$ _____, and $f(3) =$ _____.

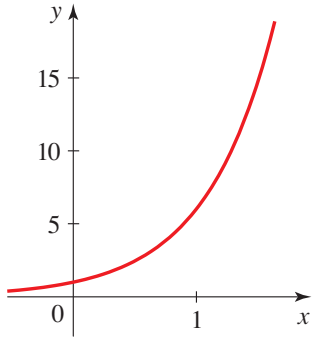
4. Match the function with its graph.

(a) $f(x) = 6^x$ **(b)** $g(x) = \log_6 x$

I



II



Think About It

5. True or false?

(a) For any $a > 1$, the logarithm function $f(x) = \log_a x$ is always increasing.

(b) For any $a > 1$, the graph of the logarithm function $f(x) = \log_a x$ has x-intercept 1.

(c) For any $a > 1$, the graph of the logarithm function $f(x) = \log_a x$ has y-intercept 1.

6. Use the fact that $1000 < 3500 < 10{,}000$ to give a rough estimate of $\log 3500$. Use a calculator to check your estimate.

SKILLS

7–8 ■ Find the given common logarithm.

7. **(a)** $\log_{10} 10{,}000$ **(b)** $\log_{10} \frac{1}{10}$ **(c)** $\log_{10} 0.01$ **(d)** $\log_{10} \sqrt[3]{10}$

8. **(a)** $\log_{10} 1{,}000{,}000{,}000{,}000$ **(b)** $\log_{10} \dfrac{1}{\sqrt{10}}$ **(c)** $\log_{10} 1$ **(d)** $\log_{10} 0.0001$

9–10 ■ Use a calculator to evaluate the given common logarithm.

9. **(a)** $\log 132$ **(b)** $\log 5000$ **(c)** $\log \sqrt{2}$ **(d)** $\log 0.3$

10. **(a)** $\log 0.145$ **(b)** $\log 16$ **(c)** $\log \frac{1}{9}$ **(d)** $\log 750$

11–14 ■ Find the logarithm of the given numbers.

11. **(a)** The diameter of a bacterium: 0.00012 centimeter
(b) The height of a human: 173 centimeters

12. **(a)** The length of an acorn: 2 centimeters
(b) The height of a fully grown oak tree: 914 centimeters

13. **(a)** The weight of a hummingbird: 4 grams
(b) The weight of a bald eagle: 5443 grams

14. **(a)** The radius of the earth: 6378 kilometers
(b) The radius of the sun: 696,000 kilometers

15–26 ■ Find the given logarithm.

15. **(a)** $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

16. **(a)** $\log_9 1$ **(b)** $\log_9 9^8$ **(c)** $\log_9 9$

17. **(a)** $\log_4 16$ **(b)** $\log_4 \frac{1}{4}$ **(c)** $\log_4 4$

18. **(a)** $\log_7 1$ **(b)** $\log_7 49$ **(c)** $\log_7 \frac{1}{49}$

19. **(a)** $\log_5 5^4$ **(b)** $\log_4 64$ **(c)** $\log_3 9$

20. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

21. **(a)** $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $4^{\log_4 5}$

22. **(a)** $9^{\log_9 18}$ **(b)** $6^{\log_6 72}$ **(c)** $10^{\log 5}$

23. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

24. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10} \sqrt{10}$ **(c)** $\log_5(0.2)$

25. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

26. **(a)** $\log_4 \sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

27–28 ■ Fill in the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 4.

27.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	
	$8^{-2} = \frac{1}{64}$

28.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4\left(\frac{1}{16}\right) = -2$	
$\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

29–32 ■ Express the equation in exponential form.

29. **(a)** $\log_5 125 = 3$ **(b)** $\log_5 1 = 0$

30. **(a)** $\log_{10} 0.1 = -1$ **(b)** $\log_8 512 = 3$

31. **(a)** $\log_8 2 = \frac{1}{3}$ **(b)** $\log_9 3 = \frac{1}{2}$

32. **(a)** $\log_3 81 = 4$ **(b)** $\log_2 \frac{1}{8} = -3$

33–36 ■ Express the equation in logarithmic form.

33. **(a)** $3^3 = 27$ **(b)** $10^{-4} = 0.0001$

34. **(a)** $10^3 = 1000$ **(b)** $81^{1/2} = 9$

35. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

36. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

37–40 ■ Use the definition of the logarithmic function to find x.

37. **(a)** $\log_2 x = 5$ **(b)** $\log_2 16 = x$

38. **(a)** $\log_5 x = 4$ **(b)** $\log_{10} 0.1 = x$

39. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

40. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

41–42 ■ Fill in the table and sketch the graph of the function by plotting points.

41. $f(x) = \log_3 x$

x	$f(x)$
$\frac{1}{27}$	
$\frac{1}{9}$	
$\frac{1}{3}$	
1	
3	
9	
27	

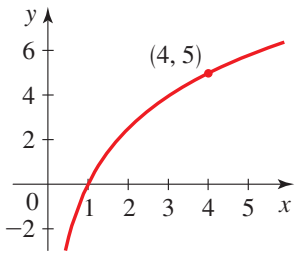
42. $f(x) = \log_4 x$

x	$f(x)$
$\frac{1}{16}$	
$\frac{1}{4}$	
1	
4	
16	

43–46 ■ Match the function with its graph.

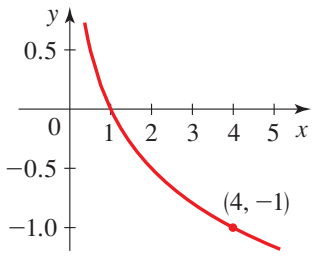
43. $f(x) = \log_4 x$

I



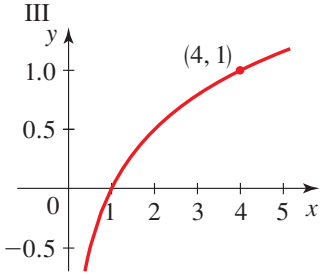
44. $f(x) = -\log_4 x$

II



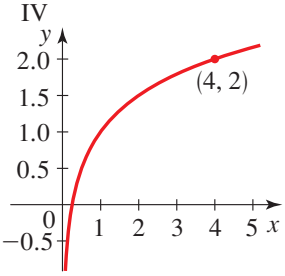
45. $f(x) = \log_4 4x$

III



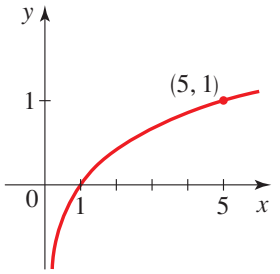
46. $f(x) = 5 \log_4 x$

IV

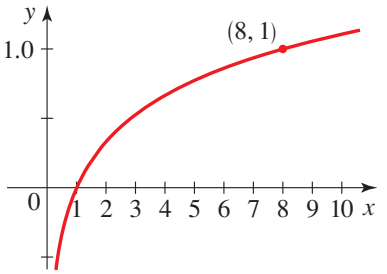


47–50 ■ The graph of a logarithm function $f(x) = \log_a x$ is shown. Find the base a.

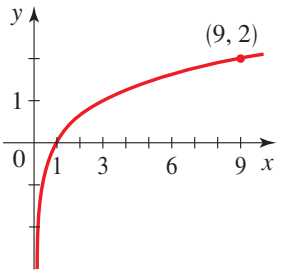
47.



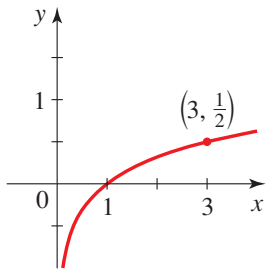
48.



49.



50.



51. Graph the family of logarithmic functions $f(x) = \log_a x$ for $a = 3, 5, 7$. How are the graphs related?

52. Graph the functions $f(x) = \log_{10} x$ and $g(x) = \log_{10}(10x)$. How are the graphs related?

4.2 Laws of Logarithms

- Laws of Logarithms
- Expanding and Combining Logarithmic Expressions
- Change of Base Formula

IN THIS SECTION... we study the Laws of Logarithms. These laws are the properties of logarithms that correspond to the Rules of Exponents.

The Laws of Logarithms are the key properties of logarithms. They allow us to use logarithms to compare large numbers (Section 4.3) and to solve exponential equations (Section 4.5). We need to solve exponential equations to answer questions such as "*When* will my bank account reach a million dollars?" or "*When* will the population of the world reach 10 billion?"

Laws of Logarithms

The Rules of Exponents are reviewed in Algebra Toolkit A.3, page T14.

Since logarithms are exponents, the Rules of Exponents give us useful rules for working with logarithms. For example, we know that

"To find the product of two powers with the same base, we *add* exponents."

For example, $10^x \cdot 10^y = 10^{x+y}$. To see what this rule tells us about logarithms, let's express the exponents as logarithms. So let

$$A = 10^x \qquad B = 10^y \qquad AB = 10^{x+y}$$

Writing these equations in logarithmic form, we get

$$\log_{10} A = x \qquad \log_{10} B = y \qquad \log_{10} AB = x + y$$

It follows that $\log_{10} AB = \log_{10} A + \log_{10} B$. We can express this rule in words:

"To find the logarithm of a product, we *add* the logarithms of the factors."

This explains the first of the following "laws." Laws 2 and 3 follow from the corresponding rules for exponents. In these laws A and B are positive real numbers, and C is any real number.

Laws of Logarithms

Law	In Words
1. $\log_a AB = \log_a A + \log_a B$	The logarithm of a product is the sum of the logarithms.
2. $\log_a \dfrac{A}{B} = \log_a A - \log_a B$	The logarithm of a quotient is the difference of the logarithms.
3. $\log_a A^C = C \log_a A$	The logarithm of a power is the exponent times the logarithm of the base.

example 1 Evaluating Logarithmic Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3} \log 8$

Solution

To find the logarithm of a number, it's helpful to express the number as a power of the base.

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3} \log 8 = \log 8^{-1/3}$ Law 3

$= \log\left(\frac{1}{2}\right)$ Rules of Exponents

≈ -0.301 Calculator

$8^{-1/3} = \dfrac{1}{8^{1/3}} = \dfrac{1}{2}$

■ NOW TRY EXERCISE 9 ■

Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as a sum or difference of logarithms. This process, called **expanding a logarithmic expression**, is illustrated in the next example.

example 2 Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2 6x$ **(b)** $\log_4 \dfrac{z^2}{y}$ **(c)** $\log_5 x^3y^6$ **(d)** $\log \dfrac{ab}{\sqrt[3]{c}}$

Solution

(a) $\log_2 6x = \log_2 6 + \log_2 x$ Law 1

(b) $\log_4 \dfrac{z^2}{y} = \log_4 z^2 - \log_4 y$ Law 2

$= 2 \log_4 z - \log_4 y$ Law 3

(c) $\log_5 x^3y^6 = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3 \log_5 x + 6 \log_5 y$ Law 3

(d) $\log \dfrac{ab}{\sqrt[3]{c}} = \log ab - \log \sqrt[3]{c}$ Law 2

$= \log a + \log b - \log c^{1/3}$ Law 1

$= \log a + \log b - \frac{1}{3} \log c$ Law 3

Recall that $\log x$ is the common logarithm and is the same as $\log_{10} x$.

NOW TRY EXERCISES 11 AND 15

The Laws of Logarithms also allow us to reverse the process of expanding. That is, we can write sums and differences of logarithms as a single logarithm. This process, called **combining logarithmic expressions**, is illustrated in the next example.

example 3 Combining Logarithmic Expressions

Combine the given expression into a single logarithm.

(a) $3 \log x + 2 \log(x - 5)$ **(b)** $3 \log s - \frac{1}{2} \log(t + 1)$

Solution

(a) $3 \log x + 2 \log(x - 5) = \log x^3 + \log(x - 5)^2$ Law 3

$= \log(x^3(x - 5)^2)$ Law 1

(b) $3 \log s - \frac{1}{2} \log(t + 1) = \log s^3 - \log(t + 1)^{1/2}$ Law 3

$= \log\left(\dfrac{s^3}{\sqrt{t + 1}} \right)$ Law 2

NOW TRY EXERCISE 19

example 4 Comparing Logarithms

Find the difference in the logarithms of the given weights (subtract the smaller logarithm from the larger one).

(a) The weight of a beetle is 10 times that of an ant.

(b) The weight of a mouse is a thousandth that of a wolf.

Solution

(a) Let A be the weight of the ant. Then the weight of the beetle is $10A$. The difference in the logarithms is

$$\begin{aligned} \log(10A) - \log A &= \log\left(\frac{10A}{A}\right) && \text{Law 2} \\ &= \log 10 && \text{Simplify} \\ &= 1 && \text{Because } 10 = 10^1 \end{aligned}$$

The difference in the logarithms is 1.

(b) Let W be the weight of the wolf. Then the weight of the mouse is $W/1000$. The difference in the logarithms is

$$\begin{aligned} \log(W) - \log\left(\frac{W}{1000}\right) &= \log W - (\log W - \log 1000) && \text{Law 2} \\ &= \log 1000 && \text{Simplify} \\ &= 3 && \text{Because } 1000 = 10^3 \end{aligned}$$

The difference in the logarithms is 3.

■ **NOW TRY EXERCISES 25 AND 29** ■

Change of Base Formula

So far, we have been calculating logarithms by inspection or by using the calculator. That's fine when the numbers involved are exact powers of the base—for example, $\log_2 16 = 4$ because we know that $2^4 = 16$. But what if we need to find, say, $\log_3 17$? Scientific calculators in general have only two keys that calculate logarithms: LOG for finding logarithms base 10 and LN for finding logarithms base e. (We'll learn about LN in Section 4.4.) We can use a calculator to find logarithms for other bases by using the Change of Base Formula.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

To prove this formula, we'll start with the logarithm that we want to find. Let

$$y = \log_b x$$

We write this equation in exponential form and take the logarithm, with base a, of each side.

$$\begin{aligned} b^y &= x && \text{Exponential form} \\ \log_a(b^y) &= \log_a x && \text{Take } \log_a \text{ of each side} \\ y \log_a b &= \log_a x && \text{Law 3} \\ y &= \frac{\log_a x}{\log_a b} && \text{Divide by } \log_a b \end{aligned}$$

Since $y = \log_b x$, this completes the proof.

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms and then using a calculator.

example 5 Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula to evaluate the logarithm $\log_8 5$, correct to five decimal places.

Solution

We use the Change of Base Formula with $b = 8$, $a = 10$, and $x = 5$.

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \qquad \text{Change of Base Formula}$$

$$\approx 0.77398 \qquad \text{Calculator}$$

NOW TRY EXERCISE 33

Graphing calculators have a LOG key for calculating logarithms base 10. We can use a graphing calculator to draw a graph $f(x) = \log_a x$ for any base a by using the Change of Base Formula. For example, to draw a graph of $f(x) = \log_4 x$, we note that

$$f(x) = \log_4 x = \frac{\log x}{\log 4} = \left(\frac{1}{\log 4}\right) \log x \approx (1.66) \log x$$

This shows that $\log_4 x$ is a constant multiple of $\log x$, that is, $\log_4 x \approx (1.66) \log x$. We use this fact to sketch graphs of logarithm functions in the next example.

example 6 Graphing Logarithmic Functions Using the Change of Base Formula

Draw the graph of the family of logarithmic functions $f(x) = \log_a x$ for $a = 2, 3, 5, 10$, all on the same graph.

Solution

To draw a graph of $y = \log_2 x$ using a graphing calculator, we first use the Change of Base Formula:

$$\log_2 x = \frac{\log x}{\log 2} \qquad \text{Change of Base Formula}$$

$$= \left(\frac{1}{\log 2}\right) \log x \qquad \text{Rewrite the fraction}$$

$$\approx (3.32) \log x \qquad \text{Calculator}$$

Similarly, you can check that

$$f(x) = \log_3 x \approx (2.10) \log x$$

$$f(x) = \log_5 x \approx (1.43) \log x$$

Using a graphing calculator, we sketch these graphs in Figure 1. Notice how increasing the value of the base a affects the graph.

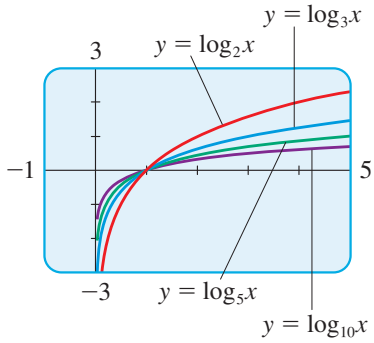


figure **1** A family of logarithmic functions

NOW TRY EXERCISES 37 AND 39

4.2 Exercises

CONCEPTS

Fundamentals

1. The logarithm of a product of two numbers is the same as the _____ of the logarithms of these numbers. So $\log_5(25 \cdot 125) =$ _____ + _____.

2. The logarithm of a quotient of two numbers is the same as the _____ of the logarithms of these numbers. So $\log_5\left(\frac{25}{125}\right) =$ _____ − _____.

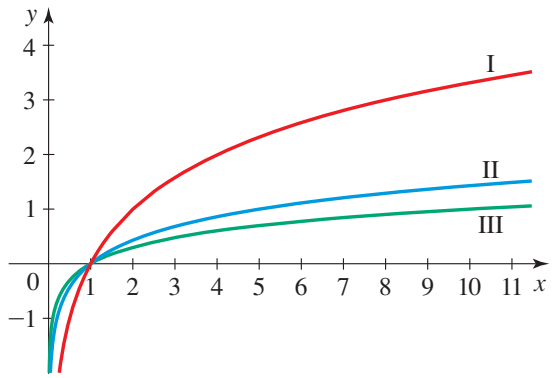
3. The logarithm of a number raised to a power is the same as the logarithm of the number multiplied by the _____. So $\log_5(25^{10}) =$ _____.

4. Most calculators can find logarithms with base _____. To find logarithms with different bases, we use the _____ Formula. To find $\log_7 12$, we write

$$\log_7 12 = \frac{\log \square}{\log \square} = \text{_____}.$$

5. Match the logarithmic function with its graph.

(a) $f(x) = \log_{10} x$ **(b)** $f(x) = \log_2 x$ **(c)** $f(x) = \log_5 x$



6. When using a graphing calculator to draw the graph of the function $f(x) = \log_{13} x$, first rewrite f as a constant multiple of the common logarithm function, using the _____ Formula:

$$f(x) = \left(\frac{1}{\log \square}\right) \log x$$

Think About It

7. True or false? (Assume that x, y, a, b are positive numbers.)

(a) $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$ **(b)** $\log a - \log b = \log\left(\frac{a}{b}\right)$

(c) $\log x + \log y = \log xy$ **(d)** $\log(x + y) = \log x + \log y$

(e) $\frac{\log a}{\log b} = \log a - \log b$ **(f)** $\log_2(x - y) = \log_2 x - \log_2 y$

(g) $(\log x)^2 = \log x^2$ **(h)** $2 \log x = \log x^2$

8. How are the graphs of $f(x) = \log(10x)$ and $g(x) = 1 + \log x$ related? Answer this question without using a graphing calculator. [*Hint:* Use the Laws of Logarithms.]

SKILLS

9–10 ■ Evaluate the given expression.

9. **(a)** $\log_{10} 4 + \log_{10} 25$ **(b)** $\log_2 160 - \log_2 5$ **(c)** $-\frac{1}{2} \log_2 64$

10. **(a)** $\log_{12} 9 + \log_{12} 16$ **(b)** $\log_4 192 - \log_4 3$ **(c)** $\frac{1}{2} \log_5 625 - 3 \log_5 \sqrt{5}$

11–18 ■ Use the Laws of Logarithms to expand the given expression.

11. **(a)** $\log_2 2x$ **(b)** $\log_3 \frac{x}{y^3}$

12. **(a)** $\log_5 \frac{x}{2}$ **(b)** $\log_3(x\sqrt{y})$

13. **(a)** $\log_2(AB^2)$ **(b)** $\log\left(\frac{1}{\sqrt{z}}\right)$

14. **(a)** $\log_3(5a)$ **(b)** $\log_5 \frac{2a}{b}$

15. **(a)** $\log_2 s^2\sqrt{t}$ **(b)** $\log \frac{r^3 s^4}{\sqrt[4]{t}}$

16. **(a)** $\log(w^2 z)^{10}$ **(b)** $\log_7 \frac{\sqrt[3]{wz}}{x}$

17. **(a)** $\log(xy)^{10}$ **(b)** $\log_5\left(\frac{x^2}{yz^3}\right)$

18. **(a)** $\log \sqrt[3]{3r^2 s}$ **(b)** $\log\left(\frac{x^3 y^4}{z^6}\right)$

19–24 ■ Use the Laws of Logarithms to combine the given expression.

19. **(a)** $\log_2 A + \log_2 B - 2 \log_2 C$

(b) $4 \log_6 y - \frac{1}{4} \log_6 z$

20. **(a)** $4 \log_2 x - \frac{1}{3} \log_2(x^2 + 1)$

(b) $\log 5 + 2 \log x + 3 \log(x^2 + 5)$

21. **(a)** $3 \log_2 A + 2 \log_2(B + 1)$

(b) $4 \log_3(2x - 1) - \frac{1}{2}\log_3(x + 1)^2$

22. **(a)** $2 \log_8(x + 1) + 2 \log_8(x - 1)$

(b) $\log_5(x^2 - 1) - \log_5(x - 1)$

23. **(a)** $\frac{1}{2}\log_4(y + 1) - \frac{1}{2}\log_4(y - 1)$

(b) $4 \log x - \frac{1}{3}\log(x^2 + 1) + 2 \log(x - 1)$

24. **(a)** $\log_7(s + 1) - \frac{1}{3}\log_7(s - 1)$

(b) $\log(a + b) + \log(a - b) - 2 \log c$

25–28 ■ Find the difference in the logarithms of the given weights. (Subtract the smaller logarithm from the larger one.)

25. The weight of an elephant is 100,000 times as much as the weight of a mouse.

26. The weight of an average star is 10^{27} times as much as the weight of a hippopotamus.

27. The weight of a bacterium is about 10^{15} times as much as the weight of an atom.

28. The weight of an electron is one thousandth the weight of an atom.

29–32 ■ Find the difference in the logarithms of the given heights or lengths. (Subtract the smaller logarithm from the larger one.)

29. The length of a ladybug is one thousandth of the height of a giraffe.

30. The height of a human is 10^6 times the diameter of a bacterium.

31. The height of a tree is 100 times the height of a rabbit.

32. The radius of the earth is one hundredth of the radius of the sun.

33–36 ■ Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places.

33. **(a)** $\log_2 5$ **(b)** $\log_5 2$

34. **(a)** $\log_3 16$ **(b)** $\log_6 92$

35. **(a)** $\log_7 2.61$ **(b)** $\log_6 532$

36. **(a)** $\log_4 125$ **(b)** $\log_{12} 2.5$

37–38 ■ Use the Change of Base Formula to draw a graph of the function (see Example 6).

37. **(a)** $f(x) = \log_5 x$ **(b)** $f(x) = \log_9 x$

38. **(a)** $g(x) = \log_6 x$ **(b)** $g(x) = \log_8 x$

39–40 ■ Draw graphs of the family of logarithmic functions $f(x) = \log_a x$, for the given values of a, all in the same viewing rectangle (see Example 6). How are these graphs related?

39. $a = 2, 4, 6$ **40.** $a = 3, 5, 7$

41–42 ■ A family of functions is given.

(a) Graph the family for $c = 1, 2, 3, 4$.

(b) How are the graphs in part (a) related?

41. $G(x) = \log(cx)$ **42.** $H(x) = c \log x$

4.3 Logarithmic Scales

- Logarithmic Scales
- The pH Scale
- The Decibel Scale
- The Richter Scale



AP Images

IN THIS SECTION... we see how logarithms allow us to efficiently compare real-world phenomena that vary over extremely large ranges (logarithmic scales).

Many real-world quantities involve a huge disparity in size. For example, an average-sized human is many times larger than an atom but trillions of times smaller than the smallest star, which in turn is insignificantly small in comparison to the average galaxy. So how can we meaningfully compare these very different sizes? How can we represent these differing sizes graphically?

Even everyday activities can involve enormously varying sizes. For example, our ears are sensitive to huge variations in sound intensity; we can hear a soft whisper as well as the almost deafening roar of a jet engine. The sound of a jet engine is a million billion times more intense than a whisper. We'll see in this section that the key to comparing and visualizing these vast disparities in size is to use logarithms.

Logarithmic Scales

We have observed that when numbers vary over a large range, their logarithms vary over a much smaller range. For example, the logarithms of the numbers between 10^{-3} and 10^{6} vary between -3 and 6. So to manage and understand large ranges of numbers, we represent them on a "logarithmic ruler" or logarithmic scale. An ordinary ruler uses a linear scale: Each successive mark on the ruler adds a fixed constant. On a "logarithmic ruler" each successive mark multiplies by a fixed factor. (See Figure 1.)

Linear scale: Each successive mark on the scale corresponds to *adding* a fixed constant.

Logarithmic scale: Each successive mark on the scale corresponds to *multiplying* by a fixed factor.

For example, for a logarithmic scale in base 10, moving up one unit on the scale corresponds to multiplying by 10. So the marks on the logarithmic ruler are the logarithms of the numbers they represent. The marks at 2, 3, and 4 on the ruler represent 100, 1000, and 10000, respectively. (Logarithmic scales can be made in any base, but the most common base is 10.)

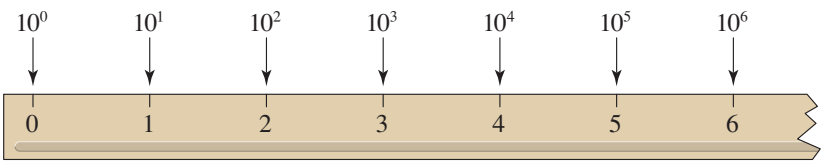


figure **1** The marks on the "logarithmic ruler" are the logarithms of the number they represent.

example 1 Logarithmic Scale

The weights of a particular ant, elephant, and whale are given. Compare the weights on a logarithmic scale (see Example 3 in Section 4.1, page 326).

Ant	0.000003 kilograms
Elephant	4000 kilograms
Whale	170,000 kilograms

Solution

On a logarithmic scale, these weights are represented by their logarithms. So on a logarithmic scale, the weights are represented by -5.5, 3.6, and 5.2, respectively.

NOW TRY EXERCISE 7

In Figure 2 we represent the weights of the ant, elephant, and whale of Example 1 graphically. On a linear scale, the weight of the ant is indistinguishable from 0 and from the weight of the elephant. On a logarithmic scale we can more easily gauge their relative sizes.

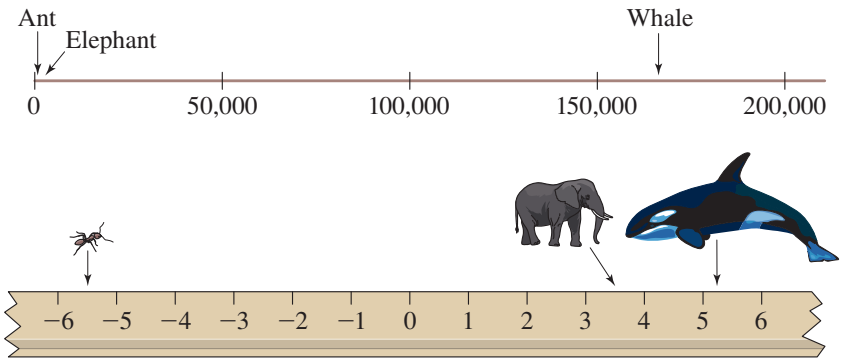


figure 2 A linear scale (top) and a logarithmic scale (bottom).

The pH Scale

pH for some common substances

Substance	pH
Milk of magnesia	10.5
Seawater	8.0–8.4
Human blood	7.3–7.5
Crackers	7.0–8.5
Hominy (lye)	6.9–7.9
Cow's milk	6.4–6.8
Spinach	5.1–5.7
Tomatoes	4.1–4.4
Oranges	3.0–4.0
Apples	2.9–3.3
Limes	1.3–2.0
Battery acid	1.0

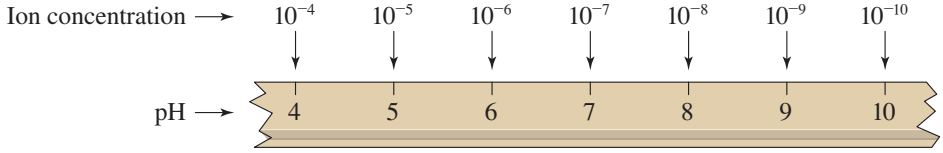
Chemists measure the acidity of a solution by giving its hydrogen ion concentration $[H^+]$ in moles per liter (M). The hydrogen ion concentration varies greatly from substance to substance and involves huge numbers. In 1909 Sorensen proposed using a logarithmic scale to measure hydrogen ion concentration. He defined

$$\text{pH} = -\log[H^+]$$

He did this to avoid very small numbers and negative exponents. For instance,

$$\text{If} \quad [H^+] = 10^{-4}\,\text{M} \quad \text{then} \quad \text{pH} = -\log_{10}(10^{-4}) = -(-4) = 4$$

In other words, the pH scale is a "logarithmic ruler" for measuring ion concentration.



Solutions with a pH of 7 are defined as neutral, those with pH < 7 are acidic, and those with pH > 7 are basic. Notice that when the pH increases by one unit, $[H^+]$ decreases by a factor of 10.

example 2 pH Scale and Hydrogen Ion Concentration

(a) The hydrogen ion concentration of a sample of human blood was measured to be $[H^+] = 3.16 \times 10^{-8}$ M. Find the pH and classify the blood as acidic or basic.

(b) The most acidic rainfall ever measured occurred in Scotland in 1974; its pH was 2.4. Find the hydrogen ion concentration.

Solution

(a) The definition of pH gives

$$\begin{aligned} \text{pH} &= -\log[H^+] && \text{Definition of pH} \\ &= -\log(3.16 \times 10^{-8}) && \text{Replace } H^+ \text{ by } 3.16 \times 10^{-8} \\ &\approx 7.5 && \text{Calculator} \end{aligned}$$

So the pH is 7.5. Since this is greater than 7, the blood is basic.

(b) We use the definition of pH and express it in exponential form.

$$\begin{aligned} \log[H^+] &= -\text{pH} && \text{From the definition of pH} \\ [H^+] &= 10^{-\text{pH}} && \text{Exponential form} \\ [H^+] &= 10^{-2.4} && \text{Replace pH by 2.4} \\ &\approx 0.0039 && \text{Calculator} \end{aligned}$$

The hydrogen ion concentration is approximately 4.0×10^{-3} moles per liter.

■ NOW TRY EXERCISES 11, 13, AND 17 ■

The Decibel Scale



Scientists model human responses to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k \log\left(\frac{I}{I_0}\right)$$

where S is the subjective intensity of the stimulus, I is the physical intensity, and I_0 is the threshold physical intensity (the intensity at which the stimulus is barely felt). The constant k depends on the sensory stimulus (sound, light, or pressure).

The ear is sensitive to an extremely wide range of sound intensities. The threshold intensity is $I_0 = 10^{-12}$ W/m^2 (watts per square meter) at a frequency of 1000 Hz (hertz), which measures a sound that is just barely audible. The psychological sen-

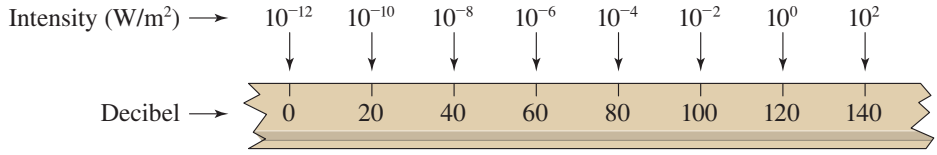
sation of loudness varies with the logarithm of the intensity, so the intensity level B, measured in decibels (dB), is defined as

$$B = 10 \cdot \log \frac{I}{I_0}$$

The decibel level of the barely audible threshold sound is

$$B = 10 \cdot \log \frac{I_0}{I_0} = 10 \cdot \log 1 = 0 \text{ dB}$$

So the decibel scale is a logarithmic scale for measuring sound intensity, with 0 decibels corresponding to an intensity of 10^{-12} W/m^2.



The table in the margin lists decibel levels for some common sounds ranging from the threshold of human hearing to the jet takeoff of Example 3. The threshold of pain is about 120 decibels.

Decibel levels of common sounds

Source of sound	dB
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10–20
Threshold of hearing	0

example 3 Sound Intensity of a Jet Takeoff

Find the decibel level of a jet engine during takeoff if the intensity was measured at 100 W/m^2.

Solution

From the definition of decibel level we see that

$$\begin{aligned} B &= 10 \log \frac{I}{I_0} && \text{Definition of decibel} \\ &= 10 \log \frac{10^2}{10^{-12}} && \text{Replace } I \text{ by } 10^2 \text{ and } I_0 \text{ by } 10^{-12} \\ &= 10 \log 10^{14} && \text{Rules of Exponents} \\ &= 140 && \text{Calculate} \end{aligned}$$

Thus, the decibel level is 140 dB.

■ NOW TRY EXERCISES 15 AND 21 ■

The Richter Scale

In 1935 the American geologist Charles Richter (1900–1984) defined the magnitude M of an earthquake to be

$$M = \log \frac{I}{S}$$

Largest earthquakes

Location	Date	Magnitude
Chile	1960	9.5
Alaska	1964	9.2
Sumatra	2004	9.1
Alaska	1957	9.1
Kamchatka	1952	9.0
Ecuador	1906	8.8
Alaska	1965	8.7
Sumatra	2005	8.7
Tibet	1950	8.6
Kamchatka	1923	8.5

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ centimeter). The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 794,000,000, so the Richter scale provides more manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

example 4 Magnitude of Earthquakes



Roger Ressmeyer/Corbis

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border and was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

Solution

If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$\begin{aligned} M &= \log \frac{4I}{S} && \text{Definition of magnitude} \\ &= \log 4 + \log \frac{I}{S} && \text{Laws of Logarithms} \\ &= \log 4 + 8.3 && \text{Replace } \log (I/S) \text{ by } 8.3 \\ &\approx 8.9 && \text{Calculator} \end{aligned}$$

■ NOW TRY EXERCISE 23 ■

example 5 Intensity of Earthquakes

The 2004 earthquake in Sumatra that caused a giant tsunami had an estimated magnitude of 9.1 on the Richter scale. The 2008 earthquake in Sichuan, China, had an estimated magnitude of 7.9 on the Richter scale.

(a) Express the intensity I of an earthquake in terms of the magnitude M and the standard intensity S.

(b) Use the results of part (a) to find how many times more intense the Sumatra earthquake was than the one in China.

Solution

(a) We use the definition for magnitude and solve for I:

$$M = \log \frac{I}{S} \qquad \text{Definition of earthquake magnitude}$$

$$10^M = \frac{I}{S} \qquad \text{Exponential form}$$

$$I = S \cdot 10^M \qquad \text{Multiply by } S \text{ and switch sides}$$

(b) If I_1 and I_2 are the intensities of the Sumatra and China earthquakes, then we are required to find I_1/I_2. Using the results of part (a), we get

$$\frac{I_1}{I_2} = \frac{S \cdot 10^{9.1}}{S \cdot 10^{7.9}} \qquad \text{From part (a)}$$

$$= 10^{9.1-7.9} \qquad \text{Rules of Exponents}$$

$$\approx 16 \qquad \text{Calculator}$$

The Sumatra earthquake was about 16 times as intense as the China earthquake.

■ **NOW TRY EXERCISE 25** ■

4.3 Exercises

CONCEPTS

Fundamentals

1. On a logarithmic scale, a number is represented by its _____. So on a logarithmic scale (base 10), the number 1000 is represented by _____, and the number $1/100$ is represented by _____.
2. On the pH scale, the higher the hydrogen ion concentration, the _____ (higher/lower) the pH.
3. On the decibel scale, the greater the sound intensity, the _____ (higher/lower) the decibel level.
4. A magnitude 8 earthquake is _____ times more intense than a magnitude 5 earthquake.

Think About It

5. If we graph 10 and 1000 on a linear scale, how many units apart are they? How many units apart are they on a logarithmic scale?
6. The distance between two numbers A and B on a logarithmic scale is the difference between their logarithms. Explain why this distance is the same as the logarithm of the *ratio*.

SKILLS

7–10 ■ Three quantities are given. Graph the quantities on a logarithmic scale.

7. Weight of the smallest primate: 1 ounce
   Weight of a monkey: 12 pounds
   Weight of a gorilla: 350 pounds

8. Radius of a hydrogen atom: 5.3×10^{-13} m
Length of an amoeba: 7.1×10^{-4} m
Height of a human: 1.8 m

9. Number of stars in the Andromeda galaxy: 5.0×10^{11}
Number of cells in a human body: 4.9×10^{13}
Number of atoms in a drop of water: 3.3×10^{19}

10. Length of a redwood tree seed: 0.17 cm
Height of a redwood tree seedling: 20 m
Height of a mature redwood tree: 122 m

11–12 ■ The hydrogen ion concentration of a substance is given. Calculate the pH of the substance.

11. **(a)** Lemon juice: $[H^+] = 5.0 \times 10^{-3}$ M **(b)** Tomato juice: $[H^+] = 3.2 \times 10^{-4}$ M

12. **(a)** Seawater: $[H^+] = 5.0 \times 10^{-9}$ M **(b)** Vinegar: $[H^+] = 1.98 \times 10^{-7}$ M

13–14 ■ The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.

13. **(a)** Soda: pH = 2.6 **(b)** Milk: pH = 6.5

14. **(a)** Beer: pH = 4.6 **(b)** Water: pH = 7.3

15–16 ■ The intensity of a sound is given. Calculate the decibel level of the sound.

15. **(a)** Power mower: 1.3×10^{-3} W/m^2 **(b)** Circular saw: 7.9×10^{-3} W/m^2

16. **(a)** Rock concert: 1.6×10^{-1} W/m^2 **(b)** Alarm clock: 6.3×10^{-5} W/m^2

CONTEXTS

17. pH of Wine The pH of wines varies from 2.8 to 3.8. If the pH of a wine is too high, say, 4.0 or above, the wine becomes unstable and has a flat taste.

(a) A certain California red wine has a pH of 3.8, and a certain Italian white wine has a pH of 2.8. Find the corresponding hydrogen ion concentrations of the two wines.

(b) Which wine has the lower hydrogen ion concentration?

18. pH of Saliva The pH of saliva is normally in the range of 6.4 to 7.0. However, when a person is ill, the person's saliva becomes more acidic.

(a) When Marco is sick, he tests the pH of his saliva and finds that it is 5.5. What is the hydrogen ion concentration of his saliva?

(b) Will the hydrogen ion concentration in Marco's saliva increase or decrease as he gets better?

(c) After Marco recovers, he tests the pH of his saliva, and it is 6.5. Was the saliva more acidic or less acidic when he was sick?

19. pH of Blood The pH of your blood can drop when you exercise a lot. Under normal circumstances your body copes with this and brings the pH back up. However if the pH of the blood gets too low (below 7.4), a condition known as acidosis results. A serious case of acidosis can result in death.

(a) After a training session for a race, Hannali measures the hydrogen ion concentration in her blood and finds it to be $[H^+] = 1.98 \times 10^{-7}$ M. What is the pH of her blood? Is there cause for concern?

(b) Should her doctor try to raise or lower the hydrogen ion concentration to bring the pH back to normal?

20. pH of Soil A citrus tree can't get the nutrients it needs from the soil if its pH is above 6.5.

(a) The hydrogen ion concentration of a soil sample is $[H^+] = 1.26 \times 10^{-7}$ M. What is the pH of the soil? Is this soil suitable for citrus trees?

(b) After the soil is amended, the pH is 5.6. Is the amended soil more acidic or less acidic than the original soil?

21. Hearing Loss at Work The National Campaign for Hearing Health states that hearing loss results from prolonged exposure to noise of 85 dB or higher. People who work with power tools must take precautionary measures, like wearing ear muffs, to protect their hearing.

(a) Mika uses a snow blower to clean yards. The intensity of the sound from the blower is measured at 3.2×10^{-2} W/m^2. Find the intensity level in decibels.

(b) Mika's friend Jerri uses a hair dryer in her salon. The intensity of the sound from her dryer is measured at 3.1×10^{-4} W/m^2. Find the intensity level in decibels.



22. Hearing Loss from MP3 Players Recent research has shown that the use of earbud-style headphones packaged with MP3 players can cause permanent hearing loss. According to one study, two-thirds of young people who regularly use earbud-style headphones face premature hearing loss.

(a) The intensity of the sound from a certain MP3 player (without earbuds) is 3.1×10^{-5} W/m^2. Find the intensity level in decibels.

(b) The intensity of the sound of the MP3 player in part (a) with earbuds is 1.0×10^{-3} W/m^2. Find the intensity level in decibels.

23. Earthquake Magnitude The 1985 Mexico City earthquake had a magnitude of 8.1 on the Richter scale. The 1976 earthquake in Tangshan, China, was 1.26 times as intense. What was the magnitude of the Tangshan earthquake?

24. Earthquake Magnitude The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9 caused only minor damage. How many times more intense was the San Francisco earthquake than the Japanese earthquake?

25. Earthquake Magnitude The Gansu, China, earthquake of 1920 had a magnitude of 8.6 on the Richter scale. How many times more intense was this than the 1906 San Francisco earthquake? (See Exercise 24.)

26. Earthquake Magnitude The Northridge, California, earthquake of 1994 had a magnitude of 6.8 on the Richter scale. A year later, a 7.2-magnitude earthquake struck Kobe, Japan. How many times more intense was the Kobe earthquake than the Northridge earthquake?

27. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log\left(\frac{B}{B_0}\right)$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less is its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less than Albiero.

28. Absorption of Light A spectrophotometer measures the concentration of a substance dissolved in water by shining a light through the solution and recording the amount of light that emerges. If we know the amount of light that is absorbed, we can calculate the concentration of the sample. For a certain substance the concentration (in moles per liter) is given by the formula

$$C = -5756 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.





29. Fitt's Law The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitt's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5 mm wide with the difficulty of clicking on one that is 10 mm wide. In each case, assume that the mouse is 100 mm from the icon.

30. Charging a Battery The rate at which a battery charges is slower, the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k \log\left(1 - \frac{C}{C_0}\right)$$

where k is a constant that depends on the battery. For a certain battery, $k = 0.58$. If this battery is fully discharged, how long will it take to charge to 95% of its maximum charge C_0?

4.4 The Natural Exponential and Logarithmic Functions

- **What Is the Number *e*?**
- **The Natural Exponential and Logarithmic Functions**
- **Continuously Compounded Interest**
- **Instantaneous Rates of Growth or Decay**
- **Expressing Exponential Models in Terms of *e***

IN THIS SECTION... we introduce the natural exponential and logarithmic functions. We use these functions to calculate continuously compounded interest and to model continuous growth and decay.

Recall that any positive number a can be used as a base for the exponential function. In Sections 3.1 and 3.2 we saw the effect of changing the base. Some bases occur frequently in applications, such as base 2 and 10, but the most important base is the number denoted by the letter e, which we study in this section. Using the base e allows us to find exponential models for continuously changing quantities in terms of the "instantaneous" rate of growth (or decay).

What Is the Number *e*?

table **1**
Estimating *e*

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828
10,000,000	2.71828

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made precise through the concept of a "limit.") Table 1 shows the values of $(1 + 1/n)^n$ for increasingly large values of n. It appears that correct to five decimal places, $e \approx 2.71828$; the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It turns out that the number e is irrational, so we cannot write its exact value using decimals.

Why use such a strange base for an exponential function? It may seem at first that a base such as 10 is much easier to work with. But the number e works best in situations in which there is continuous change, as in many of the examples we've studied in this chapter.

The number e is programmed into most calculators. In the next example we use a calculator to work with e.

example 1 Working with *e* on a Calculator

Evaluate the following expressions using a calculator.

(a) $5e$ **(b)** $e^{2.8}$ **(c)** $2e^{-0.53}$ **(d)** $2 + 3e^{-2}$

Solution

We show how to evaluate these expressions using a TI-83 calculator.

	Calculator Keystrokes	Output
(a) $5e$	5 * e^x 1 ENTER	13.591409
(b) $e^{2.8}$	e^x 2 . 8 ENTER	16.444647
(c) $2e^{-2.53}$	2 e^x − 2 . 5 3 ENTER	0.159318
(d) $2 + 3e^{-2}$	2 + 3 e^x − 2 ENTER	2.406006

NOW TRY EXERCISE 9

The Natural Exponential and Logarithmic Functions

We define the exponential function with base e, called the *natural exponential function*. Because this function is the best growth model for so many situations, it is sometimes referred to as *the* exponential function.

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e.

Since $2 < e < 3$, it follows that for all numbers x we have

$$2^x < e^x < 3^x$$

This means that the graph of the exponential function lies between the graphs of the functions $y = 2^x$ and $y = 3^x$ as shown in Figure 1.

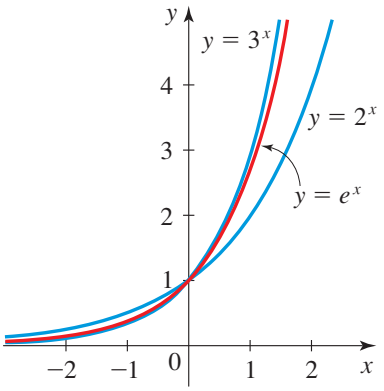


figure **1** Graph of the natural exponential function

example 2 Graphing Exponential Functions

Sketch graphs of the following exponential functions. Comment on the similarities and differences between the graphs.

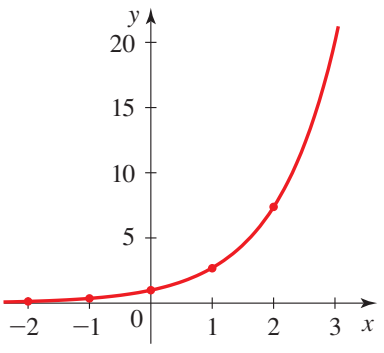
(a) $f(x) = e^x$ **(b)** $g(x) = e^{-x}$

Solution

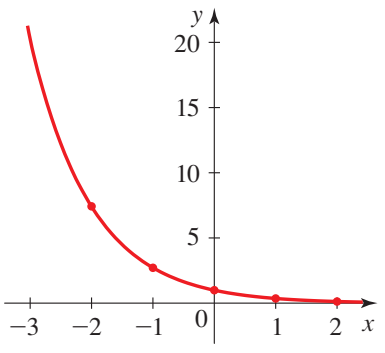
We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 2. Both graphs are graphs of exponential functions, so they have horizontal asymptote $y = 0$. The function $f(x) = e^x$ is an increasing function, whereas $g(x) = e^{-x}$ is a decreasing function.

x	−2	−1	0	1	2
$f(x)$	0.14	0.37	1	2.72	7.39

x	−2	−1	0	1	2
$g(x)$	7.39	2.72	1	0.37	0.14



(a) Graph of $f(x) = e^x$



(b) Graph of $g(x) = e^{-x}$

figure **2**

■ **NOW TRY EXERCISE 11** ■

We also define the logarithm function with base e, called the *natural logarithm function*.

The Natural Logarithm Function

The **natural logarithm function** is the logarithm function with base e, which is denoted by ln:

$$\ln x = \log_e x$$

So from the definition of logarithms we have

$$\ln x = y \quad \text{if and only if} \quad e^y = x$$

Now, since $2 < e < 3$, it follows that the graph of the natural logarithm function lies between the graphs of the functions $y = \log_2 x$ and $y = \log_3 x$ as shown in Figure 3.

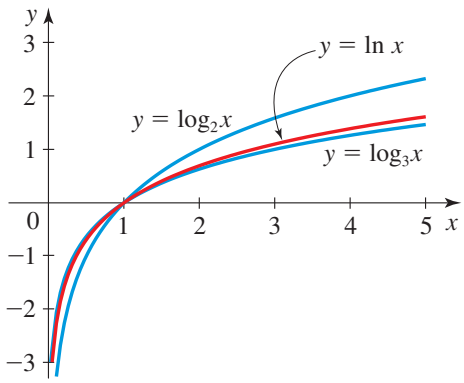


figure **3** Graph of the natural exponential function

From the definition of logarithms we get the following properties.

Basic Properties of the Natural Logarithm

Property	Reason
1. $\ln 1 = 0$	We raise e to the exponent 0 to get 1.
2. $\ln e = 1$	We raise e to the exponent 1 to get e.
3. $\ln e^x = x$	We raise e to the exponent x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the exponent to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

example 3 Evaluating Natural Logarithms

Evaluate the following.

(a) $\ln e^8$ **(b)** $\ln\left(\dfrac{1}{e^2}\right)$ **(c)** $\ln 5$ **(d)** $\ln 0.3$

Solution

(a) $\ln e^8 = 8$ Property 3: $\ln e^x = x$

(b) $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2}$ Rules of Exponents

$= -2$ Property 3: $\ln e^x = x$

(c) $\ln 5 \approx 1.609$ Calculator

(d) $\ln 0.3 \approx -1.204$ Calculator

NOW TRY EXERCISE 13

Continuously Compounded Interest

Some banks offer continuously compounded interest; that is, the interest is compounded at each instant (as opposed to each month, week, or day). To calculate continuously compounded interest, we need the exponential function. Remember from Section 3.1 that compound interest is calculated by using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Suppose that \$5000 is invested in a 3-year CD that pays 5.5% interest (see Example 6 in Section 3.1). Table 2 shows how the amount of the investment depends on the number of compounding periods.

table **2**
Compound interest

Number of compounding periods per year	Calculation	Amount
1	$5000(1 + 0.055)^3$	\$5871.21
2	$5000\left(1 + \frac{0.055}{2}\right)^{2\cdot 3}$	\$5883.84
4	$5000\left(1 + \frac{0.055}{4}\right)^{4\cdot 3}$	\$5890.34
12	$5000\left(1 + \frac{0.055}{12}\right)^{12\cdot 3}$	\$5894.74
365	$5000\left(1 + \frac{0.055}{365}\right)^{365\cdot 3}$	\$5896.89
1000	$5000\left(1 + \frac{0.055}{1000}\right)^{1000\cdot 3}$	\$5896.93
100,000	$5000\left(1 + \frac{0.055}{10^5}\right)^{10^5\cdot 3}$	\$5896.97
1,000,000	$5000\left(1 + \frac{0.055}{10^6}\right)^{10^6\cdot 3}$	\$5896.97

So the value of the investment increases as the number of compounding periods increases, but the increase in value levels off as the number of compounding periods gets very large. This same phenomenon occurs for any interest rate r. To see why, let $m = n/r$. Then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt}$$ Because $nt = (n/r) \cdot rt$

$$= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$ Because $m = n/r$, so $r/n = 1/m$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus, the amount approaches $A(t) = Pe^{rt}$. This expression gives the amount when interest is compounded at "every instant."

Continuously Compounded Interest

If an amount P is invested at an annual interest rate r **compounded continuously**, then the amount $A(t)$ of the investment after t years is given by the formula

$$A(t) = Pe^{rt}$$

example **4** **Calculating Continuously Compounded Interest**

Find the amount after 3 years if \$5000 is invested at an interest rate of 5.5% per year, compounded continuously.

Solution

The amount we obtained for continuous compounding is the same as the last few entries in Table 2, because for these entries there were many compounding periods in the year.

We use the formula for continuously compounded interest with $P = \$5000$, $r = 0.055$, and $t = 3$ to get

$$A(t) = 5000e^{0.055(3)} = 5896.97$$

Thus the amount is \$5896.97. Compare this amount with the amounts in Table 2.

■ NOW TRY EXERCISE 43 ■

Instantaneous Rates of Growth or Decay

In Section 3.1 we modeled the growth of populations using exponential functions, where the base of the exponential function is the growth factor. But in reality, the population is continually changing, so the rate of change of the population is itself continually changing. Likewise, we may know the half-life of a radioactive substance, but in reality the substance is continuously decaying, so the rate of change of the amount of radioactive material is itself continually changing. So it is reasonable to model such quantities in terms of an "instantaneous" growth rate. It can be shown by using calculus that exponential growth can be modeled by a function of the form $f(x) = Ce^{rt}$, where r is the "instantaneous" growth rate. This is identical to the situation we encountered with continuously compounded interest.

Exponential Growth and Decay

The exponential function

$$f(t) = Ce^{rt}$$

models **exponential growth** (if $r > 0$) or **exponential decay** (if $r < 0$).

- The variable t represents time.
- The constant C is the **initial value** of f (the value when $t = 0$).
- The number r is the **instantaneous growth rate** (or **instantaneous decay rate**) expressed as a proportion of the population per time unit.

Bacteria usually grow in colonies that become visible to the naked eye. So microbiologists often count the number of bacteria colonies in a portion of a sample and then estimate the bacteria count. In this case the bacteria count is given in terms of **colony-forming units** per milliliter (CFU/mL).

example 5 Modeling Exponential Growth

The initial bacteria count in a culture is 500 CFU/mL. The instantaneous growth rate of the bacteria is 40% per hour.

(a) Find an exponential function $f(t) = Ce^{rt}$ that models the number of bacteria after t hours.

(b) What is the estimated bacteria count after 10 hours?

(c) Draw the graph of the function found in part (a).

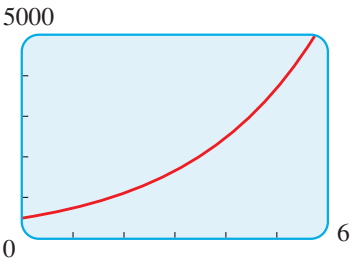


figure **4** Graph of $f(t) = 500e^{0.4t}$

Solution

(a) We use the exponential growth model with $C = 500$ and $r = 0.4$ to get

$$f(t) = 500e^{0.4t}$$

where t is measured in hours.

(b) Using the function in part (a), we find that the bacteria count after 10 hours is

$$f(10) = 500e^{0.4 \cdot 10} = 500e^4 \approx 27{,}300 \text{ CFU/mL}$$

(c) The graph is shown in Figure 4.

NOW TRY EXERCISE 27

If $r < 0$, then the function $f(t) = Ce^{rt}$ models exponential decay. So a negative instantaneous "growth" rate r corresponds to an **instantaneous decay rate**.

example 6 Modeling Exponential Decay

The initial bacteria count in a culture is 1000 CFU/mL. In the presence of an antibiotic the bacteria count has an instantaneous decay rate r of -0.12 per hour.

(a) Is the bacteria count increasing or decreasing?

(b) Find an exponential function $f(t) = Ce^{rt}$ that models the bacteria count, where t is measured in hours.

(c) Sketch the graph of the function found in part (a).

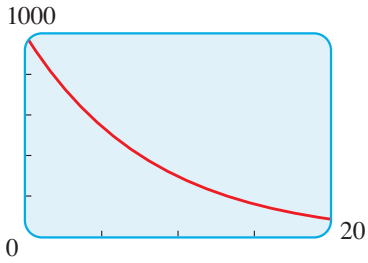


figure **5** Graph of $f(t) = 1000e^{-0.12t}$

Solution

(a) Since the instantaneous rate is negative, the bacteria count is decreasing.

(b) We use the exponential growth model with $C = 1000$ and $r = -0.12$ to get

$$f(t) = 1000e^{-0.12t}$$

where t is measured in hours.

(c) The graph is shown in Figure 5.

NOW TRY EXERCISES 29 AND 49

Expressing Exponential Models in Terms of *e*

In studying population growth, scientists measure the growth rate over a fixed time period, not the instantaneous growth rate. For example, a biologist may find that a bacteria culture increases by 40% in 1 hour. Or a physicist may find that it takes 10 minutes for a radioactive substance to decay to half its mass. This type of information allows us to find models in terms of growth or decay rates. We can also use this information to find the instantaneous growth rate and express these models in terms of the natural exponential function. Let's see how this is done.

Suppose that a bacteria culture is modeled by $f(t) = 100 \cdot 2^t$. To find the instantaneous growth rate r, let's compare the two models

$$f(x) = 100 \cdot 2^x \quad \text{and} \quad f(x) = 100 \cdot e^{rx}$$

For these models to be the same, we must have

$$\begin{aligned} 100 \cdot 2^x &= 100 \cdot e^{rx} && \text{Set the models equal} \\ 2^x &= e^{rx} && \text{Divide by 100} \\ \ln 2^x &= \ln e^{rx} && \text{Take ln of each side} \\ x \ln 2 &= rx && \text{Properties of ln} \\ \ln 2 &= r && \text{Divide by } x \end{aligned}$$

In general, we have the following.

Expressing an Exponential Model in Terms of *e*

The exponential growth or decay model

$$f(x) = Ca^x \quad \text{(growth or decay factor } a \text{ per time period)}$$

is equivalent to the exponential model

$$f(x) = Ce^{rx} \quad \text{(instantaneous growth or decay rate } r \text{ per time period)}$$

where the instantaneous growth or decay rate is

$$r = \ln a$$

example **7** **Expressing Exponential Models in Terms of *e***

Express the model $f(x) = 350(1.4)^x$ in terms of the base e.

Solution

In the model $f(x) = 350(1.4)^x$ the growth factor is 1.4, so the instantaneous growth rate is

$$r = \ln 1.4 \approx 0.34$$

So the model that we seek is

$$f(x) = 350e^{0.34x}$$

NOW TRY EXERCISE 31

example 8 Expressing Exponential Models in Terms of Instantaneous Growth Rate

A sample contains 22 micrograms of a radioactive substance, and the amount of the substance is decreasing by 10% per day.

(a) Find an exponential function $f(x) = Ca^x$ that models the amount of the substance, where x is measured in days.

(b) Find the instantaneous decay rate (in decimal form) per day, and find an exponential function $g(x) = Ce^{rx}$ that models the amount of the substance, where x is measured in days.

(c) Sketch the graphs of the functions f and g.

Solution

(a) Since the amount of the substance is decreasing by 10% per day, the decay rate is -0.10, so the decay factor is $a = 1 + (-0.10) = 0.90$. The exponential growth model that we seek is

$$f(x) = 22(0.90)^x$$

where x is measured in days.

(b) The instantaneous decay rate is

$$r = \ln a = \ln(0.90) \approx -0.105$$

We use the exponential decay model, where C is 22 and r is -0.015, to get

$$g(x) = 22e^{-0.105x}$$

where x is measured in days.

(c) The graphs of f and g are the same, as shown in Figure 6.

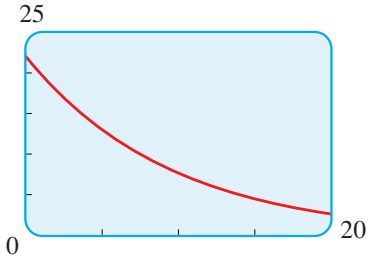


figure **6** Graphs of $f(x) = 22(0.90)^x$ and $g(x) = 22e^{-0.105x}$

NOW TRY EXERCISES 35 AND 53

4.4 Exercises

CONCEPTS

Fundamentals

1. **(a)** The function $f(x) = e^x$ is called the _____ exponential function. The number e is approximately equal to _____ (to five decimal places).

(b) The natural logarithm function is $f(x) = \ln x = \log_{\square} x$, so $\ln e =$ _____.

2. In the formula $A(t) = Pe^{rt}$ for *continuously* compounded interest, the letters P, r, and t stand for _____, _____, and _____, respectively, and $A(t)$ stands for _____. So if \$100 is invested at an interest rate of 6% compounded continuously, then the amount after 2 years is _____.

3. If the growth factor for a population is a, then the instantaneous growth rate is $r =$ _____. So if the growth factor for a population is 1.5, then the instantaneous growth rate is _____.

4. If a population P has a instantaneous growth rate r per year and the initial population is C, then an exponential model for the population after t years is given by $f(t) = C \cdot \square^{\square}$.

Think About It

5. Suppose that the bacteria count in one culture doubles every 10 minutes and that in another culture increases by 40% every 8 minutes. Which culture is growing faster? Explain how finding the instantaneous growth rate of each culture helps us to answer this question.

6. Suppose that \$100 is invested at the beginning of a year. Which (if either) of the following methods of growing the investment results in a larger amount at the end of the year?

(i) The investment grows at an interest rate of 5% per year, compounded continuously.

(ii) The investment grows at an instantaneous growth rate of 5% per year.

7. Recall that a function $f(x) = Ca^x$ models growth if $a > 1$ and decay if $0 < a < 1$. How does the formula $r = \ln a$ show that an instantaneous growth rate r is always *positive*, whereas an instantaneous decay rate r is always *negative*?

8. A bacteria culture starts with 1 bacterium. Find a model for the bacteria population after t hours under the given conditions. How do these models differ?

(a) One bacterium is added to the population each hour.

(b) The population growth factor is 2.

(c) The population increases by 100% per hour.

(d) The population has an instantaneous growth rate of 1.

SKILLS

9–10 ■ Evaluate the expression using a calculator.

9. **(a)** $3e$ **(b)** $e^{1.4}$ **(c)** $3e^{-3.1}$ **(d)** $5 + 2e^2$

10. **(a)** $4e^2$ **(b)** $e^{3.9}$ **(c)** $4.1e^{-6.2}$ **(d)** $3.2 - 2e^3 + e^{-1}$

11–12 ■ Complete the table and sketch the graph of the functions f and g on the same axes.

11. $f(x) = e^{3x}$, $g(x) = e^{-3x}$

x	-2	-1	0	1	2
$f(x)$					

x	-2	-1	0	1	2
$g(x)$					

12. $f(x) = e^{-0.5x}$, $g(x) = e^{0.5x}$

x	-2	-1	0	1	2
$f(x)$					

x	-2	-1	0	1	2
$g(x)$					

13–14 ■ Evaluate the expression. Use a calculator if necessary.

13. **(a)** $\ln e^4$ **(b)** $\ln(1/e)$ **(c)** $\ln 2$ **(d)** $5 + \ln(e - 2)$

14. **(a)** $\ln(6e)$ **(b)** $\dfrac{1}{\ln e^2}$ **(c)** $3 \ln 3$ **(d)** $\ln(1 + e)$

15–16 ■ Complete the table, and sketch the graph of the function by plotting points.

15. $f(x) = 4 \ln x$

x	$\frac{1}{e^4}$	$\frac{1}{e^3}$	$\frac{1}{e^2}$	$\frac{1}{e^1}$	1	e^1	e^2	e^3	e^4
$f(x)$									

16. $f(x) = -3 \ln x$

x	$\frac{1}{e^4}$	$\frac{1}{e^3}$	$\frac{1}{e^2}$	$\frac{1}{e^1}$	1	e^1	e^2	e^3	e^4
$f(x)$									

17–18 ■ Use a graphing calculator to draw a graph of the family of logarithmic functions $f(x) = \log_a x$ for the given values of a, all on the same graph.

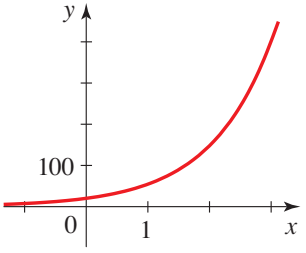
17. $a = 2, e, 4$ **18.** $a = 2, e, 3$

19–22 ■ Match the function with its graph.

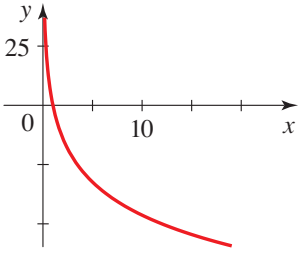
19. $f(x) = 20 \cdot e^x$ **20.** $f(x) = 20 \cdot e^{-x}$

21. $f(x) = 20 \ln x$ **22.** $f(x) = -20 \ln x$

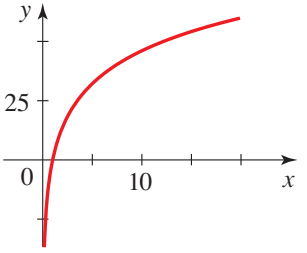
I



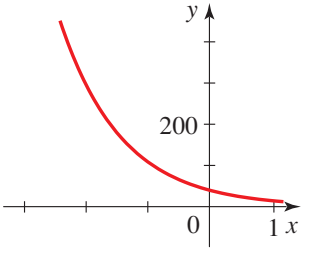
II



III



IV



23–26 ■ An exponential growth or decay model is given.

(a) Determine whether the model represents growth or decay.

(b) Find the instantaneous growth or decay rate.

23. $f(t) = 500 \cdot e^{0.5t}$ **24.** $g(t) = 200 \cdot e^{-0.25t}$

25. $M(t) = 25 \cdot e^{-0.3t}$ **26.** $h(t) = 300 \cdot e^{1.5t}$

27–30 ■ The initial size of a population is 1000, and it has the given instantaneous growth or decay rate per year.

(a) Find an exponential model $f(x) = Ce^{rx}$ for the population, where t is measured in years.

(b) Sketch the graph of the function found in part (a).

27. 2.4%

28. 0.90

29. −0.36

30. −75%

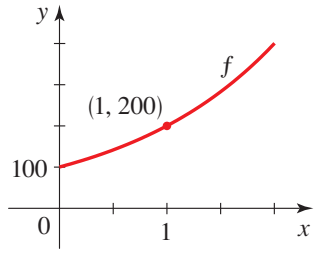
31–32 ■ Express the given model in terms of the base e. What is the instantaneous growth rate?

31. **(a)** $f(t) = 2500 \cdot 2^t$ **(b)** $g(x) = 2500 \cdot 2^{-x}$

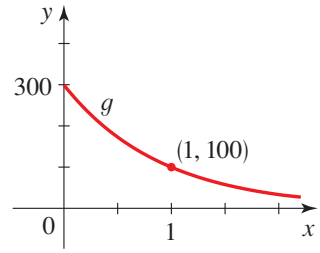
32. **(a)** $h(T) = 13(1.72)^T$ **(b)** $k(x) = 210(0.78)^x$

33–34 ■ The graph of a function that models exponential growth or decay is shown. Find the initial population and instantaneous growth rate.

33.



34.



35–38 ■ The initial size of a population is 200 and increases or decreases by the given percentage per year.

(a) Find an exponential model $f(t) = Ca^t$ for the population, where a is the growth or decay factor and t is measured in years.

(b) Find an exponential model $g(t) = Ce^{rt}$ for the population, where r is the instantaneous growth or decay rate and t is measured in years.

(c) Use a graphing calculator to graph the models f and g. Are the graphs the same?

35. 3.4%

36. −1.2%

37. −5.4%

38. 2.9%

39–42 ■ The initial size of a population is 3000, and after 5 years the population is the given amount.

(a) Find an exponential model $f(t) = Ca^t$ for the population, where a is the growth or decay factor and t is measured in years.

(b) Find an exponential model $g(t) = Ce^{rt}$ for the population, where r is the instantaneous growth or decay rate and t is measured in years.

(c) Use a graphing calculator to graph the models f and g. Are the graphs the same?

39. 5000

40. 2500

41. Half the initial population

42. Double the initial population

CONTEXTS

43. Compound Interest If $25,000 is invested at an interest rate of 7% per year, find the amount of the investment at the end of 5 years for the following compounding methods.

(a) Semiannual **(b)** Quarterly **(c)** Monthly **(d)** Continuously

44. Compound Interest If $12,000 is invested at an interest rate of 10% per year, find the amount of the investment at the end of 3 years for the following compounding methods.

(a) Semiannual **(b)** Quarterly **(c)** Monthly **(d)** Continuously

45. Compound Interest Madeleine invests $11,000 at an interest rate of 10%, compounded continuously.

(a) What is the instantaneous growth rate of the investment?

(b) Find the amount of the investment after 5 years.

(c) If the investment was compounded only quarterly, what would be the amount after 5 years?

46. Compound Interest Shalan invests $8000 at an interest rate of 9.875%, compounded continuously.

(a) What is the instantaneous growth rate of the investment?

(b) Find the amount of the investment after 7 years.

(c) If the investment was compounded only quarterly, what would the amount be after 7 years?

47. Bacteria Culture The number of bacteria in a culture is modeled by the function

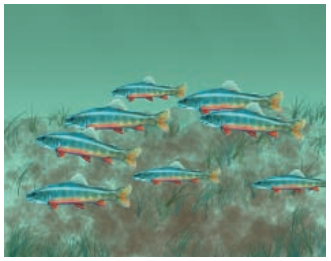
$$f(t) = 500e^{0.45t}$$

where t is measured in hours.

(a) What is the initial number of bacteria?

(b) Is the number of bacteria increasing or decreasing? Find the instantaneous growth or decay rate. Express your answer as a percentage.

(c) How many bacteria are in the culture after 3 hours?



48. Fish Population The number of fish in a Minnesota lake is modeled by the function

$$f(t) = 12e^{0.012t}$$

where t is measured in years and $f(t)$ is measured in millions.

(a) What is the initial number of fish?

(b) Is the number of fish increasing or decreasing? Find the instantaneous growth or decay rate. Express your answer as a percentage.

(c) What will the fish population be after 5 years?

49. Population of Minneapolis The population of Minneapolis was 383,000 in 2000 with an instantaneous decay rate r of -0.0052 per year.

(a) Express the instantaneous decay rate in percentage form. Is the population increasing or decreasing?

(b) Find an exponential model $f(t) = Ce^{rt}$ for the population t years after 2000.

(c) Sketch a graph of the function you found in part (b).

50. Atmospheric Pressure The atmospheric pressure of the air at sea level is 14.7 PSI (pounds per square inch), with an instantaneous decay rate r of -0.0029 per 1000 inches above sea level.

(a) Express the instantaneous decay rate in percentage form. As height increases above sea level, does the atmospheric pressure increase or decrease?

(b) Find an exponential model $f(x) = Ce^{rx}$ that models the atmospheric pressure, where x is measured in thousands of inches above sea level.

(c) Denver is one mile above sea level. Use the model found in part (b) to find the atmospheric pressure in Denver. (Remember to convert miles to inches.)

(d) Sketch a graph of the function you found in part (b).

51. Bacterial Infection The bacterium *Streptococcus pyogenes* (*S. pyogenes*) is the cause of many human diseases, the most common being strep throat. This bacterium is found in many dairy products, and its growth rate depends on the temperature. A biologist finds that the bacteria count of *S. pyogenes* is 20 colony-forming units per milliliter (CFU/mL) in a sample of sour cream stored at 25°C and that the doubling time is 30 minutes under these conditions.

(a) Find the hourly growth factor a, and find an exponential model $f(t) = Ca^t$ for the bacteria count in the sour cream, where t is measured in hours.

(b) Find the instantaneous growth rate r, and find an exponential model $g(t) = Ce^{rt}$ for the bacteria count t hours later.

(c) Use a graphing calculator to graph the functions f and g. Are the graphs the same?

52. Bacteria Infection Food poisoning is most often caused by *E. coli* bacteria. To test for the presence of *E. coli* in a pot of beef stew, a biologist performs a bacteria count on a small sample of the stew kept at 25°C. She determines that the count is 5 colony-forming units per milliliter (CFU/mL) and that the doubling time is 40 minutes under these conditions.

(a) Find the hourly growth factor a, and find an exponential model $f(t) = Ca^t$ for the bacteria count in the beef stew, where t is measured in hours.

(b) Find the instantaneous growth rate r, and find an exponential model $g(t) = Ce^{rt}$ for the bacteria count t hours later.

(c) Use a graphing calculator to graph the functions f and g. Are the graphs the same?

53. Internet Usage in China Internet World Stats reports that the number of Internet users in China increased by 1024% from 2000 to 2008 (so the growth rate is 10.24 and the growth factor is 11.24 for this time period). The number of Internet users in 2000 was about 22.5 million. Assume that the number of Internet users increases exponentially.

(a) Find the yearly growth factor a, and find an exponential model $f(t) = Ca^t$ for the number of Internet users in China t years since 2000, where the number of Internet users is measured in millions.

(b) Find the instantaneous growth rate r, and find an exponential model $g(t) = Ce^{rt}$ for the number of Internet users t years since 2000.

(c) Use each of the models you found to predict the number of Internet users in China in 2010. Do your models give the same result? Does this result seem reasonable?

54. Cell Phone Usage With rapid economic expansion, India is experiencing a cell phone boom that doesn't show any signs of slowing down. In 2004 India had 34 million cell phone subscribers, and in 2008 this number had increased to 246 million. Assume that the number of cell phone subscribers continues to increase at this rate.

(a) Find an exponential growth model $f(t) = Ca^t$ for the number of cell phone subscribers, where a is the yearly growth factor, t is the number of years since 2004, and the number of subscribers is measured in millions.

(b) Find an exponential growth model $g(t) = Ce^{rt}$ for the number of cell phone subscribers, where r is the instantaneous growth rate, t is the number of years since 2004, and the number of subscribers is measured in millions.

(c) Use each of the models found in parts (a) and (b) to predict the number of cell phone subscribers in the year 2010. Do the two models make the same prediction?

(d) Search the Internet to find the reported number of cell phone subscribers in India in 2010 and compare with your result in part (c).



55. The Beer-Lambert Law As sunlight passes through the waters of lakes and oceans, the light is absorbed, and the deeper it penetrates, the more its intensity diminishes. The light intensity I at depth x is given by the Beer-Lambert Law:

$$I = I_0 e^{-kx}$$

where I_0 is the light intensity (in lumens) at the surface and k is a constant that depends on the murkiness of the water. A biologist uses a photometer to investigate light penetration in a northern lake, obtaining the data in the table on the next page.

(a) Use a graphing calculator to find an exponential function of the form $I = ab^x$ that models the data.

(b) Express the function in part (a) in the form $I = I_0 e^{-kx}$.

(c) What is the "murkiness" constant k for this lake?

(d) Use the model to estimate the light intensity I_0 at the surface.

(e) Make a scatter plot of the data, and graph the function that you found in part (a) on your scatter plot.

Depth (ft)	Light intensity (lm)	Depth (ft)	Light intensity (lm)
5	13.0	25	1.8
10	7.6	30	1.1
15	4.5	35	0.5
20	2.7	40	0.3

4.5 Exponential Equations: Getting Information from a Model

- **Solving Exponential and Logarithmic Equations**
- **Getting Information from Exponential Models: Population and Investment**
- **Getting Information from Exponential Models: Newton's Law of Cooling**
- **Finding the Age of Ancient Objects: Radiocarbon Dating**

IN THIS SECTION... we learn how to get information from an exponential model about the situation being modeled. Getting this information requires us to solve exponential equations.



Robert Voets/CBS Photo Archive via Getty Images

CSI Miami

The applications of algebra range over every field of human endeavor in which quantitative problems arise. In this section we see how algebra is used in crime scene investigations. One of the key pieces of evidence in any homicide case is the time of death; this is often determined by comparing the temperature of the victim's body with normal body temperature. As a warm object cools, its temperature decreases with time according to an exponential model. Using this model, we can find the temperature of a cooling object if we know the initial temperature and the length of time it has been cooling. When a homicide victim is discovered, we know the initial temperature of the body (normal body temperature) and can measure the current body temperature; we need to find the time when the body started to cool (the time of death). So here is the situation:

- **The model** gives us the temperature of a body if we know how long the body has been cooling.
- **We want** to find out how long the body has been cooling if we know its current temperature.

We can get the information we want from the model; to do so, we need to solve an equation involving exponential functions. So we start this section by learning how to solve such equations.

Solving Exponential and Logarithmic Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$2^x = 9$$

is an exponential equation. To solve this equation, we take the logarithm of each side and then use the Laws of Logarithms to "bring down x" from the exponent as follows.

$$\begin{aligned} 2^x &= 9 && \text{Given equation} \\ \log 2^x &= \log 9 && \text{Take the log of each side} \\ x \log 2 &= \log 9 && \text{Law 3: "Bring down the exponent"} \\ x &= \frac{\log 9}{\log 2} && \text{Divide by log 2} \\ x &\approx 3.17 && \text{Calculator} \end{aligned}$$

The method we used to solve this equation is typical of how we solve exponential equations in general.

example 1 Solving Exponential Equations

Solve the exponential equation $5 \cdot 2^x = 36$ for the unknown x.

Solution

We first put the exponential term alone on one side of the equation.

$$\begin{aligned} 5 \cdot 2^x &= 36 && \text{Given equation} \\ 2^x &= 7.2 && \text{Divide by 5} \\ \log 2^x &= \log 7.2 && \text{Take the log of each side} \\ x \log 2 &= \log 7.2 && \text{Law 3: "Bring down the exponent"} \\ x &= \frac{\log 7.2}{\log 2} && \text{Divide by log 2} \\ x &\approx 2.848 && \text{Calculator} \end{aligned}$$

✓ CHECK Substituting $x = 2.848$ into the original equation and using a calculator, we get $5 \cdot 2^{2.848} \approx 36$.

NOW TRY EXERCISE 7

example 2 Solving Exponential Equations

Solve the exponential equation $8^x = 512 \cdot 3^x$ for the unknown x.

Solution

To put the exponential term alone on one side of the equation, we divide each side by 3^x.

$$8^x = 512 \cdot 3^x \qquad \text{Given equation}$$

$$\frac{8^x}{3^x} = 512 \qquad \text{Divide each side by } 3^x$$

$$\left(\tfrac{8}{3}\right)^x = 512 \qquad \text{Rules of Exponents}$$

$$\log\left(\tfrac{8}{3}\right)^x = \log 512 \qquad \text{Take the log of each side}$$

$$x \log \tfrac{8}{3} = \log 512 \qquad \text{Law 3: "Bring down the exponent"}$$

$$x = \frac{\log 512}{\log\left(\frac{8}{3}\right)} \qquad \text{Solve for } x$$

$$x \approx 6.36 \qquad \text{Calculator}$$

Rules of Exponents are reviewed in Algebra Toolkits A.3 and A.4, pages T14 and T20.

✓ **CHECK** We substitute $x = 6.36$ into each side of the original equation and use a calculator:

$$\text{LHS:} \quad 8^x = 8^{6.36} \approx 554{,}000$$

$$\text{RHS:} \quad 512 \cdot 3^x = 512 \cdot 3^{6.36} \approx 554{,}000$$

$$\text{LHS} = \text{RHS} \quad ✓$$

NOW TRY EXERCISE 15

example 3 Solving Exponential Equations

Solve the exponential equation $e^{3-2x} = 4$ for the unknown x.

Solution 1 Algebraic

Since the exponential term is alone on one side, we begin by taking the logarithm of each side. (We take the natural logarithm because the base in the equation is e.)

$$e^{3-2x} = 4 \qquad \text{Given equation}$$

$$\ln e^{3-2x} = \ln 4 \qquad \text{Take ln of each side}$$

$$3 - 2x = \ln 4 \qquad \text{Property of ln}$$

$$-2x = -3 + \ln 4 \qquad \text{Subtract 3}$$

$$x = -\tfrac{1}{2}(-3 + \ln 4) \qquad \text{Multiply by } -\tfrac{1}{2}$$

$$x \approx 0.807 \qquad \text{Calculator}$$

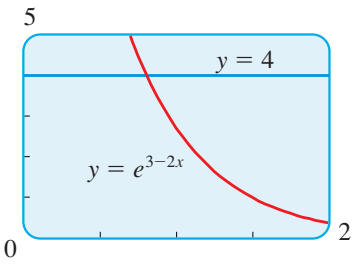


figure 1

Solution 2 Graphical

We graph the equations

$$y = e^{3-2x} \quad \text{and} \quad y = 4$$

in the same viewing rectangle, as in Figure 1. The solution occurs where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.807$.

NOW TRY EXERCISE 21

We can also solve logarithmic equations, that is, equations in which a logarithm of the unknown occurs. For example, $\log_2(x + 2) = 5$ is a logarithmic equation. To solve for x, we write the equation in exponential form:

$$\begin{aligned} \log_2(x + 2) &= 5 && \text{Given equation} \\ x + 2 &= 2^5 && \text{Exponential form} \\ x &= 32 - 2 = 30 && \text{Solve for } x \end{aligned}$$

The first step in solving logarithmic equations is to combine all logarithm terms in the equation into one term (by using the Laws of Logarithms).

example 4 Solving Logarithmic Equations

Solve the following logarithmic equations for the unknown x.

(a) $2 \log x = 3$

(b) $\ln(x + 1) + \ln 5 = 1$

(c) $\log(x + 1) - \log x = 2$

Solution

(a) We first isolate the logarithm term.

$$\begin{aligned} 2 \log x &= 3 && \text{Given equation} \\ \log x &= \frac{3}{2} && \text{Divide by 2} \\ x &= 10^{3/2} && \text{Exponential form} \\ x &\approx 31.62 && \text{Calculator} \end{aligned}$$

(b) We first combine the logarithm terms using the Laws of Logarithms.

$$\begin{aligned} \ln(x + 1) + \ln 5 &= 1 && \text{Given equation} \\ \ln 5(x + 1) &= 1 && \text{Laws of Logarithms} \\ 5(x + 1) &= e && \text{Exponential form} \\ x + 1 &= \frac{e}{5} && \text{Divide by 5} \\ x &= \frac{e}{5} - 1 && \text{Subtract 1} \\ x &\approx -0.46 && \text{Calculator} \end{aligned}$$

✓ **CHECK** We substitute $x = -0.46$ into each side of the original equation:

LHS: $\ln(-0.46 + 1) + \ln 5 = \ln 0.54 + \ln 5 = \ln(0.54 \cdot 5) = \ln 2.7 \approx 1$

RHS: 1

LHS = RHS ✓

(c) We first combine the logarithm terms using the Laws of Logarithms.

$$\log(x+1) - \log x = 2 \qquad \text{Given equation}$$

$$\log \frac{x+1}{x} = 2 \qquad \text{Laws of Logarithms}$$

$$\frac{x+1}{x} = 10^2 \qquad \text{Exponential form}$$

$$x + 1 = 100x \qquad \text{Multiply by } x$$

$$99x = 1 \qquad \text{Subtract } x \text{ and switch sides}$$

$$x = \frac{1}{99} \qquad \text{Divide by 99}$$

✓ CHECK We substitute $x = \frac{1}{99}$ into each side of the original equation:

LHS: $\log\left(\frac{1}{99} + 1\right) - \log \frac{1}{99} = \log \frac{100}{99} - \log \frac{1}{99} = \log\left(\frac{100}{99} / \frac{1}{99}\right) = \log 100 = 2$

RHS: 2

LHS = RHS ✓

NOW TRY EXERCISES 29, 35, AND 37

The equation in the next example cannot be solved algebraically, because it is impossible to put x alone on one side of the equation, so we can only solve the equation graphically.

example 5 Solving a Logarithmic Equation Graphically

Solve the equation $x^2 = 2\ln(x + 2)$ for the unknown x.

Solution

We first move all terms to one side of the equation:

$$x^2 - 2\ln(x + 2) = 0$$

Then we graph

$$y = x^2 - 2\ln(x + 2)$$

as in Figure 2. The solutions are the x-intercepts of the graph. Zooming in on the x-intercepts, we see that there are two solutions: $x \approx -0.71$ and $x \approx 1.60$.

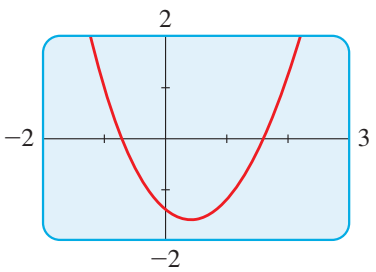


figure 2

NOW TRY EXERCISE 41

Getting Information from Exponential Models: Population and Investment

We can use logarithms to determine when quantities that grow or decay exponentially reach a given level. In the next two examples we solve exponential equations involving interest on an investment and population growth.

example 6 Investment Growth

Helen invests $10,000 in a high-yield uninsured certificate of deposit that pays 12% interest annually, compounded every 6 months.

(a) Find a formula for the amount A of the certificate after t years.

(b) What is the amount after 3 years?

(c) How long will it take for her investment to grow to $25,000?

Solution

(a) We use the formula for compound interest (Section 3.1, page 265), where the principal P is 10,000, the interest rate r is 0.12, and the number of compounding periods n is 2:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Compound interest formula}$$

$$A(t) = 10{,}000\left(1 + \frac{0.12}{2}\right)^{2t} \qquad \text{Substitute given values}$$

$$A(t) = 10{,}000(1.06)^{2t} \qquad \text{Simplify}$$

(b) Using the formula we found in part (a) and replacing t by 3, we get

$$\begin{aligned} A(t) &= 10{,}000(1.06)^{2(3)} && \text{Replace } t \text{ by } 3 \\ &= 14{,}185.20 \end{aligned}$$

So in 3 years the certificate is worth $14,185.20.

(c) We want the amount A to reach $25,000, and we need to find t, the time needed to achieve this amount. We use the formula from part (a):

$$\begin{aligned} A(t) &= 10{,}000(1.06)^{2t} && \text{Formula} \\ 25{,}000 &= 10{,}000(1.06)^{2t} && \text{Replace } A(t) \text{ by } 25{,}000 \\ 2.5 &= (1.06)^{2t} && \text{Divide by } 10{,}000 \\ \log 2.5 &= \log(1.06)^{2t} && \text{Take the log of each side} \\ \log 2.5 &= 2t \log(1.06) && \text{Law 3: “Bring down the exponent”} \\ \frac{\log 2.5}{2 \log(1.06)} &= t && \text{Divide by } 2 \log(1.06) \\ t &\approx 7.863 && \text{Calculator} \end{aligned}$$

It will take almost 8 years for Helen’s investment to grow to $25,000.

NOW TRY EXERCISE 49

example 7 Doubling the Population of the World

On January 1, 2007, the population of the world was approximately 6.6 billion and was increasing by 1.36% every year. Assume that this rate of increase continues.

(a) Find a function P that models the population after t years.

(b) What does the model predict for the population in 2015?

(c) In what year will the population of the world have doubled?

Solution

(a) We use the exponential growth model (Section 3.1, page 250)

$$P(t) = Ca^t$$

where C is the initial population and a is the growth factor. Since the population increases by 1.36% every year, the growth factor is $a = 1 + 0.0136 = 1.0136$. So the model we seek is

$$P(t) = 6.6(1.0136)^t \qquad \text{Exponential growth model}$$

where t is the number of years after 2007.

(b) The year 2015 is eight years after 2007. Replacing t by 8 in the model we get

$$P(8) = 6.6(1.0136)^8 \approx 7.35$$

So the model predicts a population of about 7.35 billion in 2015.

(c) We want to know when the population will double to $2(6.6) = 13.2$ billion. In other words, we want to find the value of t for which $P(t)$ is 13.2 billion. Substituting these values into the formula gives the following:

$$\begin{aligned}
P(t) &= 6.6(1.0136)^t && \text{Exponential growth model}\\
13.2 &= 6.6(1.0136)^t && \text{Replace } P(t) \text{ by } 13.2\\
2 &= (1.0136)^t && \text{Divide by } 6.6\\
\log 2 &= \log(1.0136)^t && \text{Take the log of each side}\\
\log 2 &= t \log 1.0136 && \text{Law 3: "Bring down the exponent"}\\
\frac{\log 2}{\log 1.0136} &= t && \text{Divide by } \log 1.0136\\
t &\approx 51.3 && \text{Calculator}
\end{aligned}$$

So it will take about 51 years for the population to double. This would happen in the year $2007 + 51 = 2058$.

■ **NOW TRY EXERCISE 55** ■

■ Getting Information from Exponential Models: Newton's Law of Cooling



The rate at which a hot cup of coffee cools is proportional to the temperature difference between the cup and its surroundings.

When a hot object, such as a cup of coffee, is left to cool, its temperature decreases continuously at a rate proportional to the temperature difference between the object and its surroundings. Since this difference is continually changing, the rate at which the object cools is also continually changing. **Newton's Law of Cooling** states that the temperature T of a cooling object is modeled by

$$T = A + (I - A)e^{-kt}$$

where t is the time since the object began cooling, I is the initial temperature of the object, A is the ambient temperature (the temperature of the surroundings), and k is a constant that depends on the type of object; k is called the **heat transfer coefficient**.

Newton's Law of Cooling is a powerful tool in many crime scene investigations. For example, in the infamous 1994 O.J. Simpson case two key pieces of evidence—a cup of ice cream and a bathtub full of warm water—could have provided compelling evidence for the time of death. According to court testimony, "the first offi-

cer on the scene did not test the temperature of the water in . . . [the] bathtub or pick up the cup of melting ice cream." In the next example we investigate how bathtub temperature could help to determine the time when the tub was filled.

example 8 Crime Scene Investigation

Police officers arrive at a crime scene and find a tub full of warm water. A thermometer shows that the water temperature is 76°F and the air temperature is 70°F. It is known that most people fill a tub with water at 100°F.

(a) Use Newton's Law of Cooling to model the temperature of the water in the tub. (Measure t in minutes and use the heat transfer coefficient $k = 0.018$.)

(b) How long has the bathtub been cooling?

Solution

(a) We use Newton's Law of Cooling, where the initial temperature I is 100°F, the ambient temperature A is 70°F, and the heat transfer constant k is 0.018.

$$T = A + (I - A)e^{-kt}$$ Newton's Law of Cooling

$$T = 70 + (100 - 70)e^{-0.018t}$$ Replace A by 70, I by 100, and k by 0.018

$$T = 70 + 30e^{-0.018t}$$ Simplify

(b) We know that the temperature of the bathtub water is 76°F, so we use the model from part (a), replace T by 76, and solve the resulting exponential equation for t.

$$T = 70 + 30e^{-0.018t}$$ Model

$$76 = 70 + 30e^{-0.018t}$$ Replace T by 76

$$6 = 30e^{-0.018t}$$ Subtract 70

$$0.20 = e^{-0.018t}$$ Divide by 30

$$\ln 0.20 = -0.018t$$ Take ln of each side

$$t = \frac{\ln 0.20}{-0.018}$$ Divide by 0.018 and switch sides

$$t \approx 89.41$$ Calculator

So the tub has been cooling for about 89 minutes.

NOW TRY EXERCISE 57

Finding the Age of Ancient Objects: Radiocarbon Dating

In the next example we examine how exponential equations are used to determine the ages of ancient artifacts. Before the twentieth century archeologists had a hard time determining the ages of the objects they found. It's difficult to tell whether a shard of pottery was dropped on the ground 50 years ago or 50 centuries ago. So archeologists generally used the principle that the deeper they had to dig to find an object, the older it probably was and that objects found at the same level, or stratum,

were close in age. This was helpful in comparing the *relative* ages of two items, but how could they get the *actual* age? And how could they compare the ages of items found at sites 50 miles apart? The discovery that radioactive decay is exponential allowed scientists to determine the age of ancient objects by comparing the amount of radioactive carbon-14 in the object to the amount it normally would have today.

IN CONTEXT ➤

A dramatic use of radiocarbon dating occurred with the discovery of the "iceman." On September 19, 1991, Erika and Helmut Simon were hiking in the Alps near the Austrian-Italian border. As they approached an ice-filled depression, they were surprised to see the frozen body of a man sticking halfway out of the ice. They reported the find to authorities, thinking that it was a recent tragic accident. But the authorities were baffled by the find because of the frozen man's unusual goatskin leather clothing and the bronze ax and quiver of arrows strapped to his body. Who was this "iceman"? Where did he come from? To answer these questions, the primary piece of information needed was the date of the iceman's demise. The surprising answer, which we calculate in the next example, was that he lived in the Neolithic Age (Late Stone Age).



© dpa/Corbis

The iceman

How does radiocarbon dating work? Plants absorb radioactive carbon-14 from the atmosphere, which then makes its way into animals through the food chain. Thus, all living creatures contain the same fixed proportion of carbon-14 to nonradioactive carbon-12 as is in the atmosphere. After an organism dies, it stops assimilating carbon-14, and the carbon-14 in it begins to decay exponentially. We can then determine the time elapsed since death by measuring the amount of carbon-14 left in the relic.

example 9 Dating the Iceman

Tissue samples from the iceman indicated that his body had 57.67% of the carbon-14 that is present in a living person. Use the fact that the half-life of carbon-14 is 5730 years to estimate how long ago the iceman died.

Solution

The formula for exponential decay is

$$m(t) = C\left(\tfrac{1}{2}\right)^{t/h}$$

where C is the initial mass of the radioactive substance, h is its half-life, and $m(t)$ is the mass remaining at time t (see Example 9 in Section 3.1). We are looking for the time t at which $m(t)$ is 57.67% of the initial mass of carbon-14 in the iceman's body, that is, $m(t)$ is $0.5767C$. We now substitute the known values into the formula and solve for t.

$$\begin{aligned}
m(t) &= C\left(\tfrac{1}{2}\right)^{t/h} && \text{Radioactive decay formula}\\
0.5767C &= C\left(\tfrac{1}{2}\right)^{t/5730} && \text{Substitute given values}\\
0.5767 &= \left(\tfrac{1}{2}\right)^{t/5730} && \text{Divide by } C\\
\log 0.5767 &= \log\left(\tfrac{1}{2}\right)^{t/5730} && \text{Take the log of each side}\\
\log 0.5767 &= \tfrac{t}{5730}\log\left(\tfrac{1}{2}\right) && \text{Law 3: "Bring down the exponent"}\\
\frac{5730\log 0.5767}{\log\left(\frac{1}{2}\right)} &= t && \text{Multiply by 5730, divide by } \log\left(\tfrac{1}{2}\right)\\
t &\approx 4550 && \text{Calculator}
\end{aligned}$$

So the iceman died about 4550 years ago.

■ **NOW TRY EXERCISE 65** ■

4.5 Exercises

CONCEPTS

Fundamentals

1. Let's solve the exponential equation $2e^x = 50$.

(a) First, we isolate e^x to get the equivalent equation _____.

(b) Next, we take ln of each side to get the equivalent equation _____.

(c) Now we use a calculator to find $x =$ _____.

2. Let's solve the logarithmic equation $\log 3 + \log(x - 2) = \log x$.

(a) First, we combine the logarithms to get the equivalent equation _____.

(b) Next, we write each side in exponential form to get the equivalent equation _____.

(c) Now we find $x =$ _____.

Think About It

3. To solve the exponential equation $2^{3x} = 2^{x+1}$, we first take logarithms of both sides. What base logarithm leads to the easiest solution? Why does this base work so nicely?

4. Logarithms with any base work equally well to solve the equation $3^x = 4 \cdot 5^x$. Why? First solve the equation using common logarithms, and then solve the equation using natural logarithms. Now use a logarithm with a different base to solve the equation. (You will need to use the Change of Base Formula in the solution.) Do you always get the same answer?

SKILLS

5–18 ■ Solve the exponential equation for the unknown x.

5. $10^x = 25$

6. $e^x = 7$

7. $25 \cdot 3^x = 7$

8. $6 \cdot 10^x = 4$

9. $2e^{12x} = 17$

10. $9 \cdot 4^{3x} = 12$

11. $3^{1-4x} = 2$

12. $e^{2x+1} = 200$

13. $2^{3x+1} = 34$

14. $8^{4x-1} = 5$

15. $6^x = 21 \cdot 2^x$

16. $5^x = 50 \cdot 3^x$

17. $5^x = 4^{x+1}$

18. $(3.2)^x = 35 \cdot (2.6)^x$

19–26 ■ Solve the exponential equation (a) algebraically and (b) graphically.

19. $e^{2x} = 5$

20. $3 \cdot e^x = 10$

21. $2^{1-x} = 3$

22. $2^{3x} = 34$

23. $10^{1-x} = 6^x$

24. $7^{x/2} = 5^{1-x}$

25. $5^x = 212 \cdot 9^x$

26. $5^x = 3 \cdot 2^x$

27–40 ■ Solve the logarithmic equation for the unknown x.

27. $\log_2 x = 10$

28. $\log x = 7$

29. $3 \log x = 12$

30. $2 \log_9 x = 1$

31. $\log_2 8x = -2$

32. $\ln 3x = -4$

33. $\log(x - 4) = 3$

34. $4 \log_5(3x + 5) = 2$

35. $\log(2x + 1) + \log 2 = 2$

36. $\log_2(3x - 1) + \log_2 8 = 5$

37. $\log_2 x - \log_2(x - 3) = 2$

38. $\log_5(x + 1) - \log_5(x - 1) = 2$

39. $\ln 5 + \ln(x - 2) = \ln(3x)$

40. $\log 2x = \log 2 + \log(3x - 4)$

41–48 ■ Use a graphing calculator to find all solutions of the equation.

41. $\ln x = 3 - x$

42. $\log x = x^2 - 2$

43. $e^x = -x$

44. $2^{-x} = x - 1$

45. $x^3 - x = \log(x + 1)$

46. $x = \ln(4 - x^2)$

47. $4^{-x} = \sqrt{x}$

48. $e^{x^2} - 2 = x^3 - x$

CONTEXTS

49. Compound Interest Aviel invests $6000 in a high-yield uninsured certificate of deposit that pays 4.5% interest per year, compounded quarterly.

(a) Find a formula for the amount A of the certificate after t years.

(b) What is the amount after 2 years?

(c) How long will it take for his investment to grow to $8000?

50. Compound Interest Ping invests $4000 in a saving certificate that has an interest rate of 2.75% per year, compounded semiannually.

(a) Find a formula for the amount A of the certificate after t years.

(b) What is the amount after 4 years?

(c) How long will it take for her investment to grow to $5000?

51. Compound Interest Suzanne is planning to invest $2000 in a certificate of deposit. How long does it take for the investment to grow to $3000 under the given conditions?

(a) The certificate of deposit pays $3\frac{1}{2}\%$ interest annually, compounded every month.

(b) The certificate of deposit pays $2\frac{7}{8}\%$ interest annually, compounded continuously.

52. Compound Interest Masako is planning to invest $5000 in a certificate of deposit. How long does it take for the investment to grow to $8000 under the given conditions?

(a) The certificate of deposit pays 3.55% interest annually, compounded every month.

(b) The certificate of deposit pays 3.05% interest annually, compounded continuously.

53. *Salmonella* Bacteria Count Although cooking meat kills the microorganisms in it, they may survive gentle frying and roasting, especially if the meat was not properly defrosted before preparation. Inspectors for the U.S. Department of Agriculture test for *Salmonella* in a sample of chicken at a meat-packing plant. The sample is found to have a bacteria count of 15 colony-forming units per milliliter (CFU/mL). The sample is kept at a temperature of 80°F, and 6 hours later the count is 20,000 CFU/mL.

(a) Find the instantaneous growth rate r for the bacteria count in the sample.

(b) Find an exponential model $f(t) = Ce^{rt}$ for the bacteria count in the sample, where t is measured in hours.

(c) What does the model predict the bacteria count will be after 4 hours?

(d) How long will it take for the bacteria count to reach 10 million CFU/mL?

(e) Find the doubling time for the *Salmonella* bacteria.

54. *E. coli* Bacteria Count Inspectors for the U.S. Department of Agriculture test a sample of ground beef for the bacterium *E. coli*. The sample is found to have a bacteria count of 100 colony-forming units per milliliter (CFU/mL). The sample is kept at a temperature of 100°F, and 2 hours later the meat has a count of 13,300 CFU/mL.

(a) Find the instantaneous growth rate r (per hour) for the bacteria count in the sample.

(b) Find an exponential model $f(t) = Ce^{rt}$ for the bacteria count in the sample, where t is measured in hours.

(c) What does the model predict the bacteria count will be after 3 hours?

(d) How long will it take for the bacteria count to reach one million CFU/mL?

(e) Find the doubling time for the population of *E. coli* bacteria.

55. Population of Ethiopia In 2003 the United Nations estimated that the population of Ethiopia was about 70.7 million, with an annual growth rate of 2.9%. Assume that this rate of growth continues.

(a) Find the yearly growth factor a.

(b) Find an exponential growth model $f(t) = Ca^t$ for the population t years since 2003.

(c) How long will it take for the population to double?

(d) Use the model found in part (b) to predict the year in which the population will reach 90 million.

56. Population of Germany In 2004 the population of Germany was about 82.5 million, with an annual growth rate of 0.02%. Assume that this rate of growth continues.

(a) Find the yearly growth factor a.

(b) Find an exponential growth model $f(t) = Ca^t$ for the population t years since 2004.

(c) How long will it take for the population to double?

(d) Use the model found in part (b) to predict the year in which the population will reach 83 million.



57. Pot of Chili Angela prepares a large pot of chili the night before a church potluck. The temperature of the chili is 212°F, and it must cool down to 70°F before it can be stored in the refrigerator. Assume that the ambient temperature is 65°F and the heat transfer coefficient is $k = 2.895$.

(a) Find a model for the temperature T of the pot of chili t hours after cooling.

(b) How long will it take for the pot of chili to cool down to the desired temperature of 70°F?

(c) Graph the function T to confirm your answers to parts (a) and (b).

58. Time of Death Newton's Law of Cooling is used in homicide investigations to determine the time of death. Suppose that a body is discovered in a location whose ambient temperature is 60°F. The police determine that the heat transfer coefficient in this case is $k = 0.1947$. (The heat transfer coefficient depends on many factors, including the size of the body and the amount of clothing.) Normal body temperature is 98.6°F.

(a) Find a model for the temperature T of the body t hours after death.

(b) When the body was found, it had a temperature of 72°F. Find the length of time the victim has been dead.

(c) Graph the function T to confirm your answers to parts (a) and (b).

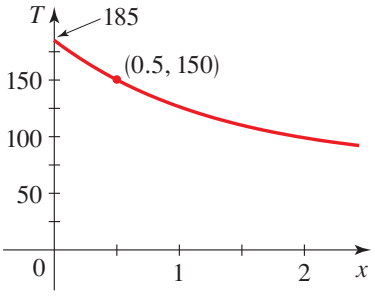
59. Boiling Water A kettle of water is brought to a boil in a room where the temperature is 20°C. After 15 minutes the temperature of the water has decreased from 100°C to 75°C.

(a) Find the heat transfer coefficient k, and find a model for the temperature T of the water t hours after it is brought to a boil.

(b) Use the model to predict the temperature of the water after 25 minutes. Illustrate by graphing the temperature function.

(c) How long will it take the water to cool to 40°C?

(d) Graph the function T to confirm your answers to parts (b) and (c).



60. Cooling Turkey A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows the temperature of the turkey after x hours.

(a) Find a model for the temperature T of the turkey t hours after it is taken out of the oven.

(b) Use the model to predict the temperature of the turkey after 45 minutes.

(c) How long will it take the turkey to cool to 100°F?

61. Radioactive Radium The half-life of radium-226 is 1600 years. Suppose we have a 22-mg sample.

(a) Find the yearly growth factor a.

(b) Find an exponential model $m(t) = Ca^t$ for the mass remaining after t years.

(c) How much of the sample will remain after 4000 years?

(d) After how long will only 18 mg of the sample remain?

62. Radioactive Cesium The half-life of cesium-137 is 30 years. Suppose we have a 10-gram sample.

(a) Find the yearly growth factor a.

(b) Find an exponential model $m(t) = Ca^t$ for the mass remaining after t years.

(c) How much of the sample will remain after 80 years?

(d) After how long will only 2 mg of the sample remain?

63. Plutonium-239 A nuclear power plant produces radioactive plutonium-239, which has a half-life of 24,110 years. Because of its long half-life, plutonium-239 must be disposed of safely. How long does it take 100 grams of radioactive waste from the nuclear power plant to decay to 10 grams?

64. Strontium-90 One radioactive material that is produced in atomic bombs is the isotope strontium-90, which has a half-life of 28 years. If a person is exposed to strontium-90, it can collect in human bone tissue, where it can cause leukemia and other cancers. If an atomic bomb test site is contaminated by strontium-90, how long will it take for the radioactive material to decay to 10% of the original amount?

65. Carbon-14 Dating Archeologists find an ancient shard of pottery and use some burnt olive pits found in the same layer of the site to determine the age of the shard. The archeologists determine that the olive pits contain 69.32% of the carbon-14 that is present in a living olive. (The half-life of carbon-14 is 5730 years.) How old is the shard of pottery?



66. Carbon-14 Dating A donkey bone is estimated to contain 73% of the carbon-14 that it contained originally. How old is the donkey bone? (The half-life of carbon-14 is 5730 years.)

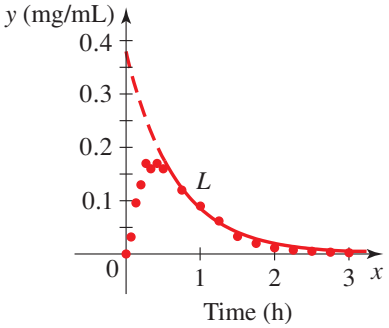
67. Dead Sea Scrolls The Dead Sea Scrolls are documents that contain some of the oldest known texts of parts of the Hebrew bible. They were discovered between 1947 and 1956 in several caves near the ruins of the ancient settlement of Khirbet Qumran on the northwest shore of the Dead Sea. Archeologists determine that a sample taken from the scrolls contains 79.10% of the carbon-14 that it originally contained. (The half-life of carbon-14 is 5730 years.)

(a) Estimate the age of the Dead Sea Scrolls.

(b) Search the Internet for the historical date of the writing of the Dead Sea Scrolls. How does this date compare with your estimate?



Donald Parry



68. Algebra and Alcohol The table in the Prologue (page P2) gives the alcohol concentration at different times following the consumption of 15 mL of alcohol. A scatter plot of the data (see the margin) shows that the alcohol concentration reaches a maximum in about half an hour and then begins to decay exponentially. The function

$$L(t) = 0.38e^{-1.5t}$$

closely models the decay part of the alcohol concentration. Use this function to determine the time at which the concentration decays to 0.05 mg/mL.

4.6 Working with Functions: Composition and Inverse

- Functions of Functions
- Reversing the Rule of a Function
- Which Functions Have Inverses?
- Exponential and Logarithmic Functions as Inverse Functions

IN THIS SECTION... we study a method of combining functions called *composition*. We also study inverse functions and see how the logarithm and exponential functions are inverses of each other.

Functions of Functions

In many real-world situations we need to apply several functions in a row, that is, the output of one function is used as the input to the next function. For example, suppose you work for \$8 an hour, and you pay tax at the rate of 5%. The function that gives your pay for working x hours is $P(x) = 8x$, and the function that calculates the tax on an income of x dollars is $T(x) = 0.05x$.

Function	In Words
$P(x) = 8x$	"Pay for working x hours"
$T(x) = 0.05x$	"Tax on x dollars"

Let's say that we want to find how much tax you pay when you work x hours. To do this, we must apply the functions P and T in the appropriate order. We start with the input x hours, apply the function P to get the pay, and then apply the function T to get the tax on that pay. For example, if you work 2 hours, then your pay is $P(2) = 8(2) = 16$ dollars. Then we apply the function T to this output to get $T(16) = 0.05(16) = 0.80$. The diagram in Figure 1 shows how we apply these two functions.

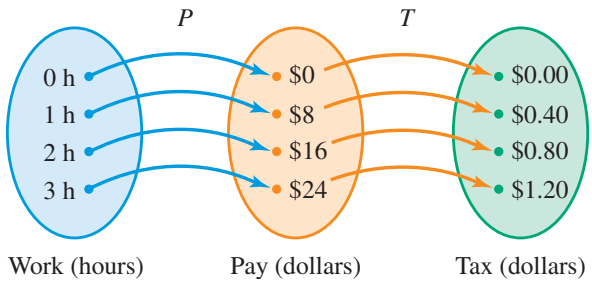


figure **1** From "hours worked" to "pay received" to "tax paid"

table **1**

x	$P(x)$	$T(P(x))$
0	0	0
1	8	0.40
2	16	0.80
3	24	1.20
4	32	1.60
5	40	2.00
6	48	2.40
7	56	2.80

What we did was start with an input of x hours, apply the function P to get $P(x)$, then apply the function T to the result to get $T(P(x))$. We can express this as a single function $L(x)$ whose input x is the number of hours worked and whose output is the tax owed. Several values of $T(P(x))$ are given in Table 1. In general we call the process of applying one function to the output of another *composition of functions*.

The Composition of Functions

Given two functions f and g, we can make a new function h called **the composition of f with g**, defined by

$$h(x) = f(g(x))$$

Of course, to be able to compose f with g, the range of g must be contained in the domain of f. In the next example we find an expression for the composition of the "Tax" and "Pay" functions described above.

example 1 Composing Functions

Find the composition of the tax function $T(x) = 0.05x$ with the pay function $P(x) = 8x$.

Solution

We use L as the name of the new composite function. Then

$$\begin{aligned} L(x) &= T(P(x)) && \text{Definition of the new function } L \\ &= T(8x) && \text{Definition of } P \\ &= 0.05(8x) && \text{Definition of } T \\ &= 0.40x && \text{Simplify} \end{aligned}$$

So the new function is $L(x) = 0.40x$. You can check that this function gives the values in Table 1.

■ NOW TRY EXERCISE 15 ■

example 2 A Function of a Function

Let $f(x) = x^2$ and $g(x) = x + 1$. Give a verbal description of each of the following composite functions, and then find an algebraic expression for the function.

(a) $M(x) = g(f(x))$

(b) $N(x) = f(g(x))$

Solution

(a) The function $M(x) = g(f(x))$ first applies the rule f and then applies the rule g. So M is the rule "Square, then add 1." In symbols we have

$$\begin{aligned} M(x) &= g(f(x)) && \text{Definition of the function } M \\ &= g(x^2) && \text{Definition of } f \\ &= x^2 + 1 && \text{Definition of } g \end{aligned}$$

So $M(x) = x^2 + 1$.

(b) The function $N(x) = f(g(x))$ first applies the rule g and then applies the rule f. So N is the rule "Add 1, then square." In symbols we have

$$\begin{aligned} N(x) &= f(g(x)) && \text{Definition of the function } N \\ &= f(x+1) && \text{Definition of } g \\ &= (x+1)^2 && \text{Definition of } f \end{aligned}$$

So $N(x) = (x + 1)^2$.

NOW TRY EXERCISE 17

Reversing the Rule of a Function

x	$P(x)$
0	12
1	14
2	16
3	18

x	$P^{-1}(x)$
12	0
14	1
16	2
18	3

We often need to reverse the rule of a function. In other words, we might want to find the rule that takes us from the output of a function back to the corresponding input.

For example, suppose that at your local pizzeria the daily special is \$12 for a plain cheese pizza plus \$2 for each additional topping. We can express the price of the pizza as a function $P(x) = 2x + 12$ whose input x is the number of toppings and whose output is the price. The function that "reverses" the rule of P is the function whose input is the price and whose output is the number of toppings. You can check that the rule for getting the number of toppings from the price is "Subtract 12 from the price, then divide by 2." This rule "reverses" the rule P; the notation for this new rule is P^{-1} (read "P inverse"). So P^{-1} is the function $P^{-1}(x) = (x - 12)/2$. (The tables in the margin give several values of P and P^{-1}.)

Function	In Words
$P(x) = 2x + 12$	"Price of a pizza with x toppings"
$P^{-1}(x) = \dfrac{x - 12}{2}$	"Number of toppings of a pizza with price x"

The rule P is "Multiply by 2, then add 12," and the rule for P^{-1} is "Subtract 12, then divide by 2." Do you see how the rule P^{-1} reverses the rule for P? Try starting with 5 toppings. Apply P to get 22, then apply P^{-1} to 22 to get back to 5.

$$\begin{array}{lccccc} P & 5 & \xrightarrow{\text{Multiply by 2}} & 10 & \xrightarrow{\text{Add 12}} & 22 \\ P^{-1} & 5 & \xleftarrow{\text{Divide by 2}} & 10 & \xleftarrow{\text{Subtract 12}} & 22 \end{array}$$

The diagrams in Figure 2 show the relationship between P and P^{-1}.

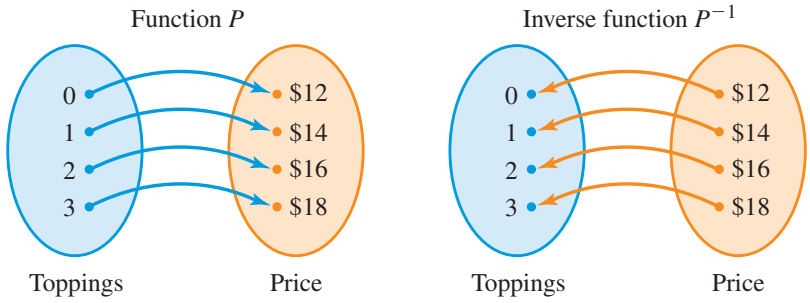


figure **2** The function P^{-1} "reverses" the function P.

In general, the inverse of a function is defined as follows.

The Inverse of a Function

If a function f has domain A and range B then its inverse function (if it exists) is the function f^{-1} with domain B and range A defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

for any y in B.

example **3**

Finding the Values of f^{-1} Graphically

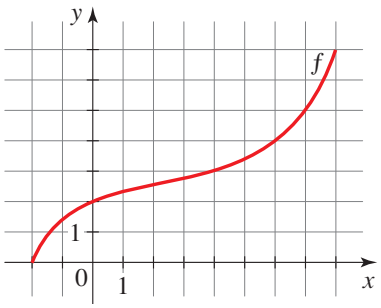


figure **3** Graph of f

The graph of a function f is given in Figure 3. Use the graph to find the following.

(a) $f^{-1}(3)$ **(b)** $f^{-1}(5)$

Solution

(a) From the graph we see that $f(4) = 3$, so $f^{-1}(3) = 4$. (See Figure 4.)

(b) From the graph we see that $f(7) = 5$, so $f^{-1}(5) = 7$. (See Figure 4.)

■ **NOW TRY EXERCISE 35** ■

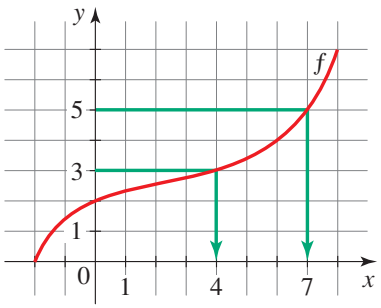


figure **4** Finding values of f^{-1} from the graph of f

Expressing a function in equation form is studied in Section 1.5.

Let's see how we can find an equation for the inverse of a function f. From the definition we see that if we can solve the equation $y = f(x)$ for x, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we get $y = f^{-1}(x)$, which is the function we want. Thus, we have the following procedure for finding the inverse of a function.

How to Find the Inverse of a Function

To find the inverse of a function f:

Step 1 Write the function in equation form: $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

In the next example we use these steps to find the inverse of the pizza function P described earlier in this section.

example **4**

Finding the Inverse of a Function

Find the inverse of the function $P(x) = 2x + 12$.

Solution

First we write P in equation form: $y = 2x + 12$

Then we solve this equation for x:

$$y = 2x + 12 \qquad \text{Equation}$$

$$y - 12 = 2x \qquad \text{Subtract 12}$$

$$x = \frac{y - 12}{2} \qquad \text{Divide by 2 and switch sides}$$

In Example 4, notice how the rule P^{-1} reverses the rule P: The function P is the rule "Multiply by 2, then add 12." The function P^{-1} is the rule "Subtract 12, then divide by 2."

Finally, we interchange x and y:

$$y = \frac{x - 12}{2}$$

So the inverse function is $P^{-1}(x) = \frac{x - 12}{2}$.

NOW TRY EXERCISE 57

example 5 Finding the Inverse of a Function

Find the inverse of the function $f(x) = 2x^3 + 1$.

Solution

First we write f in equation form: $y = 2x^3 + 1$

Then we solve this equation for x:

$$y = 2x^3 + 1 \qquad \text{Equation}$$

$$y - 1 = 2x^3 \qquad \text{Subtract 1}$$

$$\frac{y - 1}{2} = x^3 \qquad \text{Divide by 2}$$

$$x = \sqrt[3]{\frac{y - 1}{2}} \qquad \text{Take the cube root and switch sides}$$

In Example 5, notice how the rule f^{-1} reverses the rule f: The function f is the rule "Cube x, multiply by 2, then add 1." The function f^{-1} is the rule "Subtract 1, divide by 2, then take the cube root."

Finally, we interchange x and y:

$$y = \sqrt[3]{\frac{x - 1}{2}}$$

So the inverse function is $f^{-1}(x) = \sqrt[3]{\frac{x - 1}{2}}$.

NOW TRY EXERCISE 61

It is not always possible to give a simple verbal description of the inverse of a function. The next example shows why.

example 6 Finding the Inverse of a Function

Find the inverse of the function $f(x) = \frac{3x}{x + 2}$.

Solution

First we write f in equation form:

$$y = \frac{3x}{x + 2}$$

Then we solve this equation for x.

$$\begin{aligned} y &= \frac{3x}{x+2} && \text{Equation} \\ y(x+2) &= 3x && \text{Multiply by } x+2 \\ yx + 2y &= 3x && \text{Distributive Property} \\ 2y &= 3x - yx && \text{Bring the } x\text{-terms to one side} \\ 2y &= x(3-y) && \text{Factor } x \\ x &= \frac{2y}{3-y} && \text{Divide by } 3-y \text{ and switch sides} \end{aligned}$$

Finally, we interchange x and y:

$$y = \frac{2x}{3-x}$$

So the inverse function is $f^{-1}(x) = \dfrac{2x}{3-x}$.

NOW TRY EXERCISE 63

How can we check whether our answers to the preceding two examples are correct? By definition the inverse f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. We can express these facts as follows.

Inverse Function Properties

Let f be a function with domain A and range B. The function g is the inverse of f if and only if the following properties are satisfied:

$$g(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(g(x)) = x \quad \text{for every } x \text{ in } B$$

We can now use these properties to check whether two functions are inverses of each other.

example 7 Checking That f and g Are Inverses of Each Other

Show that $f(x) = \dfrac{x-2}{5}$ and $g(x) = 5x + 2$ are inverses of each other.

Solution

We need to check that $f(g(x)) = x$ and $g(f(x)) = x$. We have

$$\begin{aligned} f(g(x)) &= f(5x+2) && \text{Definition of } g \\ &= \frac{(5x+2)-2}{5} && \text{Definition of } f \\ &= \frac{5x}{5} && \text{Simplify} \\ &= x && \text{Cancel 5} \end{aligned}$$

Also

$$
\begin{aligned}
g(f(x)) &= g\left(\frac{x-2}{5}\right) && \text{Definition of } f \\
&= 5\left(\frac{x-2}{5}\right) + 2 && \text{Definition of } g \\
&= (x-2) + 2 && \text{Cancel 5} \\
&= x && \text{Simplify}
\end{aligned}
$$

This shows that f and g are inverses of each other.

NOW TRY EXERCISE 37

Which Functions Have Inverses?

table **2**

Input: *toppings*	Output: *price*
0	12
1	12
2	12
3	14

It's not always possible to find the inverse of a function. For example, suppose your local pizzeria has a coupon that gives you "up to two toppings for the same $12 price as the plain cheese pizza—and $2 for each additional topping after that." Table 2 gives the price of a pizza for different numbers of toppings (if you use the coupon). So we have a function P whose input is the number of toppings and whose output is the price.

If Jason tells you that he paid $12 for a pizza, you can't tell how many toppings the pizza had. Jason could have chosen to get one free topping, two free toppings, or no free topping. The problem in finding an inverse of this function is that one output corresponds to two (or more) different inputs. The diagram in Figure 5 illustrates the situation. When we reverse the arrows, we get a relation that is not a function. So there is no inverse function for this function.

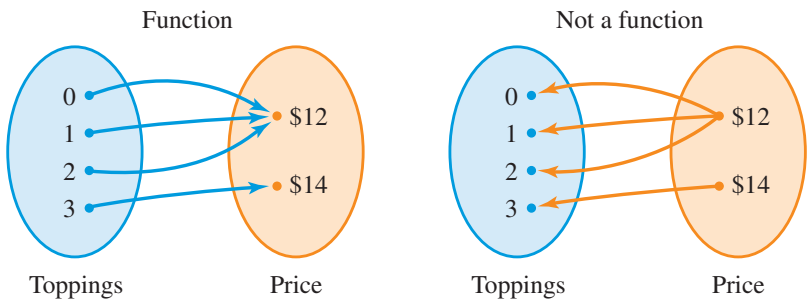


figure **5**

For a function to have an inverse, different inputs must correspond to different outputs. (We can't have two different inputs giving the same output.) Functions with this property are called *one-to-one*, as in the following definition.

Definition of One-to-One Function

A function f is called **one-to-one** if no two different inputs have the same output; that is,

$$\text{if} \quad x_1 \neq x_2 \quad \text{then} \quad f(x_1) \neq f(x_2)$$

example 8 A Function That Is Not One-to-One

Is the function $f(x) = x^2$ one-to-one?

Solution

The function f is not one-to-one because, for example,

$$f(2) = 4 \qquad \text{and} \qquad f(-2) = 4$$

So the inputs 2 and -2 have the same output 4.

NOW TRY EXERCISE 53

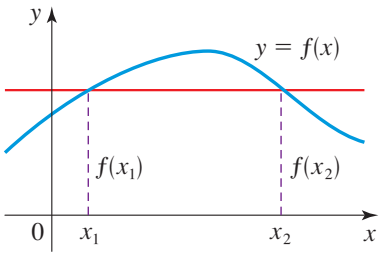


figure **6** Horizontal Line Test

The graph of a function can help us to easily decide whether it is one-to-one. If a horizontal line intersects the graph of f at more than one point, then we see from Figure 6 that there are inputs x_1 and x_2 whose corresponding outputs $f(x_1)$ and $f(x_2)$ are the same (because they are the same height above the x-axis). This means that f is not one-to-one.

These observations lead us to the following graphical test for deciding whether a function is one-to-one.

Horizontal Line Test

A function is **one-to-one** if and only if no horizontal line intersects its graph more than once.

We use the Horizontal Line Test in the following examples.

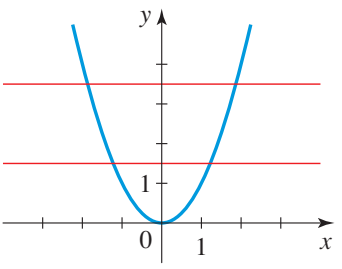
example 9 Using the Horizontal Line Test

Determine whether the function is one-to-one.

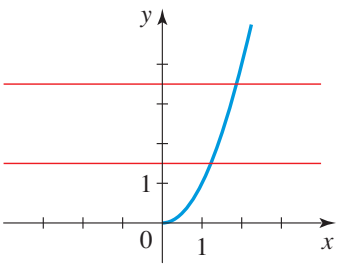
(a) $f(x) = x^2$ **(b)** $g(x) = x^2$ for $x \geq 0$ **(c)** $h(x) = x^3$

Solution

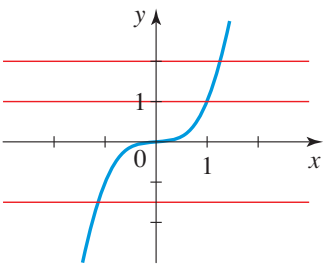
We first sketch graphs of each of these functions as in Figure 7. Notice the difference between the graph of f (whose domain is all real numbers) and the graph of g (whose domain is the nonnegative real numbers).



(a) $f(x) = x^2$
Not one-to-one



(b) $g(x) = x^2$ $(x \geq 0)$
One-to-one



(c) $h(x) = x^3$
One-to-one

figure **7**

(a) From Figure 7(a) we see that there are horizontal lines that intersect the graph of $f(x) = x^2$ more than once. So by the Horizontal Line Test, f is not one-to-one.

(b) From Figure 7(b) we see that no horizontal line intersects the graph of $g(x) = x^2$ $(x \geq 0)$ more than once. So by the Horizontal Line Test, g is one-to-one.

(c) From Figure 7(c) we see that no horizontal line intersects the graph of $h(x) = x^3$ more than once. So by the Horizontal Line Test, h is one-to-one.

NOW TRY EXERCISE 71

The function $f(x) = x^2$ of Example 9 is not one-to-one, so there is no inverse function for f. You can check that the inverse function for $g(x) = x^2$ $(x \geq 0)$ is the function $g^{-1}(x) = \sqrt{x}$ $(x \geq 0)$. Also, the inverse function of $h(x) = x^3$ is the function $h^{-1}(x) = \sqrt[3]{x}$.

Exponential and Logarithmic Functions as Inverse Functions

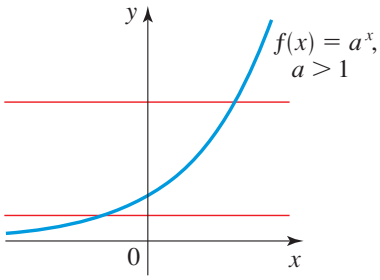


figure **8** $f(x) = a^x$ is one-to one.

Every exponential function $f(x) = a^x$, with $a > 0$, $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 8 for the case in which $a > 1$). So every exponential function has an inverse function. We can check that the logarithm function $g(x) = \log_a x$ is the inverse of f by the Inverse Function Property (as in Example 7):

$$\begin{aligned} g(f(x)) &= g(a^x) && \text{Definition of } f \\ &= \log_a a^x && \text{Definition of } g \\ &= x && \text{Property of logarithms} \end{aligned}$$

Also,

$$\begin{aligned} f(g(x)) &= f(\log_a x) && \text{Definition of } g \\ &= a^{\log_a x} && \text{Definition of } f \\ &= x && \text{Property of logarithms} \end{aligned}$$

So g is the inverse of f by the Inverse Function Property.

Inverse Functions of Exponential Functions

The exponential function $f(x) = a^x$ and the logarithmic function $g(x) = \log_a x$ are inverse functions of each other. So

$$\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x$$

example **10** **Checking That *f* and *g* Are Inverses of Each Other**

Show that $f(x) = \log x^2$ and $g(x) = 10^{x/2}$ are inverses of each other.

Solution

We need to check that $f(g(x)) = x$ and $g(f(x)) = x$. We have

$$\begin{aligned} f(g(x)) &= f\left(10^{x/2}\right) && \text{Definition of } g \\ &= \log\left(10^{x/2}\right)^2 && \text{Definition of } f \\ &= \log 10^x && \text{Rules of Exponents} \\ &= x && \log x \text{ and } 10^x \text{ are inverse functions} \end{aligned}$$

Also

$$\begin{aligned} g(f(x)) &= g\left(\log x^2\right) && \text{Definition of } f \\ &= 10^{(\log x^2)/2} && \text{Definition of } g \\ &= 10^{(2\log x)/2} && \text{Laws of Logarithms} \\ &= 10^{\log x} && \text{Simplify} \\ &= x && \log x \text{ and } 10^x \text{ are inverse functions} \end{aligned}$$

This shows that f and g are inverses of each other.

NOW TRY EXERCISE 41

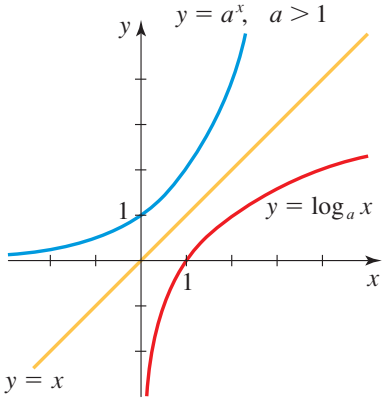


figure **9** Graphs of $f(x) = a^x$ and $g(x) = \log_a x$

You may have noticed that the graphs of the exponential and logarithm functions have similar shapes. In fact, the graph of $g(x) = \log_a x$ is the mirror image of the graph of $f(x) = a^x$, where the "mirror" is the line $y = x$ (Figure 9).

In general, the graph of f^{-1} is the reflection of the graph of f through the line $y = x$. This is because if $f(s) = t$, then $f^{-1}(t) = s$, so if (s, t) is on the graph of f, then (t, s) is on the graph of f^{-1}.

4.6 Exercises

CONCEPTS

Fundamentals

1. If $g(2) = 5$ and $f(5) = 12$, then $f(g(2)) =$ _____.

2. If the rule of the function f is "Add 1" and the rule of the function g is "Multiply by 2," then the rule of the function $M(x) = f(g(x))$ is "_____," and the rule of the function $N(x) = g(f(x))$ is "_____." Now express these functions algebraically:

$f(x) =$ _____ $\quad g(x) =$ _____

$M(x) =$ _____ $\quad N(x) =$ _____

3. A function f is one-to-one if different inputs produce _____ outputs. You can tell from the graph that a function is one-to-one by using the _____ _____ Test.

4. **(a)** For a function to have an inverse, it must be _____. So which one of the following functions has an inverse? $f(x) = x^2$, $g(x) = x + 2$

(b) What is the inverse of the function that you chose in part (a)?

5. If the rule for the function f is "Multiply by 3, add 5, and then take the third power," then the rule for f^{-1} is "_____." We can express these functions algebraically as:

$$f(x) = _____ \qquad f^{-1}(x) = _____$$

6. The inverse of $f(x) = e^x$ is the function $f^{-1}(x) = _____$.

Think About It

7. If f and g are linear functions, what kind of function is f composed with g? What kind of function is f^{-1}?

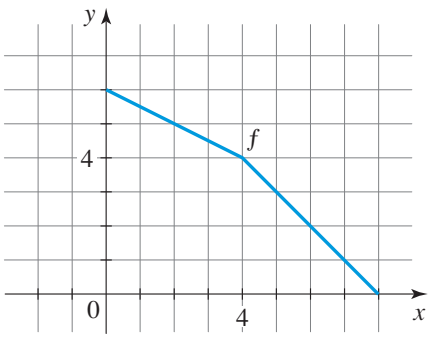
8. True or false?

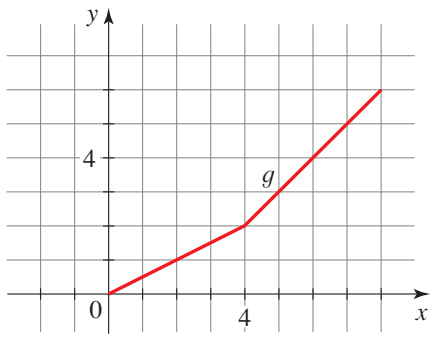
(a) If f has an inverse, then $f^{-1}(x)$ is the same as $\dfrac{1}{f(x)}$.

(b) If f has an inverse, then $f^{-1}(f(x)) = x$.

SKILLS

9–12 ■ The graphs of two functions f and g are given. Use the graphs to evaluate the indicated expressions.





9. **(a)** $f(g(4))$ **(b)** $g(f(4))$

10. **(a)** $f(g(0))$ **(b)** $g(f(0))$

11. **(a)** $f(f(6))$ **(b)** $g(g(6))$

12. **(a)** $f(g(f(8)))$ **(b)** $g(f(g(8)))$

13–14 ■ Two functions f and g are given by tables. Complete the tables for the functions "f composed with g" and "g composed with f."

13.

x	$f(x)$
1	5
2	1
3	2
4	2
5	4

x	$g(x)$
1	3
2	5
3	4
4	1
5	3

x	$f(g(x))$
1	2
2	
3	
4	
5	

x	$g(f(x))$
1	3
2	
3	
4	
5	

14.

x	$f(x)$
-3	0
-2	1
-1	-1
0	2
1	-2
2	3
3	-3

x	$g(x)$
-3	3
-2	3
-1	1
0	0
1	-1
2	-3
3	-3

x	$f(g(x))$
-3	-3
-2	
-1	
0	
1	
2	
3	

x	$g(f(x))$
-3	0
-2	
-1	
0	
1	
2	
3	

15–22 ■ Two functions f and g are given.

(a) Give a verbal description of the functions $M(x) = f(g(x))$ and $N(x) = g(f(x))$.

(b) Find algebraic expressions for the functions $M(x)$ and $N(x)$.

(c) Evaluate $M(3)$ and $N(-2)$.

15. $f(x) = x + 2, \quad g(x) = 3x$

16. $f(x) = 5x, \quad g(x) = x - 1$

17. $f(x) = 2x^2, \quad g(x) = x - 1$

18. $f(x) = x - 2, \quad g(x) = x^2$

19. $f(x) = e^x, \quad g(x) = x + 3$

20. $f(x) = 2x - 4, \quad g(x) = 10^x$

21. $f(x) = 3x + 5, \quad g(x) = \ln x$

22. $f(x) = \log x, \quad g(x) = x^4 + 1$

23–28 ■ A verbal description of the rule of a function f is given.

(a) Give a verbal description of the inverse function f^{-1}.

(b) Find algebraic expressions for $f(x)$ and $f^{-1}(x)$.

(c) Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

23. Multiply the input by 3, then subtract 6.

24. Multiply the input by 5, then add 2.

25. Add 4 to the input, then multiply by $\frac{1}{2}$.

26. Subtract 2 from the input, then divide by 4.

27. Square the input, then multiply by -2 and add 3.

28. Square the input, add 1, then divide by 7.

29–32 ■ Assume that f is a one-to-one function.

29. If $f(2) = 7$, find $f^{-1}(7)$.

30. If $f(-5) = 18$, find $f^{-1}(18)$.

31. If $f^{-1}(3) = -1$, find $f(-1)$.

32. If $f^{-1}(4) = 2$, find $f(2)$.

33–34 ■ A table of values for a one-to-one function f is given. Find

(a) $f(3)$ **(b)** $f^{-1}(5)$ **(c)** $f^{-1}(f(4))$ **(d)** $f^{-1}(10)$

33.

x	$f(x)$
0	7
1	−3
2	5
3	12
4	−1
5	10
6	6

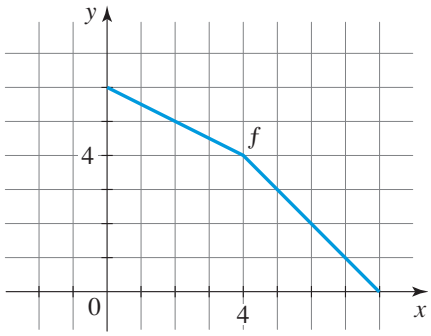
34.

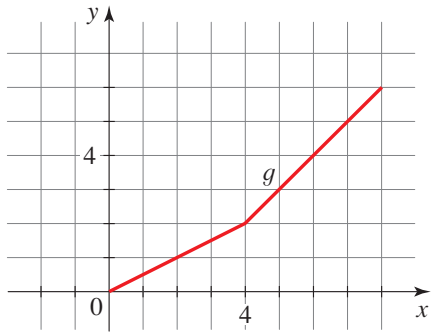
x	$f(x)$
−1	3
0	5
1	−2
2	0
3	10
4	1
5	6

35–36 ■ The graph of a function is given. Use the graph to find the indicated values.

35. **(a)** $f^{-1}(2)$, **(b)** $f^{-1}(4)$, **(c)** $f^{-1}(6)$

36. **(a)** $g^{-1}(2)$, **(b)** $g^{-1}(5)$, **(c)** $g^{-1}(6)$





37–44 ■ Show that f and g are inverses of each other (as in Example 6).

37. $f(x) = 2x - 5$; $g(x) = \dfrac{x + 5}{2}$

38. $f(x) = \dfrac{3 - x}{4}$; $g(x) = 3 - 4x$

39. $f(x) = x^3 + 1$; $g(x) = \sqrt[3]{x - 1}$

40. $f(x) = \dfrac{1}{x - 1}$; $g(x) = \dfrac{1}{x} + 1$

41. $f(x) = 10 \cdot 4^x$; $g(x) = \log_4\left(\dfrac{x}{10}\right)$

42. $f(x) = 10^{x/3}$; $g(x) = \log x^3$

43. $f(x) = \log_5 x^2$; $g(x) = 5^{x/2}$

44. $f(x) = \ln x + 2$; $g(x) = e^{x-2}$

45–56 ■ A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether the function is one-to-one.

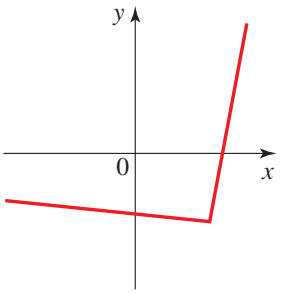
45.

x	0	1	2	3	4	5
$f(x)$	1	$\frac{1}{3}$	3	9	27	81

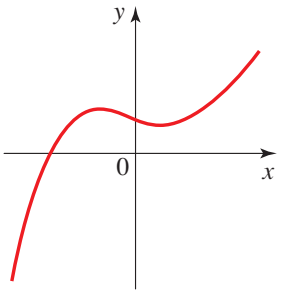
46.

x	0	1	2	3	4	5
$f(x)$	1.1	2.4	5.8	3.6	2.1	2.4

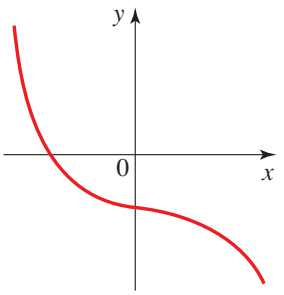
47.



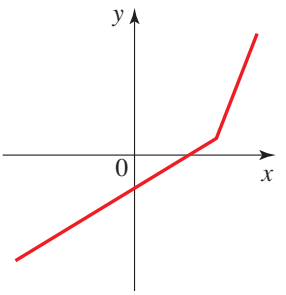
48.



49.



50.



51. $f(x) = -2x + 4$

52. $f(x) = 3x - 2$

53. $g(x) = x^2 - 2x$

54. $g(x) = x^4 + 5$

55. $h(t)$ is the height of a basketball t seconds after it is thrown in a free-throw shot.

56. $S(x)$ is your shoe size at age x.

57–70 ■ Find the inverse function of f.

57. $f(x) = 4x + 7$

58. $f(x) = 3 - 5x$

59. $f(x) = \dfrac{x}{2}$

60. $f(x) = \dfrac{1}{x}$

61. $f(x) = x^3 - 4$

62. $f(x) = \sqrt[3]{x + 2}$

63. $f(x) = \dfrac{x - 2}{x + 2}$

64. $f(x) = \dfrac{1 + x}{3 - x}$

65. $f(x) = \log_2(x + 1)$

66. $f(x) = 10^{3x}$

67. $f(x) = e^{0.5x}$

68. $f(x) = \log_3(2x)$

69. $f(x) = \ln(x - 3)$

70. $f(x) = 2^{x^3}$

71–76 ■ Draw the graph of f and use it to determine whether the function is one-to-one.

71. $f(x) = x^3 - x$

72. $f(x) = x^3 + x$

73. $f(x) = \dfrac{x + 12}{x - 6}$

74. $f(x) = \sqrt{x^3 - 4x + 1}$

75. $f(x) = |x| - |x - 6|$

76. $f(x) = x \cdot |x|$

CONTEXTS

77. Multiple Discounts You have a $50 coupon from the manufacturer good for the purchase of a cell phone. The store where you are purchasing your cell phone is offering a 20% discount on all cell phones. Let x represent the sticker price of the cell phone.

(a) Suppose that only the 20% discount applies. Find a function f that models the purchase price of the cell phone as a function of the sticker price x.

(b) Suppose that only the $50 coupon applies. Find a function g that models the purchase price of the cell phone as a function of the sticker price x.

(c) If you can use the coupon and the discount, then the purchase price is either $f(g(x))$ or $g(f(x))$, depending on the order in which they are applied to the sticker price. Find both $f(g(x))$ and $g(f(x))$. Which composition gives the lower price?

78. Multiple Discounts An appliance dealer advertises a 10% discount on all his washing machines. In addition, the manufacturer offers a $100 rebate on the purchase of a washing machine. Let x represent the sticker price of the washing machine.

(a) Suppose that only the 10% discount applies. Find a function f that models the purchase price of the washer as a function of the sticker price x.

(b) Suppose that only the $100 rebate applies. Find a function g that models the purchase price of the washer as a function of the sticker price x.

(c) Find $f(g(x))$ and $g(f(x))$. What do these functions represent? Which is the better deal?

79. Carbon Dioxide Levels Many scientists believe that the increase of carbon dioxide (CO_2) in the atmosphere is a major contributor to global warming. The EPA estimates that 1 gallon of gas produces on average about 19 pounds of CO_2. Raul owns a car that gets 30 miles per gallon. Raul is concerned about how his driving habits affect the environment, so he decides to calculate his car's CO_2 emissions for several different mileages.

(a) Find a rule for the function g that gives the number of gallons $g(x)$ that Raul uses in driving x miles.

(b) Find a rule for the function f that gives the weight of CO_2 emissions $f(x)$ when x gallons of gasoline are used.

(c) Evaluate the composite function $f(g(x))$. What does this represent?

(d) Use your answers to parts (a)–(c) to complete the table.

Miles driven x	Gallons $g(x)$	$f(g(x))$
12,000	400	7600
24,000		
36,000		
50,000		
100,000		



KAMBOU SIA/AFP/Getty Images

80. Biodiesel Fuels Scientists have begun exploring the use of biodiesel fuel (plant-based diesel fuel) as a substitute for fossil fuel. One possible source for biodiesel is oil extracted from the seeds of the species *Jatropha curcsa*, a drought-resistant bushy tree that is indigenous to India. In a study assessing the viability of using these trees as a biofuel, scientists gathered data on the number of seeds per tree at various elevations and the oil content of those seeds. Some of their data are shown in the tables on the following page. The first table gives the average seed yield per tree at each elevation, and the second gives the oil content of a seed produced by a tree bearing the given number of seeds. Note that as elevation increases, so does seed production per tree, but the more seeds a tree produces, the lower the oil content of those seeds.

Let $S(x)$ be the average number of seeds per tree at elevation x meters, and let $C(n)$ be the amount of oil in each seed from a tree producing n seeds.

(a) Let $D(x) = C(S(x))$. What does the function D represent? Complete the third table, giving the values of $D(x)$.

(b) Explain why the product $D(x) \cdot S(x)$ gives the amount of oil produced per tree at elevation x.

Elevation (m) x	Seeds/tree $S(x)$
400	330
600	360
750	410
900	600
1000	670

Seeds/tree n	Oil/seed (dL) $C(n)$
330	4.3
360	4.1
410	3.8
600	3.3
670	3.1

Elevation (m) x	Oil/seed (dL) $D(x) = C(S(x))$
400	4.3
600	
750	
900	
1000	

81. Multiple Discounts A car dealership advertises a 15% discount on all its new cars. In addition, the manufacturer offers a \$1000 rebate on the purchase of a new car. Let x represent the sticker price of the car.

(a) Suppose that only the 15% discount applies. Find a function f that models the purchase price of the car as a function of the sticker price x.

(b) Suppose that only the \$1000 rebate applies. Find a function g that models the purchase price of the car as a function of the sticker price x.

(c) Find a formula for $H(x) = f(g(x))$.

(d) Find H^{-1}. What does H^{-1} represent?

(e) Find $H^{-1}(13{,}000)$. What does your answer represent?

82. Health Care Spending The function $f(x) = 2813(1.0578)^x$ is an exponential model for annual U.S. health care expenditures per capita, where x is the number of years since 1990 (see Exercise 39 in Section 3.1).

(a) Find f^{-1}. What does f^{-1} represent?

(b) Use the inverse function f^{-1} to predict when annual U.S. health care expenditures per capita reaches \$7000.

(c) Search the Internet to find when the reported annual health care expenditures reached \$7000. Compare with your results in part (b).

83. Population of India The function $f(x) = 846.3(1.031)^x$ is an exponential model for the population of India, where x is the number of years since 1990 and the population is measured in millions (see Exercise 45 in Section 3.2).

(a) Find f^{-1}. What does f^{-1} represent?

(b) Use the inverse function f^{-1} to predict when the population of India reaches 1.1 billion.

(c) Search the Internet to find when the population of India actually did reach 1.1 billion. Compare with your results in part (b).

84. Service Fee For his services, a private investigator charges a \$500 retention fee plus \$80 per hour. Let x represent the number of hours the investigator spends on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

85. Torricelli's Law A tank holds 100 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 40 minutes. Torricelli's Law relates the volume $F(x)$ of water remaining in the tank after x minutes by the function

$$F(x) = 100\left(1 - \frac{x}{40}\right)^2$$

(a) Find F^{-1}. What physical quantities (volume or time) do the inputs and outputs of F^{-1} represent?

(b) Find $F^{-1}(15)$. What does your answer represent?

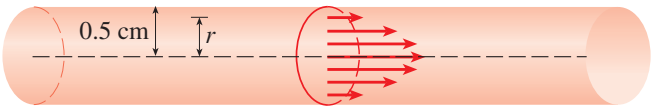


86. Blood Flow As blood moves through a vein or artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure below). For an artery with radius 0.5 cm, v (in cm/s) is given as a function of r by

$$v(r) = 4.8(0.25 - r^2)$$

(a) Find v^{-1}. What does v^{-1} represent?

(b) Find $v^{-1}(0.9)$. What does your answer represent?



CHAPTER 4 REVIEW

CHAPTER 4 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

4.1 Logarithmic Functions

The **logarithm base *a*** of a number is the exponent to which we must raise the base a to get the number. In symbols,

$$\log_a x = y \qquad \text{means} \qquad a^y = x$$

When the base is 10, we usually don't write the a:

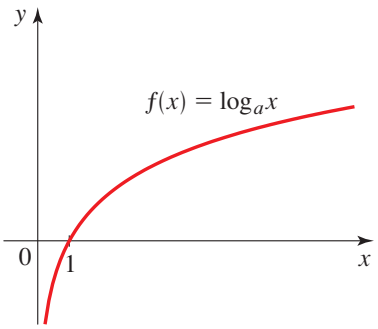
$$\log x = y \qquad \text{means} \qquad 10^y = x$$

Logarithms have the following **basic properties**:

1. $\log_a 1 = 0$ **2.** $\log_a a = 1$ **3.** $\log_a a^x = x$ **4.** $a^{\log_a x} = x$

The **logarithmic function with base *a*** is the function

$$f(x) = \log_a x \qquad a > 0, \quad a \neq 1$$



Its domain is $(0, \infty)$, and the y-axis is a vertical asymptote of its graph. The graph for $a > 1$ has the shape shown in the margin.

4.2 Laws of Logarithms

Logarithms obey the following **Laws of Logarithms**:

1. $\log_a(AB) = \log_a A + \log_a B$
2. $\log_a(A/B) = \log_a A - \log_a B$
3. $\log_a A^C = C \log_a A$

We can use the Laws of Logarithms to **expand** or **combine logarithmic expressions**.

Calculators have a LOG key to evaluate logarithms base 10. To evaluate other logarithms, we use the **Change of Base Formula**:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

4.3 Logarithmic Scales

When quantities vary over a very large range, we often use a **logarithmic scale** to make the range of measurements more manageable. Some examples are as follows:

- **pH** scale, for measuring acidity
- **decibel** scale, for measuring sound intensity
- **Richter** scale, for measuring earthquake intensity

4.4 The Natural Exponential and Logarithmic Functions

The **number e** is the value that $(1 + 1/n)^n$ approaches as n becomes large. The value of e is approximately 2.71828. The **natural exponential function** is the function $f(x) = e^x$.

The **natural logarithm function** is the logarithm function with base e; it is denoted by ln, so $\ln x = \log_e x$.

The natural exponential and logarithm functions have the following **basic properties:**

1. $\ln 1 = 0$ 2. $\ln e = 1$ 3. $\ln e^x = x$ 4. $e^{\ln x} = x$

The natural exponential function can be used to calculate **continuously compounded interest**. If a principal P is invested at an annual interest rate r compounded continuously, then the amount $A(t)$ to which it grows after t years is given by

$$A(t) = Pe^{rt}$$

The exponential function $f(t) = Ce^{rt}$ models **exponential growth** if $r > 0$ or **exponential decay** if $r < 0$. Any growth or decay model $f(x) = Ca^x$ can be expressed in terms of e by writing $f(t) = Ce^{rt}$, where $r = \ln a$.

4.5 Exponential Equations: Getting Information from a Model

An **exponential equation** is one in which the unknown appears in the exponent of one or more terms or factors in the equation. For example, $8^x = 512$ is an exponential equation in the unknown x. To **solve** an exponential equation, we take logarithms of both sides and then use the fact that $\log a^x = x \log a$ to "bring down the exponent."

A **logarithmic equation** is one in which the unknown appears inside a logarithm function. For example, $\log(x + 1) - \log x = 2$ is a logarithmic equation. To solve a logarithmic equation, we first combine all the logarithm terms using the Laws of Logarithms and then rewrite the equation in exponential form.

Exponential equations are important because they arise when we use exponential growth or decay models to determine when a certain event has happened or will happen. For instance, we can determine how old a historical artifact is by using the decay model for radioactive carbon-14.

4.6 Working with Functions: Composition and Inverse

Given two functions f and g, we can make a new function h called the **composition of f with g**, defined by

$$h(x) = f(g(x))$$

The **inverse of a function** f is a function f^{-1} that "undoes the rule" of f; that is, if we apply f to the input x and then apply f^{-1} to the result, we get back x again as the final output. We have

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x$$

Only one-to-one functions have inverses. A function is **one-to-one** if no two different inputs have the same output.

To tell from its graph whether a function is one-to-one we use the **Horizontal Line Test**: If no horizontal line intersects the graph more than once, then the function is one-to-one.

Exponential and logarithmic functions are inverses of each other:

- If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.
- If $g(x) = \log_a x$ then $g^{-1}(x) = a^x$.

CHAPTER 4 REVIEW EXERCISES

SKILLS

1–2 ■ Find the exact value of each logarithm.

1. **(a)** $\log_4 16$ **(b)** $\log_4 \frac{1}{64}$ **(c)** $\log 1000$ **(d)** $\log(0.001)$

2. **(a)** $\log_5 \frac{1}{25}$ **(b)** $\log_5 125$ **(c)** $\log \sqrt{10}$ **(d)** $\log 10^{25}$

3–4 ■ Complete the table of values for each logarithmic function, and then use your table to graph the function.

3. $f(x) = \log_4 x$

x	$f(x)$
0.25	
0.5	
1	
4	
8	

4. $g(x) = \log_9 x$

x	$g(x)$
$\frac{1}{9}$	
$\frac{1}{3}$	
1	
3	
9	

5–6 ■ Use a calculator to find the value of the logarithm correct to two decimals. Use the Change of Base Formula if necessary.

5. **(a)** $\log_6 27$ **(b)** $\log_2 1000$ **(c)** $\log 94$ **(d)** $\ln(12.4)$

6. **(a)** $\log_8 67$ **(b)** $\log_2(0.33)$ **(c)** $\log 50$ **(d)** $\ln 50$

7. An equation is given in logarithmic form. Express it in exponential form.

(a) $\log_5 25 = 2$ **(b)** $\log_2 x = 3$ **(c)** $\ln y = 4$

8. An equation is given in exponential form. Express it in logarithmic form.

(a) $9^{1/2} = 3$ **(b)** $3^x = 16$ **(c)** $4^y = 7$

9–10 ■ Use the Laws of Logarithms to expand the given expression.

9. **(a)** $\log_2(3ab)$ **(b)** $\log \dfrac{5x}{y^2}$

10. **(a)** $\log_4 \dfrac{a^3}{b}$ **(b)** $\ln(xyz^2)$

11–12 ■ Use the Laws of Logarithms to combine the given expression.

11. **(a)** $3 \log_2 X + \log_2 Y$ **(b)** $\ln(x + 1) - 2 \ln y$

12. **(a)** $\log_5(b + 1) - 4 \log_5(b + 3)$ **(b)** $\log(a - 2) + 3 \log a$

13–14 ■ Use a graphing calculator and the Change of Base Formula to draw a graph of the logarithmic function.

13. $f(x) = \log_6 x$ **14.** $g(x) = \log_{15} x$

15. The hydrogen ion concentration in one cell of a car battery is found to be 7.3×10^{-2} M. What is the pH of the acid in this cell? What is it about the pH value that tells you that it is indeed acidic, not basic?

16. Your upstairs neighbors are having a noisy party, to which you were not invited. You measure the sound level at 55 W/m^2. Express this in decibels.

17–18 ■ Use a graphing calculator to draw a graph of the exponential function.

17. $f(x) = \frac{1}{2} e^{0.3x}$ **18.** $g(x) = e^{-0.5x}$

19–20 ■ A growth or decay model is given. Express the model in terms of the base e.

19. $f(t) = 2400 \cdot 3^t$ **20.** $g(t) = 12\left(\frac{1}{4}\right)^t$

21–22 ■ Find an exponential model of the form $f(t) = Ce^{rt}$ for the population that is described (where t is measured in hours).

21. The initial population is 1500, and it triples every hour.

22. The initial population is 600, and it increases by 50% every hour.

23–30 ■ Solve the equation for the unknown x.

23. $3^{2x+1} = 27$ **24.** $7 \cdot 2^{3x} = 56$

25. $5^{x-1} = 47$ **26.** $17e^{3x} = 12$

27. $\log(x + 1) - \log x = 1$ **28.** $\log_2 x = 3 + \log_2(x - 1)$

29. $\log_4 x = 2 + \log_4(2x - 3)$ **30.** $\ln(x + 3) = 5 + \ln x$

31–34 ■ Two functions f and g are given.

(a) Find algebraic expressions for the functions $f(g(x))$ and $g(f(x))$.

(b) Evaluate $f(g(2))$ and $g(f(-3))$.

31. $f(x) = 2x - 4, \quad g(x) = 3x + 1$

32. $f(x) = x^2, \quad g(x) = 1 - 2x$

33. $f(x) = 2x^2, \quad g(x) = x^3$

34. $f(x) = \log x, \quad g(x) = 10^{3x}$

35–38 ■ Find the inverse function of f.

35. $f(x) = 3x - 7$

36. $f(x) = \dfrac{2}{x + 1}$

37. $f(x) = 8e^x$

38. $f(x) = \log\left(\dfrac{x}{6}\right)$

39–40 ■ Use a graphing calculator to draw the graph of f. Then use the graph to determine whether the function is one-to-one.

39. $f(x) = x^3 + 4x$

40. $f(x) = x^3 - 4x$

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

41. Drug Metabolism Herman is prescribed a blood pressure medication, of which he must take an 80-milligram dose at the same time every morning. Research has shown that in a man of Herman's size, 10% of the drug is eliminated from the body in each 4-hour period. In this problem we consider what happens to his first dose of the drug.

(a) Find an exponential function of the form $M(x) = Ca^x$ that models the amount $M(x)$ of the drug in Herman's body x 4-hour time periods after he ingests it.

(b) Rewrite your model from part (a) as a function of the form $N(t) = Cb^t$, where t is the number of *hours* since Herman ingests the drug. What is the 1-hour decay factor? The 1-hour decay rate?

(c) For best therapeutic effect, the drug level should always be above 50 milligrams. Use a graphing calculator to graph the function N over the 24-hour period between doses. Does it appear that Herman always has 50 milligrams or more of the drug in his body? If not, determine the time at which the drug falls to a level of 50 milligrams. (Find your solution both graphically and algebraically.)

(d) Express the model using the natural exponential function: $N(t) = Ce^{rt}$. What is the instantaneous decay rate?

(e) How much of the drug is still in Herman's system after 24 hours (when he is ready to take his next dose)?

(f) If we want to construct a model for the second day's drug levels, what change will we have to make to the function N from part (b)? (Keep in mind that on the second day Herman starts out with some of the drug already in his system, from the preceding day's pill.) Find a function $S(t)$ that models the drug levels on the second day.

(g) Graph the function you found in part (f). Will the drug level fall below 50 milligrams on the second day?

42. Dating an Ancient Relic During the times of the Roman Empire much of present-day northern Germany was inhabited by Germanic tribes that were never conquered by Rome. It is not uncommon to find relics from these ancient peoples in bogs and lake bottoms.

While digging a foundation for his new house in Mecklenburg-Vorpommern, Dietmar discovers a well-preserved rowboat carved out of beechwood. A test at the local university reveals that the wood has 73% of the carbon-14 that exists in current plant life.

(a) Use the fact that carbon-14 has a half-life of 5370 years to find a model of the form $A(t) = C\left(\frac{1}{2}\right)^{t/h}$ for the amount $A(t)$ of carbon-14 remaining after t years from a sample with initial mass C.

(b) Convert your model from part (a) to a function of the form $A(t) = Ce^{rt}$. The value of r should be negative. What does this mean?

(c) What percentage of the original carbon-14 should remain in a sample that is 2000 years old?

(d) Assume that the initial amount C is 100 grams in the function of part (b). Find a formula for the function A^{-1}. What does $A^{-1}(x)$ represent (if x is a mass measured in grams)?

(e) How old is the rowboat that Dietmar dug up? Could it be a relic from the ancient Germanic tribes?

CONTEXTS

43. pH of Soil The acidity of soil has a major effect on the types of plants that can be grown in it. Beans do not grow well in soil that is strongly acidic, with a pH of less than 5.3.

(a) A chemist tests the soil in her back yard and finds that its hydrogen ion concentration is $[H^+] = 3.7 \times 10^{-7}$ M. What is the pH of the soil? Is it suitable for growing beans?

(b) The chemist amends the soil to change its pH to 5.9. What is the hydrogen ion concentration of the amended soil?

44. Earthquake Magnitude Seismographs were not in common use until the 20th century, so it is difficult to gauge the strength of earthquakes that occurred earlier than that. Nevertheless, scientists estimate that the Lisbon earthquake of November 1, 1755, which completely destroyed the Portuguese capital and caused a massive tsunami, had a magnitude of 9.0 on the Richter scale. A prior earthquake in Lisbon in 1714 had only 2% of the intensity of the 1755 event. What was the magnitude of the 1714 earthquake on the Richter scale?

45. Compound Interest Jennifer invests $250,000 in a three-year CD that pays 4.25% annual interest, compounded continuously.

(a) What is the instantaneous growth rate of her CD?

(b) What will the value of the CD be when it matures after three years?

(c) If the CD had compounded the interest annually instead of continuously, how much would it have been worth at maturity?

46. Compound Interest Zacky has $5000 that he wants to invest in a secure account. His bank offers him the choice of two different guaranteed interest savings accounts. How long will it take for his deposit to grow to $7000 in each of these accounts?

(a) The account pays 3.75% annual interest, compounded monthly.

(b) The account pays 3.62% annual interest, compounded continuously.

47. Size of a Turtle Most organisms have genetic limiters on their growth, so they increase in size only when they are young. Dogs, for example, reach their maximum size in the first 15 months of life and stop growing after that. However, some reptiles grow throughout their life, a phenomenon known as *attenuated growth*.

Hortense is a loggerhead turtle in the Norristown zoo. Her weight (in pounds) t years since she was acquired by the zoo in 2000 is modeled by the function

$$W(t) = 23 \cdot 2^{t/6}$$

(a) How many years does it take for Hortense to double her weight?

(b) Express Hortense's weight as a function of the form $W(t) = Ce^{rt}$. What is the instantaneous growth rate of her weight?

(c) When will Hortense weigh 150 pounds?

(d) Loggerhead turtles sometimes live 150 years or more. If Hortense lives to the year 2100, how much will she weigh then, according to the model? Does this seem reasonable? What kind of model might be a better long-term predictor of her weight?

48. Time of Death The deer-hunting season in Wappaloosa County opens at noon on October 15. A game warden interviews a hunter who has a deer carcass in his pickup truck at 3:00 P.M. on opening day. Normal body temperature for deer is 102°F, and the air temperature on this day is 68°F. Under these circumstances, Newton's Law of Cooling indicates that t hours after death, a deer will have cooled to T degrees, where

$$T = 68 + (102 - 68)e^{-0.095t}$$

(a) The warden measures the deer's temperature and finds it to be 85°F. How long ago did the deer die?

(b) Was the deer killed before the official season opening, or was it shot legally?

49. Multiple Coupons A grocery store distributes gasoline coupons of two types to its customers, based on the amount of their grocery purchase. The Type A coupon provides a 5% discount, and the Type B coupon provides a $2.00 discount on the total gasoline purchase. If you buy a tankful of gasoline priced at x dollars, let $f(x)$ be your final cost when using a Type A coupon, and let $g(x)$ be your final cost when using a Type B coupon.

(a) Find formulas for the functions f and g.

(b) The gasoline retailer permits the use of one of each type of coupon on a single purchase. What do the functions $f(g(x))$ and $g(f(x))$ represent?

(c) Find formulas for both $f(g(x))$ and $g(f(x))$.

(d) Find $f(g(40))$ and $g(f(40))$. Does the order in which the coupons are applied to your purchase make a difference?

50. Cost of Services A house cleaner decides to charge his customers using an exponential scale, to encourage them to maintain their homes more neatly between cleanings. For a cleaning that takes him x hours, he charges $C(x) = 20(2.1)^x$ dollars.

(a) How much would a cleaning cost if it takes 1 hour? 3 hours?

(b) Find a formula for $C^{-1}(x)$. What does this function represent?

(c) A customer receives a bill for $388.96. How long did it take to clean this person's home?

CHAPTER 4 TEST

1. Find the exact value of the logarithm.

(a) $\log_4\left(\frac{1}{16}\right)$ **(b)** $\log_9 27$ **(c)** $\log 10^{100}$

2. Let $f(x) = \log_3 x$.

(a) Use the Change of Base Formula to complete the table of values below, correct to two decimal places.

x	0.1	0.5	1	3	5
$f(x)$					

(b) Use the table of values to sketch a graph of f.

3. Use the Laws of Logarithms to perform the indicated operation on the expression.

(a) Expand: $\log\left(\dfrac{5x^4}{6x - 11}\right)$

(b) Combine as a single logarithm: $\ln 5 + 2 \ln x - \ln(x^3 + 7)$

4. If the hydrogen ion concentration in a solution is $[H^+]$ moles/liter (M), then its pH is $-\log[H^+]$.

(a) Find the pH of a yogurt dip whose hydrogen ion concentration is 4.3×10^{-7} M.

(b) Find the hydrogen ion concentration of a glass of lemonade with a pH of 4.3.

5. Karl-Heinz is trying to decide where to invest his tax refund of $2325. Calculate the amount his investment will grow to after 3 years under the following CD choices offered by competing banks. Which is the better investment?

(a) 3% annual interest, compounded daily

(b) 2.95% annual interest, compounded continuously

6. Rhodium-106 has a half-life of 30 seconds, so its decay is modeled by the equation $A(t) = C\left(\frac{1}{2}\right)^{t/30}$, where $A(t)$ is the amount remaining after t seconds of a sample that initially weighted C grams.

(a) Express the model in the form $A(t) = Ce^{rt}$. What is the instantaneous decay rate?

(b) How long will it take for a 1000-gram sample of rhodium-106 to decay to 1 gram?

7. Solve the equation for the unknown x. Express your answer correct to two decimal places.

(a) $7 \cdot 5^{3x} = 105$ **(b)** $2 \log_4 x - \log_4(x^2 + x - 4) = 0$

8. An online book club charges its members a shipping fee of $2.95 for the first book ordered plus $1.00 for each additional book.

(a) Let $S(x)$ be the shipping fee on an order of x books. Find a formula for the function S.

(b) Find a formula for S^{-1}. What does $S^{-1}(x)$ represent?

(c) If a customer's order has a shipping fee of $12.95, how many books were ordered?

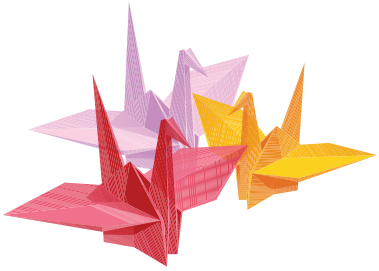
9. Let $f(x) = 10^{x+2}$, and let $g(x) = \log(x - 1)$.

(a) Find formulas for the compositions $f(g(x))$ and $g(f(x))$.

(b) Evaluate $f(g(0))$ and $g(f(0))$.

1 Super Origami

OBJECTIVE To explore how logarithms are used to solve exponential equations.

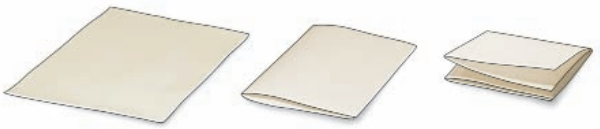


Slash331/Shutterstock.com 2009

Origami is the traditional Japanese art of paper folding. The Japanese word *origami* literally means "paper folding" (from *oru*, meaning "folding," and *kami*, meaning "paper"). The goal of this art is to create an illustration of an object by folding a sheet of paper. Most of us have learned how to make paper airplanes or simple paper boats. One of the most common origami constructions is the crane shown here. But origami masters can fold paper into creative forms, including animals, birds, flowers, buildings, and towers. In this exploration we investigate the simplest type of origami: just folding a sheet of paper over and over again.

I. How to Make a Paper Tower

Take an ordinary sheet of paper and fold it in half. Now fold it in half again. Keep folding the paper in half. How many times are you able to fold it?



0 folds 1 fold 2 folds

1. **(a)** When you fold the sheet of paper once, the result is a stack with two layers of paper; folding it twice results in a four-layer stack, and so on. Complete the table for the number of layers after x folds.

Number of folds x	0	1	2	3	4	5
Layers $f(x)$	1					

(b) The number of layers grows exponentially. What is the growth factor? What is the initial value? Find an exponential function $f(x) = Ca^x$ that models the number of layers after x folds.

$$f(x) = \square \cdot \square^{x}$$

2. Suppose the sheet of paper we started with is 1/1000 of an inch thick.

(a) How thick is the stack of paper after 10 folds?

Number of layers: $2^{\square}$

Thickness in inches: _____

(b) How thick is the stack of paper after 20 folds?

Number of layers: $2^{\square}$

Thickness in inches: _____

Thickness in feet: _____

(c) How thick is the stack of paper after 50 folds?

Number of layers: $2^{\square}$

Thickness in miles: _____

II. How High a Tower?

How many times do you need to fold the paper to make a tower as tall as you are? Or as tall as the Empire State Building? To answer these questions, let's start with a sheet of paper that is 1/1000 of an inch thick.

1. How many sheets of paper are needed to form a stack of paper with the given height?

1 inch: _____

1 foot: _____

1 mile: _____

2. How many times do you need to fold the paper to get a stack of paper as tall as you are? Answer this question in the following steps:

Express your height in inches: $h =$ _____

Express your height in sheets of paper: $N =$ _____

Solve the equation $2^x = N$ for x. Explain why your answer is the number of folds you need to get a stack your height. Can you actually fold the paper that many times?

3. How many times do we need to fold the paper to get a stack with the following heights?

(a) The height of the Empire State Building (1250 feet)

(b) The height of Mount Everest (29,029 feet)

(c) The distance to the moon (240,000 miles)

(d) The distance to the sun (93,000,000 miles)

2 Orders of Magnitude



A still from the film *Powers of Ten*

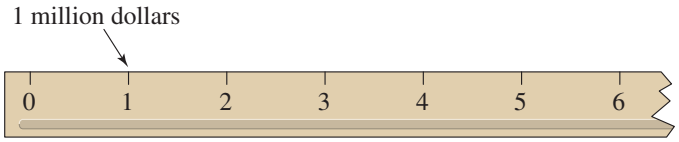
OBJECTIVE To compare sizes of objects using logarithms or orders of magnitude.

Our universe presents us with enormous disparities in size—from incredibly tiny subatomic particles to unbelievably huge galaxies. How can we meaningfully compare these sizes? The 1977 short documentary film *Powers of Ten,* created by Ray and Charles Eames, uses zoom photography to visually compare the relative sizes of different objects in the universe. (This delightful film is available for viewing on several Internet sites.) We also encounter disparities in size in our everyday life—for example, our own income in comparison to trillion dollar national budgets (or deficits). But to bring these disparities in size down to earth, let's just try to fathom the wealth of the billionaire next door.

I. How Rich Is a Billionaire?

A millionaire is rich; he or she must have at least a million dollars in the bank. A billionaire is richer. Let's compare the relative wealth of a millionaire and a billionaire graphically and numerically.

1. Let's use the **millionaire measuring tape** to measure wealth. Each mark on the tape represents one million dollars. So the mark at 1 represents one million dollars, the mark at 2 represents two million dollars, and so on.



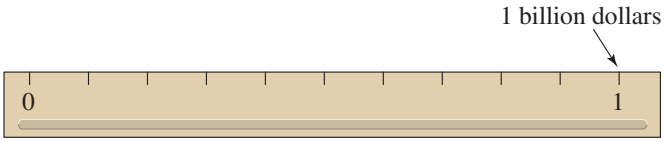
The millionaire measuring tape

(a) At what number on the tape is the one billion dollar mark? _____

(b) If the marks on the tape are one inch apart, how many inches to the right of zero is the billion dollar mark? _____

(c) How many *feet* to the right of zero is the billion dollar mark? _____

2. The billion dollar mark on the millionaire measuring tape in Question 1 is way off the page. So let's use the **billionaire ruler**, on which the mark at 1 represents one billion dollars.



The billlionaire ruler

(a) Where is the 100 million dollar mark on the ruler?

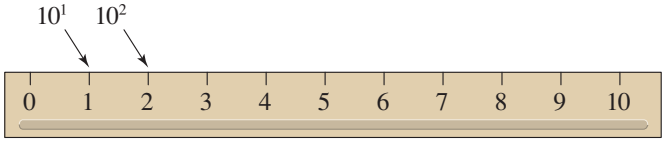
(b) Place marks on the ruler at 1 million and 2 million dollars. Is it easy to visually distinguish the two marks?

(c) Place marks at one thousand and ten thousand dollars on the ruler.

3. The "millionaire measuring tape" and the "billionaire ruler" are not so easy to use.

- The millionaire measuring tape is just too long to handle easily.
- The billionaire ruler can't be easily used to measure small amounts.

So let's try the "logarithm ruler." Each mark on the **logarithm ruler** is the logarithm (base 10) of the number it represents. So the mark at 1 represents 10^1 dollars, the mark at 2 represents 10^2 dollars, and so on. Note that each mark on this ruler represents a number that is *ten times* the number of the preceding mark.



The logarithm ruler

(a) Where is the one thousand dollar mark on the logarithm ruler? Where is the ten thousand dollar mark?

(b) Where are the one million and one billion dollar marks on the ruler?

(c) Is there a one dollar mark on the ruler?

(d) Is the logarithm ruler too long to handle easily? Can the ruler be easily used to measure small numbers of dollars?



cbarnesphotography/Shutterstock.com 2009

Money under the mattress

4. In Questions 1–3 we used graphical methods to help us understand the difference in size between a million dollars and a billion dollars. Now let's use another method.

(a) Suppose you have one million dollars under your mattress (you don't trust banks). Each day you take out one thousand dollars and spend it. For how many years would you be able to maintain this lifestyle before you become totally broke?

(b) If you have a billion dollars (instead of a million) in part (a), how many years would pass by before you would be broke?

II. How Small Is a Bacterium?

We've talked a lot about bacteria in this book. Let's compare the size of a bacterium with other, more familiar things. Recall that a centimeter (cm) is one hundredth of a meter (m), a millimeter (mm) is a thousandth of a meter, and a micrometer (μm) is a millionth of a meter.

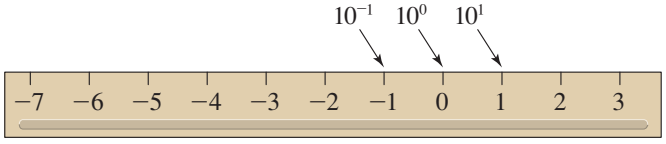
Small child	1 m
Housefly	1 cm
Mite	1 mm
Bacterium	1 μm

1. To visually compare the size of the living things in the above list, let's use a **meter stick**. Mark the size of each one on the meter stick below.



Meter stick

2. Some of the marks you put on the meter stick in Question 1 are not easily distinguishable. Let's use the **logarithmic meter stick** shown below.



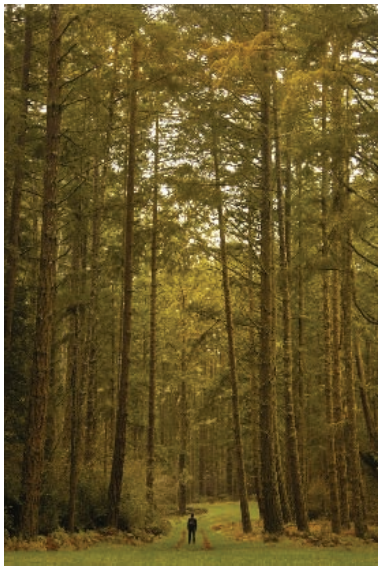
Logarithmic meter stick

(a) Express the size (in meters) of each of the living things on the list on the preceding page in exponential notation.

(b) Label the size of each object on the logarithm meter stick.

(c) Where would a 10-meter tree and a 100-meter tree land on the logarithmic meter stick?

III. Orders of Magnitude



Cate Frost/Shutterstock.com

A tree is one order of magnitude taller than a human.

Sometimes we want to make rough size comparisons. To do this, we round off numbers to the nearest power of 10. For example, for a man who is 6 feet tall, we round his height to 10 feet. In other words, the height of the man is closer to 10^1 feet than to 10^0 feet (or 1 foot). For a ladybug that is 9.0×10^{-3} feet long, we round its length to 10^{-2} feet. A number rounded to the nearest power of 10 is called an **order of magnitude**. The idea is that an object that is 10 times or so larger than another object is in a different category or "order of magnitude."

To compare the height of a human being (10^1 feet) to the height of a ladybug (10^{-2} feet), we find the ratio:

$$\frac{\text{human height}}{\text{ladybug length}} = \frac{10^1}{10^{-2}} = 10^{1-(-2)} = 10^3$$

So a human being is roughly 10^3 or 1000 times larger than a ladybug. We say that a human being is 3 orders of magnitude (three powers of 10) greater than a ladybug.

1. Express the following in terms of order of magnitude.

Object	Size (m)	Order of magnitude
Gold atom	2.8×10^{-10}	_____
Bacterium	3.5×10^{-6}	_____
Mite	2.0×10^{-3}	10^{-3}
Ladybug	7.0×10^{-3}	10^{-2}
Human	1.8×10^{0}	_____
Earth	1.3×10^{7}	10^{7}
Sun	1.4×10^{9}	10^{9}
Betelgeuse	8.3×10^{11}	_____

2. Perform the following order of magnitude comparisons using the results of Question 1.

(a) The sun is ___2___ orders of magnitude larger than the earth.

(b) A bacterium is _____ orders of magnitude larger than a gold atom.

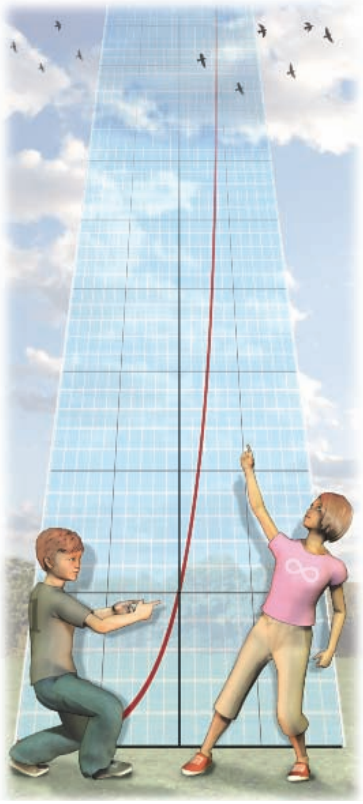
(c) A lady bug is _____ orders of magnitude smaller than the earth.

(d) Betelgeuse is _____ orders of magnitude larger than a bacterium.

3 Semi-Log Graphs

OBJECTIVE To make semi-log graphs and use them to find models of exponential data.

In Chapter 3 we sketched graphs of exponential functions. But in each case we sketched only a small portion of the graph. This is because exponential functions grow so rapidly that we would need a sheet of paper larger than the pages of this book to sketch them, even for relatively small input values. In this exploration we see how logarithms can be used to help us manage the size of such large graphs. In the process we'll discover how logarithms allow us to "straighten" exponential graphs, helping us to analyze their properties as easily as we analyze lines.



I. How Big Is the Graph?

Let's see what size sheet of paper we need to sketch a graph of

$$f(x) = 10^x$$

for x between 0 and 8.

1. For the function f, the input values of 0 and 8 correspond to what output values?

Input value	Output value
0	_____
8	_____

2. Let's make each unit on our graph to be 1 inch. So we need just 8 inches for the x-axis, which is the width of the pages of this book.

(a) How many inches would we need for the y-axis?

(b) To get a better idea of how long the y-axis needs to be, convert your answer to part (b) into miles.

(c) Would you be able to see the whole graph at once?

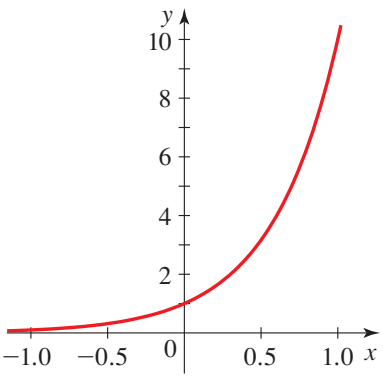
3. What if we decide to let each inch on the y-axis represent 1000 (or 10^3) units?

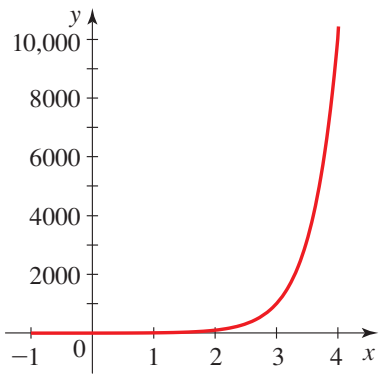
(a) How many inches would we need for the y-axis?

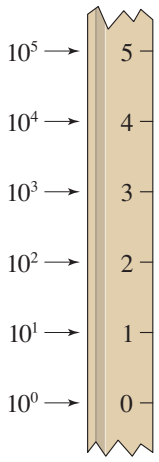
(b) Convert your answer to part (b) into miles.

(c) Would you be able to see the whole graph at once?

4. Two graphs of f are shown below. Which graph can be used to estimate the values of $f(0.5)$? $f(2.3)$? $f(3.1)$? $f(-0.9)$?







The marks on the "logarithmic ruler" are the logarithms of the numbers they represent.

II. Semi-Log Graphs

None of the graphs we considered in Part I give a satisfactory representation of an exponential function. One way around this dilemma is to use a "logarithmic ruler" or *logarithmic scale* on the y-axis. (See the figure in the margin.)

When we graph a function and use a logarithmic scale on one of the axes, the resulting graph is called a **semi-log graph** or **semi-log plot**. This is equivalent to graphing the points $(x, \log y)$.

- To draw a **graph** of f, we plot the points (x, y).
- To draw a **semi-log graph** of f, we plot the points $(x, \log y)$.

Let's sketch a semi-log graph of the exponential function

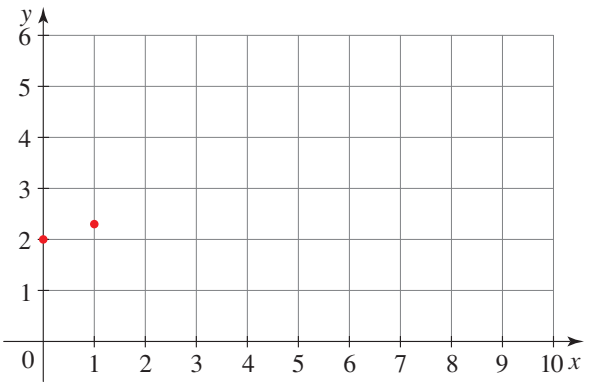
$$y = 100 \cdot 2^x$$

for x between 0 and 10.

1. Complete the table for the values of y and $\log y$

x	y	$\log y$	x	y	$\log y$
0	100	2	6		
1	200	2.3	7		
2			8		
3			9		
4			10		
5					

2. Draw a semi-log graph of f by plotting the points $(x, \log y)$ from the table in Question 1.



3. Does the semi-log graph appear to be a line? If so, estimate the slope and the y-intercept from the graph.

Slope: _____ y-intercept: _____

4. Let's show that the graph in Question 2 is a line by finding an algebraic formula for $\log y$.

(a) Supply the missing reasons.

$$y = 100 \cdot 2^x \qquad \text{Definition of } y$$
$$\log y = \log(100 \cdot 2^x) \qquad \text{Take log of each side}$$
$$\log y = \log 100 + x \log 2 \qquad \underline{\qquad\qquad}$$
$$\log y \approx 2 + 0.3x \qquad \underline{\qquad\qquad}$$

(b) The last equation in part (a) shows that $\log y$ is a linear function of x. What is the slope? What is the y-intercept? Do your answers agree with the line in the graph you sketched in Question 2?

III. Linearizing Data

A semi-log graph of an exponential function $y = Ca^x$ is a line. We can see this from the following calculations:

$$y = Ca^x \qquad \text{Exponential function}$$
$$\log y = \log Ca^x \qquad \text{Take the log of each side}$$
$$\log y = \log C + x \log a \qquad \text{Laws of Logarithms}$$

To see that $\log y$ is a linear function of x, let $Y = \log y$, $M = \log a$, and $B = \log C$; then

$$Y = B + Mx$$

So if we make a semi-log graph of exponential data, we would get a line with the following properties:

$$\text{Slope: } M = \log a$$
$$y\text{-intercept: } B = \log C$$

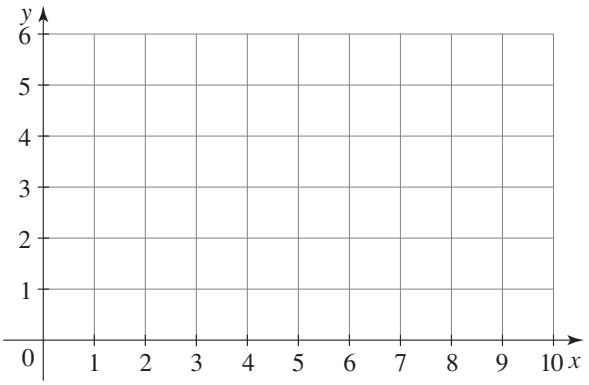
Let's see how we can use this information to obtain a function that models exponential data.

1. The following data are exponential. Note that the inputs are *not* equally spaced.

x	y	$\log y$
0	10	
0.5	20	
2.0	160	
3.0	640	
4.5	5120	

(a) Complete the log y column in the table.

(b) Make a semi-log graph of the data.



2. Let's find a function of the form $y = Ca^x$ that fits the data. (In other words, we need to find a and C.)

(a) Estimate the slope and y-intercept from the graph.

Slope: $M =$ _____

y-intercept: $B =$ _____

(b) From the above we know that $M = \log a$ and $B = \log C$. So

$\log a =$ _____

$\log C =$ _____

(c) Solve the equations in part (b).

$a =$ _____

$C =$ _____

(d) So a function that fits the data is

$y = \square \cdot \square^x$

(e) Check that the values of the function you found in part 2(d) agree with the data.

4 The Even-Tempered Clavier

OBJECTIVE To learn how musical scales are related to exponential functions and how exponential functions determine the pitch to which musical instruments are tuned.

Poets, writers, philosophers, and even politicians have extolled the virtues of music—its beauty, its power to communicate emotion, and even its healing power. The philosopher Nietzsche said that "without music life would be a mistake." Perhaps that's why



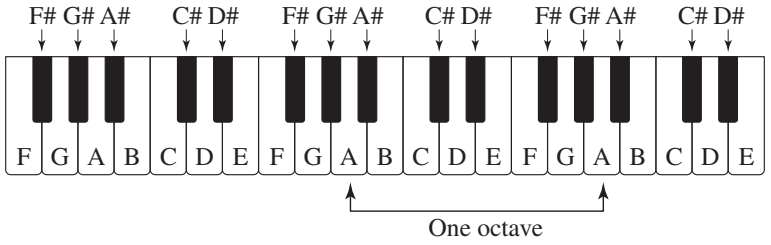
ROBERT VOS/AFP/Getty Images

Classical pianist Lang Lang

some musicians—from rock stars to classical pianists—can command huge fees for performing their art.

What is music, exactly? Is it just "noise" that happens to sound nice? In general, music consists of *tones* played by instruments or sung by voices, either sequentially (as a melody) or simultaneously (as a chord). *Tones*, the building blocks of music, are sounds that have one dominant frequency.

The tones that we are familiar with from our everyday listening can all be reproduced by the white and black keys of any properly tuned piano. The strings of a piano produce specific sound frequencies; for instance, middle A has a fundamental frequency of 440 cycles per second, or 440 Hertz (Hz). The other "A" keys on the piano are either higher-sounding, with frequencies of 880 Hz, 1760 Hz, and so on, or lower-sounding, with frequencies of 220 Hz, 110 Hz, 55 Hz, and so on. To get from one A to the next, you just need to double or halve the frequency. Every pair of notes with the same letter name "sounds" the same to most listeners. Such notes are said to be separated by "octaves" on the musical scale. In this Exploration we learn how exponential functions allow us to create all the notes in the musical scale.



I. Frequencies of Notes

We divide the interval between two notes that are an octave apart into 12 parts in such a way that each note's frequency is a fixed multiple of the preceding one. This frequency interval is called a *semitone* in music theory. Since an octave involves multiplying the frequency by 2, each semitone therefore involves multiplying the frequency by $2^{1/12}$. This means that the frequencies are evenly spaced on a logarithmic scale. The distance between semitones on a logarithmic scale is

$$\log 2^{1/12} = \tfrac{1}{12} \log 2 \approx 0.025$$

So the keys on a piano are "evenly tempered" on a logarithmic scale.

1. The note immediately above middle A is A sharp (A# or B♭), then the note above A# is B, and so on. The frequency of each note is obtained by multiplying 440 (the frequency of middle A) by an appropriate power of $2^{1/12}$. We have

A $\quad 440 \cdot (2^{1/12})^0 = 440$ Hz

A# $\quad 440 \cdot (2^{1/12})^1 \approx 466.164$ Hz

B $\quad 440 \cdot (2^{1/12})^2 \approx 493.883$ Hz

Complete the following table by calculating the frequencies of the notes produced by the keys in one octave on a piano, from middle A (known by piano tuners as A4 or A440) to the next A (called A5).

Key	Frequency
A	440.000
A# B♭	466.164
B	
C	
C# D♭	
D	
D# E♭	
E	
F	
F# G♭	
G	
G# A♭	
A	

2. The lowest-sounding key on a piano is an A (called A0 or Double Pedal A). This key produces a note four octaves below middle A (A440).

(a) What fundamental frequency does the lowest piano key have?

(b) Many people lose the ability to hear low frequencies as they age. Elderly people often can't hear frequencies lower than 40 Hz. Will they be able to hear the fundamental tone produced by the A0 key?

3. The highest A on the piano is A7, three octaves above middle A (A440). The highest-sounding key of all on the piano is the 88th, called C8. Its frequency is three semitones above A7.

(a) What is the frequency of A7?

(b) What is the frequency of C8, the highest key on the piano?

(c) All people with normal hearing can hear sounds up to at least 15,000 Hz. Can these people hear the sound of the highest key on the piano?

II. Intervals, Frequencies, and Dissonance

The "equally tempered" tuning of modern instruments described above did not come into wide use until the late 17th century. Before that, musicians used many tunings that sounded good to their ears and to their listeners'. Johann Sebastian Bach published a set of preludes and fugues in 1722, called *The Well-Tempered Clavier,* which was designed to popularize the new tuning system for keyboard instruments. Each of the pieces in this work was written in one of the 24 major and minor keys in the "well-tempered" tuning. Some listeners found the new tuning harsh and unmusical for reasons that we now explore.

1. When two or more keys on the piano are pressed simultaneously, the resulting mixed sound is called a *chord*. When the ratios of the frequencies of the notes in the chord involve small numbers, such as 2:1, 3:2, or 4:3, the chord sounds pleasant and musical to most ears. But other ratios may sound unpleasant or dissonant.

(a) A perfect fifth is an interval in a chord between two notes whose frequencies are in the ratio 3/2. On the piano, this is approximated by two notes that are seven semitones apart (for instance, C and G). Use your table from Problem I.1 to determine the ratio between the frequencies of G and C. How far does this differ from the ideal ratio of 3/2?

(b) A perfect fourth is an interval between two notes with frequencies in the ratio 4/3. On the piano a fourth is approximated by two notes that are five semitones apart (for instance, C and F). What is the ratio between the frequencies of F and C on the piano? How far does this differ from the ideal ratio of 4/3?

(c) Why do you think some people who are accustomed to perfect fifths and fourths might find the modern tuning of a piano to be dissonant?

2. In an octave the two notes have a frequency ratio of 2/1. So to divide an octave into 12 semitones, the frequency of each note is $2^{1/12}$ times the frequency of the preceding note:

$$\text{frequency of note} = \left(\tfrac{2}{1}\right)^{1/12} \times \text{frequency of preceding note}$$

(a) In a perfect fifth the two notes have a frequency ratio of 3/2. If we divide a perfect fifth into seven semitones, explain why the frequency of each note is $(3/2)^{1/7}$ times the frequency of the preceding note:

$$\text{frequency of note} = \left(\tfrac{3}{2}\right)^{1/7} \times \text{frequency of preceding note}$$

(b) In a perfect fourth the two notes have a frequency ratio of 4/3. To divide a perfect fourth into five semitones, by what factor must the frequency of each note be multiplied to get the frequency of the next note?

$$\text{frequency of note} = \left(\frac{\square}{\square}\right)^{\square} \times \text{frequency of preceding note}$$

(c) Calculate the factors in parts (a) and (b), and note that they are not equal. Explain why this means that we can't tune a piano (with even tempering) so that it has exact perfect fifths as well as exact perfect fourths.

5 Quadratic Functions and Models



© image100/Corbis

5.1 Working with Functions: Shifting and Stretching

5.2 Quadratic Functions and Their Graphs

5.3 Maxima and Minima: Getting Information from a Model

5.4 Quadratic Equations: Getting Information from a Model

5.5 Fitting Quadratic Curves to Data

EXPLORATIONS

1 Transformation Stories
2 Torricelli's Law
3 Quadratic Patterns

Secret connections? When a volleyball is hit, it goes flying through the air, but it always comes back down. The function that describes how the volleyball travels is a quadratic function—it's the function that describes how gravity pulls down on objects. Of course, this same function also determines how high a rocket reaches or how your keys fall when they are dropped. But we'll see in this chapter that a quadratic function also models how long your car tires will last (as a function of the tire pressure) or how much grain a farm produces (as a function of the amount of rainfall). So what's the connection between the volleyball and crop yield? The connection is a secret that has been discovered by using algebra. The volleyball goes up and then down, but so does crop yield; the more rain, the more crop is produced—up to a point. If there is too much rain, crop yield begins to decrease. That these very different processes can be described by the same type of function is the secret to the usefulness of algebra.

5.1 Working with Functions: Shifting and Stretching

- Shifting Graphs Up and Down
- Shifting Graphs Left and Right
- Stretching and Shrinking Graphs Vertically
- Reflecting Graphs

IN THIS SECTION... we learn how certain transformations of a function affect its graph. The transformations that we study are shifting, reflecting, and stretching; these are used in the next section to help us graph quadratic functions.

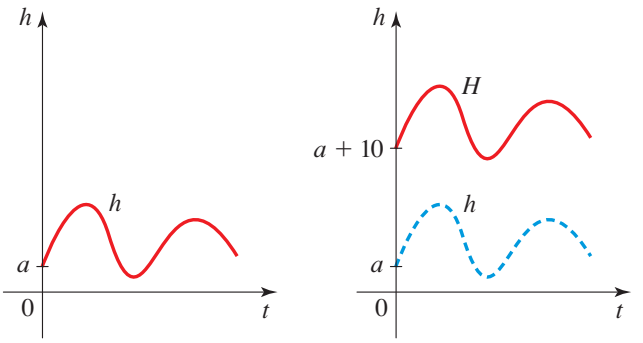
GET READY... by reviewing function notation in Section 1.5.

Shifting Graphs Up and Down



Sonya Etchison/Shutterstock.com 2009

The function h graphed in Figure 1(a) gives the height at time t of the feet of a child jumping on a trampoline, where a is the height of the trampoline. If the trampoline is set on a 10-foot-high platform, the corresponding function H is graphed in Figure 1(b). How are the functions h and H related graphically? How are they related algebraically?



(a) Graph of h (b) Graph of H

figure **1** Height above the ground

We can see that the graph of H is the same as the graph of h but shifted 10 units upward. Algebraically, H is obtained from h by adding 10 to the output values: $H(t) = h(t) + 10$.

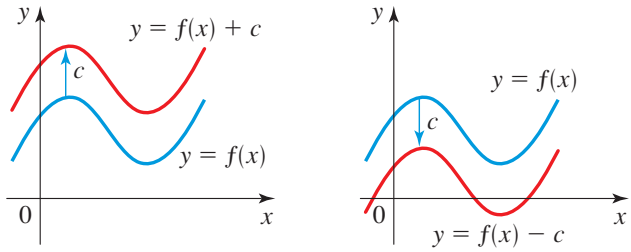
The above example shows that adding a constant to a function shifts its graph vertically. In general, we have the following.

Vertical Shifts of Graphs

Recall that the graph of the function f is the same as the graph of the equation $y = f(x)$.

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.
To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



example **1** **Vertical Shifts of Graphs**

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

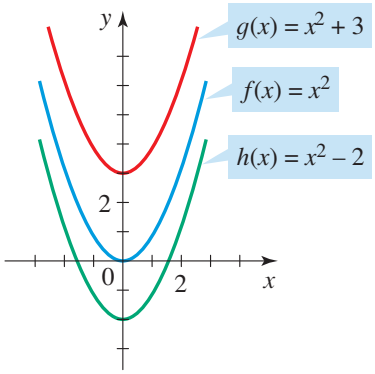


figure **2** Graphs of $g(x) = x^2 + 3$ and $h(x) = x^2 - 2$

Solution

The function $f(x) = x^2$ was graphed in Example 3 in Section 1.6. It is sketched again in Figure 2.

(a) Observe that

$$g(x) = x^2 + 3 = f(x) + 3$$

So the y-coordinate of each point on the graph of g is 3 units above the y-coordinate of the corresponding point on the graph of f. This means that to graph g, we shift the graph of f upward 3 units, as in Figure 2.

(b) Similarly, $h(x) = f(x) - 2$, so to graph h, we shift the graph of f downward 2 units, as shown in Figure 2.

■ **NOW TRY EXERCISE 7** ■

example **2** **Combining Speeds**

A train is traveling at 1320 feet per minute. A man inside the train is walking in the same direction as that in which the train is moving. The man's speed at time t relative to the train is $v(t) = 10\sqrt{t}$ feet per minute. Find a function V that gives the man's speed relative to the ground. How are the graphs of v and V related?

Solution

Since the man and the train are moving in the same direction, we add the train's speed to his speed. So the man's speed relative to the ground is

$$V(t) = 10\sqrt{t} + 1320$$

The graph of V is obtained from the graph of v by shifting up 1320 units.

■ **NOW TRY EXERCISE 65** ■

Shifting Graphs Left and Right

Suppose that Madeleine leaves home for work early in the morning. Her roommate Jennifer leaves an hour later. To find out where each of them is at any given time, we need to model each roommate's distance from home using a *common starting time*. We'll do this in the next example.

example 3 Changing the Starting Time


Kristian Sekulic/Shutterstock.com 2009

Jennifer leaves home an hour after Madeleine and drives at a constant speed of 60 miles per hour on a straight road.

(a) Find a function j that models Jennifer's distance from home t hours after she started her trip.

(b) Find a function J that models Jennifer's distance from home t hours after Madeleine started her trip.

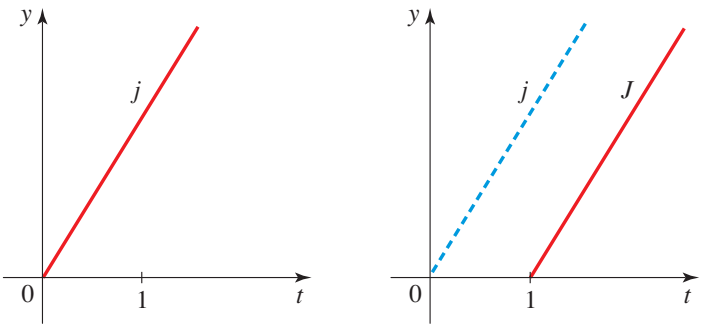
(c) Graph the functions j and J. How are the graphs related?

Solution

(a) Since Jennifer drives at 60 miles per hour, her distance t hours after she started her trip is $j(t) = 60t$.

(b) If we take time 0 to be the time when Madeleine started her trip, then Jennifer should just be starting her trip when t is 1 hour. In other words, in terms of this new starting time, Jennifer's distance at time t is given by $J(t) = 60(t - 1)$ (for $t \geq 1$).

(c) Graphs of these functions are shown in Figure 3. We see that the graph of J is the same as the graph of j but shifted one unit to the right.



(a) Graph of $j(t) = 60t$ (b) Graph of $J(t) = 60(t - 1)$

figure 3 Distance traveled

NOW TRY EXERCISE 67

Let's compare the functions j and J of Example 3:

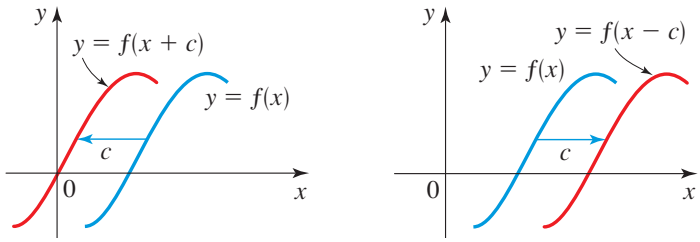
$$j(t) = 60t \qquad J(t) = 60(t - 1)$$

The function J is the same as j evaluated at $t - 1$; that is, $J(t) = j(t - 1)$. As we saw in the example, this means that the graph of J is obtained from the graph of j by shifting 1 unit to the right. In general we have the following.

Horizontal Shifts of Graphs

Suppose $c > 0$.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.
To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.



example 4 Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$

(b) $h(x) = (x - 2)^2$

Solution

We start with the graph of f (see Figure 4).

(a) To graph g, we shift the graph of f to the left 4 units.

(b) To graph h, we shift the graph of f to the right 2 units.

The graphs of g and h are sketched in Figure 4.

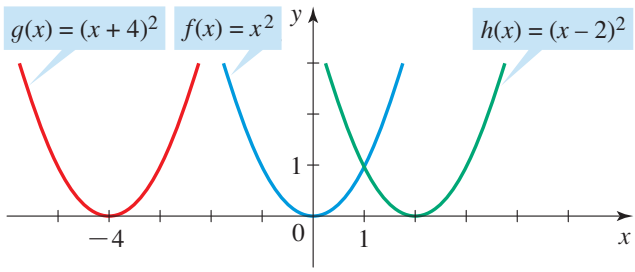


figure 4 Graphs of $g(x) = (x + 4)^2$ and $h(x) = (x - 2)^2$

■ NOW TRY EXERCISE 9 ■

example 5 Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = \sqrt{x - 3} + 4$.

Solution

We start with the graph of $y = \sqrt{x}$ (see Example 4 in Section 1.6, page 66) and shift it to the right 3 units to obtain the graph of $y = \sqrt{x - 3}$. Then we shift the resulting

graph upward 4 units to obtain the graph of $f(x) = \sqrt{x - 3} + 4$ shown in Figure 5.

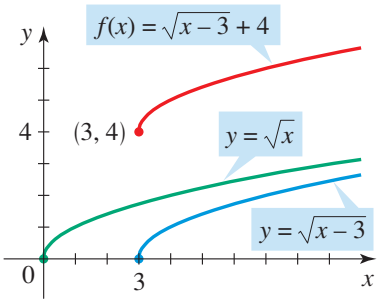


figure **5** Graph of $f(x) = \sqrt{x - 3} + 4$

NOW TRY EXERCISE 13

Stretching and Shrinking Graphs Vertically

In the next example we examine how a function changes if its outputs are multiplied by a fixed constant.

example 6 Doubling the Speed

Alan is on a trip to the dentist. The function $y = f(t)$ graphed in Figure 6 gives his distance from home (in miles) t hours after he started.

(a) Graph the function $y = 2f(t)$. How is the graph of the new function related to the graph of the original function?

(b) How does the trip described by the new function differ from the original trip?

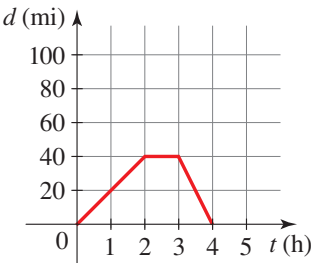


figure **6** Graph of $y = f(t)$

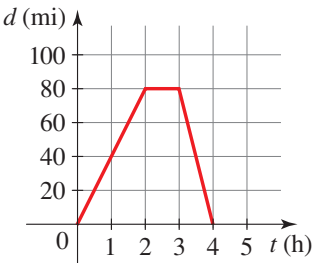


figure **7** Graph of $y = 2f(t)$

Solution

(a) To graph $y = 2f(t)$, we multiply each y-value of the original function by 2. This has the effect of stretching the graph of the original function vertically by a factor of 2. The graph is shown in Figure 7.

(b) In the new trip Alan doubles the distance he drives in any given time period, so he doubles his driving speed.

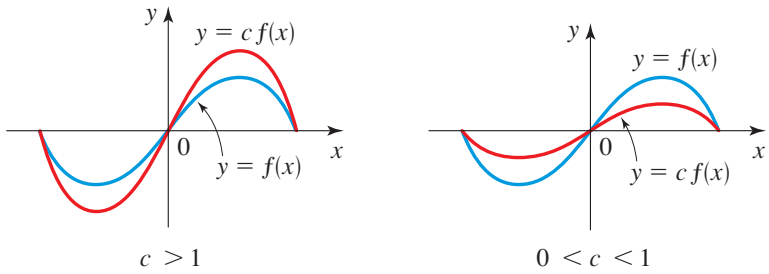
NOW TRY EXERCISE 69

Example 6 is a special case of the following general situation.

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.
If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



example **7** **Vertical Stretching and Shrinking of Graphs**

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ **(b)** $h(x) = \frac{1}{3}x^2$

Solution

We start with the graph of f (see Figure 8).

(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrower parabola in Figure 8.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the wider parabola in Figure 8.

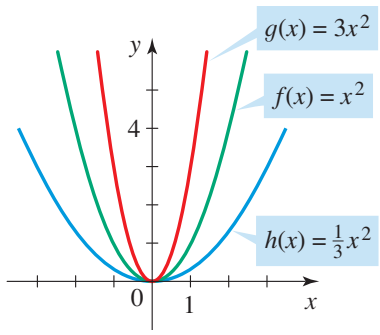


figure **8** Graphs of $g(x) = 3x^2$ and $h(x) = \frac{1}{3}x^2$

■ **NOW TRY EXERCISES 33 AND 35** ■

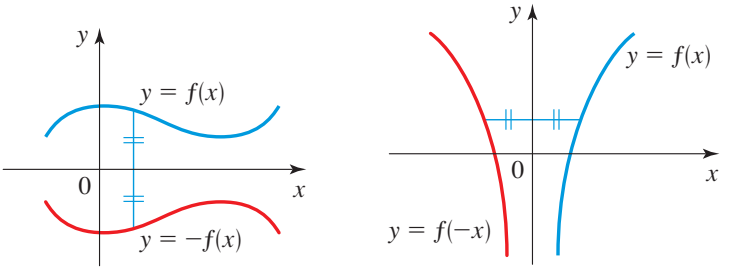
Reflecting Graphs

Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$? The y-coordinate of each point on the graph of $y = -f(x)$ is simply the negative of the y-coordinate of the corresponding point on the graph of

$y = f(x)$. So the desired graph is the reflection of the graph of $y = f(x)$ in the x-axis. On the other hand, the value of $y = f(-x)$ at x is the same as the value of $y = f(x)$ at $-x$, so the desired graph here is the reflection of the graph of $y = f(x)$ in the y-axis. The following box summarizes these observations.

Reflecting Graphs

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



example **8** **Reflecting Graphs**

Sketch the graph of each function.

(a) $f(x) = -x^2$ **(b)** $g(x) = \sqrt{-x}$

Solution

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected in the x-axis (see Figure 9).

(b) We start with the graph of $y = \sqrt{x}$ (see Example 4 in Section 1.6). The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected in the y-axis (see Figure 10). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \le 0\}$.

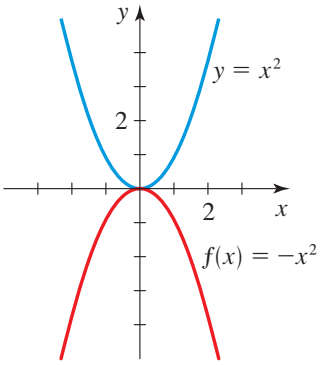


figure **9** Graph of $f(x) = -x^2$

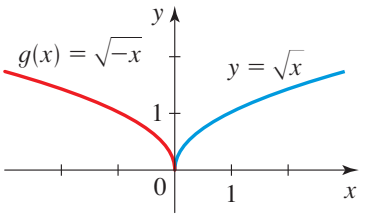


figure **10** Graph of $g(x) = \sqrt{-x}$

■ **NOW TRY EXERCISES 37 AND 39** ■

example **9** **Transformations of Exponential Functions**

Use the graph of $f(x) = 2^x$ to sketch the graph of each function.

(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$

Solution

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward one unit. Notice from Figure 11(a) that the line $y = 1$ is now the horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 11(b).

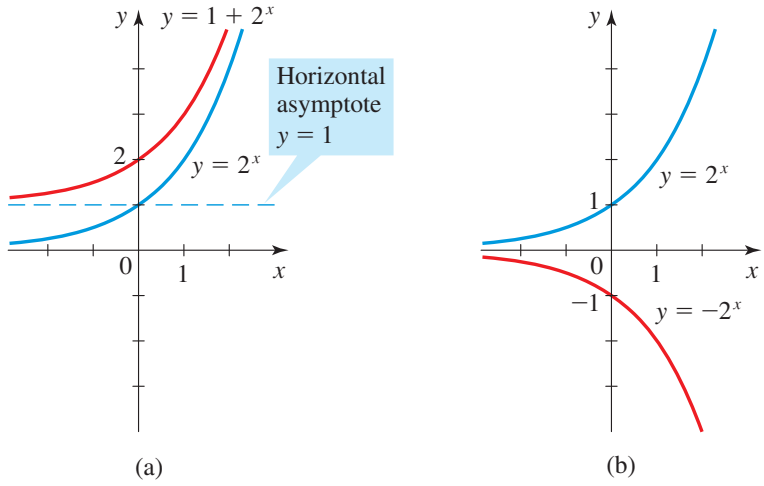


figure **11**

NOW TRY EXERCISES 45 AND 47

example 10 Transformations of Logarithmic Functions

Use the graph of $f(x) = \log_{10} x$ to sketch the graph of each function.

(a) $g(x) = \log_{10}(x - 3)$ **(b)** $h(x) = \log_{10}(-x)$

Solution

(a) To obtain the graph of $g(x) = \log_{10}(x - 3)$, we start with the graph of $f(x) = \log_{10} x$ and shift it to the right 3 units, as shown in Figure 12. Notice that the line $x = 3$ is a vertical asymptote.

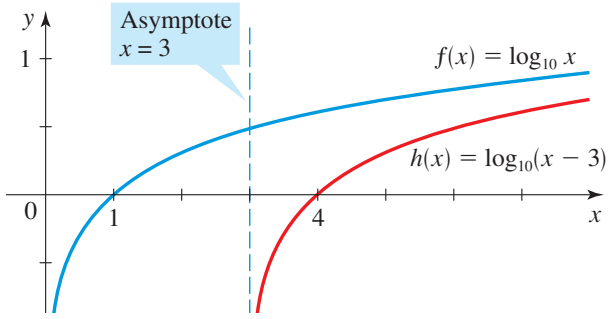


figure **12**

(b) Again we start with the graph of $f(x) = \log_{10} x$, but here we reflect in the y-axis to get the graph of $h(x) = \log_{10}(-x)$ in Figure 13.

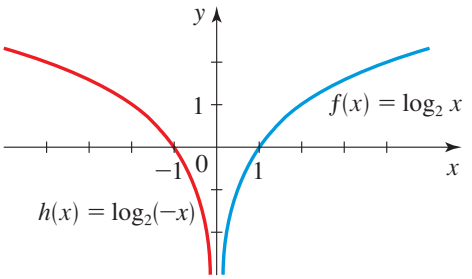


figure **13**

NOW TRY EXERCISES 49 AND 51

5.1 Exercises

CONCEPTS

Fundamentals

1. Fill in the blank with the appropriate direction (left, right, up, or down).

(a) The graph of $y = f(x) + 3$ is obtained from the graph of $y = f(x)$ by shifting _____ 3 units.

(b) The graph of $y = f(x + 3)$ is obtained from the graph of $y = f(x)$ by shifting _____ 3 units.

2. Fill in the blank with the appropriate direction (left, right, up, or down).

(a) The graph of $y = f(x) - 3$ is obtained from the graph of $y = f(x)$ by shifting _____ 3 units.

(b) The graph of $y = f(x - 3)$ is obtained from the graph of $y = f(x)$ by shifting _____ 3 units.

3. Fill in the blank with the appropriate axis (x-axis or y-axis).

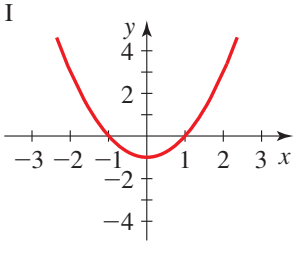
(a) The graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by reflecting in the _____.

(b) The graph of $y = f(-x)$ is obtained from the graph of $y = f(x)$ by reflecting in the _____.

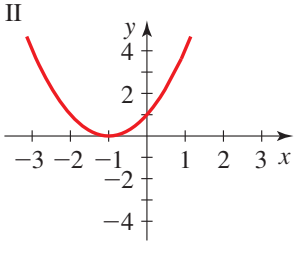
4. Match the graph with the function.

(a) $y = (x + 1)^2$ **(b)** $y = (x - 1)^2$ **(c)** $y = x^2 - 1$ **(d)** $y = -x^2$

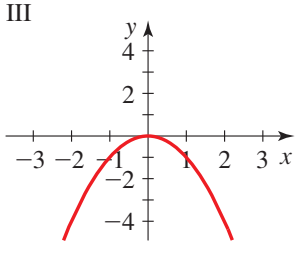
I



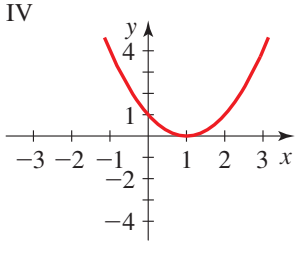
II

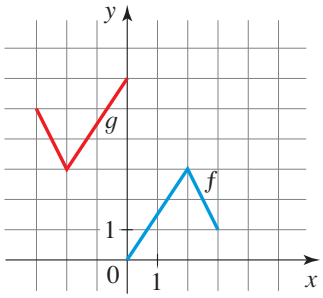


III



IV





Think About It

5–6 ■ Can the function g be obtained from f by transformations? If so, describe the transformations needed.

5. f and g are described algebraically: $f(x) = (x + 2)^2, \quad g(x) = (x - 2)^2 + 5$

6. f and g are described graphically in the figure in the margin.

SKILLS

7–12 ■ Use the graph of $f(x) = x^2$ to graph the following.

7. **(a)** $g(x) = x^2 - 4$ **(b)** $g(x) = x^2 + 2$

8. **(a)** $g(x) = x^2 - 1$ **(b)** $g(x) = x^2 + 6$

9. **(a)** $g(x) = (x + 2)^2$ **(b)** $g(x) = (x - 4)^2$

10. **(a)** $g(x) = (x - 7)^2$ **(b)** $g(x) = (x + 3)^2$

11. **(a)** $g(x) = (x + 2)^2 - 1$ **(b)** $g(x) = (x - 3)^2 + 5$

12. **(a)** $g(x) = (x - 5)^2 - 3$ **(b)** $g(x) = (x + 3)^2 + 2$

13. Use the graph of $f(x) = \sqrt{x}$ to graph the following.

(a) $g(x) = \sqrt{x + 4}$ **(b)** $g(x) = \sqrt{x} + 1$ **(c)** $g(x) = \sqrt{x + 2} + 2$

14. Use the graph of $f(x) = |x|$ (see Example 9 of Section 1.6, page 70) to graph the following.

(a) $g(x) = |x - 3|$ **(b)** $g(x) = |x| + 1$ **(c)** $g(x) = |x - 2| + 2$

15. Use the graph of $f(x) = 3^x$ to graph the following.

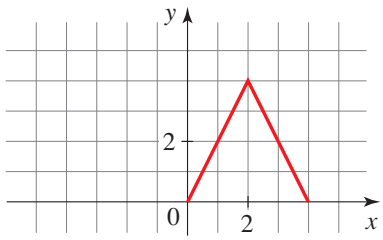
(a) $g(x) = 3^x - 2$ **(b)** $g(x) = -3^x$ **(c)** $g(x) = 1 - 3^x$

16. Use the graph of $f(x) = \log_3 x$ to graph the following.

(a) $g(x) = \log_3(x + 1)$ **(b)** $g(x) = -\log_3 x$ **(c)** $g(x) = -\log_3(x - 4)$

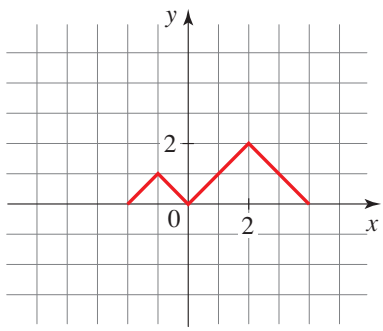
17. The graph of $y = f(x)$ is given. Sketch graphs of the following functions.

(a) $y = f(x - 2)$ **(b)** $y = f(x) - 2$



18. The graph of $y = g(x)$ is given. Sketch graphs of the following functions.

(a) $y = g(x) - 2$ **(b)** $y = -g(x) - 1$



19–24 ■ A function f is given by the table. Complete the table for the given transformation.

19.

x	$f(x)$	$f(x) + 1$
−3	20	21
−2	27	
−1	53	
0	42	
1	39	
2	70	
3	21	

20.

x	$f(x)$	$f(x) - 1$
−3	9	8
−2	28	
−1	15	
0	20	
1	31	
2	27	
3	19	

21.

x	$f(x)$	$f(x + 2)$
−6	105	99
−4	99	
−2	82	
0	53	
2	20	
4	6	
6	2	—

22.

x	$f(x)$	$f(x - 2)$
−6	13	—
−4	29	13
−2	38	
0	49	
2	55	
4	62	
6	75	

23.

x	$f(x)$	$f(-x)$
−6	105	2
−4	99	
−2	82	
0	53	
2	20	
4	6	
6	2	

24.

x	$f(x)$	$1 - f(x)$
−6	105	−104
−4	99	
−2	82	
0	53	
2	20	
4	6	
6	2	

25–28 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a basic function and applying a vertical shift.

25. $f(x) = x^2 - 1$

26. $f(x) = x^3 + 5$

27. $f(x) = \sqrt{x} + 1$

28. $f(x) = |x| - 1$

29–32 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a basic function and applying a horizontal shift.

29. $f(x) = (x - 5)^2$

30. $f(x) = (x + 1)^2$

31. $f(x) = |x - 3|$

32. $f(x) = \sqrt{x + 4}$

33–36 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a basic function and applying vertical stretching or shrinking.

33. $f(x) = \frac{1}{4}x^2$

34. $f(x) = 3|x|$

35. $f(x) = 2x^4$

36. $f(x) = \frac{1}{5}x^3$

37–52 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a basic function and applying transformations.

37. $f(x) = 1 - x^2$

38. $f(x) = -2 - x^2$

39. $f(x) = -|x|$

40. $f(x) = -\sqrt{x}$

41. $f(x) = -(x - 3)^2 + 5$

42. $f(x) = -(x + 4)^2 - 3$

43. $f(x) = 6 - \sqrt{x + 4}$

44. $f(x) = 3 - |x - 2|$

45. $f(x) = 2^x - 3$

46. $f(x) = 3^x + 5$

47. $f(x) = -3^x$

48. $f(x) = 1 + 2^{-x}$

49. $f(x) = \log_2(x - 4)$

50. $f(x) = \log_2(x + 2)$

51. $f(x) = -\log_5(-x)$

52. $f(x) = -\log_{10}(x + 2)$

53–60 ■ A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

53. $f(x) = x^2$; shift upward 3 units

54. $f(x) = x^2$; shift downward 1 unit

55. $f(x) = \sqrt{x}$; shift 2 units to the left

56. $f(x) = \sqrt{x}$; shift 1 unit to the right

57. $f(x) = |x|$; shift 3 units to the right and shift upward 1 unit

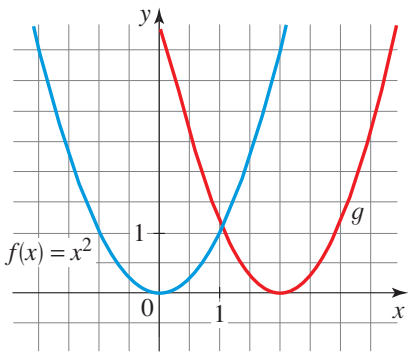
58. $f(x) = x^2$; shift 2 units to the left and reflect in the x-axis

59. $f(x) = 2^x$; shift downward 2 units

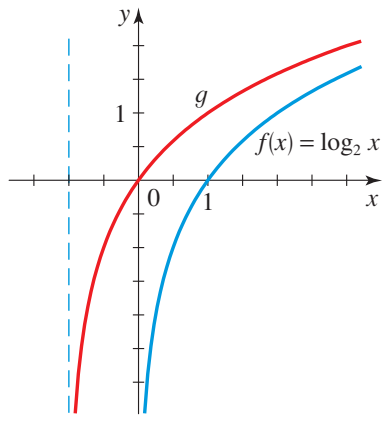
60. $f(x) = \log_3 x$; shift 2 units to the right and reflect in the y-axis

61–64 ■ The graphs of f and g are given. Find a formula for the function g.

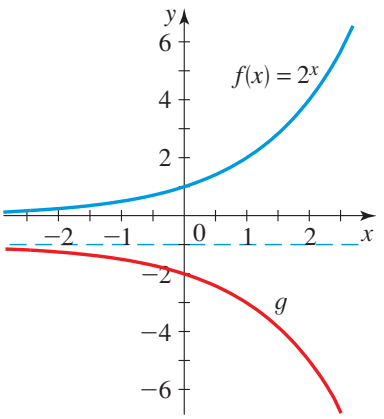
61.



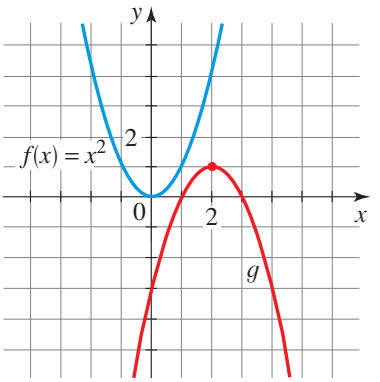
62.



63.



64.



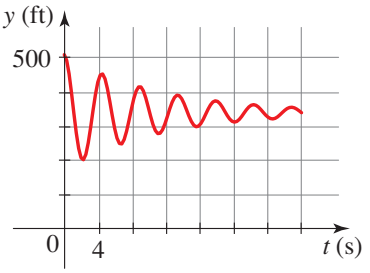
CONTEXTS

65. Bungee Jumping Luisa goes bungee jumping from a 500-foot-high bridge. The graph shows Luisa's height $h(t)$ (in feet) after t seconds.

(a) Describe in words what the graph indicates about Luisa's bungee jump.

(b) Suppose Luisa goes bungee jumping from a 400-foot-high bridge. Sketch a new graph that shows Luisa's height $H(t)$ after t seconds.

(c) What transformation must be performed on the function h to obtain the function H? Express the function H in terms of h.



66. Train Ride Liam is riding a train that is traveling at a constant speed of 20 m/s. Liam is inside the train walking in the direction opposite to that in which the train is moving. Liam's speed relative to the train at time t (in seconds) is $v(t) = 4\sqrt{5t}$ m/s. Find a function V that gives Liam's speed relative to the ground. How are the graphs of v and V related?

67. Distance Alessandra leaves home 30 minutes after her brother Alberto and drives at a constant speed of 90 km/h on a straight road.

(a) Find a function d that models Alessandra's distance from home t hours after she started her trip.

(b) Find a function D that models Alessandra's distance from home t hours after Alberto started his trip.

(c) Graph the functions d and D. How are the graphs related?

68. Sales Growth The annual sales of a certain company can be modeled by the function $f(t) = 4 + 0.01t^2$, where t represents years since 1990 and $f(t)$ is measured in millions of dollars.

(a) What shifting and shrinking operations must be performed on the function $y = t^2$ to obtain the function $y = f(t)$?

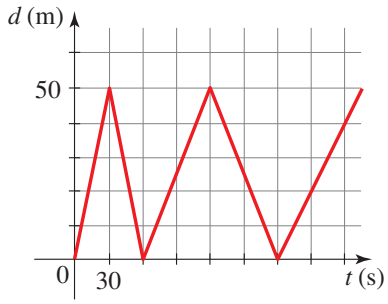
(b) Suppose you want t to represent years since 2000 instead of 1990. What transformation would you have to apply to the function $y = f(t)$ to accomplish this? Write the new function $y = g(t)$ that results from this transformation.

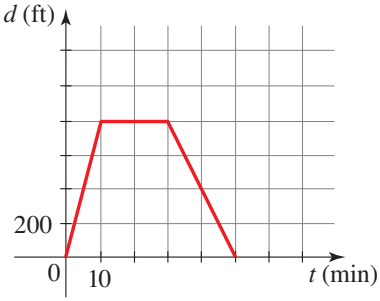
69. Swimming Laps Miyuki practices swimming laps with her team. The function $y = f(t)$ graphed below gives her distance (in meters) from the starting edge of the pool t seconds after she starts her laps.

(a) Describe in words Miyuki's swim practice. What is her average speed for the first 30 seconds?

(b) Graph the function $y = 1.2f(t)$. How is the graph of the new function related to the graph of the original function?

(c) What is Miyuki's new average speed for the first 30 seconds?





70. Field Trip A class of fourth graders walks to a park on a field trip. The function $y = f(t)$ graphed in the margin gives their distance from school (in feet) t minutes after they left school.

(a) What is the average speed going to the park? How long was the class at the park? How far away is the park?

(b) Graph the function $y = 0.50f(t)$. How is the graph of the new function related to the graph of the original function? What is the average speed going to the new park? How far away is the new park?

(c) Graph the function $y = f(t - 10)$. How is the graph of the new function related to the graph of the original function? How does the field trip described by this function differ from the original trip?

71. Bacterial Infection A culture of the bacteria *Streptococcus pneumoniae* (*S. pneumoniae*) initially has 10 bacteria and increases at a rate of 5% per minute.

(a) Find a function $f(t)$ that represents the number of bacteria in the culture after t minutes.

(b) Suppose the culture initially has 50 bacteria and $g(t)$ is the number of bacteria after t minutes. Describe the stretching or shrinking operations that must be performed on the function $y = f(t)$ to obtain the function $y = g(t)$. Write the new function $y = g(t)$ that results from this transformation.



72. Rabbit Population A population of 20 rabbits is introduced to a small island. The population increases at a rate of 60% per year.

(a) Find a function $f(t)$ that represents the number of rabbits on the island after t years.

(b) Suppose 100 rabbits were introduced to the island and $g(t)$ is the number of rabbits after t years. Describe the stretching or shrinking operations that must be performed on the function $y = f(t)$ to obtain the function $y = g(t)$. Write the new function $y = g(t)$ that results from this transformation.

73. Newton's Law of Cooling A cup of coffee has a temperature of 190°F and is placed in a room that has a temperature of 75°F. The temperature of the coffee after t minutes is modeled by the function

$$T(t) = 75 + 115(0.95)^t$$

(a) Describe the transformations you would apply to the function $y = (0.95)^t$ to obtain the function $y = T(t)$.

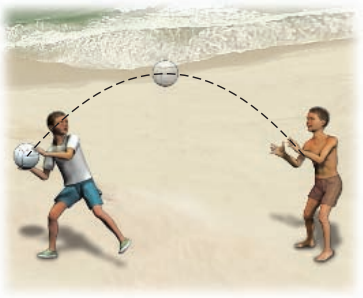
(b) Graph the function T by applying transformations to the graph of $y = (0.95)^t$.

5.2 Quadratic Functions and Their Graphs

- The Squaring Function
- Quadratic Functions in General Form
- Quadratic Functions in Standard Form
- Graphing Using the Standard Form

IN THIS SECTION... we study quadratic functions and compare them to linear functions. We learn how to express quadratic functions in standard form and then use the standard form to graph such functions.

GET READY... by reviewing how to expand expressions and how to complete the square in **Algebra Toolkits B.1** and **B.2**. Test your understanding by doing the **Algebra Checkpoint** at the end of this section.



Every time you throw a ball, it travels according to a quadratic function.

Some of the most important functions in modeling the real world are functions that involve squaring a quantity. Such functions are called *quadratic functions*. For example, you are familiar with the function that gives the area of a square, $A(x) = x^2$, or the area of a circle, $A(r) = \pi r^2$. One of the most fundamental functions in our everyday experience is the quadratic function $d(t) = 16t^2$. This function gives the distance d that a falling object travels in t seconds under the force of gravity. Every time you throw a ball, drop your keys, or ride a roller coaster, you are experiencing this function in action. In this section we begin our study of this important family of functions.

The Squaring Function

A **squaring function** is a function of the form

$$f(x) = Cx^2$$

where C is a number different from zero. We'll see in this section that the squaring function is the prototype quadratic function.

We all know that when we drop an object from a high building, the object falls faster and faster. In fact, the distance d (in feet) it falls in t seconds is given by the function $d(t) = 16t^2$. But suppose that you make a mistake in your physics class and you write this function as $d(t) = 16t$ (not squared). This apparently small error makes a huge difference in reality. If objects fell according to this rule, we would have quite a different world. For one thing, we would be able to leap over tall trees with ease, and we wouldn't need a parachute to go skydiving. We compare these two different functions in the next example.

example 1 Comparing Linear and Squaring Functions

Dave and his friend Harriet (who has superpowers) go skydiving together. In t seconds Dave falls $D(t) = 16t^2$, but Harriet falls $H(t) = 16t$ feet.

(a) Make a table of values for the functions D and H, and find their average rate of change on intervals of length 1 second, for t between 0 and 5 seconds.

(b) What do the tables in part (a) indicate about how fast Dave and Harriet are falling? Which skydiver do you think will have the softer landing?



The average rate of change of distance with respect to time is the average speed. Average speed is studied in Section 2.1.

figure **1** Graphs of D and H

(c) Sketch a graph of the functions D and H. What information can we get from these graphs?

Solution

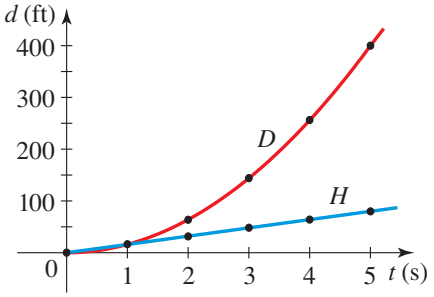
(a) The tables below give the values and average rates of change of the functions D and H.

t	$D(t)$	Rate of change
0	0	—
1	16	16
2	64	48
3	144	96
4	256	128
5	400	160

t	$H(t)$	Rate of change
0	0	—
1	16	16
2	32	16
3	48	16
4	64	16
5	80	16

(b) Since H is a linear function, its rate of change is constant (Section 2.2). So Harriet falls at a constant speed of 16 ft/s. Dave falls a lot faster; from the table we see that his average speed increases on each succeeding one-second interval. Clearly, Harriet will have the softer landing.

(c) The graphs in Figure 1 indicate that the gap between how far Dave has fallen and how far Harriet has fallen keeps widening with passing time.



NOW TRY EXERCISE 43

Example 1 shows that a squaring function grows much faster than a linear function. On the other hand, exponential functions grow much faster than squaring functions, as we will see in Example 1 of Section 6.2, page 494.

Quadratic Functions in General Form

A quadratic function is a combination of a squaring function and a linear function.

Quadratic Functions

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers and $a \neq 0$. This form of a quadratic function is called the **general form**.

example 2 Identifying Quadratic Functions

Determine whether the given function is a quadratic function.

(a) $f(x) = 0.1x^2 + 3$ **(b)** $g(x) = x^3 - 2x + 1$ **(c)** $h(x) = 3^2x + 1$

Solution

(a) Notice that $f(x) = 0.1x^2 + 0 \cdot x + 3$, so f is a quadratic function with $a = 0.1$, $b = 0$, and $c = 3$.

(b) The function g is not a quadratic function, since g has a term involving x^3.

(c) The function h is not a quadratic function, since h does not have a nonzero term involving x^2.

NOW TRY EXERCISE 7

example 3 Expressing a Quadratic Function in General Form

Express the function $f(x) = (2x + 8)(x - 12)$ in the general form of a quadratic function.

Solution

We study how to multiply algebraic expressions using the distributive property in Algebra Toolkit B.1, page T25.

We use the distributive property to multiply the factors of f:

$$\begin{aligned} f(x) &= (2x + 8)(x - 12) && \text{Given function} \\ &= 2x^2 + 8x - 24x - 96 && \text{Distributive property} \\ &= 2x^2 - 16x - 96 && \text{Combine like terms} \end{aligned}$$

The general form is $f(x) = 2x^2 - 16x - 96$.

NOW TRY EXERCISE 9

example 4 Expressing a Quadratic Function in General Form

Express the function $f(x) = 3(x - 4)^2 + 1$ in the general form of a quadratic function.

Solution

We multiply out the squared term of f using Special Product Formula 1 (see **Algebra Toolkit B.1**):

$$\begin{aligned} f(x) &= 3(x - 4)^2 + 1 && \text{Given function} \\ &= 3(x^2 - 8x + 16) + 1 && \text{Special Product Formula 1} \\ &= 3x^2 - 24x + 48 + 1 && \text{Multiply} \\ &= 3x^2 - 24x + 49 && \text{Combine constant terms} \end{aligned}$$

The general form is $f(x) = 3x^2 - 24x + 49$.

NOW TRY EXERCISE 13

Quadratic Functions in Standard Form

In Example 4 we saw that the function $f(x) = 3(x - 4)^2 - 1$ can be expressed in the general form of a quadratic function. This is an example of a quadratic function in standard form according to the following definition.

Standard Form of a Quadratic Function

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square.

The next example shows how "completing the square" allows us to express a quadratic function in standard form.

example 5 Expressing Quadratic Functions in Standard Form

Express the quadratic function $f(x) = x^2 + 16x + 24$ in standard form.

Solution

To get f in standard form we "complete the square" by first taking half of the coefficient of x and squaring it: $\left(\frac{16}{2}\right)^2 = 64$. We then add and subtract the result.

$$\begin{aligned} f(x) &= x^2 + 16x + 24 && \text{Given function} \\ &= \underbrace{(x^2 + 16x + 64)}_{\text{perfect square}} - 64 + 24 && \text{Complete the square: Add 64 inside parentheses, and subtract 64 outside} \\ &= (x + 8)^2 - 40 && \text{Factor and simplify} \end{aligned}$$

The standard form is $f(x) = (x + 8)^2 - 40$.

We study how to factor perfect squares in Algebra Toolkit B.2, page T33.

■ NOW TRY EXERCISE 21 ■

example 6 Expressing a Quadratic Function in Standard Form

Express the quadratic function $f(x) = 2x^2 - 12x + 23$ in standard form.

Solution

Since the coefficient of x^2 is not 1, we must factor this coefficient from the terms involving x *before* completing the square.

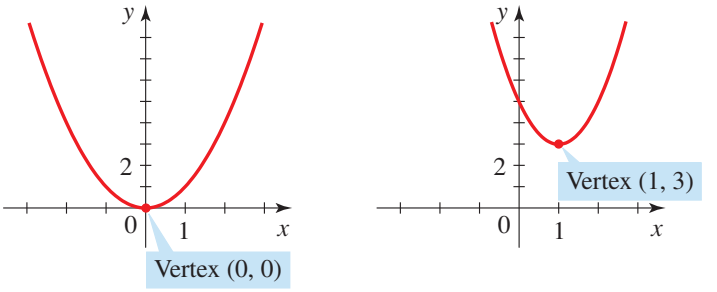
$$\begin{aligned} f(x) &= 2x^2 - 12x + 23 && \text{Given function} \\ &= 2(x^2 - 6x) + 23 && \text{Factor 2 from } x\text{-terms} \\ &= 2\underbrace{(x^2 - 6x + 9)}_{\text{perfect square}} - 2 \cdot 9 + 23 && \text{Complete the square: Add 9 inside parentheses, and subtract } 2 \cdot 9 \text{ outside} \\ &= 2(x - 3)^2 + 5 && \text{Factor and simplify} \end{aligned}$$

The standard form is $f(x) = 2(x - 3)^2 + 5$.

■ NOW TRY EXERCISE 23 ■

Graphing Using the Standard Form

Recall that the graph of the squaring function $f(x) = x^2$ is a U-shaped curve called a **parabola**. The **vertex** of this parabola is at the point (0, 0) (see Example 3 in Section 1.6). The graph of f is shown in Figure 2(a).



(a) Graph of $f(x) = x^2$ (b) Graph of $f(x) = 2(x - 1)^2 + 3$

figure **2**

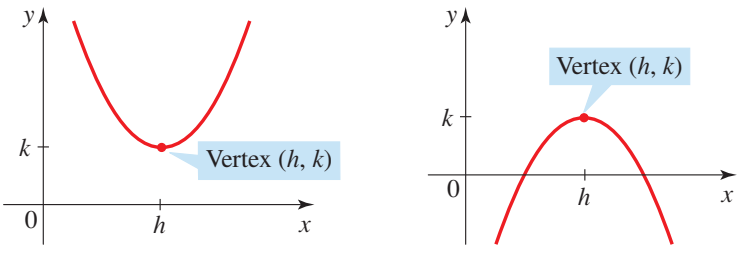
If we transform f by shifting 1 unit to the right, stretching by a factor of 2, and then shifting upward 3 units, we get the function

$$f(x) = 2(x - 1)^2 + 3$$

We recognize this as a quadratic function in standard form (see Figure 2(b)). Since any quadratic function can be put into standard form, we see that the graph of any quadratic function is a parabola obtained by transforming the basic squaring function $f(x) = x^2$. Notice that the standard form of a quadratic function allows us to quickly identify the location of the vertex. In this example the vertex is at (1, 3) because the original vertex (0, 0) has been shifted 1 unit to the right and 3 units upward.

Graphs of Quadratic Functions

The graph of the quadratic function $f(x) = a(x - h)^2 + k$ in standard form is a **parabola** with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k, \quad a > 0$ $f(x) = a(x - h)^2 + k, \quad a < 0$

To graph a quadratic function, we first express it in standard form.

example 7 Graphing a Quadratic Function

Let $f(x) = 2x^2 - 12x + 23$.

(a) Express f in standard form.

(b) Sketch a graph of f.

Solution

(a) In Example 6 we found that the standard form of f is

$$f(x) = 2(x - 3)^2 + 5$$

(b) The standard form tells us that we get the graph of f by taking the parabola $y = x^2$ and performing the following transformations: Shift to the right 3 units, stretch by a factor of 2, then shift upward 5 units. The vertex of the parabola is at (3, 5). The parabola opens upward because the coefficient 2 is positive.

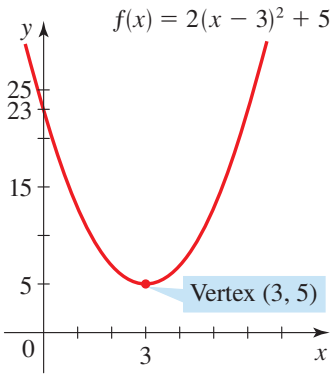


figure 3

Vertex (3, 5)

$$f(x) = 2(x - 3)^2 + 5$$

Opens upward

The y-intercept is obtained by evaluating f at 0. This is more easily done by using the general form of a quadratic function. In this case

$$f(0) = 2(0)^2 - 12(0) + 23 = 23$$

So the y-intercept is 23. The graph is shown in Figure 3.

NOW TRY EXERCISE 29

example 8 Graphing a Quadratic Function

Let $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch the graph of f.

Solution

(a) To express this quadratic function in standard form, we complete the square.

$$\begin{aligned} f(x) &= -x^2 + x + 2 && \text{Given function} \\ &= -(x^2 - x) + 2 && \text{Factor } -1 \text{ from the } x\text{-terms} \\ &= -\underbrace{\left(x^2 - x + \tfrac{1}{4}\right)}_{\text{perfect square}} + 2 - (-1)\cdot\tfrac{1}{4} && \text{Complete the square: Add } \tfrac{1}{4} \text{ inside parentheses, and subtract } \left(-1\cdot\tfrac{1}{4}\right) \text{ outside} \\ &= -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{9}{4} && \text{Factor and simplify} \end{aligned}$$

(b) From the standard form we see that the graph is a parabola with vertex $\left(\frac{1}{2}, \frac{9}{4}\right)$.

Vertex $\left(\frac{1}{2}, \frac{9}{4}\right)$

$$f(x) = -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{9}{4}$$

Opens downward

The parabola opens downward because the coefficient of the square term is -1, which is negative. Since $f(0) = 2$, the y-intercept is 2. The graph is shown in Figure 4.

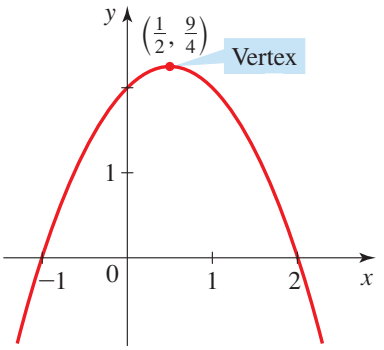


figure **4** Graph of $f(x) = -x^2 + x + 2$

NOW TRY EXERCISE 33

example **9** **Finding a Quadratic Function from Its Graph**

Find the quadratic function f whose graph is shown in Figure 5.

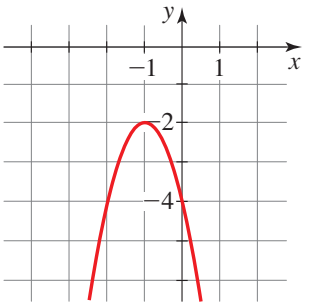


figure **5**

Solution

We observe from the graph that the vertex is $(-1, -2)$, so replacing h by -1 and k by -2 in the standard form of a quadratic function, we have

$$f(x) = a(x - h)^2 + k \qquad \text{Standard form}$$
$$f(x) = a(x - (-1))^2 - 2 \qquad \text{Replace } h \text{ by } -1, k \text{ by } -2$$
$$f(x) = a(x + 1)^2 - 2 \qquad \text{Simplify}$$

We also observe from the graph that the y-intercept is -4, and hence $f(0)$ is -4. We can use this to find the value of a:

$$f(0) = a(0 + 1)^2 - 2 \qquad \text{Replace } x \text{ by } 0$$
$$-4 = a - 2 \qquad \text{Replace } f(0) \text{ by } -4$$
$$a = -2 \qquad \text{Add 2 and switch sides}$$

So the function we seek is $f(x) = -2(x + 1)^2 - 2$.

NOW TRY EXERCISE 39

Algebra Checkpoint ✓

Check your knowledge of expanding quadratic expressions and completing the square by doing the following problems. You can review these topics in **Algebra Toolkits B.1** and **B.2** on pages T25 and T33.

1. Expand each expression.

(a) $(x + 2)(x - 3)$ **(b)** $(2t - 5)(t + 1)$

(c) $(3r - 2)(2r - 3)$ **(d)** $(6 + 3v)(6 - 3v)$

2. Determine whether the given expression is a perfect square.

(a) $x^2 + 6x + 9$ **(b)** $t^2 - 10t + 100$ **(c)** $t^2 - 20t + 100$ **(d)** $w^2 + 2w + \frac{4}{9}$

3. Factor the given perfect square.

(a) $x^2 + 14x + 49$ **(b)** $t^2 - 16t + 64$ **(c)** $w^2 - w + \frac{1}{4}$ **(d)** $s^2 + 5s + \frac{25}{4}$

4. Add the appropriate constant to make a perfect square.

(a) $x^2 + 10x + ____ = (x + ____)^2$

(b) $t^2 - 20t + ____ = (t - ____)^2$

(c) $3u^2 + 18u + ____ = 3(u + ____)^2$

(d) $4v^2 - 20v + ____ = 4(v - ____)^2$

5. Add and subtract the appropriate constants to complete the square.

(a) $x^2 + 6x = (x^2 + 6x + ____) - ____$

(b) $y^2 - y = (y^2 - y + ____) - ____$

(c) $2w^2 + 8w = 2(w^2 + 4w + ____) - ____$

(d) $3s^2 - s + 1 = 3(s^2 - \frac{1}{3}s + ____) - ____ + 1$

5.2 Exercises

CONCEPTS

Fundamentals

1. To put the quadratic function $f(x) = ax^2 + bx + c$ in standard form, we complete the _____.

2. The quadratic function $f(x) = a(x - h)^2 + k$ is in standard form.

(a) The graph of f is a parabola with vertex (_____, _____).

(b) If $a > 0$, the graph of f opens _____ (upward/downward).

(c) If $a < 0$, the graph of f opens _____ (upward/downward).

3. The graph of $f(x) = 2(x - 3)^2 + 5$ is a parabola that opens _____, with its vertex at (_____, _____).

4. The graph of $f(x) = -2(x + 3)^2 + 5$ is a parabola that opens _____, with its vertex at (_____, _____).

Think About It

5. Consider the quadratic function $y = (x - 2)(x - 4)$.

(a) In general form, $y =$ _____.

(b) From the general form we see that the y-intercept is _____.

6. Consider the quadratic function $y = (x - m)(x - n)$.

(a) In general form, $y =$ _____.

(b) From the general form we see that the y-intercept is _____.

(c) Using the formula from part (b), we see that the y-intercept of $y = (x - 3)(x - 5)$ is _____.

SKILLS

7–8 ■ Determine whether the given function is a quadratic function.

7. **(a)** $f(x) = \frac{1}{2}x^2 + 2x + 1$ **(b)** $g(x) = 3x^3 + 2x^2 + 1$ **(c)** $f(x) = 5^2x + 1$

8. **(a)** $f(x) = \sqrt{2}x^2 - 3x + \sqrt{2}$ **(b)** $g(x) = 5x^4 + 2x^2 + x$ **(c)** $f(x) = 0.7x^2 + \sqrt{5}$

9–16 ■ Express the function in the general form of a quadratic function.

9. $f(x) = (2x + 3)(4x - 1)$

10. $f(x) = (5x - 7)(3x - 2)$

11. $f(x) = (2x + 5)(2x - 5)$

12. $f(x) = (3x - 2)(3x + 2)$

13. $f(x) = (x - 1)^2 + 5$

14. $f(x) = (x + 2)^2 - 3$

15. $f(x) = 4(5x + 3)^2 + 7$

16. $f(x) = -(4x - 1)^2 + 10$

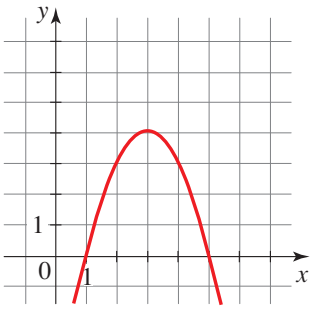
17–20 ■ A quadratic function f and its graph are given.

(a) Express the function in standard form.

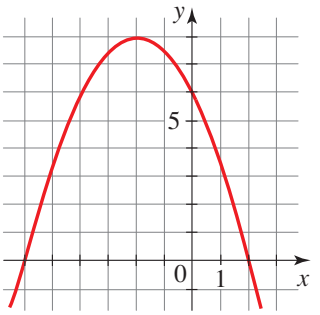
(b) Find the coordinates of the vertex from the standard form.

(c) Confirm your answer to part (b) from the graph.

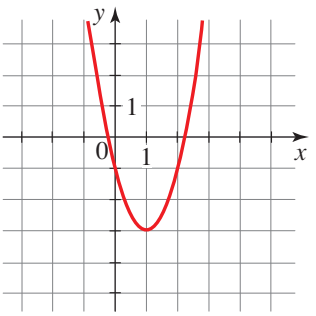
17. $f(x) = -x^2 + 6x - 5$



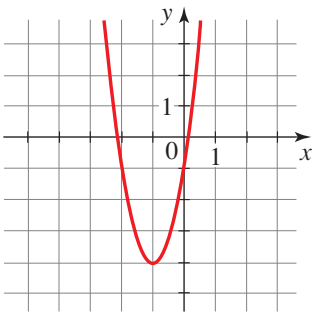
18. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



19. $f(x) = 2x^2 - 4x - 1$



20. $f(x) = 3x^2 + 6x - 1$



21–24 ■ Express the function in the standard form of a quadratic function.

21. $f(x) = x^2 + 2x - 5$

22. $f(x) = x^2 - 4x + 2$

23. $f(x) = 2x^2 + 20x + 1$

24. $f(x) = 3x^2 - 18x - 5$

25–34 ■ A quadratic function is given.

(a) Express the quadratic function in standard form.

(b) Find its vertex.

(c) Sketch its graph.

25. $f(x) = x^2 - 6x$

26. $f(x) = x^2 + 8x$

27. $f(x) = -x^2 + 10x$

28. $f(x) = 2x^2 + 6x$

29. $f(x) = 2x^2 + 4x + 3$

30. $f(x) = x^2 - 2x + 2$

31. $f(x) = -x^2 + 6x + 4$

32. $f(x) = -x^2 - 4x + 4$

33. $f(x) = -x^2 - 3x + 3$

34. $f(x) = 2x^2 + 10x - 1$

35–38 ■ Find a function f whose graph is a parabola with the given vertex and that passes through the given point.

35. Vertex $(-2, -3)$; passes through the point $(-1, -8)$

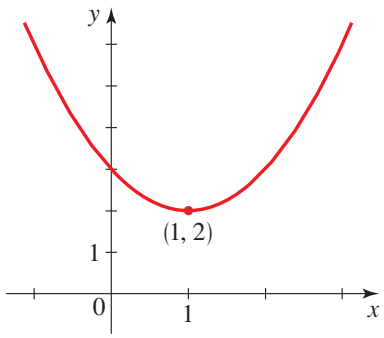
36. Vertex $(1, -2)$; passes through the point $(4, 16)$

37. Vertex $(3, 4)$; passes through the point $(1, -8)$

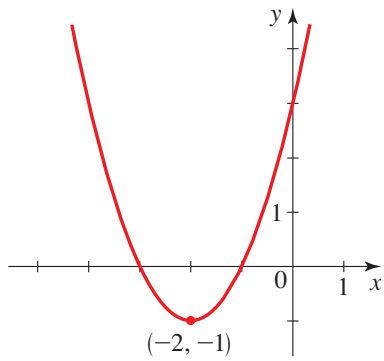
38. Vertex $(-5, 6)$; passes through the point $(-3, 30)$

39–42 ■ Find a quadratic function f whose graph is shown.

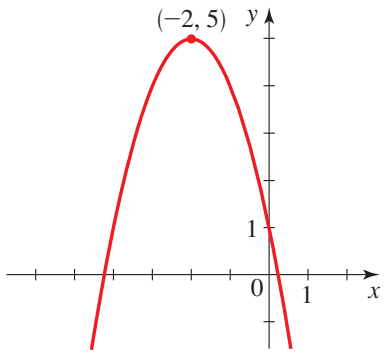
39.



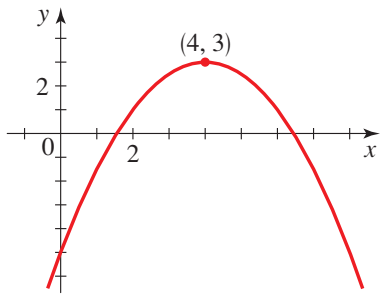
40.



41.



42.



43–44 ■ A linear function f and a quadratic function g are given.

(a) Complete the table, and find the average rate of change of each function on intervals of length 1 for x between 0 and 5.

(b) Sketch a graph of each function.

43. $f(x) = 3x$; $g(x) = 3x^2$

x	$f(x)$	Rate of change
0	0	—
1	3	3
2		
3		
4		
5		

x	$g(x)$	Rate of change
0	0	—
1	3	3
2		
3		
4		
5		

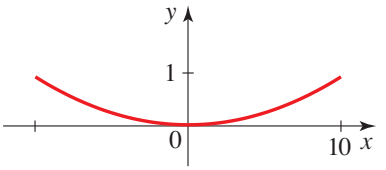
44. $f(x) = 20x$; $g(x) = 15x^2$

x	$f(x)$	Rate of change
0	0	—
1	20	20
2		
3		
4		
5		

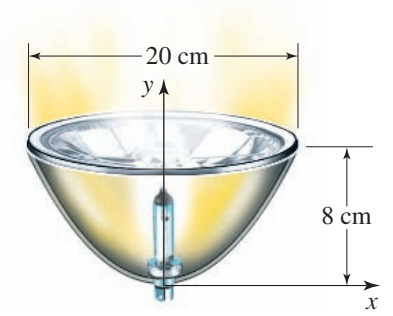
x	$g(x)$	Rate of change
0	0	—
1	15	15
2		
3		
4		
5		

CONTEXTS

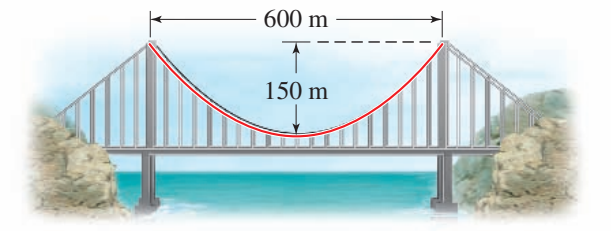
45. Satellite Dish A reflector for a satellite dish is parabolic in cross section. The reflector is 1 ft deep and 20 ft wide from rim to rim (see the figure). Find a quadratic function that models the parabolic part of the dish.



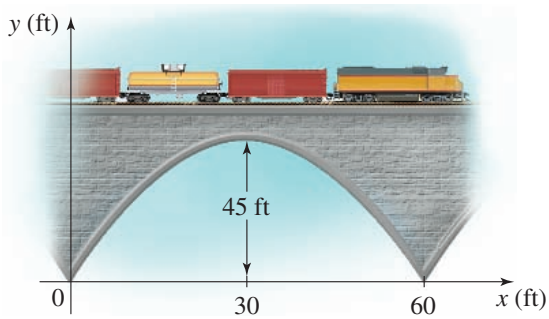
46. Parabolic Reflector A headlamp of a car headlight has a reflector that is parabolic in cross section. The reflector is 8 cm deep and 20 cm wide from rim to rim, as shown in the figure. Find a quadratic function that models the parabolic part of the reflector, placing the origin of the coordinate axes at the vertex.



47. Suspension Bridge In a suspension bridge, the shape of the suspension cables is parabolic. The bridge shown in the figure has towers that are 600 m apart, and the lowest point of the suspension cables is 150 m below the top of the towers, as shown in the figure. Find a quadratic function that models the parabolic part of the cables, placing the origin of the coordinate axes at the vertex.



48. Parabolic Truss Bridge The shape of a suspension bridge can be turned upside down to create a bridge where each supporting truss is in the shape of a parabola. The truss of the bridge shown in the figure is 45 feet high at its highest point, and the two points where it touches the ground are 60 feet apart, as shown in the figure. Find a quadratic function that models the parabolic truss.



5.3 Maxima and Minima: Getting Information from a Model



istockphoto.com/Robert Launch

Amateur rocket science

- **Finding Maximum and Minimum Values**
- **Modeling with Quadratic Functions**

IN THIS SECTION... we learn how to find the maximum or minimum value of a quadratic function. We then model real-world situations with quadratic functions and use the maximum or minimum value to get information about the thing being modeled.

The human race has always been fascinated by projectiles—throwing balls, hurling spears, or launching rockets. Today, interest in rocketry is very high, as is indicated by the large number of amateur rocket clubs. One of the main questions in "rocket science" is: How high will a rocket reach?

In Section 4.2 we observed that gravity pulls down on objects according to a squaring rule: An object falls $16t^2$ feet in t seconds. Now suppose a rocket is launched at an initial speed of 800 ft/s. If there were no gravity, the rocket would continue traveling at this same speed indefinitely, so it would reach a height of $800t$ feet after t seconds. But because of the pull of gravity, the rocket reaches a height of only $h = 800t - 16t^2$, where h is measured in feet and t in seconds. Now, to find the

maximum height the rocket reaches, we need to find the maximum value of the quadratic function h.

- **The model** gives us the height the rocket reaches at any time.
- **Our goal** is to find the maximum height the rocket reaches.

In this section we solve this problem and explore several other real-world problems that require us to find a maximum (or minimum) value of a quadratic function (see Example 2). So we begin by developing algebraic formulas for finding these values.

Finding Maximum and Minimum Values

The maximum value of a function occurs at the highest point on the graph of the function, and the minimum value occurs at the lowest point on the graph of the function. So to find the maximum value (or minimum value) of a quadratic function $f(x) = ax^2 + bx + c$, we look at the graph of the function. If $a > 0$, then the graph of f is a parabola that opens upward, so the function has a minimum value at the vertex (see Figure 1(a)). On the other hand, if $a < 0$, then the graph of f opens downward, so the function has a maximum value at the vertex (see Figure 1(b)). If f is expressed in standard form $f(x) = a(x - h)^2 + k$, then the vertex is (h, k), and the maximum or minimum value of f occurs at $x = h$. Also,

- If $a > 0$, then the minimum value of f is $f(h) = k$.
- If $a < 0$, then the maximum value of f is $f(h) = k$.

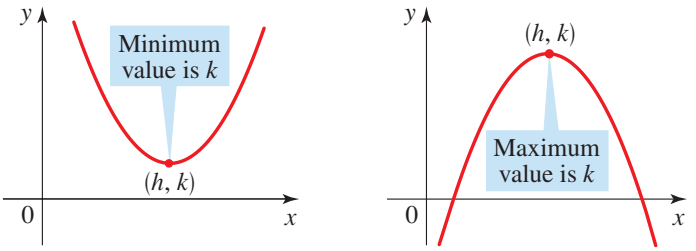


figure **1** (a) (b)

If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{Factor } a \text{ from the } x\text{-terms} \end{aligned}$$

To complete the square, we add $\frac{b^2}{4a^2}$ inside the parentheses and subtract $a\left(\frac{b^2}{4a^2}\right)$ outside. We get

$$\begin{aligned} f(x) &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \qquad \text{Complete the square} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \qquad \text{Factor} \end{aligned}$$

This equation is in standard form with $h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$.

Since the maximum or minimum value occurs at $x = h$, we have the following result.

Maximum or Minimum Value of a Quadratic Function

The x-coordinate of the maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\dfrac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\dfrac{b}{2a}\right)$.

example **1** **Finding Maximum and Minimum Values of Quadratic Functions**

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ **(b)** $f(x) = -2x^2 + 4x - 5$

Solution

(a) This is a quadratic function where a is 1 and b is 4. Thus the maximum or minimum value occurs at

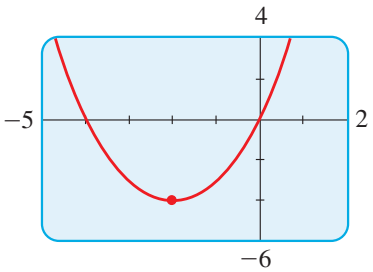
$$x = -\frac{b}{2a} \qquad \text{Formula}$$

$$= -\frac{4}{2 \cdot 1} \qquad \text{Replace } a \text{ by 1 and } b \text{ by 4}$$

$$= -2 \qquad \text{Calculate}$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$



The minimum value occurs at $x = -2$.

(b) This is a quadratic function where a is -2, b is 4, and c is -5. Thus the maximum or minimum value occurs at

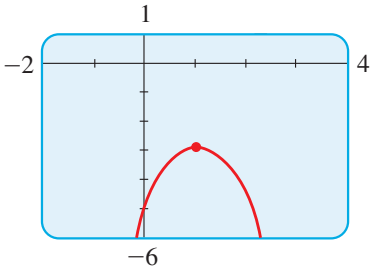
$$x = -\frac{b}{2a} \qquad \text{Formula}$$

$$= -\frac{4}{2(-2)} \qquad \text{Replace } a \text{ by } -2 \text{ and } b \text{ by 4}$$

$$= 1 \qquad \text{Calculate}$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$



The maximum value occurs at $x = 1$.

■ NOW TRY EXERCISES 9 AND 11 ■

Modeling with Quadratic Functions

Now we study real-world phenomena that are modeled by quadratic functions.

example 2 Maximum Height for a Model Rocket

A model rocket is shot straight upward with an initial speed of 800 ft/s, and the height of the rocket is given by

$$h = 800t - 16t^2$$

where h is measured in feet and t in seconds. What is the maximum height of the rocket, and after how many seconds is this height attained?

Solution

The function h is a quadratic function where a is -16 and b is 800. Thus the maximum value occurs when

$$t = -\frac{b}{2a} \qquad \text{Formula}$$

$$= -\frac{800}{2(-16)} \qquad \text{Replace } a \text{ by } -16 \text{ and } b \text{ by } 800$$

$$= 25 \qquad \text{Calculate}$$

The maximum value is $h = -16(25)^2 + 800(25) = 10{,}000$. So the rocket reaches a maximum height of 10,000 feet after 25 seconds.

NOW TRY EXERCISE 23

example 3 Maximum Gas Mileage for a Car

Most cars get their best gas mileage when traveling at a relatively moderate speed. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{28}s^2 + 3s - 31 \qquad 15 \le s \le 70$$

where s is the speed in miles per hour and M is measured in miles per gallon. What is the car's best mileage, and at what speed is it attained?

Solution

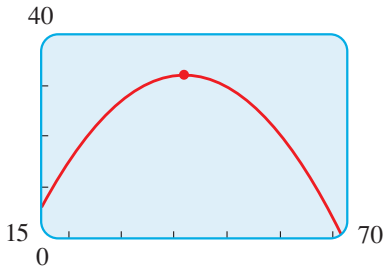
The function M is a quadratic function where a is $-\frac{1}{28}$ and b is 3. Thus the maximum value occurs when

$$s = -\frac{b}{2a} \qquad \text{Formula}$$

$$= -\frac{3}{2\left(-\frac{1}{28}\right)} \qquad \text{Replace } a \text{ by } -\tfrac{1}{28} \text{ and } b \text{ by } 3$$

$$= 42 \qquad \text{Calculate}$$

The maximum value is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal, when it is traveling at 42 mi/h.



The maximum gas mileage occurs at 42 mi/h.

NOW TRY EXERCISE 25

example 4 Fencing a Garden

A gardener has 140 feet of fencing to fence in a rectangular vegetable garden.

(a) Find a function that models the area of the garden in terms of the width of the garden.

(b) Find the largest area she can fence and the dimensions of that area.

Solution



figure **2** Area is $A(x) = x(70 - x) = 70x - x^2$

(a) In Example 5 of Section 1.8 (page 94) we found that the function that models the area she can fence is

$$A(x) = 70x - x^2$$

where x is the width of the garden in feet, and $70 - x$ is the length in feet. (See Figure 2.)

(b) In Section 1.8 we found the maximum value of the function A graphically. Here we find the maximum area algebraically.

The function A is a quadratic function where a is -1 and b is 70. Thus the maximum value occurs when

$$x = -\frac{b}{2a} \qquad \text{Formula}$$

$$= -\frac{70}{2(-1)} \qquad \text{Replace } a \text{ by } -1 \text{ and } b \text{ by } 70$$

$$= 35 \qquad \text{Calculate}$$

Compare with graphical solution in Example 5 of Section 1.8 (page 94).

When the width is 35, the length is $70 - 35 = 35$, and the maximum area is $A(35) = 70(35) - (35)^2 = 1225$. So the largest area she can fence is 1225 square feet, and the dimensions are 35 by 35 feet.

NOW TRY EXERCISE 27

example 5 Maximum Revenue from Ticket Sales

A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

Solution

(a) The model that we want is a function that gives the revenue for any ticket price.

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$

There are two varying quantities: ticket price and attendance. Since the function we want depends on price, we let

$$x = \text{ticket price}$$

Next, we express attendance in terms of x.

In Words	In Algebra
Ticket price	x
Amount ticket price is lowered	$14 - x$
Increase in attendance	$1000(14 - x)$
Attendance	$9500 + 1000(14 - x)$

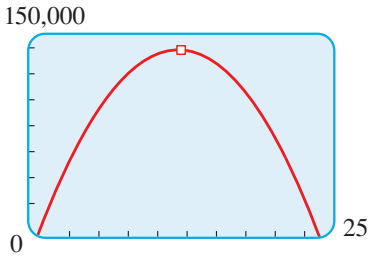
The model that we want is the function R that gives the revenue for a given ticket price x.

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$
$$R(x) = x \times [9500 + 1000(14 - x)]$$
$$R(x) = x(23{,}500 - 1000x)$$
$$R(x) = 23{,}500x - 1000x^2$$

(b) Since R is a quadratic function where a is -1000 and b is 23,500, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{23{,}500}{2(-1000)} = 11.75$$

So a ticket price of \$11.75 gives the maximum revenue.



Maximum revenue occurs when the ticket price is \$11.75.

NOW TRY EXERCISE 33

5.3 Exercises

CONCEPTS

Fundamentals

1. The quadratic function $f(x) = ax^2 + bx + c$ is in general form.

(a) The maximum or minimum value of f occurs at $x = \frac{\square}{\square}$

(b) If $a > 0$, then f has a _____ (maximum/minimum) value.

(c) If $a < 0$, then f has a _____ (maximum/minimum) value.

2. (a) The quadratic function $f(x) = 2x^2 - 12x + 5$ has a _____ (maximum/minimum) value of $f\left(\frac{\square}{\square}\right)$ = _____.

(b) The quadratic function $f(x) = -2x^2 - 12x + 5$ has a _____ (maximum/minimum) value of $f\left(\frac{\square}{\square}\right)$ = _____.

Think About It

3. Consider the quadratic function $y = (x - 2)(x - 4)$.

(a) In general form, y = _____.

(b) The graph is a parabola that opens _____ (up/down).

(c) From the general form we see that the minimum value occurs at $x = \frac{\square}{\square}$ = _____.

4. Consider the quadratic function $y = (x - m)(x - n)$.

(a) In general form, $y =$ _____.

(b) The graph is a parabola that opens _____ (up/down).

(c) From the general form we see that the minimum value occurs at $x =$ $\frac{\square}{\square}$.

(d) Using the formula in part (c), the minimum value of $y = (x + 3)(x - 5)$ occurs when $x =$ _____.

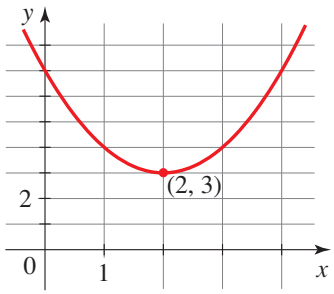
SKILLS

5–8 ■ A graph of a quadratic function is shown.

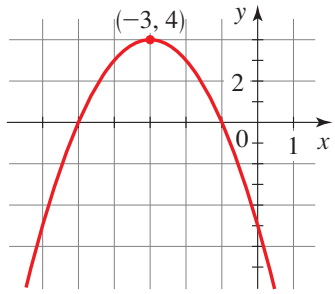
(a) Does the function have a minimum or a maximum value? What is that value?

(b) Find the x-value at which the minimum or maximum value occurs.

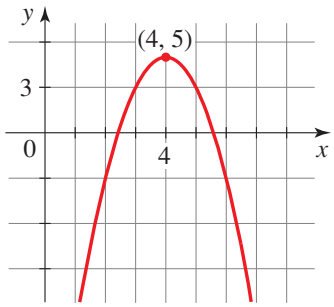
5.



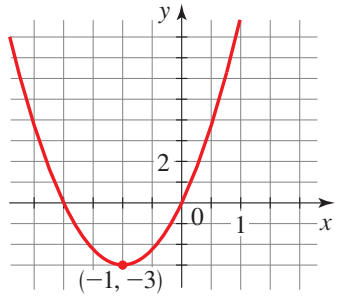
6.



7.



8.



9–14 ■ Find the minimum or maximum value of the quadratic function, and find the x-value at which the minimum or maximum value occurs.

9. $f(x) = -\frac{x^2}{3} + 2x + 7$

10. $g(x) = 2x(x - 4) + 7$

11. $f(x) = x^2 + x + 1$

12. $f(x) = 1 + 3x - x^2$

13. $h(x) = \frac{1}{2}x^2 + 2x - 6$

14. $f(x) = 3 - x - \frac{1}{2}x^2$

15–18 ■ A quadratic function is given.

(a) Sketch its graph.

(b) Find the minimum or maximum value of f, and find the x-value at which the minimum or maximum value occurs.

15. $f(x) = x^2 + 2x - 1$

16. $f(x) = x^2 - 8x + 8$

17. $f(x) = -x^2 - 3x + 3$

18. $f(x) = 1 - 6x - x^2$

19–22 ■ A quadratic function is given.

(a) Use a graphing device to find the maximum or minimum value of the quadratic function f, correct to two decimal places.

(b) Find the maximum or minimum value of f, and compare it with your answer to part (a).

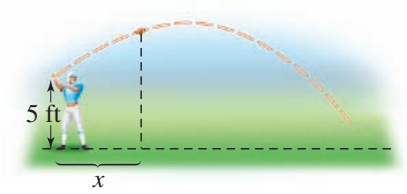
19. $f(x) = x^2 + 1.79x - 3.21$

20. $f(x) = x^2 - 3.2x + 4.1$

21. $f(x) = 1 + x - \sqrt{2}x^2$

22. $f(x) = 3 - 2x - \sqrt{3}x^2$

CONTEXTS

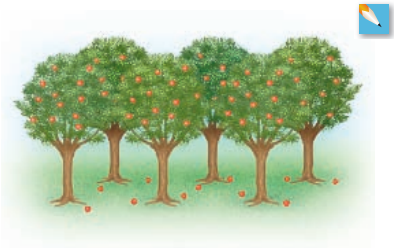


23. Height of a Ball If a ball is thrown directly upward with a velocity of 12 m/s, its height (in meters) after t seconds is given by $y = 12t - 4.9t^2$. What is the maximum height attained by the ball, and after how many seconds is that height attained?

24. Path of a Ball A ball is thrown across a playing field from a height of 5 ft above the ground at an angle of 45° to the horizontal and at a speed of 20 ft/s. It can be deduced from physical principles that the path of the ball is modeled by the function

$$y = -\frac{32}{(20)^2}x^2 + x + 5$$

where x is the distance in feet that the ball has traveled horizontally and y is the height in feet. What is the maximum height attained by the ball, and at what horizontal distance does this occur?



25. Agriculture The number of apples produced by each tree in an apple orchard depends on how densely the trees are planted. If n trees are planted on an acre of land, then each tree produces $900 - 9n$ apples. So the number of apples produced per acre is

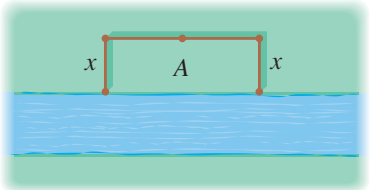
$$A(n) = n(900 - 9n)$$

What is the maximum yield of the trees, and how many trees should be planted per acre to obtain the maximum yield of apples?

26. Agriculture At a certain vineyard it is found that each grape vine produces about 10 pounds of grapes in a season when about 700 vines are planted per acre. For each additional vine that is planted, the production of each vine decreases by about 1%. So the number of pounds of grapes produced per acre is modeled by

$$A(n) = (700 + n)(10 - 0.01n)$$

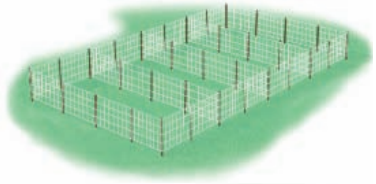
where n is the number of additional vines planted. What is the maximum yield of the grape vines, and how many vines should be planted to maximize grape production?



27. Fencing a Field A farmer has 2400 feet of fencing with which he wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river (see the figure).

(a) Find a function A that models the area of the field in terms of one of its sides x.

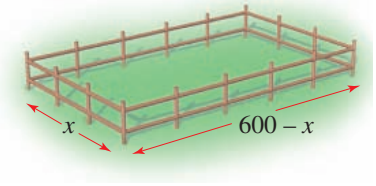
(b) What is the largest area that he can fence, and what are the dimensions of that area? [Compare to your graphical solution in Exercise 29, Section 1.8.]



28. Dividing a Pen A rancher with 750 feet of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Find a function A that models the total area of the four pens.

(b) Find the largest possible total area of the four pens and the dimensions of that area. [Compare to your graphical solution in Exercise 30, Section 1.8.]



29. Fencing a Horse Corral Carol has 1200 feet of fencing to fence in a rectangular horse corral.

(a) Find a function A that models the area of the corral in terms of the width of the corral.

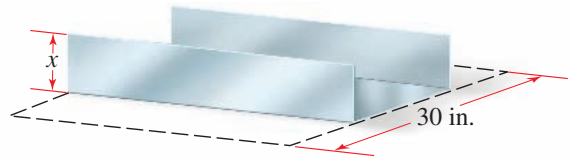
(b) Find the dimensions of the rectangle that maximize the area of the corral.

30. Making a Rain Gutter A rain gutter is formed by bending up the sides of a 30-inch-wide rectangular metal sheet as shown in the figure.

(a) Find a function A that models the cross-sectional area of the gutter in terms of x.

(b) Find the value of x that maximizes the cross-sectional area of the gutter.

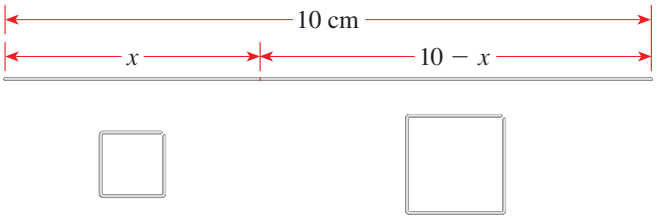
(c) What is the maximum cross-sectional area for the gutter?



31. Minimizing Area A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$, as shown in the figure. Each piece is bent into the shape of a square.

(a) Find a function A that models the total area enclosed by the two squares.

(b) Find the value of x that minimizes the total area of the two squares.





32. Light from a Window A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure. A Norman window with perimeter 30 feet is to be constructed.

(a) Find a function that models the area of the window.

(b) Find the dimensions of the window that admits the greatest amount of light.

33. Stadium Revenue A baseball team plays in a stadium that holds 55,000 spectators. With the ticket price at \$10, the average attendance at recent games has been 27,000. A market survey indicates that for every dollar the ticket price is lowered, the attendance increases by 3000.

(a) Find a function R that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

34. Maximizing Profit A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost \$6, and the society sells an average of 20 feeders per week at a price of \$10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it loses 2 sales per week.

(a) Find a function P that models weekly profit in terms of price per feeder.

(b) What price should the society charge for each feeder to maximize profits? What is the maximum weekly profit?

5.4 Quadratic Equations: Getting Information from a Model

- Solving Quadratic Equations: Factoring
- Solving Quadratic Equations: The Quadratic Formula
- The Discriminant
- Modeling with Quadratic Functions

IN THIS SECTION... we study how to solve quadratic equations. Solving quadratic equations helps us get information from quadratic models.

GET READY... by reviewing how to factor expressions in **Algebra Toolkit B.2.** Test your understanding by doing the **Algebra Checkpoint** at the end of this section.

In Section 4.3 we found that the height a rocket reaches is modeled by a quadratic function. In this section we use the model to answer the question "*When* does the rocket reach a given height?" (see Example 8).

- **The model** gives us the height the rocket reaches at any time.
- **Our goal is** to find the time at which the rocket reaches a given height.

We *can* get the information we want from the model; to do so, we need to solve a quadratic equation. So we start this section by learning how to solve such equations.

Solving Quadratic Equations: Factoring

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

Zero-Product Property

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

example 1 Solving a Quadratic Equation by Factoring

Solve the equation $x^2 + 5x = 24$.

Solution

We must first rewrite the equation so that the right-hand side is 0.

$$x^2 + 5x = 24 \qquad \text{Given equation}$$
$$x^2 + 5x - 24 = 0 \qquad \text{Subtract 24}$$
$$(x - 3)(x + 8) = 0 \qquad \text{Factor}$$
$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0 \qquad \text{Zero-Product Property}$$
$$x = 3 \qquad\qquad x = -8 \qquad \text{Solve}$$

The solutions are $x = 3$ and $x = -8$.

✓ CHECK We substitute $x = 3$ into the original equation:

$$(3)^2 + 5(3) = 9 + 15 = 24 \quad ✓$$

We substitute $x = -8$ into the original equation:

$$(-8)^2 + 5(-8) = 64 - 40 = 24 \quad ✓$$

NOW TRY EXERCISE 7

Do you see why one side of the equation must be 0 in Example 1? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $\left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

example 2 Graphing a Quadratic Function

Let $f(x) = x^2 + 5x - 24$.

(a) Find the x-intercepts of the graph of f.

(b) Sketch the graph of f, and label the x- and y-intercepts and the vertex.

Solution

***x*- and *y*-intercepts are reviewed in Algebra Toolkit D.2, page T71.**

(a) To find the x-intercepts, we solve the equation

$$x^2 + 5x - 24 = 0$$

The equation was solved in Example 1. The x-intercepts are 3 and -8.

(b) To find the y-intercept, we set x equal to 0:

$$f(0) = 0 + 5 \cdot 0 - 24 = -24$$

So the y-intercept is -24.

The function f is a quadratic function where a is 1 and b is 5. So the x-coordinate of the vertex occurs at

$$x = -\frac{b}{2a} \qquad \text{Formula}$$
$$= -\frac{5}{2 \cdot 1} \qquad \text{Replace } a \text{ by 1 and } b \text{ by 5}$$
$$= -\frac{5}{2} \qquad \text{Calculate}$$

The y-coordinate of the vertex is

$$y = f\left(-\frac{b}{2a}\right) \qquad \text{Formula}$$

$$= f\left(-\frac{5}{2}\right) \qquad a \text{ is 1 and } b \text{ is 5}$$

$$= \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) - 24 \qquad \text{Definition of } f$$

$$= -\frac{121}{4} \qquad \text{Calculate}$$

The graph is a parabola with vertex at $\left(-\frac{5}{2}, -\frac{121}{4}\right)$. The parabola opens upward because $a > 0$. The graph is shown in Figure 1.

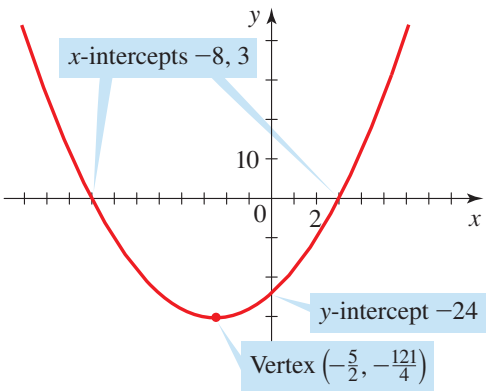


figure **1** Graph of $f(x) = x^2 + 5x - 24$

NOW TRY EXERCISE 53

example **3**

Finding a Quadratic Function from a Graph

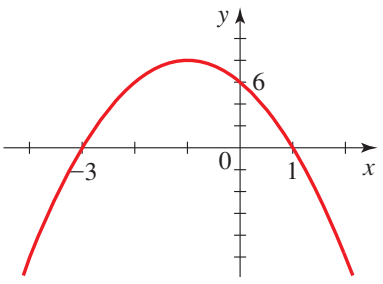


figure **2**

The graph of a quadratic function f is shown in Figure 2. Find an equation that represents the function f.

Solution

We observe from the graph that the x-intercepts are -3 and 1. So we can express the function f in factored form:

$$f(x) = a(x - 1)(x + 3)$$

We need to find a. From the graph we see that the y-intercept is 6, so $f(0) = 6$. We have

$$f(0) = a(0 - 1)(0 + 3) \qquad \text{Replace } x \text{ by } 0$$

$$6 = a(0 - 1)(0 + 3) \qquad y\text{-intercept is } 6$$

$$6 = -3a \qquad \text{Simplify}$$

$$a = -2 \qquad \text{Divide by } -3 \text{ and switch sides}$$

It follows that $f(x) = -2(x - 1)(x + 3)$.

NOW TRY EXERCISE 17

Solving Quadratic Equations: The Quadratic Formula

If a quadratic equation does not factor readily, we can solve it by using the **quadratic formula**. This formula gives the solutions of the general quadratic equation $ax^2 + bx + c = 0$ and is derived by using the technique of completing the square (see **Algebra Toolkit C.3**, page T62).

The Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

example 4 Using the Quadratic Formula

Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ **(b)** $4x^2 + 12x + 9 = 0$ **(c)** $x^2 + 2x + 2 = 0$

Solution

(a) In this quadratic equation a is 3, b is -5, and c is -1.

$$3x^2 - 5x - 1 = 0$$

($b = -5$, $a = 3$, $c = -1$)

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \quad \text{and} \quad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

(b) Using the Quadratic Formula where a is 4, b is 12, and c is 9 gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) Using the Quadratic Formula where a is 1, b is 2, and c is 2 gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution.

■ NOW TRY EXERCISES 27, 31, AND 33 ■

The Discriminant

The quantity $b^2 - 4ac$ that appears under the square root sign in the Quadratic Formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D. If D is negative, then $\sqrt{b^2 - 4ac}$ is undefined, so the quadratic equation has no real solution. If D is zero, then the equation has exactly one real solution. If D is positive, then the equation has two distinct real solutions. The following box summarizes these observations.

The Discriminant

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, is $D = b^2 - 4ac$.

1. If $D < 0$, then the equation has no real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D > 0$, then the equation has two distinct real solutions.

We will see in the next example that we can use the discriminant to determine whether the graph of a quadratic function has two x-intercepts, one x-intercept, or no x-intercepts.

example 5 Using the Discriminant to Find the Number of x-Intercepts

A quadratic function $f(x) = ax^2 + bx + c$ is given. Find the discriminant of the equation $ax^2 + bx + c = 0$, and use it to determine the number of x-intercepts of the graph of f. Sketch a graph of f to confirm your answer.

(a) $f(x) = x^2 - 6x + 8$

(b) $f(x) = x^2 - 6x + 9$

(c) $f(x) = x^2 - 6x + 10$

Solution

(a) The equation is $x^2 - 6x + 8 = 0$, so a is 1, b is -6, and c is 8. The discriminant is

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 8 = 32 > 0$$

Since the discriminant is positive, there are two distinct solutions, and hence there are two x-intercepts for the graph of f. The graph of f in Figure 3(a) confirms this.

(b) The equation is $x^2 - 6x + 9 = 0$, so a is 1, b is -6, and c is 9. The discriminant is

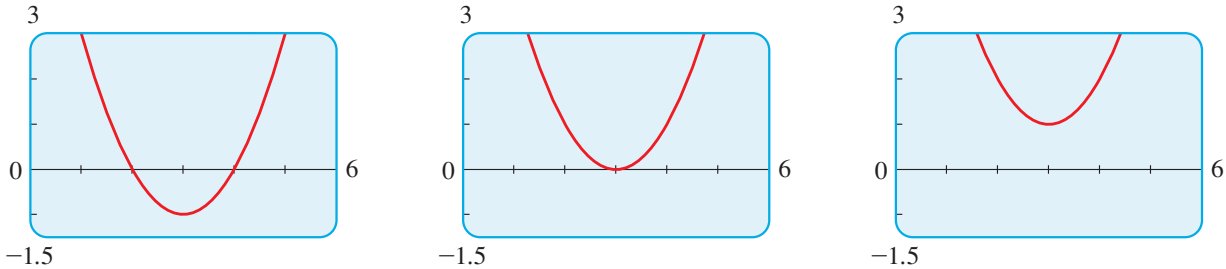
$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 9 = 0$$

Since the discriminant is zero, there is exactly one solution and hence one x-intercept for the graph of f. The graph in Figure 3(b) confirms this.

(c) The equation is $x^2 - 6x + 10 = 0$, so a is 1, b is -6, and c is 10. The discriminant is

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 10 = 36 - 40 = -4 < 0$$

Since the discriminant is negative, there are no solutions, and hence there are no x-intercepts for the graph of f. The graph in Figure 3(c) confirms this.



(a) $f(x) = x^2 - 6x + 8$ (b) $f(x) = x^2 - 6x + 9$ (c) $f(x) = x^2 - 6x + 10$

figure **3**

NOW TRY EXERCISES 43, 45, AND 47

Modeling with Quadratic Functions

We now study real-world phenomena that are modeled by quadratic functions, and we solve quadratic equations to get information from the model.

example 6 Dimensions of a Lot

A rectangular building lot is 8 feet longer than it is wide.

(a) Find a function that models the area of the lot for any width.

(b) If the lot has area of 2900 square feet, find the dimensions of the lot.

Solution

(a) We want a function A that models the area of the lot in terms of the width. Let

$$w = \text{width of lot}$$

We translate the information given in the problem into the language of algebra (see Figure 4).

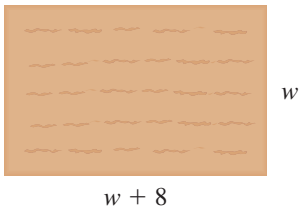


figure **4**

In Words	In Algebra
Width of lot	w
Length of lot	$w + 8$

Now we set up the model:

$$\text{area of lot} = \text{width of lot} \times \text{length of lot}$$

$$A(w) = w(w + 8)$$

(b) Suppose the lot has an area of 2900 square feet. Then

$$\begin{aligned} A(w) &= w(w + 8) && \text{Model} \\ 2900 &= w(w + 8) && \text{Area of lot is 2900} \\ 2900 &= w^2 + 8w && \text{Expand} \\ 0 &= w^2 + 8w - 2900 && \text{Subtract 2900} \\ 0 &= (w - 50)(w + 58) && \text{Factor} \end{aligned}$$

$w = 50$ or $w = -58$ Zero-Product Property

Since the width of a lot must be a positive number, we conclude that the width is 50 feet. The length of the lot is $w + 8 = 50 + 8 = 58$ feet.

NOW TRY EXERCISE 65

example 7 Ticket Sales

A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) At what ticket price is the revenue \$100,000?

(c) What ticket price is so high that no one attends and so no revenue is generated?

Solution

(a) In Example 5 of Section 5.3 we found the following function that models the revenue R in terms of the ticket price x:

$$R(x) = -1000x^2 + 23{,}500x$$

(b) We want to find the ticket price for which $R(x) = 100{,}000$.

$$\begin{aligned} R(x) &= -1000x^2 + 23{,}500x && \text{Model} \\ 100{,}000 &= -1000x^2 + 23{,}500x && \text{Revenue is 100,000} \\ 100 &= -x^2 + 23.5x && \text{Divide by 1000} \\ x^2 - 23.5x + 100 &= 0 && \text{Subtract 100} \end{aligned}$$

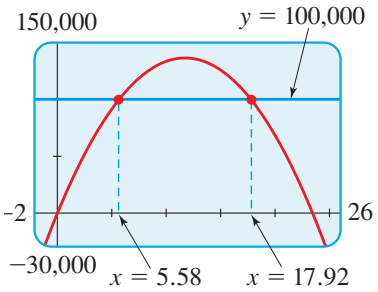
The expression on the left-hand side of this equation does not factor easily, so we use the Quadratic Formula where a is -1, b is 23.5, and c is -100.

$$\begin{aligned} x &= \frac{-23.5 \pm \sqrt{(-23.5)^2 - 4(-1)(-100)}}{2(-1)} \\ &= \frac{-23.5 \pm \sqrt{152.25}}{-2} \end{aligned}$$

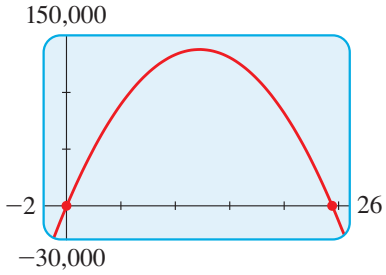
Using a calculator, we find

$$x \approx 17.92 \quad \text{or} \quad x \approx 5.58$$

So the revenue is \$100,000 when the ticket price is \$17.92 or \$5.58.



A graph of the revenue function R shows that revenue is \$100,000 when the ticket price is about \$5.58 and \$17.92.



A graph of the revenue function R shows that $R(x) = 0$ when the ticket price x is 0 or \$23.50 (the x-intercepts of the graph).

(c) We want to find the ticket price for which $R(x) = 0$.

$$R(x) = -1000x^2 + 23{,}500x \quad \text{Model}$$
$$0 = -1000x^2 + 23{,}500x \quad \text{Revenue is 0}$$
$$0 = -x^2 + 23.5x \quad \text{Divide by 1000}$$
$$0 = x(-x + 23.5) \quad \text{Factor}$$
$$x = 0 \quad \text{or} \quad x = 23.5 \quad \text{Zero-Product Property}$$

According to this model, a ticket price of \$23.50 is just too high; at that price, no one attends the hockey game. (Of course the revenue is also zero if the ticket price is zero.)

NOW TRY EXERCISE 67

example **8**

Model Rocket

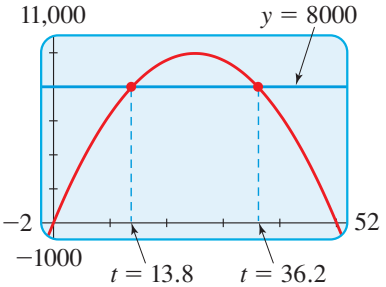
A model rocket is shot straight upward with an initial speed of 800 ft/s, and the height of the rocket is given by

$$h = 800t - 16t^2$$

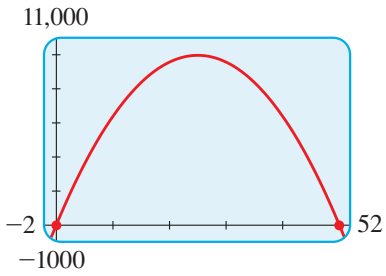
where h is measured in feet and t in seconds.

(a) How many seconds does it take for the rocket to reach a height of 8000 feet?

(b) How long will it take the rocket to hit the ground?



A graph of the height function h shows that the rocket reaches a height of 8000 ft at times of about 13.8 s and 36.2 s.



A graph of the height function h shows that $h(t) = 0$ when t is about 0 or 50 seconds (the t-intercepts of the graph).

Solution

(a) We want to find the time t such that $h = 8000$.

$$h = 800t - 16t^2 \quad \text{Model}$$
$$8000 = 800t - 16t^2 \quad \text{Height is 8000}$$
$$10 = t - 0.02t^2 \quad \text{Divide by 800}$$
$$0.02t^2 - t + 10 = 0 \quad \text{Subtract 10 and switch sides}$$

The expression on the left-hand side does not factor easily, so we use the Quadratic Formula where a is 0.02, b is -1, and c is 10.

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(0.02)(10)}}{2(0.02)}$$
$$= \frac{1 \pm \sqrt{0.20}}{0.04}$$

Using a calculator, we find

$$t \approx 13.8 \quad \text{or} \quad t \approx 36.2$$

So the rocket is at a height of 8000 feet after about 13.8 seconds and again after 36.2 seconds.

(b) At ground level the height is 0, so we must solve the equation

$$h = 800t - 16t^2 \quad \text{Model}$$
$$0 = -16t^2 + 800t \quad \text{Height is 0}$$
$$0 = 16t(-t + 50) \quad \text{Factor}$$
$$t = 0 \quad \text{or} \quad t = 50 \quad \text{Zero-Product Property}$$

According to this model, the rocket hits the ground after 50 seconds. (Of course the rocket is also at ground level at time 0.)

NOW TRY EXERCISE 63

Notice that the model in Example 8 is valid only for $0 \le t \le 50$. For values of t outside this interval the height h would be negative (that is, the rocket would be traveling below ground level).

Algebra Checkpoint ✓

Check your knowledge of factoring quadratic expressions by doing the following problems. You can review this topic in **Algebra Toolkit B.2** on page T33.

1. Factor each expression by factoring out common factors.

(a) $x^2 + x$ **(b)** $4t^2 - 6t$

(c) $2(x + 1)^2 + (x + 1)$ **(d)** $(2r + 3)(3r + 4) - 5(3r + 4)$

2. Factor the given expression using the Difference of Squares Formula.

(a) $x^2 - 36$ **(b)** $4t^2 - 16$

(c) $(u + 1)^2 - 36$ **(d)** $4(w - 2)^2 - 49$

3. Factor the perfect square.

(a) $x^2 + 8x + 16$ **(b)** $t^2 - 12t + 36$

(c) $9r^2 - 24r + 16$ **(d)** $(u + 2)^2 + 10(u + 2) + 25$

4. Factor each expression by trial and error.

(a) $x^2 + 5x + 6$ **(b)** $t^2 - t - 12$

(c) $2s^2 - s - 3$ **(d)** $6u^2 - 7u - 3$

5.4 Exercises

CONCEPTS

Fundamentals

1. The Quadratic Formula gives us the solutions of the equation $ax^2 + bx + c = 0$.

(a) State the Quadratic Formula: $x =$ _____.

(b) In the equation $\frac{1}{2}x^2 - x - 4 = 0$, $a =$ _____, $b =$ _____, and $c =$ _____. So the solution of the equation is $x =$ _____.

2. To solve the quadratic equation $x^2 - 4x - 5 = 0$, we can do the following.

(a) Factor the equation as $(x -$ ____$)\cdot(x +$ ____$) = 0$, so the solutions are _____ and _____.

(b) Use the Quadratic Formula to get $x =$ _____, so the solutions are _____ and _____.

3. Let $f(x) = ax^2 + bx + c$, and let $D = b^2 - 4ac$.

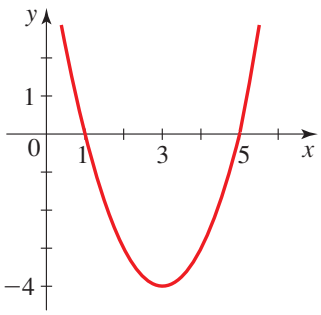
(a) If $D > 0$, then the number of x-intercepts for the graph of f is _____.

(b) If $D = 0$, then the number of x-intercepts for the graph of f is _____.

(c) If $D < 0$, then the number of x-intercepts for the graph of f is _____.

4. The graph of a quadratic function $f(x) = ax^2 + bx + c$ is shown at the top of the next page.

(a) The x-intercepts are _____.



Graph for Exercise 4

(b) The discriminant of the equation $ax^2 + bx + c = 0$ is _____ (positive, 0, or negative).

(c) The solution(s) to the equation $ax^2 + bx + c = 0$ is (are) _____.

Think About It

5. Consider the quadratic function $y = (x - 2)(x - 4)$.

(a) The graph is a parabola that opens _____.

(b) The x-intercepts are _____ and _____.

(c) From the location of the vertex between the x-intercepts, we see that the x-coordinate of the vertex is _____.

6. Consider the quadratic function $y = (x - m)(x - n)$.

(a) The graph is a parabola that opens _____.

(b) The x-intercepts are _____ and _____.

(c) From the location of the vertex between the x-intercepts, we see that the x-coordinate of the vertex is _____.

(d) Using the formula we found in part (c), we find that the x-coordinate of the vertex of $y = (x - 3)(x - 5)$ is _____.

SKILLS

7–16 ■ Solve the equation by factoring.

7. $x^2 + x = 12$

8. $x^2 + 3x = 4$

9. $t^2 - 7t + 12 = 0$

10. $t^2 + 8t + 12 = 0$

11. $3s^2 - 5s - 2 = 0$

12. $4x^2 - 4x - 15 = 0$

13. $2y^2 + 7y + 3 = 0$

14. $4w^2 = 4w + 3$

15. $6x^2 + 5x = 4$

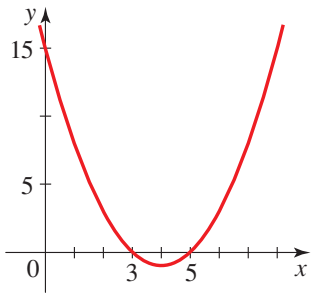
16. $3x^2 + 1 = 4x$

17–20 ■ A graph of a quadratic function f is given.

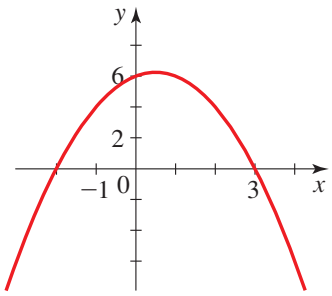
(a) Find the x-intercepts.

(b) Find an equation that represents the function f (as in Example 3).

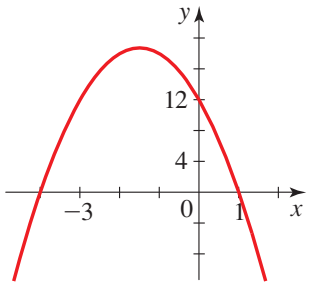
17.



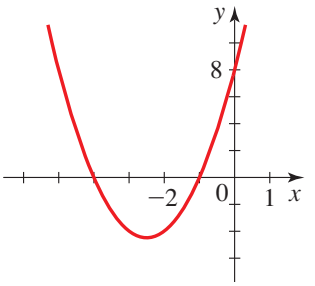
18.



19.



20.



21–26 ■ Solve the equation by both factoring and using the Quadratic Formula.

21. $x^2 - 2x - 15 = 0$

22. $x^2 + 5x - 6 = 0$

23. $x^2 - 7x + 10 = 0$

24. $x^2 + 30x + 200 = 0$

25. $2x^2 + x - 3 = 0$

26. $3x^2 + 7x + 4 = 0$

27–38 ■ Find all real solutions of the equation.

27. $t^2 + 3t + 1 = 0$

28. $2t^2 - 8t + 4 = 0$

29. $y^2 + 12y - 27 = 0$

30. $8y^2 - 6y - 9 = 0$

31. $s^2 - \frac{3}{2}s + \frac{9}{16} = 0$

32. $s^2 - 6s + 1 = 0$

33. $w^2 = 3(w - 1)$

34. $2 + 2z + 3z^2 = 0$

35. $x^2 - \sqrt{5}x + 1 = 0$

36. $5x^2 - 7x + 5 = 0$

37. $10y^2 - 16y + 5 = 0$

38. $25y^2 + 70y + 49 = 0$

39–42 ■ Solve the quadratic equation algebraically and graphically, correct to three decimal places.

39. $x^2 - 0.011x - 0.064 = 0$

40. $x^2 - 2.450x + 1.500 = 0$

41. $x^2 - 2.450x + 1.501 = 0$

42. $x^2 - 1.800x + 0.810 = 0$

43–48 ■ A quadratic function $f(x) = ax^2 + bx + c$ is given.

(a) Find the discriminant of the equation $ax^2 + bx + c = 0$. How many real solutions does this equation have?

(b) Use the answer to part (a) to determine the number of x-intercepts for the graph of the function $f(x) = ax^2 + bx + c$, and then graph the function to confirm your answer.

43. $f(x) = x^2 - 6x + 1$

44. $f(x) = x^2 - 6x + 9$

45. $f(x) = x^2 + 2.20x + 1.21$

46. $f(x) = 0.1x^2 - 0.38x + 0.361$

47. $f(x) = 32x^2 + 40x + 13$

48. $f(x) = 9x^2 - 4x + \frac{4}{9}$

49–52 ■ A graph of a quadratic function $f(x) = ax^2 + bx + c$ is shown.

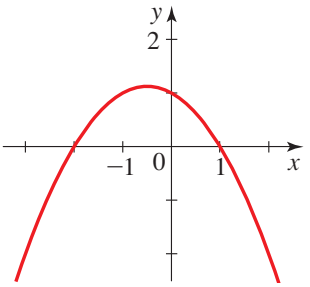
(a) Find the x-intercept(s), if there are any.

(b) Is the discriminant $D = b^2 - 4ac$ positive, negative, or 0?

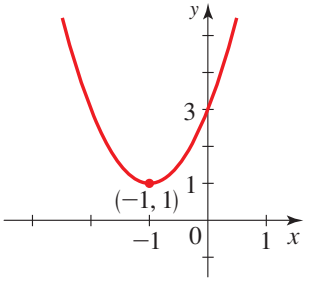
(c) Find the solution(s) to the equation $ax^2 + bx + c = 0$.

(d) Find an equation that represents the function f.

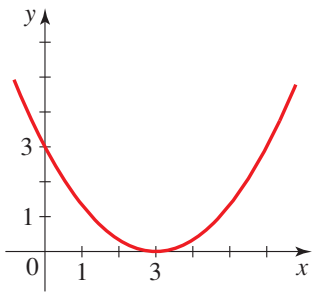
49.



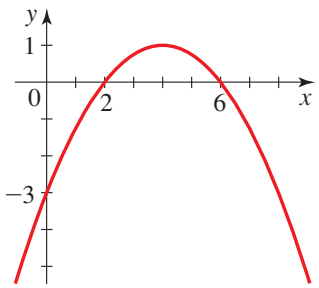
50.



51.



52.



53–60 ■ A quadratic function $f(x) = ax^2 + bx + c$ is given.

(a) Find the x-intercepts of the graph of f.

(b) Sketch the graph of f and label the x- and y-intercepts and the vertex.

53. $f(x) = x^2 + 2x - 1$

54. $f(x) = x^2 - 8x + 8$

55. $f(x) = 3x^2 - 6x + 1$

56. $f(x) = 5x^2 + 30x + 4$

57. $f(x) = -x^2 - 3x + 3$

58. $f(x) = 1 - x - x^2$

59. $f(x) = 3x^2 - 12x + 13$

60. $f(x) = 2x^2 + 8x + 11$

CONTEXTS

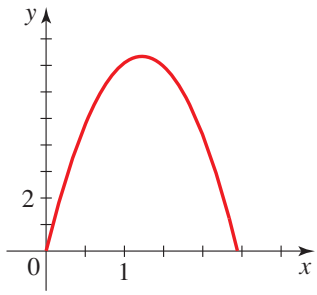
61. Height of a Ball If a ball is thrown directly upward with a velocity of 12 m/s, its height (in meters) after t seconds is modeled by $f(t) = 12t - 4.9t^2$.

(a) When does the ball reach a height of 5 meters?

(b) Does the ball reach a height of 8 meters?

(c) When does the ball hit the ground?

(d) Identify the points on the graph that correspond to your solutions to parts (a)–(c).



62. Path of a Ball A ball is thrown across a playing field from a height of 5 ft above the ground at an angle of 45° to the horizontal at a speed of 20 ft/s. It can be deduced from physical principles that the path of the ball is modeled by the function

$$f(x) = -\frac{32}{(20)^2}x^2 + x + 5$$

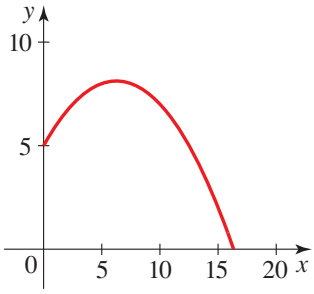
where x is the distance in feet that the ball has traveled horizontally.

(a) At what horizontal distance x is the ball 7 ft high?

(b) Does the ball reach a height of 10 ft?

(c) At what horizontal distance x does the ball hit the ground?

(d) Identify the points on the graph in the margin that correspond to your solutions to parts (a)–(c).



63. Height of a Rocket A model rocket shot straight upward with an initial velocity of v_0 m/s will reach a height of h meters in t seconds, where the height h is modeled by

$$h(t) = -4.9t^2 + 25t$$

(a) When does the rocket reach a height of 20 m?

(b) When does the rocket reach the highest point of its path?

(c) When does the rocket hit the ground?

(d) Graph the function h and identify the points on the graph that correspond to your solutions to parts (a)–(c).

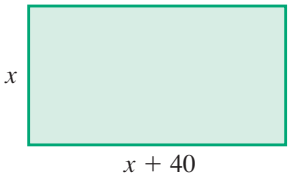
64. Height of a Ball A ball is thrown straight upward with an initial velocity of v_0 ft/s. It will reach a height of h feet in t seconds, where the height h is modeled by the function

$$h(t) = -16t^2 + v_0t$$

(a) Show that the ball reaches its maximum height when $t = v_0/32$.

(b) Show that the ball reaches a maximum height of $v_0^2/64$.

(c) How fast should the ball be thrown upward to reach a maximum height of 100 ft?



65. Dimensions of a Lot A rectangular building lot is 40 ft longer than it is wide. (See the figure to the left.)

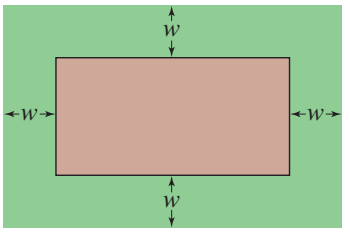
(a) Find a function A that models the area of the lot.

(b) If the lot has an area of 11,700 ft^2, then what are the dimensions of the lot?

66. Width of a Lawn A factory is to be built on a lot measuring 180 ft by 240 ft. A local building code specifies that a lawn of uniform width and equal in area to the factory must surround the factory. Let w be the width of the lawn, as shown in the figure.

(a) Find a function A that models the area of the factory in terms of w.

(b) What must the width of the lawn be, and what are the dimensions of the factory?



67. Stadium Revenue A baseball team plays in a stadium that holds 55,000 spectators. With the ticket price at $10, the average attendance at recent games has been 27,000. A market survey indicates that for every dollar the ticket price is lowered, attendance increases by 3000. (See Exercise 33 in Section 5.3.)

(a) Find a function that models the revenue in terms of ticket price.

(b) What ticket price is so high that no revenue is generated?

(c) The baseball team hopes to have a revenue of $250,000. What ticket price should they charge to meet this goal?

68. Bird Feeders A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost $6, and the society sells an average of 20 feeders per week at a price of $10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it loses two sales per week. (See Exercise 34 in Section 5.3.)

(a) Find a function P that models weekly profit in terms of price per feeder.

(b) What ticket price is so high that no profit is generated?

(c) The society needs to make a weekly profit of $90 to cover the expenses of their nature center. What price should the society charge per feeder to meet this goal?

5.5 Fitting Quadratic Curves to Data

■ Modeling Data with Quadratic Functions

IN THIS SECTION... we learn how to model data using the quadratic function of best fit.



Proper tire inflation is particularly important in driving on roads with tight curves, such as this section of Pacific Coast Highway in California.

The importance of proper tire inflation cannot be overestimated. A car's tendency to skid as well as the tire's potential to overheat and explode are some of the safety factors associated with tire inflation. Proper tire inflation also affects a car's ability to handle tight curves, which is the reason race car drivers pay special attention to tire pressure. Today, many passenger cars have a Tire Pressure Monitoring (TPM) system, which continually monitors tire pressure and warns the driver of unsafe pressure levels. The first passenger vehicle to adopt a TPM system was the Porsche 959 in 1986.

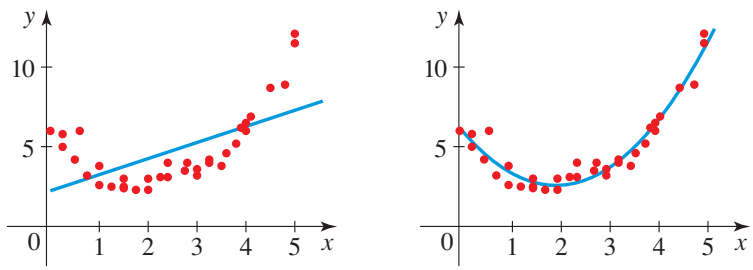
Gas mileage and tire wear are also affected by tire inflation. So what exactly is proper pressure? And how can we determine the proper tire pressure? To find the tire pressure that maximizes tire life, researchers perform experiments in which tires under different inflation pressures are rolled along special surfaces. The data exhibit an increasing and then decreasing pattern, so a quadratic function is appropriate for modeling the data.

- **The data** give tire life for different inflation pressures.
- **The model** is a quadratic function that best fits the data.
- **Our goal** is to use the model to predict the inflation pressure that gives the longest tire life.

We'll see in this section how this modeling technique works, and we'll solve the tire inflation problem in Example 2.

■ Modeling Data with Quadratic Functions

We have learned how to fit a line to data (Section 2.5). The line models the increasing or decreasing trend of the data. If the data exhibit more variability, such as an increase followed by a decrease, then to model the data we need a curve rather than a line. Figure 1 shows a scatter plot with two possible models that appear to fit the data. Which model appears to fit the data better?



(a) Linear model (b) Quadratic model

figure **1**

We can use a graphing calculator to find a quadratic model that fits any given set of data.

example 1 Finding the Quadratic Curve of Best Fit

(a) Make a scatter plot of the data in Table 1. Is it appropriate to model the data by a quadratic function?

(b) Use a graphing calculator to find the quadratic model that best fits the data. Draw a graph of the model.

(c) Use the model to predict the value of y when x is 6.

table **1**

x	0.0	0.5	1.0	1.2	1.5	2.0	2.4	2.4	3.0	3.5	3.6	3.8	4.5	5.0
y	6.0	4.2	3.1	2.5	2.4	2.3	3.5	3.1	3.6	3.8	4.6	5.2	7.1	10.2

Solution

(a) A scatter plot is shown in Figure 2. From the data we see that the y-values appear to decrease and then increase, so a quadratic model is appropriate.

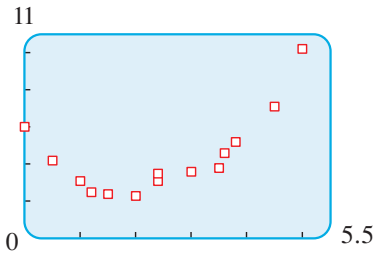
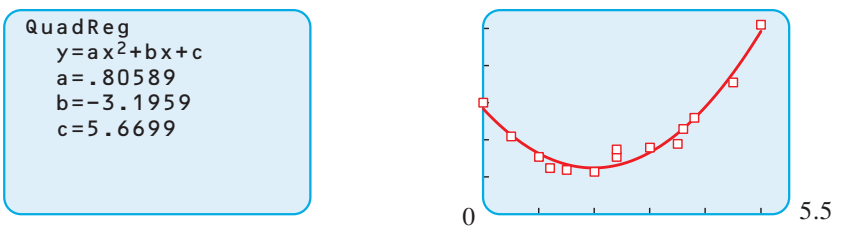


figure **2** Scatter plot of data

(b) Using the `QuadReg` command on a graphing calculator, we obtain the result in Figure 3(a). Thus, the quadratic function of best fit is

$$y = 0.80589x^2 - 3.1959x + 5.6699$$

A graph of this function together with the scatter plot is shown in Figure 3(b).



(a) Result of `QuadReg` command (b) Graph of quadratic model

figure **3**

(c) When x is 6, the model predicts the following value for y.

$$\begin{aligned} y &= 0.80589x^2 - 3.1959x + 5.6699 && \text{Model} \\ &= 0.80589(6)^2 - 3.1959(6) + 5.6699 && \text{Replace } x \text{ by } 6 \\ &= 15.507 && \text{Calculator} \end{aligned}$$

NOW TRY EXERCISE 5

example **2**

Proper Tire Inflation

table **2**

Pressure (lb/in^2)	Tire life (mi × 1000)
26	50
28	66
31	78
35	81
38	74
42	70
45	59

Car tires need to be inflated properly. Overinflation or underinflation can cause premature tread wear. The data in Table 2 show tire life (in thousands of miles) for different inflation pressures for a certain type of tire.

(a) Find the quadratic function that best fits the data.

(b) Draw a graph of the quadratic model from part (a) together with a scatter plot of the data.

(c) Use the result from part (b) to estimate the inflation pressure that gives the longest tire life.

Solution

(a) Using the `QuadReg` command on a graphing calculator, we obtain the quadratic function that best fits the data.

$$y = -0.27543x^2 + 19.7485x - 273.55$$

(See Figure 4(a).)

(b) The graph and scatter plot are shown in Figure 4(b).

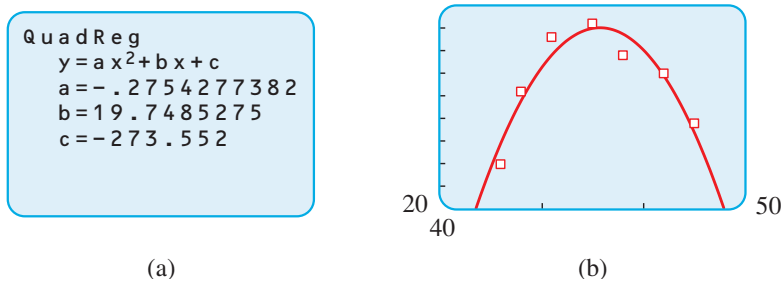


figure **4** (a) (b)

(c) We need to find when the maximum value of y occurs. Using the formula from Section 5.3, we see that the maximum value occurs at

$$\begin{aligned} x &= -\frac{b}{2a} && \text{Formula} \\ &= -\frac{19.7485}{2(-0.27543)} && \text{Replace } b \text{ by } 19.7485 \text{ and } a \text{ by } -0.27543 \\ &= 35.8503 && \text{Calculator} \end{aligned}$$

So the inflation pressure that gives the maximum tire life is about 36 lb/in^2.

NOW TRY EXERCISE 9

5.5 Exercises

CONCEPTS

Fundamentals

1. When modeling data we make a _____ plot to help us visually determine whether a line or some other curve is appropriate for modeling the data.

2. If the y-values of a set of data increase and then decrease, then a _____ function may be appropriate to model the data.

Think About It

3–4 ■ The following data are obtained from the function $f(x) = (x - 3)^2$.

x	0	1	2	3	4	5	6	7	8
$f(x)$	9	4	1	0	1	4	9	16	25

3. If you use your calculator to find the quadratic function that best fits the data, what function would you expect to get? Try it.

4. Find the *line* of best fit for this data. Graph the line and a scatter plot of the data on the same screen. How well does "the line of best fit" fit the data?

SKILLS

5–8 ■ A set of data is given.

(a) Make a scatter plot of the data. Is it appropriate to model the data by a quadratic function?

(b) Use a graphing calculator to find the quadratic model that best fits the data. Draw a graph of the model.

(c) Use the model to predict the value of y when x is 7.

5.

x	0.1	0.8	0.8	1.2	1.7	2.3	2.6	3.1	3.3	3.4	3.9	4.1	5.2	5.9
y	101.2	106.7	105.2	110.1	112.7	114.6	113.3	113.1	109.1	110.4	109.2	107.1	97.6	95.5

6.

x	0.2	0.3	0.9	1.4	1.5	1.8	2.4	2.6	3.9	4.1	4.7	5.2	5.5	6.3
y	37.8	41.7	45.8	46.2	48.1	49.9	47.3	45.1	42.4	36.2	32.5	29.3	27.9	21.8

7.

x	0.5	0.7	1.3	1.9	2.3	2.8	2.9	3.3	3.7	4.1	4.1	4.4	4.9	5.5
y	52.1	48.2	45.3	40.8	39.7	35.5	34.1	32.5	31.2	32.7	33.4	36.1	38.3	41.2

8.

x	0.0	0.2	0.7	1.2	1.2	2.1	2.9	3.1	3.5	3.7	4.2	4.3	5.5	6.1
y	13.1	12.2	10.1	9.2	9.6	8.7	8.6	9.1	10.7	10.9	11.8	13.4	16.9	18.2

CONTEXTS



Ted Wood/Getty Images

9. Rainfall and Crop Yield Rain is essential for crops to grow, but too much rain can diminish crop yields. The data give rainfall and cotton yield per acre for several seasons in a certain county.

(a) Make a scatter plot of the data. Does a quadratic function seem appropriate for modeling the data?

(b) Use a graphing calculator to find the quadratic model that best fits the data. Draw a graph of the model.

(c) Use the model that you found to estimate the yield if there are 25 inches of rainfall.

Season	Rainfall (in.)	Yield (kg/acre)
1	23.3	5311
2	20.1	4382
3	18.1	3950
4	12.5	3137
5	30.9	5113
6	33.6	4814
7	35.8	3540
8	15.5	3850
9	27.6	5071
10	34.5	3881

10. Too Many Corn Plants per Acre? The more corn a farmer plants per acre, the greater is the yield the farmer can expect—but only up to a point. Too many plants per acre can cause overcrowding and decrease yields. The data give crop yields per acre for various densities of corn plantings, as found by researchers at a university test farm.

(a) Use a graphing calculator to find the quadratic model that best fits the data.

(b) Draw a graph of the model you found together with a scatter plot of the data.

(c) Use the model that you found to estimate the yield for 37,000 plants per acre.

Density (plants/acre)	Crop yield (bushels/acre)
15,000	43
20,000	98
25,000	118
30,000	140
35,000	142
40,000	122
45,000	93
50,000	67

11. Height of a Baseball A baseball is thrown upward, and its height is measured at 0.5-second intervals using a strobe light. The resulting data are given in the table.

(a) Make a scatter plot of the data. Does a quadratic function seem appropriate for modeling the data?

(b) Use a graphing calculator to find the quadratic model that best fits the data. Draw a graph of the model on the scatter plot of the data.

(c) Find the times when the ball is 6 meters above the ground.

(d) What is the maximum height attained by the ball?

Time (s)	Height (m)
0.0	1.28
0.5	7.96
1.0	12.22
1.5	14.02
2.0	13.38
2.5	10.27
3.0	4.82

Time (min)	Height (ft)
0	5.0
4	3.1
8	1.9
12	0.8
16	0.2

12. Torricelli's Law Water in a tank will flow out of a small hole in the bottom faster when the tank is nearly full than when it is nearly empty. According to Torricelli's Law, the height $h(t)$ of water remaining at time t is a quadratic function of t. A certain tank is filled with water and allowed to drain. The height of the water is measured at different times as shown in the table.

(a) Use a graphing calculator to find the quadratic model that best fits the data.

(b) Draw a graph of the model, together with a scatter plot of the data.

(c) Use the model that you found to estimate how long it takes for the tank to drain completely.

CHAPTER 5 REVIEW

CHAPTER 5 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

5.1 Working with Functions: Shifting and Stretching

Shifting Graphs Up and Down If $c > 0$, then:

- To graph $y = f(x) + c$, shift the graph of $y = f(x)$ **upward** by c units.
- To graph $y = f(x) - c$, shift the graph of $y = f(x)$ **downward** by c units.

Shifting Graphs Left and Right If $c > 0$, then:

- To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the **left** by c units.
- To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the **right** by c units.

Stretching and Shrinking Graphs Vertically To graph $y = cf(x)$:

- If $c > 1$, **stretch** the graph of $y = f(x)$ vertically by a factor of c.
- If $0 < c < 1$, **shrink** the graph of $y = f(x)$ vertically by a factor of c.

Reflecting Graphs

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the ***x*-axis**.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the ***y*-axis**.

5.2 Quadratic Functions and Their Graphs

A **squaring function** is a function of the form $f(x) = Cx^2$, where $C \neq 0$.

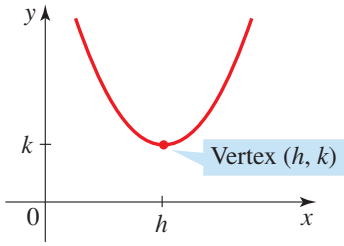
A **quadratic function** is a function that has been derived from a squaring function by applying one or more of the transformations described in Section 5.1.

Every quadratic function can be described in either of the following forms:

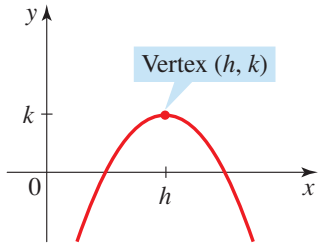
- **General form:** $f(x) = ax^2 + bx + c$, where $a \neq 0$
- **Standard form:** $f(x) = a(x - h)^2 + k$, where $a \neq 0$

A quadratic function written in general form can be expressed in standard form by **completing the square**.

The graph of a quadratic function is most easily obtained from the standard form. The graph of $f(x) = a(x - h)^2 + k$ is a **parabola** with **vertex** (h, k); it opens **upward** if $a > 0$ and **downward** if $a < 0$.



$f(x) = a(x - h)^2 + k, \quad a > 0$



$f(x) = a(x - h)^2 + k, \quad a < 0$

5.3 Maxima and Minima: Getting Information from a Model

If a quadratic function is given in standard form $f(x) = a(x - h)^2 + k$, then we have the following:

- If $a > 0$, then f has the **minimum** value $f(h) = k$ at the vertex of its graph.
- If $a < 0$, then f has the **maximum** value $f(h) = k$ at the vertex of its graph.

If a quadratic function is given in general form $f(x) = ax^2 + bx + c$, then its minimum or maximum value occurs at $x = -b/(2a)$.

- If $a > 0$, then f has the **minimum** value $f(-b/(2a))$.
- If $a < 0$, then f has the **maximum** value $f(-b/(2a))$.

5.4 Quadratic Equations: Getting Information from a Model

To get information from a quadratic model, we often need to solve a **quadratic equation**, which is an equation of the form

$$ax^2 + bx + c = 0$$

To **solve** a quadratic equation, we use one of the two following methods:

- **Factoring:** If the left-hand side of the equation factors readily, factor it, set each factor equal to zero, and solve the resulting simpler equations.
- **Quadratic formula:** If factoring is not practical, use the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** D of a quadratic equation is the number that appears inside the square root in the quadratic formula, that is, $D = b^2 - 4ac$.

- If $D > 0$, then the equation has two distinct real solutions.
- If $D = 0$, then the equation has exactly one real solution.
- If $D < 0$, then the equation has no real solutions.

5.5 Fitting Quadratic Curves to Data

If the scatter plot of a data set indicates that the data seem to be best modeled by a quadratic function, then we can use the `QuadReg` feature on a graphing calculator to find the **quadratic curve**

$$y = ax^2 + bx + c$$

that best fits the data.

CHAPTER 5 REVIEW EXERCISES

SKILLS

1–6 ■ Sketch the graph of f. Then use shifting, stretching, and/or reflecting to sketch the graph of g on the same axes.

1. $f(x) = x^2, \quad g(x) = x^2 - 2$

2. $f(x) = x^2, \quad g(x) = -x^2 + 4$

3. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x + 9}$

4. $f(x) = \log x, \quad g(x) = \log(x - 3)$

5. $f(x) = x^3, \quad g(x) = -(x - 1)^3 + 4$

6. $f(x) = \sqrt{x}, \quad g(x) = 2\sqrt{x + 4}$

7–10 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a basic function and applying transformations.

7. $f(x) = -(x + 3)^2$

8. $g(x) = (x - 2)^2 - 4$

9. $h(x) = 3\sqrt{x - 2} + 1$

10. $k(x) = -2\sqrt{x + 9} + 1$

11–12 ■ A quadratic function is given.

(a) Express the function in general form.

(b) Express the function in standard form.

(c) Find the x- and y-intercepts of its graph.

11. $f(x) = (x + 4)(x - 2)$

12. $g(x) = (x - 3)(x - 1) - 8$

13–18 ■ A quadratic function in general form is given.

(a) Express the function in standard form.

(b) What is the vertex of the graph of the function?

(c) Sketch the graph.

(d) Find the average rate of change of the function between $x = 1$ and $x = 3$.

13. $f(x) = x^2 + 6x$

14. $f(x) = -x^2 + 4x$

15. $f(x) = -x^2 + 4x + 5$

16. $f(x) = x^2 - 3x + \frac{1}{2}$

17. $f(x) = \frac{1}{2}x^2 - x - 3$

18. $f(x) = -2x^2 + 6x + 7$

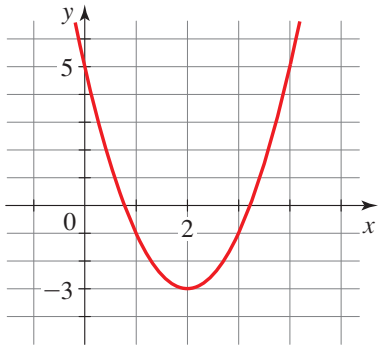
19–20 ■ Find an equation for the parabola with the given properties.

19. The vertex is $(-2, 6)$ and the y-intercept is 3.

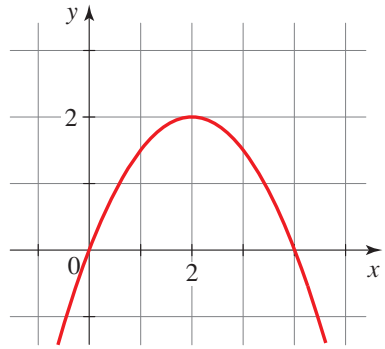
20. The vertex is $(3, -4)$ and it passes through the point $(1, 4)$.

21–22 ■ Find the quadratic function $f(x) = ax^2 + bx + c$ whose graph is shown.

21.



22.



23–28 ■ Find the maximum or minimum value of the function, and find the x-coordinate of the point at which the maximum or minimum occurs.

23. $f(x) = x^2 + 4x - 7$

24. $f(x) = -2x^2 + 10x + 65$

25. $f(x) = -\frac{1}{3}x^2 - x + 1$

26. $f(x) = 2 - 6x + 3x^2$

27. $f(x) = 15x - 5x^2$

28. $f(x) = \frac{1}{4}x^2 + 2x - 3$

29–32 ■ Solve the equation by factoring.

29. $x^2 + 7x = 0$

30. $x^2 - 6x + 8 = 0$

31. $2x^2 + 5x = 3$

32. $3x^2 = x + 2$

33–38 ■ A quadratic equation is given.

(a) Find the discriminant of the equation. How many real solutions does the equation have?

(b) Solve the equation (if it has real solutions).

33. $x^2 + 3x - 4 = 0$

34. $4x^2 - 3x - 1 = 0$

35. $4x^2 + 1 = 4x$

36. $x^2 + 2\sqrt{2}x + 2 = 0$

37. $x^2 + 2x + 5 = 0$

38. $3x^2 + 10 = 10x$

39–40 ■ A set of data is given.

(a) Use a graphing calculator to find the quadratic model that best fits the data.

(b) Make a scatter plot of the data, and graph the function you found in part (a) on the scatter plot. Does the model seem to fit the data well?

39.

x	0.5	1.0	1.7	2.2	2.5	3.0
y	6.0	3.4	2.0	2.3	3.2	5.6

40.

x	0	2	6	7	11	20
y	5	8	24	23	18	37

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

41. The Form of a Quadratic Function In this chapter we have worked with three different ways of writing the equation for a quadratic function:

- General form: $f(x) = ax^2 + bx + c$
- Standard form: $f(x) = a(x - h)^2 + k$
- Factored form: $f(x) = a(x - m)(x - n)$

(a) What does the number c represent in the general form?

(b) What do the numbers h and k represent in the standard form?

(c) What do the numbers m and n represent in the factored form?

(d) The graph of a quadratic function has x-intercepts 2 and 5 and y-intercept -20. On the basis of the fact that you have been given the x-intercepts, which of the three forms should you use to find the equation of the function? Use that form to find the equation.

(e) The graph of a quadratic function has vertex $(4, -6)$ and one x-intercept at 8. Which of the three forms should you use to find the equation of the function? Use that form to find the equation.

(f) The graph of the quadratic function f is the same as that of the function $g(x) = 3x^2 + 6x$, except that it has been shifted downward so that $f(2) = 5$. Which of the three forms should you use to find the equation of f? Use that form to find the equation.

(g) Find the maximum or minimum value of each of the functions you found in parts (d), (e), and (f).

42. The Parts of a Parabola In this problem we'll see how the vertex and the x-intercepts of a parabola are related to each other.

(a) A parabola has the equation $y = x^2 - 6x$. Find the x-intercepts of the parabola.

(b) Find the vertex of the parabola in part (a). How is the x-coordinate of the vertex related to the x-intercepts?

(c) Show that the relationship that you discovered in part (b) between the x-coordinate of the vertex and the x-intercepts also holds for the parabola $y = x^2 + 4x - 5$.

(d) If a parabola has x-intercepts 1 and 7, what must the x-coordinate of its vertex be? Find an equation for such a parabola with y-intercept -3.

(e) Find an equation for the quadratic function whose graph has x-intercepts 3 and 9 and whose maximum value is 27.

CONTEXTS

43. Height of a Stone Two stones are dropped simultaneously, one from the 24th floor and the other from the 32nd floor of a high-rise building. After t seconds, the one dropped from the 24th floor is at a height $h_1(t) = 240 - 16t^2$ above the ground, and the one dropped from the 32nd floor is at a height $h_2(t) = 320 - 16t^2$ above the ground. (The heights are measured in feet.)

(a) Sketch graphs of h_1 and h_2 on the same coordinate axes.

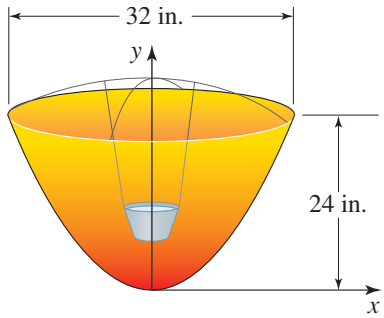
(b) What transformation would you need to perform on the graph of h_1 to get the graph of h_2? Express h_2 in terms of h_1.

44. Height of a Stone (Refer to Exercise 43.) Suppose that another stone is dropped from the 24th floor 5 seconds after the first one.

(a) What transformation would you have to perform on the function h_1 to obtain a function H that models the height of this new stone above the ground t seconds after the first stone was dropped (where $t \geq 5$)?

(b) Have the first two stones already hit the ground when the last one is dropped, or is one (or both) still in the air?

45. Parabolic Cooker *Solar cookers* are parabolic reflectors that concentrate the light and heat of sunlight at one spot, where a pot of food can be placed to cook. They are sometimes sold in camping stores and are also increasingly used in impoverished areas of the world where cooking fuel is scarce. The figure shows a parabolic bowl-shaped cooker in cross-section; the width is 32 inches, and the depth is 24 inches. Find a squaring function that models the shape of the cooker.

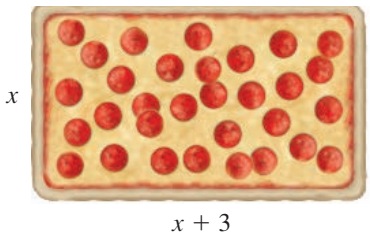


46. Parabolic Cooker (Refer to Exercise 45.) The width of a solar cooker determines how much solar energy it gathers. Its depth determines how close the "hot spot" is to the vertex: The deeper the bowl, the closer to the vertex the cooking pot can be placed. A new variety of solar cooker has the same width, 32 inches, as the one in Exercise 45, but its shape is modeled by the function $f(x) = \frac{1}{8}x^2$ instead. How deep is this new cooker?

47. Dimensions of a Rectangular Pizza Squarehead Pizza Parlor bakes its pizzas in the shape of rectangles that are all 3 inches longer than they are wide.

(a) Find a function A that models the area of a pizza in terms of its width x.

(b) If a small pizza has an area of 88 square inches, what are its dimensions?



48. Maximizing Revenue Miriam sells souvenir caps to tourists in a beach town. She observes that if she charges x dollars per cap, then she sells about caps per day. (Notice that the higher her price, the fewer caps she sells.)

(a) Explain why her revenue per day is given by the function $R(x) = x(48 - 1.6x)$.

(b) What price should she charge so that her revenue is as large as possible?

49. Horticulture Experiment A student performs an experiment to determine how much of a certain fertilizer should be applied to wheat seedlings to produce the maximum yield. He plants wheat in six identical containers and applies different amounts of fertilizer to each container. After the wheat reaches maturity, he harvests it and weighs each "mini-crop." His results are shown in the table.

Container	Fertilizer (g)	Yield (oz)
1	2.0	14.3
2	2.8	18.5
3	3.6	20.8
4	4.4	22.0
5	5.2	21.8
6	6.0	20.1

(a) Use a graphing calculator to find the quadratic model that best fits the data.

(b) Draw a scatter plot of the data, together with the quadratic function you found in part (a). Does the model seem to fit the data well?

(c) Use the model to determine how much fertilizer should be applied for the best yield.

CHAPTER 5 TEST

1. Use the graph of $f(x) = \sqrt{x}$ to graph the following functions.

(a) $g(x) = \sqrt{x - 2}$

(b) $h(x) = 2 - \sqrt{x}$

(c) $k(x) = 3\sqrt{x + 2}$

2. Let $f(x) = -x^2 + 4x + 12$.

(a) Express f in the standard form of a quadratic function.

(b) What is the vertex of the graph of f?

(c) Find the x-intercepts of the graph of f.

(d) Sketch the graph of f.

(e) Does f have a maximum or a minimum value? What is that value?

3. Find a quadratic function f whose graph has vertex $(4, -6)$ and whose y-intercept is 2.

4. For each equation, use the discriminant to determine how many real solutions each equation has, and then find all real solutions.

(a) $3x^2 + 5x - 2 = 0$

(b) $x^2 - 6x + 7 = 0$

(c) $2x^2 = 3x - 2$

5. **(a)** Use a calculator to find the quadratic function that best models the data in the table.

(b) Make a scatter plot of the data, and graph the function you found in part (a) on the scatter plot. Does the function seem to fit the data well?

x	0.3	0.5	1.9	2.0	2.4	2.6	2.6	2.9
y	3.2	4.3	5.8	5.7	5.9	5.2	5.4	3.9

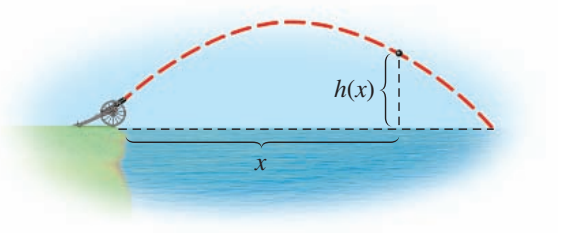
6. A cannonball that is fired out to sea from a shore battery follows a parabolic trajectory given by the graph of the equation

$$h(x) = 10x - 0.01x^2$$

where $h(x)$ is the height of the cannonball above the water when it has traveled a horizontal distance of x feet.

(a) What is the maximum height that the cannonball reaches?

(b) How far does the cannonball travel horizontally before splashing into the water?



1 Transformation Stories

OBJECTIVE To explore the relationship between graphical, algebraic, and verbal descriptions of transformations of functions in a real-world context.



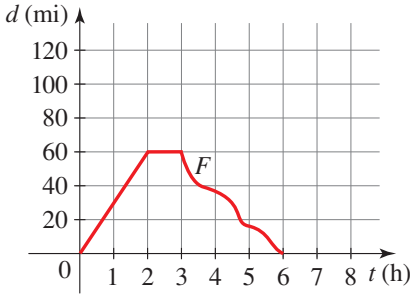
Vladimir Mucibabic/Shutterstock.com

Traveling salesman

In Exploration 3 following Chapter 1 we investigated how a graph of a real-world situation tells a story about that situation. Now that we have studied about transformations of functions, we can use graphs to tell even more elaborate stories.

In this exploration we investigate traveling salesman Jerome's original trip and some of its transformations. The graph in the margin shows Jerome's distance $F(t)$ in miles from his home t hours after 12:00 noon (time 0). We assume that Jerome travels along a straight road from home to work and back.

Here is the story this graph tells about Jerome's trip: He starts at noon and travels at a steady speed of 30 miles per hour until 2:00 P.M. (We know this because the graph is a straight line with slope 30.) He stops for the next hour (from 2:00 to 3:00 P.M.), perhaps for a business meeting, at a location 60 miles from his home. He starts driving back at 3:00 P.M. and arrives home at 6:00 P.M. He travels at various speeds, but his average speed for the trip back home is 20 miles per hour.

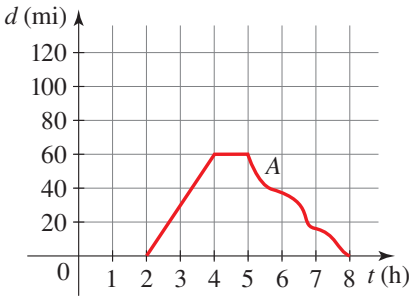


Traveling salesman Jerome's original trip

I. Reading a Transformation Story from a Graph

Let's look at some graphs of transformations of Jerome's trip. For each graph, state what the transformation is in words, then in symbols, and then describe the story that the graph tells about Jerome's transformed trip.

1. The graph below gives Jerome's distance $A(t)$ from home at time t.

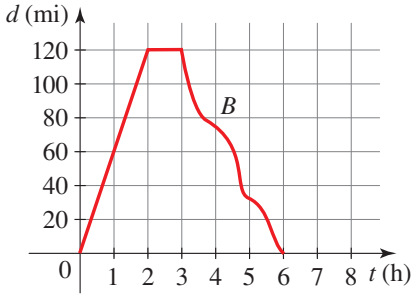


Verbal: Shift to the right 2 units

Symbolic: A(t) = F(t − 2)

Story: The trip is exactly the same as the original trip, but he started two hours later (at 2 P.M.).

2. The graph gives Jerome's distance $B(t)$ from home at time t.

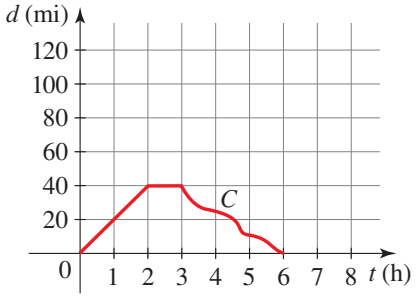


Verbal: _____

Symbolic: _____

Story: _____

3. The graph gives Jerome's distance $C(t)$ from home at time t.

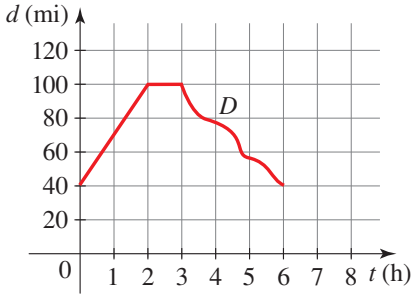


Verbal: _____

Symbolic: _____

Story: _____

4. The graph gives Jerome's distance $D(t)$ from home at time t.

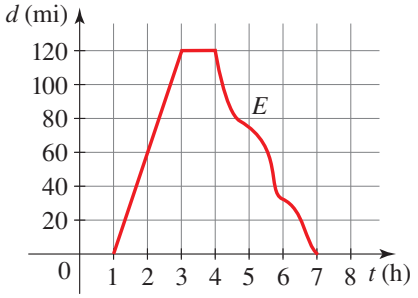


Verbal: _____

Symbolic: _____

Story: _____

5. The graph gives Jerome's distance $E(t)$ from home at time t.



Verbal: _____

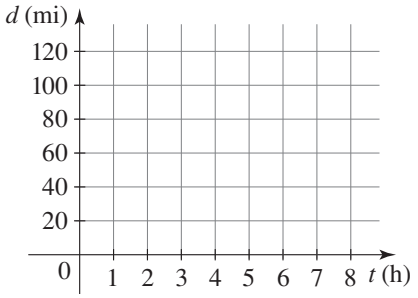
Symbolic: _____

Story: _____

II. Making a Transformation Story

Let's make up a story about Jerome's trip by using transformations.

1. Make a graph and describe a story that corresponds to the verbal description of a transformation of Jerome's original trip:

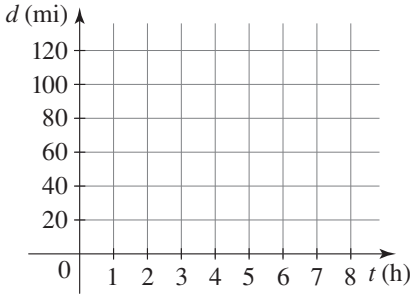


Verbal: Shift 2 units to the right and two units up.

Symbolic: _____

Story: _____

2. Make up your own transformations and story about Jerome's trip. Provide verbal, symbolic, and graphical descriptions of the transformations you have used, and then give a real-world description of the trip.



Verbal: _____

Symbolic: _____

Story: _____

2 Torricelli's Law

OBJECTIVE To obtain an expression for Torricelli's Law by fitting a quadratic function to data obtained from a simple experiment.



© mediacolor's/Alamy

Evangelista Torricelli

Evangelista Torricelli (1608–1647) was an Italian mathematician and scientist. He is best known for his invention of the barometer. In Torricelli's time it was known that suction pumps were able to raise water to a limit of about 9 meters, and no higher. The explanation at the time was that the vacuum in the pump could support the weight of only so much water. In studying this problem, Torricelli thought of using mercury, which is 14 times heavier than water, to test this theory. He made a one-meter-long tube sealed at the top end, filled it with mercury, and set it vertically in a bowl of mercury. The column of mercury fell to about 76 cm, leaving a vacuum at the top of the tube. In an impressive leap of insight, Torricelli realized that the column of mercury is held up not by the vacuum at the top of the tube, but rather by the air pressure outside the tube pressing down on the mercury in the bowl. He wrote:

> I claim that the force which keeps the mercury from falling . . . comes from outside the tube. On the surface of the mercury which is in the bowl rests the weight of a column of fifty miles of air. Is it a surprise that . . . [the mercury] should rise in a column high enough to make equilibrium with the weight of the external air which forces it up?

The device Torricelli made was the first barometer for measuring air pressure.

Another of Torricelli's discoveries, based on the same principle but applied to water pressure, is that the speed of a fluid through a hole at the bottom of a tank is related to the height of the fluid in the tank. The precise relationship is known as Torricelli's Law.

I. Collecting the Data

In this exploration we use easily obtainable materials to conduct an experiment and collect data on the speed of water leaking through a hole at the bottom of a cylindrical tank. To do this, we measure the height of the water in the tank at different times.



Richard Le Borne

You will need:

- An empty 2-liter plastic soft-drink bottle
- A method of drilling a small (4 mm) hole in the plastic bottle
- Masking tape
- A metric ruler or measuring tape
- An empty bucket

Procedure:

This experiment is best done as a classroom demonstration or as a group project with three students in each group: a *timekeeper* to call out seconds, a *bottle keeper* to estimate the height every 10 seconds, and a *record keeper* to record these values.

1. Drill a 4 mm hole near the bottom of the cylindrical part of a 2-liter plastic soft-drink bottle. Attach a strip of masking tape marked in centimeters from 0 to 10, with 0 corresponding to the top of the hole.
2. With one finger over the hole, fill the bottle with water to the 10-cm mark. Place the bottle over the bucket.

Time (s)	Height (cm)
10	
20	
30	
40	
50	
60	

3. Take your finger off the hole to allow the water to leak into the bucket. Record the values of height of the water $h(t)$ for $t = 10, 20, 30, 40, 50$, and 60 seconds.

II. Testing the Theory

Torricelli's Law states that when a fluid leaks through a hole at the bottom of a cylindrical tank, the height h of fluid in the tank is related to the time t the fluid has been leaking by a quadratic function:

$$h(t) = at^2 + bt + c$$

where the coefficients a, b, and c depend on the type of fluid, the radii of the cylinder and the hole, and the initial height of the fluid.

1. Use a graphing calculator to find the quadratic function that best fits the data.

$$h(t) = \square\, t^2 + \square\, t + \square$$

2. Graph the function you found together with a scatter plot of the data in the same viewing rectangle. Does the graph appear to fit the data well?

III. Another Method

Torricelli's Law actually gives more information about the form of the quadratic function model. It can be shown that the function defining the height of the fluid can be expressed as

$$h(t) = (C - kt)^2$$

where the coefficients C and k depend on the type of fluid, the radii of the cylinder and the hole, and the initial height of the fluid.

1. Let's find C and k for the experiment we conducted in Part I.

   (a) Use the fact that the height $h(t)$ of the water is 10 cm when the time t is 0 to solve for C in the above equation.

   $$C = _____$$

   (b) Use the height $h(t)$ you obtained in the experiment when the time t is 60 seconds and the value of C you obtained in part (a) to solve for k in the above equation.

   $$k = _____.$$

   (c) Use the results of parts (a) and (b) to find an expression for $h(t)$.

   $$h(t) = (\square - \square\, t)^2$$

2. Expand the expression for $h(t)$ that you obtained in 1(c).

$$h(t) = \square\, t^2 + \square\, t + \square$$

How does this model compare with the model you got in Part II?

3 Quadratic Patterns

OBJECTIVE To recognize quadratic data and find quadratic functions that fit the data exactly.

In Exploration 2 of Chapter 2 (page 233) we discussed the importance of finding patterns, and we found many linear patterns. In Exploration 2 of Chapter 3 (page 315) we found exponential patterns. In this exploration we find quadratic patterns. We've already seen how quadratic functions model many real-world phenomena. But quadratic patterns also occur in unexpected places, such as in cutting a pizza, as we'll see in this exploration.

I. Recognizing Quadratic Data

In this exploration we assume that the inputs are the equally spaced numbers

$$0, 1, 2, 3, \ldots$$

We want to find properties of the outputs that guarantee that there is a quadratic function that exactly models the data. So let's consider the quadratic function

$$f(x) = ax^2 + bx + c$$

We find the first differences of the outputs and the **second differences**—that is, the differences of the first differences. In the next question you are asked to perform the calculations that confirm the entries in the following table. In particular, *the second differences are constant*.

table **1**
Second differences for $f(x) = ax^2 + bx + c$

x	0	1	2	3	4
$f(x)$	c	$a + b + c$	$4b + 2b + c$	$9b + 3b + c$	$16b + 4b + c$
First difference	—	$a + b$	$3a + b$	$5a + b$	$7a + b$
Second difference	—	—	$2a$	$2a$	$2a$

1. For the quadratic function $f(x) = ax^2 + bx + c$, let's consider the inputs 0, 1, 2, 3, . . .

(a) Find the outputs corresponding to the inputs 0, 1, 2, 3, ...

$f(0) =$ ___c___, $f(1) =$ _____, $f(2) =$ _____, $f(3) =$ _____

(b) Use your answers to part (a) to find the first differences:

$f(1) - f(0) =$ ___a + b___, $f(2) - f(1) =$ _____, $f(3) - f(2) =$ _____

(c) Use your answers to part (b) to find the second differences:

___2a___, _____, _____

(d) Do your answers to parts (a)–(c) match the entries in Table 1?

2. A data set is given in the table.

x	0	1	2	3	4	5	6
y	5	10	19	32	49	70	95
First difference	—						
Second difference	—	—					

(a) Fill in the entries for the first and second differences.

(b) Observe that the second differences are constant, so there is a quadratic function

$$f(x) = ax^2 + bx + c$$

that models the data. To find a, b, and c, let's compare the entries in this table to those in Table 1. Comparing the output corresponding to the input 0 in each of these tables, we conclude that

$$c = _____$$

Comparing the second differences in each of these tables, we conclude that

$$2a = _____$$

Comparing the first differences in each of these tables, we conclude that

$$a + b = _____$$

So $a =$ _____, $b =$ _____, and $c =$ _____.

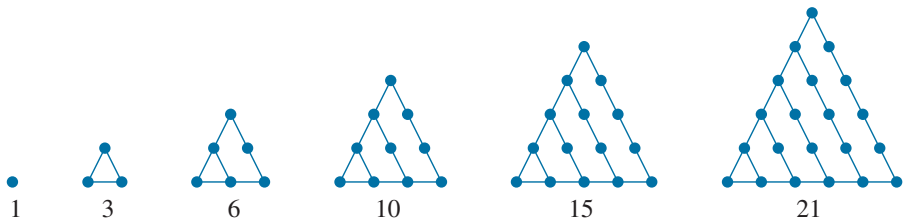
(c) A quadratic function that models the data is

$$f(x) = ____x^2 + ____x + ____$$

(d) Check that the function f you found in part (b) matches the data. That is, check that $f(0), f(1), f(2), \ldots$ match the y-values in the data.

II. The Triangular Numbers

Many real-world situations have quadratic patterns. Here, we experiment with the triangular numbers. The **triangular numbers** are the numbers obtained by placing dots in a triangular shape as shown.



1. Let's find a formula for the nth triangular number.

(a) Complete the table for the first and second differences of the triangular numbers

n	0	1	2	3	4	5
Triangular number $f(n)$	1	3	6	10	15	21
First difference	—					
Second difference	—	—				

(b) Observe that the second differences are constant, so there is a quadratic function

$$f(n) = an^2 + bn + c$$

that models the data. Find a, b, and c by comparing the entries in the table in part (a) to those in Table 1. Comparing the output corresponding to the input 0 in each of these tables, we conclude that

$$c = _____$$

Comparing the second differences in each of these tables, we conclude that

$$2a = _____$$

Comparing the first differences in each of these tables, we conclude that

$$a + b = _____$$

So $a =$ _____, $b =$ _____, and $c =$ _____.

(c) A quadratic function that models the data is

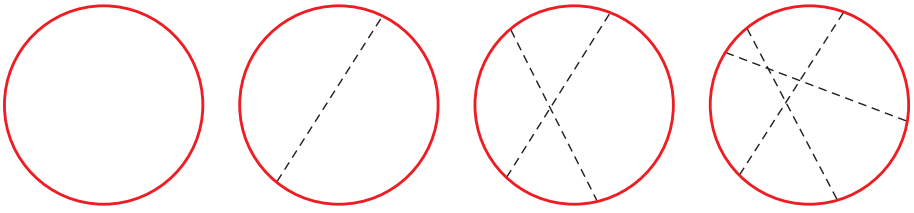
$$f(n) = ____n^2 + ____n + ____$$

Check that the function f models the data exactly. That is, check that $f(0), f(1), f(2), \ldots$ are the first, second, third, . . . triangular numbers.

(d) Use the model you found in part (c) to find the 100th triangular number (when n is 100).

III. Cutting a Pizza

Let's cut up a pizza—but not in the usual way. We want to cut the pizza in such a way that we get the maximum possible number of pieces after each cut. The diagram shows the number of pieces we can get by making 0, 1, 2, and 3 cuts.



1. Let's find a formula for the maximum number of pieces of pizza we can get by making n cuts.

(a) Show that by making one more cut we can get 11 pieces of pizza.

(b) Complete the table for the first and second differences for the number of pieces of pizza.

Number of cuts n	0	1	2	3	4
Number of pieces $f(n)$	1	2	4	7	11
First difference	—				
Second difference	—	—			

(c) Observe that the second differences are constant, so there is a quadratic function

$$f(n) = an^2 + bn + c$$

that models the data. Find a, b, and c by comparing the entries in the table in part (b) to those in Table 1.

$$a = _____, \quad b = _____, \quad c = _____$$

(d) A quadratic function that models the data is

$$f(n) = ____n^2 + ____n + ____$$

Check that the function f models the data exactly. That is, check that $f(0), f(1), f(2), \ldots$ give the number of pizza pieces if we make 1, 2, 3, . . . cuts.

(e) Use the model you found in part (d) to find the number of pieces of pizza obtained by making 100 cuts.

6 Power, Polynomial, and Rational Functions



Hulton Archive/Getty Images

6.1 Working with Functions: Algebraic Operations

6.2 Power Functions: Positive Powers

6.3 Polynomial Functions: Combining Power Functions

6.4 Fitting Power and Polynomial Curves to Data

6.5 Power Functions: Negative Powers

6.6 Rational Functions

EXPLORATIONS

1 Only in the Movies?
2 Proportionality: Shape and Size
3 Managing Traffic
4 Alcohol and the Surge Functions

Only in the movies? Movies have some pretty fantastic creatures. Giant apes as tall as skyscrapers, planet-sized spaceships, and a 50-foot-tall woman (yes, there is such a movie: "Attack of the 50 Ft. Woman," starring Daryl Hannah in the 1993 remake of the 1958 cult classic). These creatures are so large that they dwarf even giant prehistoric dinosaurs. Can such gargantuan creatures actually exist? Should we fear that someday a real-life King Kong might attack our towns? To answer such questions, we need to understand the relationship between shape and size. Shape and size (length, area, and volume) are related to each other by power functions, the main topic of this chapter. We'll see how power functions allow us to calculate the weight of a 500-foot ape without having to actually meet one (see Exploration 1, page 554).

6.1 Working with Functions: Algebraic Operations

- Adding and Subtracting Functions
- Multiplying and Dividing Functions

IN THIS SECTION... we study how functions can be combined using algebraic operations: addition, subtraction, multiplication, and division.



Image copyright originalpunkt, 2009. Used under license from Shutterstock.com

It would be fun to throw a ball up into the air and have it keep going up indefinitely. That's exactly what would happen if there were no gravity. In that case, if you threw a ball up at a speed of 48 feet per second, it would continue traveling at this same speed indefinitely, so it would reach a height of $f(t) = 48t$ feet in t seconds. But because of gravity, the ball is pulled down a distance $g(t) = 16t^2$ feet in t seconds. To find out how high the ball would actually reach in the presence of gravity, we need to *subtract* the distance gravity pulls down on the ball from the distance the ball travels up under the force of the initial throw. So $f(t) - g(t)$ gives the height the ball reaches at time t. The situation is as follows.

- Throwing a ball pushes it **up** a distance $f(t)$ feet in t seconds, and at the same time gravity pulls it **down** a distance $g(t)$ feet.
- The height the ball **actually reaches** in t seconds is given by $f(t) - g(t)$.

So the actual height the ball reaches at time t is modeled by a combination of the two functions f and g (see Example 3). In this section we will see several other real-world situations in which a model is found by combining functions using arithmetic operations.

Adding and Subtracting Functions

Two functions f and g can be combined to form new functions $f + g$ and $f - g$ in a manner similar to the way in which we add and subtract numbers. For example, we define the function $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

The new function $f + g$ is called the **sum** of the functions f and g; its value at x is $f(x) + g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. So if the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$. We define the **difference** $f - g$ of the functions f and g in a similar way, as described in the following box.

Sums and Differences of Functions

Let f and g be functions with domains A and B respectively. We define new functions as follows:

- The **sum** of f and g is the function $f + g$ defined by
$$(f + g)(x) = f(x) + g(x)$$
- The **difference** of f and g is the function $f - g$ defined by
$$(f - g)(x) = f(x) - g(x)$$

Each of these functions has the domain $A \cap B$.

example 1 Adding and Subtracting Functions

Let $f(x) = x^2 + 4$ and $g(x) = 2x + 8$.

(a) Find the functions $f + g$ and $f - g$ and their domains.

(b) Find $(f + g)(1)$ and $(f - g)(1)$.

(c) Sketch graphs of f, g, $f + g$, and $f - g$.

Solution

(a) The functions f and g have domain $\mathbb{R}$, so the domain of $f + g$ is also $\mathbb{R}$.

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) && \text{Definition of } f + g\\ &= (x^2 + 4) + (2x + 8) && \text{Definitions of } f \text{ and } g\\ &= x^2 + 2x + 12 && \text{Simplify}\end{aligned}$$

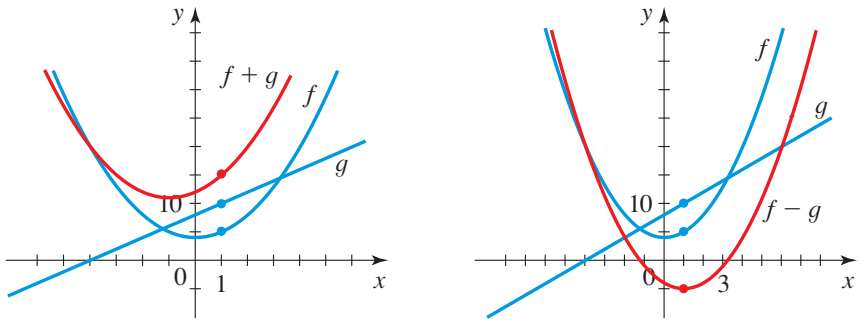
The domain of $f - g$ is also $\mathbb{R}$. We have

$$\begin{aligned}(f - g)(x) &= f(x) - g(x) && \text{Definition of } f - g\\ &= (x^2 + 4) - (2x + 8) && \text{Definitions of } f \text{ and } g\\ &= x^2 - 2x - 4 && \text{Distributive Property, simplify}\end{aligned}$$

(b) Each of these values exists because $x = 1$ is in the domain of both functions. From part (a) we have

$$(f + g)(1) = 1^2 + 2 \cdot 1 + 12 = 15$$
$$(f - g)(1) = 1^2 - 2 \cdot 1 - 4 = -5$$

(c) The graphs are shown in Figure 1.



(a) Graph of $f + g$ (b) Graph of $f - g$

figure **1**

■ **NOW TRY EXERCISES 13 AND 15** ■

In general, the graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

example **2**

Graphical Addition

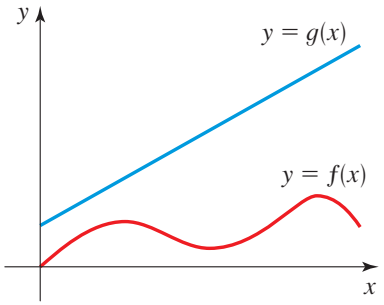


figure **2**

The graphs of f and g are shown in Figure 2. Use graphical addition to graph the function $f + g$.

Solution

We obtain the graph of $f + g$ by "graphically adding" the values of $f(x)$ and $g(x)$ as shown in Figure 3. This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f + g$.

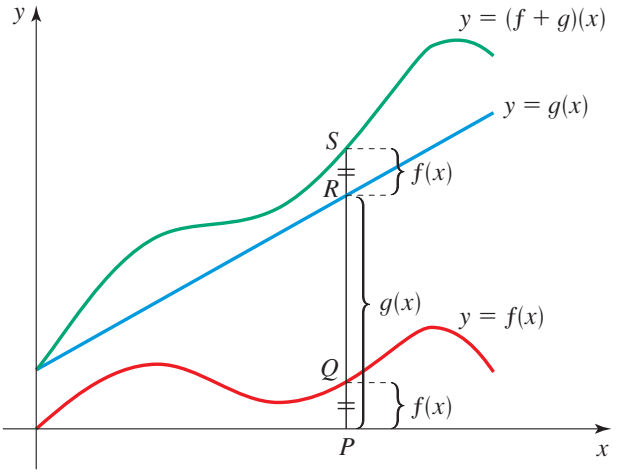


figure **3** Graphical addition of f and g

■ **NOW TRY EXERCISE 21** ■

example **3**

Graphical Subtraction

A ball is thrown straight up with a velocity of 48 feet per second. In the absence of gravity the ball reaches a height $f(t) = 48t$ in t seconds. Gravity pulls the ball down $g(t) = 16t^2$ feet in t seconds.

(a) Find a function h that models the height of the ball t seconds after it is thrown.

(b) Sketch a graph of the functions f, g, and h. On what interval is the model h valid?

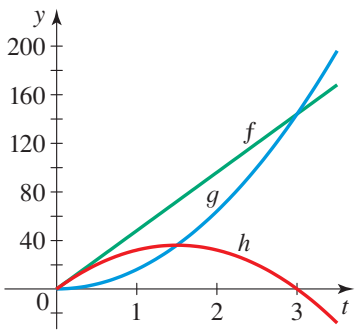


figure **4** Graphs of f, g, and h

Solution

(a) The function we want is

$$h(t) = f(t) - g(t) \qquad \text{Ball moves up } f(t) \text{ and down } g(t)$$
$$= 48t - 16t^2 \qquad \text{Definitions of } f \text{ and } g$$

(b) The functions f, g, and h are graphed in Figure 4. The ball is at ground level when $h(t) = 0$. From the graph we see that $h(t) = 0$ at times 0 and 3 seconds. The model is valid on the interval $[0, 3]$; for numbers outside this interval we see from the graph that the ball would be below ground level.

■ **NOW TRY EXERCISE 23** ■

Combining functions occurs in many business models. For example, if the revenue a company receives as a result of selling x units of their product is given by $R(x)$ and the cost of producing these x units is $C(x)$, then the profit $P(x)$ is the difference of these two functions.

$$P(x) = R(x) - C(x)$$

The next example illustrates the process.

example **4** **Finding Profit**

A doughnut shop sells bear claws for 80 cents each, so the revenue from selling x bear claws is $R(x) = 0.80x$ dollars. The cost of producing x bear claws is modeled by $C(x) = 50 + 0.40x - 0.0002x^2$. These models are valid for x in the interval $[0, 1000]$.

The cost function C includes \$50 of fixed costs and \$0.40 per unit produced. The term $-0.0002x^2$ reflects the reduction in cost per item when large quantities are produced.

(a) Find a function P that models the profit when x bear claws are produced and sold.

(b) Sketch graphs of the functions R, C, and P. How many bear claws must be sold before revenue exceeds cost?

Solution

(a) The profit function is the difference between the revenue and cost functions.

$$\begin{aligned} P(x) &= R(x) - C(x) && \text{Definition of profit} \\ &= 0.80x - (50 + 0.40x - 0.0002x^2) && \text{Definitions of } R \text{ and } C \\ &= 0.80x - 50 - 0.40x + 0.0002x^2 && \text{Distributive Property} \\ &= -50 + 0.40x + 0.0002x^2 && \text{Simplify} \end{aligned}$$

(b) Since the models are valid in the interval $[0, 1000]$, we sketch graphs of R, C, and P on that interval (see Figure 5).

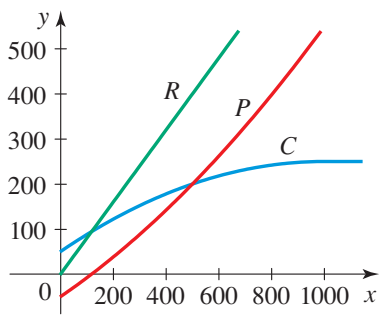


figure **5** Graphs of R, C, and P

We see from the graph that revenue exceeds cost when x is greater than about 120. So about 120 bear claws must be produced and sold before a profit is made. (Note from the graph that the values of P are positive when x is greater than 120.)

■ **NOW TRY EXERCISE 35** ■

Multiplying and Dividing Functions

The functions f and g can be combined to form new functions $f \cdot g$ and f/g in a manner similar to the way in which we defined the sum and difference of two functions.

Products and Quotients of Functions

Let f and g be functions with domains A and B. We define new functions as follows:

- The **product** of f and g is the function fg defined by

$$(fg)(x) = f(x) \cdot g(x)$$

The domain of fg is $A \cap B$.

- The **quotient** of f and g is the function f/g defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$; that is, we must remove the values of x where the denominator $g(x)$ is 0.

example 5 Products and Quotients of Functions

Let $f(x) = x + 1$ and $g(x) = \sqrt{x}$.

(a) Find the functions fg and f/g and their domains.

(b) Find $(fg)(4)$ and $(f/g)(4)$.

Solution

(a) The domain of f is all real numbers, and the domain of g is $\{x \mid x \geq 0\}$, so their intersection is $\{x \mid x \geq 0\}$. We have

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x) && \text{Definition of } fg \\ &= (x + 1)\sqrt{x} && \text{Definitions of } f \text{ and } g\end{aligned}$$

The domain of fg is $\{x \mid x \geq 0\}$. Also,

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{Definition of } fg \\ &= \frac{x + 1}{\sqrt{x}} && \text{Definitions of } f \text{ and } g\end{aligned}$$

The domain of f/g is $\{x \mid x > 0\}$. Note that in the domain of f/g we exclude 0 because $g(0)$ is 0.

(b) We can calculate each of these values because 4 is in the domain of each function. Using the results of part (a), we have

$$(fg)(4) = (4 + 1)\sqrt{4} = 10$$

$$\left(\frac{f}{g}\right)(4) = \frac{4 + 1}{\sqrt{4}} = \frac{5}{2}$$

■ NOW TRY EXERCISE 31 ■

example 6 Products of Functions

An employee invests part of his monthly paycheck in his company's stock. The table gives the price $P(t)$ (in dollars) of the stock on the purchase day and the number $N(t)$ of shares the employee buys.

(a) What is the meaning of the product function $R(t) = N(t)P(t)$?

(b) Calculate the values of $R(t)$ for the indicated months. What do you conclude?

Month t	1	2	3	4	5	6	7	8	9	10	11	12
Price $P(t)$	18	18	20	21	23	23	23	22	20	18	15	15
Number $N(t)$	10	10	12	12	12	12	12	15	20	25	30	30

Solution

(a) The function $R(t)$ gives the amount the employee invests in his company's stock each month.

(b) The value of $R(t)$ at any t is the product of $P(t)$ and $N(t)$. Therefore $R(1) = N(1)P(1) = 18 \cdot 10 = 180$. The other entries in the table below are calculated similarly.

Month t	1	2	3	4	5	6	7	8	9	10	11	12
$R(t)$	180	180	240	252	276	276	276	330	400	450	450	450

It appears that this employee is investing more money every month in his company's stock.

■ **NOW TRY EXERCISE 39** ■

6.1 Exercises

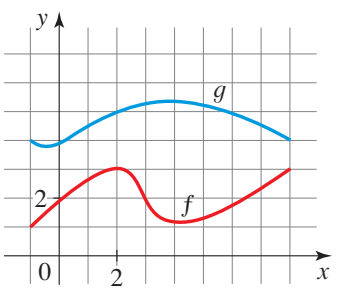
CONCEPTS

Fundamentals

1. Let f and g be functions defined for all real numbers.

(a) By definition, $(f + g)(x) =$ _____ and $(f - g)(x) =$ _____.
So if $f(3) = 5$ and $g(3) = 7$, then $(f + g)(3) =$ _____ and $(f - g)(3) =$ _____.

(b) By definition, $(fg)(x) =$ _____ and $(f/g)(x) =$ _____. So if $f(1) = 3$ and $g(1) = 6$, then $(fg)(1) =$ _____ and $(f/g)(1) =$ _____.



2. Use the graphs of f and g in the figure in the margin to find the following.

$(f + g)(2) =$ _____ $(f - g)(2) =$ _____

$(fg)(2) =$ _____ $(f/g)(2) =$ _____

Think About It

3–4 ■ True or false?

3. If $f(x) > 0$ for all x, then the graph of $f + g$ is above the graph of g for all x.

4. If $f(x) < 0$ for all x, then the graph of fg is below the graph of g for all x.

SKILLS

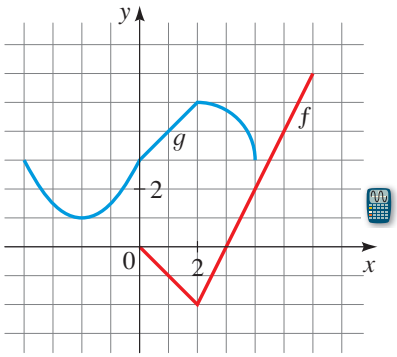
5–6 ■ Functions f and g are given in the table. Complete the table to find $f + g$ and $f - g$.

5.

t	0	1	2	3	4
$f(t)$	30	24	22	17	13
$g(t)$	294	312	331	352	370
$(f + g)(t)$					
$(f - g)(t)$					

6.

t	0	1	2	3	4
$f(t)$	131	282	291	332	504
$g(t)$	459	376	367	274	209
$(f + g)(t)$					
$(f - g)(t)$					



7–12 ■ Use the graphs of f and g shown in the margin to evaluate the expression.

7. $(f + g)(2)$

8. $(f - g)(0)$

9. $(fg)(1)$

10. $(g/f)(0)$

11. $(f/g)(0)$

12. $(fg)(4)$

13–14 ■ The functions f and g are given.

(a) Find the function $f + g$. Draw graphs of f, g, and $f + g$ in the same viewing rectangle.

(b) Find the function $f - g$. Draw graphs of f, g, and $f - g$ in the same viewing rectangle.

13. $f(x) = 2x^2 + 1, \quad g(x) = 3x + 2$

14. $f(x) = 3x^2 + 5, \quad g(x) = 4x + 7$

15–20 ■ The functions f and g are given.

(a) Find the functions $f + g$ and $f - g$ and their domains.

(b) Evaluate the functions $f + g$ and $f - g$ at the indicated value of x, if defined.

15. $f(x) = x^2, \quad g(x) = x - 3; \quad 1$

16. $f(x) = 0.5x^2, \quad g(x) = x + 2; \quad -1$

17. $f(x) = x^2 - 5x, \quad g(x) = 2 - 7x^2; \quad 2$

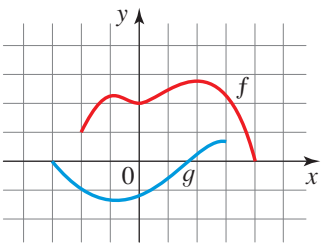
18. $f(x) = x^2 + 2x, \quad g(x) = 3x^2 - 1; \quad 1$

19. $f(x) = \sqrt{7 + x}, \quad g(x) = \sqrt{3 + x}; \quad 3$

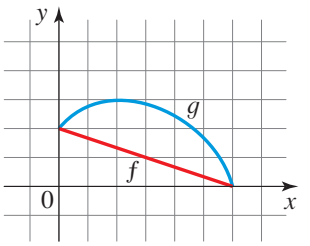
20. $f(x) = \sqrt{4 - x}, \quad g(x) = \sqrt{1 + x}; \quad 5$

21–22 ■ Use graphical addition to sketch the graph of $f + g$.

21.



22.



23–26 ■ The functions f and g are given.

(a) Draw the graphs of f, g, and $f + g$ on a common screen to illustrate graphical addition.

(b) Draw the graphs of f, g, and $f - g$ on a common screen to illustrate graphical subtraction.

23. $f(x) = x^2, \quad g(x) = x$

24. $f(x) = x^4, \quad g(x) = x^2$

25. $f(x) = \sqrt{1 + 0.5x}, \quad g(x) = \sqrt{1 - 0.5x}$

26. $f(x) = x^2, \quad g(x) = 0.5x$

27–28 ■ Functions f and g are given in the table. Use the table to find fg and f/g, if defined.

27.

t	0	1	2	3	4
$f(t)$	−3	−0.5	3	5	6
$g(t)$	2	6	7	−5	−2
$(fg)(t)$					
$(f/g)(t)$					

28.

t	0	1	2	3	4
$f(t)$	−1	9	0	5	−2
$g(t)$	−21	−25	−11	0	1
$(fg)(t)$					
$(f/g)(t)$					

29–34 ■ The functions f and g are given.

(a) Find the functions fg and f/g and their domains.

(b) Evaluate the functions fg and f/g at the indicated value, if defined.

29. $f(x) = 3x + 4, \quad g(x) = x + 5; \quad 2$

30. $f(x) = 2x - 1, \quad g(x) = 5x + 10; \quad -1$

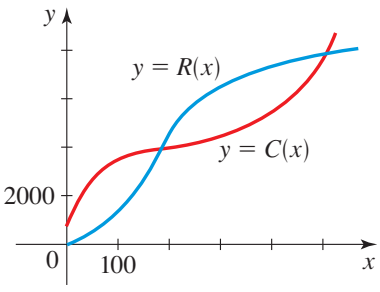
31. $f(x) = x - 3, \quad g(x) = x^2; \quad 0$

32. $f(x) = 3x^2, \quad g(x) = 4x + 8; \quad -2$

33. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{4}{x + 4}; \quad 4$

34. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = \dfrac{x}{x + 1}; \quad 0$

CONTEXTS



35. Revenue, Cost, and Profit A store sells a certain type of digital camera. The revenue from selling x cameras is modeled by $R(x) = 450x - 0.5x^2$. The cost of producing x cameras is modeled by $C(x) = 9300 + 150x - 0.1x^2$. These models are valid for x in the interval $[0, 450]$.

(a) Find a function that models the profit from producing and selling x cameras.

(b) Sketch a graph of the revenue, cost and profit functions. How many cameras must be sold in one week before revenue exceeds cost.

36. Revenue, Cost, and Profit The figure in the margin shows graphs of the cost and revenue functions reported by a manufacturer of tires.

(a) Identify on the graph the value of x for which the profit is 0.

(b) Use graphical addition to sketch a graph of the profit function.

37–38 ■ Revenue, Cost, and Profit A print shop makes bumper stickers for election campaigns. If x stickers are ordered (where $x < 10{,}000$), then the price per sticker is $0.15 - 0.000002x$ dollars, and the total cost of producing the order is $0.095x - 0.0000005x^2$ dollars.

37. Use the fact that revenue = price per item × number of items sold to express $R(x)$, the revenue from an order of x stickers, as a product of two functions of x.

38. Use the fact that profit = revenue − cost to express $P(x)$, the profit on an order of x stickers, as a difference of two functions of x.

39. Home Sales Let $f(t)$ be the number of existing homes sold in the United States in month t of 2007, and let g be the same function as f but for 2008. The table gives the values of f and g.

(a) What is the meaning of the function $h(t) = (f - g)(t)$?

(b) What is the meaning of the function $R(t) = \dfrac{h(t)}{f(t)}$?

(c) Calculate the values of $R(t)$ for each month. What do you conclude?

Month t	1	2	3	4	5	6	7	8	9	10	11	12
2007 sales: $f(t)$ (thousands)	532	550	509	494	494	479	480	458	426	422	418	409
2008 sales: $g(t)$ (thousands)	408	419	412	408	416	404	418	409	428	409	370	396

40. Labor Force The following table shows the number of workers $L(t)$ in the U.S. labor force and the number $E(t)$ of employed workers for each year from 2001 to 2008, as reported by the Bureau of Labor Statistics from the Current Population Survey.

(a) What is the meaning of the function $U(t) = L(t) - E(t)$?

(b) What is the meaning of the quotient functions $R_1(t) = \dfrac{E(t)}{L(t)}$ and $R_2(t) = \dfrac{U(t)}{L(t)}$?

(c) Calculate the values of $R_1(t)$ and $R_2(t)$ for the indicated years. What do you conclude?

t	2001	2002	2003	2004	2005	2006	2007	2008
Labor force $L(t)$ (thousands)	143,734	144,863	146,510	147,401	149,320	151,428	153,124	154,287
Employed $E(t)$ (thousands)	136,933	136,485	137,736	139,252	141,730	144,427	146,047	145,362

41. Smoking in the United States The table below shows estimated numbers of smokers as well as the total population of the United States from 1970 to 2007.

(a) What is the meaning of the quotient function $R(t) = \dfrac{S(t)}{P(t)}$?

(b) Calculate the values of $R(t)$ for the indicated years. What do you conclude?

t	1970	1980	1990	2000	2007
Smokers $S(t)$ (thousands)	76,035	75,212	63,421	65,571	62,737
Population $P(t)$ (thousands)	203,302	226,542	248,710	281,422	301,621

42. Crude Oil Imports The table shows the annual consumption as well as the annual imports of crude oil and petroleum products from 2002 to 2008, as reported by the Energy Information Administration of the U.S. Department of Energy.

(a) What is the meaning of the quotient function $R(t) = \dfrac{I(t)}{C(t)}$?

(b) Calculate the values of $R(t)$ for the indicated years. What do you conclude?

t	2002	2003	2004	2005	2006	2007	2008
Barrels imported per day $I(t)$ (thousands)	10,546	11,238	12,097	12,549	12,390	12,036	11,113
Barrels consumed per day $C(t)$ (thousands)	19,761	20,034	20,731	20,802	20,687	20,680	19,498

6.2 Power Functions: Positive Powers

- Power Functions with Positive Integer Powers
- Direct Proportionality
- Fractional Positive Powers
- Modeling with Power Functions

IN THIS SECTION... we study power functions with positive powers, and their graphs. We compare them with exponential functions and see how they arise in geometry and science.

GET READY... by reviewing the rules of exponents in **Algebra Toolkits A.3** and **A.4**, and solving power equations in **Algebra Toolkit C.1**. Test your understanding by doing the **Algebra Checkpoint** at the end of this section.

Some of the most familiar examples of functions are power functions. In geometry the formulas for the area of a square ($A = x^2$) or a circle ($A = \pi r^2$) and the volume of a cube ($V = x^3$) or a sphere ($V = \frac{4}{3}\pi r^3$) all involve power functions. In physics, power functions arise in Galileo's formula ($d = 16t^2$) for the distance traveled by a falling object and in Kepler's Third Law ($T = Cd^{3/2}$) for the period of revolution of a planet around the sun. (See Exercise 16, Section 6.4.). We will also see that power functions occur in biology, where they help us to determine the number of species that a habitat can support or how fast an animal walks naturally.

Power Functions with Positive Integer Powers

A *power function* is one in which the variable is raised to a fixed power.

Power Functions

A **power function** is a function of the form

$$f(x) = Cx^p$$

where the constant C and the power p are any nonzero numbers.

In this section we discuss the case where the power p is positive, and in Section 6.5 we will deal with negative powers.

We are already familiar with the cases in which the power is $p = 1$ or $p = 2$. If $C = 1$ and $p = 1$, the power function is just $f(x) = x$, whose graph is the line $y = x$. If $C = 1$ and $p = 2$, we have the squaring function $f(x) = x^2$, whose graph is the parabola $y = x^2$. Figure 1 shows the graphs of the power functions with $C = 1$ for the first several positive integer powers.

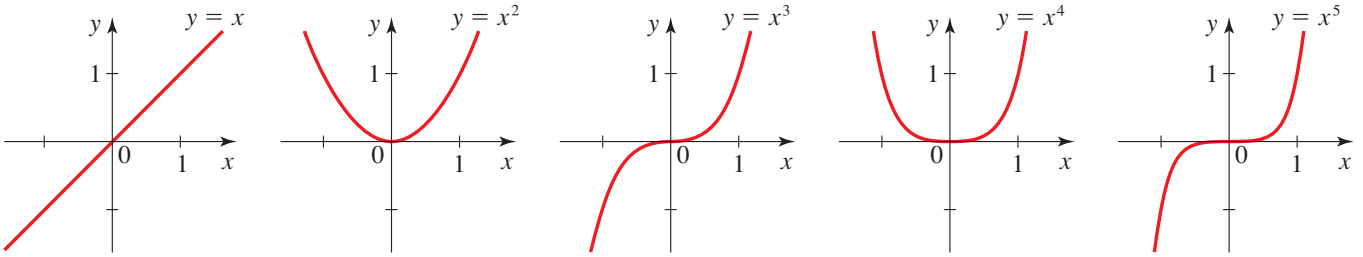


figure **1** Graphs of $f(x) = x^p$ for $p = 1, 2, 3, 4$, and 5

Notice from Figure 1 that the general shape of the graph of the power function $f(x) = x^p$ depends on whether p is odd or even. If p is an even positive integer, then the graph of $f(x) = x^p$ is similar to the parabola $y = x^2$. If p is an odd positive integer, then the graph of $f(x) = x^p$ is similar to that of $f(x) = x^3$. Observe from Figure 2, however, that as p increases, the graph of $y = x^p$ becomes flatter near 0 and steeper when $x > 1$ and when $x < -1$.

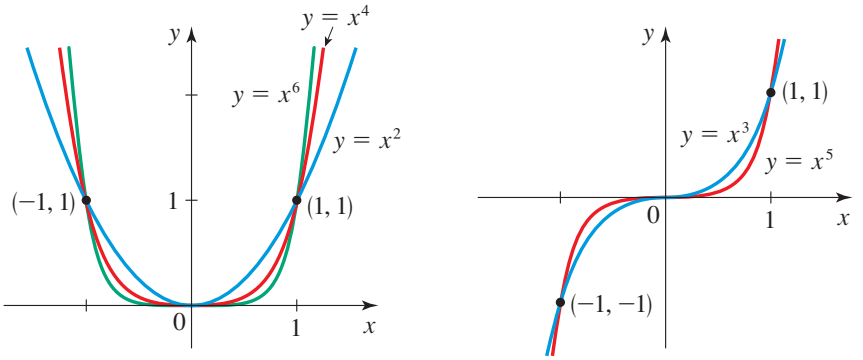


figure **2** Families of power functions

We see from Figures 1 and 2 that for $x > 1$, power functions with larger exponents grow faster than power functions with smaller exponents. In the following example we compare growth rates of a power function and an exponential function.

example 1 Comparing Power and Exponential Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 3]$ by $[0, 8]$

(b) $[0, 6]$ by $[0, 25]$

(c) $[0, 20]$ by $[0, 1000]$

Solution

(a) Figure 3(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 3(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 3(c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

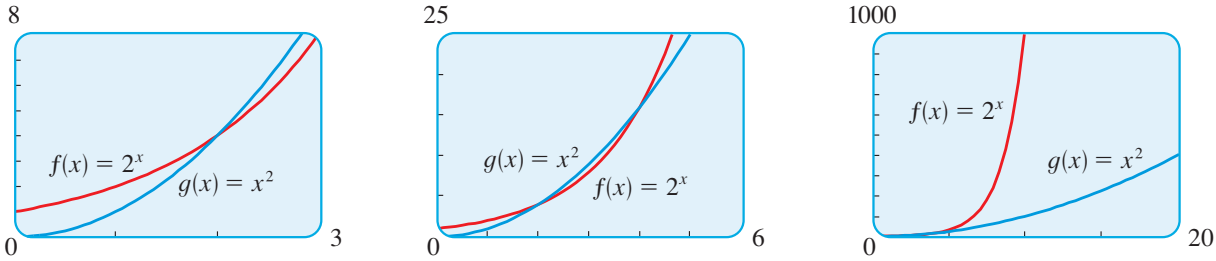


figure **3** (a) (b) (c)

NOW TRY EXERCISE 17

example 2 Transformations of Power Functions

Sketch graphs of the following functions.

(a) $P(x) = -x^3$ **(b)** $Q(x) = (x - 2)^4$ **(c)** $R(x) = -2x^5 + 4$

Solution

We use the graphs in Figure 2 and transform them using the methods of Section 5.1.

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis, as shown in Figure 4(a).

(b) The graph of $Q(x) = (x - 2)^4$ is the graph of $y = x^4$ shifted to the right 2 units, as shown in Figure 4(b).

(c) We begin with the graph of $y = x^5$. The graph of $y = -2x^5$ is obtained by stretching the graph vertically and reflecting it in the x-axis (see the dashed blue graph in Figure 4(c)). Finally, the graph of $R(x) = -2x^5 + 4$ is obtained by shifting upward 4 units (see the red graph in Figure 4(c)).

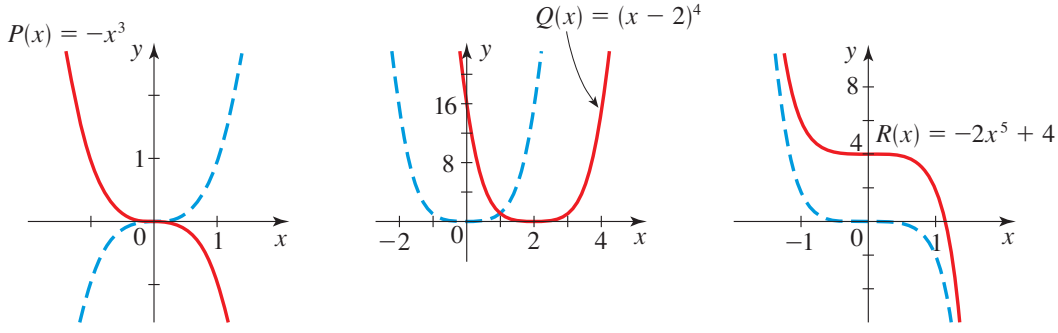


figure **4** (a) (b) (c)

NOW TRY EXERCISES 7 AND 9

Direct Proportionality

We studied the concept of direct proportionality in Section 2.4. In general, two quantities are directly proportional if one is a constant multiple of the other.

Direct Proportionality

If the quantities Q and R are related by the equation

$$Q = kR$$

for some nonzero constant k, we say that Q is **directly proportional** to R. The constant k is called the **constant of proportionality**.

It often happens that a quantity is proportional to a power function.

example 3 Finding the Proportionality Constant

Suppose that L is proportional to the fifth power of w.

(a) Express the proportionality as an equation.

(b) If it is known that $L = 80$ when $w = 2$, find the proportionality constant.

Solution

(a) We are given that L is proportional to w^5, so we can write

$$L = kw^5$$

where k is a constant and $k \neq 0$.

(b) To find k, we use the given information that $L = 80$ when $w = 2$. So

$$L = hw^5 \qquad L \text{ is proportional to } w^5$$

$$80 = k(2^5) \qquad \text{Replace } L \text{ by 80 and } w \text{ by 2}$$

$$k = \frac{80}{2^5} \qquad \text{Divide by } 2^5\text{, and switch sides}$$

$$k = \frac{5}{2} \qquad \text{Calculate}$$

The proportionality constant is $k = 2.5$, so $L = 2.5w^5$.

NOW TRY EXERCISE 23

example 4 The Effect of Doubling the Radius of a Ball

The volume of a sphere with radius r is given by the power function

$$V(r) = \tfrac{4}{3}\pi r^3$$

(a) Describe the relationship between the volume and radius as a proportionality.

(b) What happens to the volume of a ball when you double the radius?

Solution

(a) From the function $V(r) = \frac{4}{3}\pi r^3$ we see that the volume V is directly proportional to the cube of the radius r. The constant of proportionality is $\frac{4}{3}\pi$.

(b) If you double the radius, it becomes $2r$, and the corresponding volume is

$$\begin{aligned} V(2r) &= \tfrac{4}{3}\pi(2r)^3 && \text{Replace } r \text{ by } 2r \\ &= \tfrac{4}{3}\pi \cdot 2^3 r^3 && \text{Rules of Exponents} \\ &= 8(\tfrac{4}{3}\pi r^3) && \text{Group factors} \\ &= 8 \cdot V(r) && \text{Replace } \tfrac{4}{3}\pi r^3 \text{ by } V(r) \end{aligned}$$

So if you double the radius, the volume $V(2r)$ of the ball is 8 times as large as the volume $V(r)$ of the original ball.

Rules of Exponents are reviewed in Algebra Toolkits A.3 and A.4, pages T14 and T20.

NOW TRY EXERCISE 35

Fractional Positive Powers

If the power p in a power function is a fraction of the form, $p = m/n$, where m and n are positive integers, then

$$f(x) = x^p = x^{m/n} = \sqrt[n]{x^m}$$

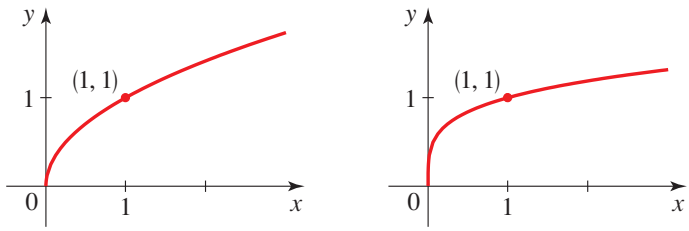
In particular, if $p = 1/n$, it becomes

$$f(x) = x^{1/n} = \sqrt[n]{x}$$

and the power function is a **root function**.

Fractional exponents are reviewed in Algebra Toolkit A.4, page T20.

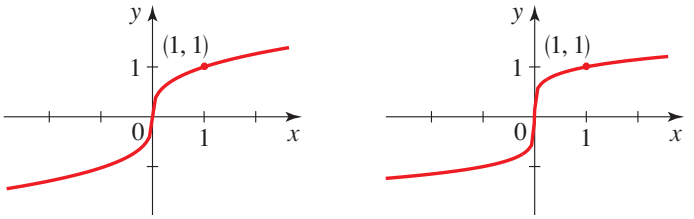
For $n = 2$ the root function is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is shown in Figure 5(a). (This graph is actually the upper half of the parabola $x = y^2$.) For other even values of n the graph of $f(x) = \sqrt[n]{x}$ is similar to the graph of the square root function, as in Figure 5(b).



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[4]{x}$

figure **5** Graphs of root functions $f(x) = \sqrt[n]{x}$ with n even

For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$, whose domain is the whole real line. (Recall that every real number has a cube root.) The graphs of $f(x) = \sqrt[n]{x}$ for other odd values of n are similar. (See Figure 6.)



(a) $f(x) = \sqrt[3]{x}$ (b) $f(x) = \sqrt[5]{x}$

figure **6** Graphs of root functions $f(x) = \sqrt[n]{x}$ with n odd

The following example shows how the square root function is used to model the falling time of a dropped object.

example 5 The Inverse Function of a Distance Function

If a stone is dropped from a cliff, Galileo's Law says that the distance d (in feet) traveled by the stone after t seconds is given by the power function $f(t) = 16t^2$. (So the distance is proportional to the square of the time.)

Inverse functions are studied in Section 4.6.

(a) Find the inverse function of f.

(b) Interpret the inverse function.

Solution

(a) First we write the function f in equation form, and then we solve for t.

$$d = 16t^2 \qquad \text{Equation form}$$

$$\frac{d}{16} = t^2 \qquad \text{Divide by 16}$$

$$\sqrt{\frac{d}{16}} = t \qquad \text{Take square roots}$$

$$t = \frac{\sqrt{d}}{4} \qquad \text{Simplify, and switch sides}$$

So the inverse function is $f^{-1}(d) = \frac{1}{4}\sqrt{d}$.

(b) The interpretation is that the time required for the stone to fall a distance of d feet is

$$t = f^{-1}(d) = \tfrac{1}{4}\sqrt{d}$$

This says that the falling time is proportional to the square root of the distance fallen.

■ NOW TRY EXERCISE 37 ■

Modeling with Power Functions

We have seen in Example 5 how power functions arise in physics. And in Section 6.4 you will be asked to use data concerning the planets to model the period of revolution of a planet as a power function of its distance from the sun.

Here we describe how quantities in the biological sciences are modeled by power functions. In the exercises, power function models are given for the wingspan of a bird as a function of its weight and for the natural walking speed of an animal as a function of its leg length. (These models will be derived from data in Section 6.4.) The next example is concerned with the species-area relationship. It's certainly plausible that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the relationship as a direct proportionality involving power functions.

example 6 The Species-Area Relation in a Bat Cave

The number of species S of bats living in caves in central Mexico has been related to the surface area A of the interior of the caves by the equation $S = 0.14A^{0.64}$. (In Example 1 in Section 6.4 this model will be derived from data.)

(a) The cave called Mision Imposible near Puebla, Mexico, has a surface area of $A = 60\ \text{m}^2$. How many species of bats would you expect to find in that cave?

(b) If you discover that four species of bats live in a cave, estimate the area of the cave.



Grigory Kubatyan/Shutterstock.com 2009

Bats in a cave

Solution

(a) According to the model $S = 0.14A^{0.64}$, the expected number of species in a cave with surface area $A = 60\ \text{m}^2$ is

$$S = 0.14(60)^{0.64} \approx 1.92$$

So we would expect there to be two species of bats in the Mision Imposible cave.

(b) For a cave with four species of bats we have

$$S = 0.14A^{0.64} \qquad \text{Model}$$

$$4 = 0.14A^{0.64} \qquad \text{Replace } S \text{ by } 4$$

$$A^{0.64} = \frac{4}{0.14} \qquad \text{Divide by } 0.14\text{, and switch sides}$$

$$A = \left(\frac{4}{0.14}\right)^{1/0.64} \qquad \text{Raise each side to the power } 1/0.64$$

$$A \approx 188 \qquad \text{Calculator}$$

We predict that a cave with four species of bats would have a surface area of approximately $188\ \text{m}^2$.

Solving power equations is reviewed in Algebra Toolkit C.1, page T47.

■ NOW TRY EXERCISE 39 ■

Algebra Checkpoint ✓

Check your knowledge of solving equations involving a power of the variable by doing the following problems. You can review these topics in **Algebra Toolkit C.1** on page T47.

1. Simplify each expression.

(a) $5^3 \cdot 5^2$ **(b)** $\left(\frac{3}{5}\right)^3$ **(c)** $x^{1/2}x^{3/2}$ **(d)** $\dfrac{x^2}{x^{1/2}}$

2. The given equation involves a power of the variable. Find all real solutions of the equation.

(a) $x^2 = 25$ **(b)** $x^3 = -27$ **(c)** $(3t)^3 = 6$ **(d)** $10.1(1.1z)^5 = 203$

3. The given equation involves a power of the variable. Find all real solutions of the equation.

(a) $x^{1/2} = 5$ **(b)** $A^{4/3} = 81$ **(c)** $(16y)^{3/4} = 16$ **(d)** $2.1S^{1.03} = 17.2$

4. An equation in two variables is given. Solve the equation for the indicated variable.

(a) $x = 2y^2$; y **(b)** $\dfrac{1}{3A} = \left(\dfrac{2}{3}S\right)^2$; S

(c) $9w = z^{1/5}$; z **(d)** $2a_1 = 3(a_2)^{2.12}$; a_2

6.2 Exercises

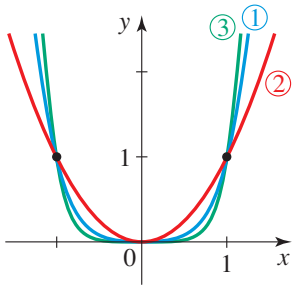
CONCEPTS

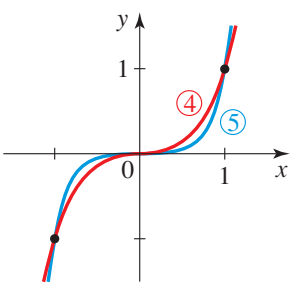
Fundamentals

1. **(a)** When p is an even positive integer, then the graph of the power function $f(x) = x^p$ is similar to the graph of $y =$ _____.

(b) When p is an odd positive integer, then the graph of the power function $f(x) = x^p$ is similar to the graph of $y =$ _____.

2. Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Match each function with its graph.





3. If the quantities x and y are related by the equation $y = 7x^5$, then we say that y is _____ proportional to the _____ power of x and the constant of proportionality is _____.

4. If y is directly proportional to the third power of x and if the constant of proportionality is 4, then x and y are related by the equation $y =$ □ $x^{□}$.

Think About It

5. True or false?

(a) For $x > 1$ the power function $f(x) = x^4$ is greater than the power function $g(x) = x^3$.

(b) For $0 < x < 1$ the power function $f(x) = x^4$ is greater than the power function $g(x) = x^3$.

6. Graph the functions $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$ for $-1 \le x \le 1$ on the same coordinate axes. What do you think the graph of $y = x^{100}$ would look like on this same interval? What about $y = x^{101}$?

SKILLS

7–10 ■ Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^p$ from Figure 1, page 494. Indicate all x- and y-intercepts.

7. **(a)** $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$

8. **(a)** $R(x) = -(x + 2)^3$ **(b)** $S(x) = \frac{1}{2}(x - 1)^3 + 4$

9. **(a)** $P(x) = x^{1/4} + 2$ **(b)** $Q(x) = x^{1/4} - 8$

10. **(a)** $R(x) = (x + 2)^{1/5}$ **(b)** $S(x) = (x - 1)^{1/5}$

11–16 ■ Graph the family of functions in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

11. $R(x) = x^c$; $c = 1, 3, 5, 7$

12. $S(x) = x^c$; $c = 2, 4, 6, 8$

13. $P(x) = cx^3$; $c = 1, 2, 5, \frac{1}{2}$

14. $Q(x) = x^4 + c$; $c = -1, 0, 1, 2$

15. $F(x) = x^c$; $c = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$

16. $G(x) = x^c$; $c = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$

17. Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 5]$ by $[0, 20]$ **(b)** $[0, 25]$ by $[0, 10^7]$ **(c)** $[0, 50]$ by $[0, 10^8]$

18. Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[-4, 4]$ by $[0, 20]$ **(b)** $[0, 10]$ by $[0, 5000]$ **(c)** $[0, 20]$ by $[0, 10^5]$

19–22 ■ Write an equation that expresses the statement.

19. V is directly proportional to the third power of x.

20. E is directly proportional to the square of c.

21. z is directly proportional to the square root of y.

22. R is directly proportional to the cube root of t.

23–26 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

23. A is directly proportional to the third power of x. If $x = 2$, then $A = 40$.

24. P is directly proportional to the fourth power of T. If $T = 3$, then $P = 27$.

25. z is directly proportional to square root of t. If $t = 64$, then $z = 2$.

26. S is directly proportional to the fifth root of r. If $r = \frac{1}{32}$, then $S = 12$.

27–28 ■ Use the given information to solve the problem.

27. y is directly proportional to the square of x. If $k = 0.5$ is the constant of proportionality, then find y when $x = 4$.

28. P is directly proportional to the square root of t. If $k = 1.5$ is the constant of proportionality, then find P when $t = 64$.

29. Determine whether the variable w is directly proportional to the third power of x. If so, write the equation of proportionality.

(a)

x	w
0	0
1	2
2	16
3	54
4	128

(b)

x	w
0	0
1	2.1
2	16.8
3	56.7
4	134.4

(c)

x	w
0	0
1	1.4
2	2.8
3	5.6
4	11.2

30. Determine whether the variable w is related to the variable x by a linear, exponential, or power function. In each case, find the function.

(a)

x	r
0	0
1	5
2	80
3	405
4	1280

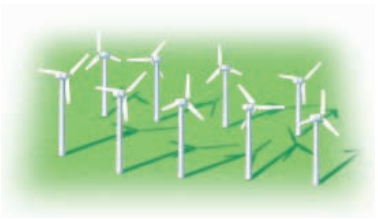
(b)

x	w
0	51
1	153
2	459
3	1377
4	4131

(c)

x	t
0	0
1	2.3
2	4.6
3	6.9
4	9.2

CONTEXTS



31. Power from a Windmill The power P that can be obtained from a windmill is directly proportional to the cube of the wind speed s.

(a) Write an equation that expresses this proportionality.

(b) Find the constant of proportionality for a windmill that produces 96 watts of power when the wind is blowing at 20 mi/h.

(c) How much power will this windmill produce if the wind speed increases to 30 mi/h?

32. Walking Speed The natural walking speed s of a person or animal is proportional to the square root of their leg length l.

(a) Write an equation that expresses this proportionality.

(b) A person whose legs are 3 ft long has a natural walking speed of 3.7 ft/s. Find the constant of proportionality.

(c) What is the natural walking speed of a person whose legs are 2.75 ft long?



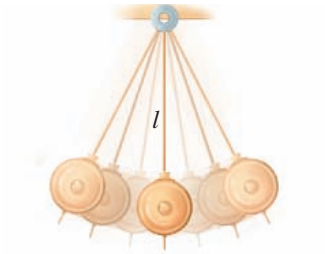
33. Skidding in a Curve A 1000-kilogram car is traveling on a curve that forms a circular arc. The centripetal force F that pushes the car out is directly proportional to the square of the speed s.

(a) If the car is traveling around the curve at a speed of 15 meters per second, the centripetal force is 2500 newtons. Find the constant of proportionality, and write an equation that expresses this proportionality.

(b) The car will skid if the centripetal force is greater than the frictional force holding the tires to the road. For this road the maximum frictional force is 5880 newtons. If the car travels around the curve at a speed of 20 meters per second, will it skid?

34. A Jet of Water The power P of a jet of water from a fire hose 3 inches in diameter is directly proportional to the cube of the water velocity v. If the velocity is cut in half, by what factor will the power decrease?





35. Radiation Energy The radiation energy E emitted by a heated surface (per unit surface area) is directly proportional to the fourth power of the absolute temperature T of the surface.

(a) Express this relationship by writing an equation.

(b) The temperature is 6000 K at the surface of the sun and 300 K at the surface of the earth. How many times more radiation energy per unit area is produced by the sun than by the earth?

36. Law of the Pendulum The period of a pendulum (the time elapsed during one complete swing of the pendulum) is directly proportional to the square root of the length of the pendulum.

(a) Express this relationship by writing an equation.

(b) To double the period, how would we have to change the length l?



37. Stopping Distance The stopping distance D of a car after the brakes have been applied is directly proportional to the square of the speed s. The stopping distance of a car on dry asphalt is given by the function $f(s) = 0.033s^2$, where s is measured in mi/h.

(a) Find the inverse function of f.

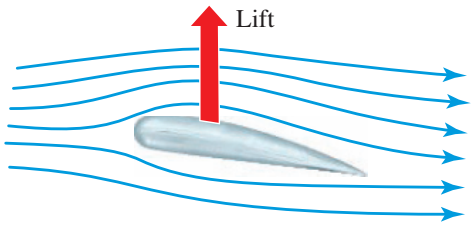
(b) Interpret the inverse function.

(c) If a car skids 50 ft on dry asphalt, how fast was it moving when the brakes were applied?

38. Aerodynamic Lift If a certain airplane wing has an area of 500 square feet, then the lift L of the airplane wing at takeoff is proportional to the square of the speed s of the plane in miles per hour. The lift of the airplane is given by the function $f(s) = 0.00136s^2$.

(a) Find the inverse function of f.

(b) Interpret the inverse function.



39. Ostrich Flight? The weight W (in pounds) of a bird (that can fly) has been related to the wingspan L (in inches) of the bird by the equation $L = 30.6 \cdot W^{0.3952}$. (In Exercise 19 of Section 6.4 this model will be derived from data.)

(a) The bald eagle has a wingspan of about 90 inches. Use the model to estimate the weight of the bald eagle.

(b) An ostrich weighs about 300 pounds. Use the model to estimate what the wingspan of an ostrich should be in order for it to be able to fly.

(c) The wingspan of an ostrich is about 72 inches. Use your answer to part (b) to explain why ostriches can't fly.



40. Biodiversity The number N of species of reptiles and amphibians inhabiting Caribbean islands has been related to the area A (in square meters) of an island by the equation $N = 3.1046A^{0.3080}$.

(a) Use the model to estimate the number of species found in an area that is 65 square meters.

(b) If you discover that 20 species of reptiles and amphibians live on an island, estimate the area of the island.

6.3 Polynomial Functions: Combining Power Functions

- Polynomial Functions
- Graphing Polynomial Functions by Factoring
- End Behavior and the Leading Term
- Modeling with Polynomial Functions



pzAxe/Shutterstock.com 2009

Polynomial functions at the post office

IN THIS SECTION... we study polynomial functions as sums of power functions. We learn how to graph a polynomial function using factoring.

GET READY... by reviewing how to factor polynomial expressions in **Algebra Toolkit B.2**. Test your understanding by doing the **Algebra Checkpoint** at the end of this section.

Polynomial functions are obtained by combining power functions using the algebraic operations on functions we studied in Section 6.1. In this section we'll encounter an application of polynomial functions to the ordinary task of mailing a package at the post office. The U.S. Postal Service has restrictions on the size of packages that it will mail. The rule is: The length of the package plus the distance around the package must not exceed a specified number of inches. To find out the dimensions of the largest box that the Postal Service will mail requires using polynomial functions, as we'll see in Example 6. But first we must learn how to graph polynomial functions.

Polynomial Functions

A *polynomial function* is a function that is a sum of power functions. The following are examples of polynomial functions:

$$f(x) = 3x^3 + 2x - 1 \qquad g(x) = x^4 + 2x^3 \qquad h(x) = 3x + 2$$

Recall that the degree of a polynomial is the highest power of the variable that appears in the polynomial. So the degrees of the above polynomials are 3, 4, and 1, respectively. In general, we have the following definition.

Polynomial Functions

A polynomial function is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a nonnegative integer.

- If $a_n \neq 0$, then the polynomial has **degree n**.
- The term $a_n x^n$ is the **leading term** of the polynomial.

One way to graph a polynomial function is to graph the terms of the polynomial (they are power functions), then use graphical addition.

example 1 Graphing a Polynomial with Graphical Addition

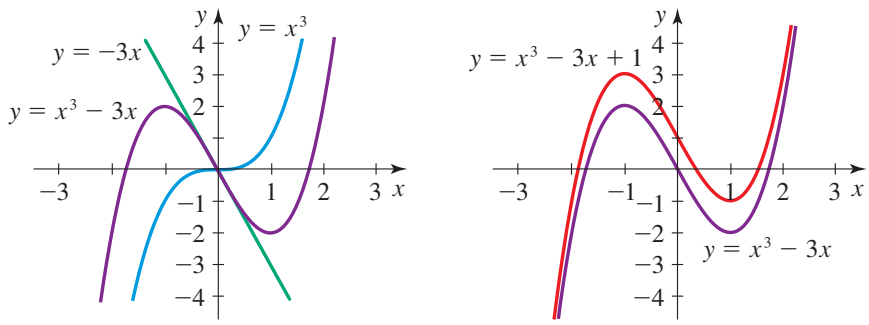
Graph the polynomial function $f(x) = x^3 - 3x + 1$ by expressing it as a sum of power functions and then using graphical addition.

Solution

The polynomial function f is the sum of the three power functions

$$y = x^3 \qquad y = -3x \qquad y = 1$$

We first graph $y = x^3$ and $y = -3x$; then we use graphical addition to graph their sum $y = x^3 - 3x$, as in Figure 1(a). We now shift this graph up 1 unit to obtain the graph of $y = x^3 - 3x + 1$, as in Figure 1(b).



(a) Graph of $y = x^3 - 3x$ (b) Graph of $y = x^3 - 3x + 1$

figure 1 Graphing a polynomial function with graphical addition

NOW TRY EXERCISE 7

Graphing Polynomial Functions by Factoring

If a polynomial function can be factored into linear factors, then an effective way to graph the polynomial is to first use the zero-product property to find the x-intercepts. Between consecutive x-intercepts the polynomial must be always positive or always negative; to decide which sign is the correct one, we use test points. The process is illustrated in the next example.

example 2 Graphing a Polynomial Function

Let $f(x) = x^3 - 3x^2 - 4x + 12$.

(a) Show that $f(x) = (x - 3)(x + 2)(x - 2)$.

(b) Find the x-intercepts of the graph of f.

(c) Find the sign of f on each of the intervals determined by the x-intercepts.

(d) Sketch a graph of f.

Intercepts are reviewed in Algebra Toolkit D.2, page T71.

Solution

(a) We expand the product to see whether it is the same expression as the expression defining f.

$$\begin{aligned}(x-3)(x+2)(x-2) &= (x-3)(x^2-4) && \text{Sum and difference of terms}\\ &= x^2(x-3)-4(x-3) && \text{Distributive Property}\\ &= x^3-3x^2-4x+12 && \text{Distributive Property}\end{aligned}$$

This last expression is the same as the expression defining f.

(b) The x-intercepts are the solutions to the equation $f(x)=0$. We have

$$\begin{aligned} x^3-3x^2-4x+12 &= 0 && \text{Set } f(x) \text{ equal to 0}\\ (x-3)(x+2)(x-2) &= 0 && \text{Factored form}\\ x-3=0 \quad\text{or}\quad x+2=0 \quad\text{or}\quad x-2 &= 0 && \text{Zero-Product Property}\\ x=3 \quad\text{or}\quad x=-2 \quad\text{or}\quad x &= 2 && \text{Solve}\end{aligned}$$

So the x-intercepts are 3, -2, and 2.

(c) The x-intercepts separate the real line into four intervals (see Figure 2):

$$(-\infty,-2),\quad (-2,2),\quad (2,3),\quad \text{and}\quad (3,\infty)$$

The function f is either positive or negative on each of these intervals. To decide which it is, we pick a test point in the interval and evaluate f at that point. For example, in the interval $(-\infty,-2)$, let's pick the test point -3. Evaluating f at -3, we get

It is usually easier to evaluate a polynomial by using the factored form. In Example 1 we used the factored form to evaluate f at -3.

$$\begin{aligned} f(x) &= (x-3)(x+2)(x-2) && \text{Factored form of } f\\ f(-3) &= (-3-3)(-3+2)(-3-2) && \text{Evaluate } f \text{ at } -3\\ &= (-6)(-1)(-5)=-30 && \text{Calculate}\end{aligned}$$

Since -30 is less than 0, f is negative at each point in the entire interval $(-\infty,-2)$. In Figure 2 we choose test points in each of the four intervals and determine the sign of the function f on each interval.

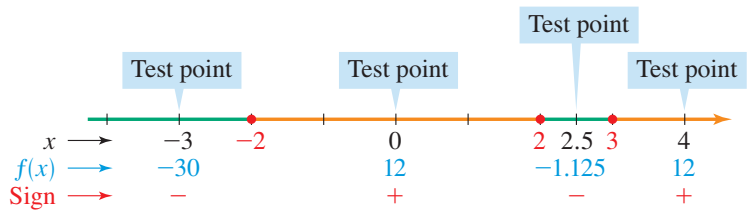


figure **2** Intercepts and test points for $f(x)=x^3-3x^2-4x+12$

(d) We first plot the x-intercepts as well as the other points calculated in Figure 2. The graph of f is above the x-axis on intervals where f is positive and below the x-axis where f is negative. From Figure 2 we know the sign of f on each interval, so we can complete the graph as in Figure 3.

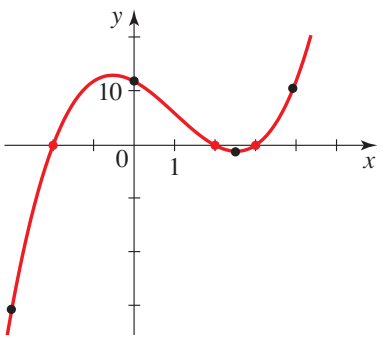


figure **3** Graph of $f(x)=x^3-3x^2-4x+12$

■ **NOW TRY EXERCISE 9** ■

example 3 Graphing a Polynomial Function

Let $f(x) = -2x^4 - x^3 + 3x^2$.

(a) Express the function f in factored form.

(b) Find the x-intercepts of the graph of f.

(c) Find the sign of f on each of the intervals determined by the x-intercepts.

(d) Sketch a graph of f.

Solution

(a) We factor the polynomial completely.

$$\begin{aligned} f(x) &= -2x^4 - x^3 + 3x^2 && \text{Definition of } f \\ &= -x^2(2x^2 + x - 3) && \text{Factor } -x^2 \\ &= -x^2(2x + 3)(x - 1) && \text{Factor trinomial} \end{aligned}$$

(b) The x-intercepts are the solutions to the equation $f(x) = 0$. We use the factored form of f to solve this equation.

$$-x^2(2x + 3)(x - 1) = 0 \qquad \text{Set } f(x) \text{ equal to 0}$$

$$-x^2 = 0 \quad \text{or} \quad 2x + 3 = 0 \quad \text{or} \quad x - 1 = 0 \qquad \text{Zero-Product Property}$$

$$x = 0 \quad \text{or} \quad x = -\tfrac{3}{2} \quad \text{or} \quad x = 1 \qquad \text{Solve}$$

So the x-intercepts are 0, $-\frac{3}{2}$, and 1.

(c) The x-intercepts separate the real line into four intervals (see Figure 4):

$$(-\infty, -\tfrac{3}{2}), \quad (-\tfrac{3}{2}, 0), \quad (0, 1), \quad \text{and} \quad (1, \infty)$$

To determine the sign of f on each of these intervals, we use test points as shown in the diagram in Figure 4.

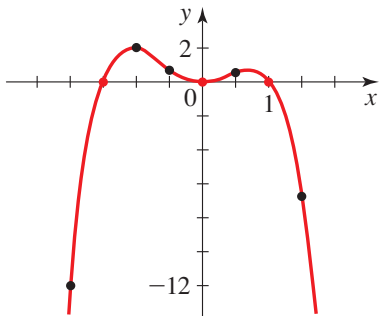


figure 5 Graph of $f(x) = -2x^4 - x^3 + 3x^2$

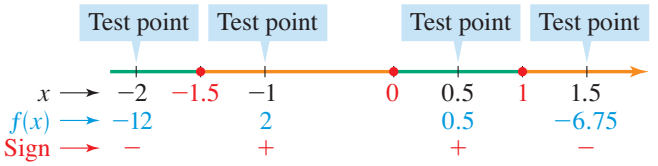


figure 4 Intercepts and test points for $f(x) = -2x^4 - x^3 + 3x^2$

(d) We plot the x-intercepts as well as the test points that we calculated in the diagram in Figure 4. The graph of f is above the x-axis on intervals where f is positive and below the x-axis where f is negative, as shown in Figure 5.

■ **NOW TRY EXERCISE 19** ■

example 4 Graphing a Polynomial Function

Let $f(x) = x^3 - 2x^2 - 4x + 8$.

(a) Express the function f in factored form.

(b) Find the x-intercepts of the graph of f.

(c) Find the sign of f on each of the intervals determined by the x-intercepts.

(d) Sketch a graph of f.

Solution

(a) We factor the polynomial by grouping.

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 4x + 8 && \text{Definition of } f \\ &= (x^3 - 2x^2) + (-4x + 8) && \text{Group} \\ &= x^2(x - 2) - 4(x - 2) && \text{Factor each term} \\ &= (x^2 - 4)(x - 2) && \text{Factor } x - 2 \text{ from each term} \\ &= (x + 2)(x - 2)(x - 2) && \text{Factor difference of squares} \end{aligned}$$

(b) The x-intercepts are the solutions to the equation $f(x) = 0$. We use the factored form of f to solve this equation.

$$\begin{aligned} & & (x + 2)(x - 2)(x - 2) &= 0 && \text{Set } f(x) \text{ equal to 0} \\ x + 2 = 0 \quad &\text{or} \quad x - 2 = 0 \quad \text{or} \quad & x - 2 &= 0 && \text{Zero-Product Property} \\ x = -2 \quad &\text{or} \quad x = 2 \quad \text{or} \quad & x &= 2 && \text{Solve} \end{aligned}$$

So the x-intercepts are -2 and 2.

(c) The x-intercepts separate the real line into three intervals (see Figure 6):

$$(-\infty, -2), \quad (-2, 2), \quad \text{and} \quad (2, \infty)$$

To determine the sign of f on each of these intervals, we use test points as shown in the diagram in Figure 6.

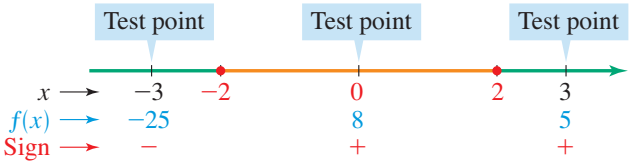


figure **6** Intercepts and test points for $f(x) = x^3 - 2x^2 - 4x + 8$

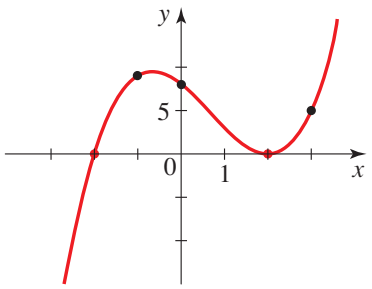


figure **7** Graph of $f(x) = x^3 - 2x^2 - 4x + 8$

(d) We plot the x-intercepts as well as the test points that we calculated in the diagram in Figure 6. The graph of f is above the x-axis on intervals where f is positive and below the x-axis where f is negative, as shown in Figure 7.

■ **NOW TRY EXERCISE 23** ■

End Behavior and the Leading Term

The end behavior of a polynomial function is a description of what happens to the values of the function as x becomes large in the positive or negative direction. To explain end behavior, we use the following arrow notation.

Arrow Notation

$x \to \infty$ means "x becomes large in the positive direction"
$x \to -\infty$ means "x becomes large in the negative direction"

For example, the power functions $y = x^2$ and $y = x^3$ have the end behavior shown in Figure 8.

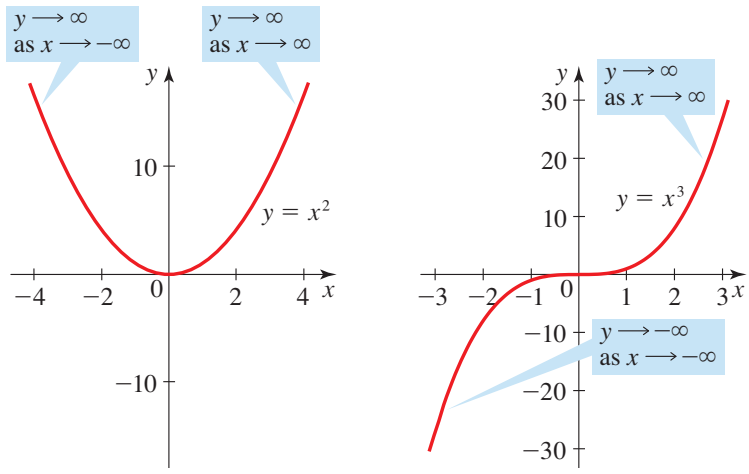


figure **8** End behavior of $y = x^2$ and $y = x^3$

Power functions with even powers have end behavior like that of $y = x^2$ or $y = -x^2$, and power functions with odd powers have end behavior like that of $y = x^3$ or $y = -x^3$. But how do we determine the end behavior of any polynomial? For any polynomial function the end behavior is completely determined by the leading term.

End Behavior

A polynomial function has the same end behavior as the power function consisting of its leading term.

So the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has the same end behavior as the power function $Q(x) = a_n x^n$. The reason is that for large values of x the other terms are relatively insignificant in size when compared to the leading term. The next example illustrates this relationship.

example **5** **Graphing a Polynomial Function**

Let $P(x) = 3x^5 - 5x^3 + 2x$.

(a) Determine the end behavior of the leading term of P.

(b) Confirm that P and its leading term have the same end behavior by comparing their values for large values of x.

(c) Confirm that P and its leading term have the same end behavior by graphing them together.

Solution

(a) The leading term of P is the function $Q(x) = 3x^5$. Since Q is a power function with odd power, its end behavior is as follows:

$$y \to \infty \text{ as } x \to \infty \qquad \text{and} \qquad y \to -\infty \text{ as } x \to -\infty$$

(b) We make a table of values of P and Q for large values of x. We see that when x is large, P and Q have approximately the same value (in each case $y \to \infty$). This confirms that they have the same end behavior as $x \to \infty$.

x	$P(x) = 3x^5 - 5x^3 + 2x$	$Q(x) = 3x^5$
15	2,261,280	2,278,125
30	72,765,060	72,900,000
50	936,875,100	937,500,000

(c) Figure 9 shows the graphs of P and Q in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior.

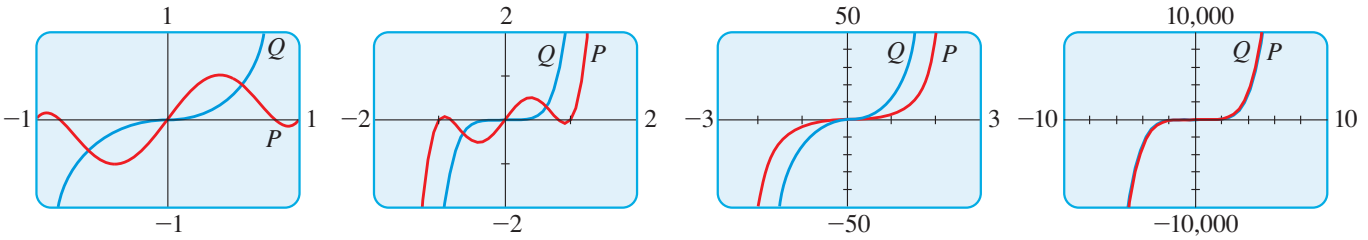


figure **9** Graphs of P and Q in progressively larger viewing rectangles

NOW TRY EXERCISE 27

Modeling with Polynomial Functions

We have seen that the graphs of polynomial functions often have local maximum and minimum values. In the applications of polynomials these points often play an important role, as we'll see in the next example.

Local maxima and minima of functions are studied in Section 1.7.

example **6** **Finding Maximum Volume**

Your local post office will mail a package only if its length plus girth is 108 inches (or less). April wants to mail a package in the shape of a rectangular box with a square base. To get the largest volume, she wants the length plus the girth to be exactly 108 inches.

(a) Express the volume V of the package as a function of the side x of the base.

(b) Use a graphing calculator to graph V in the viewing rectangle [0, 30] by [0, 12,000]. For what value of x does the maximum volume occur?

(c) What are the dimensions of the box with greatest volume that April can mail at the post office?

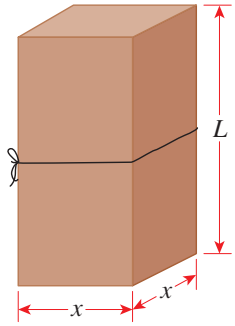


figure **10** Girth is the "distance around."

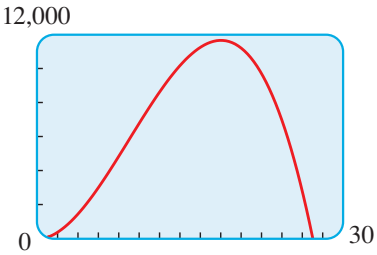


figure **11** Graph of $V = 108x^2 - 4x^3$

Solution

(a) Let x be the side of the base and let L be the length, as shown in Figure 10. The volume of the box is

$$V = x^2L \qquad \text{Volume of box}$$

To express V as a function of the variable x alone, we need to eliminate L. We know that $4x + L = 108$, so $L = 108 - 4x$. Using this expression for L, we can now express the function V as a function of x alone.

$$V = x^2L \qquad \text{Volume of box}$$
$$V = x^2(108 - 4x) \qquad \text{Replace } L \text{ by } 108 - 4x$$
$$V = 108x^2 - 4x^3 \qquad \text{Distributive property}$$

(b) The graph is shown in Figure 11. From the graph we see that the maximum value occurs when x is 18.

(c) To obtain the box with maximum volume, the side x of the base must be 18. Let's find the length of the box.

$$L = 108 - 4x \qquad \text{From part (a)}$$
$$= 108 - 4 \cdot 18 \qquad \text{Replace } x \text{ by } 18$$
$$= 36 \qquad \text{Calculate}$$

So the dimensions of the box with maximum volume are $18 \times 18 \times 36$ inches.

NOW TRY EXERCISE 47

example **7**

Solving a Polynomial Equation Graphically

If the box in Example 6 is to have a volume of 8000 cubic inches, what are the dimensions of the box?

Solution

From Example 6 we know that the volume of the box is modeled by the function

$$V = 108x^2 - 4x^3$$

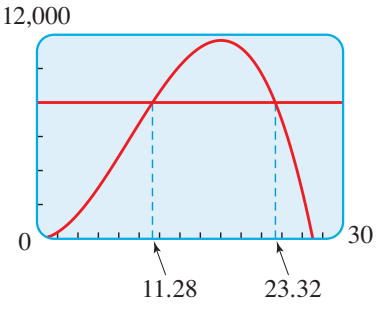


figure **12**

We graph the function $V = 108x^2 - 4x^3$ and the line $V = 8000$ in Figure 12. From the graph we see that the volume is 8000 when x is approximately 11.28 or 23.32. To find the corresponding lengths L, we use the equation $L = 108 - 4x$, which we found in Example 6.

For $x \approx 11.28$ we have $L \approx 108 - 4(11.28) = 62.88$.

For $x \approx 23.32$ we have $L \approx 108 - 4(23.32) = 14.72$.

So there are two possible boxes of the given shape having volume 8000 cubic inches. Their dimensions are approximately

$$11.28 \times 11.28 \times 62.88 \quad \text{or} \quad 23.32 \times 23.32 \times 14.72$$

where all distances are measured in inches.

NOW TRY EXERCISE 49

Algebra Checkpoint ✓

Check your knowledge of factoring polynomial expressions by doing the following problems. You can review these topics in **Algebra Toolkit B.2** on page T33.

1. Factor out the common factor.
   (a) $6x^2 + 3$ **(b)** $r^2 - 3r$ **(c)** $t^4 + 4t^2$ **(d)** $2y^3 - 6y^2 + 4y$
2. Factor the quadratic.
   (a) $x^2 - 81$ **(b)** $25m^2 + 10m + 1$ **(c)** $t^2 + 3t - 28$ **(d)** $3n^2 - n - 2$
3. Factor the expression completely.
   (a) $3t^5 - 12t^3$ **(b)** $w^4 - 16$ **(c)** $x^5 + 3x^4 - 28x^3$ **(d)** $r^6 - 12r^3 + 36$
4. Factor the expression by grouping terms.
   (a) $x^3 + 3x^2 + x + 3$ **(b)** $7z^5 - 14z^3 + 5z^2 - 10$

6.3 Exercises

CONCEPTS

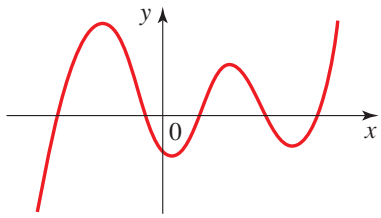
Fundamentals

1. **(a)** A polynomial function is a sum of _____ functions.
   (b) The polynomial $P(x) = 3x^4 - 2x^3 + 4$ is the sum of the power functions $y =$ _____, $y =$ _____, and $y =$ _____. The leading term of P is _____, and the degree of P is _____.
2. **(a)** To find the zeros of a polynomial function, we first _____ the polynomial and then apply the Zero-Product Property. The zeros of a polynomial are the ____-intercepts of the graph of the polynomial.
   (b) The polynomial function $f(x) = (x - 2)(x + 1)(x - 3)$ is in factored form. The zeros of f are ____, ____, ____. The x-intercepts of the graph of f are ____, ____, ____.
3. Between consecutive x-intercepts the values of a polynomial must always be _____ or always be _____. To decide which sign is the appropriate one, we use _____ points.
4. Every nonconstant polynomial has one of the following end behaviors:
   (i) $y \to \infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$
   (ii) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$
   (iii) $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$
   (iv) $y \to -\infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

   For each polynomial, choose the appropriate description of its end behavior from the list above.
   (a) $y = x^3 - 8x^2 + 2x - 15$; end behavior: _____.
   (b) $y = -x^3 + 8x^2 - 2x + 15$; end behavior: _____.
   (c) $y = 5x^6 + 3x^3 - 9x - 12$; end behavior: _____.
   (d) $y = -5x^6 - 3x^3 + 9x + 12$; end behavior: _____.

Think About It

5. Can the polynomial whose graph is shown have a degree that is even?



6. A function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$.

(a) Explain why a power function that contains only odd powers of x is an odd function.

(b) Explain why a power function that contains only even powers of x is an even function.

(c) Explain why a power function that contains both odd and even powers of x is neither an odd nor an even function

SKILLS

7–8 ■ Graph the given polynomial function by expressing it as a sum of power functions, and then using graphical addition.

7. $f(x) = -x^3 + 2x^2 + 5$

8. $f(x) = x^4 + 2x - 1$

9. Let $f(x) = x^3 + x^2 - x - 1$.

(a) Show that $f(x) = (x + 1)^2(x - 1)$.

(b) Find the x-intercepts of the graph of f.

(c) Find the sign of f on each of the intervals determined by the x-intercepts.

(d) Sketch a graph of f.

10. Let $f(x) = x^3 + 3x^2 - 4x - 12$.

(a) Show that $f(x) = (x + 3)(x - 2)(x + 2)$.

(b) Find the x-intercepts of the graph of f.

(c) Find the sign of f on each of the intervals determined by the x-intercepts.

(d) Sketch a graph of f.

11–18 ■ Sketch a graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

11. $P(x) = (x - 1)(x + 2)$

12. $P(x) = (x - 1)(x + 1)(x - 2)$

13. $P(x) = x(x - 3)(x + 2)$

14. $P(x) = (2x - 1)(x + 1)(x + 3)$

15. $P(x) = (x - 3)(x + 2)(3x - 2)$

16. $P(x) = \frac{1}{5}x(x - 5)^2$

17. $P(x) = (x - 1)^2(x - 3)$

18. $P(x) = (x - 3)^2(x + 1)^2$

19–26 ■ A polynomial function P is given.

(a) Express the function P in factored form.

(b) Sketch a graph of P.

19. $P(x) = x^3 - x^2 - 6x$

20. $P(x) = x^3 + 2x^2 - 8x$

21. $P(x) = -x^3 + x^2 + 12x$

22. $P(x) = -2x^3 - x^2 + x$

23. $P(x) = x^4 - 3x^3 + 2x^2$

24. $P(x) = x^5 - 9x^3$

25. $P(x) = x^4 - 3x^2 - 4$

26. $P(x) = x^6 - 2x^3 + 1$

27–32 ■ Determine the end behavior of P. Compare the graphs of P and Q on large and small viewing rectangles as in Example 5.

27. $P(x) = 3x^3 - x^2 + 5x + 1, \quad Q(x) = 3x^3$

28 $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x, \quad Q(x) = -\frac{1}{8}x^3$

29. $P(x) = x^4 - 7x^2 + 5x + 5, \quad Q(x) = x^4$

30 $P(x) = -x^5 + 2x^2 + x, \quad Q(x) = -x^5$

31. $P(x) = x^{11} - 9x^9, \quad Q(x) = x^{11}$

32 $P(x) = 2x^2 - x^{12}, \quad Q(x) = -x^{12}$

33–38 ■ Match the polynomial function with one of the graphs I–VI. Give reasons for your choice.

33. $P(x) = x(x^2 - 4)$

34. $Q(x) = -x^2(x^2 - 4)$

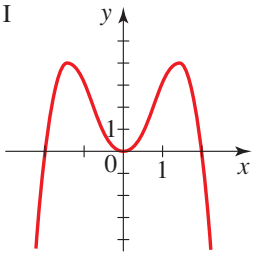
35. $R(x) = -x^5 + 5x^3 - 4x$

36. $S(x) = \frac{1}{2}x^6 - 2x^4$

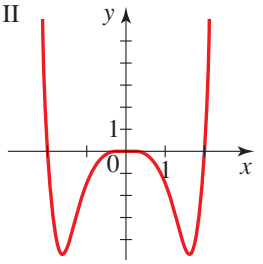
37. $T(x) = x^4 + 2x^3$

38. $U(x) = -x^3 + 2x^2$

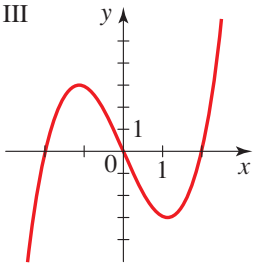
I



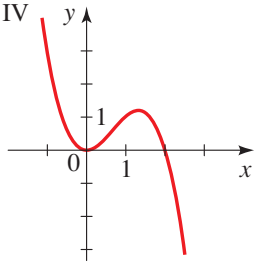
II



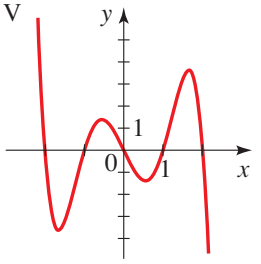
III



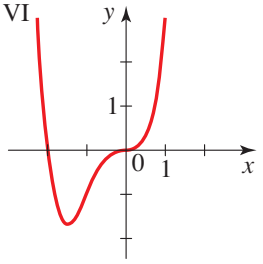
IV



V



VI



39–42 ■ A polynomial function P is given.

(a) Graph the polynomial P in the given viewing rectangle.

(b) Find all the local maxima and minima of P.

(c) Find all solutions of the equation $P(x) = 0$.

39. $P(x) = x^3 - 3x^2 - 4x + 12; \quad [-4, 4]$ by $[-15, 15]$

40. $P(x) = x^4 - 5x^2 + 4; \quad [-4, 4]$ by $[-30, 30]$

41. $P(x) = x^5 - 5x^2 + 6; \quad [-3, 3]$ by $[-5, 10]$

42. $P(x) = 2x^3 - 8x^2 + 9x - 9; \quad [-4, 6]$ by $[-40, 40]$

CONTEXTS

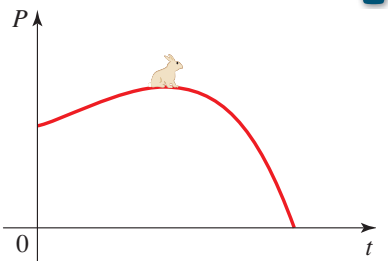
43. Market Research A market analyst working for a small appliance manufacturer finds that if the firm produces and sells x blenders annually, a model for the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.001x^3 - 372$$

Graph the function P in an appropriate viewing rectangle, and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?



44. Population Change The rabbit population on a small island is observed to be modeled by the function.

$$P(x) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began. Graph the function P in an appropriate viewing rectangle, and use the graph to answer the following questions.

(a) When is the maximum population attained, and what is that maximum population?

(b) When does the rabbit population disappear from the island?

45. Depth of Snowfall Snow began falling at noon on Sunday. The amount of snow on the ground at a certain location at time t was modeled by the function.

$$h(t) = 11.60t - 12.41t^2 + 6.20t^3 - 1.58t^4 + 0.20t^5 - 0.01t^6$$

where t is measured in days from the start of the snowfall and $h(t)$ is the depth of snow in inches. Graph this function in an appropriate viewing rectangle, and use your graph to answer the following questions.

(a) What happened shortly after noon on Tuesday?

(b) Were there ever more than 5 inches of snow on the ground? If so, on what day(s)?

(c) On what day and at what time (to the nearest hour) did the snow disappear completely?

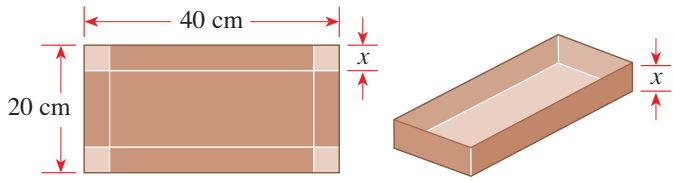
46. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

(a) Show that the volume of the open box is given by the function

$$V(x) = 4x^3 - 120x^2 + 800x$$

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

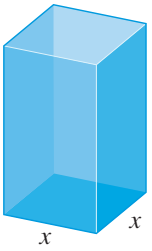
(c) Draw a graph of the function V, and use it to estimate the maximum volume for such a box.



47. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure on the following page. The total length of all 12 edges of the box is 144 inches.

(a) Show that the volume of the open box is given by the function

$$V(x) = 2x^2(18 - x)$$



(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V, and use it to estimate the maximum volume for such a box.

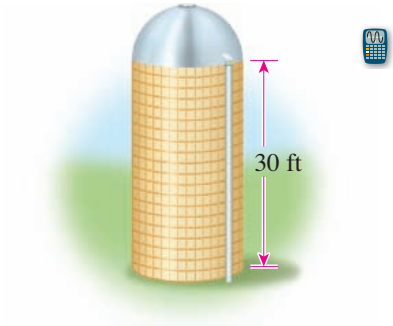
48. Girth of Box Acme Postal Service will mail a package only if the length plus girth of the package is 150 inches or less. Ranil wants to mail a package in the shape of a rectangular box with a square base. To get the largest volume, he wants the length plus the girth to be exactly 150 inches.

(a) Express the volume V of the package as a function of the side x of the base.

(b) Use a graphing calculator to graph V in the viewing rectangle [0, 40] by [0, 35,000]. For what value of x does the maximum volume occur?

(c) What are the dimensions of the box with greatest volume that Ranil can mail with Acme Postal Service?

49. Girth of Box If the box in Exercise 48 is to have a volume of 12,000 cubic inches, what are its dimensions?



50. Volume of a Silo A grain silo consists of a 30-foot-tall cylindrical main section and a hemispherical roof.

(a) Show that the volume V of the silo is given by the function

$$V = \tfrac{2}{3}\pi r^3 + 30\pi r^2$$

where r is the radius of the cylinder.

(b) If the total volume of the silo (including the part inside the roof section) is 15,000 cubic feet, what is the radius?

6.4 Fitting Power and Polynomial Curves to Data

- Fitting Power Curves to Data
- A Linear, Power, or Exponential Model?
- Fitting Polynomial Curves to Data

IN THIS SECTION... we model data using power and polynomial functions.



TheSupe87/Shutterstock.com 2009

The number of different bat species in a cave is related to the size of the cave by a power function.

Power and polynomial functions are the fundamental tools for modeling many real-world problems. One of the examples we study in this section is the species-area relation—the relation between the area of an "island" and the number of species inhabiting it. The term *island* simply means any well-defined ecosystem, such as a forest or a mountaintop. The species-area relation is one of the most consistently verified principles in ecology. Understanding this relation helps ecologists set conservation standards that take into account the biodiversity of an ecosystem. To model the species-area relation, scientists collect data on the number of species of a particular type of plant or animal in areas of different size. We'll see that such data are often best modeled by a power function. Once the model has been found, the expected number of species in a given area can be determined by using the model. The process is summarized in the following steps.

- **Data are collected** on the areas of different regions and the number of species in each region.
- **A power model** that best fits the data is made.

- **The model is used** to estimate the number of species in a region with a given area.

This species-area model is described in Example 1 for cave bats.

Fitting Power Curves to Data

If a scatter plot of the data we are studying resembles the graph of a power function, then we seek a power model, that is, a function of the form

$$f(x) = Cx^p$$

where C is a positive constant and p is any real number. In the next example we seek a power model for the species-area relation for cave bats.

example 1 Modeling the Species-Area Relation

Table 1 gives the areas of several caves in central Mexico and the number of bat species that live in each cave.*

(a) Make a scatter plot of the data. Is a linear model appropriate?

(b) Find a power function that models the data.

(c) Draw a graph of the function you found and a scatter plot of the data on the same graph. Does the model fit the data well?

(d) The cave called El Sapo near Puebla, Mexico, has a surface area of $A = 205$ m^2. Use the model to estimate the number of bat species you would expect to find in that cave.

table 1
Species-area data

Cave	Area (m^2)	Number of species
La Escondida	18	1
El Escorpion	19	1
El Tigre	58	1
Mision Imposible	60	2
San Martin	128	5
El Arenal	187	4
La Ciudad	344	6
Virgen	511	7

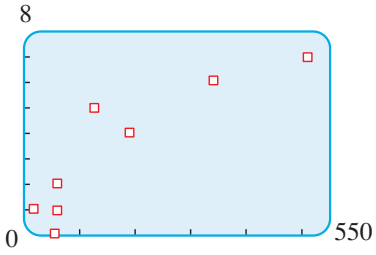


figure 1 Scatter plot of cave-bat data

Solution

(a) The scatter plot shown in Figure 1 indicates that the plotted points do not lie along a straight line, so a linear model is not appropriate.

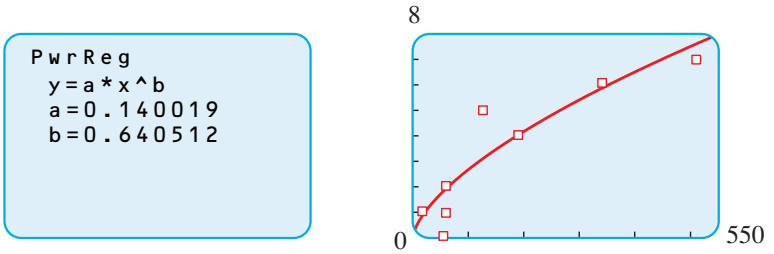
(b) Using a graphing calculator and the `PwrReg` command (see Figure 2(a) on the next page), we get the power model

$$S = 0.14A^{0.64}$$

(where we have rounded to two decimals).

*A. K. Brunet and R. A. Medallin, "The Species-Area Relationship in Bat Assemblages of Tropical Caves," *Journal of Mammalogy*, *82*(4):1114–1122, 2001.

(c) The graph is shown in Figure 2(b). The model appears to fit the data well.



(a) `PwrReg` output (b) Graph of $S = 0.14A^{0.64}$

figure **2** Power model for species-area relationship

(d) The area A is 205 m^2, so our model gives

$$S = 0.14A^{0.64} \qquad \text{Model}$$

$$S = 0.14 \cdot 205^{0.64} \qquad \text{Replace } A \text{ by } 205$$

$$S \approx 4.22 \qquad \text{Calculator}$$

The El Sapo cave actually does have four species of bats.

We would expect about 4 bat species in the El Sapo cave.

NOW TRY EXERCISE 15

A Linear, Power, or Exponential Model?

In Section 2.5 we learned that if a scatter plot of data lies approximately on a line, then a linear model is appropriate. But suppose that a scatter plot shows that the data increase rapidly. Should we use an exponential function or a power function to model the data? To help us decide, let's recall from Exploration 3 following Chapter 4 (page 406) that if the data points (x, y) lie on an *exponential curve*, then a plot of the points $(x, \log y)$, called a **semi-log plot**, lies on a straight line. We now show that if the data points (x, y) lie on a *power curve*, then a plot of the points $(\log x, \log y)$, called a **log-log plot**, would lie on a line.

Suppose the data points (x, y) lie on a power curve $y = Cx^p$. Let's take the logarithm of each side of this equation.

$$y = Cx^p \qquad \text{Given equation}$$

$$\log y = \log Cx^p \qquad \text{Take log of each side}$$

$$\log y = \log C + \log x^p \qquad \text{Property of log}$$

$$\log y = \log C + p \log x \qquad \text{Property of log}$$

To see that $\log y$ is a linear function of $\log x$, let $Y = \log y$, $X = \log x$, and $A = \log C$. Then the last equation becomes

$$Y = A + pX$$

We recognize that Y is a linear function of X, so the points (X, Y) or $(\log x, \log y)$ lie on a straight line. We now summarize our observations.

Linear, Power, or Exponential Model?

To determine whether a linear, exponential, or power model is appropriate, we make a scatter plot, a semi-log plot, and a log-log plot.

- If the **scatter plot** of the data lie approximately on a line, then a linear model is appropriate.
- If the **semi-log plot** of the data lie approximately on a line, then an exponential model is appropriate.
- If the **log-log plot** of the data lie approximately on a line, then a power model is appropriate.

To determine whether the plots lie on a line, we graph the regression line for each plot, as we show in the next example.

example 2 A Linear, Power, or Exponential Model?

table **2**

x	y	$\log x$	$\log y$
1	3.0	0	0.48
2	17.1	0.30	1.23
3	46.8	0.48	1.67
4	96.0	0.60	1.98
5	167.7	0.70	2.22
6	264.6	0.78	2.42
7	388.9	0.85	2.59
8	543.0	0.90	2.73

Data points (x, y) are given in Table 2. (For convenience we've also included the values of $\log x$ and $\log y$.)

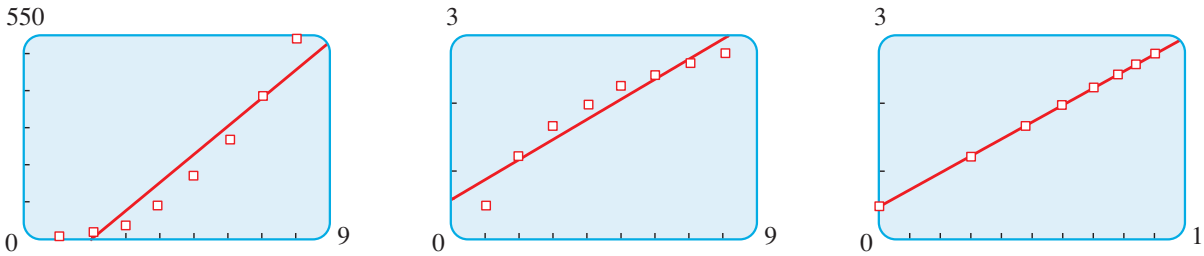
(a) Make a scatter plot, a semi-log plot, and a log-log plot of the data. Also graph the regression line for each set of data.

(b) Is a linear, power, or exponential model appropriate?

(c) Find an appropriate model for the data, and then graph the model together with a scatter plot of the data.

Solution

(a) For a scatter plot of the data we graph the points (x, y). For the semi-log plot we graph the points $(x, \log y)$, and for the log-log plot we graph the points $(\log x, \log y)$. We use the calculator to find the regression line for each data set. Each data set together with its regression line is shown in Figure 3.



(a) Scatter plot $y = -150 + 75.763x$

(b) Semi-log plot $y = 0.573 + 0.299x$

(c) Log-log plot $y = 0.477 + 2.50x$

figure **3**

(b) From Figure 3 we see that the log-log plot lies along a line much more closely than the other plots, so we conclude that a power model is appropriate.

(c) Using the `PwrReg` command on a graphing calculator, we get the power model

$$y = 3.0 \cdot x^{2.5}$$

The graph of this function and the original data points are shown in Figure 4.

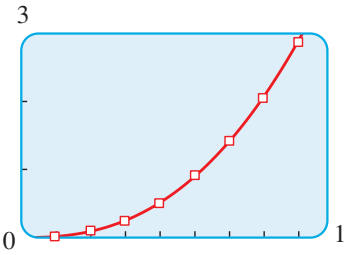


figure **4** Scatter plot and model

NOW TRY EXERCISE 11

Fitting Polynomial Curves to Data

We have learned how to fit a line to data (Section 2.6) or a power function to data. If the data exhibit more variability than either of these two models, we may need a different type of function to model the data. Figure 5 shows a scatter plot with three possible models that appear to fit data. Which model fits the data best?

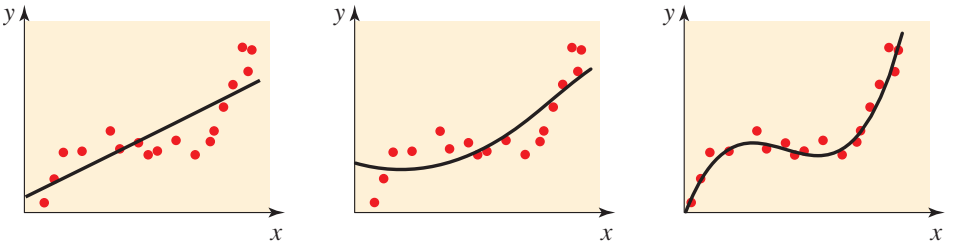
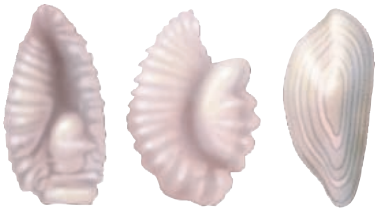


figure **5** Different types of models for the data

Linear model | Quadratic model | Cubic model

Although we can *always* find the linear model or the power model that best fits these data, it is clear from Figure 5 that the cubic model fits the data better than the others do. In fact, polynomial functions are ideal for modeling data for which the scatter plot has peaks or valleys (that is, local maxima or minima). For example, if the data have a single peak, then it may be appropriate to use a quadratic polynomial to model. The more peaks or valleys the data exhibit, the higher the degree of the polynomial needed to model the data (see Figure 5).

IN CONTEXT ➤



Cod | Redfish | Hake

Marine biologists need to be able to determine the ages of fish so that they can track fish populations and help to protect and manage this valuable resource. The age of a fish can be determined by the otoliths ("ear stones") in the head of the fish. These tiny structures have microscopic growth rings, not unlike growth rings of trees, which record the age of a fish. The art in the margin shows examples of otoliths of some fish species. Using otoliths, scientists collect data on the age and lengths of a particular species of fish.

We'll see in the next example that data on the length and age of fish are often best modeled by cubic polynomials. Once the model has been found, the age of a fish

can be estimated from its length (without further need to examine the otoliths of each fish). One of the oldest fish ever recorded was caught off the coast of Alaska in 2007. Its age was estimated through the otoliths to be between 90 and 115 years.



Photo by Karna McKinney, AFSC, NOAA Fisheries.

90-year old rockfish

example 3 Length-at-Age Data for Fish

The table gives the lengths of rock bass caught at different ages, as determined by the otoliths. Scientists have proposed a cubic polynomial to model the data.

(a) Use a graphing calculator to find the cubic polynomial of best fit for the data. Draw a graph of the polynomial function you found, together with a scatter plot of the data.

(b) Use the model to estimate the length of a 5-year-old rock bass.

(c) A fisherman catches a rock bass that is 20 inches long. Use the model to estimate its age.

table **3**
Length-at-age data

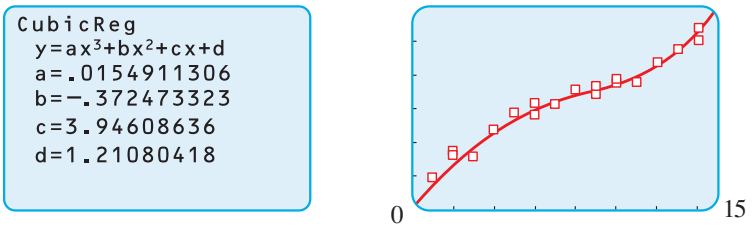
Age (yr)	Length (inches)	Age (yr)	Length (inches)
1	4.8	9	18.2
2	8.8	9	17.1
2	8.0	10	18.8
3	7.9	10	19.5
4	11.9	11	18.9
5	14.4	12	21.7
6	14.1	12	21.9
6	15.8	13	23.8
7	15.6	14	26.9
8	17.8	14	25.1

Solution

(a) Using the `CubicReg` command on a graphing calculator (see Figure 6(a) on the next page), we find the cubic polynomial of best fit:

$$y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21$$

Here, x is age in years and y is length in inches. A scatter plot of the data together with a graph of the cubic polynomial model is shown in Figure 6(b).



(a) `CubicReg` output (b) Scatter plot and model

figure **6** Cubic polynomial model for length-at-age data

(b) We replace x by 5 in the model.

$$y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21 \qquad \text{Model}$$

$$y = 0.0155(5)^3 - 0.372(5)^2 + 3.95(5) + 1.21 \qquad \text{Replace } x \text{ by } 5$$

$$y \approx 13.6 \qquad \text{Calculator}$$

So a 5-year-old rock bass would be approximately 13.6 inches long.

(c) Moving the cursor along the graph of the polynomial in Figure 6(b), we find that y is 20 when x is approximately 10.8. So the fish is about 10.8 years old.

■ **NOW TRY EXERCISE 21** ■

6.4 Exercises

CONCEPTS

Fundamentals

1. (a) When modeling data, we make a _____ plot to help us visually determine whether a line or some other curve is appropriate for modeling the data.

(b) If the y-values of a set of data increase rapidly, then an exponential function or a _____ function may be appropriate for modeling the data.

2. To determine whether an exponential function or a power function is appropriate for modeling a set of data, we make a semi-log plot and a log-log plot of the data.

(a) If the semi-log plot lies approximately along a line, then a _____ model is appropriate.

(b) If the log-log plot lies approximately along a line, then a _____ model is appropriate.

Think About It

3–4 ■ The following data are obtained from the function $f(x) = 5x^3$.

x	1	2	3	4	5
$f(x)$	5	40	135	320	625

3. If you use your calculator to find the power function that best fits the data, what function would you expect to get? Try it.

4. If you use your calculator to find the cubic polynomial that best fits the data, what function would you expect to get? Try it.

SKILLS

5–8 ■ A set of data is given.

(a) Use your calculator to find the indicated model for the data.

(b) Graph a scatter plot of the data along with your model. Does the model fit the data?

5. Power:

x	$f(x)$
0.1	10
1.0	254
1.4	298
1.9	480
2.0	584
2.6	698
3.1	822
3.7	1141

6. Linear:

x	$f(x)$
6	314
9	401
10	869
22	1126
29	1352
37	2251
62	3112
114	3551

7. Cubic:

x	$f(x)$
0.6	24.2
1.5	32.8
2.0	34.0
2.4	33.2
3.9	24.9
4.8	20.6
5.2	22.7
5.6	25.8

8. Exponential:

x	$f(x)$
0.5	30
1.1	51
2.3	72
3.0	89
3.2	115
4.1	168
4.9	206
5.5	342

Data for Exercise 9

x	$f(x)$
1.0	4
1.5	16.5
2.0	45.3
2.3	73.8
3.0	187.1
3.5	320.9
3.7	389.7
4.1	558.2
4.4	714.7
5.0	1118.0

Data for Exercise 10

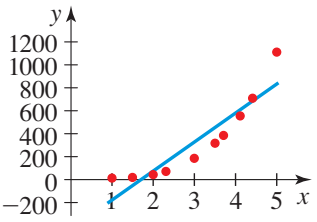
x	$f(x)$
0.3	1.50
1.0	1.72
1.2	1.79
1.7	1.95
2.1	2.08
2.4	2.17
2.8	2.30
3.0	2.37
3.4	2.50
4.0	2.69

9–10 ■ A scatter plot, a semi-log plot, and a log-log plot for the set of data given in the margin are shown below. Each graph also shows the regression line.

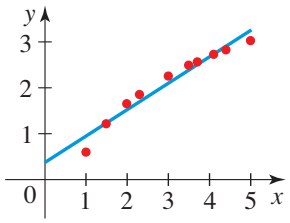
(a) Is a linear, exponential, or power function more appropriate for modeling the data?

(b) Find the model that is most appropriate, and graph the scatter plot along with your model. Does the model fit the data?

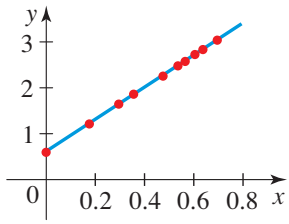
9.



Scatter plot

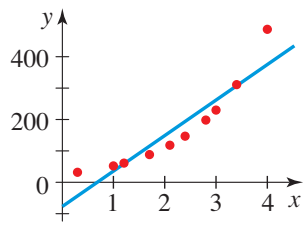


Semi-log plot

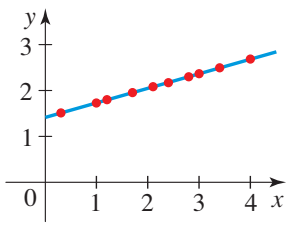


Log-log plot

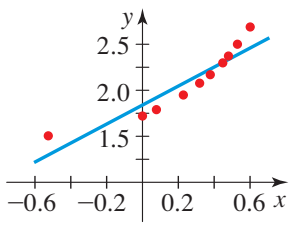
10.



Scatter plot



Semi-log plot



Log-log plot

11–14 ■ Data points (x, y) are given (see the next page).

(a) Draw a scatter plot of the data.

(b) Make semi-log and log-log plots of the data.

(c) Is a linear, power, or exponential function appropriate for modeling these data?

(d) Find an appropriate model for the data, and then graph the model together with a scatter plot of the data.

11.

x	y
2	0.08
4	0.12
6	0.18
8	0.25
10	0.36
12	0.52
14	0.73
16	1.06

12.

x	y
10	29
20	82
30	151
40	235
50	330
60	430
70	546
80	669
90	797
100	935

13.

x	y
1	1.5
2	5.1
3	11.3
4	20.1
5	32.4
6	45.5
7	62.1
8	81.1
9	102.8
10	127.0

14.

x	y
3	0.003
6	0.012
9	0.042
12	0.153
15	0.551
18	1.992
21	7.239
24	26.279

CONTEXTS

15. A Falling Ball In a physics experiment a lead ball is dropped from a height of 5 meters. The students record the distance the ball has fallen every one-tenth of a second. (This can be done by using a camera and a strobe light.)

Time (s)	Distance (m)
0.1	0.048
0.2	0.197
0.3	0.441
0.4	0.882
0.5	1.227
0.6	1.765
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.902

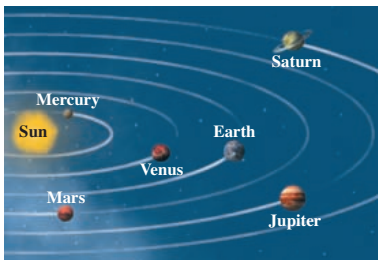
(a) Make a scatter plot of the data.

(b) Use a graphing calculator to find a power function of the form $d = at^b$ that models the distance d that the ball has fallen after t seconds.

(c) Draw a graph of the function you found and the scatter plot on the same graph. How well does the model fit the data?

(d) Use your model to predict how far a dropped ball would fall in 3 seconds.

16. Kepler's Law for Periods of the Planets For each planet, its mean distance d from the sun [in astronomical units (AU)] and its period T (in years) are given in the table. (Although Pluto is no longer considered a planet, we include it in the table because the distance-period relationship we study here applies to any body orbiting the sun.)



(a) Make a scatter plot of the data. Is a linear model appropriate?

(b) Find a power function that models the data.

(c) Draw a graph of the function you found and the scatter plot on the same graph. How well does the model fit the data?

(d) Use the model that you found to calculate the period of an asteroid whose mean distance from the sun is 5 AU.

Distance and period

Planet	d (AU)	T (yr)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto	39.507	248.350

17. Lead Emissions The table at the left gives U.S. lead emissions into the environment in millions of metric tons for 1970–1992.

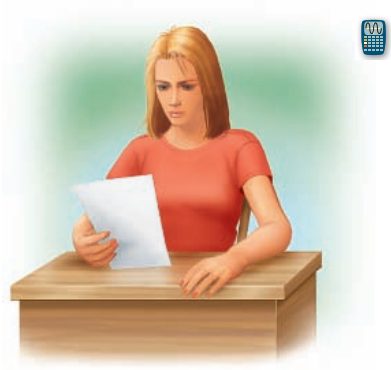
Year	Lead emissions
1970	199.1
1975	143.8
1980	68.0
1985	18.3
1988	5.9
1989	5.5
1990	5.1
1991	4.5
1992	4.7

(a) Make a scatter plot of the data.

(b) Find an exponential model for these data. (Use $t = 0$ for the year 1970.)

(c) Find a fourth-degree polynomial that models the data.

(d) Which of these curves gives a better model for the data? Use graphs of the two models to decide.



18. Experimenting with "Forgetting" Curves Every one of us is all too familiar with the phenomenon of forgetting. Facts that we clearly understood at the time we first learned them sometimes fade from our memory by the time the final exam rolls around. Psychologists have proposed several ways to model this process. To develop her own model, a psychologist performs an experiment on a group of volunteers by asking them to memorize a list of 100 related words. She then tests how many of these words they can recall after various periods of time. The average results for the group are shown in the table.

(a) Use a graphing calculator to find a power function of the form $y = at^b$ that models the average number of words y that the volunteers remember after t hours. Then find an exponential function of the form $y = ab^t$ to model the data.

(b) Make a scatter plot of the data, and graph both functions that you found in part (a) on your scatter plot.

(c) Which of these curves gives a better model for the data? Use graphs of the two models to decide.

Time	Words recalled
15 min	64.3
1 h	45.1
8 h	37.3
1 day	32.8
2 days	26.9
3 days	25.6
5 days	22.9

19. Bird Flight Ornithologists measured and catalogued the wingspans and weights of many different species of birds that can fly. The table below shows the wingspan L for a bird of weight W.

(a) Use a graphing calculator to find a power function of the form $L = aW^b$ to model the data.

(b) Use a graphing calculator to find an exponential function of the form $L = ab^W$ to model the data.

(c) Make a scatter plot of the data, and graph the models that you found in parts (a) and (b) on your scatter plot. Which of these curves gives a better model for the data?

(d) The extinct dodo weighed about 45 pounds and had a wingspan of about 20 in. Use the model chosen in part (c) to find the wingspan of a 45-pound (20 kg) bird. Use this to show why a dodo bird couldn't fly.



The dodo (now extinct)

Study of a Dodo (oil on canvas), Hart, F (19th Century)/ Royal Albert Memorial Museum, Exeter, Devon, UK/ The Bridgeman Art Library International

Bird	Weight (lb)	Wingspan (in.)
Turkey vulture	4.40	69
Bald eagle	6.82	84
Great horned owl	3.08	44
Cooper's hawk	1.03	28
Sandhill crane	9.02	79
Atlantic puffin	0.95	24
California condor	17.82	109
Common loon	7.04	48
Yellow warbler	0.022	8
Common grackle	0.20	16
Wood stork	5.06	63
Mallard	2.42	35

Tree species of the Pasoh Forest of Malaysia

Area (m^2)	Observed number of species
3.81	3
7.63	3
15.26	12
30.52	13
61.04	31
122.07	70
244.14	112
488.28	134
976.56	236

20. Biodiversity To test for the biodiversity of trees in a tropical rain forest, biologists collected data in the Pasoh Forest Reserve of Malaysia. The table in the margin shows the number of tree species S found for a given area A in the rain forest.*

(a) Use a graphing calculator to find a power function of the form $S = aA^b$ that models the number of tree species S that are in an area of size A. Then find an exponential function of the form $S = ab^A$ to model the data.

(b) Make a scatter plot of the data, and graph both functions that you found in part (a) on your scatter plot.

(c) Which of these curves gives a better model for the data? Use graphs of the two models to decide.



21. How Fast Can You Name Your Favorite Things? If you are asked to make a list of objects in a certain category, the speed with which you can list them follows a predictable pattern. For example, if you try to name as many vegetables as you can, you'll probably think of several right away—for example, carrots, peas, beans, corn, and so on. Then after a pause you might think of ones you eat less frequently—perhaps zucchini, eggplant, and asparagus. Finally, more exotic vegetables might come to mind—artichokes, jicama, bok choy, and the like. A psychologist performs this experiment on a number of subjects. The table below gives the average number of vegetables that the subjects named within a given number of seconds.

(a) Find the cubic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the number of vegetables that subjects would be able to name in 40 seconds.

(d) According to the model, how long (to the nearest 0.1 second) would it take a person to name five vegetables?

Seconds	Number of vegetables
1	2
2	6
5	10
10	12
15	14
20	15
25	18
30	21

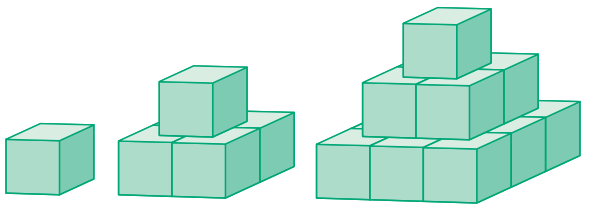
22. Polynomial Pattern The figure shows a sequence of pyramids made of cubic blocks.

(a) Complete the table for the number of blocks in a pyramid with n layers.

(b) Find the cubic polynomial P that best fits the data you obtained.

(c) Compare the values $P(1), P(2), P(3), \ldots$ with the data in the table. Does the polynomial you found model the data exactly?

Layers	1	2	3	4	5	6
Blocks	1	5	14			



*K. M. Kochummen, J.V. LaFrankie, and N. Manokaran, "Floristic Composition of Pasoh Forest Reserve, a Lowland Rain Forest in Peninsular Malaysia," *Journal of Tropical Forest Science*, *3:*1–13, 1991.

6.5 Power Functions: Negative Powers

- The Reciprocal Function
- Inverse Proportionality
- Inverse Square Laws

IN THIS SECTION... we study power functions with negative powers, their graphs, and their applications, particularly to inverse square laws.

GET READY... by reviewing the rules for working with exponents in **Algebra Toolkits A.3** and **A.4**. Test your skills in working with exponents by doing the **Algebra Checkpoint** at the end of this section.

In this section we investigate the behavior of the power function $f(x) = Cx^p$ when the power p is a negative number. We will see that the cases in which p is -1 or -2 give rise to important applications.

The Reciprocal Function

If C is 1 and p is -1, the power function is

$$f(x) = x^{-1} = \frac{1}{x}$$

which is called the **reciprocal function**.

example 1 The Graph of the Reciprocal Function

Sketch a graph of the reciprocal function $f(x) = 1/x$.

Solution

The function f is not defined when x is zero. The following tables show that when x is close to zero, the magnitude of $f(x)$ is large, and the closer x gets to zero, the larger the magnitude of $f(x)$.

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	$-100{,}000$

Approaching 0^- Approaching $-\infty$

x	$f(x)$
0.1	10
0.01	100
0.00001	100,000

Approaching 0^+ Approaching ∞

The first table shows that as x approaches zero *from the left*, the values of $y = f(x)$ become large in the negative direction. The second table shows that as x approaches zero *from the right,* the values of x become large in the positive direction. We describe this behavior in symbols and in words as follows.

$y \to -\infty$ as $x \to 0^-$ y approaches negative infinity as x approaches zero from the left

$y \to \infty$ as $x \to 0^+$ y approaches infinity as x approaches zero from the right

The next two tables show how the values of $y = f(x)$ change as x becomes large in the positive or negative direction.

x	$f(x)$
-10	-0.1
-100	-0.01
$-100{,}000$	-0.00001

Approaching $-\infty$ | Approaching 0

x	$f(x)$
10	0.1
100	0.01
100,000	0.00001

Approaching ∞ | Approaching 0

These tables show that as x becomes large, the values of $y = f(x)$ get closer and closer to zero. We describe this situation in symbols as follows:

$$y \to 0 \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad y \to 0 \quad \text{as} \quad x \to \infty$$

Using the information in these tables and plotting a few additional points, we obtain the graph shown in Figure 1.

x	x^{-1}
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

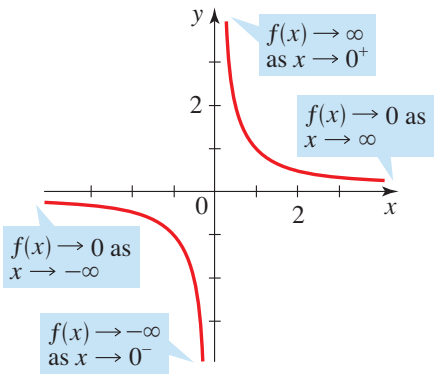


figure **1** Graph of $y = 1/x$

NOW TRY EXERCISES 5 AND 13

In Example 1 we used the following arrow notation.

Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right

The line $x = 0$ is called a **vertical asymptote** of the graph in Figure 1, and the line $y = 0$ is called a **horizontal asymptote**. Informally, an asymptote of a function is a line to which the graph of the function gets closer and closer as one travels along the line.

Inverse Proportionality

A quantity Q is inversely proportional to a quantity R if it is proportional to the reciprocal of R.

Inverse Proportionality

If the quantities Q and R are related by the equation

$$Q = \frac{k}{R}$$

for some nonzero constant k, we say that Q is **inversely proportional** to R. The constant k is called the **constant of proportionality**.

So if y is inversely proportional to x, the equation $y = k/x$ says that y is proportional to the reciprocal function.

example **2** **Inverse Proportionality**

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas.

(a) Write the equation that expresses this inverse proportionality.

(b) Suppose the pressure of a sample of air that occupies 0.106 cubic meters at 25°C is 50 kilopascals.

(c) If the sample expands to a volume of 0.3 cubic meters, find the new pressure.

Solution

(a) Let P be the pressure of the sample of gas, and let V be its volume. Then, by the definition of inverse proportionality we have

$$P = \frac{k}{V}$$

where k is a constant.

(b) To find k, we use the fact that P is 50 when V is 0.106.

$$P = \frac{k}{V} \qquad \text{Equation of proportionality}$$

$$50 = \frac{k}{0.106} \qquad \text{Replace } P \text{ by 50 and } V \text{ by 0.106}$$

$$k = (50)(0.106) \qquad \text{Multiply by 0.106, and switch sides}$$

$$k = 5.3 \qquad \text{Calculate}$$

Putting this value of k into the proportionality equation, we have $P = \dfrac{5.3}{V}$.

(c) We use the equation that we found, replacing V by 0.3.

$$P = \frac{5.3}{V} \qquad \text{Equation of proportionality}$$

$$P = \frac{5.3}{0.3} \qquad \text{Replace } V \text{ by 0.3}$$

$$P \approx 17.7 \qquad \text{Calculator}$$

So the new pressure is 17.7 kPa.

■ **NOW TRY EXERCISES 25 AND 31** ■

The equation $P = 5.3/V$ from Example 2 shows that P is a constant multiple of the reciprocal function, so the graph of P as a function of V has the same shape as the right half of Figure 1. (See Figure 2.)

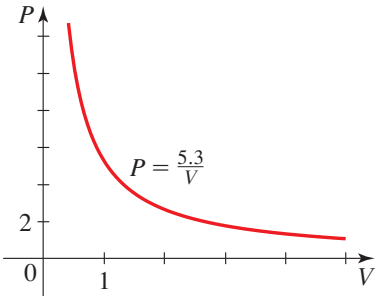


figure **2** Graph of $P = 5.3/V$

Inverse Square Laws

Among the remaining negative powers for the power function $f(x) = Cx^p$, by far the most important is when p is -2. In fact, some of the best-known scientific laws state that one quantity is inversely proportional to the square of another quantity. In other words, the first quantity is modeled by a function of the form

$$f(x) = \frac{k}{x^2}$$

and we refer to an **inverse square law**. Before considering such a law, let's see what the graph of such a function looks like.

example **3** **Graphing an Inverse Square Function**

Sketch a graph of the function $f(x) = 1/x^2$, and compare it with the graph of the reciprocal function.

Solution

We calculate values of $f(x)$, noting that $f(-x) = f(x)$.

x	x^{-2}
± 1	1
± 0.1	100
± 0.01	10,000

x	x^{-2}
± 10	0.01
± 100	0.0001
± 1000	0.000001

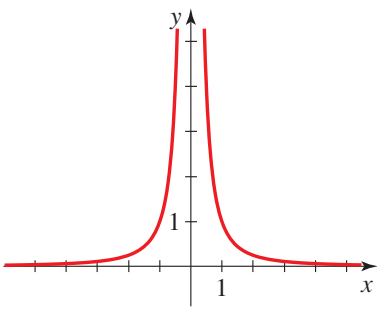


figure **3** Graph of $y = 1/x^2$

When we plot the graph of f in Figure 3, we notice that, as with $y = 1/x$, the x-axis and y-axis are asymptotes. But $1/x^2 \to 0$ faster than $1/x$ as $x \to \pm\infty$. And because $f(-x) = f(x)$, the graph of f is symmetric about the y-axis.

■ **NOW TRY EXERCISES 7 AND 15** ■

Many physical quantities are connected by inverse square laws. In the next example we use the fact that the illumination of an object by a light source is inversely proportional to the square of the distance from the source. Other inverse square laws model gravitational force (Exercise 36), loudness of sound (Exercise 33), and the electrostatic force between two charged particles.

example 4 An Inverse Square Law

Suppose that after dark you are in a room with just one lamp and you are trying to read a book. The light is too dim, so you move halfway to the lamp. How much brighter is the light?

Solution

If $I(x)$ is the illumination when your distance from the lamp is x, then $I(x)$ is inversely proportional to the square of x:

$$I(x) = \frac{k}{x^2}$$

where k is a constant.

If you are at a distance d from the lamp, then the illumination is

$$I(d) = \frac{k}{d^2} \qquad \text{Replace } x \text{ by } d$$

Rules of Exponents are reviewed in Toolkits A.3 and A.4, pages T14 and T20.

If you move halfway to the lamp, your new distance is $\frac{1}{2}d$, so the illumination is now

$$I\left(\tfrac{1}{2}d\right) = \frac{k}{\left(\frac{1}{2}d\right)^2} \qquad \text{Replace } x \text{ by } \tfrac{1}{2}d$$

$$= \frac{k}{\frac{1}{4}d^2} \qquad \text{Property of Exponents}$$

$$= 4\left(\frac{k}{d^2}\right) \qquad \text{Property of Fractions}$$

$$= 4I(d) \qquad \text{Replace } k/d^2 \text{ by } I(d)$$

Thus, the light is now four times as bright.

■ **NOW TRY EXERCISE 37** ■

There is a geometric reason that many of the laws of nature are inverse square laws. Imagine a force or energy originating from a point source and spreading its influence equally in all directions, just like the light shining from a light bulb in Example 4 or the gravitational force exerted by a planet. The influence of the force or energy at a distance r from the source is spread out over the surface of a

sphere of radius r, which has area $A = 4\pi r^2$ (see Figure 4). So the intensity I at a distance r from the source is equal to the source strength S divided by the area A of the sphere:

$$I = \frac{S}{4\pi r^2} = \frac{k}{r^2}$$

where k is the constant $S/(4\pi)$. Thus point sources of light, sound, gravity, electromagnetic fields, and radiation must all obey inverse square laws, simply because of the geometry of space.

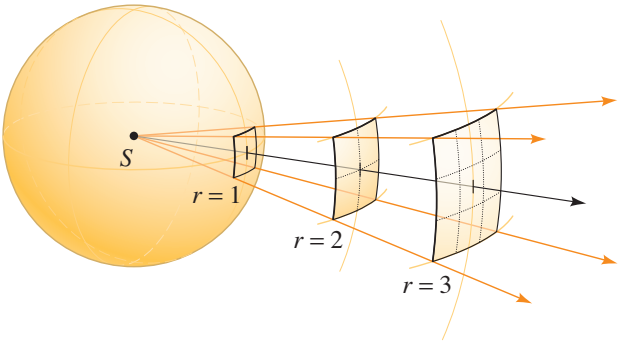


figure **4** Energy from a point source S

Algebra Checkpoint ✓

Check your knowledge of rules of exponents by doing the following problems. You can review these topics in **Algebra Toolkit A.3** and **A.4** on pages T14 and T20.

1. Evaluate each expression.

(a) 2^{-3} **(b)** $\left(\frac{1}{2}\right)^{-1}$

(c) $\dfrac{5^4}{5^{-2}}$ **(d)** $\left(\frac{5}{2}\right)^{-2}$

2. Simplify each expression, and eliminate any negative exponents.

(a) x^3x^{-4} **(b)** $(2y^{-2})^3$

(c) $(-b)^{-3}$ **(d)** $(-b)^{-2}$

3. Simplify each expression, and eliminate any negative exponents.

(a) $4^{-3/2}$ **(b)** $\left(\frac{1}{8}\right)^{-1/3}$

(c) $\dfrac{a^{-1/2}}{a^{3/2}}$ **(d)** $a^{5/2}a^{-1/2}$

4. Solve the equation.

(a) $x^{-1} = 5$ **(b)** $2x^{-2} = \frac{1}{18}$

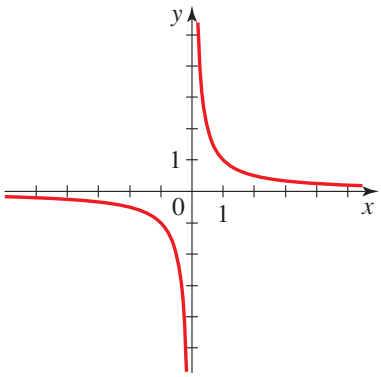
(c) $x^{-1/3} = 2$ **(d)** $3x^{-1/2} = 15$

6.5 Exercises

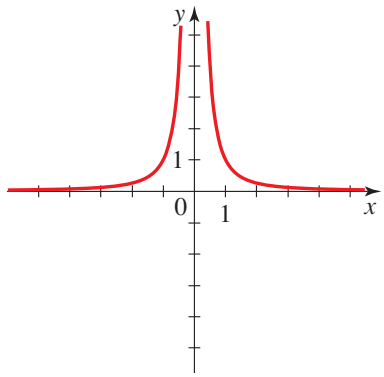
CONCEPTS

Fundamentals

1. Graphs of $y = x^{-1}$ and $y = x^{-2}$ are shown. Identify each graph.



$y =$ _____



$y =$ _____

2. **(a)** If the quantities x and y are related by the equation $y = 7/x^5$, then we say that y is _____ proportional to the _____ power of x and the constant of proportionality is _____.

(b) If y is inversely proportional to the third power of x and if the constant of proportionality is 4, then x and y are related by the equation $y = \square \cdot \frac{1}{x^{\square}}$.

Think About It

3. Graph the function $y = x^{-1}$ in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Explain why you get a blank screen.

4. Suppose that a skunk is perched at the top of a tall flag pole. Do you think that the intensity of its smell is inversely proportional to the square of the distance from the skunk? (Read the explanation of the geometry of inverse square laws on page 531.)

SKILLS

5–8 ■ A power function f is given.

(a) Complete each table for the values of the function.

(b) Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2.

(c) Determine the behavior of the function near its horizontal asymptote, based on Tables 3 and 4.

table **1**

x	$f(x)$
−0.1	
−0.01	
−0.001	
−0.00001	

table **2**

x	$f(x)$
0.1	
0.01	
0.001	
0.00001	

table **3**

x	$f(x)$
10	
50	
100	
100,000	

table **4**

x	$f(x)$
−10	
−50	
−100	
−100,000	

5. $f(x) = x^{-3}$ **6.** $f(x) = x^{-5}$ **7.** $f(x) = x^{-4}$ **8.** $f(x) = x^{-6}$

9–12 ■ Graph the family of functions in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

9. $f(x) = x^{-c}; \quad c = 1, 3, 5, 7$

10. $g(x) = x^{-c}; \quad c = 2, 4, 6, 8$

11. $F(x) = cx^{-1}; \quad c = 1, 2, 5, \frac{1}{2}$

12. $G(x) = x^{-2} + c; \quad c = -1, 0, 1, 2$

13–20 ■ Sketch a graph of the given function.

13. $f(x) = 3x^{-1}$

14. $f(x) = 10x^{-2}$

15. $f(x) = 5x^{-4}$

16. $f(x) = 2x^{-7}$

17. $f(x) = \dfrac{4}{x^2}$

18. $f(x) = \dfrac{6}{x}$

19. $f(x) = \dfrac{1}{x} + 1$

20. $f(x) = \dfrac{1}{x^2} - 1$

21–24 ■ Write an equation that expresses the statement.

21. P is inversely proportional to T.

22. R is inversely proportional to the third power of c.

23. y is inversely proportional to the square root of t.

24. E is inversely proportional to the fourth root of t.

25–28 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

25. z is inversely proportional to t. If t is 3, then z is 2.

26. R is inversely proportional to s. If s is 4, then R is 3.

27. s is inversely proportional to the square root of t. If t is 9, then s is 15.

28. W is inversely proportional to the square of r. If r is 6, then W is 10.

29–30 ■ Use the given information to solve the problem.

29. V is inversely proportional to the cube of x. If the constant of proportionality k is 1.5, then find V when x is 2.

30. C is inversely proportional to the square of t. If the constant of proportionality k is 5, then find C when t is 7.

CONTEXTS

31. Boyle's Law The pressure P of a sample of oxygen gas that is compressed at a constant temperature is inversely proportional to the volume V of gas.

(a) Find the constant of proportionality if a sample of oxygen gas that occupies 0.671 m^3 exerts a pressure of 39 kPa at a temperature of 293 K (absolute temperature measured on the Kelvin scale). Write the equation that expresses the inverse proportionality.

(b) If the sample expands to a volume of 0.916 m^3, find the new pressure.



32. Skidding in a Curve A car weighing 1000 kilograms is traveling at a speed of 15 meters per second on a curve that forms a circular arc. The centripetal force F that pushes the car outward is inversely proportional to the radius r (measured in meters) of the curve.

(a) If the curve has a radius of 90 meters, then the force required to keep the car from skidding is 2500 newtons. Write an equation that expresses the inverse proportionality.

(b) The car will skid if the centripetal force is greater than the frictional force holding the tires to the road. For this road the maximum frictional force is 5880 newtons. What is the radius of the tightest curve the car can maneuver before it starts slipping?

33. Loudness of Sound The loudness L of a sound measured in decibels (dB) is inversely proportional to the square of the distance d from the source of the sound.

(a) A person who is 10 ft from a lawn mower experiences a sound level of 70 dB. Find the constant of proportionality, and write the equation that expresses the inverse proportionality.

(b) How loud is the lawn mower when the person is 100 ft away?

34. Electrical Resistance The resistance R of a wire of a fixed length is inversely proportional to the square of its diameter.

(a) A wire that is 1.2 meters long and 0.005 meter in diameter has a resistance of 140 ohms. Find the constant of proportionality, and write the equation that expresses the inverse proportionality.

(b) Find the resistance of a wire made of the same material and of the same length as the wire in part (a) but with a diameter of 0.008 meter.

35. Growing Cabbages In the short growing season of the Canadian arctic territory of Nunavut, some gardeners find it possible to grow gigantic cabbages in the midnight sun. Assume that the cabbages receive a constant amount of nutrients and that the final size of a cabbage is inversely proportional to the number of other cabbages surrounding it.

(a) A cabbage that had 12 other cabbages around it grew to 30 pounds. Find the constant of proportionality, and write the equation that expresses the inverse proportionality.

(b) What size would this cabbage grow to if it had only 5 cabbage "neighbors"?

36. Newton's Law of Gravitation The gravitational force between two objects is inversely proportional to the square of the distance between them. How does the gravitational force between two objects change if the distance between them is (a) tripled? (b) halved?

37. Heat of Campfire The heat experienced by a hiker at a campfire is inversely proportional to the cube of his distance from the fire.

(a) Write an equation that expresses the inverse proportionality.

(b) If the hiker is too cold and moves halfway closer to the fire, how much more heat does the hiker experience?



38. Frequency of Vibration The frequency f of vibration of a violin string is inversely proportional to its length L. The constant of proportionality k is positive and depends on the tension and density of the string.

(a) Write an equation that represents this proportionality.

(b) What effect does doubling the length of the string have on the frequency of its vibration?

6.6 Rational Functions

- Graphing Quotients of Linear Functions
- Graphing Rational Functions



Lena Small/Shutterstock.com 2009

Focusing distance is a rational function of the focal length of a lens.

IN THIS SECTION... we study rational functions, which are quotients of polynomial functions.

GET READY... by reviewing rational expressions and long division of polynomials in **Algebra Toolkit B.3**. Check your understanding of these topics by doing the **Algebra Checkpoint** at the end of this section.

We have encountered many real-world situations in which one quantity is directly or inversely proportional to another. But the relationship between two quantities may be more complicated, combining aspects of both direct and inverse dependence. In many such cases the function relating the two quantities is a ratio of two polynomial functions. Such functions are called *rational functions*.

For example, camera manufacturers use rational functions to model the relationship between the focusing distance and the focal length of a lens. Modern camera lenses are really lens assemblies, made up of several individual lenses working together. Combining such lenses makes it possible to manufacture telephoto lenses that have adjustable focal lengths. The basic principle underlying the design of such complex lenses is a rational function relating focal length and focusing distance (see Example 2).

Graphing Quotients of Linear Functions

A **rational function** is a function of the form $r(x) = p(x)/q(x)$, where p and q are polynomial functions. We assume that the polynomials $p(x)$ and $q(x)$ have no factor in common. We first study rational functions that are quotients of two linear functions:

$$y = \frac{a + bx}{c + dx}$$

To graph such a function, we can use "graphical division" as in Section 5.1. But we'll see that every such function is a transformation of the reciprocal function. So to graph the function, we need to express it as an appropriate transformation (shift or stretch) of the reciprocal function. The next example illustrates this method.

example 1 Graphing a Rational Function

Sketch a graph of each rational function.

(a) $r(x) = \dfrac{2}{x - 3}$ **(b)** $s(x) = \dfrac{3x + 5}{x + 2}$

Solution

(a) Let $f(x) = 1/x$. Then we can express r in terms of f as follows:

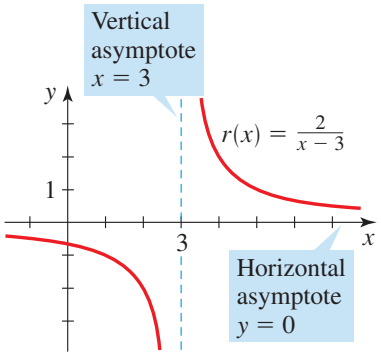


figure **1** Graph of $r(x) = 2/(x - 3)$

$$r(x) = \frac{2}{x-3} \qquad \text{Definition of } r$$

$$= 2\left(\frac{1}{x-3}\right) \qquad \text{Factor 2}$$

$$= 2f(x-3) \qquad \text{Because } f(x-3) = \frac{1}{x-3}$$

So the graph of r is obtained from the graph of f by shifting 3 units to the right and stretching vertically by a factor of 2. Thus the graph of r has a vertical asymptote $x = 3$ and a horizontal asymptote $y = 0$. A graph of r is shown in Figure 1.

(b) Let $f(x) = 1/x$. Using long division (see the margin), we can express s in terms of f as follows:

$$s(x) = \frac{3x+5}{x+2} \qquad \text{Definition of } s$$

$$= 3 - \frac{1}{x+2} \qquad \text{Long division (see margin)}$$

$$= -\frac{1}{x+2} + 3 \qquad \text{Rearrange terms}$$

$$= -f(x+2) + 3 \qquad \text{Because } f(x+2) = \frac{1}{x+2}$$

So the graph of s is obtained from the graph of f by shifting 2 units to the left, reflecting in the x-axis, and then shifting upward 3 units. Thus s has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$. The graph of s is shown in Figure 2.

$$x+2\overline{)\,3x+5}\quad\Big|\quad 3$$
$$\underline{3x+6}$$
$$-1$$

Division of polynomials is reviewed in Algebra Toolkit B.3, page T39.

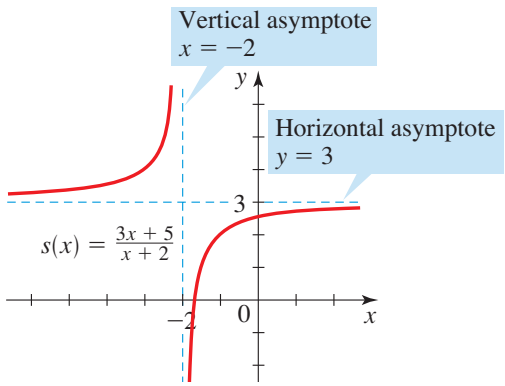


figure **2** Graph of $r(x) = (3x + 5)/(x + 2)$

■ **NOW TRY EXERCISES 29 AND 31** ■

example **2**

A CCD, or charge-coupled device, is commonly used in the light sensor of a digital camera.

Focusing Distance for Camera Lenses

If a camera has a lens of fixed focal length F, then to focus on an object located a distance x from the lens, the camera's charge-coupled device (CCD) must be at a distance y behind the lens (see Figure 3), where F, x, and y are related by the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

This equation holds for x greater than F (otherwise no image is produced). Suppose a camera has a 50-mm lens ($F = 50$).

(a) Express y as a function of x, and graph the resulting rational function.

(b) If a person is 3 m from the lens, what is the focusing distance?

(c) What happens to the focusing distance y as the object moves far away from the lens?

(d) What happens to the focusing distance y as the object moves closer to the lens?

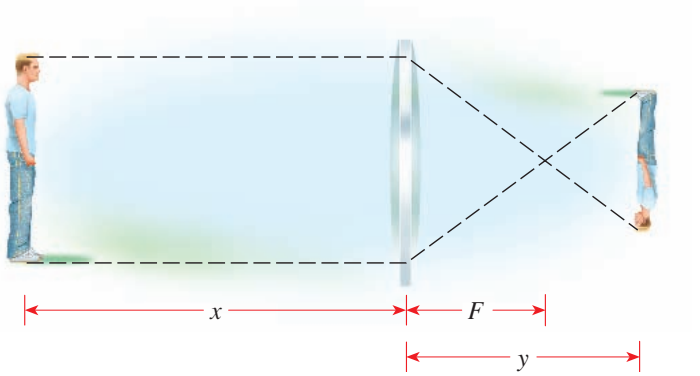


figure **3**

Solution

(a) We replace F by 50 in the equation and solve for y:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{50} \qquad \text{Replace } F \text{ by } 50$$

$$\frac{1}{y} = \frac{1}{50} - \frac{1}{x} \qquad \text{Subtract } 1/x$$

$$\frac{1}{y} = \frac{x - 50}{50x} \qquad \text{Common denominator}$$

$$y = \frac{50x}{x - 50} \qquad \text{Take the reciprocal of each side}$$

The last equation defines y as a rational function of x. To graph this function, we can use the methods of Example 1, but here we use a graphing calculator with the viewing rectangle $[0, 500]$ by $[-400, 1000]$. The graph is shown in Figure 4. Note that the focusing distance y is never negative, so only the portion of the graph to the right of the vertical asymptote $x = 50$ applies to the problem.

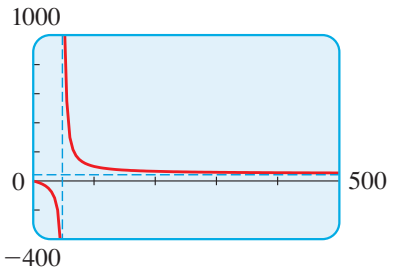


figure **4** Graph of $y = 50x/(x - 50)$

(b) Since the person is 3 m from the lens, x is 3000 mm. Using this value of x, we find the focusing distance y.

$$y = \frac{50x}{x - 50} \qquad \text{Focusing function}$$

$$= \frac{50 \cdot 3000}{3000 - 50} \qquad \text{Replace } x \text{ by } 3000$$

$$\approx 50.85 \qquad \text{Calculator}$$

So the focusing distance is 50.85 mm.

(c) From the graph we see that $y = 50$ is a horizontal asymptote, so the focusing distance y approaches 50 as x becomes large.

(d) From the graph we see that as the distance x gets close to 50, the focusing distance $y \to \infty$. (Of course, x cannot be less than 50.)

NOW TRY EXERCISE 41

Graphing Rational Functions

The most important features of the graph of a rational function are its asymptotes. In fact, once we locate all the asymptotes, we can readily complete the graph of a rational function. To find the asymptotes, we need to express the rational function in different ways. In **standard form**, a rational function is expressed as the quotient of two polynomials

$$r(x) = \frac{p(x)}{q(x)}$$

In **factored form**, the polynomials p and q are written in factored form. In the **compound fraction form**, we divide numerator and denominator by the highest power of x that appears in the polynomials p and q.

example 3 Three Forms of a Rational Function

Express the rational function $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$ in factored form and in compound fraction form.

Solution

In factored form, we factor the numerator and denominator. You can verify the following factorization:

Rational expressions are reviewed in Algebra Toolkit B.3, page T39.

$$r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$$ Factored form

The highest power of x that appears in the numerator or denominator is x^2. So to get the compound fraction form, we divide numerator and denominator by x^2.

$$r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \frac{2 + \dfrac{7}{x} - \dfrac{4}{x^2}}{1 + \dfrac{1}{x} - \dfrac{2}{x^2}}$$ Compound fraction form

NOW TRY EXERCISES 13 AND 23

Each of the forms of a rational function helps us to discover a useful property of the graph.

- We use the standard form to find the y-intercept (by replacing x by 0).
- We use the factored form to find the x-intercepts (these are the zeros of the numerator) and the asymptotes (these are determined by the zeros of the denominator).

- We use the compound fraction form to find the end behavior (by seeing what happens as x gets large).

The procedure is explained in the next example.

example 4 Graphing a Rational Function

Consider the rational function $y = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$.

(a) Find the x- and y-intercepts of the graph.

(b) Find the vertical asymptotes.

(c) Find the horizontal asymptotes.

(d) Sketch a graph.

Solution

We use the three forms of this rational function found in Example 2.

(a) To find the y-intercept, we replace x by 0 in the standard form:

$$y = \frac{2(0)^2 + 7(0) - 4}{(0)^2 + (0) - 2} = 2$$

So the y-intercept is 2. To find the x-intercept, we replace y by 0 in the factored form.

$$\frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)} = 0 \qquad \text{Replace } y \text{ by 0 in factored form}$$

$$(2x - 1)(x + 4) = 0 \qquad \text{The numerator is 0}$$

$$2x - 1 = 0 \quad \text{or} \quad x + 4 = 0 \qquad \text{Zero-Product Property}$$

$$x = \tfrac{1}{2} \quad \text{or} \quad x = -4 \qquad \text{Solve}$$

So the x-intercepts are $\frac{1}{2}$ and -4.

(b) The vertical asymptotes occur where the denominator is zero. The factored form is

$$y = \frac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$$

The denominator is zero when x is 1 or -2. So the vertical asymptotes are the vertical lines

$$x = 1 \qquad \text{and} \qquad x = -2$$

(c) A horizontal asymptote is a horizontal line that the graph approaches as x gets large. Let's write the compound fraction form and see what happens as x gets large. In this case the fractions $7/x$, $4/x^2$, $1/x$, and $2/x^2$ go to zero. So we have

$$y = \frac{2 + \dfrac{7}{x} - \dfrac{4}{x^2}}{1 + \dfrac{1}{x} - \dfrac{2}{x^2}} \quad \rightarrow \quad \frac{2 + 0 + 0}{1 + 0 + 0} = 2$$

So the graph gets closer to 2 as x gets large. This means that the horizontal asymptote is the horizontal line $y = 2$.

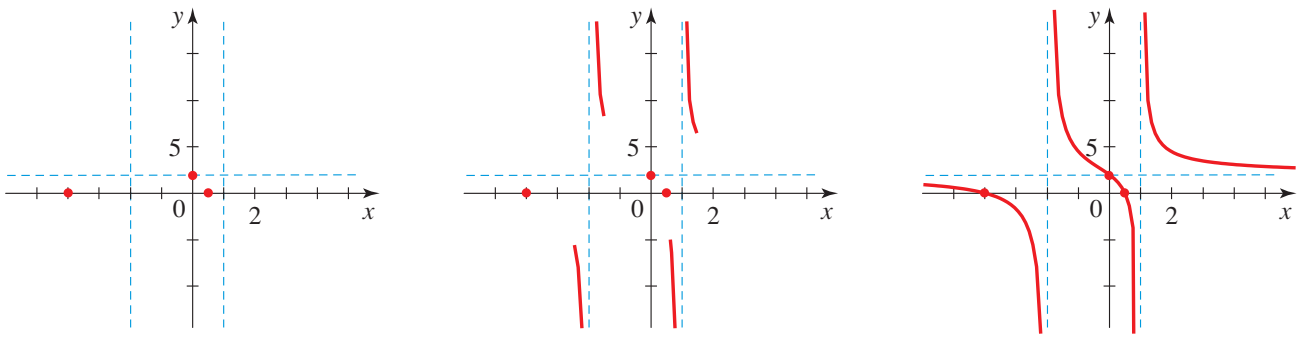
(d) We sketch the graph in three steps.

- First we sketch the asymptotes and intercepts as shown in Figure 5(a).
- Next, we examine the behavior of the graph near each vertical asymptote. We choose test points to the left and right of each asymptote to help us decide whether the graph goes up ($y \to \infty$) or down ($y \to -\infty$) as x gets near a vertical asymptote. For example, near the vertical asymptote $x = 1$ we choose the test points 0.9 and 1.1. From the factored form we can quickly decide the sign of y at the test points by examining the sign of each factor (as shown in the table below).

Test point	**Sign of** $y = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)}$	**Behavior near asymptote**
0.9	$\dfrac{(+)(+)}{(-)(+)}$ negative	$y \to -\infty$ as $x \to 1^-$
1.1	$\dfrac{(+)(+)}{(+)(+)}$ positive	$y \to \infty$ as $x \to 1^+$

The information in the table allows us to sketch the graph near the asymptote $x = 1$ as shown in Figure 5(b). A similar table allows us to sketch the graph near the asymptote $x = -2$, also shown in Figure 5(b).

- Finally, we complete the graph by connecting the parts already sketched, making sure that we approach the horizontal asymptotes as x gets large in the positive and negative directions. (See Figure 5(c).)



(a) Asymptotes and intercepts (b) Behavior near asymptotes (c) Graph of function

figure **5**
Steps in graphing
$y = (2x^2 + 7x - 4)/(x^2 + x - 2)$

NOW TRY EXERCISE 37

Algebra Checkpoint ✓

Check your working knowledge of rational expressions and division of polynomials by doing the following problems. You can review these topics in **Algebra Toolkit B.3** on page T39.

1. Simplify the rational expression.

(a) $\dfrac{5x}{x+x^2}$ **(b)** $\dfrac{x-2}{x^2-4}$ **(c)** $\dfrac{x^2+6x+8}{x^2+5x+4}$ **(d)** $\dfrac{2x^3-x^2-6x}{2x^2-7x+6}$

2. Perform the indicated operations and simplify.

(a) $\dfrac{1}{x+1}+\dfrac{1}{x-1}$ **(b)** $2+\dfrac{x}{x+3}$ **(c)** $\dfrac{1}{x+1}-\dfrac{1}{x+2}$ **(d)** $\dfrac{1}{x}+\dfrac{1}{x^2+x}$

3. Find the quotient and the remainder using long division.

(a) $\dfrac{3x+5}{x+2}$ **(b)** $\dfrac{x-3}{x+1}$ **(c)** $\dfrac{8x+5}{x-2}$ **(d)** $\dfrac{2x-5}{x-1}$

4. Find the quotient and the remainder using long division.

(a) $\dfrac{x^2-6x-8}{x-4}$ **(b)** $\dfrac{x^3-8x+2}{x+3}$

6.6 Exercises

CONCEPTS

Fundamentals

1. If the rational function $y = r(x)$ has the vertical asymptote $x = a$, then as $x \to a^+$, either $y \to$ _____ or $y \to$ _____.

2. If the rational function $y = r(x)$ has the horizontal asymptote $y = a$, then $y \to$ _____ as $x \to \pm\infty$.

3. The rational function r is shown in standard form, factored form, and compound fraction form. Identify each form of the function.

(a) $r(x) = \dfrac{(x+1)(x-2)}{(x+2)(x-3)}$: _____ form

(b) $r(x) = \dfrac{1-\frac{1}{x}-\frac{2}{x^2}}{1-\frac{1}{x}-\frac{6}{x^2}}$: _____ form

(c) $r(x) = \dfrac{x^2-x-2}{x^2-x-6}$: _____ form

4. The following questions are about the rational function r in Exercise 3.

(a) The function r has x-intercepts _____ and _____.

(b) The function r has y-intercept _____.

(c) The function r has vertical asymptotes $x =$ _____ and $x =$ _____.

(d) The function r has horizontal asymptote $y =$ _____.

Think About It

5. Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of one that has vertical asymptote $x = 3$ *and* horizontal asymptote $y = 2$. Now give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and x-intercept 4.

6. Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^2 + 10}{x^4 + 15}$$

has no x-intercept and no vertical asymptote.

SKILLS

7–10 ■ A rational function r is given.

(a) Complete each table for the function.

(b) Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2.

(c) Determine the behavior of the function near its horizontal asymptote, based on Tables 3 and 4.

table **1**

x	$r(x)$
1.5	
1.9	
1.99	
1.999	

table **2**

x	$r(x)$
2.5	
2.1	
2.01	
2.001	

table **3**

x	$r(x)$
10	
50	
100	
1000	

table **4**

x	$r(x)$
−10	
−50	
−100	
−1000	

7. $r(x) = \dfrac{x}{x - 2}$ **8.** $r(x) = \dfrac{4x + 1}{x - 2}$ **9.** $r(x) = \dfrac{3x - 10}{(x - 2)^2}$ **10.** $r(x) = \dfrac{3x^2 + 1}{(x - 2)^2}$

11–14 ■ Find the x- and y-intercepts of the rational function.

11. $r(x) = \dfrac{x - 1}{x + 4}$

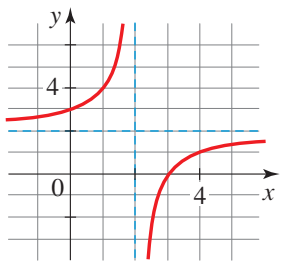
12. $s(x) = \dfrac{3x}{x - 5}$

13. $t(x) = \dfrac{x^2 - x - 2}{x - 6}$

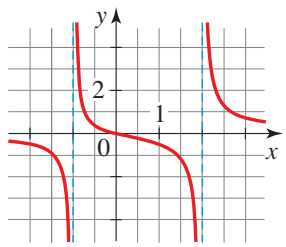
14. $r(x) = \dfrac{x^2 - 9}{x^2}$

15–18 ■ The graph of a rational function is given. Determine its x- and y-intercepts and its vertical and horizontal asymptotes.

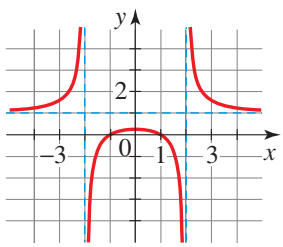
15.



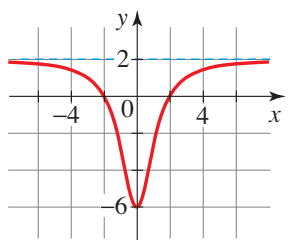
16.



17.



18.



19–26 ■ A rational function is given. Find all horizontal and vertical asymptotes of its graph.

19. $r(x) = \dfrac{5}{x - 2}$

20. $r(x) = \dfrac{2x - 3}{x^2 - 1}$

21. $r(x) = \dfrac{6x}{x^2 + 2}$

22. $r(x) = \dfrac{2x - 4}{x^2 + x + 1}$

23. $s(x) = \dfrac{6x^2 + 1}{2x^2 + x - 1}$

24. $s(x) = \dfrac{8x^2 + 1}{4x^2 + 2x - 6}$

25. $s(x) = \dfrac{(5x - 1)(x + 1)}{(3x - 1)(x + 2)}$

26. $s(x) = \dfrac{(2x - 1)(x + 3)}{(3x - 1)(x - 4)}$

27–32 ■ Use transformations of the graph of the reciprocal function $y = 1/x$ to graph the rational function, as in Example 1.

27. $r(x) = \dfrac{1}{x - 1}$

28. $r(x) = \dfrac{1}{x + 4}$

29. $s(x) = \dfrac{3}{x + 1}$

30. $r(x) = \dfrac{-2}{x - 2}$

31. $s(x) = \dfrac{2x - 3}{x - 2}$

32. $s(x) = \dfrac{3x - 3}{x + 2}$

33–40 ■ Find the intercepts and asymptotes, and then sketch a graph of the rational function. Use a graphing calculator to confirm your answer.

33. $r(x) = \dfrac{4x - 4}{x + 2}$

34. $r(x) = \dfrac{2x + 6}{-6x + 3}$

35. $s(x) = \dfrac{18}{(x - 3)^2}$

36. $s(x) = \dfrac{x - 2}{(x + 1)^2}$

37. $t(x) = \dfrac{4x - 8}{(x - 4)(x + 1)}$

38. $t(x) = \dfrac{x + 2}{(x + 3)(x - 1)}$

39. $r(x) = \dfrac{2x^2 + 10x - 12}{x^2 + x - 6}$

40. $r(x) = \dfrac{2x^2 + 2x - 4}{x^2 + x}$

CONTEXTS

41. Focusing a Camera Lens A camera has a lens of fixed focal length F. To focus on an object located a distance x from the lens, the film must be placed a distance y behind the lens, where F, x, and y are related by

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

(See Example 2.) Suppose the camera has a wide-angle 35-mm lens ($F = 35$).

(a) Express y as a function of x, and graph the function.

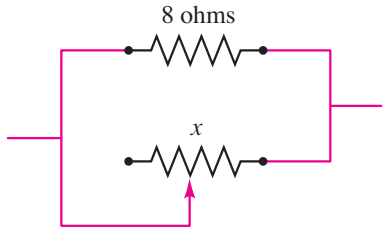
(b) What happens to the focusing distance y as the object moves farther away from the lens?

(c) What happens to the focusing distance y as the object moves closer to the lens?

42. Resistors in Parallel When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

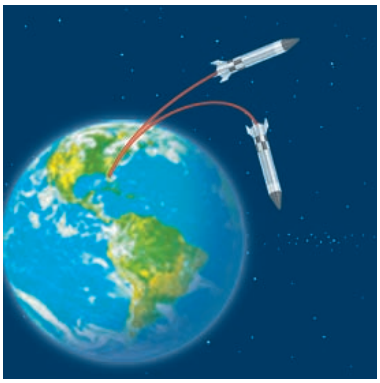
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor, as shown in the figure. If the resistance of the variable resistor is denoted by x, then the combined resistance R is a function of x.



(a) Find the function R.

(b) Draw a graph of the function $y = R(x)$. (Note that R and x must both be positive, so the viewing rectangle need not contain negative values.) Give a physical interpretation of the horizontal asymptote.



43. Flight of a Rocket Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in meters per second). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where the radius of the earth R is 6.4×10^6 m and the acceleration due to gravity g is 9.8 m/s^2.

(a) If the initial velocity of the rocket is 10,400 m/s, what is the maximum height reached by the rocket?

(b) Use a graphing calculator to draw a graph of the function $y = h(v)$. (Note that y and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?

44. The Doppler Effect As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest, because the crests of the sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch P is a function of the speed v (measured in m/s) of the train and is given by

$$P(v) = P_0\left(\frac{s_0}{s_0 - v}\right)$$

where P_0 is the actual pitch of the whistle at the source, and the speed of sound in air s_0 is 332 m/s. Suppose that a train has a whistle pitch P_0 that is 440 Hz.

(a) If the speed of the train is 30 m/s, what is the observed pitch of the train whistle?

(b) Use a graphing calculator to draw a graph of the function $y = P(v)$. What does the vertical asymptote represent physically?



CHAPTER 6 REVIEW

CHAPTER 6 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

6.1 Working with Functions: Algebraic Operations

Functions can be added, subtracted, multiplied, or divided. For the functions f and g:

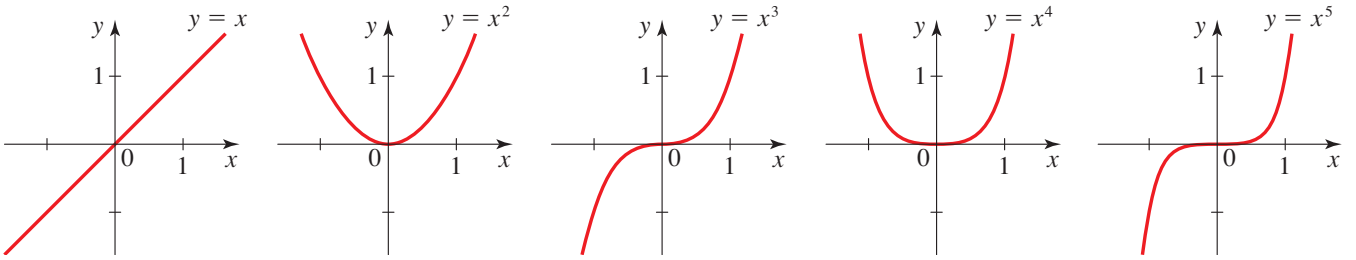
- The **sum** is the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$.
- The **difference** is the function $f - g$ defined by $(f - g)(x) = f(x) - g(x)$.
- The **product** is the function fg defined by $(fg)(x) = f(x) \cdot g(x)$.
- The **quotient** is the function f/g defined by $(f/g)(x) = f(x)/g(x)$.

The **domains** of $f + g$, $f - g$, and fg are the intersections of the domains of f and g. The **domain** of f/g is the intersection of the domain of f and g, except that we must remove all the points where $g(x)$ is 0.

6.2 Power Functions: Positive Powers

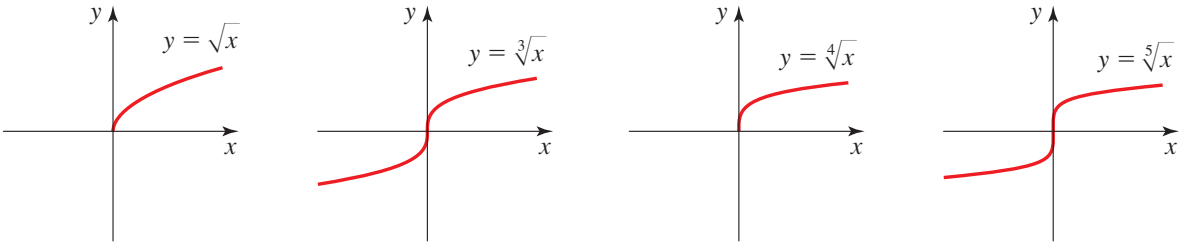
A **power function** is a function of the form $f(x) = Cx^p$, where C and p are nonzero.

If p is a **positive integer**, then the shape of the **graph** of the power function $f(x) = Cx^p$ depends on whether p is odd or even. Graphs of $f(x) = x^p$ for $p = 1$, 2, 3, 4, and 5 are shown in the figure.



If p is the **positive rational number** m/n, then the power function $f(x) = x^p$ is defined by $f(x) = x^{m/n} = \sqrt[n]{x^m}$. If n is *odd*, then the domain of f consists of all real numbers; but if n is *even*, then the domain consists of the nonnegative real numbers.

The inverse of the power function $f(x) = x^n$ with positive integer power n is the **root function** $f^{-1}(x) = x^{1/n} = \sqrt[n]{x}$.



The quantity Q is **directly proportional** to the quantity R if they are related by the equation $Q = kR$ (where k is a nonzero constant called the **constant of proportionality**). If two real-world quantities are directly proportional, then their relationship can often be modeled by a power function.

6.3 Polynomial Functions: Combining Power Functions

A **polynomial function** is a function P of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer, $a_0, a_1, \ldots, a_n$ are real numbers, and $a_n \neq 0$. The polynomial has **degree n**, and $a_n x^n$ is its **leading term**.

The **end behavior** of a polynomial function is a description of what happens to its values when x becomes large in the positive and negative directions. A polynomial function has the same end behavior as its leading term.

One way to **graph** a polynomial function is to proceed as follows:

- Factor the polynomial into linear factors.
- Find the zeros of the polynomial from its factors.
- Use test points to check the sign of the values of the polynomial on each of the intervals determined by the zeros.
- Graph the zeros and test points, and complete the graph.
- Use the end behavior of the polynomial as a final check on the overall shape of the graph.

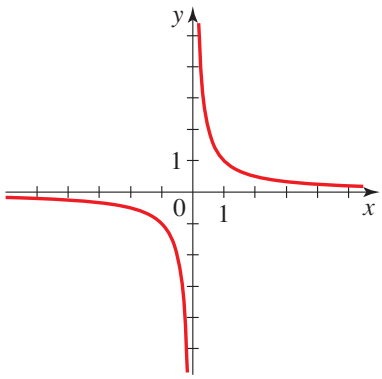
6.4 Fitting Power and Polynomial Curves to Data

Real-world data are often best modeled by a power or polynomial function. A graphing calculator or computer software is the most practical way to find such a model.

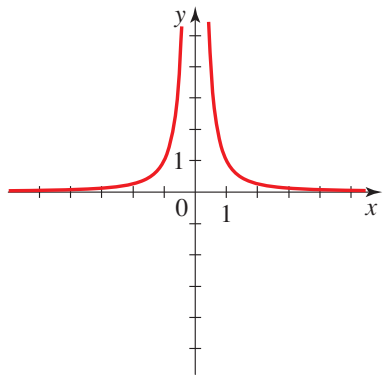
To see whether a linear, power, or exponential model is most appropriate for a given data set, we can use a **scatter plot**, a **semi-log plot**, and a **log-log plot** to help us make the appropriate choice.

6.5 Power Functions: Negative Powers

Power functions $f(x) = Cx^p$ in which the power p is negative occur often in the applications of mathematics. Two such functions (the ones where p is -1 or -2) have graphs that are typical of power functions with negative integer powers:



Graph of $f(x) = \dfrac{1}{x} = x^{-1}$



Graph of $f(x) = \dfrac{1}{x^2} = x^{-2}$

The quantity Q is **inversely proportional** to the quantity R if they are related by the equation $Q = k/R$ (where k is a nonzero constant called the **constant of proportionality**). Many of the laws of science are inverse proportionalities, often inverse square laws such as the Law of Gravity.

6.6 Rational Functions

A **rational function** r is a quotient of two polynomials:

$$r(x) = \frac{P(x)}{Q(x)}$$

Most rational functions have **horizontal** and/or **vertical asymptotes**: lines that the graph of r approaches as we move vertically or horizontally on the graph.

If P and Q are both linear functions then the graph of r can be obtained from the graph of $f(x) = \dfrac{1}{x}$ by shifting, stretching, and reflecting.

A rational function can be expressed in three forms: **standard form**, **factored form**, and **compound-fraction form**.

To graph a rational function in which P or Q are polynomials of degree 2 or higher, we first find the following information about the graph:

- the x- and y-intercepts
- the vertical asymptotes (by finding the zeros of Q)
- the horizontal asymptotes (by determining the behavior of y as $x \to \infty$)

Then we sketch a graph that shows all this information, plot some additional points, and complete the graph.

CHAPTER 6 REVIEW EXERCISES

SKILLS

1–4 ■ The functions f and g are given.

(a) Find the functions $f + g, f - g, fg,$ and f/g and their domains.

(b) Evaluate $f + g, f - g, fg$, and f/g at the indicated value, if defined.

(c) Draw the graphs of f, g, and $f + g$ on the same screen to illustrate graphical addition.

1. $f(x) = 2x, \quad g(x) = x - 1; \quad 3$

2. $f(x) = x^2, \quad g(x) = 2x + 1; \quad 0.5$

3. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x - 1}; \quad 1$

4. $f(x) = \sqrt{x + 1}, \quad g(x) = x - 3; \quad 3$

5–8 ■ Sketch the graph of the given power function f. Then sketch the graph of g on the same axes by transforming the graph of f.

5. $f(x) = x^2, \quad g(x) = (x + 1)^2$

6. $f(x) = x^3, \quad g(x) = -x^3 + 1$

7. $f(x) = x^3, \quad g(x) = \frac{1}{2}x^3 - 4$

8. $f(x) = x^{1/2}, \quad g(x) = (x + 4)^{1/2} - 2$

9–12 ■ A statement describing a proportionality is given.

(a) Express the statement as an equation.

(b) Use the given information to find the constant of proportionality.

9. E is directly proportional to the square of v. If $v = 6$, then $E = 9$.

10. F is directly proportional to the fourth power of x. If $x = 0.5$, then $F = 32$.

11. L is directly proportional to the one-third power of v. If $V = 8$, then $L = 36$.

12. h is directly proportional to the cube of w. If $w = 3$, then $h = \frac{1}{3}$.

13–14 ■ Use the proportionality from the indicated exercise to solve the problem.

13. In Exercise 9 above, find E when v is 4.

14. In Exercise 11 above, find V when L is 3.

15–22 ■ Sketch a graph of the polynomial function. Make sure your graph shows all x- and y-intercepts and exhibits the proper end behavior.

15. $P(x) = x(x + 2)(x - 1)$

16. $P(x) = -(x - 1)^2(x + 2)$

17. $P(x) = -x^3 + 3x^2$

18. $P(x) = x^3 - x^2 - 4x + 4$

19. $Q(x) = x^3 + 2x^2 - 4x - 8$

20. $Q(x) = x^4 - x^2$

21. $P(x) = x^4 + 3x^2 - 4$

22. $P(x) = x^4 - 2x^2 + 1$

23–24 ■ A polynomial function P is given.

(a) Graph the polynomial P in the given viewing rectangle.

(b) Find all the local maxima and minima of P.

(c) Find all solutions of the equation $P(x) = 0$.

23. $P(x) = 2x^3 - 3x^2 - 12x + 10$; $[-3, 4]$ by $[-15, 20]$

24. $P(x) = x^4 - 2x^2 + 3$; $[-2, 2]$ by $[-1, 5]$

25–26 ■ A set of data is given.

(a) Use your graphing calculator to find the two indicated types of models for the data.

(b) Graph a scatter plot of the data together with the models.

(c) From the graphs in part (b), which model do you think fits the data better?

25. Linear, cubic:

x	$f(x)$
0	2
1	33
2	48
3	51
4	48
5	45
6	48
7	63

26. Linear, power:

x	$f(x)$
1	10
2	32
3	65
4	105
5	154
6	210
7	273

27–28 ■ A set of data is given.

(a) Make semi-log and log-log plots of the data.

(b) From the graphs in part (a), does an exponential or a power model seem appropriate for the data?

(c) Find an appropriate model for the data, and graph the model together with a scatter plot of the data.

27.

x	y
2	15
4	44
6	80
8	124
10	173
12	229
14	288
16	352

28.

x	y
1	3.0
10	7.8
20	20.2
30	52.5
40	136.0

29–32 ■ Sketch a graph of the given power function.

29. $f(x) = \frac{1}{2}x^{-2}$ **30.** $f(x) = x^{-3}$ **31.** $f(x) = \dfrac{8}{x^3}$ **32.** $f(x) = \dfrac{1}{3x^4}$

33–34 ■ A statement describing a proportionality is given.

(a) Express the statement as an equation.

(b) Use the given information to find the constant of proportionality.

33. G is inversely proportional to the cube of r. If $r = 10$, then $G = 0.3$.

34. n is inversely proportional to the two-thirds power of t. If $t = 0.125$, then $n = 40$.

35–36 ■ Use the proportionality from the indicated exercise to solve the problem.

35. In Exercise 33 above, find G when r is 8.

36. In Exercise 34 above, find t when n is 100.

37–40 ■ Use transformations of the graph of $y = 1/x$ to graph the rational function.

37. $r(x) = \dfrac{1}{x + 2}$ **38.** $r(x) = \dfrac{2}{x - 3}$

39. $s(x) = -\dfrac{1}{x - 2}$ **40.** $s(x) = \dfrac{x}{x + 1}$

41–44 ■ A rational function is given.

(a) Find the x- and y-intercepts of the function.

(b) Find the horizontal and vertical asymptotes of the function.

(c) Sketch a graph of the function.

41. $r(x) = \dfrac{2x + 2}{x - 1}$ **42.** $r(x) = \dfrac{-x + 4}{2x + 6}$

43. $s(x) = \dfrac{x}{x^2 - x - 2}$ **44.** $s(x) = \dfrac{1}{4(x + 1)^2}$

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

45. Linear, Exponential, and Power Functions We have studied three classes of functions that are of particular importance in modeling real-life phenomena:

- Linear functions: $f(x) = b + mx$
- Exponential functions: $f(x) = Ca^x$
- Power functions: $f(x) = Cx^p$

(a) What type of function is each of the following?

(i) $f(x) = 4^x$ **(ii)** $g(x) = 4x$ **(iii)** $h(x) = x^4$

(b) One of the functions in part (a) actually belongs to two of the three categories. Which one? Explain.

(c) The tables give three sets of data. For each set, determine whether a linear, exponential, or power function is an appropriate model for the data. Explain how you arrived at your answer.

table **1**

x	y
1	2
2	5
3	8
4	11
5	14

table **2**

x	y
1	2
2	5
3	12.5
4	31.3
5	78.1

table **3**

x	y
1	2
2	5
3	8.5
4	12.5
5	16.8

(d) Find an appropriate model for each of the data sets in part (c).

46. Inverses of Functions In Section 6.2 we saw that the inverse of the power function $f(x) = x^2$ $(x > 0)$ is the power function $f^{-1}(x) = x^{1/2}$. In this problem we'll investigate the inverses of some of the other types of functions we have studied.

(a) Find the inverse of the linear function $f(x) = 3 - 5x$. Is the inverse a linear function? Will this be the case for all nonconstant linear functions?

(b) Find the inverse of the exponential function $f(x) = 3 \cdot 5^x$. Is the inverse an exponential function? If not, what kind of function is it?

(c) The rational function $f(x) = \dfrac{2 + x}{1 - x}$ is a quotient of two linear functions. Find its inverse. Is it also a quotient of two linear functions? Will this be the case for all quotients of two linear functions?

CONTEXTS

47. Revenue, Cost, and Profit Freebies, Inc. stamps company logos on pens intended for use by employees and prospective clients. The revenue from an order of x pens is $R(x) = 0.75x - 0.00002x^2$, and the cost of producing x pens is $C(x) = 0.35x - 0.00001x^2$ (both in dollars).

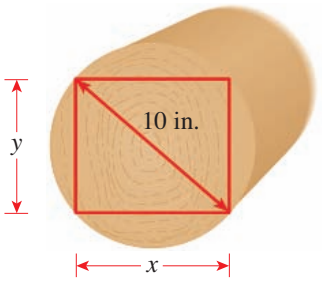
(a) What algebraic operation must you perform on the functions R and C to get P, the function that gives the profit from an order on x pens? Find $P(1000)$.

(b) What do the quotient functions $r(x) = \dfrac{R(x)}{x}$ and $c(x) = \dfrac{C(x)}{x}$ represent?

48. Range of a Projectile The range of a projectile (the horizontal distance it travels before it hits the ground) is directly proportional to the square of its speed. Jason throws a ball at 60 mi/h and it lands 242 ft away.

(a) Express the proportionality that relates range R to speed v as an equation. What is the constant of proportionality?

(b) What is Jason's range if he throws the ball at 70 mi/h?



49. Strength of a Beam The strength S of a wooden beam of width x and depth y is given by the formula $S = 13.8xy^2$. A beam is to be cut from a log of diameter 10 inches, as shown in the figure.

(a) From Pythagoras' Theorem we can see that $x^2 + y^2 = 10^2$. Use this fact to express S as a polynomial function of x only.

(b) What is the domain of S (keeping in mind that the distances x and y can't be negative)?

(c) Draw a graph of S.

(d) Find the width x that will give the beam the maximum possible strength.

50. Height of a Baseball A baseball is thrown upward, and its height in feet is measured at 0.5-second intervals by using a strobe light. The resulting data are given in the table.

(a) Find a quadratic model that best fits the data.

(b) Draw a scatter plot of the data, and graph your model from part (a) on the scatter plot.

(c) At what times does the model indicate that the ball will be 20 feet above the ground?

(d) What is the maximum height attained by the ball?

Time (s)	Height (ft)
0	4.2
0.5	26.1
1.0	40.1
1.5	46.0
2.0	43.9
2.5	33.7
3.0	15.8

51. Drug Concentration A drug is administered to a patient, and the concentration of the drug in her bloodstream is monitored. After t hours the concentration (in mg/L) is approximated by the rational function

$$c(t) = \frac{5t}{t^2 + 1}$$

(a) Does the function c have a vertical asymptote? A horizontal asymptote?

(b) From your answers in part (a), determine what happens to the concentration of the drug in the patient's bloodstream after a long period of time.

(c) Draw a graph of c. Use your graph to determine the maximum concentration of the drug in the patient's bloodstream and the time at which the maximum was reached.

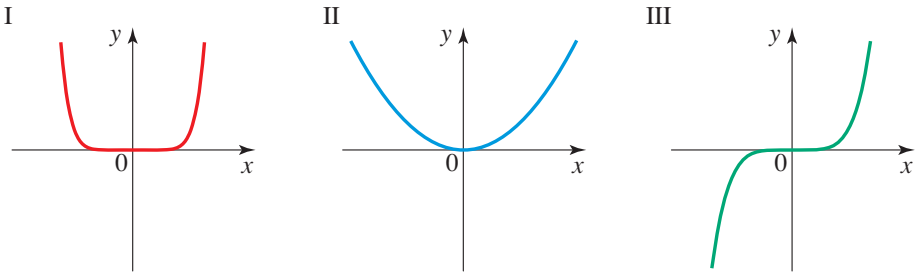
CHAPTER 6 TEST

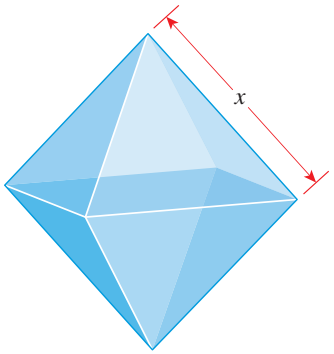
1. Let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$. Find the functions $f + g, f - g, fg$, and f/g and their domains. Then find the value of each of these functions at $x = 1$, if the value is defined.

2. The graphs of $f(x) = x^2$, $g(x) = x^5$, and $h(x) = x^8$ are shown in the figure.

(a) Match each function with its graph. Give reasons for your answers.

(b) What are the points at which all three graphs intersect?





3. A regular octahedron is a solid with eight faces, all of them equilateral triangles (see the figure). The volume V of a regular octahedron is proportional to the cube of the length x of one of its edges.

(a) Express this proportionality as an equation.

(b) A regular octahedron with edges of length 4 cm has volume 30.17 cm^3. Use this fact to find the constant of proportionality.

(c) What is the volume of an octahedron whose edges are 5 cm long?

4. Sketch a graph of the polynomial function. Make sure your graph shows all x- and y-intercepts and exhibits the proper end behavior.

(a) $P(x) = (x - 1)^2(x + 2)$ **(b)** $Q(x) = x^4 - 3x^2 - 4$

5. For the polynomial Q of Question 4(b), use a graphing calculator to find its local maximum and minimum values and the values of x at which they occur.

6. The table gives a set of data.

(a) Make semi-log and log-log plots of the data. On the basis of your graphs, determine whether an exponential model or a power model is more appropriate for the data.

(b) Use a graphing calculator to find an appropriate model for the data.

(c) Make a scatter plot of the data, and graph the function you found in part (b) on the scatter plot. Does the function seem to fit the data well?

x	1	2	5	7	23	44	58	80
y	1	3	11	19	108	287	439	710

7. **(a)** Sketch a graph of the reciprocal function $f(x) = 1/x$.

(b) By transforming the graph of the reciprocal function, sketch a graph of the rational function

$$r(x) = \frac{2x - 5}{x - 2}$$

8. Find the x- and y-intercepts and the horizontal and vertical asymptotes of the following rational function. Then sketch its graph.

$$r(x) = \frac{x + 1}{x^2 - 2x}$$

1 Only in the Movies?

OBJECTIVE To explore how power functions model relationships between shape and size.



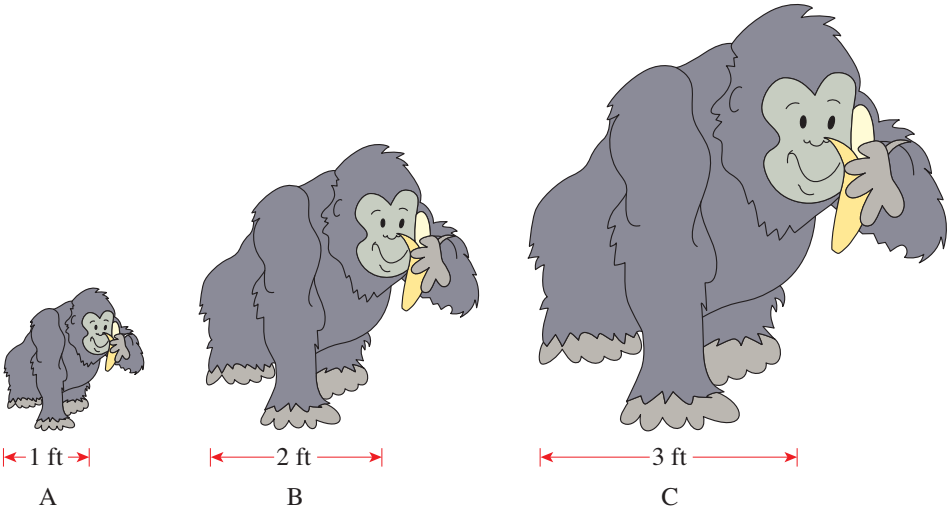
Hulton Archive/Getty Images

King Kong is similar to a normal 8-foot ape but much bigger.

King Kong is "similar" to a normal 8-foot ape but a lot bigger. In fact, he's just a blown up version of the real thing. So how does King Kong's shirt size or weight compare to that of a normal gorilla? In this exploration we investigate how areas and volumes of similar figures change as the size of the figure increases (or decreases). We'll find power functions that relate these quantities and use those functions to explore the possiblity of the existence of a real-life giant ape. We begin by finding some properties of similar figures.

I. Similar Objects

Two objects are **similar** if they have the same shape even though they may not be the same size. In geometry you learned that corresponding sides of similar triangles are proportional. This last property holds for any two similar figures, not just triangles. For example, the gorillas in the figures below are all similar. Gorilla C is three times as tall as Gorilla A, so Gorilla C's hand is three times as long, his eyes are three times as far apart, his feet are three times as long, and so on. The figures "look the same" precisely because they are similar in the mathematical sense.



© Maria Bell/Fotolia

The **scaling factor** of two similar figures is the ratio of the distance between corresponding points. So the scaling factor between Gorillas C and A is 3. This means that if two points are a distance x apart in Gorilla A, then the corresponding points are a distance $3x$ apart in Gorilla C. (We can also say that the scaling factor between gorillas A and C is $\frac{1}{3}$.)

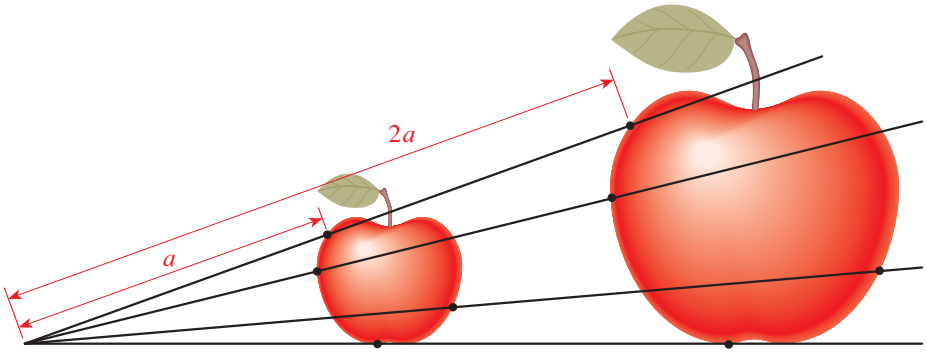
1. Find the following scaling factors:

(a) The scaling factor between Gorillas B and A is _____.

(b) The scaling factor between Gorillas A and B is _____.

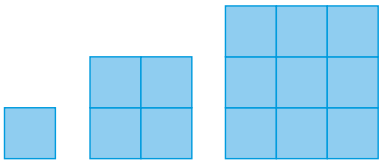
(c) The scaling factor between Gorillas B and C is _____.

2. Is a map of Chicago similar to the city of Chicago? How are distances on the map related to actual distances in the city? Is the scaling factor indicated on the map?
3. How is your photograph similar to you? Compare distances between your eyes, ears, length of nose, and so on to the corresponding distances in the photograph. What is the scaling factor?
4. The figure illustrates a method of drawing an apple twice the size of a given apple. Use the method to draw an apple one-third the size of the bigger apple shown here.

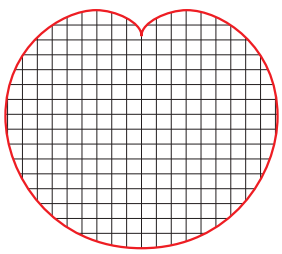


II. Areas of Similar Objects

If two objects are similar, then how are their areas related? First, let's see what happens if the objects are squares.



1. Look at the squares in the margin to help answer the following questions.
   (a) If the side of a square is doubled, then its area is multiplied by _____.
   (b) If the side of a square is tripled, then its area is multiplied by _____
   (c) If the side of a square is multiplied by s, then its area is multiplied by _____.
   (d) Let's prove the conclusion you made in part (c) algebraically.
   - If a square has side x, then its area is $A_1 =$ _____.
   - If a square has side sx, then its area is $A_2 =$ _____.
   - Conclude that for any two squares with scaling factor s, their areas satisfy $A_2 = s^2 A_1$.



2. A plane figure can always be approximated by small squares (as shown in the margin). Use what you learned about areas of squares in Question 1 to explain why the following statement is true: For any two plane figures with scaling factor s, their areas satisfy $A_2 = s^2 A_1$.
3. Any surface, whether flat or not, can be approximated by squares, so the statement in Question 2 is true for any surface. Experiment with the surface areas of cubes and spheres as follows. (Use the cubes shown in Part III on the next page to help visualize the answer to part (a).)

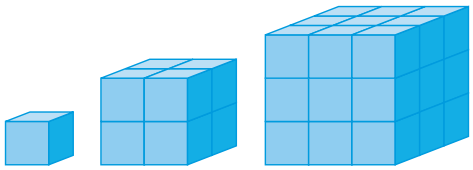
(a) If the side of a cube is doubled, then the surface area of the cube is multiplied by _____.

(b) If the radius of a ball is doubled, then the surface area of the ball is multiplied by _____.

4. Suppose an irregularly shaped object (such as a gorilla) has surface area A. If the object is magnified 10 times (scaling factor 10), then the surface area of the magnified figure is _____.

III. Volumes of Similar Objects

If two objects are similar, then how are their volumes related? First, let's see what happens if the objects are cubes.



1. Look at the cubes above to help answer the following questions.

(a) If the side of a cube is doubled, then its volume is multiplied by _____.

(b) If the side of a cube is tripled, then its volume is multiplied by _____.

(c) If the side of a cube is multiplied by s, then its volume is multiplied by _____.

(d) Let's prove the conclusion you made in part (c) algebraically.

- If a cube has side x, then its volume is $V_1 =$ _____.
- If a cube has side sx, then its volume is $V_2 =$ _____.
- Conclude that for any two cubes with scaling factor s, their volumes satisfy $V_2 = s^3 V_1$.

2. A solid figure can be "filled" with little cubes. Use what you learned about volumes of cubes in Question 1 to explain why the following statement is true: For any two solid objects with scaling factor s, their volumes satisfy $V_2 = s^3 V_1$.

3. Suppose an apple has volume V. If the apple is magnified 10 times (scaling factor 10), then the volume of the big apple is _____.

IV. Real-Life Giant Apes?

1. Let's suppose that King Kong is 10 times as tall as Joe, a normal-sized gorilla. Of course, the two gorillas are similar.

(a) The scaling factor between King Kong and Joe is _____.

(b) If Joe's foot is 20 inches long, then King Kong's foot is _____ inches long.

(c) If it takes 5 square yards of material to make a shirt for Joe, then it takes _____ square yards of material to make a shirt for King Kong.

(d) If Joe weighs 500 pounds, then King Kong weighs _____ pounds.

2. Your answer to Question 1(d) indicates that King Kong weighs half a million pounds. If he is made of normal flesh and blood, would his bones be able to support his weight? Or would his bones be crushed under his weight? In fact, no known substance is strong enough to withstand such enormous weight, so a living, moving gorilla of this size doesn't seem possible.

2 Proportionality: Shape and Size

OBJECTIVE To investigate real-world situations using the proportionality symbol $\propto$.



Luis Louro/Shutterstock.com 2009

The health of frogs is an indicator of the health of an ecosystem.

In this exploration we investigate the proportionality symbol $\propto$. We examine how this notation is used in several real-world situations. In one such situation we determine how a frog's size relates to its sensitivity to pollutants in the enviroment.

The health of frogs is a key indicator of how an ecosystem is faring. Frogs live on land and in the water, and they breathe and drink through their skin. So they are particularly sensitive to changes in any aspect of the environment; they are often the first species affected by pollutants or toxins in the land, water, or air. Of course, the effect of a toxin on a frog is proportional to the concentration of the toxin in its body. We use the proportionality symbol to discover the relationship between a frog's ability to absorb toxins (the surface area of its skin) and its weight (or its volume).

I. The Proportionality Symbol

A formula for the area of a square is $A = x^2$, and a formula for the area of a circle is $A = \pi r^2$. In each case the area is proportional to the square of a "length" measurement (see Section 6.2). It would be difficult to find formulas for the areas of irregular shapes. But for all objects of the same shape, the area A is proportional to the square of the length L. That is, there is a constant k such that $A = kL^2$. We express this proportionality by writing

$$A \propto L^2$$

The symbol $\propto$ means "is proportional to." Similarly, the volume V of an object is proportional to the cube of its length L:

$$V \propto L^3$$

In each case the constant of proportionality depends on the shape of the object and the particular "length" we are measuring.

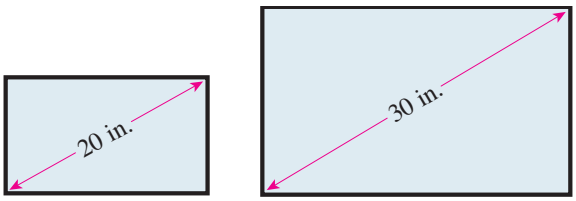
1. The "size" of a TV is given by the length L of the diagonal of the screen. But the cost of manufacturing a TV is largely determined by the *area* of the screen, not by the length of the diagonal. The figure shows a 20-inch TV and a

30-inch TV. Use the proportionality relation $A \propto L^2$ to answer the following questions:

(a) The ratio of the diagonals of the TVs is 30/20, so the ratio of the areas of the TVs is $(30/20)^2 = 2.25$

(b) If a 20-inch TV costs \$240, how much would we expect a 30-inch TV to cost?

We would expect it to cost $2.25 \times 240 = \$540$.



20-inch TV 30-inch TV

2. The "size" of a sailboat is given by its length. The cost of manufacturing a sailboat is determined by its *volume,* not by its length. Consider a 20-foot boat and a 40-foot boat. Use the relationship $V \propto L^3$ to answer the following questions:

(a) The ratio of the lengths of the two boats is _____, so the ratio of their volumes is _____.

(b) If a 20-foot boat costs \$10,000, what would you expect a similar 40-foot boat to cost?

_____

II. Proportionality: Surface Area and Volume

How much more material is needed to manufacture an industrial plastic container with twice the volume of a similar container? To answer this question, we need to find the relationship between volume and surface area. In general, we know that

$$A \propto L^2 \quad \text{and} \quad V \propto L^3$$

We can write these proportionalities as

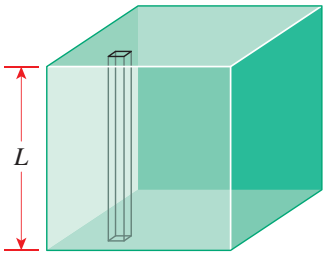
$$A^{1/2} \propto L \quad \text{and} \quad V^{1/3} \propto L$$

Since each of these quantities is proportional to L, they must be proportional to each other. That is, $A^{1/2} \propto V^{1/3}$. Squaring both sides we conclude that

$$A \propto V^{2/3}$$

Use this proportionality to answer the following questions.

1. A manufacturer of plastic containers prices its product by the amount of plastic used to make the container (surface area). A particular container costs \$12. How much would you expect a container with eight times the volume to cost?



2. The price of a child's swimming pool depends on the amount of plastic used to manufacture it. A 50-gallon pool costs $35. How much would you expect a 100-gallon pool to cost?

3. If we have a granite cube, how much *pressure* does the cube exert on its base? To calculate pressure, we divide the weight of the cube by the area of the base:

$$\text{pressure} = \frac{\text{weight}}{\text{area of base}}$$

So the pressure on the base of a 12-inch cube of granite that weighs 180 pounds is

$$\text{pressure} = \frac{180}{12^2} \approx 1.25 \text{ lb/in}^2$$

Notice that pressure is weight per unit area. From the figure in the margin we can see that the weight exerted on each square inch of the base is simply the weight of the column of granite that is directly above that square inch. So the pressure should be proportional to the side length of the cube, that is, $P \propto L$. Let's arrive at this same result using proportionality.

(a) From the above discussion we have

$$P \propto \frac{V}{A}$$

Using the fact that $V \propto L^3$ and $A \propto L^2$, we can rewrite the expression for P as

$$P \propto \frac{\square}{\square}$$

It follows that $P \propto L$.

(b) If the side of a granite cube is 10 times that of another granite cube, the pressure exerted by the larger cube on its base is _____ times the pressure exerted by the smaller cube on its base.

III. Proportionality in the Natural World

The natural world presents us with numerous creatures of greatly varying shapes and sizes, each perfectly adapted to its environment. These creatures must live by the same rules of geometry and the same relationships between length, area, and volume as we do. Let's explore some of these using the proportionalities

$$A \propto L^2 \qquad V \propto L^3 \qquad A \propto V^{2/3} \qquad P \propto \frac{V}{A}$$

R **1.** For animals with roughly the same shape, we would expect the length and volume (or weight) to satisfy the proportionality $V \propto L^3$.

(a) Look up average heights and weights for sheep and cows. Do your findings conform to the expected proportions? Make sure you use the *same* length measurements for each animal (from ground to shoulder, head to rump, etc.).

(b) Repeat part (a) for other animals.



LuckyKeeperShutterstock.com 2009

2. Frogs absorb nutrients through their skin, so they are very susceptible to toxins in the environment. The amount of toxins a frog absorbs in this way is proportional to the surface area of its skin, but the effects of the toxins on a frog are proportional to the concentration of the toxin in the frog's body. Suppose the volumes of Frogs F and G are 5 cm^3 and 40 cm^3 respectively.

(a) The ratio of the volumes of the frogs is _____, so the ratio of their surface areas is _____. (Use the fact that $A \propto V^{2/3}$.)

(b) If Frog F absorbs 0.015 mL of a certain toxin, we would expect Frog G to absorb _____ mL.

(c) The *concentration* of a toxin is the amount of the toxin divided by the volume of the frog. So

Concentration in Frog F is _____

Concentration in Frog G is _____

(d) Are toxins in the environment more harmful to smaller frogs or larger frogs?

3. Let's consider the possibility of real-life giant apes. Let's assume that Big Foot is an ape that is 10 times as tall as an ordinary ape.

(a) Complete the following about Big Foot:

Height is __10__ times that of an ordinary ape.

Weight is _____ times that of an ordinary ape.

Surface area is _____ times that of an ordinary ape.

Pressure on feet is _____ times that of an ordinary ape.

(b) Assuming that Big Foot's bones are made of the same material as those of the ordinary ape, do you think his bones can support his weight? Why or why not?

3 Managing Traffic



silver-john/Shutterstock.com 2009

How many cars can a road safely carry at different speeds?

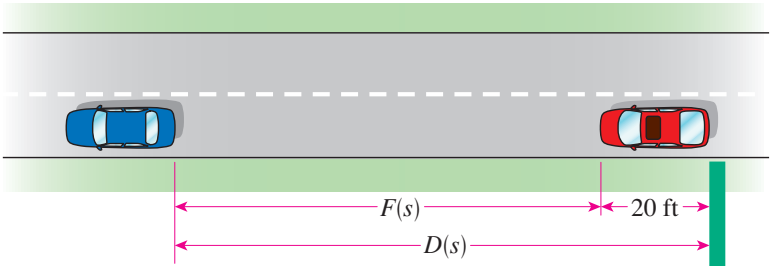
OBJECTIVE To use rational functions and proportionality to analyze the problem of road capacity.

The best way to keep traffic moving is to avoid accidents, and the best way to avoid accidents is for drivers to maintain a *safe following distance.* To give drivers enough time to react to unforeseen events, the safe following distance is greater at higher speeds. You may be familiar with the car-length rule (at least one car length for every 10 miles per hour) or the 2-second rule (the car ahead must pass some object at least 2 seconds before you do). In this exploration we investigate how algebra can help us find how many cars a road can carry at different speeds—assuming, of course, that all drivers maintain a safe following distance.

I. Carrying Capacity and Safe Following Distance

We'll investigate how many cars a road can safely carry at different speeds. Let's assume that each car is 20 feet long and that the **safe following distance** $F(s)$ is one car length for every 10 miles per hour of the speed s.

The following diagram is helpful in visualizing traffic.



1. Let's find a formula for the safe following distance at any speed.

(a) Complete the table for the safe following distance.

Speed s	10	20	30	40	50	60
Following distance F	20					

(b) So at a speed of s miles per hour, the safe following distance is

$F(s) =$ _____.

(c) At a speed of s miles per hour the **road distance** $D(s)$ that each car uses is

$$D(s) = \text{length of car} + \text{safe following distance}$$
$$= _____ + _____$$

2. Since we are measuring car lengths in feet, let's convert speeds from the familiar miles per hour to feet per minute.

(a) Use the fact that 1 mi/h = 88 ft/min to complete the table.

Speed (mi/h)	10	20	30	40	50	60
Speed (ft/min)	880					

(b) From the pattern in the table we see that s mi/h is the same as

$V(s) =$ _____ ft/min.

3. If traffic is moving at s miles per hour, then the number $N(s)$ of cars that pass a given point every minute is given by

$$N(s) = \frac{\text{distance cars travel in one minute}}{\text{distance each car uses}}$$
$$= \frac{V(s)}{D(s)} = \frac{\square}{\square}$$

We'll call the number $N(s)$ the **carrying capacity** of the road at speed s.

4. Graph the rational function you found in Question 3.

(a) What happens to the carrying capacity of the road as the speed increases?

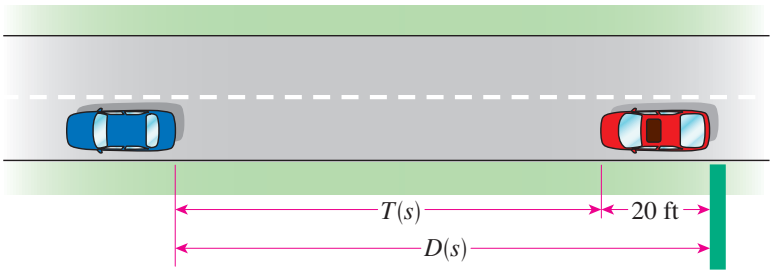
(b) At what speed is the carrying capacity greatest?

II. Using Braking Distance to Find Maximum Carrying Capacity

Note that the *stopping distance*, the actual distance required to stop a car, is greater than the braking distance because of the reaction time needed before the brakes are applied.

For a moving car the **braking distance** is the distance required to bring the car to a complete stop after the brakes are applied. We can infer from physical principles that the braking distance $T(s)$ is proportional to the square of the speed s.

Using the braking distance as the safe following distance, let's try to find the speed at which the maximum carrying capacity occurs. The following diagram is helpful.



1. Let's find a formula for the braking distance at speed s.

(a) Express the above proportionality statement as an equation:

$$T(s) = _____$$

(b) Suppose that the braking distance at 20 mi/h is 20 ft. Use this fact to find the proportionality constant. So the braking distance is given by the formula

$$T(s) = _____$$

(c) Let's assume that each car maintains the proper braking distance from the next car. In this case, at speed s the road distance $D(s)$ each car uses is

$$D(s) = \text{length of car} + \text{braking distance}$$
$$= _____ + _____$$

2. If traffic is moving at s miles per hour, then in this case the carrying capacity $N(s)$ of the road is

$$N(s) = \frac{\text{distance cars travel in 1 minute}}{\text{distance each car uses}}$$
$$= \frac{V(s)}{D(s)}$$
$$= \frac{\square}{\square}$$

3. Graph the rational function you found in Question 2.

(a) What happens to the carrying capacity as the speed increases?

(b) What is maximum carrying capacity? At what speed (in mi/h) does it occur?

4 Alcohol and the Surge Function



Martin Spurny/Shutterstock.com 2009

OBJECTIVE To model data on blood alcohol concentration using surge functions.

In the Prologue to this book (page P2) we introduced the problem of modeling the blood alcohol concentration following the consumption of different amounts of alcohol. We are now ready to experiment with finding an appropriate function to model the data.

We can see from the data that there is an initial rapid surge in the blood alcohol concentration followed by a decay as the body eliminates the alcohol. To model this behavior, we need a function that combines an increasing factor with a decaying one. Such functions are called **surge functions** and have the form

$$S(t) = at \cdot b^t$$

We need to experiment with these functions to find appropriate values of a and b that model our data.

I. Experimenting with the Surge Function: Varying *a*

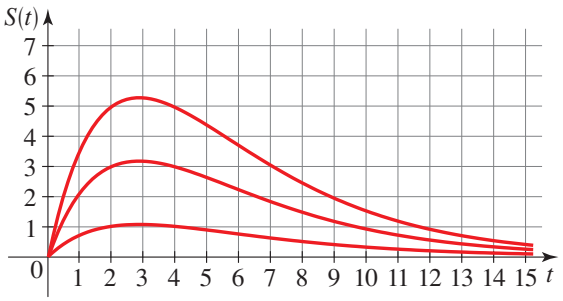
We first find out how changing the value of a affects the graph of a surge function.

1. Let's pick a particular value for b, say 0.7, and experiment with changing the value of a.

(a) Graphs of the following surge functions are shown:

$$S_1(t) = t \cdot (0.7)^t \qquad S_2(t) = 3t \cdot (0.7)^t \qquad S_3(t) = 5t \cdot (0.7)^t$$

Match each function with its graph.



(b) Use a graphing calculator to graph the surge functions $S(t) = at \cdot (0.7)^t$ for the following values of a: 2, 4, 6. Sketch the graphs you obtain on the graph in part (a).

2. **(a)** What is the maximum value of each of the six surge functions in Question 1? Where does the maximum value of each function occur? Use your answers to complete the following table.

Surge functions $S(t) = at \cdot (0.7)^t$

a	Maximum value	Maximum value occurs at t = ?
1	1.0	2.8
2		
3		
4		
5		
6		

(b) Describe how the maximum value changes as the value of a increases.

II. Experimenting with the Surge Function: Varying *b*

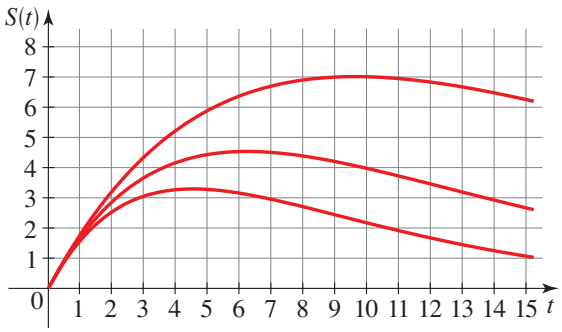
We now experiment with how changing the value of b affects the graph of a surge function.

1. Let's pick a particular value for a, say 2, and experiment with changing the value of b.

(a) Graphs of the following surge functions are shown:

$$S_1(t) = 2t \cdot (0.90)^t \qquad S_2(t) = 2t \cdot (0.85)^t \qquad S_3(t) = 2t \cdot (0.80)^t$$

Match each function with its graph.



(b) Use a graphing calculator to graph the surge functions $S(t) = 2t \cdot b^t$ for the following values of b: 0.70, 0.60, 0.50. Sketch the graphs you obtain on the graph in part (a).

2. **(a)** What is the maximum value of each of the six surge functions in Question 1? Where does the maximum value of each function occur? Use your answers to complete the following table.

Surge functions $S(t) = 2t \cdot b^t$

b	Maximum value	Maximum value occurs at t = ?
0.90	7.0	9.5
0.85		
0.80		
0.70		
0.60		
0.50		

(b) Describe how the maximum value changes as the value of b increases.

III. Modeling the Alcohol Data

Let's use a surge function to model the alcohol data in the Prologue (page P2). Those data give the blood alcohol concentration at various times following the consumption of different amounts of alcohol. The first two columns of that table are reproduced here.

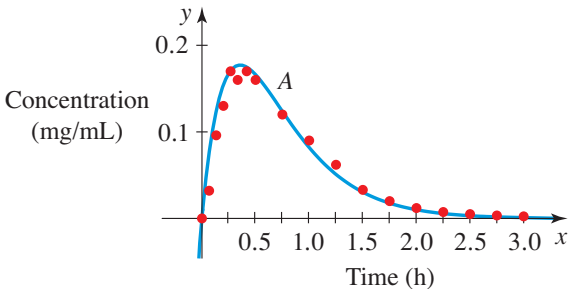
Time (h)	Concentration (mg/mL)
0	0
0.067	0.032
0.133	0.096
0.2	0.13
0.267	0.17
0.333	0.16
0.417	0.17
0.5	0.16
0.75	0.12
1.0	0.090
1.25	0.062
1.5	0.033
1.75	0.020
2.0	0.012
2.25	0.0074
2.5	0.0052
2.75	0.0034
3.0	0.0024

1. The data in the margin give the blood alcohol concentration in a three-hour period following the consumption of 15 mL of ethanol.

(a) Use a graphing calculator to make a scatter plot of the data. Where does the maximum concentration appear to occur?

(b) Try to fit a surge function $S(t) = at \cdot b^t$ to the data. Use what you learned from Parts I and II to guess at reasonable values of a and b, and then graph the function to see how well it fits the scatter plot. It may take several tries to find a good fit for the data.

(c) The graph below shows a surge function (obtained by a student) that models the data reasonably well. How does your model compare to this?



(d) It is illegal in the United States to drive with a blood alcohol concentration of 0.08 mg/mL or greater. (Many states have more restrictive laws for holders of commercial drivers' licenses.) Use the model you found to determine how long a person must wait after drinking 15 mL of alcohol before he or she can legally drive.

2. Use the data in the Prologue and follow the steps in Question 1 to find surge functions that model the blood alcohol concentration following the consumption of 30, 45, and 60 mL of ethanol.

3. In the table below, list the four models you obtained in Questions 1 and 2. Do you see a pattern? If so, find a surge function that would model the blood alcohol concentration following the consumption of 75 and 90 mL of ethanol.

Alcohol consumption (mL)	Surge function model
15	
30	
45	
60	

7 Systems of Equations and Data in Categories



Josef78/Shutterstock.com 2009

7.1 Systems of Linear Equations in Two Variables

7.2 Systems of Linear Equations in Several Variables

7.3 Using Matrices to Solve Systems of Linear Equations

7.4 Matrices and Categorical Data

7.5 Matrix Operations: Getting Information from Data

7.6 Matrix Equations: Solving Linear Systems

EXPLORATIONS

1 Collecting Categorical Data

2 Will the Species Survive?

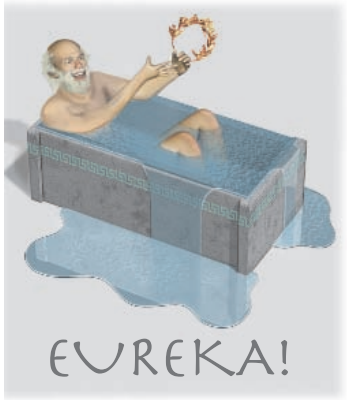
Will the species survive? In the 1970s humpback whales became a focus of controversy. Environmentalists believed that whaling threatened the whales with imminent extinction; whalers saw their livelihood threatened by any attempt to stop whaling. Are whales really threatened to extinction by whaling? What level of whaling is safe to guarantee survival of the whales? These questions motivated mathematicians to study population patterns of whales and other species more closely. How such populations grow and decline depends on many factors, such as the percentage of adults that reproduce and the number of calves that reach maturity. So to model (and forecast) the whale population, scientists must use many equations, each having many variables. Such collections of equations, called systems of equations, *work together* to describe the situation being modeled. Systems of equations with hundreds or even thousands of variables are used extensively by airlines to establish consistent flight schedules and by telecommunications companies to find efficient routings for telephone calls. In this chapter, on a smaller scale, we learn how to solve systems of equations that consist of several equations in several variables.

7.1 Systems of Linear Equations in Two Variables

- Systems of Equations and Their Solutions
- The Substitution Method
- The Elimination Method
- Graphical Interpretation: The Number of Solutions
- Applications: How Much Gold Is in the Crown?

IN THIS SECTION... we learn that it is sometimes necessary to use two or more equations to model a real-world situation. We learn how to solve such *systems* of linear equations.

GET READY... by reviewing linear equations and their graphs in Sections 2.3 and 2.7.



Archimedes discovers the solution to the crown problem.

Archimedes (287–212 B.C.) was the greatest mathematician of the ancient world. He was born in Syracuse, a Greek colony on the island of Sicily. One of his many discoveries is the law of the lever (see Exercise 33 in Section 2.6, page 209). Archimedes famously said, "Give me a place to stand and a fulcrum for my lever, and I can lift the earth." Renowned as a mechanical genius for his many engineering inventions, he designed pulleys for lifting large ships and a spiral screw for transporting water to higher levels.

King Hiero of Syracuse once suspected a goldsmith of keeping part of the gold intended for the king's crown and replacing it with an equal weight of silver. So the crown was the proper weight, but was it solid gold? The king asked Archimedes for advice. Archimedes quickly realized that he needed more information to solve this problem: He needed to know the volume of the crown. The problem contains two variables: weight and volume. But how do we find the volume of an irregularly shaped crown?

While in deep thought at a public bath, Archimedes discovered the solution when he noticed that his body's volume was the same as the volume of water it displaced from the tub. Using this insight, he was able to measure the volume of the crown. As the story is told, Archimedes ran home naked, shouting, "Eureka, eureka!" ("I have found it, I have found it!"). This incident attests to Archimedes' enormous powers of concentration—an essential element in the process of problem solving.

We solve the crown problem in Example 5. But first, we learn how to solve a system of two equations in two variables.

Systems of Equations and Their Solutions

A **system of equations** is a set of equations in which each equation involves the same variables. For example, here is a system of two equations in the two variables x and y:

$$\begin{cases} 2x - y = 5 & \text{Equation 1} \\ x + 4y = 7 & \text{Equation 2} \end{cases}$$

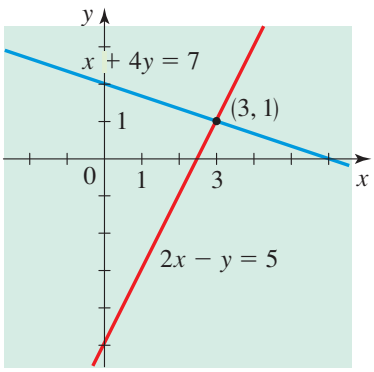


figure **1**

Lines and their graphs are studied in Chapter 2.

A **solution** of a system is a pair of values for the variables that make each equation true. We can check that if we choose x to be 3 and y to be 1 we get a solution of the above system.

Equation 1	**Equation 2**
$2x - y = 5$	$x + 4y = 7$
$2(3) - 1 = 5$ ✓	$3 + 4(1) = 7$ ✓

We can also express this solution as the ordered pair (3, 1).

Note that the graphs of Equations 1 and 2 are lines (see Figure 1). Since the solution (3, 1) satisfies each equation, the point (3, 1) lies on each line. So it is the point of intersection of the two lines.

The Substitution Method

In the **substitution method** we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

Substitution Method

1. **Solve for one variable.** Choose one equation, and solve for one variable in terms of the other variable.
2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1, and solve for the remaining variable.

example **1** **Substitution Method**

Find all solutions of the system.

$$\begin{cases} 2x + y = 1 & \text{Equation 1} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

Solution

We solve for y in the first equation.

$$2x + y = 1 \qquad \text{Equation 1}$$
$$y = 1 - 2x \qquad \text{Solve for } y$$

Now we substitute for y in the second equation and solve for x.

$$\begin{aligned} 3x + 4y &= 14 && \text{Equation 2} \\ 3x + 4(1 - 2x) &= 14 && \text{Substitute } y = 1 - 2x \\ 3x + 4 - 8x &= 14 && \text{Expand} \\ -5x + 4 &= 14 && \text{Simplify} \\ -5x &= 10 && \text{Subtract 4} \\ x &= -2 && \text{Solve for } x \end{aligned}$$

Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$.

$$y = 1 - 2(-2) = 5 \qquad \text{Back-substitute}$$

Thus, $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$. Figure 2 shows that the graphs of the two equations intersect at the point $(-2, 5)$.

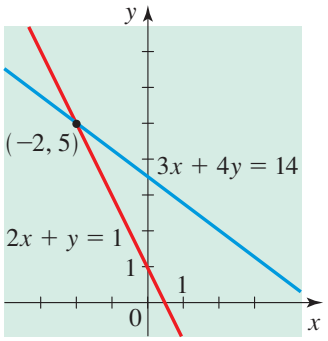


figure **2**

NOW TRY EXERCISE 17

The Elimination Method

To solve a system using the **elimination method**, we try to combine the equations using sums or differences so as to eliminate one of the variables.

Elimination Method

1. **Adjust the coefficients.** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.
2. **Add the equations.** Add the two equations to eliminate one variable, then solve for the remaining variable.
3. **Back-substitute.** Substitute the value that you found in Step 2 back into one of the original equations, and solve for the remaining variable.

example **2** **Elimination Method**

Find all solutions of the system.

$$\begin{cases} 3x + 2y = 14 & \text{Equation 1} \\ x - 2y = 2 & \text{Equation 2} \end{cases}$$

Solution

Since the coefficients of the y-terms are negatives of each other, we can add the equations to eliminate y.

$$\begin{array}{rll} \begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases} & & \text{System} \\ \hline 4x = 16 & & \text{Add} \\ x = 4 & & \text{Solve for } x \end{array}$$

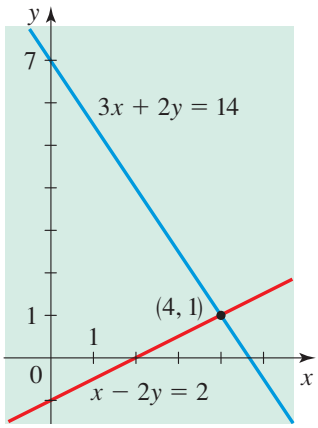


figure **3**

Now we back-substitute $x = 4$ into one of the original equations and solve for y. Let's choose the second equation because it looks simpler.

$$\begin{aligned} x - 2y &= 2 && \text{Equation 2} \\ 4 - 2y &= 2 && \text{Back-substitute } x = 4 \\ -2y &= -2 && \text{Subtract 4} \\ y &= 1 && \text{Solve for } y \end{aligned}$$

The solution is (4, 1). Figure 3 shows that the graphs of the equations in the system intersect at the point (4, 1).

NOW TRY EXERCISE 21

Graphical Interpretation: The Number of Solutions

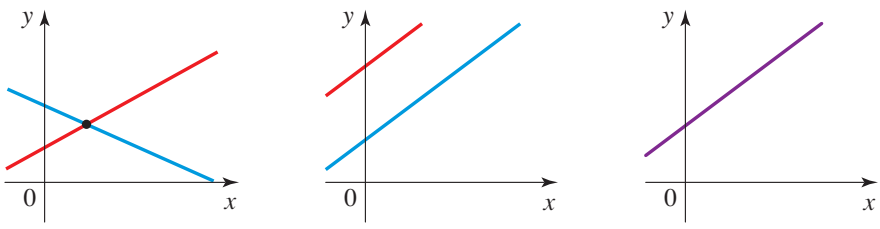
The graph of a system of two linear equations in two variables is a pair of lines. The solution of the system is the intersection point of the lines. Two lines may intersect at a single point, they may be parallel, or they may coincide, as shown in Figure 4. So there are three possible outcomes in solving such a system.

Number of Solutions of a Linear System in Two Variables

For a system of linear equations in two variables, exactly one of the following is true.

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system that has no solution is said to be **inconsistent**. A system with infinitely many solutions is called **dependent**.



One solution
Lines intersect at a single point.

No solution
Lines are parallel—they do intersect.

Infinitely many solutions
Lines coincide.

figure **4**

Examples 1 and 2 are systems with one solution. In the next two examples we solve systems that have no solution or infinitely many solutions.

example 3 A Linear System with No Solution

Solve the system, and graph the lines.

$$\begin{cases} 8x - 2y = 5 & \text{Equation 1} \\ -12x + 3y = 7 & \text{Equation 2} \end{cases}$$

Solution

We try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second equation by 2 gives

$$\begin{array}{rll} 24x - 6y = 15 & & 3 \times \text{Equation 1} \\ -24x + 6y = 14 & & 2 \times \text{Equation 2} \\ \hline 0 = 29 & & \text{Add} \end{array}$$

Adding the two equations eliminates *both* x *and* y in this case, and we end up with $0 = 29$, which is obviously false. No matter what values we assign to x and y, we cannot make this statement true, so the system has *no solution*. Figure 5 shows that the lines in the system are parallel; hence they do not intersect. The system is inconsistent.

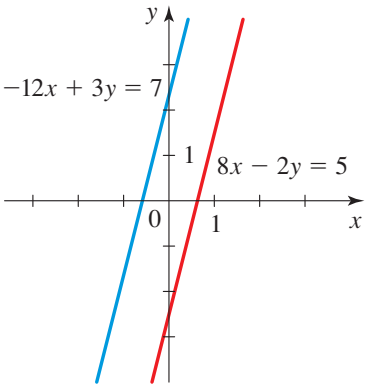


figure 5

NOW TRY EXERCISE 29

example 4 A Linear System with Infinitely Many Solutions

Solve the system.

$$\begin{cases} 3x - 6y = 12 & \text{Equation 1} \\ 4x - 8y = 16 & \text{Equation 2} \end{cases}$$

Solution

We multiply the first equation by 4 and the second by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 & 4 \times \text{Equation 1} \\ 12x - 24y = 48 & 3 \times \text{Equation 2} \end{cases}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line give a solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$
$$y = \tfrac{1}{2}t - 2$$

We can also write the solution in ordered-pair form as

$$(t, \tfrac{1}{2}t - 2)$$

where t is any real number. The system has infinitely many solutions (see Figure 6).

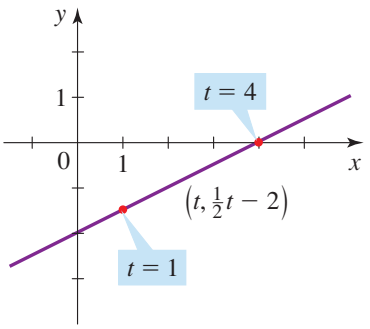


figure 6

NOW TRY EXERCISE 31

Applications: How Much Gold Is in the Crown?

We are now ready to solve the problem about the amount of gold in King Hiero's crown, as discussed at the beginning of this section. Recall that King Hiero suspected that his goldsmith had replaced part of the gold intended for the king's crown with an equal weight of silver. Archimedes was able to determine the volume of the crown by immersing it in water and observing the amount of water the crown displaced. The density of gold and silver are well known. The density D of a substance is its weight W divided by its volume V. We have

$$D = \frac{W}{V} \quad \text{or} \quad V = \frac{W}{D}$$

The densities of gold is 19.3 g/cm^3, and the density of silver is 10.5 g/cm^3. In other words a cubic centimeter of gold weighs 19.3 grams, and a cubic centimeter of silver weighs 10.5 grams.

example 5 How Much Gold Is in the Crown?

Suppose King Hiero's crown weighs 235 g and has a volume of 14 cm^3. Find the weight of the gold and the weight of the silver in the crown.

Solution

Let x and y be the weights of the gold and the silver in the crown, respectively. Since the crown weighs 235 g, we have $x + y = 235$. The volume of gold in the crown is $V = W/D = x/19.3$; similarly, the volume of silver in the crown is $y/10.5$. Since the volume of the crown is 14 cm^3, we have $x/19.3 + y/10.5 = 14$. So we have the system of equations

$$\begin{cases} x + y = 235 & \text{Weight Equation} \\ \dfrac{x}{19.3} + \dfrac{y}{10.5} = 14 & \text{Volume Equation} \end{cases}$$

We solve the system by elimination. First, to eliminate fractions, we multiply the Volume Equation by $(19.3)(10.5) = 202.65$. Then we multiply the Weight Equation by -10.5 to get

$$\begin{cases} -10.5x - 10.5y = -2467.5 & -10.5 \times \text{Weight Equation} \\ 10.5x + 19.3y = 2837.1 & 202.65 \times \text{Volume Equation} \end{cases}$$

$$8.8y = 369.6 \qquad \text{Add}$$

$$y = 42 \qquad \text{Solve for } y$$

Now we substitute for y in the Volume Equation and solve for x.

$$x + y = 235 \qquad \text{Volume Equation}$$

$$x + 42 = 235 \qquad \text{Substitute } y = 42$$

$$x = 193 \qquad \text{Subtract 42}$$

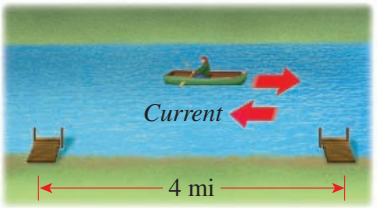
So the crown has 193 grams of gold and 42 grams of silver.

NOW TRY EXERCISE 53

Guidelines for Modeling with Systems of Equations

1. **Identify the variables.** Identify the quantities that the problem asks you to find. Introduce notation for the variables (call them x and y or some other letters).
2. **Set up a system of equations.** Find the facts in the problem that relate the quantities you identified in Step 1. Set up a system of equations (or a model) that expresses these relationships.
3. **Solve the system and interpret the results.** Solve the system you found in Step 2, and state your final answer as a sentence that answers the question posed in the problem.

example 6 A Distance-Speed-Time Problem



A woman rows a boat upstream from one point on a river to another point 4 miles away in $1\frac{1}{2}$ hours. The return trip, traveling with the current, takes only 45 minutes. How fast does she row relative to the water, and at what speed is the current flowing?

Solution

Identify the variables. We are asked to find the rowing speed and the speed of the current, so we let

$$x = \text{rowing speed (mi/h)}$$

$$y = \text{current speed (mi/h)}$$

Express unknown quantities in terms of the variables. The woman's speed when she rows upstream is her rowing speed minus the speed of the current; her speed downstream is her rowing speed plus the speed of the current. We now translate this information into the language of algebra.

In Words	In Algebra
Rowing speed	x
Current speed	y
Speed upstream	$x - y$
Speed downstream	$x + y$

The distance upstream and downstream is 4 miles, so using the fact that speed $\times$ time $=$ distance for both legs of the trip, we get

$$\text{speed upstream} \times \text{time upstream} = \text{distance traveled}$$

$$\text{speed downstream} \times \text{time downstream} = \text{distance traveled}$$

Set up a system of equations. In algebraic notation this translates into the following equations.

$$(x - y)\tfrac{3}{2} = 4 \qquad \text{Equation 1}$$

$$(x + y)\tfrac{3}{4} = 4 \qquad \text{Equation 2}$$

(The times have been converted to hours, since we are expressing the speeds in miles per *hour*.)

Solve the system. We multiply the equations by 2 and 4, respectively, to clear the denominators.

$$\begin{cases} 3x - 3y = 8 & 2 \times \text{Equation 1} \\ 3x + 3y = 16 & 4 \times \text{Equation 2} \end{cases}$$

$$6x = 24 \quad \text{Add}$$

$$x = 4 \quad \text{Solve for } x$$

Back-substituting this value of x into the first equation above (the second works just as well) and solving for y, we get

$$3x - 3y = 8 \quad \text{Equation}$$

$$3 \cdot 4 - 3y = 8 \quad \text{Back-substitute } x = 4$$

$$-3y = 8 - 12 \quad \text{Subtract 12}$$

$$y = \tfrac{4}{3} \quad \text{Solve for } y$$

The woman rows at 4 mi/h, and the current flows at $1\frac{1}{3}$ mi/h.

✓ CHECK

Speed upstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{1\frac{1}{2} \text{ h}} = 2\tfrac{2}{3} \text{ mi/h},$$

and this should equal

rowing speed − current speed

$= 4 \text{ mi/h} - \frac{4}{3} \text{ mi/h} = 2\frac{2}{3} \text{ mi/h}$ ✓

Speed downstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{\frac{3}{4} \text{ h}} = 5\tfrac{1}{3} \text{ mi/h},$$

and this should equal

rowing speed + current speed

$= 4 \text{ mi/h} + \frac{4}{3} \text{ mi/h} = 5\frac{1}{3} \text{ mi/h}$ ✓

NOW TRY EXERCISE 51

example 7 A Mixture Problem

A vintner fortifies wine that contains 10% alcohol by adding a 70% alcohol solution to it. The resulting mixture has an alcoholic strength of 16% and fills 1000 one-liter bottles. How many liters of the wine and of the alcohol solution does the vintner use?

Solution

Identify the variables. Since we are asked for the amounts of wine and alcohol, we let

x = amount of wine used (in liters)

y = amount of alcohol solution used (in liters)

Express unknown quantities in terms of the variables. From the fact that the wine contains 10% alcohol and the solution contains 70% alcohol, we get the following.

In Words	In Algebra
Amount of wine used	x
Amount of alcohol solution used	y
Amount of alcohol in wine	$0.10x$
Amount of alcohol in solution	$0.70y$

The volume of the mixture must be the total of the two volumes the vintner is adding together, so

$$x + y = 1000$$

Also, the amount of alcohol in the mixture must be the total of the alcohol contributed by the wine and by the alcohol solution, that is,

$$0.10x + 0.70y = (0.16)1000 \quad \text{Total alcohol is 16\% of 1000 L}$$
$$0.10x + 0.70y = 160 \quad \text{Simplify}$$
$$x + 7y = 1600 \quad \text{Multiply by 10 to clear decimals}$$

Set up a system of equations. Thus we get the system

$$\begin{cases} x + y = 1000 & \text{Equation 1} \\ x + 7y = 1600 & \text{Equation 2} \end{cases}$$

Solve the system. Subtracting the first equation from the second eliminates the variable x, and we get

$$6y = 600 \quad \text{Subtract Equation 1 from Equation 2}$$
$$y = 100 \quad \text{Solve for } y$$

We now back-substitute $y = 100$ into the first equation and solve for x.

$$x + y = 1000 \quad \text{Equation 1}$$
$$x + 100 = 1000 \quad \text{Back-substitute } y = 100$$
$$x = 900 \quad \text{Solve for } x$$

The vintner uses 900 L of wine and 100 L of the alcohol solution.

■ **NOW TRY EXERCISE 55** ■

7.1 Exercises

CONCEPTS

Fundamentals

1. A set of equations involving the same variables is called a _____ of equations.

2. The system of equation

$$\begin{cases} 2x + 3y = 7 \\ 5x - y = 9 \end{cases}$$

is a system of two equations in the two variables _____ and _____. To determine whether $(5, -1)$ is a solution of this system, we check whether each _____ in the system is true when x is 5 and y is -1.

3. Which of the following is a solution of the system of equations in Exercise 2?

$$(5, -1), \quad (-1, 3), \quad (2, 1)$$

4. A system of linear equations in two variables can be solved by using the _____ method, the _____ method, or the graphical method.

5. A system of two linear equations in two variables can have one solution, _____ solution, or _____ _____ solutions. How many solutions does each of the following systems have?

(a) $\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$ **(b)** $\begin{cases} x - y = 1 \\ 3x + 3y = 3 \end{cases}$

6. The following is a system of two linear equations in two variables:

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

The graph of the first equation is the same as the graph of the second equation, so the system has _____ _____ solutions. We express these solutions by writing

$$x = t$$

$$y = _____$$

where t is any real number. Some of the solutions of this system are (1, ____) (−3, ____), and (5, ____).

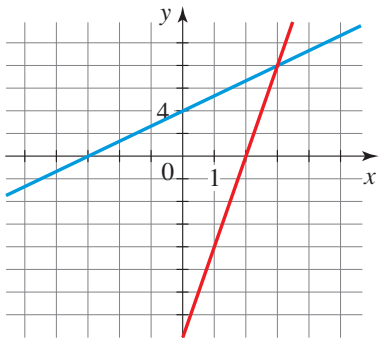
Think About It

7–8 ■ A system of equations and their graphs are given. Note that one of these systems is *not* linear.

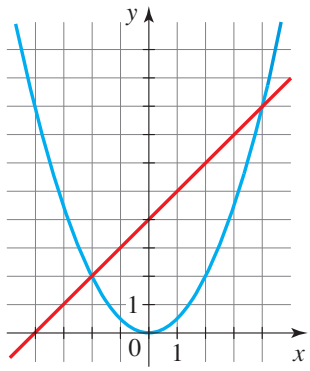
(a) Use the graph to find the solution(s) of the system.

(b) Check that the solutions you found in part (a) satisfy the system.

7. $\begin{cases} 4x - y = 8 \\ -2x + 3y = 6 \end{cases}$



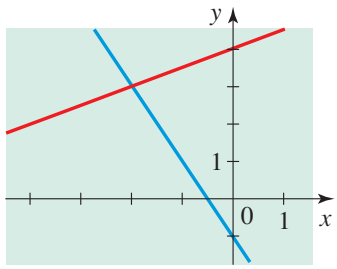
8. $\begin{cases} 2y - x^2 = 0 \\ y - x = 4 \end{cases}$



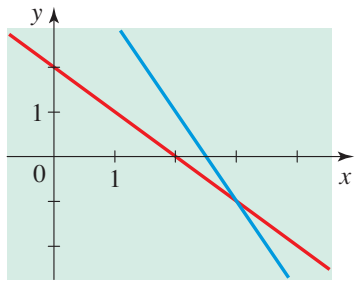
SKILLS

9–10 ■ Two equations and their graphs are given. Find the intersection point of the graphs by solving the system.

9. $\begin{cases} 2x + y = -1 \\ x - 2y = -8 \end{cases}$



10. $\begin{cases} x + y = 2 \\ 2x + y = 5 \end{cases}$



11–16 ■ Graph each linear system, either by hand or using a graphing device. Use the graph to determine whether the system has one solution, no solution, or infinitely many solutions. If there is exactly one solution, use the graph to find it.

11. $\begin{cases} x - y = 4 \\ 2x + y = 2 \end{cases}$

12. $\begin{cases} 2x - y = 4 \\ 3x + y = 6 \end{cases}$

13. $\begin{cases} 2x - 3y = 12 \\ -x + \frac{3}{2}y = 4 \end{cases}$

14. $\begin{cases} 2x + 6y = 0 \\ -3x - 9y = 18 \end{cases}$

15. $\begin{cases} -x + \frac{1}{2}y = -5 \\ 2x - y = 10 \end{cases}$

16. $\begin{cases} 12x + 15y = -18 \\ 2x + \frac{5}{2}y = -3 \end{cases}$

17–20 ■ Use the substitution method to solve the system of linear equations.

17. $\begin{cases} x + y = 4 \\ -x + y = 0 \end{cases}$

18. $\begin{cases} x - y = 3 \\ x + 3y = 7 \end{cases}$

19. $\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases}$

20. $\begin{cases} x + y = 7 \\ 2x - 3y = -1 \end{cases}$

21–24 ■ Use the elimination method to solve the system of linear equations.

21. $\begin{cases} 2x - 3y = 9 \\ 4x + 3y = 9 \end{cases}$

22. $\begin{cases} 3x + 2y = 0 \\ -x - 2y = 8 \end{cases}$

23. $\begin{cases} 3x + 2y = 8 \\ x - 2y = 0 \end{cases}$

24. $\begin{cases} 4x + 2y = 16 \\ x - 5y = 70 \end{cases}$

25–40 ■ Solve the system, or show that it has no solution. If the system has infinitely many solutions, express them in the ordered-pair form given in Example 4.

25. $\begin{cases} -x + y = 2 \\ 4x - 3y = -3 \end{cases}$

26. $\begin{cases} 4x - 3y = 28 \\ 9x - y = -6 \end{cases}$

27. $\begin{cases} x + 2y = 7 \\ 5x - y = 2 \end{cases}$

28. $\begin{cases} -4x + 12y = 0 \\ 12x + 4y = 160 \end{cases}$

29. $\begin{cases} x + 4y = 8 \\ 3x + 12y = 2 \end{cases}$

30. $\begin{cases} -3x + 5y = 2 \\ 9x - 15y = 6 \end{cases}$

31. $\begin{cases} 2x - 6y = 10 \\ -3x + 9y = -15 \end{cases}$

32. $\begin{cases} 2x - 3y = -8 \\ 14x - 21y = 3 \end{cases}$

33. $\begin{cases} 6x + 4y = 12 \\ 9x + 6y = 18 \end{cases}$

34. $\begin{cases} 25x - 75y = 100 \\ -10x + 30y = -40 \end{cases}$

35. $\begin{cases} 8s - 3t = -3 \\ 5s - 2t = -1 \end{cases}$

36. $\begin{cases} u - 30v = -5 \\ -3u + 80v = 5 \end{cases}$

37. $\begin{cases} 0.4x + 1.2y = 14 \\ 12x - 5y = 10 \end{cases}$

38. $\begin{cases} 26x - 10y = -4 \\ -0.6x + 1.2y = 3 \end{cases}$

39. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 2 \\ -8x + 6y = 10 \end{cases}$

40. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = 4 \\ 2x - 10y = -80 \end{cases}$

41–44 ■ Use a graphing device to graph both lines in the same viewing rectangle. (Note that you must solve for y in terms of x before graphing if you are using a graphing calculator.) Solve the system correct to two decimal places, either by zooming in and using TRACE or by using `Intersect`.

41. $\begin{cases} 0.21x + 3.17y = 9.51 \\ 2.35x - 1.17y = 5.89 \end{cases}$

42. $\begin{cases} 18.72x - 14.91y = 12.33 \\ 6.21x - 12.92y = 17.82 \end{cases}$

43. $\begin{cases} 2371x - 6552y = 13{,}591 \\ 9815x + 992y = 618{,}555 \end{cases}$

44. $\begin{cases} -435x + 912y = 0 \\ 132x + 455y = 994 \end{cases}$

CONTEXTS

45. Number Problem Find two numbers whose sum is 34 and whose difference is 10.

46. Number Problem The sum of two numbers is twice their difference. The larger number is 6 more than twice the smaller. Find the numbers.

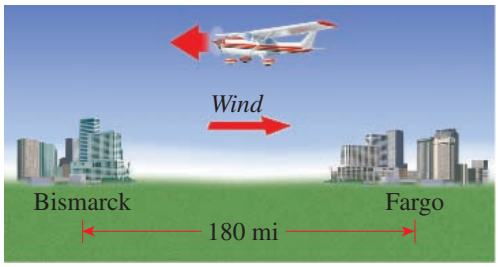
47. Value of Coins A man has 14 coins in his pocket, all of which are dimes and quarters. If the total value of his change is $2.75, how many dimes and how many quarters does he have?

48. Admission Fees The admission fee at an amusement park is $1.50 per child and $4.00 per adult. On a certain day, 2200 people entered the park, and the admission fees that were collected totaled $5050. How many children and how many adults were admitted?

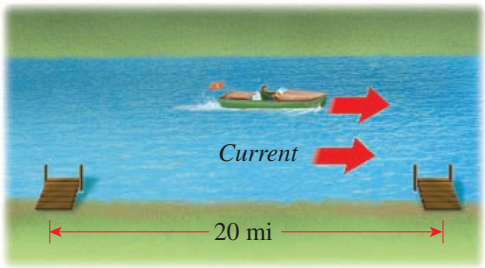
49. Gas Station A gas station sells regular gas for $2.20 per gallon and premium gas for $3.00 per gallon. At the end of a business day 280 gallons of gas were sold, and receipts totaled $680. How many gallons of each type of gas were sold?

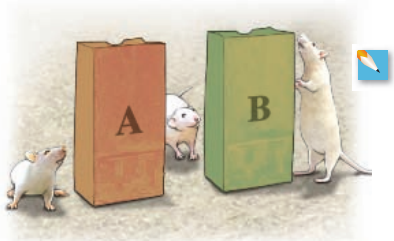
50. Fruit Stand A fruit stand sells two varieties of strawberries: standard and deluxe. A box of standard strawberries sells for $7, and a box of deluxe strawberries sells for $10. In one day the stand sells 135 boxes of strawberries for a total of $1110. How many boxes of each type were sold?

51. Airplane Speed A man flies a small airplane from Fargo to Bismarck, North Dakota—a distance of 180 miles. Because he is flying into a head wind, the trip takes him 2 hours. On the way back, the wind is still blowing at the same speed, so the return trip takes only 1 hour, 12 minutes. What is his speed in still air, and how fast is the wind blowing?



52. Boat speed A boat on a river travels downstream between two points that are 20 miles apart in 1 hour. The return trip against the current takes $2\frac{1}{2}$ hours. What is the boat's speed, and how fast does the current in the river flow?





53. Nutrition A researcher performs an experiment to test a hypothesis that involves the nutrients niacin and retinol. She feeds one group of laboratory rats a daily diet of precisely 32 units of niacin and 22,000 units of retinol. She uses two types of commercial pellet foods. Food A contains 0.12 unit of niacin and 100 units of retinol per gram. Food B contains 0.20 unit of niacin and 50 units of retinol per gram. How many grams of each food does she feed this group of rats each day?

54. Coffee Blends A customer in a coffee shop purchases a blend of two coffees: Kenyan, costing \$3.50 a pound, and Sri Lankan, costing \$5.60 a pound. He buys 3 pounds of the blend, which costs him \$11.55. How many pounds of each kind went into the mixture?

55. Mixture Problem A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second gives a mixture that is 15% acid, whereas blending 100 mL of the first with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

56. Mixture Problem A biologist has two brine solutions, one containing 5% salt and another containing 20% salt. How many milliliters of each solution should she mix to obtain 1 liter of a solution that contains 14% salt?

7.2 Systems of Linear Equations in Several Variables

- Solving a Linear System
- Inconsistent and Dependent Systems
- Modeling with Linear Systems

IN THIS SECTION... we study how to solve systems of three (or more) equations in three (or more) variables. The method of solution is an extension of the elimination method that we studied in the preceding section.

Systems with several linear equations and several variables arise in the process of modeling real-world situations that involve several varying quantities with several constraints on each variable. In Example 4 we encounter such a situation involving financial investments. We begin by learning how to solve systems with several variables and several equations.

Solving a Linear System

The following are two examples of systems of linear equations in three variables. The second system is in **triangular form**; that is, the variable x doesn't appear in the second equation, and the variables x and y do not appear in the third equation.

A system of linear equations	A system in triangular form
$\begin{cases} x - 2y - z = 1 \\ -x + 3y + 3z = 4 \\ 2x - 3y + z = 10 \end{cases}$	$\begin{cases} x - 2y - z = 1 \\ y + 2z = 5 \\ z = 3 \end{cases}$

It's easy to solve a system that is in triangular form by using back-substitution, so our goal in this section is to start with a system of linear equations and change it to a system in triangular form that has the same solutions as the original system. We begin by showing how to use back-substitution to solve a system that is already in triangular form.

example 1 Solving a Triangular System Using Back-Substitution

Solve the system using back-substitution.

$$\begin{cases} x - 2y - z = 1 & \text{Equation 1} \\ y + 2z = 5 & \text{Equation 2} \\ z = 3 & \text{Equation 3} \end{cases}$$

Solution

From the last equation we know that $z = 3$. We back-substitute this into the second equation and solve for y.

$$y + 2(3) = 5 \qquad \text{Back-substitute } z = 3 \text{ into Equation 2}$$

$$y = -1 \qquad \text{Solve for } y$$

Then we back-substitute $y = -1$ and $z = 3$ into the first equation and solve for x.

$$x - 2(-1) - (3) = 1 \qquad \text{Back-substitute } y = -1 \text{ and } z = 3 \text{ into Equation 1}$$

$$x = 2 \qquad \text{Solve for } x$$

The solution of the system is $x = 2$, $y = -1$, $z = 3$. We can also write the solution as the ordered triple $(2, -1, 3)$.

■ **NOW TRY EXERCISE 5** ■

The following operations on a linear system always lead to an equivalent system (that is, a system with the same solutions as the original system).

Operations That Lead to an Equivalant System

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the positions of two equations.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. Then we use back-substitution as in Example 1. This process is called **Gaussian elimination**.

example 2 Solving a System of Three Equations in Three Variables

Solve the system using Gaussian elimination.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ x + 2y - z = 13 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Solution

We need to change this to a triangular system, so we begin by eliminating the x-term from the second equation.

$$\begin{array}{rl} x - 2y + 3z = 1 & \text{Equation 2} \\ x + 2y - z = 13 & \text{Equation 1} \\ \hline 4y - 4z = 12 & \text{Equation 2} + (-1) \times \text{Equation 1} = \text{new Equation 2} \end{array}$$

This gives us a new, equivalent system that is one step closer to triangular form.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Now we eliminate the x-term from the third equation.

$$\begin{array}{r} 3x + 2y - 5z = 3 \\ -3x + 6y - 9z = -3 \\ \hline 8y - 14z = 0 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ 8y - 14z = 0 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation.

$$\begin{array}{r} 8y - 14z = 0 \\ -8y + 8z = -24 \\ \hline -6z = -24 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ -6z = -24 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but it will be easier to work with if we divide the second and third equations by the common factors of each term.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ y - z = 3 & \tfrac{1}{4} \times \text{Equation 2} = \text{new Equation 2} \\ z = 4 & -\tfrac{1}{6} \times \text{Equation 3} = \text{new Equation 3} \end{cases}$$

Now we use back-substitution to solve the system. From the third equation we get $z = 4$. We back-substitute this into the second equation and solve for y.

$$\begin{aligned} y - (4) &= 3 && \text{Back-substitute } z = 4 \text{ into Equation 2} \\ y &= 7 && \text{Solve for } y \end{aligned}$$

Now we back-substitute $y = 7$ and $z = 4$ into the first equation and solve for x.

$$\begin{aligned} x - 2(7) + 3(4) &= 1 && \text{Back-substitute } y = 7 \text{ and } z = 4 \text{ into Equation 1} \\ x &= 3 && \text{Solve for } x \end{aligned}$$

The solution of the system is $x = 3$, $y = 7$, $z = 4$. We can also write the solution as the ordered triple $(3, 7, 4)$.

■ **NOW TRY EXERCISES 13 AND 17** ■

Inconsistent and Dependent Systems

Just as in the case of a system with two equations in two variables (Section 7.1), a system of equations in several variables may have one solution, no solution, or infinitely many solutions. A system with no solutions is said to be **inconsistent**, and a system with infinitely many solutions is said to be **dependent**. As we see in the next example, a linear system has no solution if we end up with a *false equation* after applying elementary row operations to the system.

example 3 A System with No Solution

Solve the following system.

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ 2x + 2y - z = 2 & \text{Equation 2} \\ 3x + 4y - 3z = 5 & \text{Equation 3} \end{cases}$$

Solution

To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ -2y + 3z = 0 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ -2y + 3z = 2 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation.

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ -2y + 3z = 4 & \text{Equation 2} \\ 0 = 2 & \text{Equation 3} + (-1) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but the third equation says $0 = 2$, which is false. No matter what values we assign to x, y, and z, the third equation will never be true. This means that the system has *no solution*.

■ **NOW TRY EXERCISE 23** ■

example 4 A System with Infinitely Many Solutions

Solve the following system.

$$\begin{cases} x - y + 5z = -2 & \text{Equation 1} \\ 2x + y + 4z = 2 & \text{Equation 2} \\ 2x + 4y - 2z = 8 & \text{Equation 3} \end{cases}$$

Solution

To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x - y + 5z = -2 & \text{Equation 1} \\ 3y - 6z = 6 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ 6y - 12z = 12 & \text{Equation 3} + (-2) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation.

$$\begin{cases} x - y + 5z = -2 & \text{Equation 1} \\ 3y - 6z = 6 & \text{Equation 2} \\ 0 = 0 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The new third equation is true, but it gives us no new information, so we can drop it from the system. Only two equations are left. We can use them to solve for x and y in terms of z, but z can take on any value, so there are infinitely many solutions.

To find the complete solution of the system, we begin by solving for y in terms of z, using the new second equation.

$$\begin{aligned} 3y - 6z &= 6 && \text{Equation 2} \\ y - 2z &= 2 && \text{Multiply by } \tfrac{1}{3} \\ y &= 2z + 2 && \text{Solve for } y \end{aligned}$$

Then we solve for x in terms of z, using the first equation.

$$\begin{aligned} x - (2z + 2) + 5z &= -2 && \text{Substitute } y = 2z + 2 \text{ into Equation 1} \\ x + 3z - 2 &= -2 && \text{Simplify} \\ x &= -3z && \text{Solve for } x \end{aligned}$$

To describe the complete solution, we let t represent any real number. The solution is

$$\begin{aligned} x &= -3t \\ y &= 2t + 2 \\ z &= t \end{aligned}$$

We can also write the solution as the ordered triple $(-3t, 2t + 2, t)$.

NOW TRY EXERCISE 27

In the solution of Example 4 the variable t is called a *parameter*. To get a specific solution, we give a specific value to the parameter t. For instance, if we set $t = 2$, we get

$$\begin{aligned} x &= -3(2) = -6 \\ y &= 2(2) + 2 = 6 \\ z &= 2 \end{aligned}$$

Thus $(-6, 6, 2)$ is a solution of the system. Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	Solution $(-3t, 2t + 2, t)$
-1	$(3, 0, -1)$
0	$(0, 2, 0)$
3	$(-9, 8, 3)$
10	$(-30, 22, 10)$

You should check that each of these points satisfies the original equations. There are infinitely many choices for the parameter t, so the system has infinitely many solutions.

Modeling with Linear Systems

Linear systems are used to model situations that involve several varying quantities. In the next example we consider an application of linear systems to finance.

example 5 Modeling a Financial Problem Using a Linear System

Jason receives an inheritance of $50,000. His financial advisor suggests that he invest this in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 16%. Jason wants a total first-year return of $4000. To avoid excessive risk, he decides to invest three times as much in the money-market fund as in the high-tech stock fund. How much should he invest in each fund?

Solution

Let

x = amount invested in the money-market fund

y = amount invested in the blue-chip stock fund

z = amount invested in the high-tech stock fund

We convert each fact given in the problem into an equation.

$$\begin{cases} x + y + z = 50{,}000 & \text{Total amount invested is \$50,000} \\ 0.05x + 0.09y + 0.16z = 4000 & \text{Total investment return is \$4000} \\ x = 3z & \text{Money-market amount } = 3 \times \text{high-tech amount} \end{cases}$$

Multiplying the second equation by 100 and rewriting the third gives the following system, which we solve using Gaussian elimination.

$$\begin{cases} x + y + z = 50{,}000 & \text{Equation 1} \\ 5x + 9y + 16z = 400{,}000 & \text{Equation 2} \\ x - 3z = 0 & \text{Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \text{Equation 1} \\ 4y + 11z = 150{,}000 & \text{Equation 2} + (-5) \times \text{Equation 1} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \text{Equation 3} + (-1) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \text{Equation 1} \\ -5z = -50{,}000 & \text{Equation 2} + 4 \times \text{Equation 3} = \text{new Equation 2} \\ -y - 4z = -50{,}000 & \text{Equation 3} \end{cases}$$

$$\begin{cases} x + y + z = 50{,}000 & \text{Equation 1} \\ z = 10{,}000 & (-\tfrac{1}{5}) \times \text{Equation 2} \\ y + 4z = 50{,}000 & (-1) \times \text{Equation 3} \end{cases}$$

Interchanging Equations 2 and 3, we get:

$$\begin{cases} x + y + z = 50{,}000 & \text{Equation 1} \\ y + 4z = 50{,}000 & \text{Equation 2} \\ z = 10{,}000 & \text{Equation 3} \end{cases}$$

Now that the system is in triangular form, we use back-substitution to find that $x = 30{,}000$, $y = 10{,}000$, and $z = 10{,}000$. This means that Jason should invest

$30,000 in the money-market fund
$10,000 in the blue-chip stock fund
$10,000 in the high-tech stock fund

NOW TRY EXERCISE 31

7.2 Exercises

CONCEPTS

Fundamentals

1. The following system of equations is in triangular form.

$$\begin{cases} x + 2y - z = 6 \\ y - z = -2 \\ z = 3 \end{cases}$$

(a) From the third equation we know that $z =$ _____.

(b) Replacing z by 3 in the second equation, we get $y =$ _____.

(c) Replacing z by _____ and y by _____ in the first equation, we find that $x =$ _____.

(d) The solution of the system is (____, ____, ____).

2. Consider the following system of linear equations.

$$\begin{cases} x - y + z = 2 \\ -x + 2y + z = -3 \\ 3x + y - 2z = 2 \end{cases}$$

(a) If we add 2 times the first equation to the second equation, the second equation becomes _____ = _____.

(b) To eliminate x from the third equation, we add _____ times the first equation to the third equation. The third equation becomes _____ = _____.

Think About It

3–4 ■ Consider the following system of linear equations.

$$\begin{cases} x + y + z = 1 \\ x + y + z = 2 \\ 2x + y - z = 3 \end{cases}$$

3. Explain how the first two equations tell us that the system *cannot* have a solution.

4. Show that (1, 1, 0) is a solution of the second and third equations. Why is (1, 1, 0) not a solution of the system?

SKILLS

5–8 ■ Use back-substitution to solve the triangular system.

5. $\begin{cases} x - 2y + 4z = 3 \\ y + 2z = 7 \\ z = 2 \end{cases}$

6. $\begin{cases} x + y - 3z = 8 \\ y - 3z = 5 \\ z = -1 \end{cases}$

7. $\begin{cases} x + 2y + z = 7 \\ -y + 3z = 9 \\ 2z = 6 \end{cases}$

8. $\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$

9–12 ■ Perform an operation on the given system that eliminates the indicated variable. Write the new equivalent system.

9. $\begin{cases} x - 2y - z = 4 \\ x - y + 3z = 0 \\ 2x + y + z = 0 \end{cases}$

Eliminate the x-term from the second equation.

10. $\begin{cases} x + y - 3z = 3 \\ -2x + 3y + z = 2 \\ x - y + 2z = 0 \end{cases}$

Eliminate the x-term from the second equation.

11. $\begin{cases} 2x - y + 3z = 2 \\ x + 2y - z = 4 \\ -4x + 5y + z = 10 \end{cases}$

Eliminate the x-term from the third equation.

12. $\begin{cases} x - 4y + z = 3 \\ y - 3z = 10 \\ 3y - 8z = 24 \end{cases}$

Eliminate the y-term from the third equation.

13–16 ■ Find the solution of the linear system.

13. $\begin{cases} x - y - z = 4 \\ 2y + z = -1 \\ -x + y - 2z = 5 \end{cases}$

14. $\begin{cases} x - y + z = 0 \\ y + 2z = -2 \\ x + y - z = 2 \end{cases}$

15. $\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$

16. $\begin{cases} x + y + z = 0 \\ -x + 2y + 5z = 3 \\ 3x - y = 6 \end{cases}$

17–30 ■ Solve the system, or show that it has no solution. If the system has infinitely many solutions, express them in the ordered triple form given in Example 4.

17. $\begin{cases} x - 4z = 1 \\ 2x - y - 6z = 4 \\ 2x + 3y - 2z = 8 \end{cases}$

18. $\begin{cases} x - y + 2z = 2 \\ 3x + y + 5z = 8 \\ 2x - y - 2z = -7 \end{cases}$

19. $\begin{cases} 2x + 4y - z = 2 \\ x + 2y - 3z = -4 \\ 3x - y + z = 1 \end{cases}$

20. $\begin{cases} 2x + y - z = -8 \\ -x + y + z = 3 \\ -2x + 4z = 18 \end{cases}$

21. $\begin{cases} y - 2z = 0 \\ 2x + 3y = 2 \\ -x - 2y + z = -1 \end{cases}$

22. $\begin{cases} 2y + z = 3 \\ 5x + 4y + 3z = -1 \\ x - 3y = -2 \end{cases}$

23. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y - 4z = -3 \\ 3x + 6y - 3z = 4 \end{cases}$

24. $\begin{cases} -x + 2y + 5z = 4 \\ x - 2z = 0 \\ 4x - 2y - 11z = 2 \end{cases}$

25. $\begin{cases} 2x + 3y - z = 1 \\ x + 2y = 3 \\ x + 3y + z = 4 \end{cases}$

26. $\begin{cases} x - 2y - 3z = 5 \\ 2x + y - z = 5 \\ 4x - 3y - 7z = 5 \end{cases}$

27. $\begin{cases} x + y - z = 0 \\ x + 2y - 3z = -3 \\ 2x + 3y - 4z = -3 \end{cases}$

28. $\begin{cases} x - 2y + z = 3 \\ 2x - 5y + 6z = 7 \\ 2x - 3y - 2z = 5 \end{cases}$

29. $\begin{cases} x + 3y - 2z = 0 \\ 2x + 4z = 4 \\ 4x + 6y = 4 \end{cases}$

30. $\begin{cases} 2x + 4y - z = 3 \\ x + 2y + 4z = 6 \\ x + 2y - 2z = 0 \end{cases}$

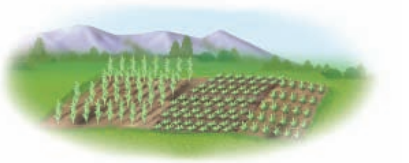
CONTEXTS

31–32 ■ An investor has $100,000 to invest in three types of bonds: short-term, intermediate-term, and long-term. How much should she invest in each type to satisfy the given conditions?

31. Finance Short-term bonds pay 4% annually, intermediate-term bonds pay 5%, and long-term bonds pay 6%. The investor wishes to realize a total annual income of 5.1%, with equal amounts invested in short- and intermediate-term bonds.

32. Finance Short-term bonds pay 4% annually, intermediate-term bonds pay 6%, and long-term bonds pay 8%. The investor wishes to have a total return of $6700 on her investment, with equal amounts invested in intermediate- and long-term bonds.

33. Agriculture A farmer has 1200 acres of land on which he grows corn, wheat, and soybeans. It costs $45 per acre to grow corn, $60 to grow wheat, and $50 to grow soybeans. Because of market demand the farmer will grow twice as many acres of wheat as corn. He has allocated $63,750 for the cost of growing his crops. How many acres of each crop should he plant?



34. Gas Station A gas station sells three types of gas: Regular for $3.00 a gallon, Performance Plus for $3.20 a gallon, and Premium for $3.30 a gallon. On a particular day 6500 gallons of gas were sold for a total of $20,050. Three times as many gallons of Regular as Premium gas were sold. How many gallons of each type of gas were sold that day?

35. Nutrition A biologist is performing an experiment on the effects of various combinations of vitamins. She wishes to feed each of her laboratory rabbits a diet that contains exactly 9 mg of niacin, 14 mg of thiamin, and 32 mg of riboflavin. She has available three different types of commercial rabbit pellets; their vitamin content (per ounce) is given in the table. How many ounces of each type of food should each rabbit be given daily to satisfy the experiment requirements?

Vitamin (mg)	Type A	Type B	Type C
Niacin	2	3	1
Thiamin	3	1	3
Riboflavin	8	5	7

36. Diet Program Nicole started a new diet that requires each meal to have 460 calories, 6 grams of fiber, and 11 grams of fat. The table shows the fiber, fat, and calorie content of one serving of each of three breakfast foods. How many servings of each food should Nicole eat to follow her diet?

Food	Fiber	Fat	Calories
Toast	2	1	100
Cottage cheese	0	5	120
Fruit	2	0	60

37. Juice Blends The Juice Company offers three kinds of smoothies: Midnight Mango, Tropical Torrent, and Pineapple Power. Each smoothie contains the amounts of juices shown in the table. On a particular day the Juice Company used 820 ounces of mango juice, 690 ounces of pineapple juice, and 450 ounces of orange juice. How many smoothies of each kind were sold that day?

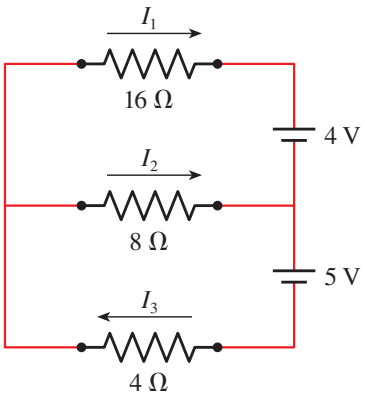
Smoothie	Mango Juice (oz)	Pineapple Juice (oz)	Orange Juice (oz)
Midnight Mango	8	3	3
Tropical Torrent	6	5	3
Pineapple Power	2	8	4

38. Appliance Manufacturing Kitchen Korner produces refrigerators, dishwashers, and stoves at three different factories. The table gives the number of each product produced at each factory per day. Kitchen Korner receives an order for 110 refrigerators, 150 dishwashers, and 114 ovens. How many days should each plant be scheduled to fill this order?

Appliance	Factory A	Factory B	Factory C
Refrigerators	8	10	14
Dishwashers	16	12	10
Stoves	10	18	6

39. Stock Portfolio An investor owns three stocks: A, B, and C. The closing prices of the stocks on three successive trading days are given in the table. Despite the volatility in the stock prices, the total value of the investor's stocks remained unchanged at $74,000 at the end of each of these three days. How many shares of each stock does the investor own?

Day	Stock A	Stock B	Stock C
Monday	$10	$25	$29
Tuesday	$12	$20	$32
Wednesday	$16	$15	$32



40. Electricity By using Kirchhoff's Laws, it can be shown that the currents I_1, I_2, and I_3 that pass through the three branches of the circuit in the figure satisfy the given linear system. Solve the system to find I_1, I_2, and I_3.

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 16I_1 - 8I_2 = 4 \\ 8I_2 + 4I_3 = 5 \end{cases}$$

7.3 Using Matrices to Solve Systems of Linear Equations

- Matrices
- The Augmented Matrix of a Linear System
- Elementary Row Operations
- Row-Echelon Form
- Reduced Row-Echelon Form
- Inconsistent and Dependent Systems

IN THIS SECTION... we study a method for solving systems of linear equations by first expressing the system as a matrix. Matrix notation allows us to use a graphing calculator to solve the system.

Matrices

A *matrix* is simply a rectangular array of numbers. Matrices* are used to organize data into categories that correspond to the rows and columns of the matrix (see Section 7.4). In this section we represent a linear system by a matrix, called the *augmented matrix* of the system. We begin by defining the various elements that make up a matrix.

Definition of a Matrix

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**. We say that the matrix has **dimension** $m \times n$. The numbers in the array are the **entries** of the matrix. The (i, j) entry is the number in the ith row and jth column.

Here are some examples of matrices.

Matrix	Dimension	
$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$	2×3	2 rows by 3 columns
$\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$	1×4	1 row by 4 columns

In the first matrix the (1, 2) entry is 3, the (2, 2) entry is 4, and the (2, 3) entry is -1.

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations.

*The plural of *matrix* is *matrices*.

Linear system		Augmented matrix
$\begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases}$	Equation 1 Equation 2	$\begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix}$

The augmented matrix contains the same information as the system but in a simpler form.

example 1 Finding the Augmented Matrix of a Linear System

Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

Solution

First we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

■ NOW TRY EXERCISE 1 ■

Elementary Row Operations

The operations that we used in Section 7.2 to solve a linear system correspond to operations on the rows of the augmented matrix of the system. For example, adding a multiple of one equation to another corresponds to adding a multiple of one row of the augmented matrix to another.

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Note that performing any of these operations on the augmented matrix of a system does not change its solution. In the next example we compare the two ways of writing systems of linear equations.

example 2 Using Elementary Row Operations to Solve a Linear System

Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Solution

Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison we write both the system of equations and its augmented matrix.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$$

Add $(-1) \times$ Equation 1 to Equation 2
Add $(-3) \times$ Equation 1 to Equation 3

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

Add $(-1) \times$ Row 1 to Row 2
Add $(-3) \times$ Row 1 to Row 3

Multiply Equation 3 by $\frac{1}{2}$

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Multiply Row 3 by $\frac{1}{2}$

Add $(-3) \times$ Equation 3 to Equation 2

$$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $(-3) \times$ Row 3 to Row 2

Interchange Equations 2 and 3

$$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Interchange Rows 2 and 3

Back-substitution is explained in Section 7.2.

Now we use back-substitution to find that $x = 2$, $y = 7$, and $z = 3$. The solution is $(2, 7, 3)$.

■ **NOW TRY EXERCISE 11** ■

Row-Echelon Form

In Example 2 we solved a system by using its augmented matrix and applying elementary row operations to arrive at a matrix in a certain form. This form is called *row-echelon form*.

Row-Echelon Form of a Matrix

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

In the following matrices the first one is not in row-echelon form. The second one is in row-echelon form

$$\begin{bmatrix} 0 & 1 & -2 & 5 \\ 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 & -3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's do not shift to the right in successive rows.

Leading 1's shift to the right in successive rows.

To put an augmented matrix into row-echelon form, we can use a graphing calculator to perform the required elementary row operations. (On the TI-83 the command is `ref`.)

example 3 Solving a System Using Row-Echelon Form

Solve the system of linear equations by using row-echelon form.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution

Augmented matrix. We first write the augmented matrix of the system.

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Row-echelon form. Using the `ref` command on a graphing calculator we put the matrix into row-echelon form:

Calculator output

```
ref([A])
   [[1  2  -1  1 ]
    [0  1  2   -3]
    [0  0  1   -2]]
```

Row-echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ y + 2z = -3 \\ z = -2 \end{cases}$$

Back-substitute. Now we use back-substitution to find that $x = -3$, $y = 1$, and $z = -2$. The solution is $(-3, 1, -2)$.

■ NOW TRY EXERCISE 15 ■

Note that the row-echelon form of a matrix is not unique. If we try to put a matrix in row-echelon form, we may get different correct answers depending on which elementary row operations we use.

Reduced Row-Echelon Form

We can use elementary row operations on the augmented matrix of a linear system to put it into *reduced* row-echelon form. This form allows us to find the solutions of the linear system without the need to use back-substitution.

Reduced Row-Echelon Form of a Matrix

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition:

4. Every number above and below each leading entry is a 0.

In the following matrices, the first one is in row-echelon form. The second one is in *reduced* row-echelon form

$$\begin{bmatrix} 1 & 6 & -2 & 5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Leading 1's do not have 0's above and below them.

Leading 1's do have 0's above and below them.

The reduced row-echelon form of a matrix is unique. To put an augmented matrix into row-echelon form, we can use a graphing calculator. (On the TI-83 the command is `rref`.) From the reduced row-echelon form we can just read off the solution to the system, as we show in the next example.

example 4 Solving a System Using Reduced Row-Echelon Form

Solve the system of linear equations using reduced row-echelon form.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution

This is the same system as in Example 3. We now solve this system by putting the system into reduced row-echelon form.

Augmented matrix. The augmented matrix of this system is

$$\begin{bmatrix} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{bmatrix}$$

Reduced row-echelon form. Using the `rref` command on a graphing calculator, we find the reduced row-echelon form of this matrix.

Calculator output

```
rref([A])
     [[1  0  0  -3]
      [0  1  0  1 ]
      [0  0  1  -2]]
```

Reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution. We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Hence we immediately arrive at the solution $(-3, 1, -2)$.

■ **NOW TRY EXERCISE 21** ■

Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1 through 4 had exactly one solution. But as we know from Section 7.2, a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the reduced row-echelon form of a system allows us to determine which of these cases applies, as described in the following box.

First we need some terminology. A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

The Solutions of a Linear System in Reduced Row-Echelon Form

Suppose the augmented matrix of a system of linear equations has been transformed into reduced row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the reduced row-echelon form contains a row that represents the equation $0 = c$, where c is not zero, then the system has no solution.
2. **One solution.** If each variable in the reduced row-echelon form is a leading variable, then the system has exactly one solution.
3. **Infinitely many solutions.** If the variables in the reduced row-echelon form are not all leading variables and if the system is not inconsistent, then it has infinitely many solutions. We solve the system by expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

Recall that a system with no solutions is called **inconsistent** and a system with infinitely many solutions is called **dependent**.

The matrices below, all in reduced row-echelon form, illustrate the three cases described above.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable

example 5 A System with No Solution

Solve the system.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution

Augmented matrix. The augmented matrix of this system is

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix}$$

Reduced row-echelon form. Using the `rref` command on a graphing calculator, we find the reduced row-echelon form of this matrix.

Calculator output

```
rref([A])
   [[1  0  5  0]
    [0  1  1  0]
    [0  0  0  1]]
```

Reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution. Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means that the system *has no solution.*

■ NOW TRY EXERCISES 25 AND 33 ■

example 6 A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x \quad\quad + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution

Augmented matrix. The augmented matrix of this system is

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix}$$

Row-echelon form. Using the `rref` command on a graphing calculator, we find the reduced row-echelon form of this matrix.

Calculator output

```
rref([A])
   [[1  0  -7  -5]
    [0  1  -3  1 ]
    [0  0  0   0 ]]
```

Reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x \quad - 7z = -5 \\ y - 3z = 1 \end{cases}$$

Leading variables

Now we solve for the leading variables x and y in terms of the nonleading variable z.

$x = 7z - 5$ Solve for x in Equation 1

$y = 3z + 1$ Solve for y in Equation 2

To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t.

$$x = 7t - 5$$

$$y = 3t + 1$$

$$z = t$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number.

■ NOW TRY EXERCISES 27 AND 37 ■

example 7 Nutritional Analysis

A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: MiniCal, LiquiFast, and SlimQuick. For the experiment it is important that the subject consume exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

Solution

Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day. This means that the subject will get $50x$ mg of potassium from MiniCal, $75y$ mg from LiquiFast, and $10z$ mg from SlimQuick, for a total of $50x + 75y + 10z$ mg of potassium. Since the potassium requirement is 500 mg, we get the first equation below. Similar reasoning for the protein and vitamin D requirements leads to the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

Augmented matrix. The augmented matrix of this system is

$$\begin{bmatrix} 50 & 75 & 10 & 500 \\ 5 & 10 & 3 & 75 \\ 90 & 100 & 50 & 1150 \end{bmatrix}$$

Row-echelon form. Using the `rref` command on a graphing calculator, we find the reduced row-echelon form of this matrix.

Calculator output

```
rref([A])
   [[1  0  0  5 ]
    [0  1  0  2 ]
    [0  0  1  10]]
```

Reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 10 \end{bmatrix}$$

Solution. From the reduced row-echelon form we see that the solution of the system is $x = 5$, $y = 2$, $z = 10$. The subject should be fed 5 ounces of MiniCal, 2 ounces of LiquiFast, and 10 ounces of SlimQuick every day.

■ **NOW TRY EXERCISE 41** ■

7.3 Exercises

CONCEPTS

Fundamentals

1. Write the augmented matrix of the following system of equations.

System

$$\begin{cases} x + y - z = 1 \\ x + 2z = -3 \\ 2y - z = 3 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

2. Write the system of equations that corresponds to the following augmented matrix.

Augmented matrix

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 2 & -1 & -2 \end{bmatrix}$$

System

$$\begin{cases} _____ = ____ \\ _____ = ____ \\ _____ = ____ \end{cases}$$

3. The matrix below is the augmented matrix of a system of linear equations in the variables x, y, and z. (It is given in reduced row-echelon form.)

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The leading variables are _____ and _____.

(b) Is the system inconsistent or dependent? _____

(c) The solution of the system is

$x =$ _____ $y =$ _____ $z =$ _____

4. The augmented matrix of a system of linear equations is given in reduced row-echelon form. Find the solution of the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$x =$ _____

$y =$ _____

$z =$ _____

(b) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x =$ _____

$y =$ _____

$z =$ _____

(c) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$x =$ _____

$y =$ _____

$z =$ _____

SKILLS

5–10 ■ **(a)** State the dimensions of the matrix.
(b) Find the (2, 1) entry of the matrix.

5. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 2 & 0 & 11 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

8. $\begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}$

9. $\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

11–14 ■ The system of linear equations has a unique solution. Find the solution using elementary row operations, as in Example 2.

11. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$

12. $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

13. $\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$

14. $\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$

15–20 ■ Use a graphing calculator to put the augmented matrix of the system into row-echelon form. Then solve the system (as in Example 3).

15. $\begin{cases} x + 3y - 2z = -2 \\ x + 4y - 3z = -3 \\ x + 3y - z = 0 \end{cases}$

16. $\begin{cases} x + y - z = 4 \\ 2x + 3y - z = 5 \\ x + y = 1 \end{cases}$

17. $\begin{cases} x + z = 1 \\ 2x - y + 4z = -6 \\ x - y + 4z = -6 \end{cases}$

18. $\begin{cases} x - 3z = -5 \\ x + y - 2z = -4 \\ x + 2y = -1 \end{cases}$

19. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$

20. $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

21–24 ■ Use a graphing calculator to put the augmented matrix of the system into reduced row-echelon form, then solve the system (as in Example 4).

21. $\begin{cases} x + 2y - z = 9 \\ 2x - z = -2 \\ 3x + 5y + 2z = 22 \end{cases}$

22. $\begin{cases} 2x + y = 7 \\ 2x - y + z = 6 \\ 3x - 2y + 4z = 11 \end{cases}$

23. $\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$

24. $\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$

25–30 ■ Use a graphing calculator to put the augmented matrix of the system into reduced row-echelon form. Determine whether the system is inconsistent or dependent. If it is dependent, find the complete solution.

25. $\begin{cases} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$

26. $\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$

27. $\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$

28. $\begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y - 20z = -26 \end{cases}$

29. $\begin{cases} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{cases}$

30. $\begin{cases} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{cases}$

31–40 ■ Solve the system of linear equations.

31. $\begin{cases} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{cases}$

32. $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

33. $\begin{cases} 2x + y + 3z = 9 \\ -x - 7z = 10 \\ 3x + 2y - z = 4 \end{cases}$

34. $\begin{cases} -4x - y + 36z = 24 \\ x - 2y + 9z = 3 \\ -2x + y + 6z = 6 \end{cases}$

35. $\begin{cases} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{cases}$

36. $\begin{cases} 3x + y = 2 \\ -4x + 3y + z = 4 \\ 2x + 5y + z = 0 \end{cases}$

37. $\begin{cases} x - y + 6z = 8 \\ x + z = 5 \\ x + 3y - 14z = -4 \end{cases}$

38. $\begin{cases} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{cases}$

39. $\begin{cases} x - 3y + 2z + w = -2 \\ x - 2y - 2w = -10 \\ z + 5w = 15 \\ 3x + 2z + w = -3 \end{cases}$

40. $\begin{cases} x + y - z - w = 6 \\ 2x + z - 3w = 8 \\ x - y + 4w = -10 \\ 3x + 5y - z - w = 20 \end{cases}$

CONTEXTS

41. Nutrition A doctor recommends that a patient take 50 milligrams each of niacin, riboflavin, and thiamin daily to alleviate a vitamin deficiency. In his medicine chest at home the patient finds three brands of vitamin pills. The amounts of the relevant vitamins per pill are given in the table. How many pills of each type should he take every day to get 50 milligrams of each vitamin?

Vitamin (mg)	VitaMax	Vitron	VitaPlus
Niacin	5	10	15
Riboflavin	15	20	0
Thiamin	10	10	10

42. Mixtures A chemist has three acid solutions at various concentrations. The first is 10% acid, the second is 20%, and the third is 40%. How many milliliters of each should she use to make 100 milliliters of 18% solution if she has to use four times as much of the 10% solution as the 40% solution?

43. Distance, Speed, and Time Amanda, Bryce, and Corey enter a race in which they have to run, swim, and cycle over a marked course. Their average speeds are given in the table. Corey finishes first with a total time of 1 hour, 45 minutes. Amanda comes in second with a time of 2 hours, 30 minutes. Bryce finishes last with a time of 3 hours. Find the distance in miles for each part of the race.

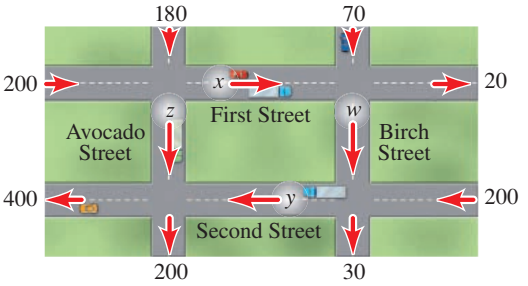
	Average speed (mi/h)		
Contestant	Running	Swimming	Cycling
Amanda	10	4	20
Bryce	$7\frac{1}{2}$	6	15
Corey	15	3	40

44. Classroom Use A small school has 100 students who occupy three classrooms: A, B, and C. After the first period of the school day, half the students in room A move to room B, one-fifth of the students in room B move to room C, and one-third of the students in room C move to room A. Nevertheless, the total number of students in each room is the same for both periods. How many students occupy each room?

45. Manufacturing Furniture A furniture factory makes wooden tables, chairs, and armoires. Each piece of furniture requires three operations: cutting the wood, assembling, and finishing. Each operation requires the number of hours given in the table. The workers in the factory can provide 300 hours of cutting, 400 hours of assembling, and 590 hours of finishing each work week. How many tables, chairs, and armoires should be produced so that all available labor-hours are used? Or is this impossible?

Task (h)	Table	Chair	Armoire
Cutting	$\frac{1}{2}$	1	1
Assembling	$\frac{1}{2}$	$1\frac{1}{2}$	1
Finishing	1	$1\frac{1}{2}$	2

46. Traffic Flow A section of a city's street network is shown in the figure. The arrows indicate one-way streets, and the numbers show how many cars enter or leave this section of the city via the indicated street in a certain 1-hour period. The variables x, y, z, and w represent the number of cars that travel along the portions of First, Second, Avocado, and Birch Streets during this period. Find x, y, z, and w, assuming that none of the cars stop or park on any of the streets shown.



7.4 Matrices and Data in Categories

- **Organizing Categorical Data in a Matrix**
- **Adding Matrices**
- **Scalar Multiplication of Matrices**
- **Multiplying a Matrix Times a Column Matrix**

IN THIS SECTION... we learn about a new form of data—data that are expressed in categories. We'll see how matrices are used in storing such categorical data.

GET READY... by reviewing one-variable and two-variable data in Section 1.1.

We have studied two types of data: one-variable data and two-variable data (Section 1.1). In each case the data we studied were *numerical*—that is, the data were real numbers. For example, age, height, and cholesterol level are numbers that can be measured with any desired precision. But in many situations data are **categorical**—that is, the data tell us whether an individual does or does not belong to a particular category. For example, a survey of your class could include categorical data such as hair color (dark, blond, red), eye color (brown, blue, green),

sex (male, female), political affiliation (Democrat, Republican, Independent), and so on. "Age" and "years of education" are numerical variables, but it is often more meaningful to convert these into categorical variables by placing individuals in age categories (0–10, 11–20, 21–30, and so on) or educational categories (elementary school, high school, college, or graduate degree). Categorical data are typically obtained from surveys, but many medical experiments also lead to categorical data. For example, when testing a new drug, we can categorize the result of the drug on individual patients (condition improved, stayed the same, got worse).

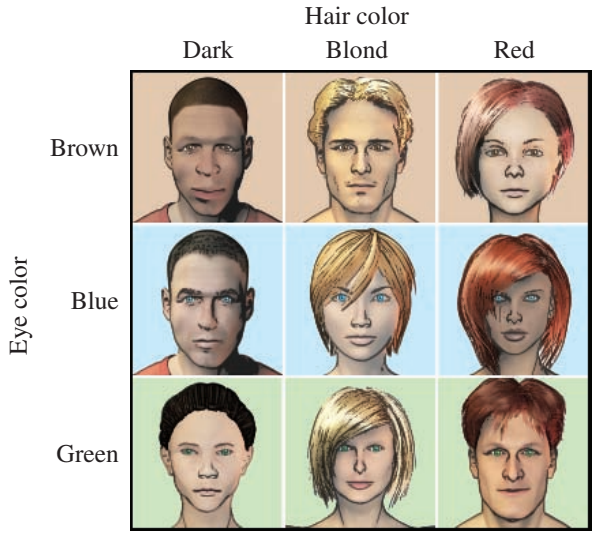


figure **1** Categories of hair and eye color

In this section we study how categorical data involving two variables can be conveniently stored in a matrix. In the next section we learn how to obtain useful information from such data by performing operations on matrices.

Organizing Categorical Data in a Matrix

A class of college algebra students is surveyed about their hair color (dark, blond, red) and eye color (brown, blue, green). (See Figure 1.) For each student a number is assigned, and the appropriate value is recorded.

Student Number	Hair Color	Eye Color
1	Dark	Brown
2	Blond	Brown
3	Dark	Blue
⋮	⋮	⋮

The results are as follows: 12 students have dark hair, 10 have blond hair, and 4 have red hair; 12 have brown eyes, 9 have blue eyes, and 5 have green eyes. We can get more precise information on hair and eye color by tabulating the result in a matrix (called a *cross-tab matrix*), as in the next example.

example 1 Tabulating Categorical Data

The results of the class survey described above are tabulated in the following data matrix.

$$A = \begin{array}{c} \text{Hair color} \\ \begin{array}{ccc} \text{Dark} & \text{Blond} & \text{Red} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 3 \\ 3 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Brown} \\ \leftarrow \text{Blue} \\ \leftarrow \text{Green} \end{array} \quad \text{Eye color}$$

(a) How many students have dark hair and blue eyes?

(b) How many students have dark hair?

(c) How many students have blue eyes?

(d) How many students were surveyed?

Solution

(a) This is the entry in the "dark hair" column and the "blue eyes" row (the (2, 1) entry in matrix A). We see that there are 0 students with dark hair and blue eyes.

(b) We add the numbers in the "dark hair" *column*.

$$9 + 0 + 3 = 12$$

There are 12 students with dark hair.

(c) We add the numbers in the "blue eyes" *row*.

$$0 + 6 + 3 = 9$$

(d) We add all the entries in the matrix.

$$9 + 3 + 0 + 0 + 6 + 3 + 3 + 1 + 1 = 26$$

Twenty-six students were surveyed.

NOW TRY EXERCISE 7

It is often more convenient to make a matrix in which the entries are *proportions* with specific characteristics rather than the actual number of individuals. For example, we can report the proportion of dark-haired students who have brown eyes, the proportion of red-haired students with green eyes, and so on. To do this, we first calculate the total number in each column of the matrix.

$$A = \begin{array}{c} \text{Hair color} \\ \begin{array}{ccc} \text{Dark} & \text{Blond} & \text{Red} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 3 \\ 3 & 1 & 1 \\ 12 & 10 & 4 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Brown} \\ \leftarrow \text{Blue} \\ \leftarrow \text{Green} \\ \leftarrow \text{Column totals} \end{array} \quad \text{Eye color}$$

example 2 Categorical Data and Proportions

(a) Express each entry in the matrix in Example 1 as a proportion of the column total. Name the resulting matrix P.

(b) What does the (1, 1) entry in P represent? What about the (3, 2) entry?

Solution

(a) The total of the first column of matrix A is 12. So the entries in the first column of matrix P are

$$\frac{9}{12} = 0.75 \qquad \frac{0}{12} = 0.00 \qquad \frac{3}{12} = 0.25$$

The other entries in matrix P are calculated similarly.

$$P = \begin{array}{c} \text{Hair color} \\ \begin{array}{ccc} \text{Dark} & \text{Blond} & \text{Red} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 0.75 & 0.30 & 0.00 \\ 0.00 & 0.60 & 0.75 \\ 0.25 & 0.10 & 0.25 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Brown} \\ \leftarrow \text{Blue} \\ \leftarrow \text{Green} \end{array} \quad \text{Eye color}$$

(b) The (1, 1) entry tells us that of the students with dark hair, 75% have brown eyes. The (3, 2) entry tells us that of the students with blond hair, 10% have green eyes.

NOW TRY EXERCISE 11

Adding Matrices

If we make the same hair and eye color survey with a different group of students, we can combine the results into one large survey using matrix addition. We **add two matrices** of the same dimension by adding corresponding entries.

example 3 Adding Matrices

A class of history students is surveyed as to hair color (dark, blond, red) and eye color (brown, blue, green). The results are tabulated in the following data matrix.

$$B = \begin{array}{c} \text{Hair color} \\ \begin{array}{ccc} \text{Dark} & \text{Blond} & \text{Red} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 12 & 4 & 1 \\ 2 & 10 & 4 \\ 0 & 2 & 2 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Brown} \\ \leftarrow \text{Blue} \\ \leftarrow \text{Green} \end{array} \quad \text{Eye color}$$

(a) To combine the results of this survey and the one in Example 1, add matrix B and matrix A (of Example 1).

(b) What is the total number of students (from both surveys) who have blond hair and blue eyes?

Solution

(a) We add matrices A and B to combine the results of the two surveys. The total in each category is simply the sum of the corresponding entries in matrices A and B.

$$A + B = \begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 3 \\ 3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 12 & 4 & 1 \\ 2 & 10 & 4 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 21 & 7 & 1 \\ 2 & 16 & 7 \\ 3 & 3 & 3 \end{bmatrix}$$

(b) From the (2, 2) entry in the matrix $A + B$, we see that 16 students have blond hair and blue eyes.

NOW TRY EXERCISE 13

Scalar Multiplication of Matrices

Suppose we double each entry in the data matrix in Example 1. This would not change any of the proportions calculated in Example 2. Doubling each entry in a data matrix simply "scales up" the size of the data. In general, multiplying a matrix by a constant is called **scalar multiplication**. The next example explains the notation.

example 4 Scalar Multiplication of a Matrix

The following questions refer to matrix A of Example 1.

(a) Find the scalar product $3A$.

(b) In this new data matrix, what is the proportion of the students with dark hair who have brown eyes?

Solution

(a) To perform the scalar multiplication, we multiply each entry in the data matrix by 3.

$$3A = 3\begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 9 & 0 \\ 0 & 18 & 9 \\ 9 & 3 & 3 \end{bmatrix}$$

(b) The total in the "dark hair" column is $27 + 0 + 9 = 36$. So the proportion of students with dark hair who have brown eyes is

$$\frac{27}{36} = 0.75$$

Notice that this is the same proportion as in the original matrix. You can check that all the other proportions are the same as in the original matrix (see Example 2).

NOW TRY EXERCISE 15

Multiplying a Matrix Times a Column Matrix

We define what it means to multiply a 3×3 matrix A times a 3×1 column matrix C. The product AC is a 3×1 column matrix that is obtained as follows.

$$\begin{bmatrix} a & b & c \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

The first entry in the product matrix ($\square$) is calculated by multiplying the entries in the first *row* of matrix A by the entries in the *column* of matrix C and adding the results.

$$a\,a' + b\,b' + c\,c' = \square$$

The other entries in the product matrix are calculated similarly, as is illustrated in the next example.

example 5 Multiplying a Matrix Times a Column Matrix



An organic farmer sells produce at an open-air market three days a week: Thursdays, Fridays, and Saturdays. He sell oranges, broccoli, and beans. Matrix A tabulates the number of pounds of produce he sold in a certain week. Matrix C gives the prices per pound (in dollars) he charged that week.

(a) Find the product AC.

(b) What was the total revenue on Friday?

(c) What was the total revenue for the week?

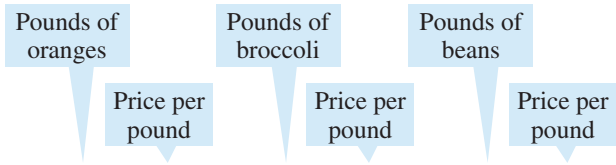
$$A = \begin{matrix} \\ \text{Thursday} \\ \text{Friday} \\ \text{Saturday} \end{matrix} \overset{\begin{matrix}\text{Oranges} & \text{Broccoli} & \text{Beans}\end{matrix}}{\begin{bmatrix} 50 & 20 & 10 \\ 70 & 35 & 30 \\ 45 & 15 & 25 \end{bmatrix}} \quad \text{Day} \qquad \text{Produce} \quad C = \begin{matrix} \\ \text{Oranges} \\ \text{Broccoli} \\ \text{Beans} \end{matrix} \overset{\text{Price}}{\begin{bmatrix} 0.90 \\ 1.20 \\ 1.50 \end{bmatrix}}$$

Solution

(a) The entries in the product matrix are obtained as explained above.

$$AC = \begin{bmatrix} 50 & 20 & 10 \\ 70 & 35 & 30 \\ 45 & 15 & 25 \end{bmatrix} \begin{bmatrix} 0.90 \\ 1.20 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 50 \cdot 0.90 + 20 \cdot 1.20 + 10 \cdot 1.50 \\ 70 \cdot 0.90 + 35 \cdot 1.20 + 30 \cdot 1.50 \\ 45 \cdot 0.90 + 15 \cdot 1.20 + 25 \cdot 1.50 \end{bmatrix} = \begin{bmatrix} 84.00 \\ 150.00 \\ 96.00 \end{bmatrix}$$

(b) The (2, 1) entry of matrix AC is calculated as follows:



This means that the (2, 1) entry of matrix AC is the sum of Friday's revenue from oranges, broccoli, and beans. So the total revenue on Friday was $150.00.

(c) We add the three entries in the column matrix AC to get the total revenue for the week.

$$84.00 + 150.00 + 96.00 = 330.00$$

So the total revenue from all three days was $330.00.

■ **NOW TRY EXERCISE 17** ■

7.4 Exercises

CONCEPTS

Fundamentals

1–4 ■ Determine whether the data described is numerical or categorical data.

1. The height of each child in a third grade class.

2. The number of laundry items tabulated by color (white, pastel, dark).

3. The number of gulls seen at a beach on a given day, tabulated by species (California, Western, Ring-billed) and maturity (immature, juvenile, adult).

4. The number of birds observed at a bird feeder each day.

CONTEXTS

5–6 ■ Organize the given set of categorical data into a data matrix.

5. Education and Income A women's club takes a survey to determine the education and income of its members. Of the women with no postsecondary education, 5 have a yearly income below $50,000, 2 have an income between $50,000 and $100,000, and 0 has an income above $100,000. Of the women with up to four years of postsecondary education, 2 have a yearly income below $50,000, 10 have an income between $50,000 and $100,000, and 4 have an income above $100,000. Of the women with more than four years of postsecondary education, 0 has a yearly income below $50,000, 3 have an income between $50,000 and $100,000, and 2 have an income above $100,000.

Postsecondary education (years)

None 0–4 More than 4

$$A = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

← Less than $50,000
← $50,000–$100,000
← More than $100,000

Income level

6. Exam Scores A physics class takes a survey of the number of hours the students slept before the exam and the exam score. Of the students who slept less than four hours, six students got a score below 60, three got a score between 60 and 80, and one got a score above 80. Of the students who slept between four and seven hours, two students got a score below 60, seven got a score between 60 and 80, and four got a score above 80. Of the students who slept more than seven hours, three students got a score below 60, nine got a score between 60 and 80, and six got a score above 80.

Hours of sleep

Less than 4 4–7 More than 7

$$A = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

← Below 60
← 60–80
← Above 80

Exam score

7. Service Ratings A Cincinnati car dealership conducted a customer satisfaction survey, and the results are tabulated in the data matrix A.

(a) How many of the customers that had a bill of less than $50 gave a fair service rating?

(b) How many customers had a bill above $100?

(c) How many customers gave an excellent service rating?

(d) How many customers were surveyed?

Amount of bill

Less than $50 | $50–100 | More than $100

$$A = \begin{bmatrix} 10 & 8 & 5 \\ 5 & 7 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

← Excellent
← Good
← Fair

Service rating

8. Service Ratings An Akron car dealership conducted a customer satisfaction survey, and the results are tabulated in the data matrix B.

(a) How many of the customers who had a bill of more than $100 gave an excellent service rating?

(b) How many customers had a bill less than $50?

(c) How many customers gave a good service rating?

(d) How many customers were surveyed?

$$B = \begin{array}{c} \text{Amount of bill} \\ \begin{array}{ccc} \text{Less than \$50} & \text{\$50–100} & \text{More than \$100} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 9 & 15 & 12 \\ 8 & 10 & 6 \\ 5 & 2 & 7 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Excellent} \\ \leftarrow \text{Good} \\ \leftarrow \text{Fair} \end{array} \quad \text{Service rating}$$

9. Health Insurance Residents of the small town of Springfield were surveyed to obtain information on the type of health insurance and the number of emergency room visits within the past year. The data are tabulated in matrix C.

(a) How many of those surveyed are members of an HMO and had four or more emergency visits in the past year?

(b) How many of those surveyed had no health insurance?

(c) How many of those surveyed had four or more emergency visits in the past year?

(d) How many people were surveyed?

$$C = \begin{array}{c} \text{Type of insurance} \\ \begin{array}{ccc} \text{HMO} & \text{Non-HMO} & \text{None} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 11 & 8 & 3 \\ 5 & 6 & 5 \\ 3 & 2 & 7 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{0–1} \\ \leftarrow \text{2–3} \\ \leftarrow \text{4 or more} \end{array} \quad \text{Emergency visits}$$

10. Health Insurance Residents of a Chicago suburb were surveyed to obtain information on the type of health insurance and the number of emergency room visits within the past year. The data are tabulated in matrix D.

(a) How many of those surveyed are members of an HMO and had four or more emergency visits in the past year?

(b) How many of those surveyed had no health insurance?

(c) How many of those surveyed had fewer than two emergency visits in the past year?

(d) How many people were surveyed?

$$D = \begin{array}{c} \text{Type of insurance} \\ \begin{array}{ccc} \text{HMO} & \text{Non-HMO} & \text{None} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 76 & 82 & 54 \\ 32 & 49 & 79 \\ 21 & 32 & 87 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{0–1} \\ \leftarrow \text{2–3} \\ \leftarrow \text{4 or more} \end{array} \quad \text{Emergency visits}$$

11. Sales A tasting booth at Joe's Specialty Foods offers pesto pizza on Monday, spinach ravioli on Tuesday, and macaroni and cheese on Wednesday. The sales distribution for these products is tabulated in matrix A.

(a) Express each entry in the matrix as a proportion of the column total. Call the resulting matrix P.

(b) What does the (1, 3) entry in P represent? What about the (3, 3) entry?

$$A = \begin{array}{c} \text{Specialty food} \\ \begin{array}{ccc} \text{Pesto pizza} & \text{Spinach ravioli} & \text{Macaroni and cheese} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 50 & 20 & 15 \\ 40 & 75 & 20 \\ 35 & 60 & 100 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Monday} \\ \leftarrow \text{Tuesday} \\ \leftarrow \text{Wednesday} \end{array} \quad \text{Day}$$

12. Politics A history class makes a survey of the students in the class, and the results are tabulated in matrix B.

(a) Express each entry in the matrix as a proportion of the column total. Call the resulting matrix P.

(b) What does the (2, 2) entry in P represent? What about the (1, 3) entry?

Political Affiliation

	Democrat	Republican	Independent	
$B =$	7	9	3	← Male
	8	10	2	← Female

Gender

13. Service Ratings The following questions refer to matrices A and B from Exercises 7 and 8.

(a) Find a data matrix for the results of the survey in Exercise 7 combined with the results of the survey in Exercise 8.

(b) What is the total number of customers (from both surveys) who had a bill between \$50 and \$100 and gave an excellent service rating?

14. Health Insurance The following questions refer to matrices C and D from Exercises 9 and 10.

(a) Find a data matrix for the results of the survey in Exercise 9 combined with the results of the survey in Exercise 10.

(b) What is the total number of those surveyed (from both surveys) that had no health insurance and had four or more emergency visits in the past year?

15. Sales The following questions refer to matrix A in Exercise 11.

(a) Find the scalar product $4A$.

(b) In this new data matrix, what proportion of the pesto pizza sales occurs on Wednesday?

16. Politics The following questions refer to matrix B in Exercise 12.

(a) Find the scalar product $5B$.

(b) In this new data matrix, what proportion of the female students are Republican?

17. Car-Manufacturing Profits A specialty car manufacturer has plants in Auburn, Biloxi, and Chattanooga. Three models are produced, with daily production given in matrix A. The profit (in dollars) per car is tabulated by model in matrix C.



(a) Find the product matrix AC.

(b) Assuming that all cars produced are sold, what is the daily profit from the Biloxi plant?

(c) What is the total daily profit (from all three plants)?

Cars produced each day

	Model K	Model R	Model W	
$A =$	12	10	0	← Auburn
	4	4	20	← Biloxi
	8	9	12	← Chattanooga

City

Profit

$$C = \begin{bmatrix} 1000 \\ 2000 \\ 1500 \end{bmatrix} \begin{matrix} \leftarrow \text{Model K} \\ \leftarrow \text{Model R} \\ \leftarrow \text{Model W} \end{matrix}$$

18. Sandwich Revenue A sandwich shop sells three types of sandwiches: Bite-size, Foot-long, and Mega-long. The sales distribution is given in matrix A. The price (in dollars) per sandwich is tabulated in matrix C.

(a) Find the product matrix AC.

(b) What is the total revenue on Friday?

(c) What is the total revenue (from all three days)?

$$A = \begin{matrix} \text{Bite-size} \\ \downarrow \end{matrix} \begin{matrix} \text{Sandwich} \\ \text{Foot-long} \\ \downarrow \end{matrix} \begin{matrix} \\ \text{Mega-long} \\ \downarrow \end{matrix}$$

$$\begin{bmatrix} 15 & 30 & 20 \\ 10 & 50 & 30 \\ 20 & 45 & 25 \end{bmatrix} \begin{matrix} \leftarrow \text{Friday} \\ \leftarrow \text{Saturday} \\ \leftarrow \text{Sunday} \end{matrix} \quad \textbf{Day}$$

$$C = \begin{matrix} \textbf{Price} \\ \end{matrix}$$

$$\begin{bmatrix} 4.00 \\ 5.00 \\ 9.00 \end{bmatrix} \begin{matrix} \leftarrow \text{Bite-size} \\ \leftarrow \text{Foot-long} \\ \leftarrow \text{Mega-long} \end{matrix}$$

7.5 Matrix Operations: Getting Information from Data

- Addition, Subtraction, and Scalar Multiplication
- Multiplying Matrices
- Getting Information from Categorical Data

IN THIS SECTION... we study addition, subtraction, scalar multiplication, and multiplication of matrices. We'll see how these operations allow us to get useful information from categorical data.

■ Addition, Subtraction, and Scalar Multiplication

Two matrices can be added or subtracted if they have the same dimension. As we saw in Section 7.4, we add matrices by adding corresponding entries. To multiply a matrix by a number (a scalar), we multiply each element of the matrix by that number.

example 1 Algebraic Operations on Matrices

Let

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

Perform the indicated operation, or explain why it cannot be performed.

(a) $A + B$ **(b)** $C - D$ **(c)** $C + A$ **(d)** $5A$

Solution

(a) Since A and B are matrices with the same dimension (3×2), we can add them.

$$A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$$

(b) Since C and D are matrices with the same dimension (2×3), we can subtract them.

$$C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$$

(c) $C + A$ is not defined because we cannot add matrices with different dimensions.

(d) To find $5A$, we multiply each entry in A by 5.

$$5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$$

■ **NOW TRY EXERCISES 7 AND 15** ■

Matrix Multiplication

In Section 7.4 we learned how to multiply a matrix times a column matrix. We now describe how to use this to define matrix multiplication in general (when the second factor has more than just one column).

First, the product AB of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. This means that if we write their dimensions side by side, the two inner numbers must match.

Matrices	A	B
Dimensions	$m \times n$	$n \times k$
	Columns in A	Rows in B

If the dimensions of A and B match in this way, then the product AB is a matrix of dimension $m \times k$. Before describing the procedure for obtaining the elements of AB, we define the *inner product* of a row of A and a column of B.

If $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ is a row of A and if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a column of B, then their **inner product** is the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. For example taking the inner product, we have

$$\begin{bmatrix} 2 & -1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ -3 \\ 2 \end{bmatrix} = 2 \cdot 5 + (-1) \cdot 6 + 0 \cdot (-3) + 4 \cdot 2 = 12$$

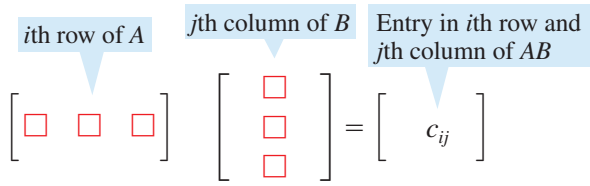
Matrix Multiplication

If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then their product AB is an $m \times k$ matrix C.

$$AB = C$$

The entries of the matrix C are calculated as follows: The entry in the ith row and jth column of the matrix C is the inner product of the ith row of A and the jth column of B.

The following diagram shows that the entry c_{ij} in the ith row and jth column of the product matrix AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results.



The next example illustrates the process.

example **2**

Multiplying Matrices

Calculate, if possible, the products AB and BA.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Solution

Inner numbers match, so product is defined.

$2 \times 2 \quad 2 \times 3$

Outer numbers give dimension of product: 2×3.

Since matrix A has dimension 2×2 and matrix B has dimension 2×3, the product AB is defined and has dimension 2×3. We can write

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in by using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B.

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1 \cdot (-1) + 3 \cdot 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows.

Entry	Inner product of	Value	Product matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1\cdot 5 + 3\cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1\cdot 2 + 3\cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot(-1) + 0\cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 5 + 0\cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1)\cdot 2 + 0\cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Thus we have

$$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

Not equal, so product not defined.

$2 \times 3 \quad 2 \times 2$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \quad \text{and} \quad 2 \times 2$$

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product.

■ **NOW TRY EXERCISES 11 AND 13** ■

Graphing calculators are capable of performing all the matrix operations. In the next example we use a graphing calculator to multiply two matrices.

example 3 Multiplying Matrices (Graphing Calculator)

Calculate the products AB and BA.

$$A = \begin{bmatrix} 3 & 0 & 5 \\ 2 & 2 & -4 \\ -1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 7 & 6 \\ 3 & 8 & -2 \end{bmatrix}$$

Solution

Using a graphing calculator, we find the product AB.

Calculator output

```
[A]*[B]
   [[18   37   5 ]
    [-10 -20 30]
    [5    24   -3]
```

Product *AB*

$$\begin{bmatrix} 18 & 37 & 5 \\ -10 & -20 & 30 \\ 5 & 24 & -3 \end{bmatrix}$$

Similarly, we find the product BA.

Calculator output

```
[B]*[A]
   [[-4  3   19 ]
    [8   20  -16]
    [27  14  -21]]
```

Product BA

$$\begin{bmatrix} -4 & 3 & 19 \\ 8 & 20 & -16 \\ 27 & 14 & -21 \end{bmatrix}$$

NOW TRY EXERCISE 19

Getting Information from Categorical Data

We will see in the next example how we can get information about categorical data by multiplying matrices. (Compare to Example 5 in Section 7.4.)

example 4 Getting Information by Multiplying Matrices

Josh sells organic produce at an open-air market three days a week: Thursdays, Fridays, and Saturdays. He sells oranges, broccoli, and beans. Matrix A tabulates the number of pounds of produce that he obtains from his suppliers every week; he is always able to sell his complete supply by the end of each day. Matrix C gives the prices per pound (in dollars) he charged on two different weeks. Find the product AC, and interpret the entries of AC.

Produce (Oranges, Broccoli, Beans)

$$A = \begin{bmatrix} 50 & 20 & 10 \\ 70 & 35 & 30 \\ 45 & 15 & 25 \end{bmatrix}$$

← Thursday
← Friday
← Saturday

Day

Prices (Week 1, Week 2)

$$C = \begin{bmatrix} 0.90 & 1.20 \\ 1.20 & 1.60 \\ 1.50 & 1.90 \end{bmatrix}$$

← Oranges
← Broccoli
← Beans

Produce

Solution

We use a graphing calculator to find the product AC.

Calculator output

```
[A]*[C]
   [[84    111   ]
    [150   197   ]
    [96    125.5]]
```

Product AC

$$\begin{bmatrix} 84.00 & 111.00 \\ 150.00 & 197.00 \\ 96.00 & 125.50 \end{bmatrix}$$

The product matrix gives us the revenue for each day of each week.

- Week 1: A total of \$84.00 of produce was sold on Thursday, \$150.00 on Friday, and \$96.00 on Saturday.
- Week 2: A total of \$111.00 of produce was sold on Thursday, \$197.00 on Friday, and \$125.50 on Saturday.

NOW TRY EXERCISE 27

7.5 Exercises

CONCEPTS

Fundamentals

1. We can add or (subtract) two matrices only if they have the same _____.

2. We can multiply two matrices only if the number of _____ in the first matrix is the same as the number of _____ in the second matrix.

3. If A is a 3×3 matrix and B is a 2×3 matrix, which of the following matrix multiplications are possible?

(a) AB **(b)** BA **(c)** AA **(d)** BB

4. Fill in the missing entries in the product matrix.

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 3 & -2 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & \square & -7 \\ 7 & -7 & \square \\ \square & -5 & -5 \end{bmatrix}$$

Think About It

5. Which of the following operations can we perform on a matrix A of *any* dimension?

(a) $A + A$ **(b)** $2A$ **(c)** AA

6. If A is a matrix for which AA is defined, what must be true about the dimension of A?

SKILLS

7–14 ■ Perform the matrix operation, or explain why the operation is not defined.

7. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

9. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

10. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

15–26 ■ Matrices A, B, C, D, E, F, G, and H are defined as follows. Perform the indicated matrix operation(s), or explain why it cannot be performed.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix} \qquad H = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$$

15. **(a)** $B + C$ **(b)** $B + F$

16. **(a)** $C - B$ **(b)** $2C - 6B$

17. **(a)** $5A$ **(b)** $C - 5A$

18. **(a)** $3B + 2C$ **(b)** $2H + D$

19. **(a)** AD **(b)** DA

20. **(a)** DH **(b)** HD

21. **(a)** AH **(b)** HA

22. **(a)** BC **(b)** BF

23. **(a)** GF **(b)** GE

24. **(a)** B^2 **(b)** F^2

25. **(a)** A^2 **(b)** A^3

26. **(a)** $(DA)B$ **(b)** $D(AB)$

CONTEXTS



27. Fast-Food Sales A small fast-food chain with restaurants in Santa Monica, Long Beach, and Anaheim sells only hamburgers, hot dogs, and milk shakes. On a certain day, sales were distributed according to matrix A. The price of each item (in dollars) is given by matrix C.

(a) Calculate the product matrix CA.

(b) What is the total profit from the Long Beach restaurant?

(c) What is the total profit (from all three restaurants)?

Number of items sold

	Santa Monica	Long Beach	Anaheim	
$A =$	4000	1000	3500	← Hamburgers
	400	300	200	← Hot dogs
	700	500	9000	← Milk shakes

Item

Item

Hamburger	Hot dog	Milk shake	
$C =$ [0.90	0.80	1.10]	← Price

28. Car-Manufacturing Profits A specialty car manufacturer has plants in Auburn, Biloxi, and Chattanooga. Three models are produced, with daily production given in matrix A. Because of a wage increase, February profits are lower than January profits. The profit (in dollars) per car is tabulated by model in matrix B.

(a) Calculate the product AB.

(b) Assuming that all cars produced were sold, what was the daily profit in January from the Biloxi plant?

(c) What was the total daily profit (from all three plants) in February?

$$A = \begin{array}{c} \text{Cars produced each day} \\ \text{Model K} \quad \text{Model R} \quad \text{Model W} \\ \downarrow \qquad \downarrow \qquad \downarrow \\ \begin{bmatrix} 12 & 10 & 0 \\ 4 & 4 & 20 \\ 8 & 9 & 12 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Auburn} \\ \leftarrow \text{Biloxi} \\ \leftarrow \text{Chattanooga} \end{array} \quad \text{City}$$

$$B = \begin{array}{c} \text{January} \quad \text{February} \\ \begin{bmatrix} 1000 & 500 \\ 2000 & 1200 \\ 1500 & 1000 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Model K} \\ \leftarrow \text{Model R} \\ \leftarrow \text{Model W} \end{array}$$

29. Canning Tomato Products Jaeger Foods produces tomato sauce and tomato paste, canned in small, medium, large, and giant sized cans. The matrix A gives the size (in ounces) of each container. The matrix B tabulates one day's production of tomato sauce and tomato paste.

(a) Calculate the product AB.

(b) Interpret the entries in the product matrix AB.

$$A = \begin{array}{c} \text{Size} \\ \text{Small} \quad \text{Medium} \quad \text{Large} \quad \text{Giant} \\ \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \\ \begin{bmatrix} 6 & 10 & 14 & 28 \end{bmatrix} \end{array} \leftarrow \text{Ounces}$$

$$B = \begin{array}{c} \text{Cans of sauce} \quad \text{Cans of paste} \\ \downarrow \qquad \downarrow \\ \begin{bmatrix} 2000 & 2500 \\ 3000 & 1500 \\ 2500 & 1000 \\ 1000 & 500 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Small} \\ \leftarrow \text{Medium} \\ \leftarrow \text{Large} \\ \leftarrow \text{Giant} \end{array}$$

30. Produce Sales A farmer's three children, Amy, Beth, and Chad, run three roadside produce stands during the summer months. One weekend they all sell watermelons, yellow squash, and tomatoes. Matrices A and B tabulate the number of pounds of each product sold by each sibling on Saturday and Sunday. Matrix C gives the price per pound (in dollars) for each type of produce that they sell. Perform each of the following matrix operations, and interpret the entries in each result.

(a) AC **(b)** BC **(c)** $A + B$ **(d)** $(A + B)C$

$$A = \begin{array}{c} \text{Saturday} \\ \text{Melons} \quad \text{Squash} \quad \text{Tomatoes} \\ \downarrow \qquad \downarrow \qquad \downarrow \\ \begin{bmatrix} 120 & 50 & 60 \\ 40 & 25 & 30 \\ 60 & 30 & 20 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Amy} \\ \leftarrow \text{Beth} \\ \leftarrow \text{Chad} \end{array}$$

$$B = \begin{array}{c} \text{Sunday} \\ \text{Melons} \quad \text{Squash} \quad \text{Tomatoes} \\ \downarrow \qquad \downarrow \qquad \downarrow \\ \begin{bmatrix} 100 & 60 & 30 \\ 35 & 20 & 20 \\ 60 & 25 & 30 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Amy} \\ \leftarrow \text{Beth} \\ \leftarrow \text{Chad} \end{array}$$

$$C = \begin{array}{c} \text{Price per pound} \\ \begin{bmatrix} 0.10 \\ 0.50 \\ 1.00 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Melons} \\ \leftarrow \text{Squash} \\ \leftarrow \text{Tomatoes} \end{array}$$

7.6 Matrix Equations: Solving a Linear System

- The Inverse of a Matrix
- Matrix Equations
- Modeling with Matrix Equations

IN THIS SECTION... we express a system of equations as a single *matrix equation* and then solve the system by solving the matrix equation.

In the preceding section we saw that when the dimensions are appropriate, matrices can be added, subtracted, and multiplied. In this section we investigate "division" of matrices. With this operation we can solve equations that involve matrices.

The Inverse of a Matrix

First, we define *identity matrices,* which play the same role for matrix multiplication that the number 1 does for ordinary multiplication of numbers; that is, $1 \cdot a = a \cdot 1 = a$ for all numbers a. In the following definition the term **main diagonal** refers to the entries of a square matrix whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

Identity Matrix

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus, the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad B \cdot I_n = B$$

whenever these products are defined.

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$

If A and B are $n \times n$ matrices, and if $AB = BA = I_n$, then we say that B is the *inverse* of A, and we write $B = A^{-1}$. The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

Inverse of a Matrix

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A.

example 1 Verifying That a Matrix Is an Inverse

Verify that matrix B is the inverse of matrix A.

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Solution

We perform the matrix multiplications to show that $AB = I$ and $BA = I$.

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 1(-5) & 2(-1) + 1\cdot 2 \\ 5\cdot 3 + 3(-5) & 5(-1) + 3\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + (-1)5 & 3\cdot 1 + (-1)3 \\ (-5)2 + 2\cdot 5 & (-5)1 + 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■ **NOW TRY EXERCISE 3** ■

The inverse of a matrix can be found by using a process that involves elementary row operations. This process is programmed into graphing calculators. On TI-83 calculators, matrices are stored in memory by using names such as [A], [B], [C], and so on. To find the inverse of [A], we key in

[A] x^{-1} ENTER

example 2 Finding the Inverse of a Matrix

Find the inverse of matrix A.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

Solution

A graphing calculator gives the following output for the inverse of matrix A.

Calculator output

```
[A]^-1 ▶Frac
    [[-3  2  0    ]
     [-4  1  -2/3]
     [1   0  1/3  ]]
```

Inverse of matrix *A*

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

We have used the ▶Frac command to display the output in fraction form rather than in decimal form.

NOW TRY EXERCISES 7 AND 13

Matrix Equations

A system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

A X B

If we let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

then this matrix equation can be written as

$$AX = B$$

Matrix A is called the **coefficient matrix**.

We solve this matrix equation by multiplying each side by the inverse of A (provided that this inverse exists).

$AX = B$	Given equation
$A^{-1}(AX) = A^{-1}B$	Multiply on left by A^{-1}
$(A^{-1}A)X = A^{-1}B$	Associative Property
$I_3X = A^{-1}B$	Property of inverses
$X = A^{-1}B$	Property of the identity matrix

Solving the matrix equation $AX = B$ is very similar to solving the simple real-number equation

$$3x = 12$$

which we do by multiplying each side by the reciprocal (or inverse) of 3:

$$\tfrac{1}{3}(3x) = \tfrac{1}{3}(12)$$

$$x = 4$$

We have proved that the matrix equation $AX = B$ can be solved by the following method.

Solving a Matrix Equation

If A is a square $n \times n$ matrix that has an inverse A^{-1}, and if X is a variable matrix and B a known matrix, both with n rows and one column, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

example 3 Solving a System as a Matrix Equation

Solve the system of equations.

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

Solution

The system is equivalent to the matrix equation $AX = B$ described on page 621.

Using a calculator, we find that

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

So because $X = A^{-1}B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$$

$X = A^{-1} B$

So the solution of the system is $(-11, -23, 7)$.

■ NOW TRY EXERCISES 19 AND 23 ■

example 4 Solving a System Using a Matrix Inverse

A system of equations is given.

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

$$\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$$

Solution

(a) We write the system as a matrix equation of the form $AX = B$.

$$\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 36 \end{bmatrix}$$

$$A \quad X = B$$

(b) Using a graphing calculator to calculate the inverse, we get

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & \frac{5}{3} \\ -1 & \frac{2}{3} \end{bmatrix}$$

So because $X = A^{-1}B$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & \frac{5}{3} \\ -1 & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 15 \\ 36 \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$$

$$X = A^{-1} \quad B$$

So the solution of the system is (30, 9).

NOW TRY EXERCISE 19

Not every square matrix has an inverse. A matrix that has no inverse is called *singular*.

example 5 A Matrix That Does Not Have an Inverse

Find the inverse of the matrix.

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution

If we try to calculate the inverse of this matrix on the TI-83 calculator, we get the error message shown below. That means that this matrix has no inverse.

Calculator output	Inverse of matrix *A*
ERR:SINGULAR MAT 1:Quit 2:Goto	The matrix A does not have an inverse.

NOW TRY EXERCISE 15

Modeling with Matrix Equations

Suppose we need to solve several systems of equations with the same coefficient matrix. Then expressing the systems as matrix equations provides an efficient way to obtain the solutions, because we need to find the inverse of the coefficient matrix only once. This procedure is particularly convenient if we use a graphing calculator to perform the matrix operations, as in the next example.

example 6 Solving a System Using a Matrix Inverse



A pet store owner feeds his hamsters and gerbils different mixtures of three types of rodent food: KayDee Food, Pet Pellets, and Rodent Chow. The protein, fat, and carbohydrates content (in mg) in one gram of each brand is given in the table below.

- Hamsters need 340 mg of protein, 280 mg of fat, and 440 mg of carbohydrates each day.
- Gerbils need 480 mg of protein, 360 mg of fat, and 680 mg of carbohydrates each day.

The pet store owner wishes to feed his animals the correct amount of each brand to satisfy their daily requirements exactly. How many grams of each food should the storekeeper feed his hamsters and gerbils daily to satisfy their nutrient requirements?

	KayDee Food	Pet Pellets	Rodent Chow
Protein (mg)	10	0	20
Fat (mg)	10	20	10
Carbohydrates (mg)	5	10	30

Solution

We let x, y, and z be the respective amounts (in grams) of KayDee Food, Pet Pellets, and Rodent Chow that the hamsters should eat and r, s, and t be the corresponding amounts for the gerbils. Then we want to solve the matrix equations

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \quad \text{Hamster Equation}$$

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \quad \text{Gerbil Equation}$$

Let

$$A = \begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \quad B = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \quad C = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad Y = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

Then we can write these matrix equations as

$$AX = B \quad \text{Hamster Equation}$$

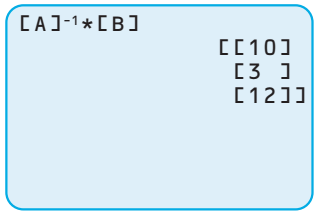
$$AY = C \quad \text{Gerbil Equation}$$

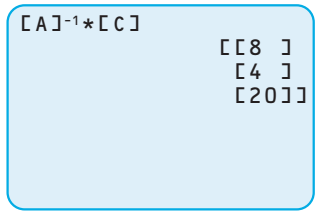
Solving for X and Y, we have

$$X = A^{-1}B \qquad \text{Hamster Equation}$$

$$Y = A^{-1}C \qquad \text{Gerbil Equation}$$

Using a graphing calculator, we find matrices X and Y.





Thus each hamster should be fed 10 g of KayDee Food, 3 g of Pet Pellets, and 12 g of Rodent Chow, and each gerbil should be fed 8 g of KayDee Food, 4 g of Pet Pellets, and 20 g of Rodent Chow daily.

NOW TRY EXERCISE 27

7.6 Exercises

CONCEPTS

Fundamentals

1. **(a)** The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called an _____ matrix.

(b) If A is a 2×2 matrix, then $A \times I =$ _____ and $I \times A =$ _____.

(c) If A and B are 2×2 matrices with $AB = I$, then B is the _____ of A.

2. **(a)** Write the following system as a matrix equation $AX = B$.

System	Matrix equation
	$A \cdot X = B$
$\begin{cases} 5x + 3y = 4 \\ 3x + 2y = 3 \end{cases}$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}\begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$

(b) The inverse of A is $A^{-1} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

(c) The solution of the matrix equation is $X = A^{-1}B$.

$$X = A^{-1} \cdot B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}\begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$$

(d) The solution of the system is $x =$ _____, $y =$ _____.

SKILLS

3–6 ■ Calculate the products AB and BA to verify that matrix B is the inverse of matrix A.

3. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix}$ $B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

7–18 ■ Use a calculator that can perform matrix operations to find the inverse of the matrix, if it exists.

7. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

10. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

11. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

19–26 ■ Solve the system of linear equations by expressing the system as a matrix equation and using the inverse of the coefficient matrix. Use the inverses you found in Exercises 7–10, and 13, 14, 17, and 18.

19. $\begin{cases} -3x - 5y = 4 \\ 2x + 3y = 0 \end{cases}$

20. $\begin{cases} 3x + 4y = 10 \\ 7x + 9y = 20 \end{cases}$

21. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$

22. $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

23. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$

24. $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

25. $\begin{cases} -2y + 2z = 12 \\ 3x + y + 3z = -2 \\ x - 2y + 3z = 8 \end{cases}$

26. $\begin{cases} x + 2y + 3w = 0 \\ y + z + w = 1 \\ y + w = 2 \\ x + 2y + 2w = 3 \end{cases}$

CONTEXTS

27. Sales Commissions An encyclopedia salesperson works for a company that offers three different grades of bindings for its encyclopedias: standard, deluxe, and leather. For each set that she sells, she earns a commission based on the set's binding grade. One week she sells one standard set, one deluxe set, and two leather sets and makes $675 in commission. The next week she sells two standard sets, one deluxe set, and one leather set for a $600 commission. The third week she sells one standard set, two deluxe sets, and one leather set, earning $625 in commission.

(a) Let x, y, and z represent the commission she earns on standard, deluxe, and leather sets, respectively. Translate the given information into a system of equations.

(b) Express the system of equations you found in part (a) as a matrix equation of the form $AX = B$.

(c) Find the inverse of the coefficient matrix A, and use it to solve the matrix equation in part (b). How much commission does the salesperson earn on a set of encyclopedias in each grade of binding?

28. Nutrition A nutritionist is studying the effects of the nutrients folic acid, choline, and inositol. He has three types of food available, and each type contains the amounts of these nutrients per ounce shown in the table.

	Type A	Type B	Type C
Folic acid (mg)	3	1	3
Choline (mg)	4	2	4
Inositol (mg)	3	2	4

Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$



(a) Find the inverse of matrix A.

(b) How many ounces of each food should the nutritionist feed his laboratory rats if he wants each rat's daily diet to contain 10 mg of folic acid, 14 mg of choline, and 13 mg of inositol?

(c) How many ounces of each food should the nutritionist feed his laboratory rats if he wants each rat's daily diet to contain 9 mg of folic acid, 12 mg of choline, and 10 mg of inositol?

CHAPTER 7 REVIEW

CHAPTER 7 CONCEPT CHECK

Make sure you understand each of the ideas and concepts that you learned in this chapter, as detailed below section by section. If you need to review any of these ideas, reread the appropriate section, paying special attention to the examples.

7.1 Systems of Linear Equations in Two Variables

A **system of equations** is a set of equations in which each equation involves the same variables. A **solution** of a system is a set of values for the variables that makes each equation true.

A system of two equations in two variables can be solved by the **substitution method** or the **elimination method**. (The details of these methods are described on pages 569 and 570.)

The graph of a system of two linear equations in two variables is a pair of lines; the solution of the system corresponds to the point of intersection of the lines. A two-variable linear system has either:

- **One solution** if the lines have different slopes.
- **No solution** if the lines are different but have the same slope. (That is, the lines are parallel.)
- **Infinitely many solutions** if the lines are the same.

7.2 Systems of Linear Equations in Several Variables

To solve a linear system involving several equations and variables, we use **Gaussian elimination** to put the system in **triangular form** and then use **back-substitution** to get the solution. The following **elementary row operations** are used in the Gaussian elimination process:

- Add a nonzero multiple of one equation to another.
- Multiply an equation by a nonzero constant.
- Interchange the positions of two equations.

A system of linear equations can have one solution, no solution, or infinitely many solutions. A system with no solution is said to be **inconsistent**, and a system with infinitely many solutions is said to be **dependent**. If we apply Gaussian elimination to a system and arrive at a false equation, then the system is inconsistent. The solution of a dependent system can be described by using one or more **parameters**.

7.3 Using Matrices to Solve Systems of Equations

A **matrix** is a rectangular array of numbers. For instance, here is a matrix with three rows and four columns:

$$\begin{bmatrix} 6 & -2 & 4 & 0 \\ -5 & 7 & 1 & -3 \\ 9 & -4 & \frac{1}{2} & 8 \end{bmatrix}$$

A matrix with m **rows** and n **columns** is said to have **dimension** $m \times n$; the above matrix has dimension 3×4. The numbers in the matrix are its **entries**; the **(i,j) entry** is the number in row i and column j. So in the above matrix the (2, 3) entry is 1, and the (3, 2) entry is -4.

The **augmented matrix** of a system of linear equations is the matrix that is obtained by writing the coefficients of the variables and the constants in each equation in matrix form, omitting the symbols for the variables and the equal signs. For example, the above matrix is the augmented matrix for the system

$$\begin{cases} 6x - 2y + 4z = 0 \\ -5x + 7y + z = -3 \\ 9x - 4y + \frac{1}{2}z = 8 \end{cases}$$

To solve a linear system, we apply elementary row operations to the augmented matrix of the system, transforming it into an equivalent system in *row-echelon* form. A matrix is in **row-echelon form** if it satisfies the following conditions:

- The first nonzero number (called the **leading entry**) is 1.

- The leading entry in each row is to the right of the leading entry in the row above it.
- All rows that consist entirely of zeros are at the bottom of the matrix.

If the matrix also satisfies the following condition, then it is in **reduced row-echelon form**:

- Every number above and below each leading entry is 0.

Graphing calculators have commands that put matrices into row-echelon and reduced row-echelon form (`ref` and `rref` on the TI-83 calculator).

If an augmented matrix in row-echelon form has a row equivalent to the equation $0 = 1$, then the system is **inconsistent**; this means the system has no solutions. If the augmented matrix in row-echelon form is not inconsistent but has fewer nonzero rows than variables, then the system is **dependent**; this means that the system has infinitely many solutions.

7.4 Matrices and Data in Categories

Matrices can be used to organize and analyze data about two different **categorical** characteristics of a population. For instance, if you gather data about the eye color (brown, blue, or green) and the hair color (black, brown, blond, or red) of the members of your class, then the numbers can be conveniently presented as a 3×4 matrix in which the rows represent eye color and the columns represent hair color. (For example, the (3, 4) entry would be the number of green-eyed redheads in the class.)

To combine analogous data matrices for two different populations, we **add** the matrices by adding corresponding entries.

7.5 Matrix Operations: Getting Information From Data

Suppose that A and B are two matrices of the same dimension $m \times n$ and that c is a real number.

- The **sum** $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries of A and B.
- The **scalar multiple** cA is the $m \times n$ matrix obtained by multiplying each entry in A by the number c.

The **product** AB of two matrices A and B is defined if the number of columns of A is the same as the number of rows of B. (The procedure for multiplying two matrices is described on pages 612–615.) Matrix products can be used to extract information from sets of categorical data.

7.6 Matrix Equations: Solving a Linear System

The **identity matrix** I_n is the square $n \times n$ matrix for which each **main diagonal** entry (from the top left corner to the bottom right corner) is a 1 and every other entry is a 0. For instance, the 3×3 identity matrix is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When a matrix is multiplied by an identity matrix I of the appropriate dimension, it remains unchanged:

$$AI = A \qquad \text{and} \qquad IB = B$$

If A is a square $n \times n$ matrix and if there exists an $n \times n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I_n$$

then we say that A^{-1} is the **inverse** of A.

A system of equations can be written as a **matrix equation** of the form $AX = B$. To solve a matrix equation, we multiply both sides by A^{-1}:

$$X = A^{-1}B$$

CHAPTER 7 REVIEW EXERCISES

SKILLS

1–4 ■ Graph each linear system, either by hand or by using a graphing device. Use the graph to determine whether the system has one solution, no solution, or infinitely many solutions. If there is exactly one solution, use the graph to find it.

1. $\begin{cases} 3x - y = 5 \\ 2x + y = 5 \end{cases}$

2. $\begin{cases} y = 2x + 6 \\ y = -x + 3 \end{cases}$

3. $\begin{cases} 6x - 8y = 15 \\ -\frac{3}{2}x + 2y = -4 \end{cases}$

4. $\begin{cases} 2x - 7y = 28 \\ y = \frac{2}{7}x - 4 \end{cases}$

5–8 ■ Solve the linear system, or show that it has no solution. (Use either the elimination or the substitution method.)

5. $\begin{cases} 2x + 3y = 7 \\ x - 2y = 0 \end{cases}$

6. $\begin{cases} 2x + 5y = 9 \\ -x + 3y = 1 \end{cases}$

7. $\begin{cases} 3x - y = 12 \\ -x + \frac{1}{3}y = -4 \end{cases}$

8. $\begin{cases} 6x + 9y = 5 \\ 4x + 6y = 3 \end{cases}$

9–16 ■ Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

9. $\begin{cases} x + y + 2z = 6 \\ 2x + 5z = 12 \\ x + 2y + 3z = 9 \end{cases}$

10. $\begin{cases} x - 2y + 3z = 1 \\ x - 3y - z = 0 \\ 2x - 6z = 6 \end{cases}$

11. $\begin{cases} x + 2y + 2z = 6 \\ x - y = -1 \\ 2x + y + 3z = 7 \end{cases}$

12. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 2y + 3z = 5 \end{cases}$

13. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 3x - y + 5z = 10 \end{cases}$

14. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y - 4z = -3 \\ 3x + 6y - 3z = 4 \end{cases}$

15. $\begin{cases} x - 2y + 3z = -2 \\ 2x - y + z = 2 \\ 2x - 7y + 11z = -9 \end{cases}$

16. $\begin{cases} x - y + z = 0 \\ 3x + 2y - z = 6 \\ x + 4y - 3z = 3 \end{cases}$

17–18 ■ The augmented matrix of a linear system is given. Use a graphing calculator to put the matrix into reduced row-echelon form, and then find the complete solution of the system. (Assume that the variables in the system are x, y, and z.)

17. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 4 & 8 \\ 1 & 0 & 2 & 7 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 1 & 1 & -3 & 4 \\ 2 & -2 & 5 & 4 \end{bmatrix}$

19–30 ■ Let A, B, C, D, E, and F be the given matrices. Carry out the indicated operations, or explain why they cannot be performed.

$$A = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{3}{2} \\ -2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad F = \begin{bmatrix} 4 & 0 & 2 \\ -1 & 1 & 0 \\ 7 & 5 & 0 \end{bmatrix}$$

19. $A + B$ **20.** $C - D$ **21.** $2C + 3D$

22. $5B - 2C$ **23.** FA **24.** AF

25. BC **26.** CB **27.** BF

28. FC **29.** $(C + D)E$ **30.** $F(2C - D)$

31–32 ■ Verify that the matrices A and B are inverses of each other by calculating the products AB and BA.

31. $A = \begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 3 & \frac{5}{2} \\ 1 & 1 \end{bmatrix}$ **32.** $A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -\frac{3}{2} & 2 & \frac{5}{2} \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

33–36 ■ Use a graphing calculator to find the inverse of the matrix.

33. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

34. $\begin{bmatrix} 7 & -15 \\ 5 & -11 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

37–40 ■ Express the system of linear equations as a matrix equation. Then solve the matrix equation by multiplying each side by the inverse of the coefficient matrix.

37. $\begin{cases} 12x - 5y = 10 \\ 5x - 2y = 17 \end{cases}$

38. $\begin{cases} 6x - 5y = 1 \\ 8x - 7y = -1 \end{cases}$

39. $\begin{cases} 2x + y + 5z = \frac{1}{3} \\ x + 2y + 2z = \frac{1}{4} \\ x + 3z = \frac{1}{6} \end{cases}$

40. $\begin{cases} 2x + 3z = 5 \\ x + y + 6z = 0 \\ 3x - y + z = 5 \end{cases}$

CONNECTING THE CONCEPTS

These exercises test your understanding by combining ideas from several sections in a single problem.

41. Solving a Linear System in Six Different Ways We have learned several different methods for solving linear systems of equations. In this problem we compare these methods by solving the following system in six different ways:

$$\begin{cases} x - 3y = 2 \\ 2x - 5y = 6 \end{cases}$$

(a) Solve the system by *graphing* both lines on a graphing calculator and then using the TRACE feature to find the intersection point of the lines.

(b) Solve the system using *substitution*.

(c) Solve the system using *elimination*.

(d) Find the augmented matrix of the system, and use *Gaussian elimination* on the matrix to solve the system.

(e) Find the *reduced row-echelon form* of your augmented matrix in part (d), and use it to solve the system.

(f) Write the system as a *matrix equation* of the form $AX = B$, and solve the system by multiplying both sides of the equation by A^{-1}.

(g) Which of these six methods did you find simplest to use? Which was the most complicated? Check that you got the same answer with all six methods.

42. Solving a Linear System in Three Different Ways Not all of the six methods described in Exercise 41, parts (a)–(f), work for a linear system that has three equations in three unknowns.

(a) Which methods do not apply? Why?

(b) Solve the following system by the three methods from Exercise 41 (a)–(f) that *do* work in solving such a system. Which method do you prefer?

$$\begin{cases} x + 2y - z = 1 \\ x - y + 2z = 2 \\ x + 2y + z = 4 \end{cases}$$

CONTEXTS

43. Children's Ages Eleanor has two children: Kieran and Siobhan. Kieran is 4 years older than Siobhan, and the sum of their ages is 22.

(a) Let x be Kieran's age, and let y be Siobhan's. Find a system of linear equations that models the facts given about the children's ages.

(b) Solve the system. How old is each child?

44. Interest on Bank Accounts Clarisse invests $60,000 in money-market accounts at three different banks. Bank A pays 2% interest per year, Bank B pays 2.5%, and Bank C pays 3%. She decides to invest twice as much in Bank B as in the other two banks. After one year, Clarisse has earned $1575 in interest.

(a) Let x, y, and z be the amounts that Clarisse invests in Banks A, B, and C, respectively. Find a system of linear equations that models the facts given about the amounts invested in each bank.

(b) Solve the system. How much did Clarisse invest in each bank?

45. A Fisherman's Catch A commercial fisherman fishes for haddock, sea bass, and red snapper. He is paid $1.25 a pound for haddock, $0.75 a pound for sea bass, and $2.00 a pound for red snapper. Yesterday he caught 560 pounds of fish worth $575. The haddock and red snapper together are worth $320. How many pounds of each fish did he catch?

46. A Vegetable Stand Rhonna and Ivan grow tomatoes, onions, and zucchini in their backyard and sell them at a roadside stand on Saturdays and Sundays. They price tomatoes at $1.50 per pound, onions at $1.00 per pound, and zucchini at 50 cents per pound. The following table shows the number of pounds of each type of produce that they sold during the last weekend in July.

	Tomatoes	Onions	Zucchini
Saturday	25	16	30
Sunday	14	12	16

(a) Let

$$A = \begin{bmatrix} 25 & 16 & 30 \\ 14 & 12 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.50 \\ 1.00 \\ 0.50 \end{bmatrix}$$

Compare these matrices to the data given in the problem, and describe what the matrix entries represent.

(b) Only one of the products AB or BA is defined. Calculate the product that *is* defined, and describe what its entries represent.

47. Automatic Teller Machine An ATM at a bank in Qualicum Beach, British Columbia, dispenses $20 and $50 bills. Jo Ann withdraws $600 from this machine and receives a total of 18 bills. Let x be the number of $20 bills, and let y be the number of $50 bills that she receives.

(a) Find a system of two linear equations in x and y that expresses the information given in the problem.

(b) Write your linear system as a matrix equation of the form $AX = B$.

(c) Find A^{-1}, and use it to solve your matrix equation in part (b). How many bills of each type did Jo Ann receive?

CHAPTER 7 TEST

1. Find all solutions of the linear system:

$$\begin{cases} 3x + 5y = 4 \\ x - 4y = 7 \end{cases}$$

2. In $2\frac{1}{2}$ hours an airplane travels 600 miles against the wind. It takes 50 minutes to travel 300 miles with the wind. Find the speed of the wind and the speed of the airplane in still air.

3–5 ■ A system of linear equations is given.

(a) Find the complete solution of the system, or show that there is no solution. You may use either Gaussian elimination or the reduced row-echelon form of the augmented matrix of the system (using a graphing calculator).

(b) State whether the system is inconsistent, dependent, or neither.

3.
$$\begin{cases} x + 2y + z = 3 \\ x + 3y + 2z = 3 \\ 2x + 3y - z = 8 \end{cases}$$

4.
$$\begin{cases} x - y + 9z = -8 \\ -4z = 7 \\ 3x - y + z = 5 \end{cases}$$

5.
$$\begin{cases} 2x - y + z = 0 \\ 3x + 2y - 3z = 1 \\ x - 4y + 5z = -1 \end{cases}$$

6. Anne, Barry, and Cathy enter a coffee shop.

- Anne orders two coffees, one juice, and two doughnuts and pays $6.25.
- Barry orders one coffee and three doughnuts and pays $3.75.
- Cathy orders three coffees, one juice, and four doughnuts and pays $9.25.

Find the price of coffee, juice, and doughnuts at this coffee shop.

7. Let A, B, C, and D be the given matrices. Carry out the indicated operations, or explain why they cannot be performed.

$$A = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & \frac{1}{2} \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} -3 & 2 \\ \frac{3}{2} & 1 \\ -2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

(a) $A + B$ **(b)** $B - C$ **(c)** $3D$

(d) BC **(e)** AD **(f)** DA

8. A system of linear equations is given:

$$\begin{cases} 4x - 3y = 10 \\ 3x - 2y = 30 \end{cases}$$

(a) Write the system as a matrix equation of the form $AX = B$.

(b) Find A^{-1} and solve the system by multiplying both sides of the matrix equation by A^{-1}.

1 Collecting Categorical Data

OBJECTIVE To experience the process of collecting categorical data and then organizing the data in a matrix.


Christopher Futcher/Shutterstock.com 2009

A society is made up of many individuals. Since each individual is unique, how can we obtain useful information about a society as a whole? Many surveys are conducted each year to learn of different trends or changes in the makeup of a society. These surveys seek to place individuals in broad categories—for example, categories of income, education, political affiliation, and so on. One of the most famous such surveys is the General Social Survey (GSS) conducted every other year by the University of Chicago. The survey collects data on demographic characteristics and attitudes of residents of the United States. The results of the survey are available to the public over the Internet. The data are widely used by sociologists for detecting social changes and trends in the population of the United States.

In this exploration we experience the process of collecting categorical data on a somewhat smaller scale—from our classmates.

I. Collecting Data

To collect data from our classmates, we take a survey.

1. Make a survey that asks any two of the following questions. (Each of the questions has categorical responses.) Or make up your own questions; you can get ideas on other questions to ask by viewing the questionnaire from the General Social Survey on the Internet. Note that each survey question can be answered in several categories (let's call them (a), (b), (c), (d), . . .).
   - How many children are in your family? (2 or fewer, between 3 and 5 (inclusive), more than 5)
   - What do you think is the ideal number of children in a family? (0, 1, 2, 3, 4 or more).
   - Are you happy with your schoolwork so far? (happy, neutral, unhappy)
   - How do you get information about events in the news? (newspapers, magazines, Internet, TV, radio)
   - How do you think the courts deal with criminals? (too harshly, just right, not harshly enough)
   - How well informed are you about economic policy? (very informed, somewhat informed, uninformed)
   - How well informed are you about foreign policy? (very informed, somewhat informed, uninformed)
   - How well informed are you about global warming? (very informed, somewhat informed, uninformed)
   - How much would it bother you if polar bears became extinct? (a great deal, somewhat, very little, not at all)
   - How much would it bother you if sea levels rose by more than 20 feet? (a great deal, somewhat, very little, not at all)
   - Do you think the universe began with a huge explosion? (yes, no, don't know)

II. Organizing the Data

We can report the data for each question separately, by reporting the number of responses for each response category [(a), (b), (c), (d), . . .]. Or we can relate the responses to the two questions by putting the results in a crosstab matrix. From this matrix we can find out, for example, the number of respondents who answered (a) to Question 1 and (c) to Question 2.

1. Report the result of your survey for each question separately. First, state each question, and then report the number of students who answered (a), (b), (c), or (d) for that question.

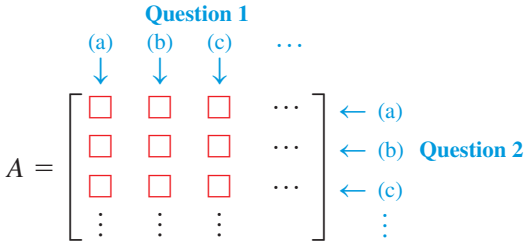
Question 1 _____

	(a)	**(b)**	**(c)**	⋯
Number of respondents				

Question 2 _____

	(a)	**(b)**	**(c)**	⋯
Number of respondents				

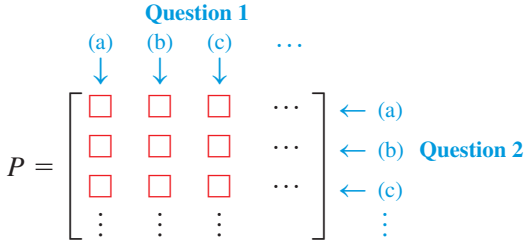
2. Use your surveys to make a cross-tab matrix of the answers to Questions 1 and 2.



3. Let's make a proportion matrix of the results of the survey.

(a) Express each entry in the matrix in Question 3 as a proportion of the column total. Call the resulting matrix P.

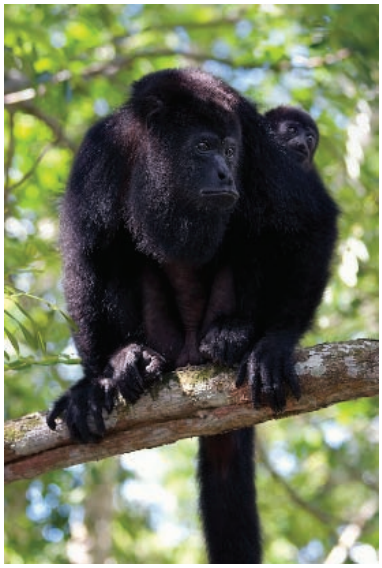
(b) What does the (1, 1) entry in matrix P represent? What about the (2, 3) entry?



(c) Use the entries in matrix P to state some conclusions that you can draw from your survey. (For example, you might find that 20% of those who think the universe began in a big explosion don't care at all if polar bears become extinct.)

2 Will the Species Survive?

OBJECTIVE Learn to use a transition matrix to predict population distributions for future generations.



Christopher Marin/Shutterstock.com 2009

Howler monkeys

To study how species survive, scientists model species populations by observing the different stages in the life of the particular species. For example, in modeling a population of howler monkeys in Mexico, scientists identified three distinct stages in the life of these monkeys: immature, juvenile, and adult.*

In general, scientists consider the stage at which an animal is fertile, the proportion of the population that reproduces, the proportion of the young that survives, the proportion of juveniles that survives to adulthood, and so on. All this information is conveniently stored in a matrix, called a *transition matrix*. By using the transition matrix, it is possible to predict population distribution in future seasons.

Modeling populations in this way is one of the most powerful tools in conservation biology. It helps scientists assess threats to species survival and intervene to reduce the risk of extinction.

I. A Transition Matrix

For a certain species there are three stages: immature, juvenile, and adult. An animal is considered immature for the first year of its life, juvenile for the second year, and an adult from then on. Conservation biologists have collected the field data for this species recorded in the matrix below.

The entries in matrix P indicate the proportion of the population that survives *to the next year*. For example, the first column describes what happens to the immature population after 1 year.

$$P = \begin{array}{c} \begin{array}{ccc} \text{Immature} & \text{Juvenile} & \text{Adult} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{Immature} \\ \leftarrow \text{Juvenile} \\ \leftarrow \text{Adult} \end{array}$$

Year 0 (Immature, Juvenile, Adult) — Year 1 (Immature, Juvenile, Adult)

- **The (1, 1) entry is 0.0.** This indicates that none of the immature remain immature.
- **The (2, 1) entry is 0.1.** This indicates that 10% of the immature survive to become juveniles.

*Salvador Mandjano and Louis A. Escobedo-Morales, "Population Viability Analysis of Howler Monkeys (*Alouatta paliata mexicana*) in a Highly Fragmented Landscape in Los Tuxtlas, Mexico," *Tropical Conservation Science, 1*: 43–62, 2008.

- **The (3, 1) entry is 0.0.** This indicates that none of the immature become adults (of course, because they first have to become juveniles).

1. Describe each entry in the Juvenile column.

- **The (1, 2) entry is 0.0.** This indicates that no juveniles become immature.
- **The (2, 2) entry is 0.0.** This indicates that
- **The (3, 2) entry is 0.3.** This indicates that

2. Describe the entries in the Adult column.

- **The (1, 3) entry is 0.4.** This indicates that
- **The (2, 3) entry is 0.0.** This indicates that
- **The (3, 3) entry is 0.8.** This indicates that

II. Predicting Population

The population of the above species in Year 0 is given in the column matrix X_0:

$$X_0 = \begin{bmatrix} 600 \\ 400 \\ 3500 \end{bmatrix} \begin{matrix} \leftarrow \text{Immature} \\ \leftarrow \text{Juvenile} \\ \leftarrow \text{Adult} \end{matrix}$$

Let $X_1 = PX_0$, $X_2 = PX_1$, $X_3 = PX_2$, and so on.

1. Find matrices $X_1 = PX_0$, $X_2 = PX_1$, $X_3 = PX_2$, and $X_4 = PX_3$.

$$X_1 = PX_0 = \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 600 \\ 400 \\ 3500 \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$X_2 = PX_1 = \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

$$X_4 = PX_3 = \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$$

2. **(a)** Explain why matrix X_1 predicts the population in Year 1.

(b) Explain why matrix X_2 predicts the population in Year 2.

(c) Do you detect any population trends from the predictions you made in Question 1?

3. Show that $X_1 = PX_0$, $X_2 = P^2X_0$, $X_3 = P^3X_0$, $X_4 = P^4X_0$, ...

To calculate a power of a matrix on a T1 graphing calculator, use the usual power key: [^]. For example, A^{50} is entered as `[A]^50`.

4. Use the observation in the preceding example and a graphing calculator to predict the population of the species in Year 50 (that is, find $X_{50} = P^{50}X_0$). Does it appear that this species will survive?

III. Management of the Environment

Suppose that proper management of the environment has resulted in an enhanced habitat for the species. In this more appropriate habitat the proportion of immatures that become juveniles each year increases to 0.3 (from 0.1), the proportion of juveniles that become adults increases to 0.7 (from 0.3), and the proportion of adults that survives to the next year increases to 0.95 (from 0.80). All other proportions in matrix P on page 637 remain the same.

1. Find the new transition matrix P for the population of this species in the improved environment.

$$P = \begin{array}{c} \begin{matrix} \text{Immature} & \text{Juvenile} & \text{Adult} \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix} 0.0 & 0.0 & 0.4 \\ \square & 0.0 & 0.0 \\ 0.0 & \square & \square \end{bmatrix} \end{array} \begin{matrix} \\ \leftarrow \text{Immature} \\ \leftarrow \text{Juvenile} \\ \leftarrow \text{Adult} \end{matrix}$$

2. Suppose that the population of the species in year 0 is given by the same column matrix X_0 as in Part I. Predict the population of the species in Year 50 (that is, find $X_{50} = P^{50}X_0$). Does it now appear that this species will survive?

Algebra Toolkit A

Working with Numbers

A.1 Numbers and Their Properties
A.2 The Number Line and Intervals
A.3 Integer Exponents
A.4 Radicals and Rational Exponents

A.1 Numbers and Their Properties

- Operations on Real Numbers
- The Distributive Property
- The Distributive Property: Expanding
- The Distributive Property: Factoring

The different types of numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below zero temperatures, rational numbers for concepts such as "half a gallon of milk," and irrational numbers for measuring certain distances, such as the diagonal of a square.

Let's review the types of numbers that make up the real number system. We start with the **natural numbers**

$$1, 2, 3, 4, \ldots$$

The **integers** consist of the natural numbers together with their negatives and 0:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

The **rational numbers** are ratios of integers. So the following are rational numbers:

$$\frac{1}{2} \qquad -\frac{3}{7} \qquad 46 = \frac{46}{1} \qquad 0.17 = \frac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions such as $\frac{0}{0}$ and $\frac{3}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. The **real numbers** consist of all the different types of numbers together.

Operations on Real Numbers

Real numbers can be combined by using the familiar **operations** of addition, subtraction, multiplication, and division. When we multiply two numbers together, each of the numbers is called a **factor** of the product. When we add two numbers together, each number is called a **term** of the sum.

$$\underbrace{2 \times 3}_{\text{Factors}} \qquad \underbrace{2 + 3}_{\text{Terms}}$$

An expression involving numbers and operations on these numbers is called an **arithmetic expression**. For instance, the expressions in the next example are arithmetic expressions. When evaluating arithmetic expressions that contain several of these operations, we use the following conventions to determine the order in which the operations are performed:

1. Perform operations inside parentheses first, beginning with the innermost pair. In dividing two expressions, the numerator and denominator of the quotient are treated as if they are within parentheses.
2. Perform all multiplications and divisions, working from left to right.
3. Perform all additions and subtractions, working from left to right.

example 1 Evaluating an Arithmetic Expression

Evaluate the expression.

(a) $7(2 \cdot 3 + 5)$

(b) $10\left(\dfrac{8 + 10}{2 \cdot 3} + 4\right)$

Solution

(a) First we evaluate the expression inside the parentheses, remembering that multiplication is performed before addition.

$$\begin{aligned} 7(2 \cdot 3 + 5) &= 7(6 + 5) && \text{Perform multiplication inside parentheses} \\ &= 7(11) && \text{Perform addition inside parentheses} \\ &= 77 && \text{Calculate} \end{aligned}$$

(b) First we evaluate the expressions inside parentheses. The numerator and denominator of the quotient are treated as if they are inside parentheses:

$$\begin{aligned} 10\left(\frac{8 + 10}{2 \cdot 3} + 4\right) &= 10\left(\frac{18}{6} + 4\right) && \text{Perform operations in fraction} \\ &= 10(3 + 4) && \text{Perform division} \\ &= 10(7) && \text{Perform addition inside parentheses} \\ &= 70 && \text{Calculate} \end{aligned}$$

■ **NOW TRY EXERCISE 7** ■

The Distributive Property

We know that $2 + 3 = 3 + 2$, $5 + 7 = 7 + 5$, and so on. In algebra we express all these (infinitely many) facts by using letters to stand for numbers: If a and b stand for any two numbers, then $a + b = b + a$.

We also know that when we add three numbers, it doesn't matter which two we add first. These properties hold for multiplication as well. So for any real numbers a, b, and c we have the following properties of numbers.

- **Associative Properties:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
- **Commutative Properties:** $a + b = b + a$ and $ab = ba$

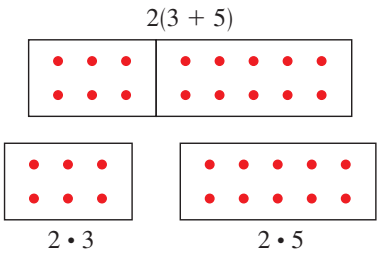
The associative and commutative properties tell us that we cannot make a mistake in the order in which we choose to multiply numbers or in the order in which we choose to add numbers.

But the really important property in algebra is the property that describes how addition and multiplication interact with each other.

The Distributive Property

If A, B, and C are any real numbers then

$$A(B + C) = AB + AC$$



In words the distributive property says that "multiplying a number by the sum of two numbers is the same as multiplying the number by each of the terms in the sum and adding the result." The diagram in the margin explains why this property works when the numbers are positive integers, but the property is true for any real numbers A, B, and C. When we use this property to multiply a number by a sum of numbers, we "distribute" the number over the sum. So to multiply $2(3 + 5)$, we distribute the 2 over the sum $3 + 5$:

$$2(3 + 5) = 2 \cdot 3 + 2 \cdot 5$$

example **2** **Using the Distributive Property**

Evaluate $2(3 + 5)$ directly, and then evaluate it using the Distributive Property. Check that the answers agree.

Solution

We first perform operations inside the parentheses.

$$\begin{aligned} 2(3 + 5) &= 2(8) && \text{Perform addition inside parentheses} \\ &= 16 && \text{Calculate} \end{aligned}$$

Using the Distributive Property, we write

$$\begin{aligned} 2(3 + 5) &= 2 \cdot 3 + 2 \cdot 5 && \text{Distributive Property} \\ &= 6 + 10 && \text{Calculate} \\ &= 16 && \text{Calculate} \end{aligned}$$

We get the same answer in both cases.

■ **NOW TRY EXERCISE 19** ■

The Distributive Property: Expanding

In algebra **we use letters to stand for numbers**; this allows us to find patterns in numbers. For example, when we write $x(3 + 2y)$, we mean that the letters x and y stand for any real numbers. To **expand** or "multiply out" $x(3 + 2y)$, we use the Distributive Property with $A = x$, $B = 3$, and $C = 2y$:

$$x(3 + 2y) = x \cdot 3 + x \cdot 2y = 3x + 2xy$$

EXPANDING →

So for *any* numbers x and y it is true that $x(3 + 2y) = 3x + 2xy$. For instance, we can check that this is true when x is 5 and y is 8:

Left-hand side: $5(3 + 2 \cdot 8) = 5(19) = 95$

Right-hand side: $3 \cdot 5 + 2 \cdot 5 \cdot 8 = 15 + 80 = 95$

example 3 Using the Distributive Property to Expand

Expand the products using the Distributive Property.

(a) $a(x + 5y)$

(b) $-(2 - x)$

(c) $7x(x + 2y + 4)$

Solution

(a) We use the Distributive Property.

$$a(x + 5y) = a \cdot x + a \cdot 5y \qquad \text{Distributive Property}$$
$$= ax + 5ay \qquad \text{Simplify}$$

(b) The minus sign is the same as multiplying by -1.

$$-(2 - x) = -1(2 - x)$$
$$= (-1)2 + (-1)(-x) \qquad \text{Distributive Property}$$
$$= -2 + x \qquad \text{Simplify}$$

Notice how the negative sign outside the parentheses switches the signs of the terms inside the parentheses.

The expression inside the parentheses in Example 3(c) has three terms.

$x + 2y + 4$

Three terms

(c) The Distributive Property is valid for three or more terms.

$$7x(x + 2y + 4) = 7x \cdot x + 7x \cdot 2y + 7x \cdot 4 \qquad \text{Distributive Property}$$
$$= 7x^2 + 14xy + 28x \qquad \text{Simplify}$$

■ NOW TRY EXERCISES 27, 31, AND 33 ■

The Distributive Property: Factoring

Factoring is the reverse of expanding. When we factor, we use the Distributive Property to write an expression as a product of simpler ones.

EXPANDING →

$$5(3 + x) = 15 + 5x$$

← FACTORING

To use the Distributive Property in reverse, we look for common factors of each of the terms in the expression we are factoring.

example 4 Using the Distributive Property to Factor

Factor the following using the Distributive Property.

(a) $3x + 3$

(b) $ax + 2ay$

(c) $2bx + 4x$

(d) $3abc + 6ab + 3bc$

Solution

(a) The common factor in each term is 3. So we factor 3 from each term using the Distributive Property.

$$3x + 3 \cdot 1 = 3(x + 1) \qquad \text{Distributive Property}$$

In Example 4(b), a is a common factor of the terms ax and ay.

$ax + 2ay$

a is a factor of each term

(b) The common factor in each term is a. So we factor a from each term using the Distributive Property.

$$ax + 2ay = a(x + 2y) \qquad \text{Distributive Property}$$

(c) The expression $2x$ is common to each term. So we factor $2x$ from each term using the Distributive Property.

$$2bx + 4x = 2x(b + 2) \qquad \text{Distributive Property}$$

$3abc + 6ab + 3bc$

$3b$ is a factor of each term

(d) The expression $3b$ is common to each term. So we factor $3b$ from each term using the Distributive Property (applied to the three terms).

$$3abc + 6ab + 3bc = 3b(ac + 2a + c) \qquad \text{Distributive Property}$$

■ **NOW TRY EXERCISES 35, 39, AND 43** ■

A.1 Exercises

CONCEPTS

1. Give an example of each of the following:

(a) A natural number

(b) An integer that is not a natural number

(c) A rational number that is not an integer

(d) An irrational number

2. Complete each statement and name the property of real numbers you used.

(a) $ab =$ _____; _____ Property

(b) $a + (b + c) =$ _____; _____ Property

(c) $a(b + c) =$ _____; _____ Property

3. **(a)** When two numbers are multiplied together, each of the numbers is called a _____ (term/factor). So in the product $3xy$ the numbers 3, x, and y are _____.

(b) When two numbers are added together, each of the numbers is a _____ (term/factor). So in the sum $3 + xy$ the expressions 3 and xy are _____.

4. (a) We use the _____ _____ to expand expressions, so to expand the expression $a(b + c)$, we multiply each term inside the parentheses by _____, and get $a(b + c) =$ _____ + _____.

(b) We use the _____ _____ to factor. The expression $ab + ac$ has a common factor _____, so to factor the expression $ab + ac$, we factor _____ from each term and get $ab + ac =$ _____.

SKILLS

5–6 ■ List the elements of the given set that are:

(a) Natural numbers

(b) Integers

(c) Rational numbers

(d) Irrational numbers

5. $\{0, -10, 50, \frac{22}{7}, 0.583, \sqrt{7}, 1.2\overline{3}, -\frac{1}{3}, \sqrt[3]{2}\}$

6. $\{1.001, 0.\overline{3}, -\pi, -11, 11, \frac{13}{15}, 3.14, \sqrt{16}, \frac{15}{3}\}$

7–10 ■ Evaluate the arithmetic expression.

7. $-2 + [4 \cdot 7 - 5(9 - \frac{8}{2})]$

8. $3(4 \cdot 6 - 2 \cdot 10) + 7(15 - 8 \cdot 2)$

9. $\dfrac{5 + 7}{3} - 6[12 - (17 - 2 \cdot 3)]$

10. $1 - 2[3 - 4(5 - 6 \cdot 7)]$

11–18 ■ State the property of real numbers being used.

11. $7 + 10 = 10 + 7$

12. $2(3 + 5) = (3 + 5)2$

13. $(x + 2y) + 3z = x + (2y + 3z)$

14. $2(A + B) = 2A + 2B$

15. $(5x + 1)3 = 15x + 3$

16. $(x + a)(x + b) = (x + a)x + (x + a)b$

17. $2x(3 + y) = (3 + y)2x$

18. $7(a + b + c) = 7(a + b) + 7c$

19–22 ■ Evaluate the given expression.

19. $\dfrac{10 - 4}{3}$

20. $\dfrac{4 - 9}{5}$

21. $\dfrac{10 - 4}{5 - 2}$

22. $\dfrac{6 - 16}{2 - 7}$

23–26 ■ Evaluate the given expression directly, then evaluate using the Distributive Property.

23. $3(10 + 2)$

24. $(20 + 14) \cdot 5$

25. $(13 - 10)(-10)$

26. $-0.3(30 - 20)$

27–34 ■ Expand the expression using the Distributive Property.

27. $3(x + 7)$

28. $8(a - 2)$

29. $(a - 3c)6$

30. $(x + 2y)7$

31. $-4(3ax - 2x)$

32. $-3c(6ab - 5bd)$

33. $4mn(2p + 3pq - 2q)$

34. $(3q - 2qr - 5r)(-2ps)$

35–42 ■ Factor the given expression using the Distributive Property.

35. $3x + 9$

36. $6y - 12$

37. $-2a - 2b$

38. $-20x + 40y$

39. $ab - 6b$

40. $2ax + 2ay$

41. $-5ab + 10bc$

42. $-30xz - 90yz$

43. $2qrs - 6qst + 12rst$

44. $\frac{1}{2}abx + \frac{1}{4}aby - \frac{1}{8}abz$

A.2 The Number Line and Intervals

- The Real Line
- Order on the Real Line
- Sets and Intervals
- Absolute Value and Distance

The Real Line

The real numbers can be represented by points on a line, as shown in Figure 1. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. Thus every real number is represented by a point on the line, and every point P on the line corresponds to exactly one real number.

The number associated with the point P is called the *coordinate* of P, and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

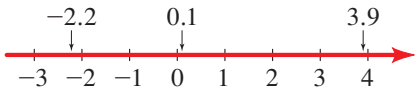


figure **1** The real line

Order on the Real Line

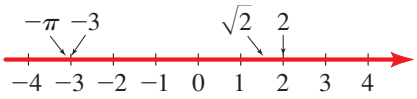


figure **2**

The real numbers are *ordered*. We say that ***a* is less than *b*** and write $a < b$ if $b - a$ is a positive number. Geometrically, this means that a lies to the left of b on the number line. (Equivalently, we can say that b is greater than a and write $b > a$.) The symbol $a \le b$ (or $b \ge a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities (see Figure 2):

$$-\pi < -3 \qquad \sqrt{2} < 2 \qquad 2 \le 2$$

example 1 Graphing Inequalities

(a) On the real line, graph all the numbers x that satisfy the inequality $x < 3$.

(b) On the real line, graph all the numbers x that satisfy the inequality $x \geq -2$.

Solution



figure **3**

(a) We must graph the real numbers that are smaller than 3: those that lie to the left of 3 on the real line. The graph is shown in Figure 3. Note that the number 3 is indicated with an open dot on the real line, since it does not satisfy the inequality.



figure **4**

(b) We must graph the real numbers that are greater than or equal to -2: those that lie to the right of -2 on the real line, including the number -2 itself. The graph is shown in Figure 4. Note that the number -2 is indicated with a solid dot on the real line, since it satisfies the inequality.

■ **NOW TRY EXERCISES 17 AND 19** ■

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S *or* T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S *and* T. In other words, $S \cap T$ is the common part of S and T. The **empty set**, denoted by $\emptyset$, is the set that contains no element.

example 2 Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

Solution

$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$ All elements in S or T

$S \cap T = \{4, 5\}$ Elements common to both S and T

$S \cap V = \emptyset$ S and V have no elements in common

■ **NOW TRY EXERCISE 31** ■

Certain sets of real numbers, called **intervals**, occur frequently in algebra and correspond geometrically to line segments. For example, if $a < b$, then the **open interval** from a to b consists of all numbers between a and b, excluding the endpoints a and b. The **closed interval** includes the endpoints. We can write these intervals in set-builder notation:

Open interval	Closed interval
$(a, b) = \{x \mid a < x < b\}$	$[a, b] = \{x \mid a \le x \le b\}$

When we graph these intervals on the real line, we use a solid dot to indicate that an endpoint is included and an open circle to indicate that an endpoint is excluded. The graphs are shown below.



We also need to consider infinite intervals. For example, all the numbers greater than 3 form the interval $(3, \infty)$. This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction. Similarly, $(-\infty, 3]$ is the interval consisting of all numbers less than or equal to 3 (the bracket on the right-hand side of the interval indicates that we include 3).

$$(-\infty, 3] = \{x \mid x \le 3\} \qquad (3, \infty) = \{x \mid 3 < x\}$$

We graph these intervals on the real line.

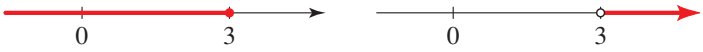


Table 1 lists the nine possible types of intervals. When these intervals are discussed, we will always assume that $a < b$.

table **1**

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

example 3 Verbal Description of an Interval

Consider the interval $(2, 5]$.

(a) Give a verbal description of the interval.

(b) Express the interval in set builder notation.

(c) Graph the interval.

Solution

(a) The interval consists of all numbers between 2 and 5, including 5 but excluding 2.

(b) $(2, 5] = \{x \mid 2 < x \leq 5\}$

(c) The graph is shown below.



NOW TRY EXERCISE 33

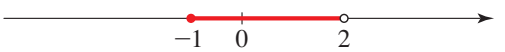
example 4 Graphing Intervals

Express each interval in set builder notation, and then graph the interval.

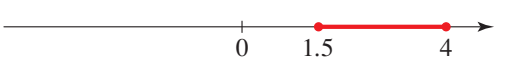
(a) $[-1, 2)$ **(b)** $[1.5, 4]$ **(c)** $(-3, \infty)$

Solution

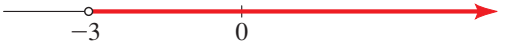
(a) $[-1, 2) = \{x \mid -1 \leq x < 2\}$



(b) $[1.5, 4] = \{x \mid 1.5 \leq x \leq 4\}$



(c) $(-3, \infty) = \{x \mid -3 < x\}$



NOW TRY EXERCISES 35 AND 37

example 5 Finding Unions and Intersections of Intervals

Graph each set.

(a) $(1, 3) \cap [2, 7]$ **(b)** $(1, 3) \cup [2, 7]$

Solution

(a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore,

$$\begin{aligned}(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \leq x \leq 7\} \\ &= \{x \mid 2 \leq x < 3\} = [2, 3)\end{aligned}$$

This set is illustrated in Figure 5.

(b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore,

$$(1, 3) \cup [2, 7] = \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\}$$
$$= \{x \mid 1 < x \le 7\} = (1, 7]$$

This set is illustrated in Figure 6.

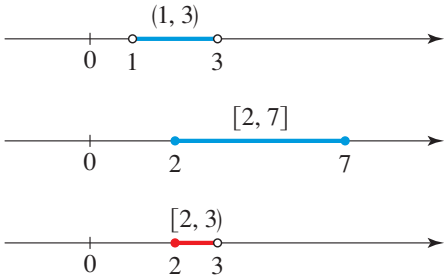


figure **5** $(1, 3) \cap [2, 7]$

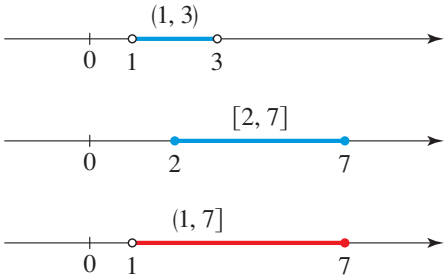


figure **6** $(1, 3) \cup [2, 7]$

NOW TRY EXERCISES 53 AND 55

Absolute Value and Distance

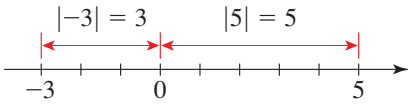


figure **7**

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line (see Figure 7). Distance is always positive or zero, so we have $|a| \ge 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

Absolute Value

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

example **6** **Evaluating Absolute Values of Numbers**

Evaluate.

(a) $|3|$ **(b)** $|-3|$ **(c)** $|0|$ **(d)** $|\sqrt{2} - 1|$

Solution

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|\sqrt{2} - 1| = \sqrt{2} - 1$ (because $\sqrt{2} > 1$, and so $\sqrt{2} - 1 > 0$)

NOW TRY EXERCISE 59

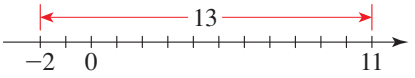


figure **8**

What is the distance on the real line between the numbers -2 and 11? From Figure 8 we see that the distance is 13. We arrive at this by finding either $|11 - (-2)| = 13$ or $|(-2) - 11| = 13$. From this observation we make the following definition (see Figure 9).

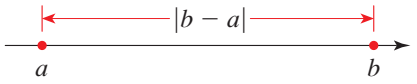


figure **9** Length of a line segment is $|b - a|$

Distance Between Points on the Real Line

If a and b are real numbers, then the distance between the points a and b on the real line is

$$d(a, b) = |b - a|$$

Note that $|b - a| = |a - b|$. This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a.

example **7** **Distance Between Points on the Real Line**

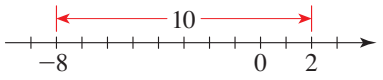


figure **10**

The distance between the numbers -8 and 2 is

$$d(a, b) = |-8 - 2| = |-10| = 10$$

We can check this calculation geometrically, as shown in Figure 10.

■ **NOW TRY EXERCISE 63** ■

A.2 Exercises

CONCEPTS

1. Explain how to graph numbers on a real number line.

2. If $a < b$, how are the points on a real line that correspond to the numbers a and b related to each other?

3. The set of numbers between but not including 2 and 7 can be written as follows:

(a) _____ in set builder notation

(b) _____ in interval notation

4. Explain the differences between the following two sets: $A = [-2, 5]$ and $B = (-2, 5)$

5. The symbol $|x|$ stands for the _____ of the number x. If x is not 0, then the sign of $|x|$ is always _____.

6. The absolute value of the difference between a and b is (geometrically) the _____ between them on the real line.

SKILLS

7–8 ■ Place the correct symbol ($<$, $>$, or $=$) in the space.

7. **(a)** $3 \square \frac{7}{2}$ **(b)** $-3 \square -\frac{7}{2}$ **(c)** $3.5 \square \frac{7}{2}$

8. **(a)** $\frac{2}{3} \square 0.67$ **(b)** $\frac{2}{3} \square -0.67$ **(c)** $|0.67| \square |-0.67|$

9–16 ■ State whether each inequality is true or false.

9. $-6 < -10$ **10.** $\sqrt{2} > 1.41$ **11.** $\frac{10}{11} < \frac{12}{13}$ **12.** $-\frac{1}{2} < -1$

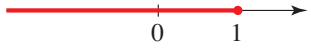
13. $-\pi > -3$ **14.** $8 \le 9$ **15.** $1.1 > 1.\overline{1}$ **16.** $8 \le 8$

17–20 ■ On a real line, graph the numbers that satisfy the inequality.

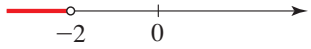
17. $x \geq 1$ **18.** $x > -4$ **19.** $x < -3$ **20.** $x \leq 0$

21–24 ■ Find the inequality whose graph is given.

21.



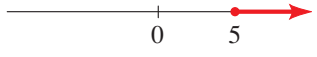
22.



23.



24.



25–26 ■ Write each statement in terms of inequalities.

25. **(a)** x is positive.
(b) t is less than 4.
(c) a is greater than or equal to -1.
(d) x is less than $\frac{1}{3}$ and is greater than -5.
(e) The distance from p to 3 is at most 5.

26. **(a)** y is negative.
(b) z is greater than 1.
(c) b is at most 8.
(d) w is positive and less than or equal to 17.
(e) y is at least 2 units away from 7.

27–30 ■ Find the indicated set if $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8\}$, and $C = \{7, 8, 9, 10\}$.

27. **(a)** $A \cup B$ **(b)** $A \cap B$

28. **(a)** $B \cup C$ **(b)** $B \cap C$

29. **(a)** $A \cup C$ **(b)** $A \cap C$

30. **(a)** $A \cup B \cup C$ **(b)** $A \cap B \cap C$

31–32 ■ Find the indicated set if $A = \{x \mid x \geq -2\}$, $B = \{x \mid x < 4\}$, and $C = \{x \mid -1 < x \leq 5\}$.

31. **(a)** $B \cup C$ **(b)** $B \cap C$

32. **(a)** $A \cup C$ **(b)** $A \cap C$

33–40 ■ Consider the given interval.
(a) Give a verbal description of the interval.
(b) Express the interval in set builder notation.
(c) Graph the interval.

33. $(-3, 0)$ **34.** $(2, 8]$

35. $[2, 8)$ **36.** $[-6, -\frac{1}{2}]$

37. $[2, \infty)$ **38.** $(-\infty, 1)$

39. $(-\infty, -2)$ **40.** $[-3, \infty)$

41–48 ■ Express the inequality in interval notation, and then graph the corresponding interval.

41. $1 \leq x \leq 2$ **42.** $3 < x < 5$

43. $-2 < x \leq 1$ **44.** $-5 < x < 2$

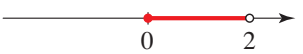
45. $x \geq -5$ **46.** $x \leq 1$

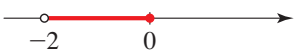
47. $x > -1$ **48.** $-5 \leq x < -2$

49–52 ■ The graph of an interval is given. Express the interval **(a)** in set-builder notation and **(b)** in interval notation.

49.


50.


51.


52.


53–58 ■ Graph the set.

53. $(-2, 0) \cup (-1, 1)$

54. $(-2, 0) \cap (-1, 1)$

55. $[-4, 6] \cap [0, 8)$

56. $[-4, 6) \cup [0, 8)$

57. $(-\infty, -4) \cup (4, \infty)$

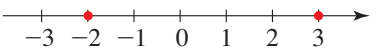
58. $(-\infty, 6) \cap (2, 10)$

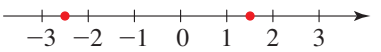
59–60 ■ Evaluate each expression.

59. **(a)** $|9|$ **(b)** $|-12|$ **(c)** $\left|\frac{-6}{24}\right|$ **(d)** $|\sqrt{5} - 5|$

60. **(a)** $|100|$ **(b)** $|-5|$ **(c)** $|(-2)\cdot 6|$ **(d)** $|10 - \pi|$

61–64 ■ Find the distance between the given numbers.

61.


62.


63. **(a)** 2 and 17 **(b)** −3 and 21 **(c)** $\frac{11}{8}$ and $-\frac{3}{8}$

64. **(a)** $\frac{7}{15}$ and $-\frac{1}{21}$ **(b)** −38 and −57 **(c)** −2.6 and −1.8

A.3 Integer Exponents

- Integer Exponents
- Rules for Working with Exponents

Integer Exponents

A product of identical numbers is usually written in exponential notation. For example,

$$9^4 = \underbrace{9 \times 9 \times 9 \times 9}_{4 \text{ factors}}$$

We say that 9 is the *base* and 4 is the *exponent*. In general, we have the following definition.

Exponential Notation

If a is any real number and n is a positive integer, then the nth power of a is

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}$$

The number a is called the base and n is called the exponent.

Let's calculate some numbers written in exponential notation.

example 1 Using Exponential Notation

Evaluate each expression.

(a) 10^6 **(b)** $(-3)^4$ **(c)** -3^4 **(d)** $\left(\frac{1}{2}\right)^5$

Solution

Using the definition of exponential notation, we can write out each number as follows.

(a) $10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1{,}000{,}000$

(b) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

(d) $\left(\frac{1}{2}\right)^5 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$

■ **NOW TRY EXERCISES 7 AND 9** ■

In Example 1, note the difference between $(-3)^4$ and -3^4. In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2:

$$5^4 \cdot 5^2 = \underbrace{(5 \cdot 5 \cdot 5 \cdot 5)}_{4 \text{ factors}}\underbrace{(5 \cdot 5)}_{2 \text{ factors}} = \underbrace{(5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5)}_{6 \text{ factors}} = 5^6 = 5^{2+4}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n, we have

$$a^m \cdot a^n = \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}}\underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdots a)}_{m+n \text{ factors}} = a^{m+n}$$

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$5^0 \cdot 5^3 = 5^{0+3} = 5^3$$

But this can happen only if $5^0 = 1$. Likewise, we want to have

$$5^3 \cdot 5^{-3} = 5^{3+(-3)} = 5^0 = 1$$

and this can be true only if $5^{-3} = \dfrac{1}{5^3}$. These observations lead to the following definition.

Zero and Negative Exponents

If a is any nonzero real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

Let's calculate some numbers with zero or negative exponents.

example 2 Zero and Negative Exponents

Evaluate the following expressions.

(a) $\left(\frac{1}{2}\right)^0$ **(b)** 5^{-4} **(c)** 5^{-1} **(d)** $(-4)^{-3}$

Solution

Using the definition of zero and negative exponents, we get the following.

(a) $\left(\frac{1}{2}\right)^0 = 1$

(b) $5^{-4} = \frac{1}{5^4} = \frac{1}{5 \cdot 5 \cdot 5 \cdot 5} = \frac{1}{625}$

(c) $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$

(d) $(-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{(-4) \cdot (-4) \cdot (-4)} = \frac{1}{-64}$

■ NOW TRY EXERCISES 13 AND 15 ■

Rules for Working with Exponents

The following rules help us to simplify expressions that involve exponents. In the table, the bases a and b are real numbers, and the exponents m and n are integers.

Rules for Working with Exponents

If a and b are any nonzero real numbers and m and n are positive integers, then

Rule	Verbal description
1. $a^m a^n = a^{m+n}$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	To raise a power to another power, multiply the exponents.
4. $(ab)^n = a^n b^n$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	To raise a quotient to a power, raise both numerator and denominator to the power.

example 3 Working with Exponents

Use the rules of exponents to write each of the following expressions in as simple a form as possible.

(a) $(10^3)^2$ **(b)** $\frac{7^5}{7^3}$ **(c)** $\frac{3^6 \cdot 3^{-2}}{3}$ **(d)** $\left(\frac{2}{3}\right)^4$

Solution

Using the rules of exponents, we can simplify these expressions as follows.

(a) Using Rule 3, we have

$$(10^3)^2 = 10^{3\cdot 2} = 10^6 = 1{,}000{,}000$$

(b) Using Rule 2, we have

$$\frac{7^5}{7^3} = 7^{5-3} = 7^2 = 49$$

(c) Using Rules 1 and 2, we have

$$\frac{3^6 \cdot 3^{-2}}{3} = \frac{3^{6+(-2)}}{3} = 3^{6+(-2)-1} = 3^3 = 27$$

(d) Using Rule 5, we have

$$\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$$

■ **NOW TRY EXERCISES 19, 25, 29, AND 31** ■

example 4 Working with Exponents

Each of the following expressions contains one or more variables. We use the rules of exponents to simplify each expression.

(a) x^5x^2 **(b)** $(xy^2)^3$ **(c)** $(-2x)^5$ **(d)** $\left(\frac{a^2}{2b}\right)^4$

Solution

(a) Using Rule 1, we have

$$x^5x^2 = x^{5+2} = x^7$$

(b) Using Rules 3 and 4, we have

$$(xy^2)^3 = x^3y^{2\cdot 3} = x^3y^6$$

(c) Using Rule 4, we have

$$(-2x)^5 = (-2)^5x^5 = -32x^5$$

(d) Using Rules 3 and 5, we have

$$\left(\frac{a^2}{2b}\right)^4 = \frac{a^{2\cdot 4}}{2^4b^4} = \frac{a^8}{16b^4}$$

■ **NOW TRY EXERCISES 35, 37, 41, AND 51** ■

The rules of exponents can be used in different ways. Sometimes one way involves fewer steps than another, even though the final simplification is the same, as we see in the next example. So as long as you use the rules correctly, you may apply them in any order you like.

example 5 Using the Rules of Exponents in Different Ways

Show two different ways to simplify the expression $\left(\frac{x^4}{5x^3}\right)^2$.

Solution 1

Here's one way: We start with Rule 5.

$$\left(\frac{x^4}{5x^3}\right)^2 = \frac{(x^4)^2}{(5x^3)^2} \quad \text{Rule 5}$$

$$= \frac{x^{4\cdot 2}}{5^2 x^{3\cdot 2}} \quad \text{Rules 3 and 4}$$

$$= \frac{x^8}{25x^6} \quad \text{Simplify}$$

$$= \frac{x^{8-6}}{25} \quad \text{Rule 2}$$

$$= \frac{x^2}{25} \quad \text{Simplify}$$

Solution 2

Here's another way that starts with Rule 2.

$$\left(\frac{x^4}{5x^3}\right)^2 = \left(\frac{x^{4-3}}{5}\right)^2 \quad \text{Rule 2}$$

$$= \left(\frac{x}{5}\right)^2 \quad \text{Simplify}$$

$$= \frac{x^2}{5^2} \quad \text{Rule 5}$$

$$= \frac{x^2}{25} \quad \text{Simplify}$$

■ **NOW TRY EXERCISE 47** ■

A.3 Exercises

CONCEPTS

1. Using exponential notation, we can write the product $5\cdot5\cdot5\cdot5\cdot5\cdot5$ as _____.

2. In the expression 3^4 the number 3 is called the _____ and the number 4 is called the _____.

3. When we multiply two powers with the same base, we _____ the exponents. So $3^4\cdot3^5 =$ _____.

4. When we divide two powers with the same base, we _____ the exponents. So $\frac{3^5}{3^2} =$ _____.

5. When we raise a power, to a new power, we _____ the exponents.

So $(3^4)^2 =$ _____.

6. Express the following numbers without using exponents.

(a) $2^{-1} =$ _____.

(b) $2^{-3} =$ _____.

(c) $\left(\frac{1}{2}\right)^{-1} =$ _____.

SKILLS

7–16 ■ Evaluate each expression.

7. **(a)** 3^4 **(b)** $\left(\frac{1}{3}\right)^5$

8. **(a)** 2^6 **(b)** $\left(\frac{1}{2}\right)^4$

9. **(a)** $(-5)^2$ **(b)** -5^2

10. **(a)** $(-10)^4$ **(b)** -10^4

11. **(a)** $(-6)^3$ **(b)** -6^3

12. **(a)** $(-3)^7$ **(b)** -3^7

13. **(a)** $\left(\frac{1}{3}\right)^0$ **(b)** -3^0

14. **(a)** 2^0 **(b)** $(-2)^0$

15. **(a)** 3^{-1} **(b)** 3^{-3}

16. **(a)** 10^{-1} **(b)** 10^{-6}

17–34 ■ Use the rules of exponents to write each expression in as simple a form as possible.

17. $5^2 \cdot 5$

18. $2^3 \cdot 2^2$

19. $(2^3)^2$

20. $(2^3)^0$

21. $(-6)^0$

22. -6^0

23. -3^2

24. $(-3)^2$

25. $\dfrac{10^7}{10^4}$

26. $\dfrac{3}{3^{-2}}$

27. $3^{-4} \cdot 3^2$

28. $5^4 \cdot 5^{-2}$

29. $\dfrac{7^5 \cdot 7^{-3}}{7^2}$

30. $\dfrac{2^{-3} \cdot 2^2}{3^0}$

31. $\left(\frac{3}{4}\right)^2$

32. $\left(\frac{2}{3}\right)^{-3}$

33. $\left(\frac{1}{2}\right)^4\left(\frac{5}{2}\right)^{-2}$

34. $\left(\frac{5}{3}\right)^0 2^{-1}$

35–52 ■ Simplify each expression, and eliminate any negative exponents.

35. x^8x^2

36. x^2x^{-6}

37. $(2y^2)^3$

38. $(8x)^2$

39. $(a^2a^4)^3$

40. $(2a^3a^2)^4$

41. $(-3a)^3$

42. $(-2w)^4$

43. $\dfrac{y^{10}y^0}{y^7}$

44. $\dfrac{x^6}{x^{10}}$

45. $\dfrac{a^9a^{-2}}{a}$

46. $\dfrac{z^2z^4}{z^3z^{-1}}$

47. $\left(\dfrac{a^2}{4a^3}\right)^3$

48. $\dfrac{3x^4}{4x^2}$

49. $(3z)^2(6z^2)^{-3}$

50. $(2z^2)^{-5}z^{10}$

51. $\left(\dfrac{3u^5}{2v^3}\right)^4$

52. $\left(\dfrac{-2x^2}{y^3}\right)^3$

A.4 Radicals and Rational Exponents

- Rational Exponents: $a^{1/n}$
- Rational Exponents: $a^{m/n}$

Rational Exponents: $a^{1/n}$

To define what is meant by a rational exponent such as $a^{1/2}$, we need to use radicals. We want to give a meaning to the symbol $a^{1/2}$ in a way that is consistent with the rules of exponents. To do this, we notice that when we square $a^{1/2}$, we get a because

$$(a^{1/2})^2 = a^{2\cdot(1/2)} = a^1 = a$$

Since $a^{1/2}$ squared is a, it follows that

$$a^{1/2} = \sqrt{a}$$

Similarly, $(a^{1/n})^n = a$, so $a^{1/n}$ must be the nth root of a.

Exponential Notation for $\sqrt[n]{a}$

If n is a positive integer, then we define

$$a^{1/n} = \sqrt[n]{a}$$

When n is even, we require that $a \geq 0$.

It is true that positive numbers have two square roots; for example, the number 9 has the two square roots 3 and -3, but the notation $\sqrt{9}$ is reserved for the positive square root (called the **principal square root**). If we want the negative square root, we must write $-\sqrt{9}$, which is -3.

If n is an even integer, then a must be nonnegative for $a^{1/n}$ to be defined. This is because we can't take the square root (or fourth root, sixth root, etc.) of negative numbers. Here are some examples of numbers with rational exponents.

example 1 Rational Exponents

Evaluate.

(a) $9^{1/2}$ **(b)** $(-125)^{1/3}$ **(c)** $\left(\frac{1}{8}\right)^{1/3}$ **(d)** $\left(\frac{16}{81}\right)^{1/4}$

Solution

(a) $9^{1/2} = 3$ Because $3^2 = 9$

(b) $(-125)^{1/3} = -5$ Because $(-5)^3 = -125$

(c) $\left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$ Because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(d) $\left(\frac{16}{81}\right)^{1/4} = \frac{2}{3}$ Because $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

■ NOW TRY EXERCISE 19 ■

Rational Exponents: $a^{m/n}$

To define the meaning of a rational exponent such as $8^{2/3}$, we simply notice that the rules of exponents tell us that $8^{2/3} = (8^{1/3})^2$ or, equivalently, $8^{2/3} = (8^2)^{1/3}$. We can compute each of these expressions:

$$8^{2/3} = (8^{1/3})^2 = 2^2 = 4$$
$$8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$$

In either case we get the same value 4 for $8^{2/3}$. The same reasoning allows us to give the appropriate meaning for $a^{m/n}$. For any rational exponent $\frac{m}{n}$ in lowest terms, where m and n are integers and $n > 0$, we have the following definition.

Exponential Notation for $\sqrt[n]{a^m}$

If m and n are positive integers, then we define

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \qquad \text{or equivalently} \qquad a^{m/n} = \sqrt[n]{a^m}$$

When n is even, we require that $a \geq 0$.

If n is an even integer, then a must be nonnegative for $a^{m/n}$ to be defined. Note that *the rules of exponents also hold for rational exponents.* Here are some examples of numbers involving rational exponents.

example 2 Rational Exponents

Calculate.

(a) $16^{3/4}$ **(b)** $(-8)^{2/3}$ **(c)** $\left(\frac{1}{32}\right)^{-3/5}$ **(d)** $\left(\frac{8}{27}\right)^{4/3}$

Solution

(a)
$$16^{3/4} = (16^{1/4})^3 \qquad \text{Rule 3: } (a^m)^n = a^{mn}$$
$$= 2^3 \qquad \text{Because } \sqrt[4]{16} = 2$$
$$= 8 \qquad \text{Calculate}$$

(b)
$$(-8)^{2/3} = ((-8)^2)^{1/3} \qquad \text{Rule 3: } (a^m)^n = a^{mn}$$
$$= 64^{1/3} \qquad \text{Because } (-8)^2 = 64$$
$$= 4 \qquad \text{Because } \sqrt[3]{64} = 4$$

(c)
$$\left(\frac{1}{32}\right)^{-3/5} = \left(\left(\frac{1}{32}\right)^{1/5}\right)^{-3} \qquad \text{Rule 3: } (a^m)^n = a^{mn}$$
$$= \left(\frac{1}{2}\right)^{-3} \qquad \text{Because } \sqrt[5]{1/32} = 1/2$$
$$= 2^3 \qquad \text{Property of Negative Exponents}$$
$$= 8 \qquad \text{Calculate}$$

(d)
$$\left(\frac{8}{27}\right)^{4/3} = \left(\left(\frac{8}{27}\right)^{1/3}\right)^4 \qquad \text{Rule 3: } (a^m)^n = a^{mn}$$
$$= \left(\frac{2}{3}\right)^4 \qquad \text{Because } \sqrt[3]{8} = 2 \text{ and } \sqrt[3]{27} = 3$$
$$= \frac{16}{81} \qquad \text{Rule 5: } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

NOW TRY EXERCISE 21

example 3 Using the Laws of Exponents with Rational Exponents

Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

(a) $a^{1/3}a^{7/3}$

(b) $\dfrac{a^{2/5}}{a^{3/5}}$

(c) $(4a^3b^4)^{3/2}$

Solution

(a)
$$a^{1/3}a^{7/3} = a^{1/3+7/3} \qquad \text{Rule 1: } a^m a^n = a^{m+n}$$
$$= a^{8/3} \qquad \text{Simplify}$$

(b)
$$\frac{a^{2/5}}{a^{3/5}} = a^{2/5-3/5} \qquad \text{Rule 2: } \frac{a^m}{a^n} = a^{m-n}$$
$$= a^{-1/5} \qquad \text{Simplify}$$
$$= \frac{1}{a^{1/5}} \qquad \text{Definition of negative exponents}$$

(c)
$$(4a^3b^4)^{3/2} = 4^{3/2}(a^3)^{3/2}(b^4)^{3/2} \qquad \text{Rule 4: } (abc)^n = a^n b^n c^n$$
$$= (2^3)a^{3(3/2)}b^{4(3/2)} \qquad \text{Rule 3: } (a^m)^n = a^{mn}$$
$$= 8a^{9/2}b^6 \qquad \text{Simplify}$$

NOW TRY EXERCISES 23, 27, AND 31

example 4 Simplifying by Writing Radicals as Rational Exponents

Simplify.

(a) $(2\sqrt{x})(3\sqrt[3]{x})$

(b) $\sqrt{x\sqrt{x}}$

Solution

(a) $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3})$ Definition of rational exponents

$= 6^{1/2+1/3}$ Rule 1: $a^m a^n = a^{m+n}$

$= 6x^{5/6}$ Simplify

(b) $\sqrt{x\sqrt{x}} = (x \cdot x^{1/2})^{1/2}$ Definition of rational exponents

$= (x^{3/2})^{1/2}$ Rule 1: $a^m a^n = a^{m+n}$

$= x^{3/4}$ Rule 3: $(a^m)^n = a^{mn}$

■ **NOW TRY EXERCISES 41 AND 47** ■

A.4 Exercises

CONCEPTS

1. Using exponential notation, we can write $\sqrt[3]{5}$ as _____.
2. Using radicals, we can write $5^{1/2}$ as _____.
3. Is there a difference between $\sqrt{5^2}$ and $(\sqrt{5})^2$? Explain.
4. Explain what $4^{3/2}$ means, and then calculate $4^{3/2}$ in two different ways:

$4^{3/2} = (4^{1/2})^{\square} =$ _____ $4^{3/2} = (4^3)^{\square} =$ _____

5. Find the missing power in the following calculation: $5^{1/3} \cdot 5^{\square} = 5$
6. Find the missing power in the following calculation: $5^{1/3} \cdot 5^{\square} = 1$

SKILLS

7–14 ■ Write each radical expression using exponents and each exponential expression using radicals.

	Radical expression	Exponential expression
7.	$\dfrac{1}{\sqrt{5}}$	_____
8.	$\sqrt[3]{7^2}$	_____
9.	_____	$4^{2/3}$
10.	_____	$11^{-3/2}$
11.	$\sqrt[5]{5^3}$	_____
12.	_____	$2^{-1/2}$
13.	_____	$a^{2/5}$
14.	$\dfrac{1}{\sqrt{x^5}}$	_____

15–22 ■ Evaluate each expression.

15. **(a)** $\sqrt{16}$ **(b)** $\sqrt[4]{16}$ **(c)** $\sqrt[4]{\frac{1}{16}}$

16. **(a)** $\sqrt{64}$ **(b)** $\sqrt[3]{-64}$ **(c)** $\sqrt[5]{32}$

17. **(a)** $\sqrt{\frac{4}{9}}$ **(b)** $\sqrt[4]{256}$ **(c)** $\sqrt[6]{\frac{1}{64}}$

18. **(a)** $\sqrt{7}\sqrt{28}$ **(b)** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **(c)** $\sqrt[4]{8}\sqrt[4]{32}$

19. **(a)** $49^{1/2}$ **(b)** $(-32)^{1/5}$ **(c)** $\left(\frac{1}{125}\right)^{1/3}$

20. **(a)** $16^{1/4}$ **(b)** $\left(-\frac{1}{27}\right)^{1/3}$ **(c)** $\left(\frac{81}{16}\right)^{1/4}$

21. **(a)** $\left(\frac{4}{9}\right)^{-1/2}$ **(b)** $(-32)^{2/5}$ **(c)** $\left(\frac{25}{64}\right)^{3/2}$

22. **(a)** $\left(\frac{1}{32}\right)^{2/5}$ **(b)** $(-27)^{-4/3}$ **(c)** $\left(\frac{1}{8}\right)^{-2/3}$

23–34 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

23. $x^{3/4}x^{5/4}$ **24.** $y^{2/3}y^{4/3}$ **25.** $(4b)^{1/2}$ **26.** $(3a^{2/3})^2$

27. $\dfrac{w^{4/3}}{w^{1/3}}$ **28.** $\dfrac{5^{5/2}}{5^{1/2}}$ **29.** $\left(\dfrac{r^{1/2}}{16}\right)^{1/4}$ **30.** $\left(\dfrac{27}{k^3}\right)^{-1/3}$

31. $(x^6y^4)^{-1/2}$ **32.** $(u^6v^3)^{1/3}$ **33.** $\left(\dfrac{2q^{3/4}}{r^{3/2}}\right)^{-4}$ **34.** $\left(\dfrac{s^{2/3}}{2t^{1/6}}\right)^6$

35–48 ■ Simplify the expression and express the answer using rational exponents. Assume that all letters denote positive numbers.

35. $\sqrt{x^3}\sqrt{x^5}$ **36.** $\sqrt[3]{5y^4}$

37. $\sqrt[4]{3s^3t^5}\sqrt[4]{27st^3}$ **38.** $\sqrt[5]{4u^2v^6}\sqrt[5]{8u^3v^4}$

39. $(\sqrt[6]{y^5})(\sqrt[3]{y^2})$ **40.** $\sqrt[4]{b^3}\sqrt{b}$

41. $(5\sqrt[3]{x})(2\sqrt[4]{x})$ **42.** $(2\sqrt{a})(\sqrt[3]{a^2})$

43. $\dfrac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$ **44.** $\dfrac{\sqrt[3]{16x^3}}{\sqrt[3]{2x^5}}$

45. $\dfrac{\sqrt[3]{8x^2}}{\sqrt{x}}$ **46.** $\dfrac{\sqrt{xy}}{\sqrt[4]{16xy}}$

47. $\sqrt[3]{y\sqrt{y}}$ **48.** $\sqrt{s\sqrt{s^3}}$

Algebra Toolkit B

Working with Expressions

B.1 Combining Algebraic Expressions
B.2 Factoring Algebraic Expressions
B.3 Rational Expressions

B.1 Combining Algebraic Expressions

- The Form and Value of an Expression
- Adding Algebraic Expressions
- Multiplying Algebraic Expressions
- Special Product Formulas
- Combining Polynomials

The Form and Value of an Expression

An **algebraic expression** is an arithmetic expression that contains letters, where the letters stand for numbers. The following are algebraic expressions:

$$x + 2 \qquad x(x + 3) \qquad \frac{x}{x + 2}$$

Recall that the letters in these expressions are called **variables** because they can "vary" over all real numbers for which the expression is defined.

What do these expressions tell us? We can describe the *form* of an expression in words. For example, the first expression says "a number plus 2." The expression has a different value when different numbers are assigned to the variable x. For example, if x is 5, then the *value* of the expression is $5 + 2 = 7$.

example 1 The Form and Value of an Expression

Consider the expression $\dfrac{x^2 + 3}{2}$.

(a) Describe the expression in words.

(b) Find the value of the expression if x is 5.

Solution

(a) The expression says "a number squared plus 3 divided by 2." Notice that the word description is ambiguous (Do we "add 3 then divide by 2," or do we add "three divided by 2"?), whereas the algebraic expression is not ambiguous—the numerator and denominator are treated as if they are in parentheses.

(b) The value of the expression is obtained by replacing x by 5.

$$\frac{x^2 + 3}{2} = \frac{5^2 + 3}{2} \qquad \text{Replace } x \text{ by } 5$$

$$= \frac{28}{2} \qquad \text{Evaluate the numerator}$$

$$= 14 \qquad \text{Calculate}$$

So if x is 5, the value of the expression is 14.

NOW TRY EXERCISE 9

One of the key features of an algebraic expression is the number of terms it has. An algebraic expression with two terms is called a **binomial**, and one with three terms is called a **trinomial**. An algebraic expression with one term is called a **monomial**.

example 2 The Form and Value of an Expression

Consider the trinomial $5ax + 3a - 8$.

(a) What are the terms of this trinomial expression?

(b) Find the value of the expression if a is 2 and x is -3.

Solution

(a) The expression has three terms: $5ax$, $3a$, and -8.

(b) To find the value of the expression, we replace a by 2 and x by -3.

$$5ax + 3a - 8 = 5(2)(-3) + 3(2) - 8 \qquad \text{Replace } a \text{ by } 2 \text{ and } x \text{ by } -3$$

$$= -32 \qquad \text{Calculate}$$

So if a is 2 and x is -3, the value of the expression is -32.

NOW TRY EXERCISE 11

Adding Algebraic Expressions

Expressions represent numbers. So we can add, subtract, multiply, and divide expressions. To **add** or **subtract** algebraic expressions, we combine "like terms." For example, to add the expressions $6ax + 7a + 2$ and $ax - 2a + 6$, we add like terms as follows:

$$(6ax + 7a + 2) + (ax - 2a + 6) = (6ax + ax) + (7a - 2a) + (2 + 6)$$

$$= 7ax + 5a + 8$$

The next example illustrates the process further.

example 3 Adding Algebraic Expressions

Perform the indicated operation and simplify.

(a) $(2cu + 2u + 1) + (8cu + u + 3)$

(b) $(2x^2 - 5x + 6) + (x^2 + 2)$

Solution

(a) We combine the terms involving cx, the terms involving x, and the constants.

$$\begin{aligned}(2cu + 2u + 1) + (8cu + u + 3) &= (2cu + 8cu) + (2u + u) + (1 + 3)\\ &= 10cu + 3u + 4\end{aligned}$$

(b) We combine the terms involving x^2 and the constants.

$$\begin{aligned}(2x^2 - 5x + 6) + (x^2 + 2) &= (2x^2 + x^2) - 5x + (6 + 2)\\ &= 3x^2 - 5x + 8\end{aligned}$$

■ NOW TRY EXERCISES 15 AND 21 ■

Multiplying Algebraic Expressions

The key to working with expressions is to understand that an algebraic expression (no matter how complicated) represents a number. So an expression can be substituted for a letter in another expression. In particular, when using the Distributive Property,

$$A(B + C) = AB + AC$$

the letters A, B, and C can be replaced by *any expressions*. For example, if we replace A by the expression $a + b$, B by x, and C by y, we get

$$(a + b)(x + y) = (a + b)\cdot x + (a + b)\cdot y$$

This process of replacing letters by expressions, called the **Principle of Substitution**, allows us to find an unlimited variety of true facts about numbers.

example 4 Multiplying Expressions Using the Distributive Property

Expand the product using the Distributive Property: $(a + b)(x + y)$.

Solution

We view the expression $a + b$ as a single number and distribute it over the terms of $x + y$.

$$\begin{aligned}(a + b)(x + y) &= (a + b)x + (a + b)y && \text{Distributive Property, } A = (a + b),\ B = x,\ C = y\\ &= (ax + bx) + (ay + by) && \text{Distributive Property (used twice)}\\ &= ax + bx + ay + by && \text{Associative Property of Addition}\end{aligned}$$

In the last step we removed the parentheses, since by the Associative Property the order of addition doesn't matter.

■ NOW TRY EXERCISE 25 ■

From Example 1 we see that to multiply two binomials we multiply each term in one factor by each term in the other factor and add the results. We can remember this using the mnemonic FOIL (see the margin).

The acronym FOIL helps us to remember that the product of two binomials is the sum of the products of the First terms, the Outer terms, the Inner terms, and the Last terms.

$$(a + b)(x + y) = ax + ay + bx + by$$

F O I L

example 5 Multiplying Binomials Using FOIL

Expand $(2x + 1)(3x - 5)$.

Solution

We use the distributive property (or FOIL).

$$(2x + 1)(3x - 5) = 6x^2 - 10x + 3x - 5 \qquad \text{Distributive Property}$$

F O I L

$$= 6x^2 - 7x - 5 \qquad \text{Combine like terms}$$

NOW TRY EXERCISE 33

Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

Special Product Formulas

If A and B are any real numbers or algebraic expressions, then

1.	$(A + B)(A - B) = A^2 - B^2$	Sum and difference of same terms
2.	$(A + B)^2 = A^2 + 2AB + B^2$	Square of a sum
3.	$(A - B)^2 = A^2 - 2AB + B^2$	Square of a difference

The key idea in using these formulas (or any other formula in algebra) is the Principle of Substitution mentioned above: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2 + y^3)^2$, we use Product Formula 2, substituting x^2 for A and y^3 for B, to get

$$(x^2 + y^3)^2 = (x^2)^2 + 2(x^2)(y^3) + (y^3)^2$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

example 6 Using the Product Formulas

Expand the expressions using the Product Formulas.

(a) $(3x + y)^2$ **(b)** $(5x + 2)(5x - 2)$

Solution

(a) We view the expression $3x$ as a single number and use Product Formula 2.

$$(3x + y)^2 = (3x)^2 + 2 \cdot 3x \cdot y + y^2 \qquad \text{Product Formula 2, } A = 3x \text{ and } B = y$$
$$= 9x^2 + 6xy + y^2 \qquad \text{Simplify}$$

Notice how the formula applies by simply replacing the letters A and B by the appropriate expressions:

$$(3x + y)^2 = (3x)^2 + 2 \cdot (3x) \cdot y + y^2$$
$$(A + B)^2 = A^2 + 2 \cdot A \cdot B + B^2$$

(b) We view the expression $5x$ as a single number and use Product Formula 1.

$$(5x + 2)(5x - 2) = (5x)^2 - 2^2 \qquad \text{Product Formula 1, } A = 5x \text{ and } B = 2$$
$$= 25x^2 - 4 \qquad \text{Simplify}$$

Again, notice how the formula applies by simply replacing the letters A and B by the appropriate expressions.

$$(5x + 2)(5x - 2) = (5x)^2 - 2^2$$
$$(A + B)(A - B) = A^2 - B^2$$

NOW TRY EXERCISES 39 AND 49

example 7 Using the Special Product Formulas

Find the product.

(a) $(2x - \sqrt{y})(2x + \sqrt{y})$ **(b)** $(x + y - 1)(x + y + 1)$

Solution

(a) Substituting $A = 2x$ and $B = \sqrt{y}$ in Product Formula 1, we get

$$(2x - \sqrt{y})(2x + \sqrt{y}) = (2x)^2 - (\sqrt{y})^2 = 4x^2 - y$$

(b) If we group $x + y$ together and think of this as one algebraic expression, we can use Product Formula 1, with $A = x + y$ and $B = 1$.

$$(x + y - 1)(x + y + 1) = [(x + y) - 1][(x + y) + 1]$$
$$= (x + y)^2 - 1^2 \qquad \text{Product Formula 1}$$
$$= x^2 + 2xy + y^2 - 1 \qquad \text{Product Formula 2}$$

NOW TRY EXERCISES 51 AND 55

Combining Polynomials

Certain types of algebraic expressions that occur frequently in algebra have special names; one of these is a polynomial. A **polynomial** in the variable x is an expression

in which each term is a number times a nonnegative integer power of x. For example, the following are polynomials:

$$4x^2 + 3x + 7 \qquad 2x^3 - 5x \qquad 7x^5 + 5x^2 - 7$$

The terms of the first polynomial above are $4x^2$, $3x$, and 7.

We **add** and **subtract** polynomials in the same way we add and subtract any other expressions: by combining like terms. In the case of polynomials, like terms are ones in which the variable is raised to the same power. In subtracting polynomials, we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:

$$-(b + c) = -b - c$$

example 8 Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

Solution

(a) $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$

$= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$ Group like terms

$= 2x^3 - x^2 - 5x + 4$ Combine like terms

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

$= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$ Distributive Property

$= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$ Group like terms

$= -11x^2 + 9x + 4$ Combine like terms

NOW TRY EXERCISES 57 AND 59

When we multiply trinomials or other polynomials with more terms, we use the Distributive Property. It is also helpful to arrange our work in table form. The next example illustrates both methods.

example 9 Multiplying Polynomials

Find the product $(2x + 3)(x^2 - 5x + 4)$.

Solution 1

Using the Distributive Property, we have

$(2x + 3)(x^2 - 5x + 4)$

$= 2x(x^2 - 5x + 4) + 3 \cdot (x^2 - 5x + 4)$ Distributive Property

$= (2x \cdot x^2 - 2x \cdot 5x + 2x \cdot 4) + (3 \cdot x^2 - 3 \cdot 5x + 3 \cdot 4)$ Distributive Property

$= (2x^3 - 10x^2 + 8x) + (3x^2 - 15x + 12)$ Laws of Exponents

$= 2x^3 - 7x^2 - 7x + 12$ Combine like terms

Solution 2

Using table form, we have

$$\begin{array}{rl} x^2 - 5x + 4 & \text{First factor} \\ \underline{\qquad\qquad 2x + 3} & \text{Second factor} \\ 3x^2 - 15x + 12 & \text{Multiply } x^2 - 5x + 4 \text{ by } 3 \\ \underline{2x^3 - 10x^2 + 8x\qquad} & \text{Multiply } x^2 - 5x + 4 \text{ by } 2x \\ 2x^3 - 7x^2 - 7x + 12 & \text{Add like terms} \end{array}$$

NOW TRY EXERCISE 61

B.1 Exercises

CONCEPTS

1. To add expressions, we add _____ terms. So

$(3a + 2b + 4) + (a - b + 1) =$ _____.

2. To subtract expressions, we subtract _____ terms. So

$(2xy + 9b + c + 10) - (xy + b + 6c + 8) =$ _____.

3. Which of the following expressions are polynomials?

(a) $2x^2 - 3x$ **(b)** $x^3 + \dfrac{2}{x}$ **(c)** $x^5 + 2x^4 + \frac{1}{3}x + 3$

4. Explain how we multiply two binomials, then perform the following multiplication:

$(x + 2)(x + 3) =$ _____.

5. The Special Product Formula for the "square of a sum" is

$(A + B)^2 =$ _____. So $(2x + 3)^2 =$ _____.

6. The Special Product Formula for the "sum and difference of the same terms" is

$(A + B)(A - B) =$ _____. So $(5 + x)(5 - x) =$ _____.

SKILLS

7–10 ■ An expression is given. Describe the expression in words, and find the value of the expression when x is 3.

7. **(a)** $3x - 5$ **(b)** $3(x - 5)$ **8.** **(a)** $\dfrac{x - 2}{5}$ **(b)** $x - \frac{2}{5}$

9. **(a)** $x^2 + 1$ **(b)** $(x + 1)^2$ **10.** **(a)** $(5 + x)2$ **(b)** $5(x + 2)$

11–14 ■ An expression is given.

(a) What are the terms of the expression?

(b) Find the value of the expression if a is -1, b is 4, x is -2, and y is 3.

11. $2ax - 10a + 1$ **12.** $axy + b$

13. $(xy)^2 + a^2$ **14.** $\dfrac{x}{2} + b + 5$

15–22 ■ Find the sum or difference.

15. $(12x - 7) - (5x - 12)$ **16.** $(5 - 3x) + (2x - 8)$

17. $(4x^2 + 2x) + (3x^2 - 5x + 6)$ **18.** $(-2x^2 - 1) + (4x - 2)$

19. $(4aw + 3w) + (5aw - 2w)$ **20.** $(5by - 2b + 7) - (2by - 3b + 2)$

21. $(6ax + 7ay - 2a) - (8ax + 4a)$ **22.** $(-2bu + 3cv + 7) + (bu - cv)$

23–26 ■ Multiply the two expressions using the Distributive Property.

23. $(a - b)(x - y)$

24. $(a + b)(x - y)$

25. $(w + a + b)(u + v)$

26. $(a + b)(u + v + w)$

27–34 ■ Multiply the algebraic expressions using the FOIL method and simplify.

27. $(x + 4)(x - 3)$

28. $(y - 1)(y + 5)$

29. $(r - 3)(r + 5)$

30. $(s + 8)(s - 2)$

31. $(3t - 2)(7t - 4)$

32. $(4s - 1)(2s + 5)$

33. $(3x + 5)(2x - 1)$

34. $(7y - 3)(2y - 1)$

35–56 ■ Multiply the algebraic expressions using a Special Product Formula and simplify.

35. $(x + 3)^2$

36. $(x - 2)^2$

37. $(3x + 4)^2$

38. $(1 - 2y)^2$

39. $(2u + v)^2$

40. $(x - 3y)^2$

41. $(2x + 3y)^2$

42. $(r - 2s)^2$

43. $(x^2 + 1)^2$

44. $(2 + y^3)^2$

45. $(x + 5)(x - 5)$

46. $(y - 3)(y + 3)$

47. $(3x - 4)(3x + 4)$

48. $(2y + 5)(2y - 5)$

49. $(x + 3y)(x - 3y)$

50. $(2u + v)(2u - v)$

51. $(\sqrt{x} + 2)(\sqrt{x} - 2)$

52. $(\sqrt{a} - b)(\sqrt{a} + b)$

53. $((x - 1) + x^2)((x - 1) - x^2)$

54. $(x + (2 + x^2))(x - (2 + x^2))$

55. $(2x + y - 3)(2x + y + 3)$

56. $(x + y + z)(x + y - z)$

57–60 ■ Find the sum or difference of the polynomials.

57. $(2x^3 - 4x^2 + 3x + 2) + (4x^3 - 6x^2 + x)$

58. $(4x^4 + 3x^2 - 1) + (2x^4 + 3x^3 - 2x)$

59. $(3x^6 + 2x^3 - 6) - (2x^3 - 2x - 5)$

60. $(4x^3 - 2x^2) - (3x^2 - 2x + 4)$

61–64 ■ Find the product of the polynomials.

61. $(x + 2)(x^2 + 2x + 3)$

62. $(x + 1)(2x^2 - x + 1)$

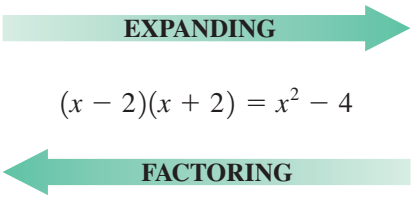
63. $(2x - 5)(x^2 - x + 1)$

64. $(1 + 2x)(x^2 - 3x + 1)$

B.2 Factoring Algebraic Expressions

- Factoring Out Common Factors
- Factoring Trinomials
- Special Factoring Formulas
- Factoring by Grouping

In **Algebra Toolkit B.1**, we used the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by factoring an expression as a product of simpler ones. For example, we can write



We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

Factoring Out Common Factors

The easiest type of factoring occurs when the terms have a common factor.

example 1 Factoring Out Common Factors

Factor each expression.

(a) $3x^2 - 6x$

(b) $8x^4y^2 + 6x^3y^3 - 2xy^4$

Solution

(a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$3x^2 - 6x = 3x(x - 2)$$

(b) We note that

8, 6, and -2 have the greatest common factor 2

x^4, x^3, and x have the greatest common factor x

y^2, y^3, and y^4 have the greatest common factor y^2

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\begin{aligned} 8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) - (2xy^2)(y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2) \end{aligned}$$

■ NOW TRY EXERCISES 7 AND 13 ■

example 2 Factoring Out Common Factors

Factor $(2x + 4)(x - 3) - 5(x - 3)$.

Solution

This expression has two terms, and each term contains the factor $x - 3$. So

$$(2x + 4)(x - 3) - 5(x - 3) = [(2x + 4) - 5](x - 3) \quad \text{Distributive Property}$$
$$= (2x - 1)(x - 3) \quad \text{Simplify}$$

NOW TRY EXERCISE 15

Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

example 3 Factoring a Trinomial

Factor $x^2 + 7x + 12$.

Solution

We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus, the factorization is

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

(Factors of 12)

CHECK To check that this factorization is correct, we multiply.

$$(x + 3)(x + 4) = x^2 + 3x + 4x + 12 = x^2 + 7x + 12 \quad ✓$$

NOW TRY EXERCISE 17

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

Therefore, we try to find numbers p, q, r, and s such that $pq = a$, $rs = c$, and $ps + qr = b$. If these numbers are all integers, then we will have a limited number of possibilities for p, q, r, and s.

example 4 Factoring a Trinomial

Factor $6x^2 + 7x - 5$.

Solution

We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and we can factor -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

Factors of 6

$$6x^2 + 7x - 5 = (3x + 5)(2x - 1)$$

Factors of -5

✓ CHECK To check that this factorization is correct, we multiply.

$$(3x + 5)(2x - 1) = 6x^2 - 3x + 10x - 5 = 6x^2 + 7x - 5 \quad ✓$$

NOW TRY EXERCISE 23

example 5 Recognizing the Form of an Expression

Factor each expression.

(a) $x^2 - 2x - 3$ **(b)** $(5a + 1)^2 - 2(5a + 1) - 3$

Solution

(a) By trial and error we find that

$$x^2 - 2x - 3 = (x - 3)(x + 1)$$

(b) This expression is of the form

$$\square^2 - 2\square - 3$$

where $\square$ represents $5a + 1$. This is the same form as the expression in part (a), so it will factor as $(\square - 3)(\square + 1)$.

$$(5a + 1)^2 - 2(5a + 1) - 3 = [(5a + 1) - 3][(5a + 1) + 1]$$
$$= (5a - 2)(5a + 2)$$

NOW TRY EXERCISE 29

Special Factoring Formulas

Some special algebraic expressions can be factored by using the following formulas. These are simply Special Product Formulas written backward.

Special Product Formulas

If A and B are any real numbers or algebraic expressions, then

1. $A^2 - B^2 = (A - B)(A + B)$ Difference of Squares
2. $A^2 + 2AB + B^2 = (A + B)^2$ Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$ Perfect square

example 6 Factoring Differences of Squares

Factor each polynomial.

(a) $4x^2 - 25$ **(b)** $(x + y)^2 - z^2$

Solution

(a) Using the Difference of Squares Formula with $A = 2x$ and $B = 5$, we have

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$$

$$A^2 - B^2 = (A - B)(A + B)$$

(b) We use the Difference of Squares Formula with $A = x + y$ and $B = z$.

$$(x + y)^2 - z^2 = (x + y - z)(x + y + z)$$

NOW TRY EXERCISES 33 AND 37

example 7 Recognizing Perfect Squares

Factor each trinomial.

(a) $x^2 + 6x + 9$ **(b)** $4x^2 - 4xy + y^2$

Solution

(a) Here $A = x$ and $B = 3$, so $2AB = 2 \cdot x \cdot 3 = 6x$. Since the middle term is $6x$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$x^2 + 6x + 9 = (x + 3)^2$$

(b) Here $A = 2x$ and $B = y$, so $2AB = 2 \cdot 2x \cdot y = 4xy$. Since the middle term is $-4xy$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$4x^2 - 4xy + y^2 = (2x - y)^2$$

NOW TRY EXERCISES 41 AND 47

When we factor an expression, the result can sometimes be factored further. In general, we first factor out common factors, then inspect the result to see whether it can be factored by any of the other methods of this section. We repeat this process until we have factored the expression completely.

example 8 Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ **(b)** $x^5y^2 - xy^6$

Solution

(a) We first factor out the power of x with the smallest exponent.

$$2x^4 - 8x^2 = 2x^2(x^2 - 4) \qquad \text{Common factor is } 2x^2$$

$$= 2x^2(x - 2)(x + 2) \qquad \text{Factor } x^2 - 4 \text{ as a Difference of Squares}$$

(b) We first factor out the powers of x and y with the smallest exponents.

$$x^5y^2 - xy^6 = xy^2(x^4 - y^4) \qquad \text{Common factor is } xy^2$$

$$= xy^2(x^2 + y^2)(x^2 - y^2) \qquad \text{Factor } x^4 - y^4 \text{ as a Difference of Squares}$$

$$= xy^2(x^2 + y^2)(x + y)(x - y) \qquad \text{Factor } x^2 - y^2 \text{ as a Difference of Squares}$$

NOW TRY EXERCISES 49 AND 53

Factoring by Grouping

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates the idea.

example 9 **Factoring by Grouping**

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ **(b)** $x^3 - 2x^2 - 3x + 6$

Solution

(a)

$$\begin{aligned} x^3 + x^2 + 4x + 4 &= (x^3 + x^2) + (4x + 4) && \text{Group terms} \\ &= x^2(x + 1) + 4(x + 1) && \text{Factor out common factors} \\ &= (x^2 + 4)(x + 1) && \text{Factor out } x + 1 \text{ from each term} \end{aligned}$$

(b)

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) + (-3x + 6) && \text{Group terms} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor out common factors} \\ &= (x^2 - 3)(x - 2) && \text{Factor out } x - 2 \text{ from each term} \end{aligned}$$

■ NOW TRY EXERCISES 57 AND 63 ■

B.2 Exercises

CONCEPTS

1. Consider the polynomial $2x^5 + 6x^4 + 4x^3$.

(a) How many terms does this polynomial have? _____.

(b) List the terms: _____.

(c) What factor is common to each term? _____.

(d) Factor the polynomial: $2x^5 + 6x^4 + 4x^3 =$ _____.

2. To factor the trinomial $x^2 + 7x + 10$, we look for two integers whose product is _____ and whose sum is _____. These integers are _____ and _____, so the trinomial factors as _____.

3. The Special Factoring Formula for the difference of squares is $A^2 - B^2 =$ _____. So $4x^2 - 25$ factors as _____.

4. The Special Factoring Formula for a perfect square is $A^2 + 2AB + B^2 =$ _____. So $x^2 + 10x + 25$ factors as _____.

SKILLS

5–16 ■ Factor out the common factor.

5. $5a - 20$

6. $-3b + 12$

7. $30x^3 + 15x^4$

8. $12x^3 + 18x$

9. $-2x^3 + 16x$

10. $6y^4 - 15y^3$

11. $5ab - 8abc$

12. $2x^4 + 4x^3 - 14x^2$

13. $2x^2y - 6xy^2 + 3xy$

14. $-7x^4y^2 + 14xy^3 + 21xy^4$

15. $y(y - 6) + 9(y - 6)$

16. $(z + 2)^2 - 5(z + 2)$

17–28 ■ Factor the trinomial.

17. $x^2 + 2x - 3$

18. $x^2 - 6x + 5$

19. $x^2 + 2x - 15$

20. $x^2 - 14x + 48$

21. $y^2 - 8y + 15$

22. $z^2 + 6z - 16$

23. $2x^2 - 5x - 7$

24. $3x^2 - 16x + 5$

25. $5x^2 - 7x - 6$

26. $2x^2 + 7x - 4$

27. $9x^2 - 36x - 45$

28. $8x^2 + 10x + 3$

29–30 ■ Factor each expression.

29. **(a)** $x^2 + 8x + 12$ **(b)** $(3x + 2)^2 + 8(3x + 2) + 12$

30. **(a)** $2x^2 + 5x - 3$ **(b)** $2(a + b)^2 + 5(a + b) - 3$

31–40 ■ Use the Difference of Squares formula to factor the expression.

31. $x^2 - 36$

32. $y^2 - 100$

33. $9a^2 - 16$

34. $4x^2 - 25$

35. $49 - 4y^2$

36. $4t^2 - 9s^2$

37. $(x + 3)^2 - 4$

38. $4(2x + 1)^2 - 9$

39. $(a + b)^2 - (a - b)^2$

40. $\left(1 + \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)^2$

41–48 ■ Factor the perfect square.

41. $x^2 + 12x + 36$

42. $y^2 + 10y + 25$

43. $t^2 - 6t + 9$

44. $t^2 + 10t + 25$

45. $16z^2 - 24z + 9$

46. $25u^2 - 10u + 1$

47. $4w^2 + 4wy + y^2$

48. $r^2 - 6rs + 9s^2$

49–56 ■ Factor the expression completely.

49. $3x^3 - 27x$

50. $x^3 + 2x^2 + x$

51. $x^4 + 2x^3 - 3x^2$

52. $2x^3 + 8x^2 + 8x$

53. $x^4y^3 - x^2y^5$

54. $18x^4y - 2x^2y^3$

55. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$

56. $(x + 1)^3x - 2(x + 1)^2x^2 + (x + 1)x^3$

57–66 ■ Factor the expression by grouping terms.

57. $x^3 + 4x^2 + x + 4$

58. $3x^3 - x^2 + 6x - 2$

59. $2r^3 + r^2 - 6r - 3$

60. $-9u^3 - 3u^2 + 3u + 1$

61. $x^3 + x^2 + x + 1$

62. $x^5 + x^4 + x + 1$

63. $y^3 - 3y^2 - 4y + 12$

64. $y^3 - y^2 + y - 1$

65. $2t^3 + 4t^2 + t + 2$

66. $3s^3 + 5s^2 - 6s - 10$

B.3 Rational Expressions

- Simplifying Rational Expressions
- Multiplying and Dividing Rational Expressions
- Adding and Subtracting Rational Expressions
- Rationalizing the Denominator
- Long Division

A **rational expression** is a fractional expression in which the numerator and denominator are both polynomials. For example, the following are rational expressions:

$$\frac{2x}{x-1} \qquad \frac{x}{x^2+1} \qquad \frac{x^3-x}{x^2-5x+6}$$

In this section we learn how to perform algebraic operations on rational expressions.

Simplifying Rational Expressions

To **simplify rational expressions**, we factor both the numerator and denominator and use the following property of fractions.

$$\frac{AC}{BC} = \frac{A}{B} \qquad C \neq 0$$

This allows us to **cancel** common factors from the numerator and denominator, as in the case of the following fraction:

$$\frac{6}{10} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{3}{5}$$

example 1 Simplifying Rational Expressions by Canceling

Simplify $\dfrac{x^2-1}{x^2+x-2}$.

Solution

$$\frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} \qquad \text{Factor}$$

$$= \frac{x+1}{x+2} \qquad \text{Cancel common factors}$$

NOW TRY EXERCISE 17

Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}$$

This says that to multiply two fractions, we multiply their numerators and denominators.

example 2 Multiplying Rational Expressions

Perform the indicated multiplication and simplify.

$$\frac{x^2+2x-3}{x^2+8x+16}\cdot\frac{3x+12}{x-1}$$

Solution

First we factor.

$$\frac{x^2+2x-3}{x^2+8x+16}\cdot\frac{3x+12}{x-1}=\frac{(x-1)(x+3)}{(x+4)^2}\cdot\frac{3(x+4)}{x-1}$$ Factor

$$=\frac{3(x-1)(x+3)(x+4)}{(x-1)(x+4)^2}$$ Multiply fractions

$$=\frac{3(x+3)}{x+4}$$ Cancel common factors

NOW TRY EXERCISE 23

To **divide rational expressions**, we use the following property of fractions.

$$\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}$$

This says that to divide a fraction by another fraction, we invert the divisor and multiply.

example 3 Dividing Rational Expressions

Perform the indicated division and simplify:

$$\frac{x-4}{x^2-4}\div\frac{x^2-3x-4}{x^2+5x+6}$$

Solution

To divide these rational expressions, we invert the divisor and multiply.

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4}$$ Invert divisor and multiply

$$= \frac{x-4}{(x-2)(x+2)} \cdot \frac{(x+2)(x+3)}{(x-4)(x+1)}$$ Factor

$$= \frac{x+3}{(x-2)(x+1)}$$ Cancel common factors

NOW TRY EXERCISE 27

Adding and Subtracting Rational Expressions

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD). The LCD is found by factoring each denominator and then taking the product of the distinct factors, using the highest power that appears in any of the factors.

example 4 Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify.

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$

(b) $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

Solution

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x+2)(x-1)}$$ Write fractions using LCD

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)}$$ Add fractions

$$= \frac{x^2+2x+6}{(x-1)(x+2)}$$ Combine terms in numerator

(b) The denominators are $x^2-1 = (x-1)(x+1)$ and $(x+1)^2$, so the LCD is $(x-1)(x+1)^2$.

$$\frac{1}{x^2-1}-\frac{2}{(x+1)^2}=\frac{1}{(x-1)(x+1)}-\frac{2}{(x+1)^2}$$ Factor

$$=\frac{(x+1)-2(x-1)}{(x-1)(x+1)^2}$$ Combine fractions using LCD

$$=\frac{x+1-2x+2}{(x-1)(x+1)^2}$$ Distributive Property

$$=\frac{3-x}{(x-1)(x+1)^2}$$ Combine terms in numerator

NOW TRY EXERCISES 35 AND 37

Rationalizing the Denominator

If a fraction has a denominator of the form $A+B\sqrt{C}$, we may "rationalize the denominator" by multiplying the numerator and denominator by the **conjugate radical** $A-B\sqrt{C}$.

example 5 Rationalizing a Denominator

Rationalize the denominator: $\dfrac{1}{1+\sqrt{2}}$.

Solution

We multiply both the numerator and the denominator by the conjugate radical of $1+\sqrt{2}$, which is $1-\sqrt{2}$.

$$\frac{1}{1+\sqrt{2}}=\frac{1}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}$$ Multiply numerator and denominator by the conjugate radical

$$=\frac{1-\sqrt{2}}{1-(\sqrt{2})^2}$$ Product Formula 1: $(a+b)(a-b)=a^2-b^2$

$$=\frac{1-\sqrt{2}}{1-2}=\frac{1-\sqrt{2}}{-1}=\sqrt{2}-1$$ Simplify

NOW TRY EXERCISE 41

Long Division

Another way to simplify a rational expression is to perform long division in much the same way that we divide numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

$$\frac{38}{7}=5+\frac{3}{7}$$

Dividend, Remainder, Divisor, Quotient

In the next example we divide rational expressions.

example 6 Long Division of a Rational Expression

Divide $6x^2 - 26x + 12$ by $x - 4$.

Solution

We begin by arranging the expressions as follows.

$$x - 4\overline{)6x^2 - 26x + 12}$$

Next we divide the leading term in the dividend by the leading term in the divisor to get the first term of the quotient: $6x^2/x = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend.

$$\begin{array}{rl} & 6x \\ x - 4\overline{)6x^2 - 26x + 12} & \text{Divide leading terms: } \frac{6x^2}{x} = 6x \\ \underline{6x^2 - 24x} & \text{Multiply: } 6x(x - 4) = 6x^2 - 24x \\ -2x + 12 & \text{Subtract and "bring down" 12} \end{array}$$

We repeat the process using the last line $-2x + 12$ as the dividend.

$$\begin{array}{rl} & 6x - 2 \\ x - 4\overline{)6x^2 - 26x + 12} & \text{Divide leading terms: } \frac{-2x}{x} = -2 \\ \underline{6x^2 - 24x} & \\ -2x + 12 & \\ \underline{-2x + 8} & \text{Multiply: } -2(x - 4) = -2x + 8 \\ 4 & \text{Subtract} \end{array}$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in the following way:

$$\frac{\overbrace{6x^2 - 26x + 12}^{\text{Dividend}}}{\underbrace{x - 4}_{\text{Divisor}}} = \overbrace{6x - 2}^{\text{Quotient}} + \overbrace{\frac{4}{x - 4}}^{\text{Remainder}}$$

■ NOW TRY EXERCISE 47 ■

B.3 Exercises

CONCEPTS

1. Which of the following are rational expressions?

(a) $\dfrac{3x}{x^2 - 1}$ **(b)** $\dfrac{\sqrt{x + 1}}{2x + 3}$ **(c)** $\dfrac{x(x^2 - 1)}{x + 3}$

2. To simplify a rational expression, we cancel *factors* that are common to the _____ and _____. So the expression $\dfrac{(x + 1)(x + 2)}{(x + 3)(x + 2)}$ simplifies to _____.

3. True or false?

(a) $\dfrac{x^2 + 3}{x^2 + 5}$ simplifies to $\dfrac{3}{5}$ **(b)** $\dfrac{3x^2}{5x^2}$ simplifies to $\dfrac{3}{5}$

4. **(a)** To multiply two rational expressions, we multiply their _____ and multiply their _____. So $\frac{2}{x+1} \cdot \frac{x}{x+3}$ is the same as _____.

(b) To divide two rational expressions, we _____ the divisor, then multiply. So $\frac{3}{x+5} \div \frac{x}{x+2}$ is the same as _____.

5. Consider the expression $\frac{1}{x} - \frac{2}{x+1} - \frac{x}{(x+1)^2}$

(a) How many terms does this expression have?

(b) Find the least common denominator of all the terms.

(c) Perform the addition and simplify.

6. True or false?

(a) $\frac{1}{2} + \frac{1}{x}$ is the same as $\frac{1}{2+x}$ **(b)** $\frac{1}{2} + \frac{1}{x}$ is the same as $\frac{x+2}{2x}$

SKILLS

7–8 ■ An expression is given. Evaluate it at the given value.

7. $\frac{2s+1}{s-4}$, 7 **8.** $\frac{2t^2-5}{3t+6}$, 1

9–20 ■ Simplify the rational expression.

9. $\frac{12x}{6x^2}$ **10.** $\frac{81x^3}{18x}$

11. $\frac{5y^2}{10y+y^2}$ **12.** $\frac{14t^2-t}{7t}$

13. $\frac{3(x+2)(x-1)}{6(x-1)^2}$ **14.** $\frac{4(x^2-1)}{12(x-2)(x-1)}$

15. $\frac{x-2}{x^2-4}$ **16.** $\frac{x^2-x-2}{x^2-1}$

17. $\frac{x^2+6x+8}{x^2+5x+4}$ **18.** $\frac{x^2+x-12}{x^2-5x+6}$

19. $\frac{y^2+y}{y^2-1}$ **20.** $\frac{y^2-3y-18}{y^2+4y+3}$

21–30 ■ Perform the multiplication or division and simplify.

21. $\frac{4x}{x^2-4} \cdot \frac{x+2}{16x}$ **22.** $\frac{x^2-25}{x^2-16} \cdot \frac{x+4}{x+5}$

23. $\frac{x^2-2x-15}{x^2-9} \cdot \frac{x+3}{x-5}$ **24.** $\frac{x^2+2x-3}{x^2-2x-3} \cdot \frac{3-x}{3+x}$

25. $\frac{t-3}{t^2+9} \cdot \frac{t+3}{t^2-9}$ **26.** $\frac{x^2-x-6}{x^2+2x} \cdot \frac{x^3+x^2}{x^2-2x-3}$

27. $\dfrac{x+3}{4x^2-9} \div \dfrac{x^2+7x+12}{2x^2+7x-15}$

28. $\dfrac{2x+1}{2x^2+x-15} \div \dfrac{6x^2-x-2}{x+3}$

29. $\dfrac{2x^2+3x+1}{x^2+2x-15} \div \dfrac{x^2+6x+5}{2x^2-7x+3}$

30. $\dfrac{4y^2-9}{2y^2+9y-18} \div \dfrac{2y^2+y-3}{y^2+5y-6}$

31–40 ■ Perform the addition or subtraction and simplify.

31. $2+\dfrac{x}{x+3}$

32. $\dfrac{2x-1}{x+4}-1$

33. $\dfrac{1}{x+5}+\dfrac{2}{x-3}$

34. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$

35. $\dfrac{1}{x+1}-\dfrac{1}{x+2}$

36. $\dfrac{x}{x-4}-\dfrac{3}{x+6}$

37. $\dfrac{x}{(x+1)^2}+\dfrac{2}{x+1}$

38. $\dfrac{5}{2x-3}-\dfrac{3}{(2x-3)^2}$

39. $\dfrac{1}{x^2}+\dfrac{1}{x^2+x}$

40. $\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}$

41–46 ■ Rationalize the denominator.

41. $\dfrac{1}{2-\sqrt{3}}$

42. $\dfrac{2}{3-\sqrt{5}}$

43. $\dfrac{2}{\sqrt{2}+\sqrt{7}}$

44. $\dfrac{1}{\sqrt{x}+1}$

45. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$

46. $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

47–52 ■ Find the quotient and remainder using long division.

47. $\dfrac{x^2-6x-8}{x-4}$

48. $\dfrac{x^3-x^2-2x+6}{x-2}$

49. $\dfrac{4x^3+2x^2-2x-3}{2x+1}$

50. $\dfrac{x^3+3x^2+4x+3}{3x+6}$

51. $\dfrac{x^6+x^4+x^2+1}{x^2+1}$

52. $\dfrac{6x^3+2x^2+20x}{2x^2+5}$

Algebra Toolkit C

Working with Equations

C.1 Solving Basic Equations
C.2 Solving Quadratic Equations
C.3 Solving Inequalities

C.1 Solving Basic Equations

- Equations
- The Distributive Property: Solving Equations
- Solving Linear Equations
- Solving Power Equations
- Solving for One Variable in Terms of Others

Equations

An **equation** is a statement that two arithmetic expressions are equal. For example $3 + 5 = 8$ is an equation. Most equations that we study in algebra contain variables. In the equation

$$4x + 7 = 19$$

the letter x is the variable. We think of x as the "unknown," and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** (or **roots**) of the equation.

Notice that an equation is a "complete sentence" (unlike algebraic expressions, which are just phrases). The above equation says:

"Four times a number plus seven equals nineteen."

Our goal is to find that number.

The process of finding the solutions of an equation is called **solving the equation**. To solve an equation, we "isolate" the variable on one side of the equation. To do this, we use the following operations on equations.

Operations on Equations

1. Add (or subtract) the same quantity to each side.
2. Multiply (or divide) each side by the same nonzero quantity.

When solving an equation, we must *perform the same operation on both sides of the equation*. Thus, if we say "*add* -7" when solving an equation, that is just a short way of saying "*add* -7 to each side of the equation."

example 1 Reading an Equation

Consider the equation $3x - 8 = 7$.

(a) Express the equation in words.

(b) Is 6 a solution of the equation?

(c) Solve the equation.

Solution

(a) The equation says "Three times a number minus eight is seven."

(b) If we replace x by 6 in the equation, we get

To check whether 6 is a solution, replace x by 6.

$3x - 8 = 7$

Replace x by 6

$$\begin{aligned} 3x - 8 &= 7 && \text{Equation} \\ 3 \cdot 6 - 8 &= 7 && \text{Replace } x \text{ by } 6 \\ 10 &= 7 && \text{False} \ \times \end{aligned}$$

So 6 is not a solution to the equation.

(c) Our goal is to isolate the unknown x on one side of the equal sign.

$$\begin{aligned} 3x - 8 &= 7 && \text{Equation} \\ 3x &= 15 && \text{Add 8 to both sides} \\ x &= 5 && \text{Divide by 3} \end{aligned}$$

So 5 is a solution to the equation.

To check whether 5 is a solution, replace x by 5.

$3x - 8 = 7$

Replace x by 5

✓ **CHECK** We replace x by 5 in the original equation:

LHS: $3 \cdot 5 - 8 = 7$ RHS: 7

LHS = RHS ✓

NOW TRY EXERCISE 5

The Distributive Property: Solving Equations

The equation in Example 1 was quite easy to solve. But sometimes it's not so easy to solve an equation. For example, in the equation

$$5(3 + x) = 9x$$

the unknown x is added to 3, then the result is multiplied by 5. We have both addition and multiplication interacting in this equation; moreover, x is on both sides of the equation. To unravel this equation and get x alone, we need to use the *Distributive Property*. Let's see how the Distributive Property is used to solve this equation.

example 2 Solving an Equation

Solve the equation $5(3 + x) = 9x$.

Solution

We use the Distributive Property to help us put x alone on one side of the equation (that is, *isolate* x):

$$\begin{aligned} 5(3 + x) &= 9x && \text{Given equation} \\ 15 + 5x &= 9x && \text{Distributive Property} \\ 15 &= 4x && \text{Subtract } 5x \text{ from each side} \\ x &= \frac{15}{4} && \text{Divide by 4 and switch sides} \\ x &= 3.75 && \text{Calculate} \end{aligned}$$

So the solution is 3.75.

✓ CHECK To check this solution, we replace x by 3.75 and see if we get a true equation.

LHS: $5(3 + 3.75) = 5(6.75) = 33.75$ RHS: $9(3.75) = 33.75$

LHS = RHS ✓

NOW TRY EXERCISE 13

Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

Linear Equations

A **linear equation** in one variable is an equation that is equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

For example, the equation $4x - 5 = 3$ is linear, but $x^2 + 2x = 8$ is not linear.

example 3 Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

Solution

We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$$\begin{aligned} 7x - 4 &= 3x + 8 && \text{Given equation} \\ 7x &= 3x + 12 && \text{Add } 4 \\ 4x &= 12 && \text{Subtract } 3x \\ x &= 3 && \text{Divide by } 4 \end{aligned}$$

✓ CHECK We substitute $x = 3$ into each side of the original equation:

LHS: $7 \cdot 3 - 4 = 17$ RHS: $3 \cdot 3 + 8 = 17$

LHS = RHS ✓

NOW TRY EXERCISE 11

When a linear equation involves fractions, solving the equation is usually easier if we first multiply each side by the LCD of the fractions, as we see in the following examples.

example 4 Solving an Equation That Involves Fractions

Solve the equation $\frac{x}{6} + \frac{2}{3} = \frac{3}{4}x$.

Solution

The LCD of the denominators (6, 3, and 4) is 12, so we first multiply each side of the equation by 12 to clear the denominators.

$$\begin{aligned} \frac{x}{6} + \frac{2}{3} &= \frac{3}{4}x && \text{Given equation} \\ 12 \cdot \left(\frac{x}{6} + \frac{2}{3} \right) &= 12 \cdot \frac{3}{4}x && \text{Multiply by LCD} \\ 2x + 8 &= 9x && \text{Distributive Property} \\ 8 &= 7x && \text{Subtract } 2x \\ x &= \frac{8}{7} && \text{Divide by 7 and switch sides} \end{aligned}$$

NOW TRY EXERCISE 17

It is always important to check your answer, even if you never make a mistake in your calculations. This is because you sometimes end up with **extraneous solutions**, which are potential solutions that do not satisfy the original equation. The next example shows how this can happen.

example 5 An Equation with No Solution

Solve the equation $2 + \frac{5}{x - 4} = \frac{x + 1}{x - 4}$.

Solution

First, we multiply each side by the common denominator, which is $x - 4$.

$$2 + \frac{5}{x-4} = \frac{x+1}{x-4} \qquad \text{Given equation}$$

$$(x-4)\left(2 + \frac{5}{x-4}\right) = (x-4)\left(\frac{x+1}{x-4}\right) \qquad \text{Multiply by } x-4$$

$$2(x-4) + 5 = x + 1 \qquad \text{Distributive Property}$$

$$2x - 8 + 5 = x + 1 \qquad \text{Distributive Property}$$

$$2x - 3 = x + 1 \qquad \text{Simplify}$$

$$2x = x + 4 \qquad \text{Add 3}$$

$$x = 4 \qquad \text{Subtract } x$$

But now if we try to substitute $x = 4$ back into the original equation, we would be dividing by 0, which is impossible. So this equation has *no solution*.

NOW TRY EXERCISE 31

The first step in the preceding solution, multiplying by $x - 4$, had the effect of multiplying by 0. (Do you see why?) Multiplying each side of an equation by an expression that contains the variable may introduce extraneous solutions. That is why it is important to check every answer.

Solving Power Equations

Linear equations have variables only to the first power. Now let's consider some equations that involve squares, cubes, and other powers of the variable. Here we just consider basic equations that can be simplified into the form $X^n = a$. Equations of this form are called **power equations** and can be solved by taking radicals of both sides of the equation.

Solving a Power Equation

The power equation $X^n = a$ has the solution

$$X = \sqrt[n]{a} \text{ if } n \text{ is odd}$$

$$X = \pm\sqrt[n]{a} \text{ if } n \text{ is even and } a \geq 0$$

If n is even and $a < 0$, the equation has no real solution.

Here are some examples of solving power equations:

The equation $x^5 = 32$ has only one real solution: $x = \sqrt[5]{32} = 2$.

The equation $x^4 = 16$ has two real solutions: $x = \pm\sqrt[4]{16} = \pm 2$.

The equation $x^5 = -32$ has only one real solution: $x = \sqrt[5]{-32} = -2$.

The equation $x^4 = -16$ has no real solutions because $\sqrt[4]{-16}$ does not exist.

example 6 Solving Power Equations

Solve each equation.

(a) $x^2 - 5 = 0$ **(b)** $(x - 4)^2 = 5$

Solution

(a) We first isolate the x^2 term and then take the square root of each side.

$$x^2 - 5 = 0 \quad \text{Given equation}$$
$$x^2 = 5 \quad \text{Add 5}$$
$$x = \pm\sqrt{5} \quad \text{Take the square root}$$

The solutions are $x = \sqrt{5}$ and $x = -\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$(x - 4)^2 = 5 \quad \text{Given equation}$$
$$x - 4 = \pm\sqrt{5} \quad \text{Take the square root}$$
$$x = 4 \pm \sqrt{5} \quad \text{Add 4}$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$. Check that each answer satisfies the original equation.

■ NOW TRY EXERCISES 35 AND 41 ■

example 7 Solving Power Equations

Find all real solutions for each equation.

(a) $x^3 = -8$ **(b)** $16x^4 = 81$

Solution

(a) Since every real number has exactly one real cube root, we can solve this equation by taking the cube root of each side.

$$x^3 = -8 \quad \text{Given equation}$$
$$x = -2 \quad \text{Take the cube root}$$

(b) After isolating the x^4 term, we take the fourth root of each side of the equation.

$$16x^4 = 81 \quad \text{Given equation}$$
$$x^4 = \tfrac{81}{16} \quad \text{Divide by 16}$$
$$(x^4)^{1/4} = \left(\tfrac{81}{16}\right)^{1/4} \quad \text{Take the fourth root}$$
$$x = \pm\tfrac{3}{2} \quad \text{Simplify}$$

■ NOW TRY EXERCISES 43 AND 45 ■

example 8 Solving Power Equations

Find all real solutions of each equation.

(a) $3 = 0.6x^{0.2}$ **(b)** $3x^{-5} = 2$

Solution

(a) After isolating the $x^{0.2}$ term, we raise both sides of the equation to the $\frac{1}{0.2}$ power.

$$\begin{aligned} 3 &= 0.6x^{0.2} && \text{Given equation} \\ \frac{3}{0.6} &= x^{0.2} && \text{Divide by } 0.6 \\ (5)^{1/0.2} &= (x^{0.2})^{1/0.2} && \text{Raise both sides to } \tfrac{1}{0.2} \text{ power} \\ x &= (5)^{1/0.2} && \text{Property of Exponents; switch sides} \\ x &= 3125 && \text{Calculator} \end{aligned}$$

(b) After isolating the x^{-5} term, we raise both sides to the $-\frac{1}{5}$ power.

$$\begin{aligned} 3x^{-5} &= 2 && \text{Given equation} \\ x^{-5} &= \tfrac{2}{3} && \text{Divide by } 3 \\ (x^{-5})^{-1/5} &= (\tfrac{2}{3})^{-1/5} && \text{Raise both sides to } -\tfrac{1}{5} \text{ power} \\ x &= (\tfrac{2}{3})^{-1/5} && \text{Property of Exponents} \\ x &\approx 1.08 && \text{Calculator} \end{aligned}$$

■ **NOW TRY EXERCISES 49 AND 51** ■

■ Solving for One Variable in Terms of Others

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In Example 10 we solve for a variable in Newton's Law of Gravity.

example 9 Solving for One Variable in Terms of Others

Solve for the variable x in the equation

$$tx = 2t + 3x$$

Solution

We first bring all terms involving x to one side of the equation, then factor x.

$$\begin{aligned} tx &= 2t + 3x && \text{Given equation} \\ tx - 3x &= 2t && \text{Subtract } 3x \\ x(t - 3) &= 2t && \text{Factor } x \\ x &= \frac{2t}{t - 3} && \text{Divide by } t - 3 \end{aligned}$$

The solution is $x = \dfrac{2t}{t - 3}$.

■ **NOW TRY EXERCISE 59** ■

example 10 Solving for One Variable in Terms of Others

Solve for the variable M in the equation

$$F = G\frac{mM}{r^2}$$

Solution

Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = G\frac{mM}{r^2} \qquad \text{Given equation}$$

$$\left(\frac{r^2}{Gm}\right)F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}M\right) \qquad \text{Multiply by } \frac{r^2}{Gm}$$

$$M = \frac{r^2F}{Gm} \qquad \text{Simplify and switch sides}$$

The solution is $M = \dfrac{r^2F}{Gm}$.

■ NOW TRY EXERCISE 63 ■

C.1 Exercises

CONCEPTS

1. Which of the following equations are linear?

(a) $\dfrac{x}{2} + 2x = 10$ **(b)** $\dfrac{2}{x} - 2x = 1$ **(c)** $x + 7 = 5 - 3x$

2. Explain why each of the following equations is not linear.

(a) $x(x + 1) = 6$ **(b)** $\sqrt{x + 2} = x$ **(c)** $3x^2 - 2x - 1 = 0$

3. True or false?

(a) Adding the same number to each side of an equation always gives an equivalent equation.

(b) Multiplying each side of an equation by the same number always gives an equivalent equation.

(c) Squaring each side of an equation always gives an equivalent equation.

4. To solve the equation $x^3 = 125$, we take the _____ root of each side. So the solution is _____.

SKILLS

5–8 ■ An equation is given.

(a) Express the equation in words.

(b) Determine whether the given value is a solution of the equation.

(c) Solve the equation.

5. $5x - 3 = 12$; 1

6. $4x + 9 = 1$; 2

7. $4x + 7 = 9x - 3$; -2

8. $2 - 5x = 8 + x$; -1

9–22 ■ Solve the equation.

9. $x - 3 = 2x + 6$

10. $4x + 7 = 9x - 13$

11. $-7w = 15 - 2w$

12. $5t - 13 = 12 - 5t$

13. $2(1 - x) = 3(1 + 2x) + 5$

14. $5(x + 3) + 9 = -2(x - 2) - 1$

15. $4(y - \frac{1}{2}) - y = 6(5 - y)$

16. $r - 2[1 - 3(2r + 4)] = 61$

17. $\frac{1}{2}y - 2 = \frac{1}{3}y$

18. $\dfrac{z}{5} = \dfrac{3}{10}z + 7$

19. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$

20. $x - \dfrac{1}{3}x - \dfrac{1}{2}x - 5 = 0$

21. $2x - \dfrac{x}{2} + \dfrac{x + 1}{4} = 6x$

22. $\dfrac{2}{3}x - \dfrac{1}{4} = \dfrac{1}{6}x - \dfrac{1}{9}$

23–32 ■ Solve the equation.

23. $\dfrac{2}{t} = \dfrac{3}{5}$

24. $\dfrac{5}{4w} = \dfrac{6}{7}$

25. $\dfrac{2x - 7}{2x + 4} = \dfrac{2}{3}$

26. $\dfrac{2y - 1}{y + 2} = \dfrac{4}{5}$

27. $\dfrac{2}{t + 6} = \dfrac{3}{t - 1}$

28. $\dfrac{6}{x - 3} = \dfrac{5}{x + 4}$

29. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$

30. $\dfrac{2}{t} - 5 = \dfrac{6}{t} + 4$

31. $\dfrac{z}{2z - 4} - 2 = \dfrac{1}{z - 2}$

32. $\dfrac{3}{r + 4} = \dfrac{1}{r} + \dfrac{6r + 12}{r^2 + 4r}$

33–56 ■ The given equation involves a power of the variable. Find all real solutions of the equation.

33. $x^2 = 49$

34. $x^2 = 18$

35. $y^2 - 24 = 0$

36. $y^2 - 7 = 0$

37. $(4t)^2 - 64 = 0$

38. $(5r)^2 - 125 = 0$

39. $s^2 + 16 = 0$

40. $6x^2 + 100 = 0$

41. $(x + 2)^2 = 4$

42. $3(x - 5)^2 = 15$

43. $A^3 = -27$

44. $z^5 + 32 = 0$

45. $6x^4 = 128$

46. $3x^6 = 45$

47. $8r^4 - 64 = 0$

48. $2r^3 + 10 = 0$

49. $1.3 = 0.5A^{3/2}$

50. $5 = 1.8S^{7/8}$

51. $2x^{-4} = 5$

52. $3x^{-6} = \frac{1}{5}$

53. $(x + 2)^4 - 81 = 0$

54. $(x + 1)^4 + 16 = 0$

55. $3(x - 3)^3 = 375$

56. $4(x + 2)^5 = 1$

57–74 ■ Solve the equation for the indicated variable.

57. $5x = 7y^2$; for y

58. $2t = 5x^3$; for x

59. $5xy - 4x = 3y$; for x

60. $2lw + 4w = 5l$; for w

61. $xy = 3y - 2x$; for x

62. $sw = 5 + w$; for w

63. $P = 2l + 2w$; for w

64. $V = \frac{4}{3}\pi r^3$; for r

65. $PV = nRT$; for n

66. $F = G\dfrac{mM}{r^2}$; for m

67. $S = 1.2A^{0.4}$; for A

68. $2.1A = 2.4S^{1.8}$; for S

69. $A = 0.5S^{-1.2}$; for S

70. $A = 1.6H^{-0.7}$; for H

71. $\dfrac{d}{t} = s$; for t

72. $\dfrac{2L}{W} = 5$; for W

73. $(3a_1)^2 = \dfrac{1}{5a_2}$; for a_1

74. $\left(\frac{1}{2}x_1\right)^3 = \dfrac{5}{4x_2}$; for x_1

C.2 Solving Quadratic Equations

- Solving Quadratic Equations by Factoring
- Completing the Square
- Solving Quadratic Equations by Completing the Square
- The Quadratic Formula

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. We study three methods of solving quadratic equations: by factoring, by completing the square, or by the Quadratic Formula.

Solving Quadratic Equations by Factoring

If a quadratic equation factors easily, then we can use the following property of real numbers to solve the equation.

Zero-Product Property

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

example 1 Solving a Quadratic Equation by Factoring

Solve the quadratic equation.

(a) $x^2 = 4x + 21$ **(b)** $6x^2 + 7x - 5 = 0$

Solution

(a) We must first rewrite the equation so that the right-hand side is 0.

$$\begin{aligned} x^2 &= 4x + 21 && \text{Given equation} \\ x^2 - 4x - 21 &= 0 && \text{Subtract } 4x \text{ and } 21 \\ (x + 3)(x - 7) &= 0 && \text{Factor} \\ x + 3 = 0 \quad &\text{or} \quad x - 7 = 0 && \text{Zero-Product Property} \\ x = -3 \quad &\qquad x = 7 && \text{Solve} \end{aligned}$$

The solutions are $x = -3$ and $x = 7$.

✓ CHECK

We substitute $x = -3$ into the equation: $(-3)^2 = 4 \cdot (-3) + 21$ ✓

We substitute $x = 7$ into the equation: $7^2 = 4 \cdot (7) + 21$ ✓

(b) The right-hand side is already 0. We can now factor the left-hand side (see Example 4 in **Algebra Toolkit B.2**):

$$\begin{aligned} 6x^2 + 7x - 5 &= 0 && \text{Given equation} \\ (3x + 5)(2x - 1) &= 0 && \text{Factor} \\ 3x + 5 = 0 \quad &\text{or} \quad 2x - 1 = 0 && \text{Zero-Product Property} \\ x = -\tfrac{5}{3} \quad &\qquad x = \tfrac{1}{2} && \text{Solve} \end{aligned}$$

The solutions are $x = -\frac{5}{3}$ and $x = \frac{1}{2}$. You can check that these are solutions to the original equations.

■ **NOW TRY EXERCISES 5 AND 9** ■

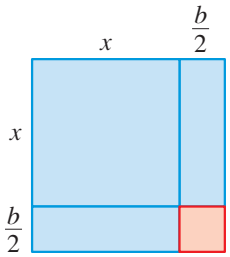
Completing the Square

Completing the Square

The area of the blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to "complete" the square.



In **Algebra Toolkit B.2** we learned how to recognize whether an algebraic expression is a perfect square. In particular, the quadratic expression $x^2 + 2ax + a^2$ is a perfect square (it is the square of $x + a$). The constant term of this expression is a^2, and the coefficient of x is $2a$. So the constant term is equal to the square of half of the coefficient of x. This observation allows us to "complete the square" for quadratic expressions of the form $x^2 + bx$. For example, to make

$$x^2 + 6x$$

a perfect square, we must add $\left(\frac{6}{2}\right)^2 = 9$. Then

$$x^2 + 6x + 9 = (x + 3)^2$$

is a perfect square. In general we have the following.

Completing the Square

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Here are some examples of completing the square.

Expression	Add	Complete the square
$x^2 + 8x$	$\left(\frac{8}{2}\right)^2 = 16$	$x^2 + 8x + 16 = (x + 4)^2$
$x^2 - 12x$	$\left(-\frac{12}{2}\right)^2 = 36$	$x^2 - 12x + 36 = (x - 6)^2$

To use the process of completing the square, the coefficient of x^2 must be 1.

example **2** **Completing a Square**

Complete the square for the following quadratic expression: $x^2 + 3x$.

Solution

The coefficient of x^2 is 1, so we can use the completing the square procedure. The coefficient of x is 3, and half of 3 is $\frac{3}{2}$. So to complete the square, we add

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

This gives the perfect square

$$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$$

■ **NOW TRY EXERCISE 13** ■

Solving Quadratic Equations by Completing the Square

In **Algebra Toolkit C.1** we learned that an equation of the form $(x \pm a)^2 = c$ can be solved by taking the square root of each side. This works because the left-hand side of this equation is a *perfect square*. So if a quadratic equation does not factor readily, then we can solve it by "completing the square." The next example illustrates how this is done.

example **3** **Solving Quadratic Equations by Completing the Square**

Solve each equation.

(a) $x^2 - 8x + 13 = 0$ **(b)** $3x^2 - 12x + 6 = 0$

Solution

(a) When completing the square, we must be careful to add the same constant to both sides of the equation.

$$x^2 - 8x + 13 = 0 \qquad \text{Given equation}$$
$$x^2 - 8x = -13 \qquad \text{Subtract 13}$$
$$x^2 - 8x + 16 = -13 + 16 \qquad \text{Complete the square: Add } \left(\frac{-8}{2}\right)^2 = 16$$
$$(x - 4)^2 = 3 \qquad \text{Perfect square}$$
$$x - 4 = \pm\sqrt{3} \qquad \text{Take square root}$$
$$x = 4 \pm \sqrt{3} \qquad \text{Add 4}$$

The solutions are $4 + \sqrt{3}$ and $4 - \sqrt{3}$.

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left-hand side to put the equation in the correct form for completing the square.

$$3x^2 - 12x + 6 = 0 \qquad \text{Given equation}$$
$$3x^2 - 12x = -6 \qquad \text{Subtract 6}$$
$$3(x^2 - 4x) = -6 \qquad \text{Factor 3 from LHS}$$

Now we complete the square for the expression inside the parentheses, by adding $(-2)^2 = 4$ *inside* the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are actually adding $3 \cdot 4 = 12$ to the left-hand side of the equation. Thus we must add 12 to the right-hand side as well.

$$3(x^2 - 4x + 4) = -6 + 3 \cdot 4 \qquad \text{Complete the square: add 4}$$
$$3(x - 2)^2 = 6 \qquad \text{Perfect square}$$
$$(x - 2)^2 = 2 \qquad \text{Divide by 3}$$
$$x - 2 = \pm\sqrt{2} \qquad \text{Take square root}$$
$$x = 2 \pm \sqrt{2} \qquad \text{Add 2}$$

The solutions are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

■ **NOW TRY EXERCISES 17 AND 21** ■

The Quadratic Formula

If we apply the technique of solving a quadratic equation by completing the square to the general quadratic equation $ax^2 + bx + c = 0$, we arrive at a *formula* that gives us the solutions of the equation in terms of the coefficients a, b, and c. So if a quadratic equation does not factor readily, we can solve it by using the **quadratic formula**.

The Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To prove this formula, we complete the square starting with the general quadratic equation. First, we move the constant term to the right-hand side and then divide each side of the equation by a.

$$ax^2 + bx + c = 0 \qquad \text{Quadratic equation}$$

$$ax^2 + bx = -c \qquad \text{Subtract } c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2} \qquad \text{Perfect square; common denominator}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Take square roots}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Subtract } \frac{b}{2a}$$

This completes the proof.

example 4 Using the Quadratic Formula

Find all solutions of each equation.

(a) $4x^2 - 3x - 6 = 0$

(b) $25x^2 - 20x + 4 = 0$

(c) $x^2 + 5x + 9 = 0$

Solution

$b = -3$

$4x^2 - 3x - 6 = 0$

$a = 4$ $\quad$ $c = -6$

(a) In this quadratic equation a is 4, b is -3, and c is -6. By the Quadratic Formula,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-6)}}{2(4)} = \frac{3 \pm \sqrt{105}}{8}$$

So the solutions are $(3 + \sqrt{105})/8$ and $(3 - \sqrt{105})/8$.

(b) Using the Quadratic Formula where a is 25, b is -20, and c is 4 gives

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 25 \cdot 4}}{2 \cdot 25} = \frac{20 \pm 0}{50} = \frac{2}{5}$$

This equation has only one solution, $x = \frac{2}{5}$.

(c) Using the Quadratic Formula where a is 1, b is 5, and c is 9 gives

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 9}}{2} = \frac{-5 \pm \sqrt{-11}}{2}$$

Since the square of any real number is nonnegative, $\sqrt{-11}$ is undefined in the real number system. The equation has no real solution.

NOW TRY EXERCISES 29, 33, AND 35

C.2 Exercises

CONCEPTS

1. The Quadratic Formula gives us the solutions of the equation $ax^2 + bx + c = 0$.

(a) State the Quadratic Formula: $x =$ _____.

(b) In the equation $\frac{1}{2}x^2 - x - 4 = 0$ the coefficient a is _____, b is _____, and c is _____. So the solution of the equation is $x =$ _____.

2. Explain how you would use each method to solve the equation $x^2 - 4x - 5 = 0$.

(a) By factoring: _____

(b) By completing the square: _____

(c) By using the Quadratic Formula: _____

SKILLS

3–10 ■ Solve the equation by factoring.

3. $x^2 + x = 12$

4. $x^2 + 3x = 4$

5. $x^2 - 7x + 12 = 0$

6. $x^2 + 8x + 12 = 0$

7. $3x^2 - 5x - 2 = 0$

8. $4x^2 - 4x - 15 = 0$

9. $2y^2 + 7y + 3 = 0$

10. $4w^2 = 4w + 3$

11–14 ■ Complete the square for the given expression.

11. $x^2 + 7x + \square = (x + \square)^2$

12. $x^2 + x + \square = (x + \square)^2$

13. $x^2 - \frac{1}{2}x + \square = (x + \square)^2$

14. $x^2 - \frac{2}{3}x + \square = (x + \square)^2$

15–22 ■ Solve the equation by completing the square.

15. $x^2 + 2x - 5 = 0$

16. $x^2 - 4x + 2 = 0$

17. $x^2 - 6x - 11 = 0$

18. $x^2 + 3x - \frac{7}{4} = 0$

19. $2x^2 + 8x + 1 = 0$

20. $3x^2 - 6x - 1 = 0$

21. $4x^2 - 20x - 2 = 0$

22. $5x^2 + 15x + 1 = 0$

23–36 ■ Solve the equation by factoring or using the Quadratic Formula.

23. $x^2 - 2x - 15 = 0$

24. $x^2 + 5x - 6 = 0$

25. $x^2 - 7x + 10 = 0$

26. $x^2 + 30x + 200 = 0$

27. $x^2 + 3x + 1 = 0$

28. $2x^2 - 8x + 4 = 0$

29. $3x^2 + 6x - 5 = 0$

30. $x^2 - 6x + 1 = 0$

31. $z^2 - \frac{3}{2}z + \frac{9}{16} = 0$

32. $2y^2 - y - \frac{1}{2} = 0$

33. $w^2 = 3(w - 1)$

34. $3 + 5z + z^2 = 0$

35. $3x^2 + 2x + 2 = 0$

36. $5x^2 - 7x + 5 = 0$

C.3 Solving Inequalities

- Solving Inequalities
- Solving Linear Inequalities
- Solving Nonlinear Inequalities

In this section we review inequalities and how to solve them *algebraically*. In **Algebra Toolkit D.4** we solve inequalities *graphically*. We will see that the graphical method helps us understand why the algebraic method works. We begin by reviewing the basic properties of inequalities.

Solving Inequalities

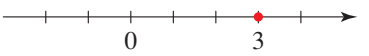
x	$4x + 7 \le 19$
1	$11 \le 19$ ✓
2	$15 \le 19$ ✓
3	$19 \le 19$ ✓
4	$23 \le 19$ ×
5	$27 \le 19$ ×

An inequality looks just like an equation, except in place of the equal sign we have one of the symbols $<$, $>$, $\le$, or $\ge$. Here is an example of an inequality:

$$4x + 7 \le 19$$

The table in the margin shows that some numbers satisfy the inequality and some numbers don't.

To solve an inequality in one variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation.

		Solution	Graph
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \le 19$	$x \le 3$	

The idea in solving an inequality is to "do the same thing to each side of the inequality." We use the following rules when working with inequalities.

Operations on Inequalities

1. Add (or subtract) the same quantity to each side.
2. Multiply (or divide) each side by the same *positive* quantity.
3. Multiply (or divide) each side by a *negative* quantity and reverse the direction of the inequality.

Pay special attention to Rules 2 and 3. Rule 2 says that we can multiply (or divide) each side of an inequality by a positive number, but Rule 3 says that if we multiply each side of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we start with the inequality $1 < 2$ and multiply by 5, we get $5 < 10$, but if we multiply by -5, we get $-5 > -10$.

Solving Linear Inequalities

An inequality is called a **linear inequality** if the corresponding equation (the equation obtained by replacing the inequality sign with an equal sign) is linear. To solve a linear inequality, we use operations on inequalities to isolate the variable on one side of the inequality.

example 1 Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$ and graph the solution set.

Solution

We use the operations on inequalities to isolate the variable x on one side of the inequality.

$$\begin{aligned} 3x &< 9x + 4 && \text{Inequality} \\ 3x - 9x &< 9x + 4 - 9x && \text{Subtract } 9x \text{ from each side} \\ -6x &< 4 && \text{Simplify} \\ \left(-\tfrac{1}{6}\right)(-6x) &> \left(-\tfrac{1}{6}\right)4 && \text{Multiply by } -\tfrac{1}{6}\text{, reverse inequality} \\ x &> -\tfrac{2}{3} && \text{Simplify} \end{aligned}$$



figure 1 The interval $\left(-\frac{2}{3}, \infty\right)$

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. It is graphed in Figure 1.

NOW TRY EXERCISE 17

example 2 Solving a Pair of Linear Inequalities

Solve the inequalities $4 \le 3x - 2 < 13$, and graph the solution set.

Solution

We use the operations on inequalities to isolate the variable x.

$$\begin{aligned} 4 &\le 3x - 2 < 13 && \text{Inequality} \\ 6 &\le 3x < 15 && \text{Add 2 to each side} \\ 2 &\le x < 5 && \text{Divide each side by 3} \end{aligned}$$



figure 2 The interval $[2, 5)$

Therefore, the solution set is the interval $[2, 5)$, as shown in Figure 2.

NOW TRY EXERCISE 21

Solving Nonlinear Inequalities

If an inequality involves squares or higher powers of the variable, it is *nonlinear*. The key idea in solving a nonlinear inequality algebraically is to first find the solutions of the corresponding equation, then use test values to determine whether the given inequality is satisfied on the intervals determined by these solutions. This method is described as follows.

Solving Nonlinear Inequalities

1. **Move all terms to one side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality symbol.
2. **Solve the corresponding equation.** Solve the equation you get by replacing the inequality with an equal sign.
3. **Find the intervals.** The solutions from Step 2 divide the real line into intervals. Use test values to decide whether the inequality is satisfied on each interval.
4. **Write down the solution.** The solution consists of those intervals on which the inequality is satisfied. Be sure to check the endpoints of each interval to see whether they also satisfy the inequality.

In the next examples we illustrate the procedure.

example 3 Solving a Nonlinear Inequality Algebraically

Solve the inequality $x^2 - 2x - 8 \le 0$.

Solution

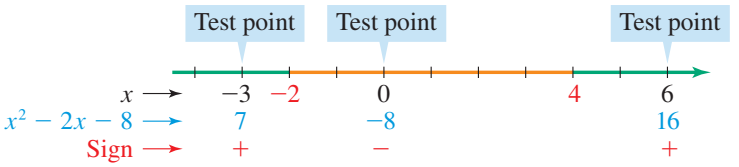
First we solve the equation $x^2 - 2x - 8 = 0$. We do this by factoring the left-hand side.

$$\begin{aligned} x^2 - 2x - 8 &= 0 && \text{Equation} \\ (x - 4)(x + 2) &= 0 && \text{Factor} \\ x = 4 \quad \text{or} \quad x &= -2 && \text{Solve} \end{aligned}$$

These solutions separate the real line into the three intervals $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$, as shown in the following diagram.



The expression $x^2 - 2x - 8$ is either positive or negative on each of these intervals. To decide which it is, we pick a test point in the interval and evaluate the expression. For example, in the interval $(-\infty, -2)$, let's pick the test point -3. Evaluating the expression at -3 gives us $(-3)^2 - 2(-3) - 8 = 7$. Since 7 is greater than 0, the expression $x^2 - 2x - 8$ is positive on the interval $(-\infty, -2)$. In the diagram below we choose test points in each of the three intervals and determine the sign of the expression on each interval.



Now that we know the sign of the expression in each interval, we can label the intervals with "+" or "−" signs as in the following diagram.



From this diagram we see that the solution of the inequality $x^2 - 2x - 8 \le 0$ is the interval $[-2, 4]$. We have included the endpoints of the interval because the inequality requires that the expression be less than *or equal to* 0.

NOW TRY EXERCISE 25

example 4 Solving a Nonlinear Inequality Algebraically

Solve the inequality $x^2 > 2x + 3$.

Solution

First we bring all terms to one side of the inequality. Subtracting $2x + 3$ gives the inequality

$$x^2 - 2x - 3 > 0$$

Next we solve the corresponding equation.

$$x^2 - 2x - 3 = 0 \qquad \text{Equation}$$

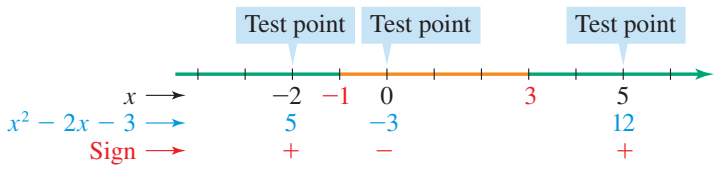
$$(x - 3)(x + 1) = 0 \qquad \text{Factor}$$

$$x = 3 \quad \text{or} \quad x = -1 \qquad \text{Solve}$$

These solutions separate the real line into the three intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$, as shown in the following diagram.



We use test points to determine the sign of the expression $x^2 - 2x - 3$ on each of these intervals:



Now that we know the sign of the expression in each interval, we can label the intervals with "+" or "−" signs as in the following diagram.



From the diagram we see that the solution of the inequality $x^2 - 2x - 3 > 0$ consists of all x-values in the set $(-\infty, -1) \cup (3, \infty)$. We did not include the endpoints -1 and 3 because they do not satisfy the inequality.

NOW TRY EXERCISE 27

C.3 Exercises

CONCEPTS

1. Fill in the blank with an appropriate inequality sign.

(a) If $x < 5$, then $x - 3$ ____ 2.

(b) If $x \le 5$, then $3x$ ____ 15.

(c) If $x \ge 2$, then $-3x$ ____ -6.

(d) If $x < -2$, then $-x$ ____ 2.

2. True or false?

(a) If $x(x + 1) > 0$, then x and $x + 1$ are either both positive or both negative.

(b) If $x(x + 1) > 5$, then x and $x + 1$ are each greater than 5.

SKILLS

3–10 ■ Let $S = \{-2, -1, 0, \frac{1}{2}, 1, \sqrt{2}, 2, 4\}$. Determine which elements of S satisfy the inequality.

3. $x - 3 > 0$

4. $x + 1 < 2$

5. $3 - 2x \le \frac{1}{2}$

6. $2x - 1 \ge x$

7. $1 < 2x - 4 \le 7$

8. $-2 \le 3 - x < 2$

9. $\dfrac{1}{x} \le \dfrac{1}{2}$

10. $x^2 + 2 < 4$

11–22 ■ Solve the linear inequality. Express the solution using interval notation, and graph the solution set.

11. $2x \le 7$

12. $-4x \ge 10$

13. $2x - 5 > 3$

14. $3x + 11 < 5$

15. $7 - x \ge 5$

16. $5 - 3x \le -16$

17. $3x + 11 \le 6x + 8$

18. $6 - x \ge 2x + 9$

19. $2 \le x + 5 < 4$

20. $5 \le 3x - 4 \le 14$

21. $-1 < 2x - 5 < 7$

22. $1 < 3x + 4 \le 16$

23–34 ■ Solve the nonlinear inequality. Express the solution using interval notation, and graph the solution set.

23. $(x + 2)(x - 3) < 0$

24. $(x - 5)(x + 4) \ge 0$

25. $x^2 - 3x - 18 \le 0$

26. $x^2 + 5x + 6 > 0$

27. $2x^2 + x \ge 1$

28. $x^2 < x + 2$

29. $x^2 < 4$

30. $x^2 \ge 9$

31. $(x + 2)(x - 1)(x - 3) \le 0$

32. $(x - 5)(x - 2)(x + 1) > 0$

33. $(x - 4)(x + 2)^2 < 0$

34. $(x + 3)^2(x + 1) > 0$

Algebra Toolkit D

Working with Graphs

D.1 The Coordinate Plane
D.2 Graphs of Two-Variable Equations
D.3 Using a Graphing Calculator
D.4 Solving Equations and Inequalities Graphically

D.1 The Coordinate Plane

- The Coordinate Plane
- The Distance Formula
- The Midpoint Formula

The Coordinate Plane

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **Cartesian plane** or the **coordinate plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. The horizontal line is the ***x*-axis** and the vertical line is the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin** O. The two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in Figure 1. (The points *on* the coordinate axes are not assigned to any quadrant.)

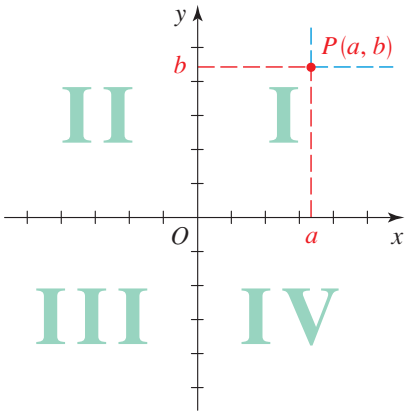


figure **1**

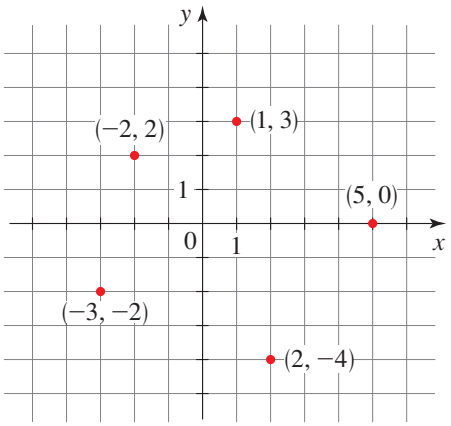


figure **2**

Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We can think of the coordinates of P as its "address," because they specify its location in the plane. Several points are labeled with their coordinates in Figure 2.

example 1 Graphing Points and Sets in the Plane

Graph the set of points in the coordinate plane.

(a) $\{(2, 3), (-1, 2), (2, -1), (-2, -2)\}$

(b) $\{(x, y) | y = 2\}$

(c) $\{(x, y) | x = -2\}$

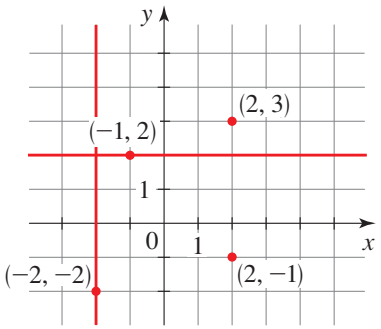


figure 3 The coordinate plane

Solution

(a) The points are graphed in Figure 3.

(b) This set consists of all points whose y-coordinate is 2. These points together form a line parallel to the x-axis (see Figure 3).

(c) This set consists of all points whose x-coordinate is -2. These points together form a line parallel to the y-axis (see Figure 3).

NOW TRY EXERCISES 5 AND 7

The Distance Formula

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from **Algebra Toolkit A.2** that the distance between points a and b on a number line is $d(a, b) = |b - a|$. So from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and that the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

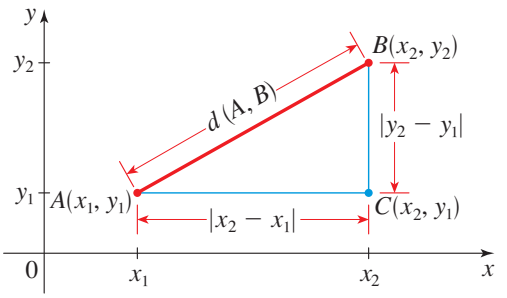


figure 4 Distance between A and B

Since triangle ABC is a right triangle, the Pythagorean Theorem gives the following Distance Formula.

The Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

example **2**

Finding the Distance Between Two Points

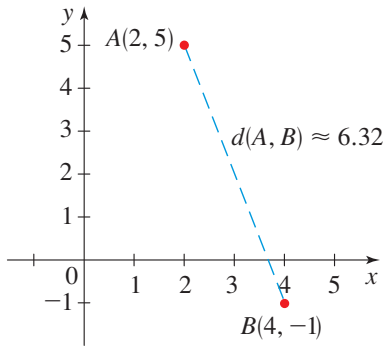


figure **5**

Find the distance between the points $A(2, 5)$ and $B(4, -1)$.

Solution

Using the Distance Formula, we have

$$\begin{aligned} d(A, B) &= \sqrt{(4-2)^2 + (-1-5)^2} \\ &= \sqrt{2^2 + (-6)^2} \\ &= \sqrt{4+36} = \sqrt{40} \approx 6.32 \end{aligned}$$

See Figure 5.

NOW TRY EXERCISE 15(b)

The Midpoint Formula

Now let's find the coordinates (x, y) of the midpoint M of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 6, notice that triangles APM and MQB are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal.

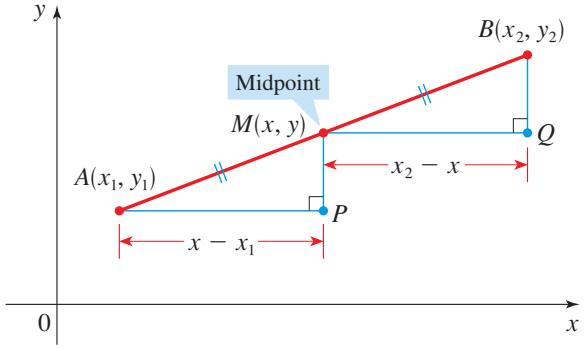


figure **6** Midpoint of the line segment AB

It follows that $d(A, P) = d(M, Q)$, so $x - x_1 = x_2 - x$. Solving for x, we get $2x = x_1 + x_2$, so $x = \dfrac{x_1 + x_2}{2}$. Similarly, $y = \dfrac{y_1 + y_2}{2}$.

Midpoint Formula

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

So the coordinates of the midpoint are the averages of the coordinates of the endpoints A and B.

example 3 Finding the Midpoint

Find the midpoint of the line segment that joins the points $(-2, 5)$ and $(4, 9)$.

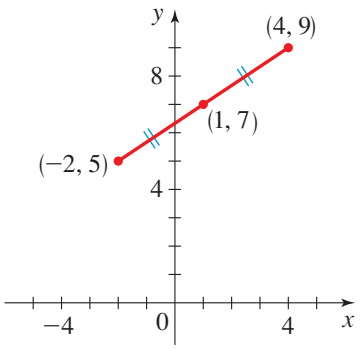


figure 7

Solution

Using the Midpoint Formula, we have

$$\left(\frac{-2+4}{2}, \frac{5+9}{2}\right) = (1, 7)$$

So the midpoint of the line segment is the point (1, 7). See Figure 7.

■ NOW TRY EXERCISE 15(c) ■

D.1 Exercises

CONCEPTS

1. The point that is 3 units to the right of the y-axis and 5 units below the x-axis has the coordinates (_____, _____).

2. If x is negative and y is positive, then the point (x, y) is in Quadrant _____.

3. The distance between the points (a, b) and (c, d) is _____. So the distance between (1, 2) and (7, 10) is _____.

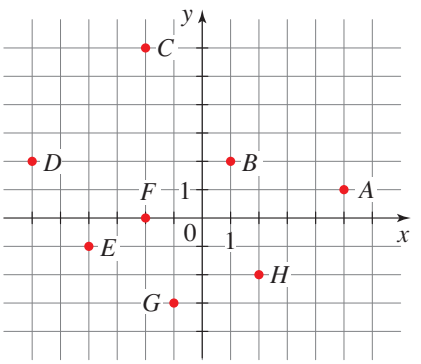
4. The point midway between (a, b) and (c, d) is _____. So the point midway between (1, 2) and (7, 10) is _____.

SKILLS

5. Plot the given points in a coordinate plane:

$$(2, 3), (-2, 3), (4, 5), (4, -5), (-4, 5), (-4, -5)$$

6. Find the coordinates of the points shown in the figure.



7–10 ■ Graph the given set of points in the coordinate plane.

7. $\{(x, y) \mid x = 3\}$

8. $\{(x, y) \mid y = -2\}$

9. $\{(x, y) \mid y = -1\}$

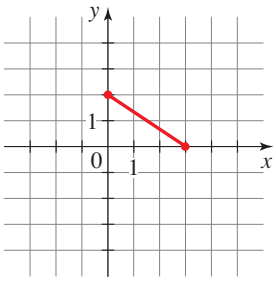
10. $\{(x, y) \mid x = 5\}$

11–14 ■ A pair of points is graphed.

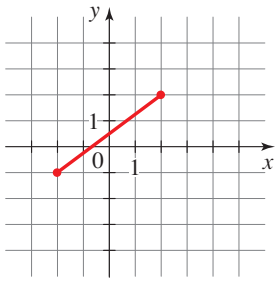
(a) Find the distance between the points.

(b) Find the midpoint of the line segment that joins the points.

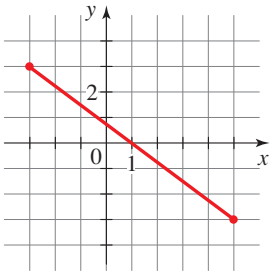
11.



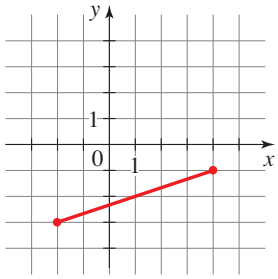
12.



13.



14.



15–24 ■ A pair of points is given.

(a) Plot the points in a coordinate plane.

(b) Find the distance between the points.

(c) Find the midpoint of the line segment that joins the points.

15. $(0, 8), (6, 16)$

16. $(-2, 5), (10, 0)$

17. $(-3, -6), (4, 18)$

18. $(-1, -1), (9, 9)$

19. $(6, -2), (-1, 3)$

20. $(-1, 6), (-1, -3)$

21. $(7, 3), (11, 6)$

22. $(2, 13), (7, 1)$

23. $(3, 4), (-3, -4)$

24. $(5, 0), (0, 6)$

D.2 Graphs of Two-Variable Equations

- Two-Variable Equations
- Graphs of Two-Variable Equations
- Finding Intercepts
- Equations of Circles

The *coordinate plane* is the link between algebra and geometry because it allows us to draw graphs of algebraic equations. The graphs, in turn, allow us to "see" the relationship between the variables in the equation.

Two-Variable Equations

A **two-variable equation** is an equation that contains two variables. For example, the equation

$$y = 3 + 2x$$

is an equation in the two variables x and y. This equation expresses a relationship between the two variables; we can read the equation as saying

"y is twice x plus 3"

Expressing the equation in words helps us to understand what the equation is telling us.

An ordered pair (a, b) is a **solution** to an equation, or **satisfies** an equation, if the equation is true when x is replaced by a and y is replaced by b. For example, $(1, 5)$ satisfies the equation $y = 3 + 2x$ because when we replace x by 1 and y by 5, we get a true equation: $5 = 3 + 2(1)$.

example 1 Reading an Equation

Consider the two-variable equation $5x - 2y = 1$.

(a) Give a verbal description of the relationship between the variables in the equation.

(b) Is the ordered pair $(2, 4)$ a solution of the equation? What about $(7, 17)$?

Solution

(a) The equation says that "subtracting two times y from five times x gives one."

To check whether $(2, 4)$ is a solution, replace x by 2 and y by 4.

$5x - 2y = 1$

Replace x by 2 and y by 4

(b) To check whether $(2, 4)$ is a solution, we replace x by 2 and y by 4 in the equation.

$5x - 2y = 1$	Equation
$5(2) - 2(4) = 1$	Replace x by 2 and y by 4
$2 = 1$ ✗	False

So $(2, 4)$ is not a solution. You can check that if we replace x by 7 and y by 17, we get a true equation, so $(7, 17)$ *is* a solution.

NOW TRY EXERCISE 5

Graphs of Two-Variable Equations

We graph an equation by plotting all the solutions of the equation in a coordinate plane.

The Graph of an Equation

The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

The graph of an equation is a curve, so to graph the equation we plot as many points as we can and then connect them by a smooth curve.

example 2 Equation, Table, Graph

Consider the equation $y = 1 + 2x$.

(a) Make a table of the solutions (x, y) of the equation for $x = 0, 1, 2, 3, 4, 5$.

(b) Sketch a graph of the equation.

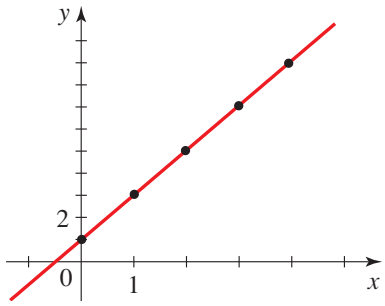


figure **1** Graph of $y = 1 + 2x$

Solution

(a) If x is 0, then $y = 1 + 2(0) = 1$, so (0, 1) is a solution. If x is 1, then $y = 1 + 2(1) = 3$, so (1, 3) is a solution. The other entries in the table below are calculated similarly.

x	0	1	2	3	4	5
y	1	3	5	7	9	11

(b) We graph the ordered pairs given in the table of part (a). Of course, the equation has many more solutions than the ones in the table, so we connect the points by a smooth curve that represents all the solutions of the equation as in Figure 1.

NOW TRY EXERCISE 9

In Example 1 we sketched the graph of the equation $y = 1 + 2x$ by plotting just a few points. To check that connecting these points by a smooth curve is correct, we can try graphing more and more points as in Figure 2. The more points we plot, the more the graph looks like a line.

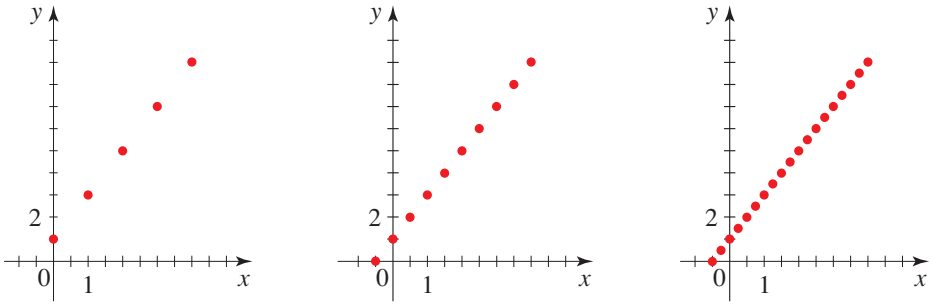


figure **2** Steps in graphing $y = 1 + 2x$

The points on the graph of an equation are solutions of the equation, so we can read solutions of an equation directly from the graph.

example **3**

Reading the Graph of an Equation

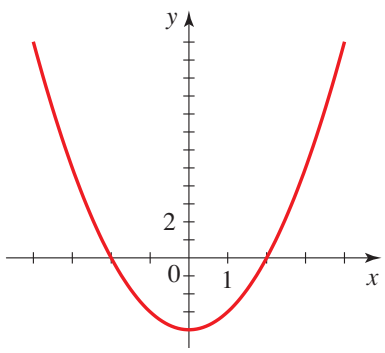


figure **3** Graph of $y = x^2 - 4$

The graph of the equation $y = x^2 - 4$ is shown in Figure 3. Answer the following questions about the solutions of this equation. Then use the equation to confirm your answer.

(a) Is (2, 2) a solution? What about (3, 5)?

(b) What are the value(s) of x when y is 5?

(c) What are the value(s) of y when x is 1?

Solution

We answer these questions from the graph and then from the equation.

(a) **From the graph:** We see that the point (2, 2) is not on the graph but the point (3, 5) is on the graph (see Figure 4(a) on the next page). So the ordered pair (3, 5) is a solution, whereas (2, 2) is not.

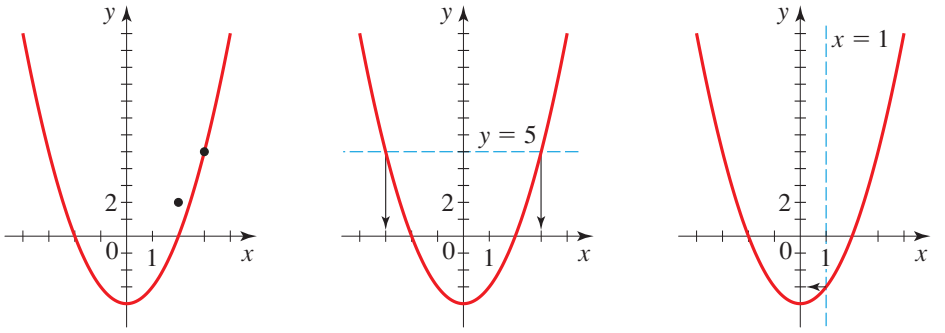


figure **4**

From the equation: We check whether these ordered pairs satisfy the equation.

$(3, 5)$	$5 = 3^2 - 4$ ✓	Solution
$(2, 2)$	$2 = 2^2 - 4$ ×	Not a solution

(b) **From the graph:** We can see that there are two points on the graph with y-coordinate 5. The x-coordinates of these points are 3 and -3 (see Figure 4(b)).

From the equation: We replace y by 5 and solve for x.

$y = x^2 - 4$	Equation
$5 = x^2 - 4$	Replace y by 5
$9 = x^2$	Add 4
$x = \pm 3$	Take square root, switch sides

So if y is 5, then x is 3 or -3.

(c) **From the graph:** We can see that there is one point on the graph with x-coordinate 1 (see Figure 4(c)). The y-coordinate of this point is -3.

From the equation: We replace x by 1 and solve for y.

$y = x^2 - 4$	Equation
$y = 1^2 - 4$	Replace x by 1
$y = -3$	Calculate

So if x is 1, then y is -3.

NOW TRY EXERCISE 25

Note the difference between an equation, a table of solutions, and a graph of the equation.

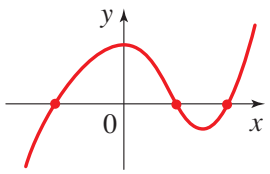
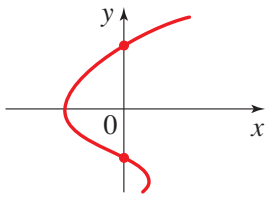
- **The equation** gives us a precise relationship between two variables.
- **A table of solutions** contains only those solutions we have calculated (usually just a few). From the equation itself we can obtain all the solutions (infinitely many).
- **The graph** gives us a "picture" of the relationship between the variables in the equation that we cannot easily see from the equation alone. From the graph we can "read" solutions of the equation.

So the equation, a table of solutions, and the graph all work together to help us understand how the variables are related.

Finding Intercepts

The x-coordinates of the points where a graph of an equation intersects the x-axis are the ***x*-intercepts** of the graph and are obtained by setting $y = 0$ in the equation. The y-coordinates of the points where a graph of an equation intersects the y-axis are the ***y*-intercepts** of the graph and are obtained by setting $x = 0$ in the equation.

The *x*- and *y*-Intercepts

Intercepts	From the equation	From the graph
***x*-intercepts:** The x-coordinates of the points where the graph intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of the points where the graph intersects the y-axis	Set $x = 0$ and solve for y	

example **4** **Finding Intercepts**

Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$.

Solution

To find the x-intercepts from the equation, we set $y = 0$ and solve for x.

$$y = x^2 - 2 \quad \text{Equation}$$
$$0 = x^2 - 2 \quad \text{Set } y = 0$$
$$x^2 = 2 \quad \text{Add 2 and switch sides}$$
$$x = \pm\sqrt{2} \quad \text{Take the square root}$$

The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$. To find the y-intercepts, we set $x = 0$ and solve for y.

$$y = x^2 - 2 \quad \text{Equation}$$
$$y = 0^2 - 2 \quad \text{Set } x = 0$$
$$y = -2 \quad \text{Simplify}$$

The y-intercept is -2. The graph of this equation is sketched in Figure 5 with the x- and y-intercepts labeled.

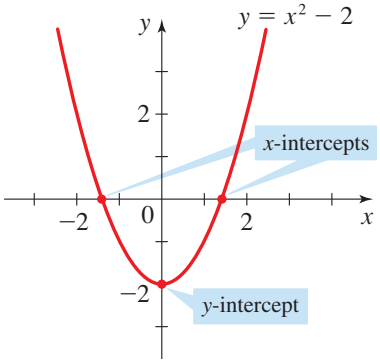


figure **5**

■ **NOW TRY EXERCISE 31** ■

Equations of Circles

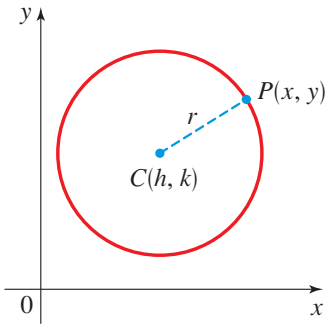


figure **6**

So far, we have discussed how to find the graph of an equation in x and y. The converse problem is to find an equation of a graph, that is, an equation that represents a given curve in the xy-plane. As an example of this type of problem, let's find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r (see Figure 6). Thus, P is on the circle if and only if $d(P, C) = r$. From the Distance Formula we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Square each side}$$

This is the equation of the circle.

Equation of a Circle

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

This form of the equation of a circle is called the *standard form.*

example **5** **Graphing a Circle**

Graph each equation.

(a) $x^2 + y^2 = 25$

(b) $(x - 2)^2 + (y + 1)^2 = 25$

Solution

(a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 7.

(b) Rewriting the equation as $(x - 2)^2 + (y - (-1))^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at $(2, -1)$. Its graph is shown in Figure 8.

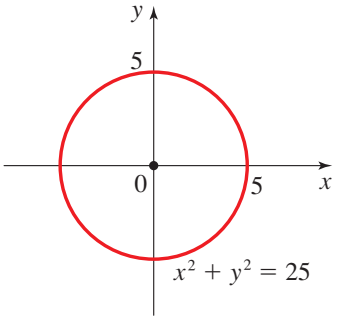


figure **7**

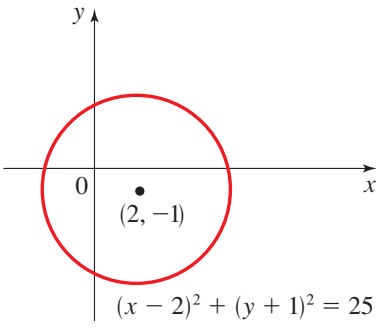


figure **8**

NOW TRY EXERCISES 35 AND 39

example 6 Finding an Equation of a Circle

Find an equation of the circle with radius 3 and center $(2, -1)$.

Solution

Using the equation of a circle with $r = 3$, $h = 2$, and $k = -1$, we get

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Equation of circle}$$

$$(x - 2)^2 + (y + 1)^2 = 3^2 \qquad \text{Replace } h \text{ by } 2,\ k \text{ by } -1,\ r \text{ by } 3$$

This is the equation of the circle we want.

NOW TRY EXERCISE 41

If we expand the equation of the circle in Example 6, we get

$$x^2 - 4x + y^2 + 2y = 4$$

From this form of the equation we can't quickly find the center and radius. To put the equation back into standard form, we need to complete the square as described in the next example.

example 7 Identifying an Equation of a Circle

Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

Solution

We first group the x-terms and y-terms. Then we complete the square within each grouping. That is, we complete the square for $x^2 + 2x$ by adding $(\frac{1}{2} \cdot 2)^2 = 1$, and we complete the square for $y^2 - 6x$ by adding $[\frac{1}{2} \cdot (-6)]^2 = 9$.

$$x^2 + y^2 + 2x - 6y + 7 = 0 \qquad \text{Given equation}$$

$$(x^2 + 2x \quad) + (y^2 - 6y \quad) = -7 \qquad \text{Group terms}$$

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \qquad \text{Complete the squares}$$

$$(x + 1)^2 + (y - 3)^2 = 3 \qquad \text{Factor and simplify}$$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r^2 = 3$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

NOW TRY EXERCISE 45

D.2 Exercises

CONCEPTS

1. If the point (3, 2) is a solution of an equation in x and y, then the equation is satisfied when we replace x by _____ and y by _____. Is the point (3, 2) a solution of the equation $2y = x + 1$?

2. If the point (1, 2) is on the graph of an equation in x and y, then the equation is satisfied when we replace x by _____ and y by _____. Is the point (1, 2) on the graph of the equation $2x - 5y = 8$?

3. (a) To find the x-intercept(s) of the graph of an equation, we set _____ equal to 0 and solve for _____. So the x-intercept(s) of the graph of $2y = x + 1$ is _____.

(b) To find the y-intercept(s) of the graph of an equation, we set _____ equal to 0 and solve for _____. So the y-intercept(s) of $2y = x + 1$ is _____.

4. The graph of the equation $(x - 1)^2 + (y - 2)^2 = 9$ is a circle with center (____, ____) and radius _____.

SKILLS

5–8 ■ An equation is given.

(a) Give a verbal description of the relationship between the variables in the equation.

(b) Determine whether the given point (x, y) is on the graph of the equation.

5. $x - y = 7$; $(3, 10)$

6. $x + y^2 = 6$; $(3, -3)$

7. $y^3 - 2x = 5$; $(11, 3)$

8. $3x + y^2 - 2y = 0$; $(-1, 3)$

9–16 ■ An equation is given.

(a) Complete the table of solutions.

(b) Draw a graph of the equation.

9. $y = 3 - 2x$

x	y
0	3
1	
2	
3	
4	
5	
6	

10. $y = -4 + 5x$

x	y
0	−4
1	
2	
3	
4	
5	
6	

11. $y = x^2$

x	y
−3	9
−2	
−1	
0	
1	
2	
3	

12. $y = x^3$

x	y
−3	−27
−2	
−1	
0	
1	
2	
3	

13. $y = x^3 - 3$

x	y
−3	−30
−2	
−1	
0	
1	
2	
3	

14. $y = x^2 + 3$

x	y
−3	12
−2	
−1	
0	
1	
2	
3	

15. $y = |x| + 3$

x	y
−3	6
−2	
−1	
0	
1	
2	
3	

16. $y = |3 - x|$

x	y
0	3
1	
2	
3	
4	
5	
6	

17–24 ■ Draw a graph of the equation.

17. $y = -x$

18. $y = 2x$

19. $y = 2x^2$

20. $y = -x^2$

21. $y = 2x - 6$

22. $y = (x + 1)^2$

23. $y = x^2 + 1$

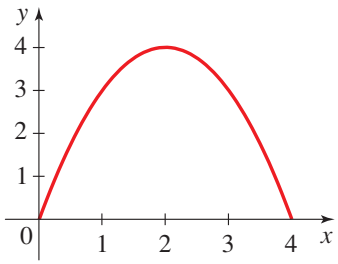
24. $y = |x| + 1$

25. The graph of the equation $y = 4x - x^2$ is given for values of x between 0 and 4. Use the graph to answer the following questions.

(a) What is the value of y when $x = 2$?

(b) For what value(s) of x is $y = 3$?

(c) What are the x- and y-intercepts?



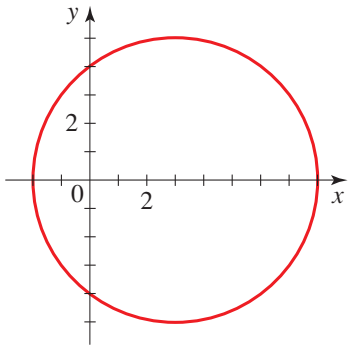
26. The graph of the equation $(x - 3)^2 + y^2 = 25$ is given. Use the graph to answer the following questions.

(a) For what value(s) of y is $x = 6$?

(b) For what value(s) of x is $y = 3$?

(c) For what value(s) of x is $y = 5$?

(d) What are the x- and y-intercepts?

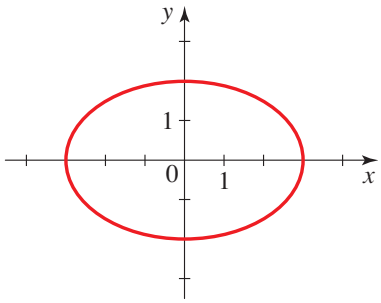


27. The graph of the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is given.

Find the x- and y-intercepts as follows.

(a) From the graph

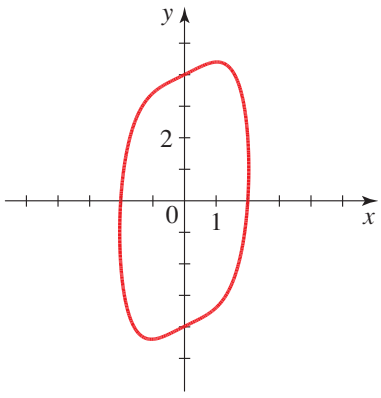
(b) From the equation



28. The graph of the equation $x^4 + y^2 - xy = 16$ is given. Find the x- and y-intercepts as follows.

(a) From the graph

(b) From the equation



29–34 ■ Find the x- and y-intercepts of the graph of the equation.

29. $y = x - 3$

30. $y = x^2 - 5x + 6$

31. $y = x^2 - 9$

32. $y - 2xy + 2x = 1$

33. $x^2 + y^2 = 4$

34. $y = \sqrt{x + 1}$

35–40 ■ Find the center and radius of the circle, and sketch its graph.

35. $x^2 + y^2 = 9$

36. $x^2 + y^2 = 5$

37. $(x - 3)^2 + y^2 = 16$

38. $x^2 + (y - 2)^2 = 4$

39. $(x + 3)^2 + (y - 4)^2 = 25$

40. $(x + 1)^2 + (y + 2)^2 = 36$

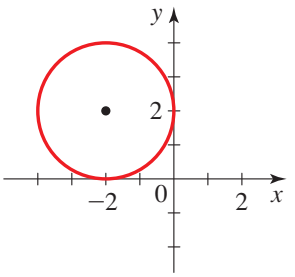
41–42 ■ Find an equation of the circle that satisfies the given conditions.

41. Center $(2, -1)$; radius 3

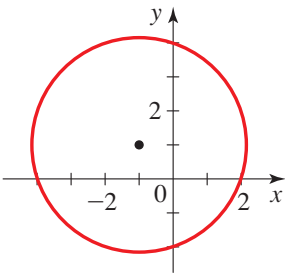
42. Center $(-1, -4)$; radius 8

43–44 ■ Find the equation of the circle shown in the figure.

43.



44.



45–48 ■ Show that the equation represents a circle, and find the center and radius of the circle.

45. $x^2 + y^2 - 2x + 4y + 1 = 0$

46. $x^2 + y^2 - 2x - 2y = 2$

47. $x^2 + y^2 - 4x + 10y + 13 = 0$

48. $x^2 + y^2 + 6y + 2 = 0$

D.3 Using a Graphing Calculator

- Choosing a Viewing Rectangle
- Two Graphs on the Same Screen
- Avoiding Extraneous Lines
- Graphing a Circle

Graphing calculators and computers provide us with the ability to quickly obtain the graph of an equation. In this section we give a few guidelines to help us use graphing devices effectively.

Choosing a Viewing Rectangle

A graphing device displays a rectangular portion of the graph of an equation in a display window or viewing screen, which we call a **viewing rectangle**. The default screen often gives an incomplete or misleading picture of the graph, so we must

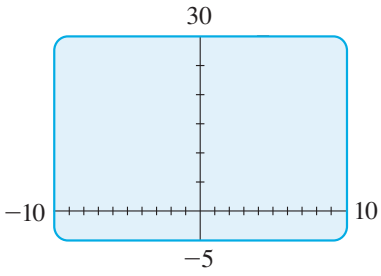


figure **1** The $[-10, 10]$ by $[-5, 30]$ viewing rectangle.

choose this rectangle carefully. To do this, we choose minimum and maximum values for x, called `Xmin` and `Xmax`, and minimum and maximum values for y, called `Ymin` and `Ymax`. For example, to obtain the viewing rectangle in Figure 1, we choose

$$\texttt{Xmin} = -10 \qquad \texttt{Ymin} = -5$$
$$\texttt{Xmax} = 10 \qquad \texttt{Ymax} = 30$$

The calculator displays the portion of the graph for the x-values in the interval $[-10, 10]$ and y-values in the interval $[-5, 30]$, so we describe this as the

$$[-10, 10] \quad \text{by} \quad [-5, 30]$$

viewing rectangle.

The graphing device produces the graph of an equation in much the same way as you would. It plots the points (x, y) for many values of x that are equally spaced between `Xmin` and `Xmax`. The machine then connects each point to the preceding plotted point to form a representation of the graph of the equation. If the equation is not defined for an x-value or if the corresponding y-value lies outside the viewing rectangle, the device ignores this value and moves on to the next x-value. The next example illustrates this process.

example 1 Choosing an Appropriate Viewing Rectangle

Graph the equation $y = x^2 + 3$ in an appropriate viewing rectangle.

Solution

Let's experiment with different viewing rectangles. We start with the viewing rectangle $[-2, 2]$ by $[-2, 2]$, so we set

$$\texttt{Xmin} = -2 \qquad \texttt{Ymin} = -2$$
$$\texttt{Xmax} = 2 \qquad \texttt{Ymax} = 2$$

The resulting graph in Figure 2(a) is blank! This is because $x^2 \geq 0$, so $x^2 + 3 \geq 3$ for all x. Thus, the graph lies entirely above the viewing rectangle, so this viewing rectangle is not appropriate. If we enlarge the viewing rectangle to $[-4, 4]$ by $[-4, 4]$ as in Figure 2(b), we begin to see a portion of the graph.

Now let's try the viewing rectangle $[-10, 10]$ by $[-5, 30]$. The graph in Figure 2(c) seems to give a more complete view of the graph. If we enlarge the viewing rectangle even further, as in Figure 2(d), the graph doesn't show clearly that the y-intercept is 3.

So the viewing rectangle $[-10, 10]$ by $[-5, 30]$ gives an appropriate representation of the graph.

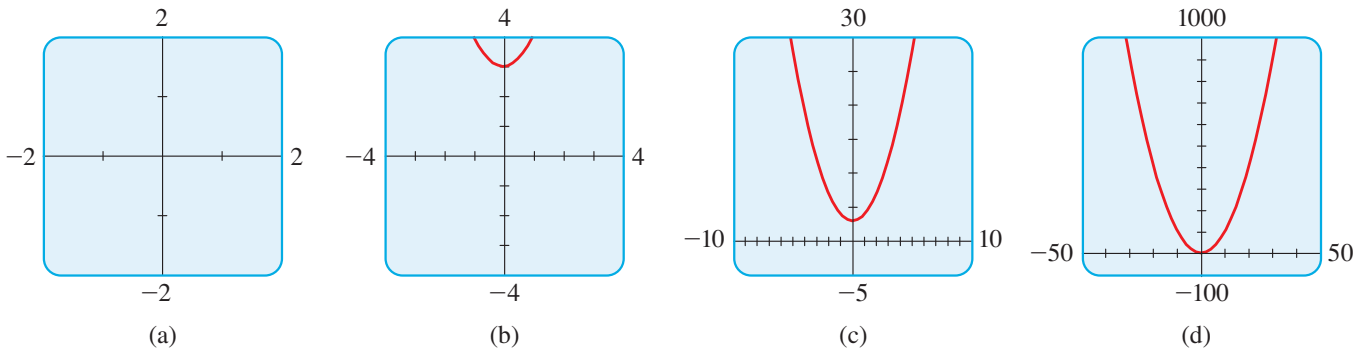


figure **2** Graphs of $y = x^2 + 3$

■ NOW TRY EXERCISES 3 AND 9 ■

You can see from Example 1 that the choice of a viewing rectangle makes a big difference in the appearance of a graph. If you want an overview of the essential features of a graph, you must choose a relatively large viewing rectangle to obtain a global view of the graph. If you want to investigate the details of a graph, you must zoom in to a small viewing rectangle that shows just the feature of interest.

Two Graphs on the Same Screen

One of the most useful features of a graphing calculator is the ability to draw more than one graph on the same screen. This allows us to more easily compare graphs and to find where two graphs intersect. But there are pitfalls to avoid here also, as the next example illustrates.

example 2 Two Graphs on the Same Screen

Graph the equations $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ together in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$. Do the graphs intersect in this viewing rectangle?

Solution

Figure 3(a) shows the essential features of both graphs. One is a parabola, and the other is a line. It looks as if the graphs intersect near the point $(1, -2)$. However, if we zoom in on the area around this point as shown in Figure 3(b), we see that although the graphs almost touch, they do not actually intersect.

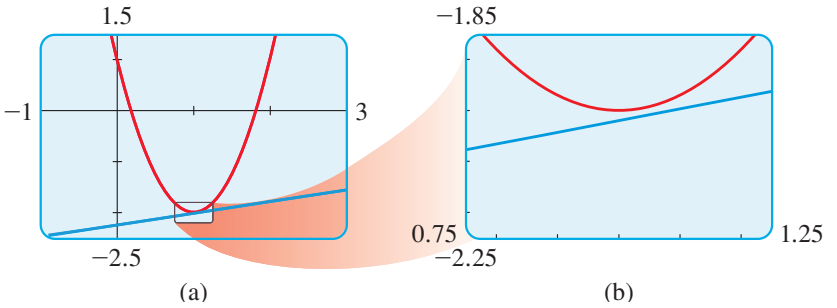


figure 3 Zooming in on part of the graph

NOW TRY EXERCISE 17

Avoiding Extraneous Lines

We have already discovered one of the pitfalls of using graphing calculators and computers: Examples 1 and 2 showed that the use of an inappropriate viewing rectangle can give a misleading representation of the graph of an equation. We also saw how to remedy the situation: We included the crucial parts of the graph by changing to a larger viewing rectangle. Another pitfall is illustrated in the next example.

example 3 Avoiding Extraneous Lines in Graphs

Draw a graph of the equation $y = \dfrac{1}{1 - x}$.

Solution

Figure 4(a) shows the graph produced by a graphing calculator with viewing rectangle $[-9, 9]$ by $[-9, 9]$. In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment should not be part of the graph. Notice that the right-hand side of the equation $y = 1/(1 - x)$ is not defined when x is 1. Sometimes we can get rid of the extraneous near-vertical line by experimenting with a change of scale. Here, for example, when we change to the smaller viewing rectangle $[-5, 5]$ by $[-5, 5]$, we obtain the much better graph in Figure 4(b).

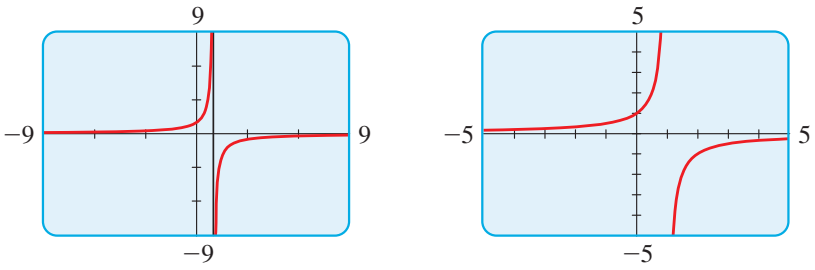


figure 4 Graphs of $y = 1/(1 - x)$

NOW TRY EXERCISE 15

Graphing a Circle

Most graphing calculators can only graph equations in which the variable y is isolated on one side of the equal sign. The next example shows how to graph equations that don't have this property.

example 4 Graphing a Circle

Graph the circle $x^2 + y^2 = 1$.

Solution

We first solve for y to isolate it on one side of the equal sign.

$$y^2 = 1 - x^2 \qquad \text{Subtract } x^2$$

$$y = \pm\sqrt{1 - x^2} \qquad \text{Take square root}$$

Therefore, the circle is described by the graphs of *two* equations:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}$$

The graph in Figure 5(c) looks somewhat flattened. Most graphing calculators allow you to set the scales on the axes so that circles really look like circles. On the TI-83, from the `ZOOM` menu, choose `ZSquare` to set the scales appropriately. On the TI-86 the command is `Zsq`.

The first equation represents the top half of the circle (because $y \geq 0$), and the second represents the bottom half of the circle (because $y \leq 0$). If we graph the first equation in the viewing rectangle $[-2, 2]$ by $[-2, 2]$, we get the semicircle shown in Figure 5(a). The graph of the second equation is the semicircle in Figure 5(b). Graphing these semicircles together in the same viewing screen, we get the full circle in Figure 5(c).

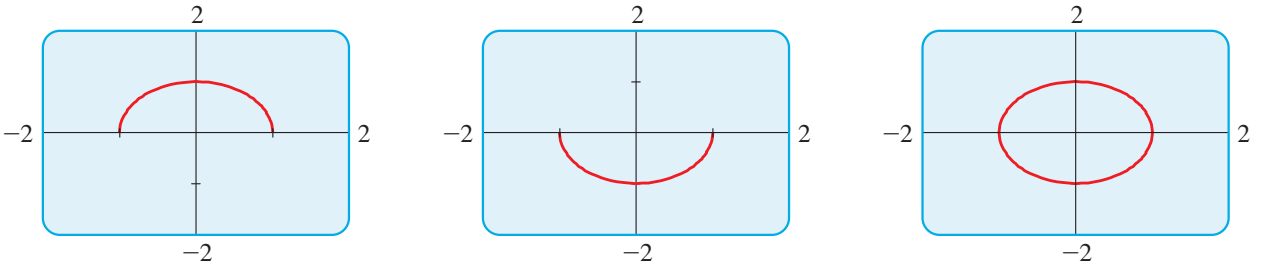


figure **5** Graphing the equation $x^2 + y^2 = 1$

■ **NOW TRY EXERCISE 21** ■

D.3 Exercises

CONCEPTS

1. A graphing calculator or computer produces a graph of an equation by plotting many points. But to get a good representation of the graph, we must choose an appropriate

_____ _____.

2. Two equations are graphed using a graphing calculator and the graphs appear to intersect. To find out whether the graphs *actually* intersect, we need to

_____ on the portion of the graph where they seem to meet.

SKILLS

3–6 ■ Use a graphing calculator or computer to decide which viewing rectangle (a)–(d) produces the most appropriate graph of the equation.

3. $y = x^4 + 2$

(a) $[-2, 2]$ by $[-2, 2]$
(b) $[0, 4]$ by $[0, 4]$
(c) $[-8, 8]$ by $[-4, 40]$
(d) $[-4, 40]$ by $[-80, 800]$

4. $y = x^2 + 7x + 6$

(a) $[-5, 5]$ by $[-5, 5]$
(b) $[0, 10]$ by $[-20, 100]$
(c) $[-15, 8]$ by $[-20, 100]$
(d) $[-10, 3]$ by $[-100, 20]$

5. $y = 100 - x^2$

(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-15, 15]$ by $[-30, 110]$
(d) $[-4, 4]$ by $[-30, 110]$

6. $y = 2x^2 - 1000$

(a) $[-10, 10]$ by $[-10, 10]$
(b) $[-10, 10]$ by $[-100, 100]$
(c) $[-10, 10]$ by $[-1000, 1000]$
(d) $[-25, 25]$ by $[-1200, 200]$

7–16 ■ Determine an appropriate viewing rectangle for the equation, and use it to draw the graph.

7. $y = 100x^2$

8. $y = -100x^2$

9. $y = 4 + 6x - x^2$

10. $y = 0.3x^2 + 1.7x - 3$

11. $y = \sqrt[4]{256 - x^2}$

12. $y = \sqrt{12x - 17}$

13. $y = 0.01x^3 - x^2 + 5$

14. $y = x(x + 6)(x - 9)$

15. $y = \dfrac{1}{x - 2}$

16. $y = \dfrac{x}{3 - x}$

17–20 ■ Do the graphs intersect in the given viewing rectangle? If they do how many points of intersection are there?

17. $y = -3x^2 + 6x - \frac{1}{2}$, $y = \sqrt{7 - \frac{7}{12}x^2}$; $[-4, 4]$ by $[-1, 3]$

18. $y = \sqrt{49 - x^2}$, $y = \frac{1}{5}(41 - 3x)$; $[-8, 8]$ by $[-1, 8]$

19. $y = 6 - 4x - x^2$, $y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$

20. $y = x^3 - 4x$, $y = x + 5$; $[-4, 4]$ by $[-15, 15]$

21. Graph the circle $x^2 + y^2 = 9$ by solving for y and graphing two equations as in Example 4.

22. Graph the circle $(y - 1)^2 + x^2 = 1$ by solving for y and graphing two equations as in Example 4.

D.4 Solving Equations and Inequalities Graphically

- Solving Equations Graphically
- Solving Inequalities Graphically

Solving Equations Graphically

To solve an equation such as

$$3x - 5 = 0$$

we use the **algebraic method**. This means that we use the rules of algebra to isolate x on one side of the equation. We view x as an *unknown,* and we use the rules of algebra to hunt it down. You can check that the solution is $x = \frac{5}{3}$.

We can also solve this equation by the **graphical method**. In this method we view x as a *variable* and sketch the graph of the equation

$$y = 3x - 5$$

Different values for x give different values for y. Our goal is to find the value of x for which $y = 0$. (That is, the solution is the x-intercept of the graph.) From the graph in Figure 1 we see that $y = 0$ when $x \approx 1.7$. Thus, the solution is about 1.7. Note that from the graph we obtain an approximate solution.

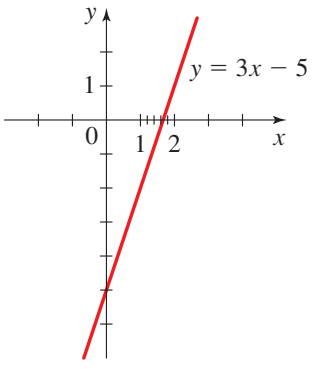


figure **1**

Solving an Equation

Algebraic method
Use the rules of algebra to isolate the unknown x on one side of the equation.

Example:

$2x = 6 - x$ Equation
$3x = 6$ Add x
$x = 2$ Divide by 3

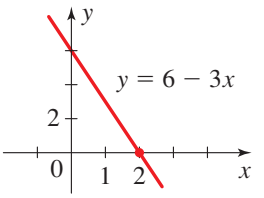
The solution is $x = 2$.

Graphical method
Move all terms to one side and set equal to y. Sketch the graph to find the value of x where $y = 0$.

Example:

$2x = 6 - x$ Equation
$0 = 6 - 3x$ Terms to one side

Set $y = 6 - 3x$ and graph.



From the x-intercept of the graph, the solution is $x \approx 2$.

Intercepts are studied in Algebra Toolkit D.2, page T75.

The algebraic method has the advantage of giving exact answers. But many equations are difficult or impossible to solve algebraically. The graphical method gives a numerical approximation to the answer. This is an advantage when a numerical answer is desired. (For example, an engineer might find an answer expressed as $x \approx 2.6$ more immediately useful than one expressed as $x = \sqrt{7}$.) Also, graphing an equation helps us to visualize how the solution is related to other values of the variable.

example 1 Solving a Quadratic Equation Algebraically and Graphically

Solve the quadratic equations algebraically and graphically.

(a) $x^2 - 4x + 2 = 0$ **(b)** $x^2 - 4x + 4 = 0$ **(c)** $x^2 - 4x + 6 = 0$

Solution 1 Algebraic

The Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

See Algebra Toolkit C.2, page T60.

We use the Quadratic Formula to solve each equation.

(a) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

There are two solutions: $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

(b) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2} = \dfrac{4 \pm \sqrt{0}}{2} = 2$

There is just one solution: $x = 2$.

(c) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 6}}{2} = \dfrac{4 \pm \sqrt{-8}}{2}$

There is no real solution.

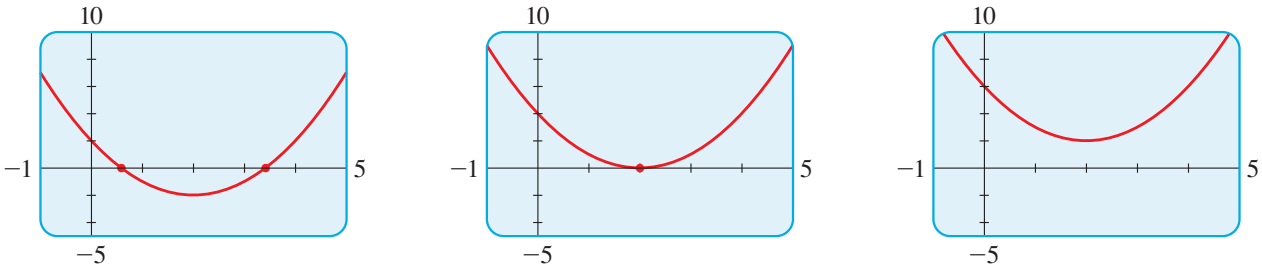
Solution 2 Graphical

We graph the equations $y = x^2 - 4x + 2$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$ in Figure 2. By determining the x-intercepts of the graphs, we find the following solutions.

(a) $x \approx 0.6$ and $x \approx 3.4$

(b) $x \approx 2$

(c) There is no x-intercept, so the equation has no solution.



(a) $y = x^2 - 4x + 2$ (b) $y = x^2 - 4x + 4$ (c) $y = x^2 - 4x + 6$

figure **2** The number of solutions of a quadratic equation

■ **NOW TRY EXERCISES 5 AND 7** ■

The graphs in Figure 2 show visually why a quadratic equation may have two solutions, one solution, or no real solution.

example **2** **Another Graphical Method**

Solve the equation algebraically and graphically: $5 - 3x = 8x - 20$.

Solution 1 Algebraic

We use the rules of algebra to isolate x on one side of the equation.

$$5 - 3x = 8x - 20 \qquad \text{Given equation}$$

$$-3x = 8x - 25 \qquad \text{Subtract 5}$$

$$-11x = -25 \qquad \text{Subtract } 8x$$

$$x = \frac{-25}{-11} \approx 2.27 \qquad \text{Divide by } -11 \text{ and simplify}$$

Solution 2 Graphical

We could move all terms to one side of the equal sign, set the result equal to y, and graph the resulting equation. We can also graph two equations instead.

$$y_1 = 5 - 3x \qquad \text{and} \qquad y_2 = 8x - 20$$

The solution of the original equation will be the value of x that makes y_1 equal to y_2; that is, the solution is the x-coordinate of the intersection point of the two graphs. Using the TRACE feature or the `intersect` command on a graphing calculator, we see from Figure 3 that the solution is $x \approx 2.27$.

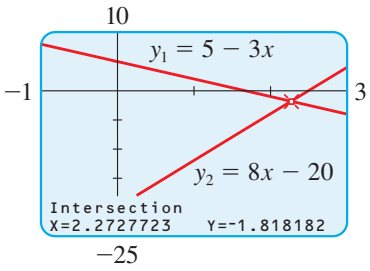


figure **3**

■ **NOW TRY EXERCISE 9** ■

In the next example we use the graphical method to solve an equation that is extremely difficult to solve algebraically.

example 3 Solving an Equation in an Interval

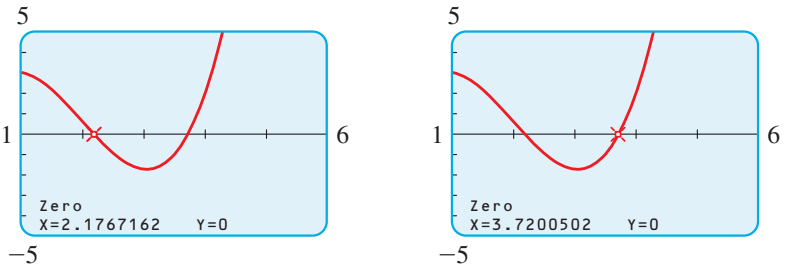
Solve the equation $x^3 - 6x^2 + 9x - \sqrt{x} = 0$ in the interval $[1, 6]$.

Solution

Since we are asked to find all solutions x that satisfy $1 \le x \le 6$, we graph the equation in a viewing rectangle for which the x-values are restricted to this interval. So we graph the equation

$$y = x^3 - 6x^2 + 9x - \sqrt{x}$$

in the viewing rectangle $[1, 6]$ by $[-5, 5]$. There are two x-intercepts in this viewing rectangle; zooming in, we see that the solutions are $x \approx 2.18$ and $x \approx 3.72$. We can also use the `zero` command to find these solutions as shown in Figures 4(a) and 4(b).



(a) (b)

figure 4 Graph of $y = x^3 - 6x^2 + 9x - \sqrt{x}$

■ NOW TRY EXERCISE 15 ■

Solving Inequalities Graphically

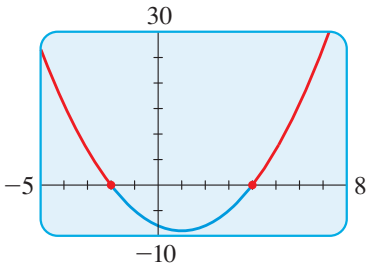


figure 5 $x^2 - 2x - 8 \le 0$

Inequalities can be solved graphically. To describe the method, we solve

$$x^2 - 2x - 8 \le 0$$

This inequality was solved algebraically in Example 3 in **Algebra Toolkit C.3**. To solve the inequality graphically, we graph the equation

$$y = x^2 - 2x - 8$$

Our goal is to find those values of x for which $y \le 0$. These are simply the x-values for which the graph lies below the x-axis. From Figure 5 we see that the solution of the inequality is the interval $[-2, 4]$.

example 4 Solving an Inequality Graphically

Solve the inequality $x^3 - 5x^2 \ge -8$.

Solution

We write the inequality as

$$x^3 - 5x^2 + 8 \ge 0$$

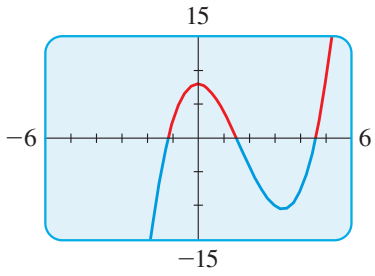


figure **6** $x^3 - 5x^2 + 8 \geq 0$

and then graph the equation

$$y = x^3 - 5x^2 + 8$$

in the viewing rectangle $[-6, 6]$ by $[-15, 15]$, as shown in Figure 6. The solution of the inequality consists of those intervals on which the graph lies on or above the x-axis. By moving the cursor to the x-intercepts, we find that, correct to one decimal place, the solution is $[-1.1, 1.5] \cup [4.6, \infty)$.

NOW TRY EXERCISE 23

example **5** **Solving an Inequality Graphically**

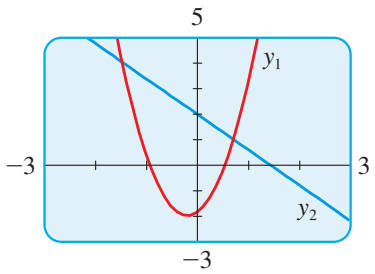


figure **7**
$y_1 = 3.7x^2 + 1.3x - 1.9$
$y_2 = 2.0 - 1.4x$

Solve the inequality $3.7x^2 + 1.3x - 1.9 \leq 2.0 - 1.4x$.

Solution

We graph the equations

$$y_1 = 3.7x^2 + 1.3x - 1.9 \qquad \text{and} \qquad y_2 = 2.0 - 1.4x$$

in the same viewing rectangle in Figure 7. We are interested in those values of x for which $y_1 \leq y_2$; these are points for which the graph of y_1 lies on or below the graph of y_2. To determine the appropriate interval, we look for the x-coordinates of points where the graphs intersect. We conclude that the solution is (approximately) the interval $[-1.45, 0.72]$.

NOW TRY EXERCISE 25

D.4 Exercises

CONCEPTS

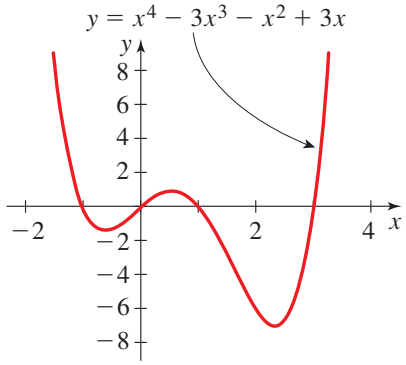
1. The solutions of the equation $x^2 - 2x - 3 = 0$ are the ____-intercepts of the graph of $y = x^2 - 2x - 3$.

2. The solutions of the inequality $x^2 - 2x - 3 > 0$ are the x-coordinates of the points on the graph of $y = x^2 - 2x - 3$ that lie _____ (above/below) the x-axis.

3. The figure shows a graph of $y = x^4 - 3x^3 - x^2 + 3x$.

(a) Find the solutions of the equation $x^4 - 3x^3 - x^2 + 3x = 0$.

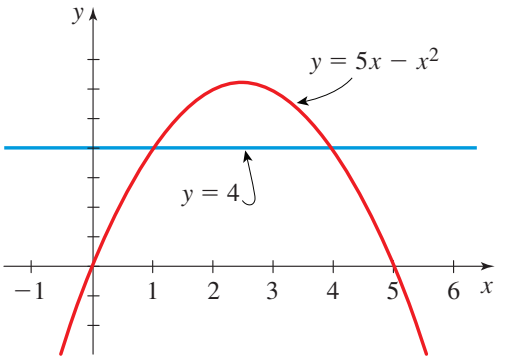
(b) Find the solutions of the inequality $x^4 - 3x^2 - x^2 + 3x \leq 0$.



4. The figure shows the graphs of $y = 5x - x^2$ and $y = 4$. Use the graphs to do the following.

(a) Find the solutions of the equation $5x - x^2 = 4$.

(b) Find the solutions of the inequality $5x - x^2 > 4$.



SKILLS

5–12 ■ Solve the equation both algebraically and graphically.

5. $x - 4 = 5x + 12$

6. $\frac{1}{2}x - 3 = 6 + 2x$

7. $x^2 - 32 = 0$

8. $x^3 + 16 = 0$

9. $x^2 + 9 = 0$

10. $x^2 + 3 = 2x$

11. $16x^4 = 625$

12. $2x^5 - 243 = 0$

13–18 ■ Solve the equation graphically in the given interval.

13. $x^2 - 7x + 12 = 0; \quad [0, 6]$

14. $x^2 - 0.75x + 0.125 = 0; \quad [-2, 2]$

15. $x^3 - 6x^2 + 11x - 6 = 0; \quad [-1, 4]$

16. $16x^3 + 16x^2 = x + 1; \quad [-2, 2]$

17. $x - \sqrt{x + 1} = 0; \quad [-1, 5]$

18. $1 + \sqrt{x} = \sqrt{1 + x^2}; \quad [-1, 5]$

19–22 ■ Use the graphical method to solve the equation in the indicated exercise from **Algebra Toolkit C.2**. These equations were solved algebraically in **Algebra Toolkit C.2**.

19. Exercise 31

20. Exercise 32

21. Exercise 33

22. Exercise 34

23–28 ■ Find the solutions of the inequality by drawing appropriate graphs.

23. $x^2 \le 3x + 10$

24. $0.5x^2 + 0.875x \le 0.25$

25. $x^3 + 11x \le 6x^2 + 6$

26. $16x^3 + 24x^2 > -9x - 1$

27. $x^{1/3} < x$

28. $\sqrt{0.5x^2 + 1} \le 2|x|$

29–32 ■ Use the graphical method to solve the inequality in the indicated exercise from **Algebra Toolkit C.3**. These inequalities were solved algebraically in **Algebra Toolkit C.3**.

29. Exercise 25

30. Exercise 26

31. Exercise 32

32. Exercise 33

Answers to Selected Exercises and Chapter Tests

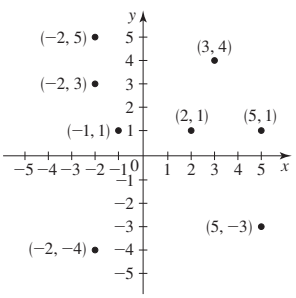
Chapter 1

1.1 Exercises ■ page 7

1. **(a)** Data with only one varying quantity. **(b)** Two-variable data have two variables associated with every data point, while one-variable data have only one variable associated with each data point. **2.** median **3.** add, n **4.** order **(a)** middle **(b)** middle **5.** A computer **7.** **(a)** 33.6 **(b)** 21 **(c)** One; two **9.** **(a)** 71.3 **(b)** 71.5 **(c)** Four; four **11.** As A increases, B increases. **13.** As A increases, B initially increases, but then levels off. **15.** 27.2 **17.** Average: 7.6; median: 8 **19.** **(a)** 7.6 **(b)** 8 **21.** **(a)** Average: \$57,286; median: \$58,000 **(b)** Four **(c)** New average: \$112,625; new median: \$58,500 Only one passenger has an income above average. In this case the median is a better measure of central tendency. **23.** **(a)** Average: \$325,800; median: \$329,000; average **(b)** Above; same **(c)** New average: \$748,167; new median: \$329,500; median **25.** **(a)** 5.7 pets **(b)** No **(c)** 2 **27.** **(a)** 12 in. **(b)** March; January **(c)** 17.5 in./month **(d)** 5 in./month **29.** **(a)** 0.6 s **(b)** 0.36 s; 0.19 s **(c)** Every additional 10 ft it falls takes less time than the preceding 10 ft. **31.** **(a)** 0.13, 0.16, and 0.033 mg/mL **(b)** After 1.0 hour **(c)** The concentration rises for the first hour and then decreases slowly over the next 2 hours.

Algebra Checkpoint 1.2 ■ page 18

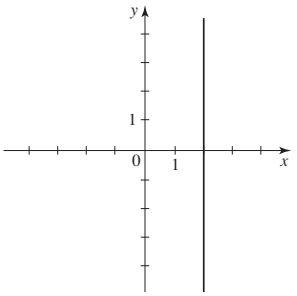
1.



2. $A\,(3, 4)$, $B\,(2, 2)$, $C\,(-1, 4)$, $D\,(-2, 2)$, $E\,(-3, -1)$, $F\,(-2, -2)$, $G\,(3, -1)$, $H\,(4, -2)$

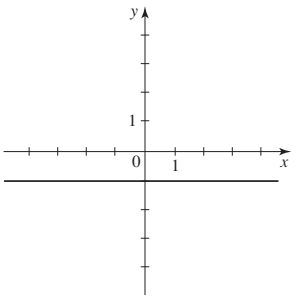
3. **(a)** The set of points with an x-coordinate of 2

(b)



4. **(a)** The set of points with a y-coordinate of -1

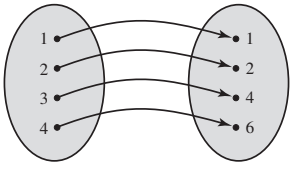
(b)



1.2 Exercises ■ page 18

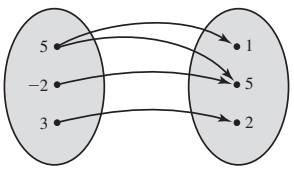
1. inputs; outputs **2.** ordered **3.** x; y **4.** **(a)** When the worker works for 10 hours, he earns \$100. **(b)** \$200 **(c)** 10 hours **5.** Input: value of coins; output: number of coins needed to have that value **6.** Input: hours of studying for a test; output: grade on the test **7.** **(a)** Domain: $\{1, 2, 3, 4\}$; range: $\{1, 2, 4, 6\}$

(b)

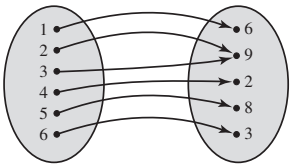


9. **(a)** Domain: $\{-2, 3, 5\}$; range: $\{1, 2, 5\}$

(b)

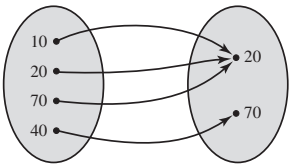


11. (a) {(1, 6), (2, 9), (3, 3), (4, 2), (5, 8), (6, 3)}
(b) Domain: {1, 2, 3, 4, 5, 6}; range: {2, 3, 6, 8, 9}
(c)



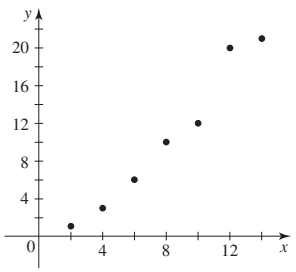
(d) 8 **(e)** 3, 6

13. (a) {(10, 20), (20, 20), (30, 20), (40, 70), (70, 20)}
(b) Domain: {10, 20, 30, 40, 70}; range: {20, 70}
(c)



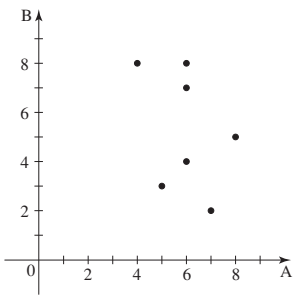
(d) 20 **(e)** 10, 20, 30, 70

15. (b) 40 **(c)** 3, 6 **17. (b)** 30 **(c)** 2, 3 **19.** Graph D
21. Graph B **23.** The y-values get larger as the x-values get larger. **25.** There is no obvious relationship between x and y.
27. (a)



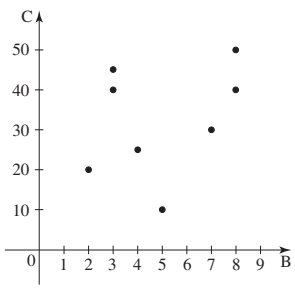
(b) The y-values get larger as the x-values get larger.

29. (A, B)
(a)



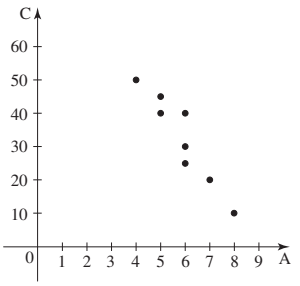
(b) There is no obvious relationship between the variables.

(B, C)
(a)



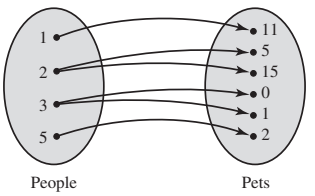
(b) There is no obvious relationship between the variables.

(A, C)
(a)



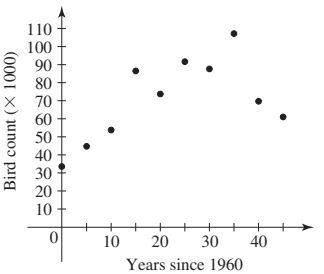
(b) The C-values get smaller as the A-values get larger.

31. (a) {(2, 5), (1, 11), (5, 2), (3, 0), (2, 15), (3, 1)}
(b)



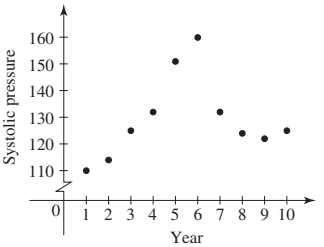
(c) Domain: {1, 2, 3, 5}; range: {0, 1, 2, 5, 11, 15}

33. (a) {(1, 49), (2, 12,), (3, 83), (4, 42), (5, 2), (6, 0), (7, 0), (8, 0), (9, 0), (10, 0), (11, 8), (12, 14)} **(c)** Domain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}; range: {0, 2, 8, 12, 14, 42, 49, 83}
35. (a) {(80,000, 400,000), (30,000, 250,000), (70,000, 300,000), (55,000, 250,000), (80,000, 450,000), (60,000, 300,000), (55,000, 300,000), (150,000, 1,500,000)}
(c) A neighborhood with a median income of $55,000 and a median home price of $250,000 **(d)** $400,000 and $450,000
(e) Domain: {30,000, 55,000, 60,000, 70,000, 80,000, 150,000}; range: {250,000, 300,000, 400,000, 450,000, 1,500,000} **37. (a)** {(2000, 78), (2004, 83), (2005, 213), (2006, 244), (2007, 350)} **(b)** In 2005, 213,000 hybrid cars were sold. **(c)** Domain: {2000, 2004, 2005, 2006, 2007}; range: {78, 83, 213, 244, 350}
39. (a)



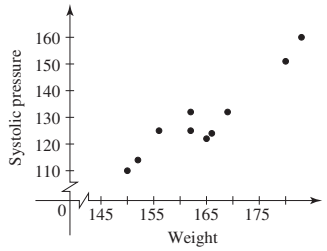
(b) The population was generally increasing until 1995, and then it decreased.

41. (a)



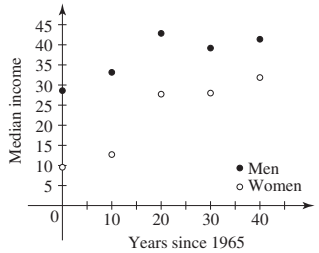
The man's systolic pressure increases for the first six years and then decreases.

(b)



The man's systolic pressure increases as his weight increases.

43. (a)



(b) In 1995 the median income of men went down, and the median income of women went up.

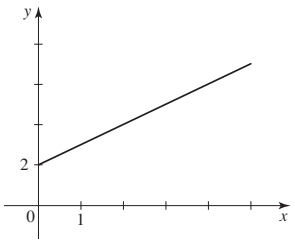
(c) The median income of men generally has been going up, though in 1995 it went down briefly. The median income of women has been going up since 1980, though it has been slowing down in the last two decades. **45. (a)** 82 in./year **(b)** 1980 **(c)** 1940 and 1950 **(d)** Yes; the pan evaporation tends to be decreasing as time goes on. **47.** The concentration of alcohol rises sharply for the first half hour, then declines gradually over the next two and a half hours. This agrees with the answer for Exercise 31 in Section 1.1.

Algebra Checkpoint 1.3 ▪ page 31

1. (a) Yes **(b)** Yes **(c)** No **2. (a)** Yes **(b)** No **(c)** Yes

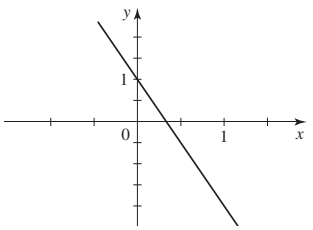
3. (a)

x	y
0	2
1	3
2	4
3	5
4	6
5	7



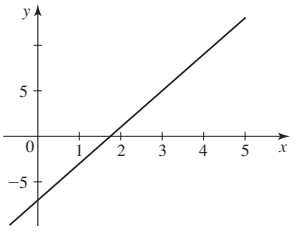
(b)

x	y
−2	7
−1	4
0	1
1	−2
2	−5
3	−8



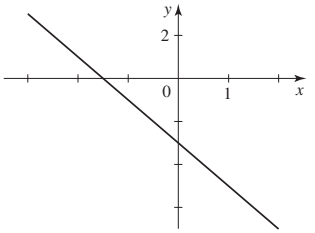
(c)

x	y
0	−7
1	−3
2	1
3	5
4	9
5	13



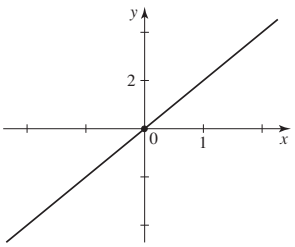
(d)

x	y
−3	3
−2	1
−1	−1
0	−3
1	−5
2	−7

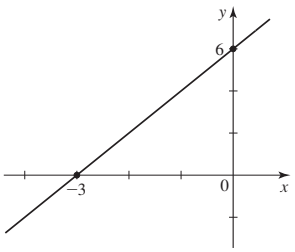


4. (a) x-intercept: 2; y-intercept: -2 **(b)** x-intercept: $-\frac{2}{3}$; y-intercept: 1 **(c)** x-intercept: 6; y-intercept: -2 **(d)** x-intercept: 2; y-intercept: $-\frac{8}{5}$

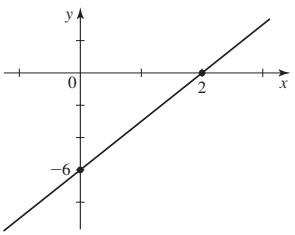
5. (a) x-intercept: 0; y-intercept: 0



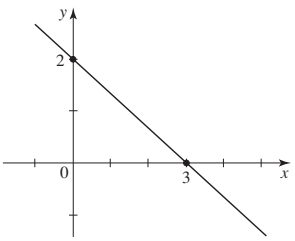
(b) x-intercept: -3; y-intercept: 6



(c) x-intercept: 2; y-intercept: -6



(d) x-intercept: 3; y-intercept: 2



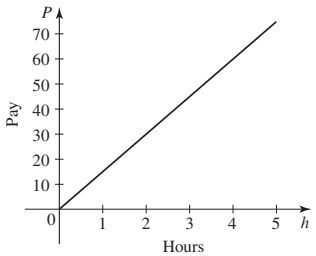
1.3 Exercises ▪ page 31

1. (a) A model is an equation that represents data.

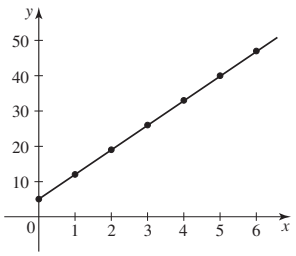
(b) (i)

Hours	Pay
1	15
2	30
3	45
4	60
5	75

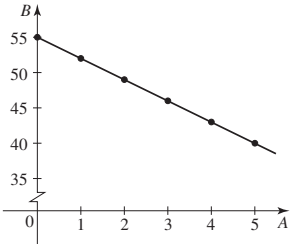
(ii)



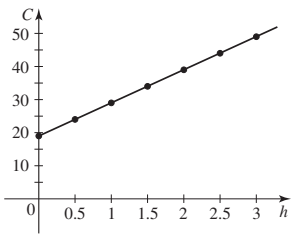
(iii) $P = 15h$ **2.** constant; -6; yes **3.** 48
4. $C = 110 + 49x$
7. (a) $y = 5 + 7x$
(b) (4, 33), (5, 40), (6, 47)
(c)



9. (a) $B = 55 - 3A$
(b) (4, 43), (5, 40), (6, 37)
(c)



11. (a) 13, 13, 13 **(b)** Yes; $y = 205 + 13x$ **(c)** (4, 257), (5, 270), (6, 283) **13. (a)** $-4, -3, -5$ **(b)** No **(c)** Not linear **15.** $y = 10 + 5x$ **17.** $y = 1000 - 1000x$
19. (a) $C = 19 + 10h$
(b)



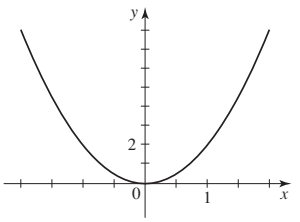
(c) $69.00
21. (b) $B = 212 - 1.8E$ **(c)** 177.188°F **23. (b)** $P = \frac{17}{20}M$ **(c)** 3825 lb **25. (a)** $S = 150 + 10y$ **(b)** $250,000

Algebra Checkpoint 1.4 ▪ page 44

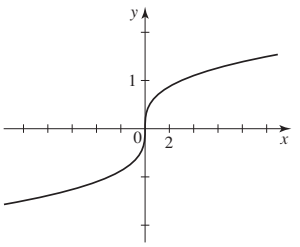
1. 3 **2.** $-\frac{1}{2}$ **3.** 10 **4.** $\frac{14}{3}$ **5.** 14; -1 **6.** 2 or -2; 3
7. 3; no solution **8.** 3 or -3; $\frac{4}{3}$ **9.** $x = \dfrac{y-5}{3}$ **10.** $T = \dfrac{P}{6}$
11. $z = \dfrac{r}{3}$ **12.** $y = 2\sqrt[3]{x}$ **13.** $t = \dfrac{2}{3x}$ **14.** $p = 4r$
15. $t = \dfrac{3s}{4+s}$ **16.** $h = \dfrac{2}{w-3}$ **17.** $y = \pm 3\sqrt{x}$
18. $R = \pm 3\sqrt{p}$

1.4 Exercises ▪ page 44

1. (a) output **(b)** x, y **2. (a)** y, one **(b)** x, y
3. $18 - 2 = 16$ **4. (a)** Vertical Line Test **(b)** (i)
5. (a) False **(b)** True **(c)** False **(d)** True
7. Yes **9.** No **11. (a)** No **(b)** Yes; y independent, x dependent **13. (a)** Yes; x independent, y dependent **(b)** No **15. (b)** -1 **(c)** -3 **17. (b)** 1 **(c)** -2
19. (a) x independent, y dependent **(b)** 14
21. (a) l independent, w dependent **(b)** 44 **23. (a)** Yes **(b)** Yes **25. (a)** No **(b)** Yes **27. (a)** Yes **(b)** Yes
29. Yes; -2 **31.** No **33.** Yes **35.** No **37.** Yes **39.** Yes
41. No
43. (a)



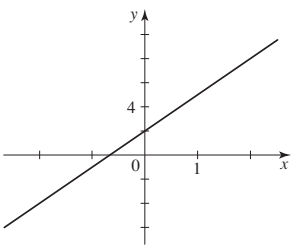
(b) Yes; $y = 2x^2$
47. (a)



(b) Yes; $y = \sqrt[3]{\dfrac{x}{3}}$
49. (a) $y = 3x + 2$
(b)

x	y
-2	-4
-1	-1
0	2
1	5
2	8

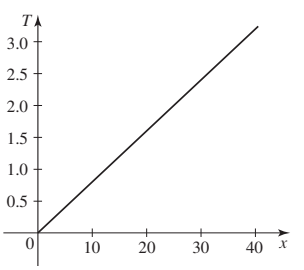
(c)



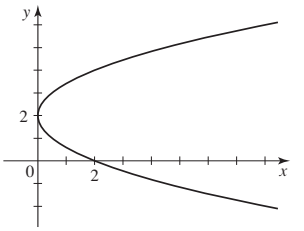
51. (a) $T = 0.08x$
(b)

x	T
0	0
10.00	0.80
20.00	1.60
30.00	2.40
40.00	3.20

(c)



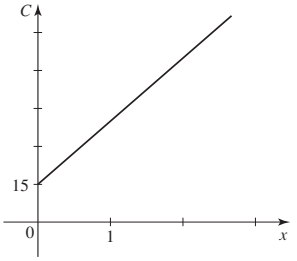
45. (a)



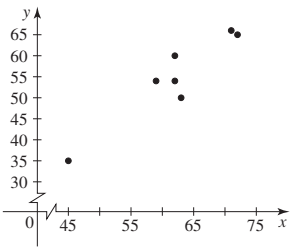
(b) No

53. **(b)** 0; 22 **(c)** \$85,000 **55.** The cost of a load of hay is \$15 plus \$25 per bale.
$C = 15 + 25x$

x	C
0	15
1	40
2	65
3	90
4	115
5	140



57. **(b)** \$1500 **(c)** $x = \dfrac{P + 5000}{15}$ **(d)** 334 tickets (x is 333.3, but you can't buy just one-third of a ticket!)
59. **(a)** $v = 24(0.25 - r^2)$ **(b)** 6 cm/s; 5.76 cm/s; 0 cm/s
(c) -0.24 cm/s; -5.76 cm/s
61. **(a)** **(b)** No



63. **(a)**

H	60	62	63	65	65	68	70	71	75	75
y	27	29	31	22	31	25	27	27	24	30

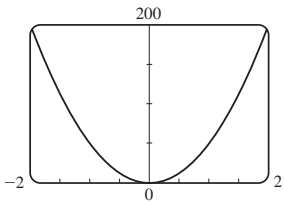
(b) (65, 22), (65, 31)
(c) No
65. **(a)** Yes **(c)** Because there can be only one measurement at any given time.

1.5 Exercises ▪ page 58

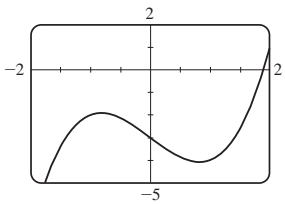
1. x, y, value **2.** 5, 1 **3.** $f(b) - f(a)$; $5 - 1 = 4$
4. domain, range **5.** **(a)** f, g **(b)** 10, 0 **6.** (2, 7), (4, 3), (6, 7) **7.** The difference between 5 squared minus 2 and -2 squared minus 2 **8.** **(a)** False **(b)** False **9.** **(a)** f **(b)** x, $2x + 1$ **(c)** "Multiply by 2, then add 1." **(d)** 21; 21 is the value of the function at 10. **11.** **(a)** h **(b)** $z, (z - 2)^2$
(c) "Subtract two, then square the difference." **(d)** 64; 64 is the value of the function at 10. **13.** $y = 5(x + 2)$
15. $y = \dfrac{x^2 + 7}{4}$ **17.** "Divide by 2, then add 7." **19.** "Square, multiply by 7, then subtract 5." **21.** "Add 3, cube the sum, then multiply by 5." **23.** **(a)** Yes; $y = 5x$ **(b)** Yes; $x = \dfrac{y}{5}$
25. **(a)** Yes; $y = \sqrt[3]{\dfrac{x}{4}}$ **(b)** Yes; $x = 4y^3$ **27.** **(a)** Yes; $y = 3x^2$ **(b)** No **29.** $y = 3x^2 - 1$; x independent, y dependent
31. $y = 3w^2 - 1$; w independent, y dependent **33.** $S = 4\pi r^2$; r independent, S dependent **35.** **(a)** $(-2, 1), (-1, -5), (0, -7), (1, -5), (2, 1)$ **(b)** 8 **37.** **(a)** $(-3, -2), (-1, 6), (0, 25), (1, 62), (2, 123)$ **(b)** 98 **39.** **(a)** 0 **(b)** $2 + 2\sqrt{2}$ **(c)** 0 **(d)** $a^2 + 2a$ **41.** **(a)** $\sqrt{3}$ **(b)** 3 **(c)** $\sqrt{5}$ **(d)** $\sqrt{2c - 1}$ **43.** **(a)** $16\pi \approx 50.27$; $36\pi \approx 113.10$ **(b)** The surface area of spheres with radii of 2 and 3, respectively **(c)** $60\pi \approx 188.50$ **45.** $\{x \mid -\infty < x < \infty\}$
47. $\{x \mid -1 \le x \le 5\}$ **49.** $\{x \mid x \ne 3\}$ **51.** $\{x \mid x \ge 5\}$
53. $\{x \mid x > 4\}$ **55.** **(a)** 1 **(b)** 0 **(c)** 2 **(d)** 1 **57.** **(a)** 9 **(b)** 0 **(c)** 4 **(d)** 2 **59.** **(a)** 4 **(b)** -1 **(c)** 0 **(d)** 0
61. **(a)** 28.14 mi; the maximum distance you can see from a height of 0.1 mi **(b)** 41.26 mi **(c)** 235.6 mi **(d)** 29.01 mi
63. **(a)** 11,400 ft, 6600 ft; the skydiver's height above the ground after 10 s and 20 s, respectively **(b)** Yes, she was 3784 ft above the ground **(c)** $-10{,}000$ ft
65. **(a)** 7.66 billion dollars, 9.49 billion dollars; the box office revenue in 2000 and 2006, respectively **(b)** 2003, 2006
(c) 1.83 billion dollars **67.** **(a)** \$0, \$960, \$2350; the tax on incomes of \$5000, \$12,000, and \$25,000, respectively
(b) \$1600 **(c)** \$176,000 **69.** **(a)** $x + 15, x$ **(b)** \$90, \$100, \$105; the cost of an order of \$75, \$100, or \$105 worth of books.
71. **(a)** $15(40 - x)$, 0, $15(x - 65)$ **(b)** \$150, \$0, \$150
(c) 25 mi/h or 80 mi/h

Algebra Checkpoint 1.6 ▪ page 70

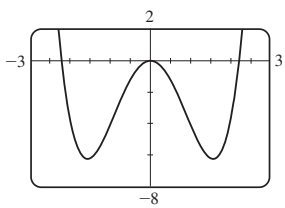
1. (iii) **2.** (iii) **3.** (iii) **4.** (iv)
5. **(a)**



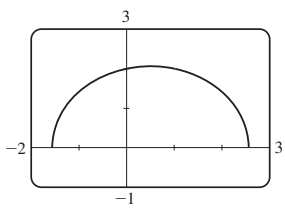
(b)



(c)



(d)

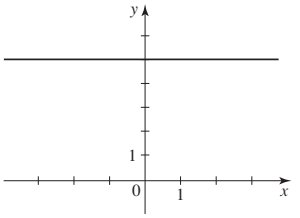


1.6 Exercises ▪ page 71

1. $f(x), x^3 + 2$, 10, 10 **2.** 3 **3.** 5 **4.** **(a)** IV **(b)** II **(c)** I **(d)** III **6.** The calculator interprets $y = x\text{^}1/3$ as $y = \dfrac{x^1}{3}$ or $y = \dfrac{1}{3}x$; the calculator interprets $y = x/x + 4$ as $y = \dfrac{x}{x} + 4$ or $y = 5$.

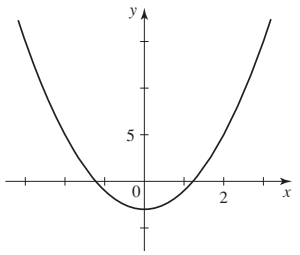
7.

x	$f(x)$
−3	5
−2	5
−1	5
0	5
1	5
2	5
3	5



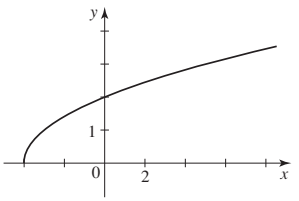
9.

x	$h(x)$
−3	15
−2	5
−1	−1
0	−3
1	−1
2	5
3	15

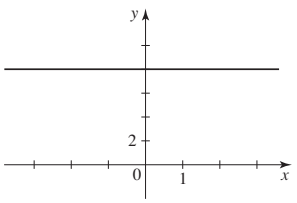


11.

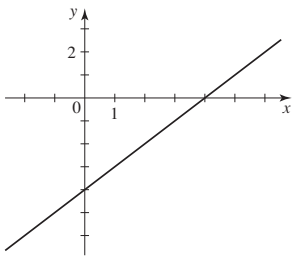
x	$F(x)$
−4	0
−2	$\sqrt{2}$
0	2
2	$\sqrt{6}$
4	$2\sqrt{2}$
8	$2\sqrt{3}$



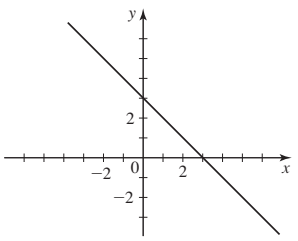
13.



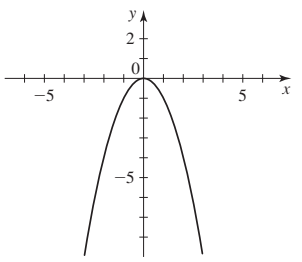
15.



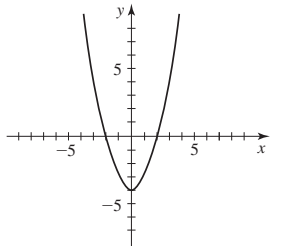
17.



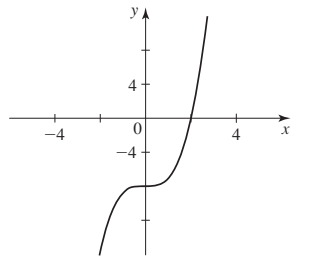
19.



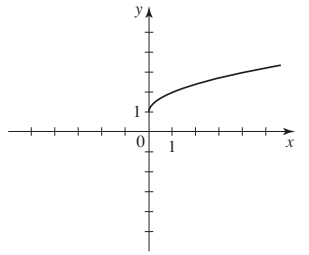
21.



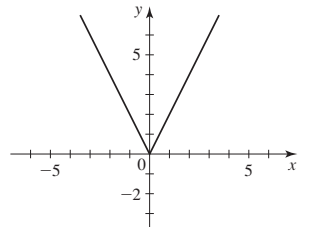
23.



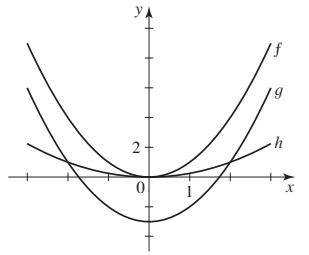
25.



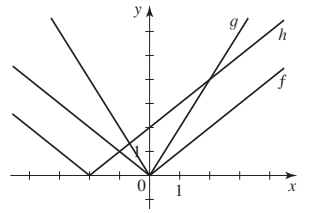
27.



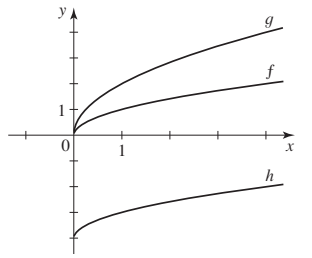
29.



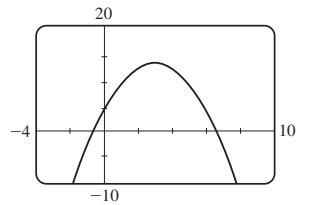
31.



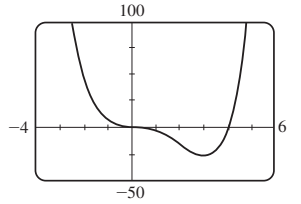
33.



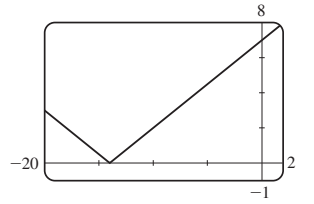
35.



37.

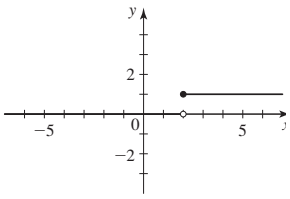


39.

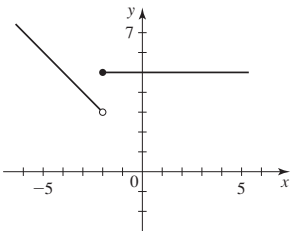


41. Yes, two

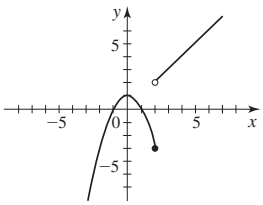
43.



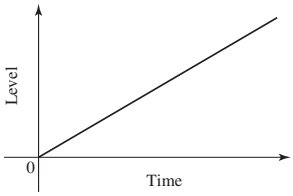
45.



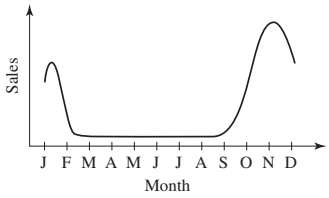
47.



49.

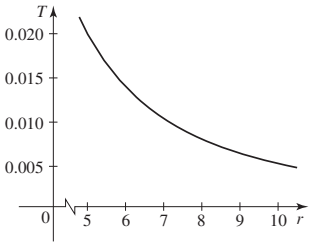


51.

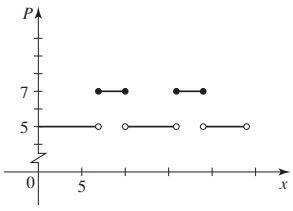


53.

r	$T(r)$
5	0.02000
6	0.01389
7	0.01020
8	0.00781
9	0.00617
10	0.00500



55. (a)



(b) The change in toll at the times that morning and afternoon rush hour start and stop

Algebra Checkpoint 1.7 ▪ page 81

1. (a) The set of values greater than or equal to 1 and less than or equal to 4 **(b)** $[1, 4]$
(c)



2. (a) The set of values greater than or equal to -3 and less than 2 **(b)** $[-3, 2)$
(c)



3. (a) The set of values greater than -10 and less than -3
(b) $(-10, -3)$
(c)



4. (a) The set of values greater than or equal to 0 **(b)** $[0, \infty)$
(c)



5. (a) The set of values greater than 2 and less than 6
(b) $\{x \mid 2 < x < 6\}$
(c)



6. (a) The set of values greater than -4 and less than or equal to 3 **(b)** $\{x \mid -4 < x \leq 3\}$
(c)



7. (a) The set of values less than 2 **(b)** $\{x \mid -\infty < x < 2\}$
(c)



8. (a) The set of values greater than or equal to -1
(b) $\{x \mid -1 \leq x < \infty\}$
(c)

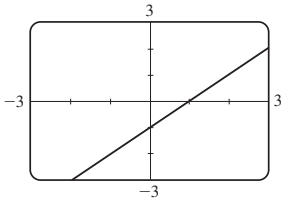


9. (a) The set of values greater than 1 and less than 5
(b) $\{x \mid 1 < x < 5\}$ **(c)** $(1, 5)$ **10. (a)** The set of values greater than or equal to -1 and less than 1
(b) $\{x \mid -1 \leq x < 1\}$ **(c)** $[-1, 1)$ **11. (a)** The set of values less than or equal to -2 **(b)** $\{x \mid -\infty < x \leq -2\}$
(c) $(-\infty, -2]$ **12. (a)** The set of values greater than or equal to -3 and less than or equal to 0 **(b)** $\{x \mid -3 \leq x \leq 0\}$
(c) $[-3, 0]$

1.7 Exercises ▪ page 82

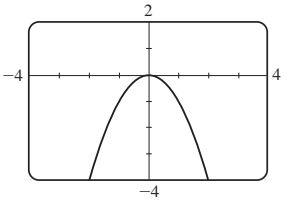
1. a, 4 **2.** x, y, [1, 6], [1, 7] **3. (a)** increase, [1, 2], [4, 5] **(b)** decrease, [2, 4], [5, 6] **4. (a)** largest, 7 or 6, 2 or 5 **(b)** smallest, 2, 4 **7. (a)** 1, -1, 3, 4 **(b)** Domain: $[-3, 4]$; range: $[-1, 4]$ **(c)** -3, 2, 4 **(d)** $[-3, 2]$ **(e)** 2
9. (a) 3, 2, -2, 1, 0 **(b)** Domain: $[-4, 4]$; range: $[-2, 3]$
11. (a) -1, 3 **(b)** $(-\infty, -1]$; $[3, \infty)$ **(c)** $(-1, 3)$
13. (a) -0.46, 1.46 **(b)** $(-\infty, -0.46]$, $[1.46, \infty)$
(c) $(-0.46, 1.46)$

15. (a)



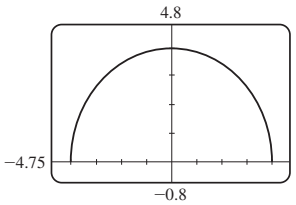
(b) Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

17. (a)



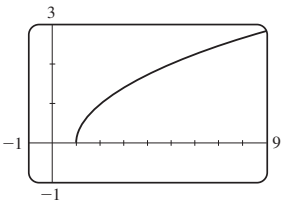
(b) Domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

19. (a)



(b) Domain: $[-4, 4]$; range: $[0, 4]$

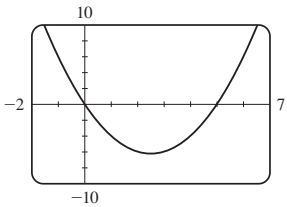
21. (a)



(b) Domain: $[1, \infty)$; range: $[0, \infty)$

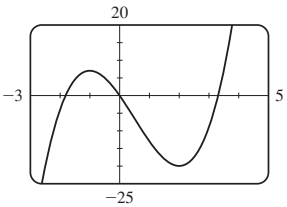
23. (a) $[-1, 1]$ and $[2, 4]$ **(b)** $[1, 2]$ **25. (a)** $[-2, -1]$ and $[1, 2]$ **(b)** $[-3, -2]$, $[-1, 1]$, $[2, 3]$

27. (a)



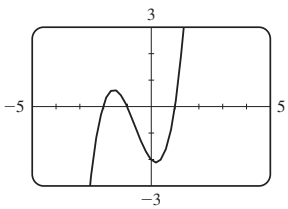
(b) Increasing: $[2.5, \infty)$; decreasing: $(-\infty, 2.5]$

29. (a)



(b) Increasing: $(-\infty, -1]$ and $[2, \infty)$; decreasing: $[-1, 2]$

31. (a)

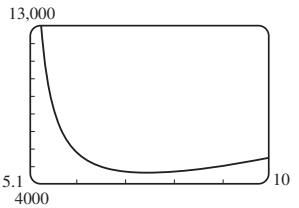


(b) Increasing: $(-\infty, -1.55]$ and $[0.22, \infty)$; decreasing: $[-1.55, 0.22]$ **33. (a)** Local maximum 2 when $x = 0$; local minimum -1 when $x = -2$, local minimum 0 when $x = 2$ **(b)** Increasing: $[-2, 0]$ and $[2, \infty)$; decreasing: $(-\infty, -2]$ and $[0, 2]$ **35. (a)** Local maximum 0 when $x = 0$, local maximum 1 when $x = 3$; local minimum -2 when $x = -2$, local minimum -1 when $x = 1$ **(b)** Increasing: $[-2, 0]$ and $[1, 3]$; decreasing: $(-\infty, -2]$, $[0, 1]$ and $[3, \infty)$ **37. (a)** Local maximum 0.38 when $x = -0.58$; local minimum -0.38 when $x = 0.58$ **(b)** Increasing: $(-\infty, -0.58]$ and $[0.58, \infty)$; decreasing: $[-0.58, 0.58]$ **39. (a)** Local maximum 5.66 when $x = 4$ **(b)** Increasing: $(-\infty, 4]$; decreasing: $[4, 6]$

41. (a) 500 MW, 720 MW **(b)** Just before noon **(c)** Between 3:00 A.M. and 4:00 A.M. **(d)** About 375 MW

43. (a) 32°F, 20°F **(b)** $T(19)$ **(c)** Day 2, days 6–10, days 12–25, day 15 **(d)** 7°F **45. (a)** Increasing: [0, 30] and [32, 66]; decreasing: [30, 32] **(b)** The person lost a lot of weight in a short period of time, then gained it back within 2 years. **47. (a)** Yes; runner B **(b)** 10 s after start **(c)** [0, 10]

49. (a)



(b) Increasing $[7.5, \infty)$; decreasing $[5.1, 7.5]$ **(c)** 4600; E is minimized when the velocity is 7.5 mi/h **51. (a)** Maximum concentration is at 0.3 h **(b)** Increasing: [0, 0.3] **(c)** Decreasing: [0.3, 3]

1.8 Exercises ▪ page 96

1. $N(w) = 7w$ **3.** $S(n) = 2n + 1$ **5.** $P(x) = 2x^2$ **7.** $A(x) = x^2 + 4x$ **9.** $V(x) = x^3$ **11.** $A(h) = h^2$

13. (b) $f(x) = x(19 - x)$ **(c)** Maximum product occurs when both numbers are 9.5. **15. (b)** $f(x) = -x(x + 24)$ **(c)** The maximum product occurs when both values are -12.

17. (a) $C(x) = 200 + 3.5x$ **(b)** \$2300

19. (a) $C(x) = \frac{32}{300}x$ **(b)** \$53.33 **(c)** 2343.75 mi

21. (a)

Number of guests	Hall	DJ	Caterer	Cake	Total
10	\$700	\$300	\$185	\$15	\$1200
20	\$700	\$300	\$370	\$30	\$1400
30	\$700	\$300	\$555	\$45	\$1600
40	\$700	\$300	\$740	\$60	\$1800
50	\$700	\$300	\$925	\$75	\$2000

(b) $C(x) = 1000 + 20x$ **(c)** \$2500 **(d)** 200 people

23. (a)

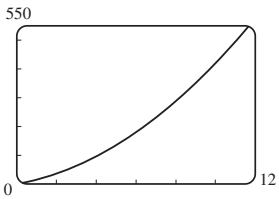
Number of frames	Number that are \$10	Number that are 1¢	Total cost
2	1	1	\$10.01
4	2	2	\$20.02
6	3	3	\$30.03
8	4	4	\$40.04
10	5	5	\$50.05

(b) $C(x) = 5.005x$ **(c)** \$100.10

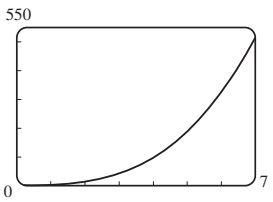
25. (a)

Width (in.)	Height (in.)	Depth (in.)	Volume (in^3)
3	7	3	63
4	8	3	96
5	9	3	135
6	10	3	180
7	11	3	231

(b) $V(x) = 3x(4 + x)$ **(c)** 420 in^3 **(d)** 8.20 in.



27. (a) $V(x) = \frac{3}{2}x^3$ **(b)** 1500 in^3 **(c)** 5.94 in.



(d) $[6.7, \infty)$
29. (b) $A(x) = x(2400 - 2x)$ **(c)** 600 ft by 1200 ft
31. (a) $V(x) = x(12 - 2x)(20 - 2x)$ **(b)** [1.17, 3.90]
(c) 262.7 in^3

1.9 Exercises ▪ page 108

1. $8 \times 12 = 96$; $2 \times 20 = 40$ **2.** $t = \frac{d}{r}$; $t = \frac{350}{55} \approx 6.4$ h

3. $R = \frac{PV}{nT}$ **5.** $R_1 = \frac{RR_2}{R_2 - R}$ **7.** $w = \frac{S - 2lh}{2(l + h)}$

9. $A = \frac{a_1 + a_2}{2}$ **11.** $S = m^2 + n^2$ **13.** $P = 2mn$

15. $t = \frac{d}{r}$ **17.** $A = wl + 2wh + 2lh$ **19.** $L = 2\pi r + 2x$

21. (a) $A = 19\frac{n}{G}$ **(b)** 13,571 lb **(c)** 5182 lb

23. (a) $P = n(p_s - p_o)$ **(b)** \$3600 **25. (a)** $x = \frac{hy}{6 - h}$
(b) 5 m **27.** 4.4 Ω **29.** 508.5 Hz

31. (a) 49,900k, 4,000,000k; the second rate is larger.
(b) 0; there is no one left to infect.

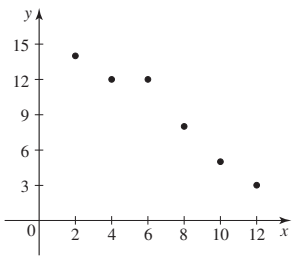
33. (a) $v = \sqrt{\frac{94{,}700w}{S}}$
(b)

Bird	w/S	V (mi/h)
Common tern	0.00342	18.00
Black-headed gull	0.00433	20.26
Common gull	0.00456	20.77
Royal tern	0.00647	24.75
Herring gull	0.00750	26.65
Great skua	0.00909	29.34
Sooty albatross	0.01189	33.55
Wandering albatross	0.02042	43.97

(c) 47.45 mi/h

Chapter 1 Review Exercises ▪ page 115

1. (a) $4\frac{1}{3}$ **(b)** 4 **3. (a)** {(2, 14), (4, 12), (6, 12), (8, 8), (10, 5), (12, 3)} **(b)** Domain: {2, 4, 6, 8, 10, 12}; range: {3, 5, 8, 12, 14} **(c)** 12
(d)

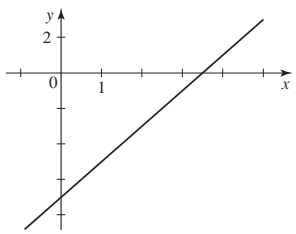


(e) Yes; the y values get smaller as the x values get larger.

5. (b) 20 **(c)** 4, 8 **(d)** Maximum: 22; minimum: 6.
7. (a) $y = 6 + 2x$ **(b)** 13; 66 **9. (a)** $y = 20 - 3x$
(b) 9.5; −70 **11. (a)** Yes **(b)** $10.7 - 2.3x$ **(c)** 1.5, −0.8, −3.1 **13.** Yes **15. (a)** No **(b)** Yes; y independent, x dependent **17. (a)** x independent, y dependent **(b)** 28
(c) 24 **19. (a)** x-intercepts: −2, 0, 2; y-intercept: 0 **(b)** Yes
(c) $y = x^3 - 4x$; 12 **21. (a)** No **(b)** Yes **23. (a)** Yes
(b) Yes
25. (a) $y = 2x - 7$
(b)

x	y
0	−7
1	−5
2	−3
3	−1
4	1

(c)



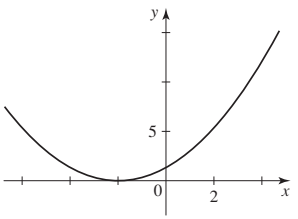
(d) $f(x) = 2x - 7$

27. **(a)** $y = \dfrac{(x+2)^2}{3}$

(b)

x	y
-5	3
-2	0
1	3
4	12
7	27

(c)



(d) $f(x) = \dfrac{(x+2)^2}{3}$

29. **(a)** Divide by 3, then add 5.

(b) $y = \dfrac{x}{3} + 5$; x independent, y dependent **(c)** $\frac{2}{3}$

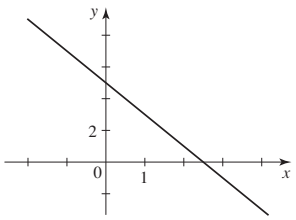
31. **(a)** Subtract three, square, then multiply by $\frac{1}{2}$.

(b) $y = \frac{1}{2}(x-3)^2$; x independent, y dependent **(c)** 2

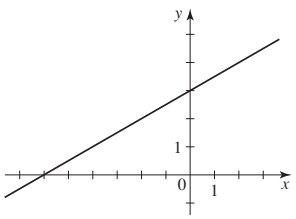
33. **(a)** 3 **(b)** 3 **(c)** 9 **(d)** $2a^2 - 4a + 3$ **35.** **(a)** 2

(b) 0 **(c)** 3 **(d)** 300

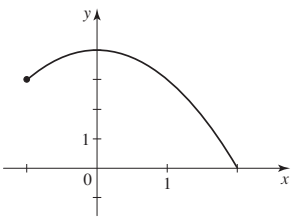
37. $\{(-2, 9), (-1, 7), (0, 5), (1, 3), (2, 1), (3, -1), (4, -3)\}$



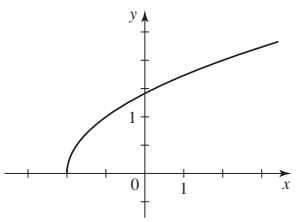
39.



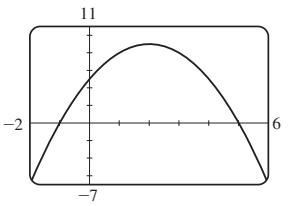
41.



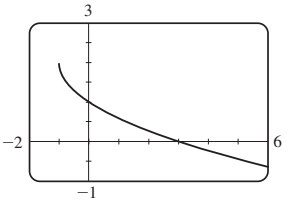
43.



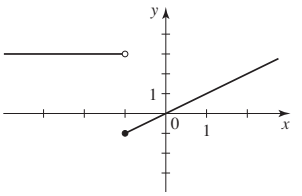
45.



47.

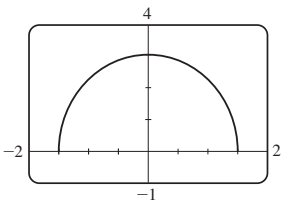


49.



51. **(a)** $-1, 1, 1$ **(b)** 3 **(c)** 0, 2 **(d)** Domain: $[-2, 2]$; range: $[-1, 2]$ **(e)** Increasing: $[-1, 1]$; decreasing: $[-2, -1]$ and $[1, 2]$

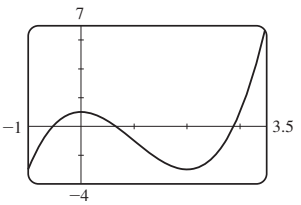
53. **(a)**



(b) Domain: $[-\frac{3}{2}, \frac{3}{2}]$; range: $[0, 3]$

(c) Increasing: $[-\frac{3}{2}, 0]$; decreasing: $[0, \frac{3}{2}]$

55. **(a)**



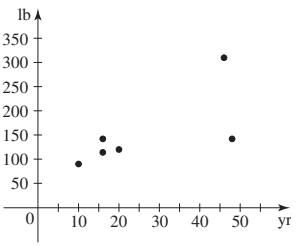
(b) Maximum 1 at $x = 0$; minimum -3 at $x = 2$

57. $H(d) = 24d$ **59.** $S = \sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3}$

61. $v = \sqrt{\dfrac{2k}{m}}$ **63.** **(a)** Average 26 years, median 18 years

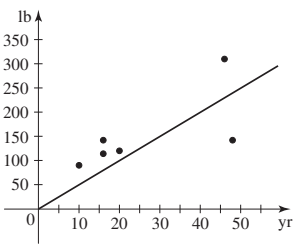
(b) Average 153 lb, median 131 lb; John brings the average up. The median is a better description. **(c)** {(10, 90), (16, 114), (16, 142), (20, 120), (46, 310), (48, 142)}; domain: {10, 16, 20, 46, 48}, range: {90, 114, 120, 142, 310}

(d) No



(e) Generally, as the age increases the weight increases.

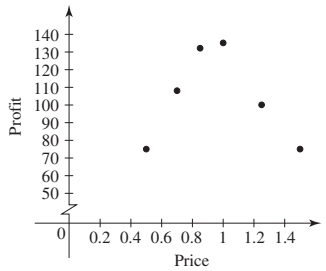
(f) All except Andrea have weights that are more than five times their age.



65. **(a)** 6 **(b)** 6 **(c)** 6.5, 6 **67.** **(a)** Average: 2.5; median: 2 **(b)** Average: 14 lb; median: 14.5 lb **(c)** Jack 4.5 lb, Helen 4.0 lb, Steve 5.5 lb, Kalpana 5.5 lb, Dieter 6.5 lb, Magda 8 lb **(d)** Magda **(e)** 5.6 lb

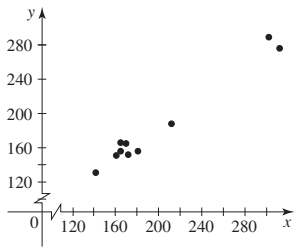
69. **(a)** 75, 108, 132, 135, 100, 75

(b)



(c) The profit increases until the price is \$1.00; then it decreases. The best price seems to be \$1.00.

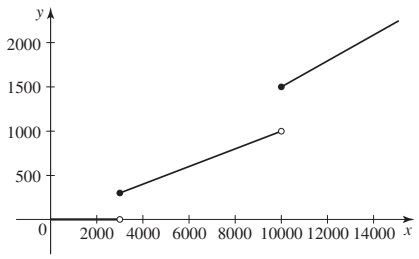
71. **(a)**



(b) No

73. **(a)** 0, 400, 3000

(b)



(c) \$8000, \$13,333.33 **(d)** \$1200

75. **(a)** Domain: [0, 365]; range: [25, 87] **(b)** Increasing: [0, 150] and [300, 365]; decreasing: [150, 300] **(c)** 75 ft **(d)** 87 ft on day 150 **77.** **(a)** $C(x) = 8x + \dfrac{7200}{x}$

(b) Width along road 30 ft; length 40 ft **(c)** 15 ft to 60 ft

79. **(a)** 5 in.

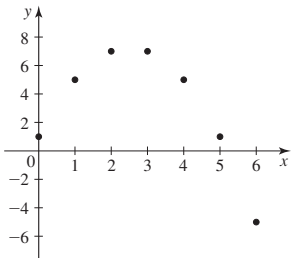
(b)

Length (in.)	Width (in.)	Height (in.)	Volume (in^3)
36	14	10	5040
40	12	10	4800
36	12	12	5184
36	16	8	4608
72	3	3	648
44	10	10	4400

Chapter 1 Test ▪ page 126

1. **(a)** 4.5, 5 **(b)** 0.4, 1.2 **2.** **(a)** {(0, 1), (1, 5), (2, 7), (3, 7), (4, 5), (5, 1), (6, −5)} **(b)** Domain: {0, 1, 2, 3, 4, 5, 6}; range: {−5, 1, 5, 7}

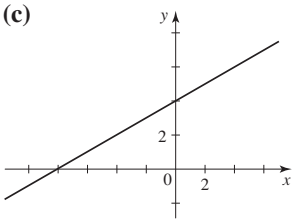
(c)



(d) Yes **(e)** Yes; y increases, then decreases as x increases. **(f)** 1, 4

3. **(a)** $y = 4 + 0.5x$ **(b)** 6.25, 9.00

(c)



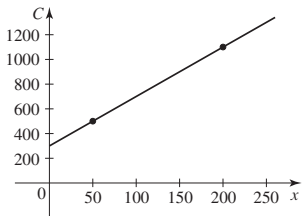
4. **(a)** No **(b)** Yes **5.** $f(x) = \dfrac{x^2 - 4}{5}$

6. **(a)** $f(x) = \begin{cases} 0.80x & \text{if } x < 20{,}000 \\ x - 5000 & \text{if } 20{,}000 \le x < 40{,}000 \\ x - 8000 & \text{if } x \ge 40{,}000 \end{cases}$

(b) 15,200, 17,000, 33,000, 32,000 **(c)** 39,000, 42,000

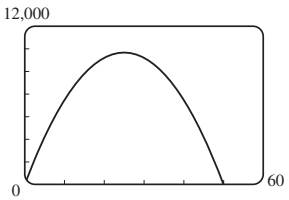
7. **(a)** $C = 300 + 4x$

(b)



(c) \$500, \$1100; the cost with 50 guests and with 200 guests **(d)** 75 guests

8. **(a)**



(b) 10,000 ft at 25 s **(c)** At $t = 50$ s **(d)** [25, 50]; falling to the ground

9. **(a)** 1056 cm^2 **(b)** $h = \dfrac{A - 2lw}{2l + 2w}$

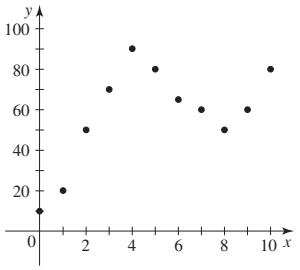
Chapter 2

2.1 Exercises ■ page 148

1. **(a)** $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{f(b) - f(a)}{b - a}$ **(b)** $\dfrac{10 - 3}{5 - 2} = \dfrac{7}{3}$ **2.** g, h, f **3.** **(a)** True **(b)** True **4.** f, g **5.** f: constant.; g: increasing; h: decreasing **7.** $\frac{2}{3}$ **9.** $-\frac{4}{5}$

11. **(a)** **(i)** 20 **(ii)** -6

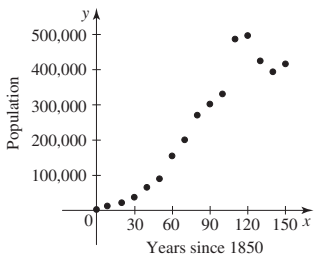
(b)



(c) $x = 1$ and $x = 2$; 30

13. 3 **15.** -2 **17.** 3 **19.** -2

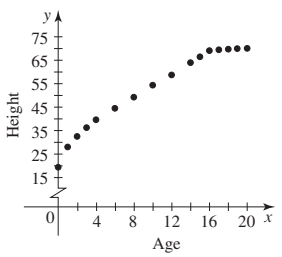
21. **(a)**



(b) **(i)** 3284 **(ii)** 1709 **(iii)** 8300 **(c)** 1950 to 1960

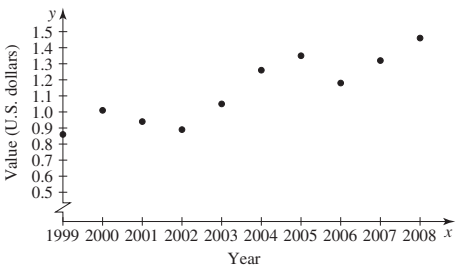
23. **(a)** 19.25 in., 39.63 in., 49.25 in. **(b)** 5.1 in./year, 2.4 in./year; the first four years

(c)



Between his 15th and 16th birthdays

25. **(a)**



(b) **(i)** 0.0667 per yr **(ii)** 0.0725 per yr **(iii)** 0.0367 per yr **(c)** From 2003 to 2004 **27.** **(a)** Skater B **(b)** A: 20 m/s, B: 10 m/s **(c)** A: 6.67 m/s, B: 20 m/s **29.** **(a)** 160 ft/s **(b)** 480 ft/s **31.** **(a)** -32 ft/s **(b)** -48 ft/s **(c)** -84 ft/s (Negative speeds because height is decreasing.)

2.2 Exercises ■ page 161

1. **(a)** linear **(b)** $f(x) = b + mx$ **(c)** line

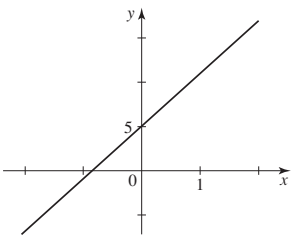
2. **(a)** -2; 7 **(b)** line; -2, 7

3. y; x; $\dfrac{5 - 1}{2 - 0} = 2$ **4.** -5; 5; 5 **5.** g

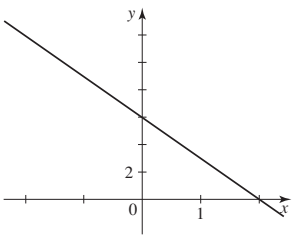
6. Positive: upward; negative: downward **7.** Yes; slope: 0, y-intercept: 2 **8.** **(a)** Increases the steepness of the line **(b)** Increases the y-intercept **9.** Yes **11.** No

13. Yes **15.** Yes

17.



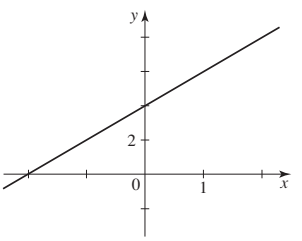
19.



21. **(a)** 2 **(b)** 2 **(c)** 2

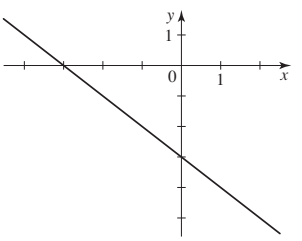
23. **(a)** -2 **(b)** -2 **(c)** -2

25. **(a)**



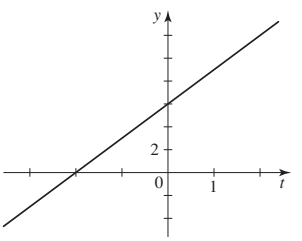
(b) 2 **(c)** 2

27. **(a)**



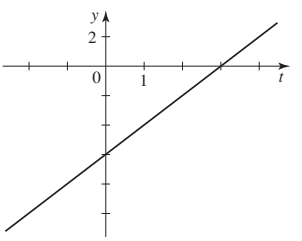
(b) -1 **(c)** -1

29. **(a)**



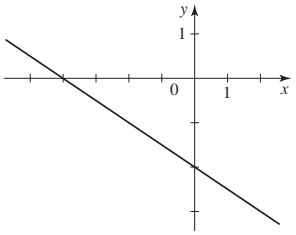
(b) 3 **(c)** 3

31. **(a)**

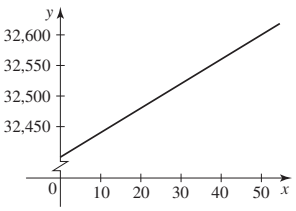


(b) 2 **(c)** 2

33. (a)



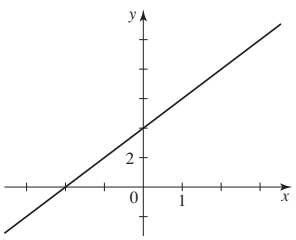
(b) -0.5 **(c)** -0.5
35. $f(x) = -3 + 7x$ **37.** $h(x) = 9 - 3x$ **39. (a)** $\frac{1}{2}$; 3
(b) $f(x) = 3 + \frac{1}{2}x$ **41. (a)** -2, 12 **(b)** $f(x) = 12 - 2x$
43. $\frac{1}{2}$ **45.** $\frac{1}{6}$ **47.** $-\frac{1}{2}$ **49. (a)** 1, 3 **(b)** $y = 3 + x$
51. (a) $-\frac{1}{2}$, 2 **(b)** $y = 2 - \frac{1}{2}x$ **53. (a)** Yes
(b) 32,400 thousand tons **(c)** 4 thousand tons per day
(d)



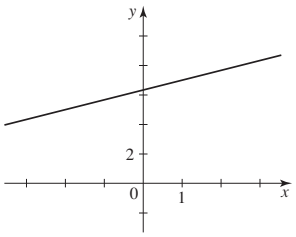
55. $V(t) = 2 + 0.5t$ **57.** Brianna: -0.05, or -5%; Meilin: -0.08, or -8% **59. (a)** Jari **(b)** Jari: 70 mi/h; Jade: 60 mi/h **(c)** Jari: $d(t) = \frac{7}{6}t$; Jade: $d(t) = 10 + t$

2.3 Exercises ▪ page 173

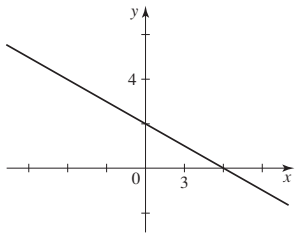
1. $y = b + mx$; $y = 4 + 2x$ **2.** $y - y_1 = m(x - x_1)$; $y - 2 = 3(x - 1)$ **3. (a)** zero; $y = 3$ **(b)** undefined; $x = 2$
4. point-slope; 2; $y - 5 = 2(x - 2)$ or $y - 7 = 2(x - 3)$
5. (a) horizontal; zero **(b)** vertical; undefined **(c)** $y = 2$
6. line; 5; 3 **7.** The temperature is increasing; decreasing; remaining the same. **8.** No **10.** $y - 2 = 3(x - 1)$ and $y - 5 = 3(x - 2)$; $y = 3x - 1$, yes **11.** $y = 2 + 5x$
13. $y = -3 - x$ **15.** $y = 5 + \frac{1}{2}x$ **17.** $y = 6 - 3x$
19. $y = -\frac{4}{3} + 3x$ **21.** $y = \frac{5}{2} - \frac{5}{4}x$ **23. (a)** $y - 4 = 2(x - 0)$
(b) $y = 4 + 2x$
(c)



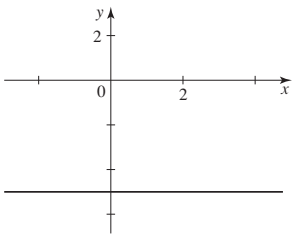
25. (a) $y - 7 = \frac{2}{3}(x - 1)$
(b) $y = \frac{19}{3} + \frac{2}{3}x$
(c)



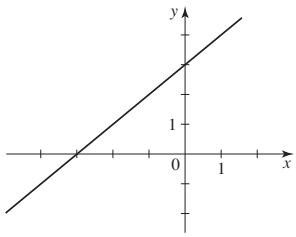
27. (a) $y - 4 = -\frac{1}{3}(x + 6)$
(b) $y = 2 - \frac{1}{3}x$
(c)



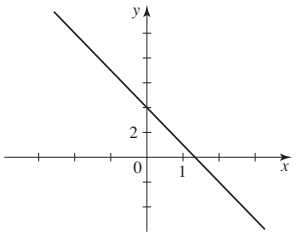
29. (a) $y + 5 = 0(x - 4)$
(b) $y = -5$
(c)



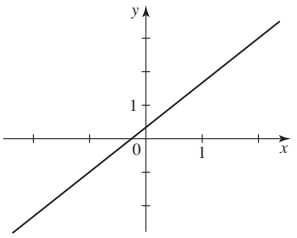
31. (a) $y - 1 = 1(x + 2)$ or $y - 7 = 1(x - 4)$
(b) $y = 3 + x$
(c)



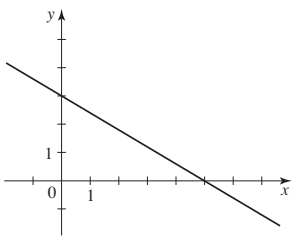
33. (a) $y - 7 = -3(x + 1)$ or $y + 2 = -3(x - 2)$
(b) $y = 4 - 3x$
(c)



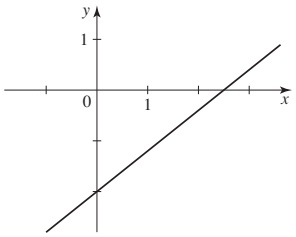
35. (a) $y - 3 = \frac{4}{3}(x - 2)$ or $y - 7 = \frac{4}{3}(x - 5)$
(b) $y = \frac{1}{3} + \frac{4}{3}x$
(c)



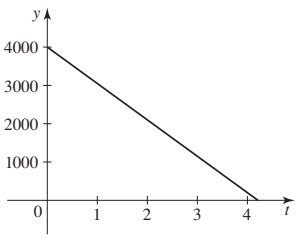
37. $y = 4 - x$ **39.** $y = -3 + \frac{3}{2}x$ **41.** $y = -7$ **43.** $y = 10$
45. $x = 8$ **47.** $x = 8$ **49.** $m = -\frac{3}{4}$, $b = 6$ **51.** $m = \frac{6}{5}$, $b = 3$ **53. (a)** $-\frac{3}{5}$ **(b)** 5, 3
(c)



55. **(a)** $\frac{4}{5}$ **(b)** $\frac{5}{2}, -2$
(c)

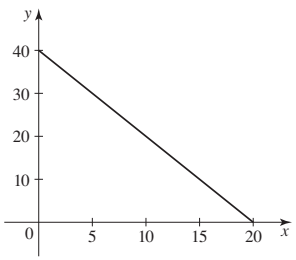


57. **(a)** $y = 45 - 4x$ **(b)** 4 mi/min or 240 mi/h
59. **(a)** $V(t) = 4{,}000 - 950t$
(b)



(c) Slope: rate of depreciation; y-intercept: initial cost of computer
(d) \$1150

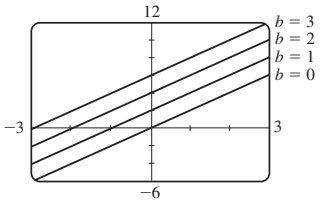
61. $V(t) = 4 + 0.5t$
63. **(a)** $y = 20 - 2(x - 10) = 40 - 2x$



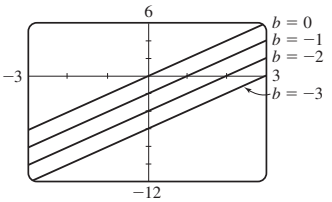
(b) 12; 28 **(c)** -2; number of sales lost for each dollar increase in the price **(d)** 40; the number of "sales" if the feeders were free **(e)** 20; the price at which no feeders would be sold

2.4 Exercises ▪ page 184

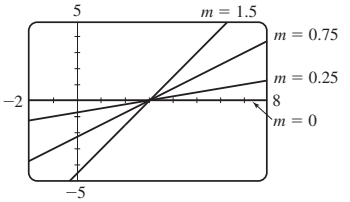
1. slope; parallel **2.** y-intercept; neither **3.** **(a)** 3 **(b)** 3 **(c)** $-\frac{1}{3}$ **4.** **(a)** directly proportional; 3 **(b)** $y = 3x$
5. **(a)** l_3 and l_4 have the greatest slope; l_3 and l_4 are parallel; none are perpendicular. **(b)** l_3 has the greatest slope; l_1 and l_2 are parallel; l_3 is perpendicular to l_1 and l_2.
7. **(a)** True **(b)** False **(c)** True **(d)** False
8. w is directly proportional to x.
9. **(a)** They have the same slope.



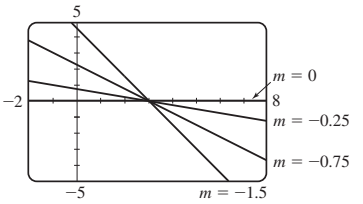
(b) They have the same slope.



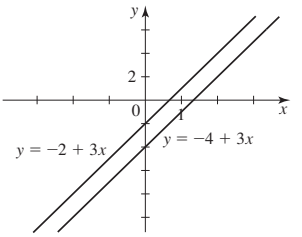
11. **(a)** They have the same x-intercept.



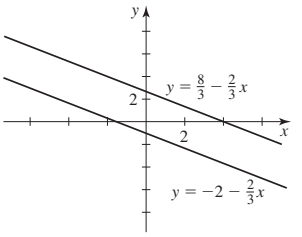
(b) They have the same x-intercept.



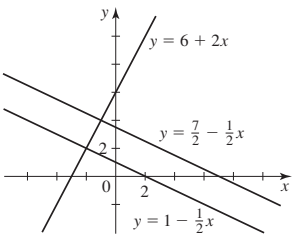
13. **(a)** $y = -2 + 3x$
(b)



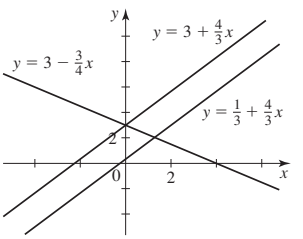
15. **(a)** $y = \frac{8}{3} - \frac{2}{3}x$
(b)



17. Yes
19. **(a)** $y = \frac{7}{2} - \frac{1}{2}x$
(b) $y = 6 + 2x$
(c)

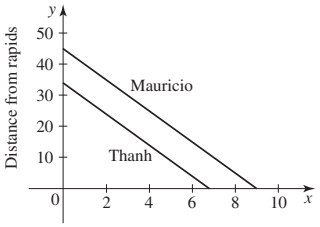


21. **(a)** $y = 3 + \frac{4}{3}x$
(b) $y = 3 - \frac{3}{4}x$
(c)



23. **(a)** $y = -6 + 3x$ **(b)** $y = -9 + 3x$ **(c)** $y = -6 - \frac{1}{3}x$
25. **(a)** $y = 4 - 2x$ **(b)** $y = 6 - 2x$ **(c)** $y = 4 + \frac{1}{2}x$
27. **(a)** l_3 **(b)** l_2 **(c)** l_1 **29.** $y = 6 + 4x$ **31.** $y = 2$
33. $x = 3$ **35.** $y = 9 - \frac{3}{2}x$ **37.** $y = \frac{19}{5} - \frac{4}{5}x$ **39.** $y = 6 + x$
41. $T = kx$ **43.** $y = 7x$ **45.** 12 **47.** $y = 5280x$
49. $y = 5376x$

51. **(a)** Mauricio: $y = 45 - 5x$; Thanh: $y = 34 - 5x$
(b)

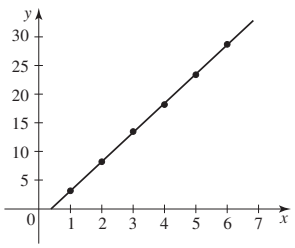


(c) No

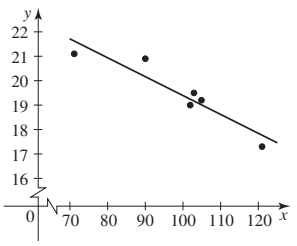
53. **(a)** $y = 0.000003x$ **(b)** 87,000 fl oz or 16.2 barrels
55. $i = 0.06P$; the interest rate **57.** **(a)** $F = kx$
(b) $k = 8$ **(c)** 32 N

2.5 Exercises ▪ page 195

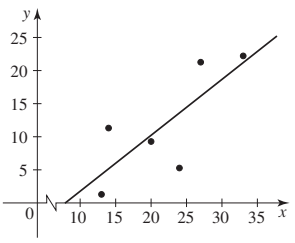
1. regression **2.** **(a)** extrapolation **(b)** interpolation
3. Yes **4.** **(a)** Regression line: $y = 4x + 6$
(b) Regression line: $y = 3.64x + 5.25$
5. **(a)**



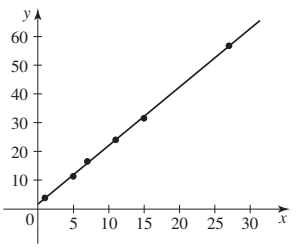
(b) $y = -1.98 + 5.0943x$
7. **(a)**



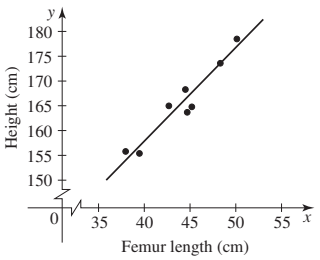
(b) $y = 27.138 - 0.07743x$
9. **(a)**



(b) $y = -6.6245 + 0.8434x$
11. **(a)**

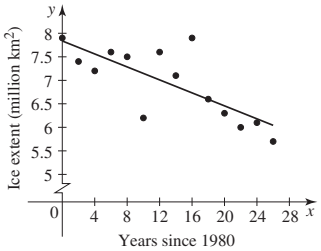


(b) $y = 1.719 + 2.0316x$
13. **(a)**

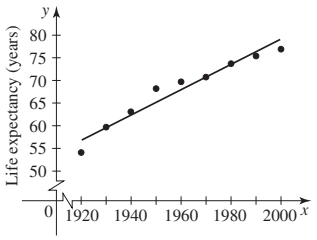


(b) $y = 82.65 + 1.8807x$ **(c)** 191.7 cm

15. **(a)**

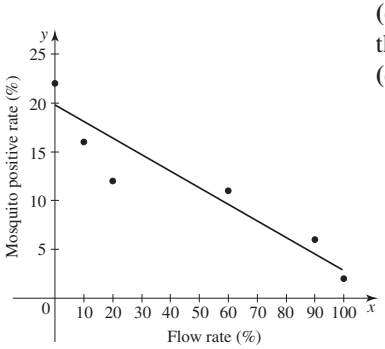


(b) $y = 7.831 - 0.0689x$ **(c)** 5.8 million km^2
17. **(a)** $y = -462.9 + 0.2708x$
(b)



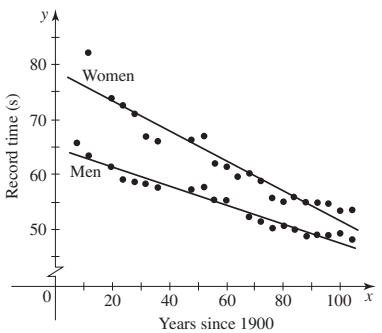
(c) The increase in life expectancy per year
(d) 80.3 yr

19. **(a)**

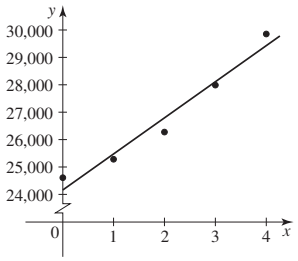


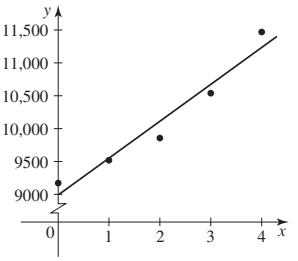
(b) $y = 19.89 - 0.168x$
(c) The mosquito rate when there is no flow
(d) 8.13%

21. **(a)** Men's: $y = 64.717 - 0.173x$; Women's: $y = 78.67 - 0.269x$
(b)



23. **(a)** $y = 241.62 + 1316.7x$ **(b)** $y = 8988.2 + 561.4x$
(c)





(d) 11,795 women in 2007

Algebra Checkpoint 2.6 ▪ page 206

1. **(a)** Yes; no **(b)** No; yes **(c)** No; yes **2.** **(a)** 12 **(b)** 10 **(c)** 2 **(d)** -40 **(e)** $\frac{95}{32}$ **(f)** $-\frac{13}{7}$ **3.** **(a)** 3 **(b)** -20 **(c)** 3 **(d)** No solution **(e)** 4

2.6 Exercises ▪ page 207

1. **(a)** two **(b)** one **2.** **(a)** $6 - 2 = 3x + 1$ **(b)** 1
3. $4000 = 500 + 10x$ **4.** $1000 - x$; $0.04x$; $0.05(1000 - x)$; $y = 0.04x + 0.05(1000 - x)$ **5.** True **6.** False **7.** -4
9. 10 **11.** 10 **13.** 250 **15.** 30 **17.** 4 **19.** 2
21. 17.7 min **23.** **(a)** About 24,650 days
(b) 24,650 days or 68 years **25.** 168 chirps/min
27. **(a)** $I = 0.04x + 0.045(12{,}000 - x)$ **(b)** \$2800 at 4% and \$9200 at 4.5% **29.** **(a)** $G = \dfrac{81}{90 + x}$ **(b)** 18 g
31. **(a)** $P = 2w + 2(w + 5)$ **(b)** $w = 4$ cm, $l = 9$ cm
33. 6.4 ft from the fulcrum

2.7 Exercises ▪ page 215

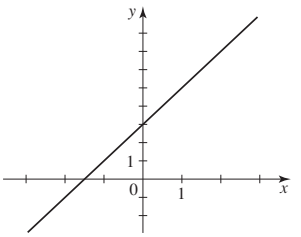
1. **(a)** $m_1x + b_1 = m_2x + b_2$ **(b)** $2x + 7 = 3x - 10$; 17, (17, 41) **2.** **(a)** equilibrium **(b)** $m_1p + b_1 = m_2p + b_2$
4. $y = 2t$; $y = 5 - 2t$ **5.** **(a)** (2.5, 0.5) approximately
(b) $y = -2 + x$; $y = 2 - \frac{2}{3}x$ **(c)** (2.4, 0.4) **7.** **(a)** 2 **(b)** 2
9. **(a)** 2 **(b)** 2 **11.** 2 **13.** 3 **15.** 2 **17.** $(-3, -1)$
19. $(1, 3)$ **21.** $(10, -9)$ **23.** $f(x) = 5 + \frac{2}{3}x$, $g(x) = 2 + 5x$; $\left(\frac{9}{13}, \frac{71}{13}\right)$ **25.** Price: \$21.81; amount: 13.82
27. **(a)** $f(x) = 10 + 0.20x$ **(b)** $g(x) = 100$
(c) 450 minutes
29. **(a)** $C(x) = 5800 + 265x$ **(b)** $S(x) = 575x$
(c) **(d)** 19 months

x	$C(x)$	$S(x)$
6	7390	3450
12	8980	6900
18	10,570	10,350
24	12,160	13,800
30	13,750	17,250
36	15,340	20,700

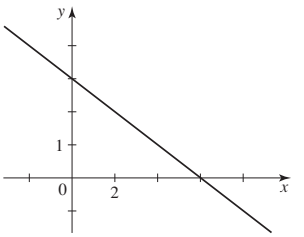
31. **(a)** Plan 1: $C(x) = 105 + 0.15x$; plan 2: $C(x) = 270$;
(b) Plan 1; plan 2; 1100 miles **33.** **(a)** $y = 72x$
(b) $y = 65\left(x + \frac{1}{6}\right)$ **(c)** 1.55 hours; 111.6 miles
35. **(a)** \$2.75 per bushel **(b)** \$18.21 per bushel
(c) \$5.88 per bushel; 13.1 billion bushels

Chapter 2 Review Exercises ▪ page 222

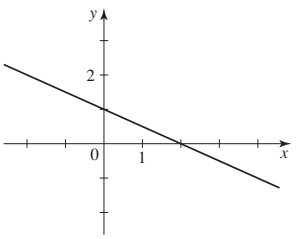
1. -1 **3.** $\frac{4}{3}$ **5.** 7 **7.** No
9. **(a)**



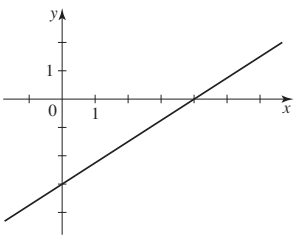
(b) 2 **(c)** 2
11. **(a)**



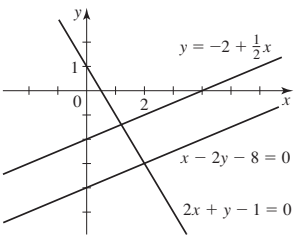
(b) $-\frac{1}{2}$ **(c)** $-\frac{1}{2}$
13. $f(x) = 10 - 5x$ **15.** $f(x) = 3 + 2x$ **17.** $f(x) = 4 - \frac{1}{2}x$
19. $y = 5x$ **21.** $y = 9 - 6x$ **23.** $y = 8 - 2x$
25. $y = 4 - \frac{4}{3}x$ **27.** $y = 26$
29. **(a)** Slope: $-\frac{1}{2}$; x-int: 2, y-int: 1
(b)



(c) $y = 1 - \frac{1}{2}x$
31. **(a)** Slope: $\frac{3}{4}$; x-int: 4, y-int: -3
(b)

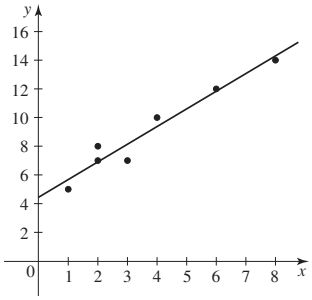


(c) $y = -3 + \frac{3}{4}x$
33. **(a)** $x - 2y - 8 = 0$
(b) $2x + y - 1 = 0$
(c)



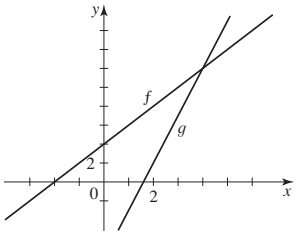
35. $y = \frac{1}{2}x$ **37.** $y = -14 + 4x$ **39.** 35 **41.** $y = 25.4x$

43. (a)



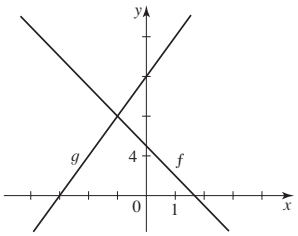
(b) $y = 4.435 + 1.229x$
(c) Yes

45. (a)



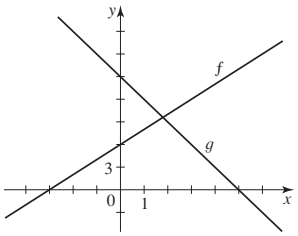
(b), (c) (4, 12)

47. (a)



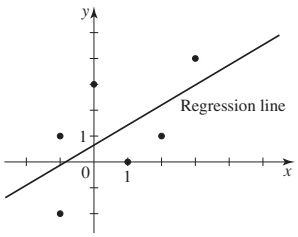
(b), (c) $(-1, 8)$

49. (a)



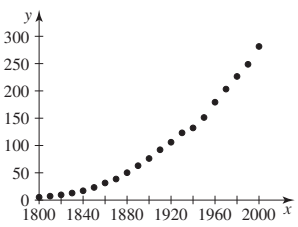
(b), (c) $\left(\frac{9}{5}, \frac{48}{5}\right)$

51. (a)



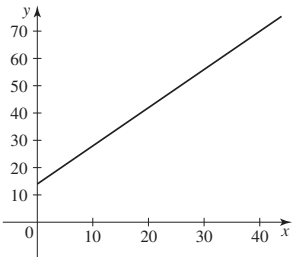
(b) $P_1, P_2; x = -1$ **(c)** $P_1, P_4; y = 1$ **(d)** $y = \frac{1}{2} - \frac{1}{2}x$
(e) $P_1, P_2; x = -1$ **(f)** $\frac{1}{3}$ **(g)** $y = 3 + 5x, y = -5 + 3x$; $(-4, -17)$ **(h)** $y = 0.65 + 0.775x$

53. (a)



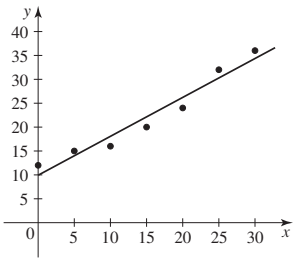
(b) Increasing **(c)** 1.502 million per year **(d)** 0.709; 20.52 million per year **(e)** 1.38 million per year **(f)** 1990–2000; 3.27 million per year

55. (a) 1.4 inches per day **(b)** $y = 14 + 1.4x$
(c) The slope is 1.4.



(d) 70 in. **(e)** 14 in. **(f)** 10 days **(g)** May 30

57. (a) $y = 9.929 + 0.8143x$
(b)

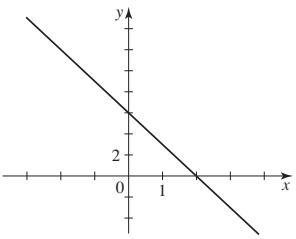


(c) 38.4 ounces **(d)** 2036

59. (a) $f(x) = 7x$; $g(x) = 24\left(x - \frac{1}{4}\right)$ **(b)** At 8:21; about 2.5 miles

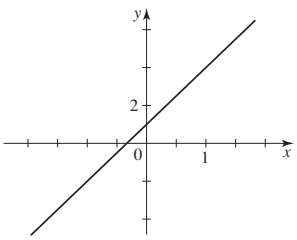
Chapter 2 Test ▪ page 228

1. (a) 2 **(b)** 5 **2. (a)** g is linear; f is not linear because it has a term involving x^2.
(b)

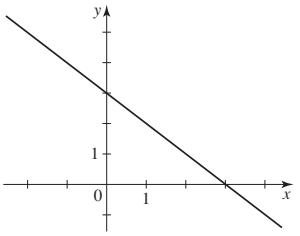


(c) −3

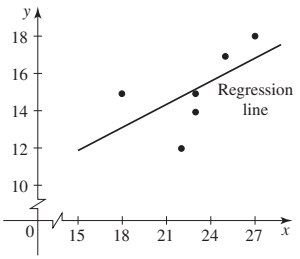
3. $y = -3 + \frac{1}{2}x$ **4. (a)** $y = -3 + 4x$ **(b)** $y = \frac{9}{5} - \frac{2}{5}x$
(c) $y = \frac{5}{2}x$ **5.** Vertical: $x = 6$; horizontal; $y = -8$
6. (a) $3x - y + 1 = 0$, $y = 1 + 3x$



(b) $x + y - 3 = 0, \quad y = 3 - x$



7. 10.5 in.
8. **(a), (b)** $y = 5.667 + 0.413x$



(c) 18 ounces
9. **(a)** (75, 200) **(b)** \$75

Chapter 3

Algebra Checkpoint 3.1 ▪ page 255

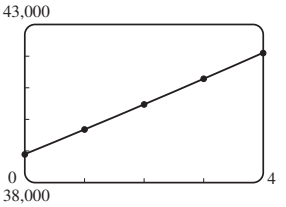
1. **(a)** 5^{10} **(b)** $\frac{1}{4^2}$ **(c)** 10^5 **(d)** 5^{18} **2.** **(a)** x^7 **(b)** $25x^6$ **(c)** z^2 **(d)** y^8 **3.** **(a)** $3^{1/2}$ **(b)** $3^{-1/2}$ **(c)** $5^{1/3}$ **(d)** $5^{4/3}$ **4.** **(a)** $\sqrt{5}$ **(b)** $\frac{1}{\sqrt{7}}$ **(c)** $\sqrt[3]{6}$ **(d)** $\sqrt[3]{6^2}$ **5.** **(a)** 2 **(b)** 6 **(c)** 4 **(d)** $\frac{5}{8}$ **6.** **(a)** x **(b)** $x^{1/2}$ **(c)** $9x^2$ **(d)** $\frac{1}{x^2y^3}$

3.1 Exercises ▪ page 256

1. **(a)** 200 bacteria **(b)** 1.8 **(c)** 0.8 **(d)** 80% **2.** **(a)** 100 g **(b)** 0.4 **(c)** -0.6 **(d)** -60% **3.** **(a)** growth **(b)** decay **4.** **(a)** decay **(b)** growth **6.** $3^{-x} = \left(\frac{1}{3}\right)^x$, so they both have a decay factor of $\frac{1}{3}$. **7.** (b), (c), and (d) **9.** **(a)** Growth **(b)** 1.2 **(c)** 0.2 **11.** **(a)** Decay **(b)** $\frac{1}{3}$ **(c)** $-\frac{2}{3}$ **13.** **(a)** Decay **(b)** 0.25 **(c)** -0.75 **15.** **(a)** Growth **(b)** 2.8 **(c)** 1.8 **17.** III **19.** II **21.** **(a)** 1.93 **(b)** 1.33 **(c)** 0.67 **(d)** 0.20 **23.** 100; 2 **25.** $P = 350 \cdot (4)^x$ **27.** $P = 350 \cdot \left(\frac{1}{2}\right)^x$ **29.** $P = 350 \cdot (0.65)^x$ **31.** $P = 350 \cdot (4)^x$ **33.** $P = 350 \cdot (1.7)^x$ **35.** **(a)** $f(t) = 10(1.26)^t$ **(b)** 31 bacteria **37.** **(a)**

Year	Years since 2003	Salary (\$)
2003	0	38,900.00
2004	1	39,678.00
2005	2	40,471.56
2006	3	41,280.99
2007	4	42,106.61

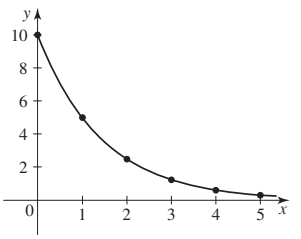
(b) $S = 38{,}900(1.02)^t$ **(c)** \$44,683.87
(d)



39. **(a)** $E(x) = 2813(1.183)^x$ **(b)** \$3940 in 1996, \$6518 in 2005
41. **(a)** $A(t) = 100(0.75)^t$ **(b)** 17.8 mg
43. **(a)** $E(x) = 232{,}300(0.911)^x$ **(b)** \$17,563
45. **(a)** $m(x) = 10(0.5)^x$ **(b)** 0.0195 g
(c)

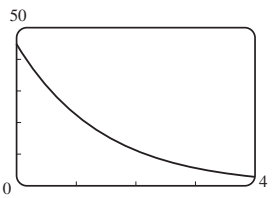
x	$m(x)$
0	10.0
1	5.0
2	2.5
3	1.25
4	0.625
5	0.3125

(d)



$m(x)$ decays exponentially

47. **(a)** 100,000 grey squirrels **(b)** $P(x) = 100{,}000(2.85)^x$ **(c)** 812,250 grey squirrels **49.** **(a)** $A(x) = 45(0.5)^x$
(b)



Algebra Checkpoint 3.2 ▪ page 267

1. 2 **2.** $\frac{1}{3}$ **3.** 2 **4.** 3 **5.** 25 **6.** 8 **7.** 64 **8.** $\frac{25}{9}$ **9.** 1.50 **10.** 1.65 **11.** 21,450.46 **12.** 16,732.12

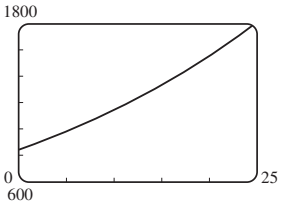
3.2 Exercises ▪ page 267

1. **(a)** a^3 **(b)** $b^{1/5}$ **(c)** $\left(\frac{1}{2}\right)^{1/h}$ **2.** principal, annual interest rate, number of times compounded per year, number of years, amount after t years **5.** **(a)** **(i)** 4 **(ii)** 1.41 **(b)** **(i)** 10 **(ii)** 1.78 **(c)** **(i)** 0.75 **(ii)** 0.93 **(d)** **(i)** 0.6 **(ii)** 0.88 **7.** 0.13 **9.** 16% **11.** 500; 0.76 **13.** **(a)** $P(x) = 2000 \cdot (2)^x$ **(b)** $P(t) = 2000 \cdot (8)^t$ **15.** **(a)** $P(x) = 2000 \cdot (2)^x$ **(b)** $P(t) = 2000 \cdot (1.59)^t$ **17.** $P(t) = 1000 \cdot (2)^t$ **19.** $P(t) = 1000 \cdot (1.06)^t$ **21.** $P(t) = 1000 \cdot (0.25)^t$ **23.** $P(t) = 1000 \cdot (0.77)^t$ **25.** **(a)** Type A: 1.69; Type B: 1.51 **(b)** Type A **27.** **(a)** Type A: 1.25; Type B: 1.22 **(b)** Type A

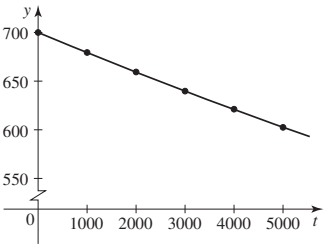
29.

Time (yr)	Amount ($)
0	5000.00
1	5203.71
2	5415.71
3	5636.36
4	5865.99
5	6104.98

31. **(a)** \$515.17 **(b)** \$530.80 **(c)** \$580.59
33. **(a)** \$4263.83 **(b)** \$4402.19 **(c)** \$4545.05
35. **(a)** \$3649.96 **(b)** \$3656.98 **(c)** \$3662.99
(d) \$3664.17 **37.** 2.531% **39.** 3.29% **41.** Option (i)
43. **(a)** Location A: 1.685; Location B: 1.119 **(b)** Location A
45. **(a)** 10-year: 1.357, annual: 1.031
(b) $P(t) = 846 \cdot (1.031)^t$ **(c)** 1558 million
(d)

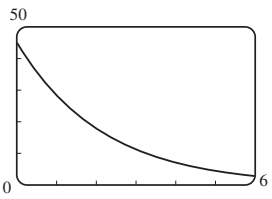


47. **(a)** $m(t) = 700 \cdot (0.99997)^t$; 0.99997 **(b)** 689.6 g
(c)



$m(t)$ decays exponentially.

49. **(a)** 665,000 **(b)** 1.052; $n(t) = 665000 \cdot (1.052)^t$
(c) 2,362,000 **51.** **(a)** $A(t) = 45(0.63)^t$
(b)



3.3 Exercises ▪ page 278

1. **(a)** varies **(b)** is constant **(c)** 400% **2.** **(a)** is constant **(b)** 200%, 2, 3 **(c)** $f(x) = 6 \cdot 3^x$ **3.** 20, 500 **4.** 25, 200
5. **(a)** True **(b)** True **7.** **(a)** Exponential **(b)** Linear **(c)** Exponential **(d)** Exponential

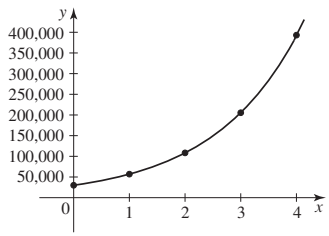
9.

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	5000	—	—
1	6500	1500	30%
2	8450	1950	30%
3	10,985	2535	30%
4	14,281	3296	30%

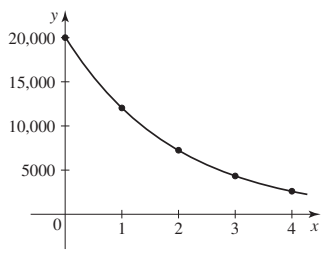
11.

Hour t	Population $f(t)$	Average rate of change	Percentage rate of change
0	30,000	—	—
1	24,000	−6000	−20%
2	19,200	−4800	−20%
3	15,360	−3840	−20%
4	12,288	−3072	−20%

13. **(a)** 27,000; 51,300; 97,470; 185,193; 90%, 90%, 90%, 90%
(b) Yes; $y = 30{,}000(1.9)^x$



15. **(a)** −8000, −4800, −2880, −1728; −40%, −40%, −40%, −40% **(b)** Yes; $y = 20{,}000(0.6)^x$

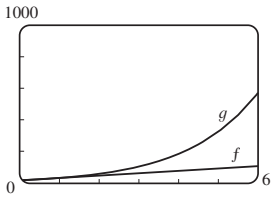


17. **(a)** −560, −560, −560, −560; −18.7%, −23.0%, −29.8%, −42.4% **(b)** No
19. **(a)**

t	$f(t)$
0	200
1	350
2	500
3	650
4	800

t	$g(t)$
0	200
1	350
2	613
3	1072
4	1876

(b) $f(t) = 200 + 150t$; $g(t) = 200(1.75)^t$
(c)



g increases much faster than f does.

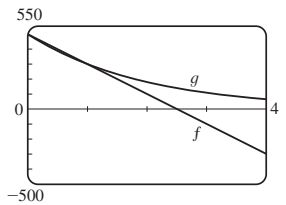
21. (a)

t	$f(t)$
0	500
1	300
2	100
3	−100
4	−300

t	$g(t)$
0	500
1	300
2	180.0
3	108.0
4	64.8

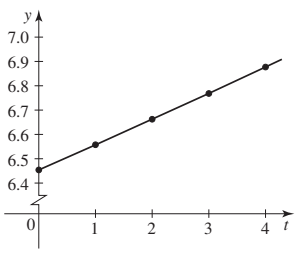
(b) $f(t) = 500 - 200t$; $g(t) = 500(0.6)^t$

(c)



g decreases to 0, while f continues to decrease indefinitely.

23. (a) Population A **(b)** Population B; 3000 **(c)** 100
25. (a) $f(t) = 6.454(1.0156)^t$; 1.56%
(b)



(c) 0.103 billion, 0.108 billion; no **(d)** 1.56%, 1.56%; yes

27. (a) 20,000 **(b)** $n(t) = 20000(1.12)^t$; 12% **(c)** 12%
29. (a)

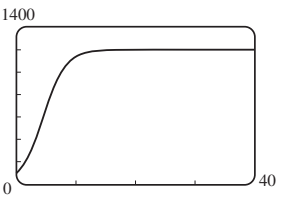
Offer A

Month	Salary ($)	Average rate of change
0	10,000	—
1	11,000	1000
2	12,000	1000
3	13,000	1000
4	14,000	1000
5	15,000	1000

Offer B

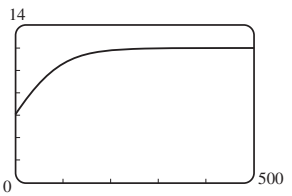
Month	Salary ($)	Percentage change
0	0.02	—
1	0.04	100%
2	0.08	100%
3	0.16	100%
4	0.32	100%
5	0.64	100%

(b) $f_A = 10{,}000 + 1000t$; linear **(c)** $f_B = 0.02(2)^t$; exponential
(d) Offer A: \$45,000, Offer B: \$687,194,767.40; Offer B
31. (a) 100 **(b)** The population stabilizes at about 1200.
(c)



From the graph it is apparent that the population of foxes increases logistically, approaching a carrying capacity of 1200.

(d) 1200, yes
33. (a) 11.79 billion; 11.97 billion
(b)

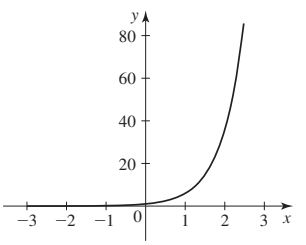


(c) The world population levels off as it reaches 12 billion.
(d) 12 billion, yes

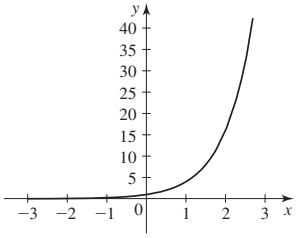
3.4 Exercises ▪ page 291

1. (a) 5 **(b)** 25, 1, $\frac{1}{25}$ **2. (a)** II **(b)** I **3.** False **4.** True
5.

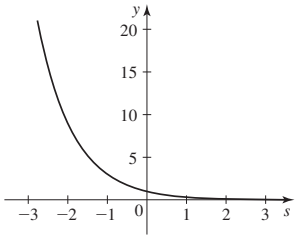
x	$f(x)$
−3	$\frac{1}{216}$
−2	$\frac{1}{36}$
−1	$\frac{1}{6}$
0	1
1	6
2	36
3	216



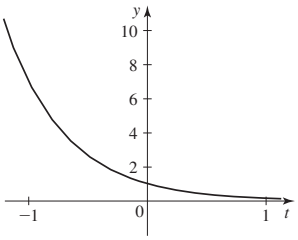
7.



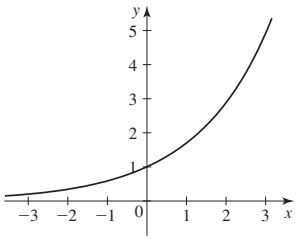
9.



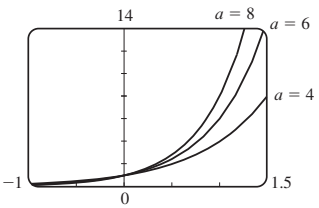
11.



13.

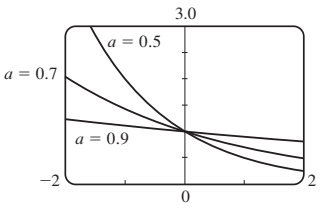


15.



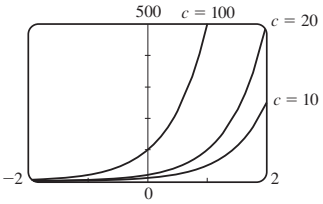
The larger the value of a, the steeper the graph of f is when $x > 0$.

17.



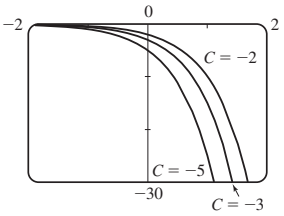
The smaller the value of a, the steeper the graph of f is for $x < 0$.

19.



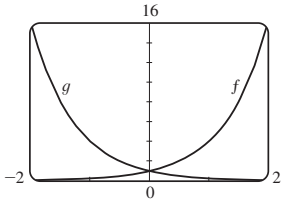
The larger the value of C, the higher the graph of f.

21.



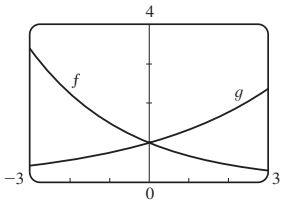
The smaller the value of C, the lower the graph of f.

23. (a)



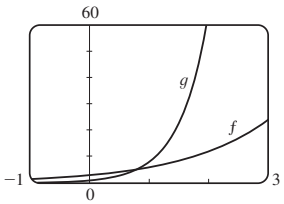
(b) Yes, (0, 1)

25. (a)



(b) Yes, (0, 1)

27. (a)

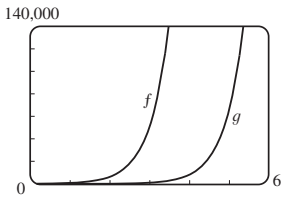


(b) The graph of g ultimately increases much more quickly than that of f; (0.79, 5.17).

29. $f(x) = 2^{-x}$ **31.** $f(x) = 8\left(\frac{3}{2}\right)^x$

33. (a) Jim's: $f(t) = 100 \cdot 8^t$; Harold's: $g(t) = 2 \cdot 8^t$

(b)



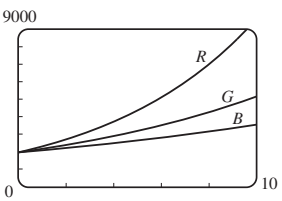
The bacteria colonies have different initial populations but grow at the same rate.

(c) Jim: 26,214,400; Harold: 524,300

35. (a)

Year	Growth fund value ($)	Bond fund value ($)	Real Estate fund value ($)
2005	2000	2000	2000
2006	2200.00	2120.00	2340.00
2007	2420.00	2247.20	2737.80
2008	2662.00	2382.03	3203.23

(b) G.F.: $G(t) = 2000(1.1)^t$; B.F.: $B(t) = 2000(1.06)^t$; R.E.F.: $R(t) = 2000(1.17)^t$

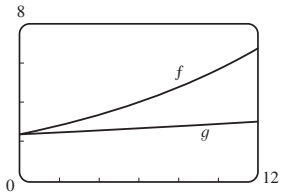


(c) Higher yearly returns result in a steeper graph.

(d) G.F.: $3543.12; B.F.: $2837.04; R.E.F.: $5130.33

37. (a) Annual rate 9%: $f(t) = 2.4(1.09)^t$; annual rate 2%: $g(t) = 2.4(1.02)^t$

(b)



The smaller annual growth rate results in a slower increase in healthcare expenditures.

(c) 9% rate: \$6.75 trillion, 2% rate: \$3.04 trillion

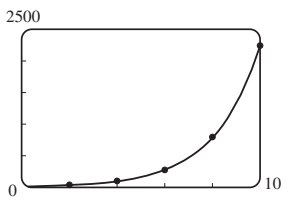
3.5 Exercises ▪ page 299

1.

x	y	Average rate of change	Percentage rate of change
0	10	—	—
1	30	20	200%
2	50	20	67%
3	70	20	40%
4	90	20	29%
5	110	20	22%

x	y	Average rate of change	Percentage rate of change
0	10	—	—
1	30	20	200%
2	90	60	200%
3	270	180	200%
4	810	540	200%
5	2430	1620	200%

2. a linear **3.** an exponential **4.** $y = 10 + 20x$; $y = 10 \cdot 3^x$
6. If the growth factor is constant, then an exponential model is appropriate. **7. (a), (c)**



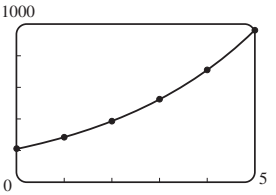
(b) $y = 12.03(1.68622)^x$

9. (a)

x	$f(x)$	Average rate of change	Percentage rate of change
0	210	—	—
1	281	71	34%
2	379	98	38%
3	512	133	36%
4	689	177	33%
5	932	243	37%

(b) Exponential; $f(x) = 209.16(1.34765)^x$

(c)

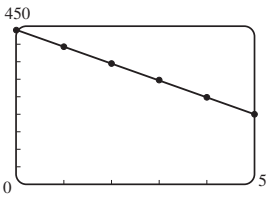


11. (a)

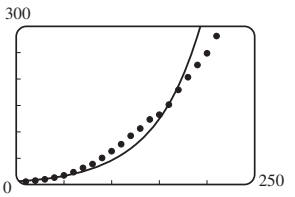
x	$f(x)$	Average rate of change	Percentage rate of change
0	441	—	—
1	392	−49	−11%
2	345	−47	−12%
3	296	−49	−14%
4	248	−48	−16%
5	200	−48	−19%

(b) Linear; $f(x) = 440.7 - 48.171x$

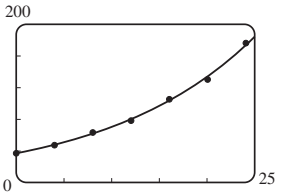
(c)



13. (a), (c)



(b) $f(x) = 6.05(1.02041)^x$
(d) 515.9 million
15. (a), (c)



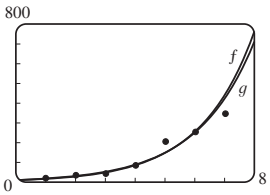
(b) $f(x) = 36.78(1.0666284)^x$
(c) The doubling time is 10.75 h.

17. (a)

Year	x	Sales in year $x+1$ / Sales in year x	Growth factor
2000	0	—	—
2001	1	20.3/9.5	2.14
2002	2	35.0/20.3	1.72
2003	3	43.4/35.0	1.24
2004	4	85.0/43.4	1.96
2005	5	205.8/85.0	2.42
2006	6	254.5/205.8	1.24
2007	7	347.1/254.5	1.36

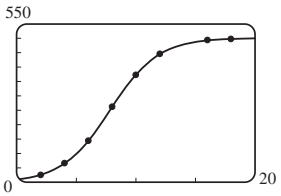
(b) 1.73 **(c)** $f(x) = 9.5(1.73)^x$ **(d)** $g(x) = 10.94(1.685)^x$

(e)



19. (a) $P(t) = \dfrac{500.9}{1 + 49.1e^{-0.498114t}}$

(b)



(c) The population increases rapidly at first, then levels off at about 500 flies.

Chapter 3 Review Exercises ▪ page 306

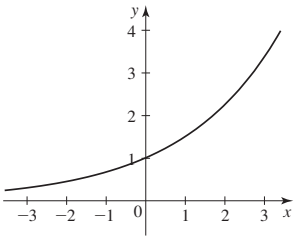
1. (a) Growth **(b)** 1.25 **(c)** 25% **(d)** 1000
3. (a) Decay **(b)** $\frac{3}{5}$ **(c)** -40% **(d)** 8
5. (a) $P(x) = 200 \cdot 2^x$ **(b)** $P(t) = 200(1.26)^t$ **(c)** 3200
7. (a) $P(x) = 1320(0.70)^x$ **(b)** $P(t) = 1320(0.888)^t$
(c) 317 **9. (a)** \$24,878.40 **(b)** \$26,732.82
(c) \$28,213.80 **11. (a)** $m(t) = 300\left(\frac{1}{2}\right)^{t/1600}$ **(b)** 296.77 kg
(c) $m(t) = 300(0.999567)^t$ **(d)** -0.000433

13.

t	$f(t)$	Average rate of change	Percentage rate of change
0	5000	—	—
1	6000	1000	20%
2	7200	1200	20%
3	8640	1440	20%

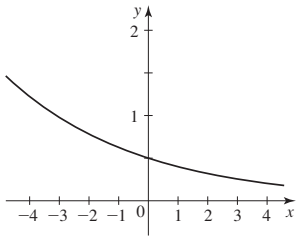
$f(t) = 5000(1.2)^t$

15.



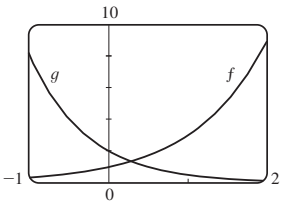
Increasing

17.



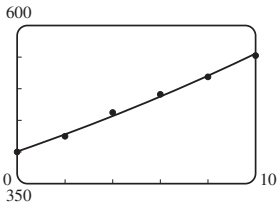
Decreasing

19. (0.28, 1.36)



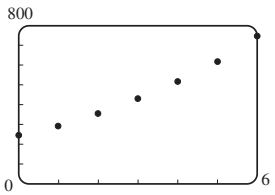
21. $f(x) = \left(\frac{2}{3}\right)^x$

23. (a), (c)

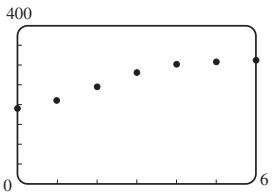


(b) $f(t) = 400.77(1.0332)^t$

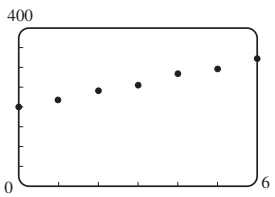
25. (a) Avalon Acres:



Coral Cay:

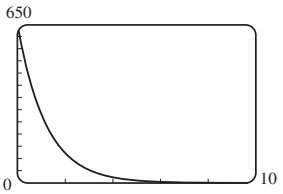


Buccaneer Beach:

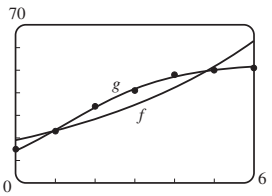


(b) Avalon Acres has exponential growth, Buccaneer Beach has linear growth, and Coral Cay has logistic growth. **(c)** Avalon: $f(x) = 248.62(1.202)^x$ Buccaneer: $f(x) = 198.9 + 20.18x$
Coral: $f(x) = \dfrac{335.5}{1 + 0.831 \cdot e^{-0.446x}}$ **(d) (i)** 20.18 **(ii)** 20.2% **(iii)** 312

27. **(a)** $A(x) = 650\left(\frac{1}{2}\right)^x$ **(b)** $A(t) = 650(0.435)^t$
(c) 1.4×10^{-6} mg
(d)



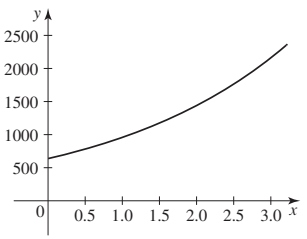
29. Plan A: \$8454.54, Plan B: \$8353.55; Plan A is better.
31. **(a)** Logistic
(b) Exponential model: $f(t) = 19.03(1.220061)^t$
Logistic model: $g(t) = \dfrac{52.86}{1 + 2.68e^{-0.76425t}}$



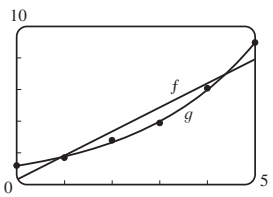
(c) The exponential model predicts 8.2 billion turtles in 100 years, while the logistic model predicts 51 turtles. The logistic model seems more reasonable.

Chapter 3 Test ▪ page 311

1. **(a)** 640 **(b)** 1.5; 50% **(c)** 7290
(d)



(e) 17.086; $P(t) = 640(17.086)^t$ **2.** **(a)** \$2024.11 after 4 months, \$2073.20 after 1 year, \$2227.74 after 3 years. **(b)** No **3.** **(a)** Average rate of change: 2; percentage rate of change: 800% **(b)** Average rate of change: 18; percentage rate of change: 800% **4.** **(a)** A logistic growth model **(b)** 125 **(c)** 457 **(d)** No, the carrying capacity is 500. **5.** Graph A: graph of h; graph B: graph of g; graph C: graph of f; graph D: graph of k **6.** **(a)** **(i)** $f(x) = 0.30952 + 1.52286x$
(ii) $g(x) = 1.1826739(1.50204)^x$
(b)



(c) The exponential model fits the data better. The mass in week 6 is predicted to be 13.6 kg.

Chapter 4

Algebra Checkpoint 4.1 ▪ page 330

1. **(a)** 64 **(b)** $\frac{1}{5}$ **(c)** $\frac{1}{25}$ **(d)** 2 **(e)** $\frac{1}{3}$ **2.** **(a)** 10^2 **(b)** 10^4 **(c)** 10^{-1} **(d)** 10^{-2} **(e)** $10^{-1/2}$ **3.** **(a)** 2^4 **(b)** 2^6 **(c)** 2^{-1} **(d)** 2^{-4} **(e)** $2^{-1/3}$ **4.** **(a)** 9^2 **(b)** 9^3 **(c)** 9^{-1} **(d)** $9^{1/2}$ **(e)** $9^{-1/2}$

4.1 Exercises ▪ page 331

1. x; 3, 2, 1, 0, −1, −2, −3, $\frac{1}{2}$, $\frac{3}{4}$ **2.** **(a)** $\log_5 125 = 3$ **(b)** $5^2 = 25$ **3.** 9; 1, 0, −1, 2, $\frac{1}{2}$ **4.** **(a)** II **(b)** I **5.** **(a)** True **(b)** True **(c)** False **7.** **(a)** 4 **(b)** −1 **(c)** −2 **(d)** $\frac{1}{3}$ **9.** **(a)** 2.121 **(b)** 3.699 **(c)** 0.151 **(d)** −0.523 **11.** **(a)** −3.921 **(b)** 2.238 **13.** **(a)** 0.602 **(b)** 3.736 **15.** **(a)** 1 **(b)** 0 **(c)** 2 **17.** **(a)** 2 **(b)** −1 **(c)** 1 **19.** **(a)** 4 **(b)** 3 **(c)** 2 **21.** **(a)** 37 **(b)** 8 **(c)** 5 **23.** **(a)** 5 **(b)** 17 **(c)** 0 **25.** **(a)** 3 **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{4}$

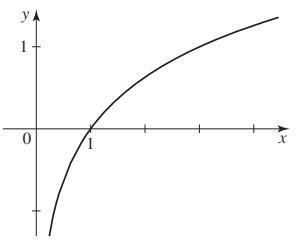
27.

Logarithmic form	Exponential form
$\log_8 8 = 1$	$8^1 = 8$
$\log_8 64 = 2$	$8^2 = 64$
$\log_8 4 = \frac{2}{3}$	$8^{2/3} = 4$
$\log_8 512 = 3$	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	$8^{-1} = \frac{1}{8}$
$\log_8\left(\frac{1}{64}\right) = -2$	$8^{-2} = \frac{1}{64}$

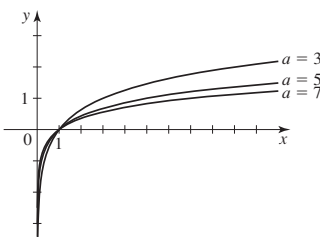
29. **(a)** $5^3 = 125$ **(b)** $5^0 = 1$ **31.** **(a)** $8^{1/3} = 2$ **(b)** $9^{1/2} = 3$ **33.** **(a)** $\log_3 27 = 3$ **(b)** $\log_{10} 0.0001 = -4$ **35.** **(a)** $\log_8\left(\frac{1}{8}\right) = -1$ **(b)** $\log_2\left(\frac{1}{8}\right) = -3$ **37.** **(a)** 32 **(b)** 4 **39.** **(a)** $\frac{1}{2}$ **(b)** 16

41.

x	$f(x)$
$\frac{1}{27}$	−3
$\frac{1}{9}$	−2
$\frac{1}{3}$	−1
1	0
3	1
9	2
27	3



43. III **45.** IV **47.** $a = 5$ **49.** $a = 3$
51.



The graphs all intersect at the point (1, 0).

4.2 Exercises ▪ page 339

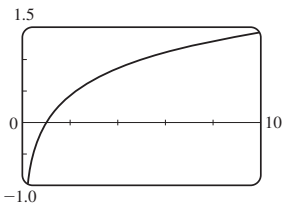
1. sum; 2, 3 **2.** difference; 2, 3 **3.** times; $10 \cdot 2$ **4.** 10; Change of Base; $\log_7 12 = \dfrac{\log_{10} 12}{\log_{10} 7} \approx 1.277$ **5.** **(a)** III **(b)** I **(c)** II **6.** Change of Base; 13 **7.** **(a)** False **(b)** True **(c)** True **(d)** False **(e)** False **(f)** False **(g)** False **(h)** True **8.** They are the same. **9.** **(a)** 2 **(b)** 5 **(c)** -3 **11.** **(a)** $1 + \log_2 x$ **(b)** $\log_3 x - 3 \log_3 y$ **13.** **(a)** $\log_2 A + 2 \log_2 B$ **(b)** $-\frac{1}{2} \log z$ **15.** **(a)** $2 \log_2 s + \frac{1}{2} \log_2 t$ **(b)** $3 \log r + 4 \log s - \frac{1}{4} \log t$ **17.** **(a)** $10 \log x + 10 \log y$ **(b)** $2 \log_5 x - \log_5 y - 3 \log_5 z$ **19.** **(a)** $\log_2 \dfrac{AB}{C^2}$ **(b)** $\log_6 \dfrac{y^4}{\sqrt[4]{z}}$

21. **(a)** $\log_2(A^3(B+1)^2)$ **(b)** $\log_3\left(\dfrac{(2x-1)^4}{x+1}\right)$

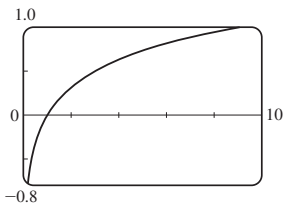
23. **(a)** $\log_4 \sqrt{\dfrac{y+1}{y-1}}$ **(b)** $\log \dfrac{x^4(x-1)^2}{\sqrt[3]{x^2+1}}$

25. 5 **27.** 15 **29.** 3 **31.** 2 **33.** **(a)** 2.321928 **(b)** 0.430676 **35.** **(a)** 0.493008 **(b)** 3.503061

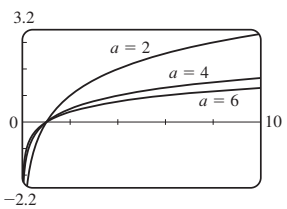
37. **(a)**



(b)

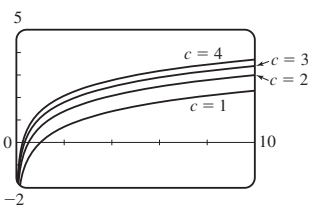


39.



The graphs all intersect at $(1, 0)$.

41.

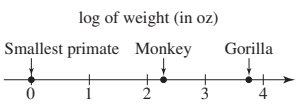


The graph of $G(x) = \log(cx)$ is the graph of $f(x) = \log x$ shifted upward by $\log c$ units.

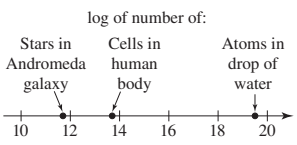
4.3 Exercises ▪ page 347

1. logarithm; 3, -2 **2.** lower **3.** higher **4.** 1000 **5.** 990; 2

7.



9.

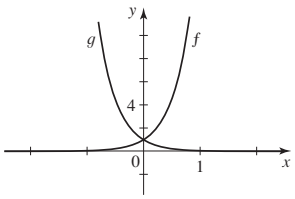


11. **(a)** 2.30 **(b)** 3.49 **13.** **(a)** 2.51×10^{-3} M **(b)** 3.16×10^{-7} M **15.** **(a)** 91 dB **(b)** 99 dB **17.** **(a)** California red: 1.58×10^{-4} M; Italian white: 1.58×10^{-3} M **(b)** California red **19.** **(a)** 6.7; yes **(b)** Lower **21.** **(a)** 105 dB **(b)** 85 dB **23.** 8.2 **25.** 2 times more intense **27.** **(a)** $M = 2.5 \log B_0 - 2.5 \log B$ **29.** For W = 5 mm, ID = 5.32; for W = 10 mm, ID = 4.32

4.4 Exercises ▪ page 358

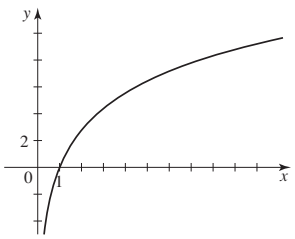
1. **(a)** natural, 2.71828 **(b)** e; 1 **2.** principal, interest rate, time in years, account value; \$112.75 **3.** $\ln a$, 0.405 **4.** $f(t) = Ce^{rt}$ **5.** The one that doubles every 10 minutes **6.** Both grow at the same rate. **8.** **(a)** $A(t) = t$ **(b)** $A(t) = 2^t$ **(c)** $A(t) = 2^t$ **(d)** $A(t) = e^t$ **9.** **(a)** 8.155 **(b)** 4.055 **(c)** 0.135 **(d)** 19.778

11.

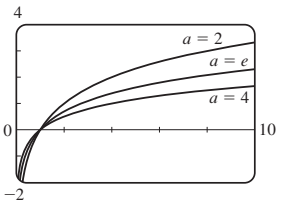


13. **(a)** 4 **(b)** -1 **(c)** 0.693 **(d)** 4.669

15.



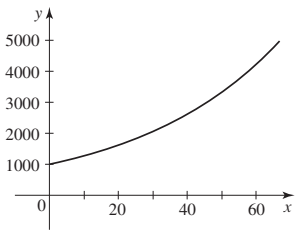
17.



19. I **21.** III **23.** **(a)** Growth **(b)** 0.50 **25.** **(a)** Decay **(b)** -0.30

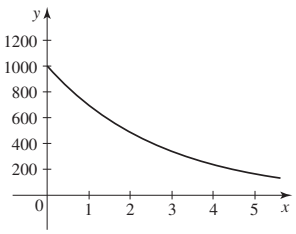
27. **(a)** $f(x) = 1000e^{0.024x}$

(b)



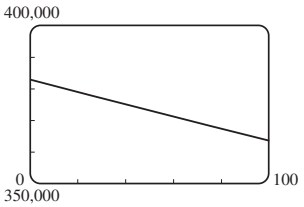
29. **(a)** $f(x) = 1000e^{-0.36x}$

(b)

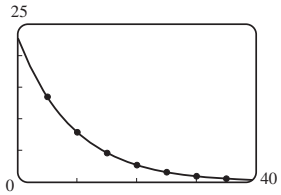


31. **(a)** $f(t) = 2500e^{0.693t}$; 0.693 **(b)** $g(x) = 2500e^{-0.693x}$; -0.693 **33.** 100; 0.693 **35.** **(a)** $f(t) = 200(1.034)^t$ **(b)** $g(t) = 200e^{0.033t}$ **(c)** The graphs are the same. **37.** **(a)** $f(t) = 200(0.946)^t$ **(b)** $g(t) = 200e^{-0.056t}$ **(c)** The graphs are the same. **39.** **(a)** $f(t) = 3000(1.1076)^t$ **(b)** $g(t) = 3000e^{0.1022t}$ **(c)** The graphs are the same. **41.** **(a)** $f(t) = 3000(0.8706)^t$ **(b)** $g(t) = 3000e^{-0.1386t}$ **(c)** The graphs are the same. **43.** **(a)** \$35,264.97 **(b)** \$35,369.45 **(c)** \$35,440.63 **(d)** \$35,476.69 **45.** **(a)** 0.10 **(b)** \$18,135.93 **(c)** \$18,024.78 **47.** **(a)** 500 bacteria **(b)** Increasing; 45% **(c)** 1928

49. **(a)** -0.52%; decreasing **(b)** $f(t) = 383{,}000e^{-0.0052t}$ **(c)**



51. **(a)** $a = 4; f(t) = 20(4)^t$ **(b)** $r = 1.386; g(t) = 20e^{1.386t}$ **(c)** The graphs are the same. **53.** **(a)** $a = 1.3531$; $f(t) = 22.5(1.3531)^t$ **(b)** $r = 0.3024; g(t) = 22.5e^{0.3024t}$ **(c)** 462.9 million; yes **55.** **(a)** $I(x) = 22.759(0.8992)^x$ **(b)** $I(x) = 22.759e^{-0.1062x}$ **(c)** $k = 0.1062$ **(d)** 22.8 lm **(e)**



4.5 Exercises ▪ page 373

1. **(a)** $e^x = 25$ **(b)** $x = \ln 25$ **(c)** 3.2189

2. **(a)** $\log\left(\dfrac{3(x-2)}{x}\right) = 0$ **(b)** $1 = \dfrac{3(x-2)}{x}$ **(c)** 3

3. Base 2 **5.** 1.398 **7.** -1.159 **9.** 0.178 **11.** 0.092 **13.** 1.363 **15.** 2.771 **17.** 6.213 **19.** 0.805 **21.** -0.585 **23.** 0.562 **25.** -9.113 **27.** 1024 **29.** 10,000 **31.** $\frac{1}{32}$ **33.** 1004 **35.** 24.5 **37.** 4 **39.** 5 **41.** 2.21 **43.** -0.57 **45.** 0, 1.14 **47.** 0.36 **49.** **(a)** $A(t) = 6000(1.01125)^{4t}$ **(b)** \$6561.75 **(c)** 6.4 yr **51.** **(a)** 11.6 yr **(b)** 14.1 yr **53.** **(a)** $r = 1.2$ **(b)** $f(t) = 15e^{1.2t}$ **(c)** 1823 **(d)** 11.18 h **(e)** 0.58 h **55.** **(a)** $a = 1.029$ **(b)** $f(t) = 70.7(1.029)^t$ **(c)** 24.2 yr **(d)** 2011 **57.** **(a)** $T = 65 + 147e^{-2.895t}$ **(b)** 1.16 h **59.** **(a)** $T = 20 + 80e^{-1.4988t}$ **(b)** 62.8°C **(c)** 0.92 h (55 min) **61.** **(a)** $a = 0.99957$ **(b)** $m(t) = 22(0.99957)^t$ **(c)** 3.94 g **(d)** 467 yr **63.** 80,089 yr **65.** 3029 yr **67.** **(a)** 1938 yr

4.6 Exercises ▪ page 386

1. 12 **2.** Multiply by 2, then add 1; add 1, then multiply by 2. $f(x) = x + 1, g(x) = 2x, M(x) = 2x + 1, N(x) = 2(x + 1)$ **3.** different; Horizontal Line **4.** **(a)** one-to-one; g **(b)** $g^{-1}(x) = x - 2$ **5.** Take the third root, subtract 5, then divide by 3. **(a)** $f(x) = (3x + 5)^3$ **(b)** $f^{-1}(x) = \dfrac{\sqrt[3]{x} - 5}{3}$

6. $\ln x$ **7.** Linear; linear **8.** **(a)** False **(b)** True **9.** **(a)** 5 **(b)** 2 **11.** **(a)** 5 **(b)** 2

13.

x	$f(g(x))$	$g(f(x))$
1	2	3
2	4	3
3	2	5
4	5	5
5	2	1

15. **(a)** $M(x)$: Multiply by 3, then add 2. $N(x)$: Add 2, then multiply by 3. **(b)** $M(x) = 3x + 2; N(x) = 3(x + 2)$ **(c)** $M(3) = 11; N(-2) = 0$ **17.** **(a)** $M(x)$: Subtract 1, square, then multiply by 2; $N(x)$: Square, multiply by 2, then subtract 1. **(b)** $M(x) = 2(x - 1)^2; N(x) = 2x^2 - 1$ **(c)** $M(3) = 8$; $N(-2) = 7$ **19.** **(a)** $M(x)$: Add 3, then raise e to that result; $N(x)$: Raise e to the input, then add 3. **(b)** $M(x) = e^{x+3}$; $N(x) = e^x + 3$ **(c)** $M(3) = 403.43; N(-2) = 3.135$ **21.** **(a)** $M(x)$: Take the natural log, multiply by 3, then add 5; $N(x)$: Multiply by 3, add 5, then take the natural log. **(b)** $M(x) = 3\ln x + 5; N(x) = \ln(3x + 5)$ **(c)** $M(3) = 8.296; N(-2)$ is not defined. **23.** **(a)** Add 6, then divide by 3. **(b)** $f(x) = 3x - 6; f^{-1}(x) = \dfrac{x + 6}{3}$

25. **(a)** Multiply by 2, then subtract 4. **(b)** $f(x) = \frac{1}{2}(x + 4)$; $f^{-1}(x) = 2x - 4$ **27.** **(a)** No inverse **(b)** $f(x) = -2x^2 + 3$ **29.** 2 **31.** 3 **33.** **(a)** 12 **(b)** 2 **(c)** 4 **(d)** 5 **35.** **(a)** 6 **(b)** 4 **(c)** 0 **45.** One-to-one **47.** Not one-to-one **49.** One-to-one **51.** One-to-one **53.** Not one-to-one **55.** Not one-to-one

57. $f^{-1}(x) = \dfrac{x - 7}{4}$ **59.** $f^{-1}(x) = 2x$

61. $f^{-1}(x) = \sqrt[3]{x + 4}$ **63.** $f^{-1}(x) = \dfrac{2(x + 1)}{1 - x}$

65. $f^{-1}(x) = 2^x - 1$ **67.** $f^{-1}(x) = 2\ln x$

69. $f^{-1}(x) = e^x + 3$ **71.** Not one-to-one **73.** One-to-one

75. Not one-to-one **77.** **(a)** $f(x) = 0.8x$ **(b)** $g(x) = x - 50$; **(c)** $f(g(x)) = 0.8(x - 50); g(f(x)) = 0.8x - 50$; $g(f(x))$ gives the lower price. **79.** **(a)** $g(x) = \dfrac{x}{30}$

(b) $f(x) = 19x$ **(c)** $f(g(x)) = 19\left(\dfrac{x}{30}\right)$; $f(g(x))$ gives the number of pounds of CO_2 emitted when x miles are driven.

(d)

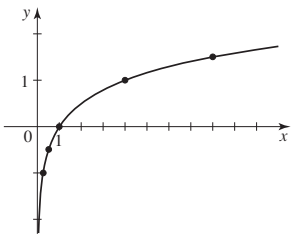
x	$g(x)$	$f(g(x))$
12,000	400	7,600
24,000	800	15,200
36,000	1200	22,800
50,000	1667	31,673
100,000	3333	63,327

81. **(a)** $f(x) = 0.85x$ **(b)** $g(x) = x - 1000$ **(c)** $H(x) = 0.85(x - 1000)$ **(d)** $H^{-1}(x) = \dfrac{x}{0.85} + 1000$; the sticker price when x is the purchase price **(e)** \$16,294; the sticker price of a car with a purchase price of \$13,000 **83.** **(a)** $f^{-1}(x) = 32.76(\ln x - 6.741)$; the number of years since 1990 as a function of the population **(b)** 1998 **85.** **(a)** $F^{-1}(x) = 40 - 4\sqrt{x}$; x represents volume, and F^{-1} represents time. **(b)** 24.5; the tank has 15 gallons left after 24.5 minutes

Chapter 4 Review Exercises ▪ page 395

1. **(a)** 2 **(b)** -3 **(c)** 3 **(d)** -3

3.



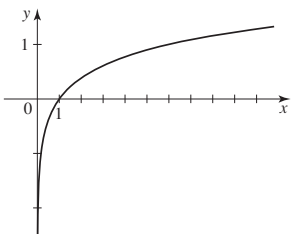
5. **(a)** 1.8394 **(b)** 9.9658 **(c)** 1.9731 **(d)** 2.5177

7. **(a)** $5^2 = 25$ **(b)** $2^3 = x$ **(c)** $e^4 = y$

9. **(a)** $\log_2 3 + \log_2 a + \log_2 b$ **(b)** $\log 5 + \log x - 2 \log y$

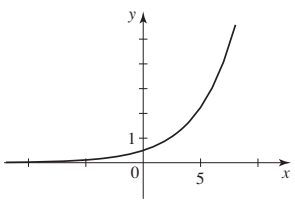
11. **(a)** $\log_2(X^3Y)$ **(b)** $\ln \dfrac{x+1}{y^2}$

13.



15. pH = 1.14; this is less than 7.

17.



19. $f(t) = 2400e^{1.0986t}$ **21.** $f(t) = 1500e^{1.0986t}$ **23.** 1 **25.** 3.3922 **27.** $\frac{1}{9}$ **29.** $\frac{48}{31}$ **31.** **(a)** $f(g(x)) = 6x - 2$; $g(f(x)) = 6x - 11$ **(b)** 10; -29

33. **(a)** $f(g(x)) = 2x^6$; $g(f(x)) = 8x^6$ **(b)** 128; 5832

35. $f^{-1}(x) = \dfrac{x+7}{3}$ **37.** $f^{-1}(x) = \ln \dfrac{x}{8}$ **39.** One-to-one

41. **(a)** $M(x) = 80(0.9)^x$ **(b)** $N(t) = 80(0.974)^t$ **(c)** No; after 17.8 hours **(d)** $N(t) = 80e^{-0.0263t}$; -0.0263 **(e)** 42.5 mg **(f)** $S(t) = 122.6e^{-0.0263t}$ **(g)** No **43.** **(a)** 6.4; yes **(b)** 1.26×10^{-6} M **45.** **(a)** 4.25% **(b)** \$283,996.22 **(c)** \$283,248.88 **47.** **(a)** 6 yr **(b)** $W(t) = 23e^{0.1155t}$; 0.1155 **(c)** 2016 **(d)** 2,386,872 lb; logistic **49.** **(a)** $f(x) = 0.95x$; $g(x) = x - 2$ **(b)** $f(g(x))$ represents taking the \$2 discount first, then the 5% discount; $g(f(x))$ represents taking the 5% discount first, then the \$2 discount. **(c)** $f(g(x)) = 0.95(x - 2)$; $g(f(x)) = 0.95x - 2$ **(d)** \$36.10; \$36.00; yes

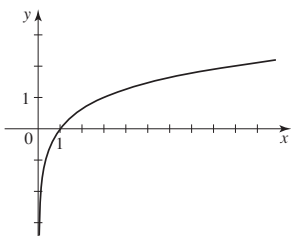
Chapter 4 Test ▪ page 400

1. **(a)** -2 **(b)** $\frac{3}{2}$ **(c)** 100

2. **(a)**

x	0.1	0.5	1	3	5
$f(x)$	-2.10	-0.63	0	1	1.47

(b)



3. **(a)** $\log 5 + 4 \log x - \log(6x - 11)$ **(b)** $\ln \dfrac{5x^2}{x^3 + 7}$

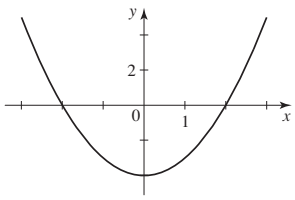
4. **(a)** 6.4 **(b)** 5.01×10^{-5} **5.** **(a)** \$2543.95 **(b)** \$2540.10; investment (a) is better. **6.** **(a)** $A(t) = Ce^{-0.0231t}$; -0.0231 **(b)** 299 s **7.** **(a)** 0.56 **(b)** 4 **8.** **(a)** $S(x) = 1.95 + x$ **(b)** $S^{-1}(x) = x - 1.95$; the number of books as a function of the shipping fee **(c)** 11 books **9.** **(a)** $f(g(x)) = 100(x - 1)$; $g(f(x)) = \log(10^{x+2} - 1)$ **(b)** $f(g(0))$ is not defined; $g(f(0)) = 1.9956$

Chapter 5

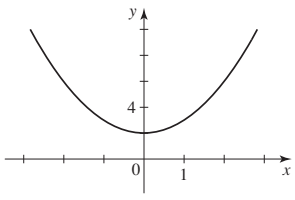
5.1 Exercises ▪ page 422

1. **(a)** up **(b)** left **2.** **(a)** down **(b)** right **3.** **(a)** x-axis **(b)** y-axis **4.** **(a)** II **(b)** IV **(c)** I **(d)** III **5.** Shift to the right 4 units, then upward 5 units. **6.** Reflect in the x-axis and in the y-axis, then shift upward 6 units.

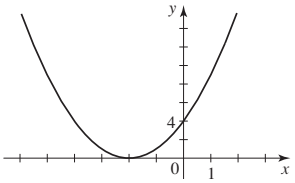
7. **(a)**



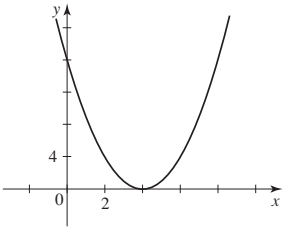
(b)



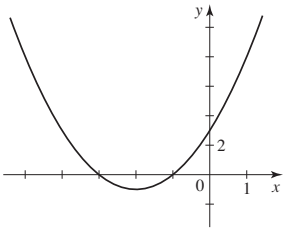
9. (a)



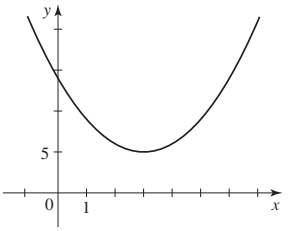
(b)



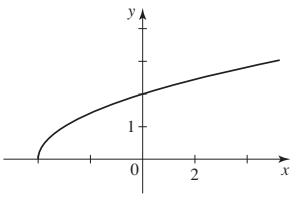
11. (a)



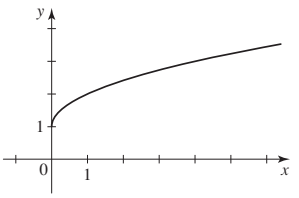
(b)



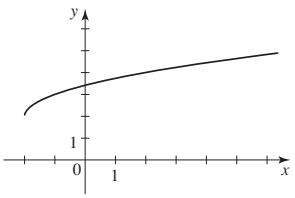
13. (a)



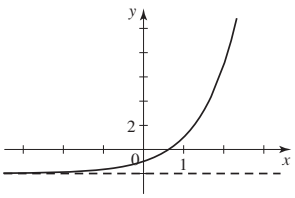
(b)



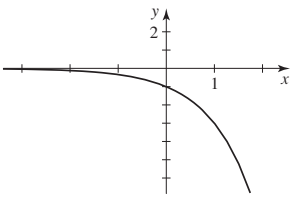
(c)



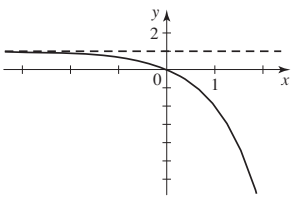
15. (a)



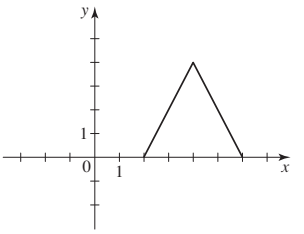
(b)



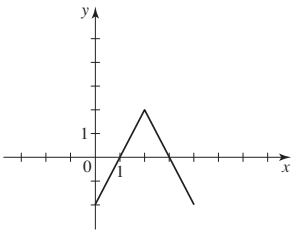
(c)



17. (a)



(b)



19.

x	$f(x)$	$f(x) + 1$
-3	20	21
-2	27	28
-1	53	54
0	42	43
1	39	40
2	70	71
3	21	22

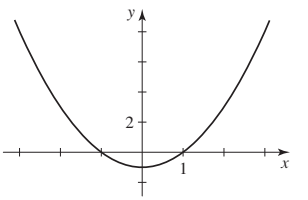
21.

x	$f(x)$	$f(x + 2)$
-6	105	99
-4	99	82
-2	82	53
0	53	20
2	20	6
4	6	2
6	2	—

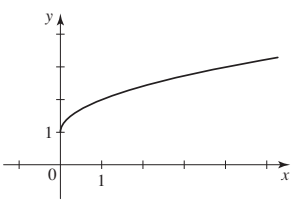
23.

x	$f(x)$	$f(-x)$
-6	105	2
-4	99	6
-2	82	20
0	53	53
2	20	82
4	6	99
6	2	105

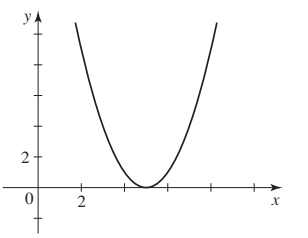
25.



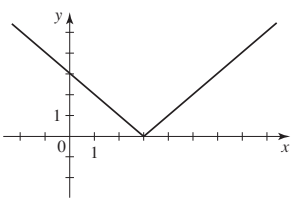
27.



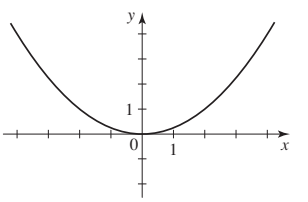
29.



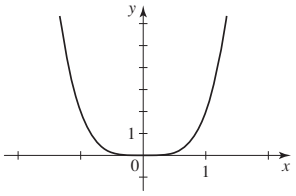
31.



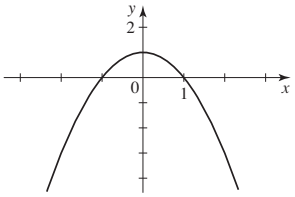
33.



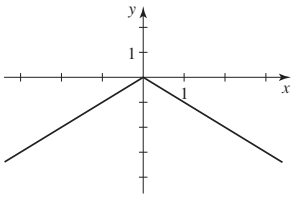
35.



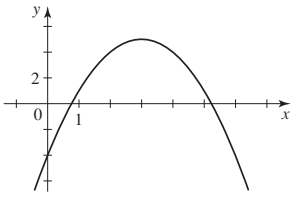
37.



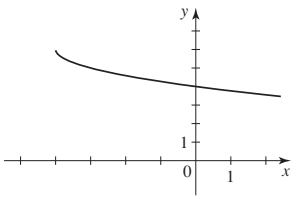
39.



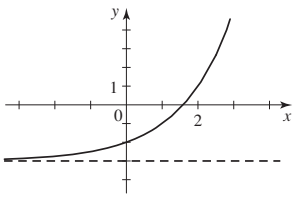
41.



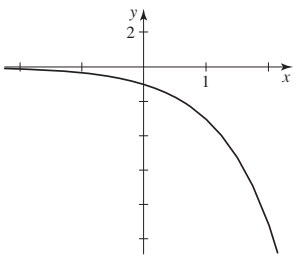
43.



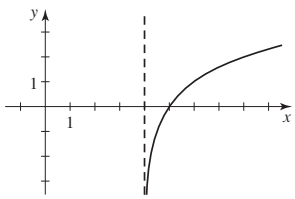
45.



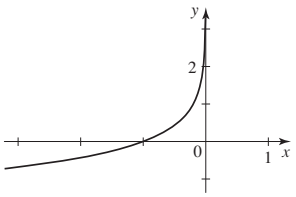
47.



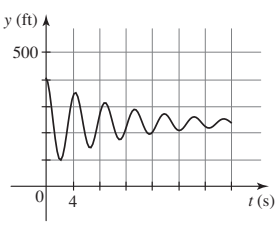
49.



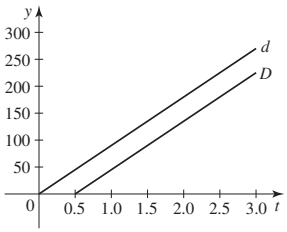
51.



53. $f(x) = x^2 + 3$ **55.** $f(x) = \sqrt{x+2}$
57. $f(x) = |x - 3| + 1$ **59.** $f(x) = 2^x - 2$
61. $g(x) = (x-2)^2$ **63.** $g(x) = -2^x - 1$ **65. (a)** Luisa drops to 200 ft, bounces up and down a bit, then settles at 350 ft.
(b) **(c)** $H(t) = h(t) - 100$

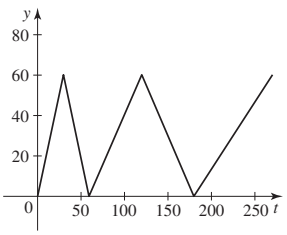


67. (a) $d(t) = 90t$ **(b)** $D(t) = 90\left(t - \frac{1}{2}\right)$
(c)



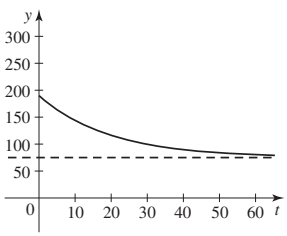
Shift the graph of d to the right 0.5 unit to get the graph of D.

69. (a) She swam two and a half laps, slowing down with each successive lap; 1.67 m/s
(b)



This graph is a vertical stretching of the original function by a factor of 1.2.

(c) 2 m/s
71. (a) $f(t) = 10(1.05)^t$ **(b)** The graph of g is a vertical stretching of the graph of f by a factor of 5; $g(t) = 50(1.05)^t$
73. (a) Stretch vertically by a factor of 115, then shift upward 75 units.
(b)



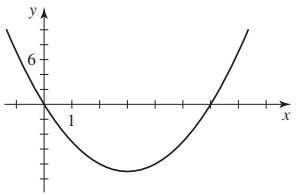
Algebra Checkpoint 5.2 ▪ page 435

1. (a) $x^2 - x - 6$ **(b)** $2t^2 - 3t - 5$ **(c)** $6r^2 - 13r + 6$ **(d)** $36 - 9v^2$ **2. (a)** Yes **(b)** No **(c)** Yes **(d)** No
3. (a) $(x+7)^2$ **(b)** $(t-8)^2$ **(c)** $\left(w - \frac{1}{2}\right)^2$ **(d)** $\left(s + \frac{5}{2}\right)^2$
4. (a) 25; 5 **(b)** 100; 10 **(c)** 27; 3 **(d)** 25; $\frac{5}{2}$ **5. (a)** 9; 9 **(b)** $\frac{1}{4}$; $\frac{1}{4}$ **(c)** 4; 8 **(d)** $\frac{1}{36}$; $\frac{1}{12}$

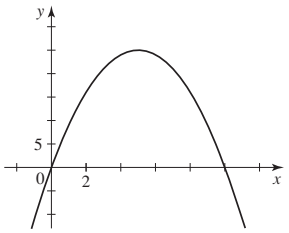
5.2 Exercises ▪ page 435

1. square **2. (a)** h, k **(b)** upward **(c)** downward
3. upward; 3, 5 **4.** downward; -3, 5 **5. (a)** $x^2 - 6x + 8$ **(b)** 8 **6. (a)** $x^2 - (m+n)x + mn$ **(b)** mn **(c)** 15
7. (a) Yes **(b)** No **(c)** No **9.** $f(x) = 8x^2 + 10x - 3$
11. $f(x) = 4x^2 - 25$ **13.** $f(x) = x^2 - 2x + 6$
15. $f(x) = 100x^2 + 120x + 43$
17. (a) $f(x) = -(x-3)^2 + 4$ **(b)** $(3, 4)$
19. (a) $f(x) = 2(x-1)^2 - 3$ **(b)** $(1, -3)$
21. $f(x) = (x+1)^2 - 6$ **23.** $f(x) = 2(x+5)^2 - 49$

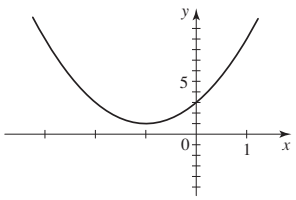
25. **(a)** $f(x) = (x - 3)^2 - 9$ **(b)** $(3, -9)$
(c)



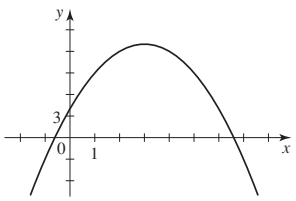
27. **(a)** $f(x) = -(x - 5)^2 + 25$ **(b)** $(5, 25)$
(c)



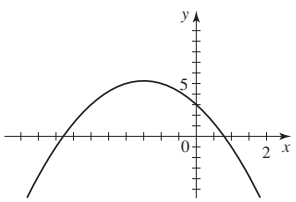
29. **(a)** $f(x) = 2(x + 1)^2 + 1$ **(b)** $(-1, 1)$
(c)



31. **(a)** $f(x) = -(x - 3)^2 + 13$ **(b)** $(3, 13)$
(c)



33. **(a)** $f(x) = -\left(x + \frac{3}{2}\right)^2 + \frac{21}{4}$ **(b)** $\left(-\frac{3}{2}, \frac{21}{4}\right)$
(c)



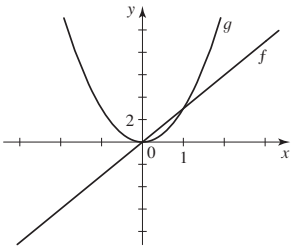
35. $f(x) = -5(x + 2)^2 - 3$ **37.** $f(x) = -3(x - 3)^2 + 4$
39. $f(x) = (x - 1)^2 + 2$ **41.** $f(x) = -(x + 2)^2 + 5$

43. **(a)**

x	$f(x)$	Rate of change
0	0	—
1	3	3
2	6	3
3	9	3
4	12	3
5	15	3

x	$g(x)$	Rate of change
0	0	—
1	3	3
2	12	9
3	27	15
4	48	21
5	75	27

(b)

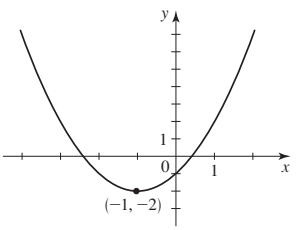


45. $f(x) = \frac{1}{100}x^2$ **47.** $f(x) = \frac{1}{600}x^2$

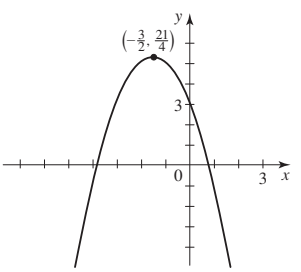
5.3 Exercises ▪ page 444

1. **(a)** $\dfrac{-b}{2a}$ **(b)** minimum **(c)** maximum **2.** **(a)** minimum; $\frac{12}{4}, -13$ **(b)** maximum; $\frac{12}{-4}, 23$ **3.** **(a)** $x^2 - 6x + 8$ **(b)** up **(c)** $-\left(\frac{-6}{2}\right) = 3$ **4.** **(a)** $x^2 - (m + n)x + mn$ **(b)** up **(c)** $\dfrac{m + n}{2}$ **(d)** 1 **5.** **(a)** Minimum; 3 **(b)** 2

7. **(a)** Maximum; 5 **(b)** 4 **9.** Maximum: $f(3) = 10$
11. Minimum: $f\left(-\frac{1}{2}\right) = \frac{3}{4}$ **13.** Minimum: $h(-2) = -8$
15. **(a)** **(b)** Minimum: $f(-1) = -2$



17. **(a)** **(b)** Maximum: $f\left(-\frac{3}{2}\right) = \frac{21}{4}$



19. **(a)** -4.01 **(b)** Minimum value: $f\left(-\frac{1.79}{2}\right) = -4.011025$

21. **(a)** 1.18 **(b)** Maximum value: $f\left(-\left(\frac{-1}{2\sqrt{2}}\right)\right) = 1.17677$

23. 7.34 m; 1.22 s **25.** 22,500; 50

27. **(a)** $A(x) = x(2400 - 2x)$ **(b)** 720,000 ft^2; 600 ft by 1200 ft **29.** **(a)** $A(x) = x(600 - x)$

(b) 300 ft by 300 ft **31.** **(a)** $A(x) = \frac{x^2}{16} + \frac{(10 - x)^2}{16}$

(b) 5 cm **33.** **(a)** $R(x) = (10 - x)(27{,}000 + 3000x)$

(b) \$9.50

Algebra Checkpoint 5.4 ▪ page 456

1. **(a)** $x(x + 1)$ **(b)** $2t(2t - 3)$ **(c)** $(x + 1)[2(x + 1) + 1]$

(d) $(3r + 4)[(2r + 3) - 5]$ **2.** **(a)** $(x + 6)(x - 6)$

(b) $4(t + 2)(t - 2)$ **(c)** $(u + 7)(u - 5)$

(d) $[2(w - 2) - 7][2(w - 2) + 7]$ **3.** **(a)** $(x + 4)^2$

(b) $(t - 6)^2$ **(c)** $(3r - 4)^2$ **(d)** $(u + 7)^2$

4. **(a)** $(x + 2)(x + 3)$ **(b)** $(t - 4)(t + 3)$

(c) $(2s - 3)(s + 1)$ **(d)** $(2u - 3)(3u + 1)$

5.4 Exercises ▪ page 456

1. **(a)** $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **(b)** $\frac{1}{2}, -1, -4$; 4 or -2

2. **(a)** 5, 1; 5 or -1 **(b)** $\frac{4 \pm \sqrt{16 - 4(1)(-5)}}{2(1)}$; 5 or -1

3. **(a)** 2 **(b)** 1 **(c)** 0 **4.** **(a)** 1 and 5 **(b)** positive

(c) 1 and 5 **5.** **(a)** upward **(b)** 2, 4 **(c)** 3 **6.** **(a)** upward

(b) m, n **(c)** $\frac{m + n}{2}$ **(d)** 4 **7.** 4, 3 **9.** 3, 4 **11.** $-\frac{1}{3}, 2$

13. $-3, -\frac{1}{2}$ **15.** $-\frac{4}{3}, \frac{1}{2}$ **17.** **(a)** 3, 5

(b) $f(x) = x^2 - 8x + 15$ **19.** **(a)** $-4, 1$

(b) $f(x) = -3x^2 - 9x + 12$ **21.** $-3, 5$ **23.** 2, 5

25. $-\frac{3}{2}, 1$ **27.** $\frac{-3 \pm \sqrt{5}}{2}$ **29.** $-6 \pm 3\sqrt{7}$ **31.** $\frac{3}{4}$

33. No real solution **35.** $\frac{\sqrt{5} \pm 1}{2}$ **37.** $\frac{8 \pm \sqrt{14}}{10}$

39. $-0.248, 0.259$ **41.** No real solution **43.** **(a)** 32; 2

(b) 2 **45.** **(a)** 0; 1 **(b)** 1 **47.** **(a)** -64; 0

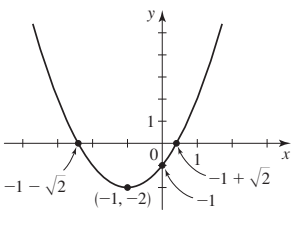
(b) None **49.** **(a)** $-2, 1$ **(b)** Positive **(c)** $-2, 1$

(d) $f(x) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$ **51.** **(a)** 3 **(b)** Zero **(c)** 3

(d) $f(x) = \frac{1}{3}x^2 - 2x + 3$

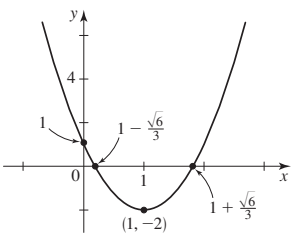
53. **(a)** $-1 \pm \sqrt{2}$

(b)



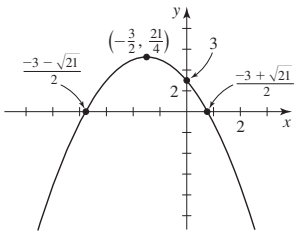
55. **(a)** $\frac{3 \pm \sqrt{6}}{3}$

(b)



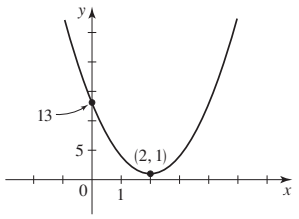
57. **(a)** $\frac{-3 \pm \sqrt{21}}{2}$

(b)



59. **(a)** None

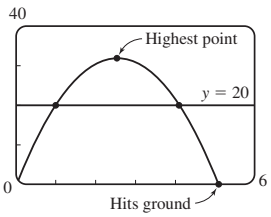
(b)



61. **(a)** 0.53 s and 1.92 s **(b)** No **(c)** 2.45 s

63. **(a)** After 0.99 s and 4.1 s **(b)** After 2.55 s **(c)** After 5.1 s

(d)



65. **(a)** $A(x) = x(x + 40)$ **(b)** 90 ft by 130 ft

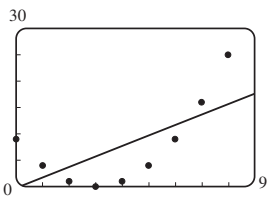
67. **(a)** $R(x) = (10 - x)(27{,}000 + 3000x)$

(b) \$19.00 **(c)** \$6.87 or \$12.13

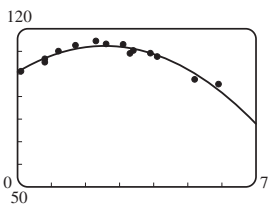
5.5 Exercises ▪ page 464

1. scatter plot **2.** quadratic **3.** $y = x^2 - 6x + 9$

4. $y = -0.33 + 2x$

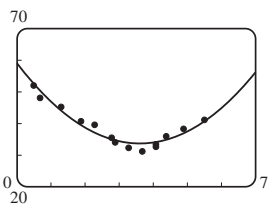


5. **(a)**



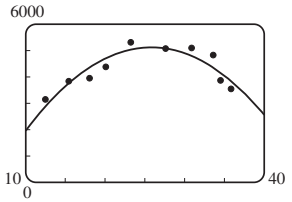
(b) $y = -1.7529x^2 + 8.983x + 100.89$ **(c)** 77.9

7. **(a)**



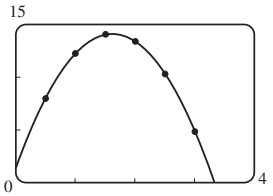
(b) $y = 1.9575x^2 - 14.082x + 59.011$ **(c)** 56.4

9. (a)



(b) $y = -12.627x^2 + 651.55x - 3283.2$ **(c)** 5114 kg/acre

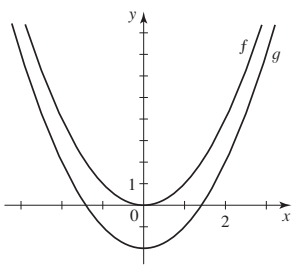
11. (a)



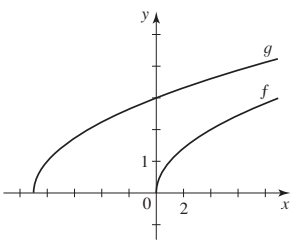
(b) $y = -4.875x^2 + 15.797x + 1.2845$
(c) 0.33 s and 2.91 s **(d)** 14.08 ft

Chapter 5 Review Exercises ▪ page 468

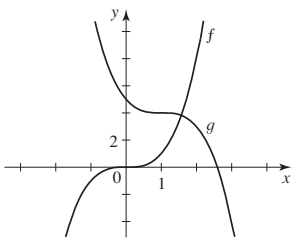
1.



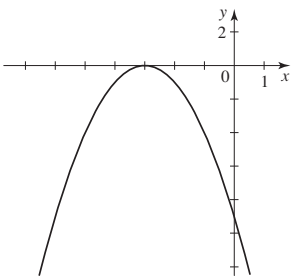
3.



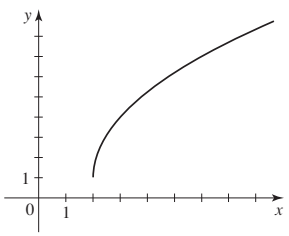
5.



7.

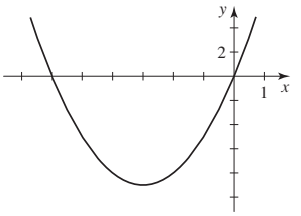


9.

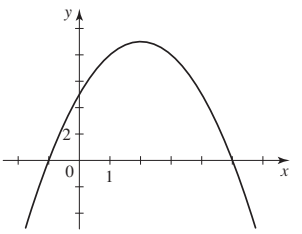


11. (a) $f(x) = x^2 + 2x - 8$ **(b)** $f(x) = (x + 1)^2 - 9$
(c) $-4, 2; -8$

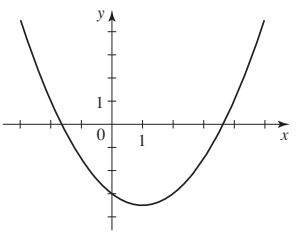
13. (a) $f(x) = (x + 3)^2 - 9$ **(b)** $(-3, -9)$
(c) **(d)** 10



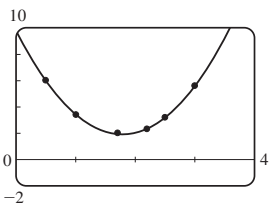
15. (a) $f(x) = -(x - 2)^2 + 9$ **(b)** $(2, 9)$
(c) **(d)** 0



17. (a) $f(x) = \frac{1}{2}(x - 1)^2 - \frac{7}{2}$ **(b)** $\left(1, -\frac{7}{2}\right)$
(c) **(d)** 1

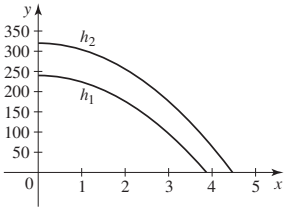


19. $y = -\frac{3}{4}(x + 2)^2 + 6$ **21.** $y = 2x^2 - 8x + 5$
23. Minimum: $f(-2) = -11$ **25.** Maximum: $f\left(-\frac{3}{2}\right) = \frac{7}{4}$
27. Maximum: $f\left(\frac{3}{2}\right) = \frac{45}{4}$ **29.** $-7, 0$ **31.** $-3, \frac{1}{2}$
33. (a) 25; two **(b)** $-4, 1$ **35. (a)** 0; one **(b)** $\frac{1}{2}$
37. (a) -16; none **(b)** No real solution
39. (a) $y = 2.4765x^2 - 8.827x + 9.7847$
(b)



41. (a) The y-intercept **(b)** The x- and y-coordinates of the vertex **(c)** The x-intercepts **(d)** Factored form; $f(x) = -2(x - 2)(x - 5)$ **(e)** Standard form; $f(x) = \frac{3}{8}(x - 4)^2 - 6$ **(f)** General form; $f(x) = 3x^2 + 6x - 19$ **(g)** Maximum: $f\left(\frac{7}{2}\right) = -\frac{9}{2}$; minimum: $f(4) = -6$; minimum: $f(-1) = -22$

43. **(a)**

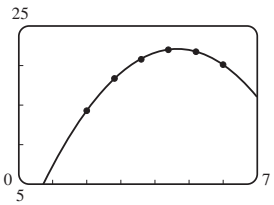


(b) h_2 is a shift of h_1 upward 80 units.

45. $f(x) = \frac{3}{32}x^2$ **47.** **(a)** $A(x) = x(x + 3)$ **(b)** 8 in. by 11 in.

49. **(a)** $f(x) = -1.1021x^2 + 10.249x - 1.7219$

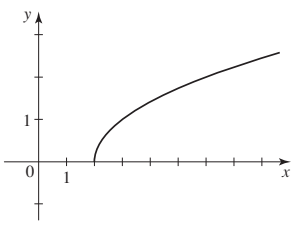
(b)



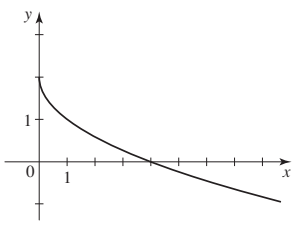
(c) 4.65 g

Chapter 5 Test ▪ page 472

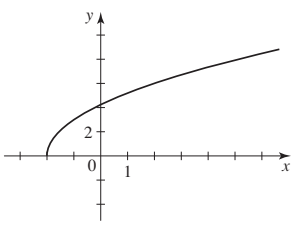
1. **(a)**



(b)



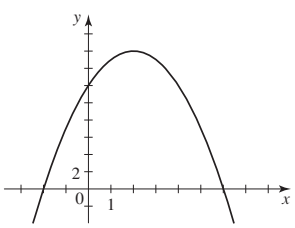
(c)



2. **(a)** $f(x) = -(x - 2)^2 + 16$

(b) $(2, 16)$ **(c)** $-2, 6$

(d)



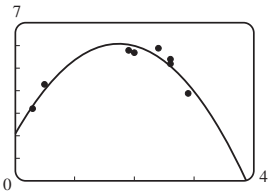
(e) Maximum: 16

3. $y = \frac{1}{2}(x - 4)^2 - 6$ **4.** **(a)** 49; two; $-2, \frac{1}{3}$

(b) 8; two; $3 \pm \sqrt{2}$ **(c)** -7; none

5. **(a)** $y = -1.3254x^2 + 4.598x + 2.0934$

(b)



6. **(a)** 2500 ft **(b)** 1000 ft

Chapter 6

6.1 Exercises ▪ page 489

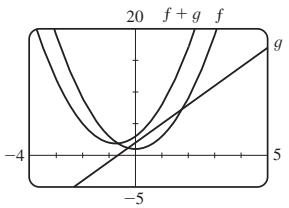
1. **(a)** $f(x) + g(x), f(x) - g(x)$; $12, -2$ **(b)** $f(x) \cdot g(x), \dfrac{f(x)}{g(x)}$; $18, \frac{1}{2}$ **2.** $8, -2, 15, \frac{3}{5}$ **3.** True **4.** False

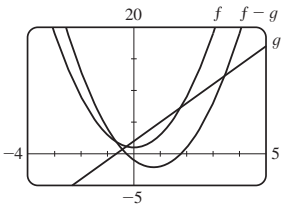
5.

t	0	1	2	3	4
$f(t)$	30	24	22	17	13
$g(t)$	294	312	331	352	370
$(f + g)(t)$	324	336	353	369	383
$(f - g)(t)$	−264	−288	−309	−335	−357

7. 3 **9.** -4 **11.** 0 **13.** **(a)** $(f + g)(x) = 2x^2 + 3x + 3$

(b) $(f - g)(x) = 2x^2 - 3x - 1$





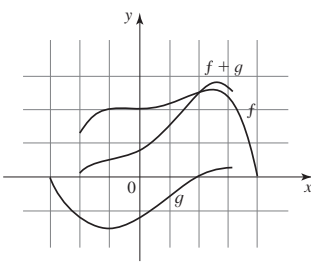
15. **(a)** $(f + g)(x) = x^2 + x - 3, (-\infty, \infty)$;
$(f - g)(x) = x^2 - x + 3, (-\infty, \infty)$ **(b)** $-1; 3$

17. **(a)** $(f + g)(x) = -6x^2 - 5x + 2, (-\infty, \infty)$;
$(f - g)(x) = 8x^2 - 5x - 2, (-\infty, \infty)$ **(b)** $-32; 20$

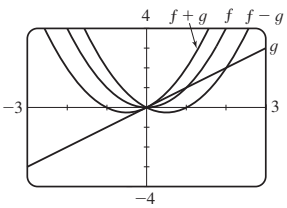
19. **(a)** $(f + g)(x) = \sqrt{7 + x} + \sqrt{3 + x}, [-3, \infty)$;
$(f - g)(x) = \sqrt{7 + x} - \sqrt{3 + x}, [-3, \infty)$

(b) $\sqrt{10} + \sqrt{6}; \sqrt{10} - \sqrt{6}$

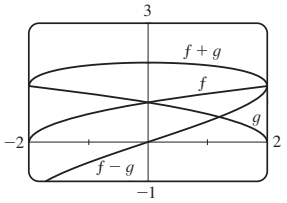
21.



23.



25.



27.

t	0	1	2	3	4
$f(t)$	−3	−0.5	3	5	6
$g(t)$	2	6	7	−5	−2
$(fg)(t)$	−6	−3	21	−25	−12
$(f/g)(t)$	−3/2	−1/12	3/7	−1	−3

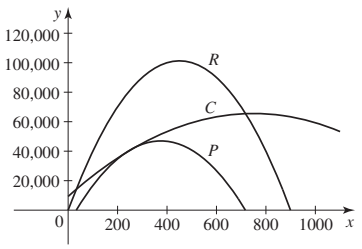
29. (a) $(fg)(x) = 3x^2 + 19x + 20, \{x \mid -\infty < x < \infty\}$; $(f/g)(x) = \dfrac{3x+4}{x+5}, \{x \mid x \neq -5\}$ **(b)** 70; $\frac{10}{7}$

31. (a) $(fg)(x) = x^3 - 3x^2, \{x \mid -\infty < x < \infty\}$; $(f/g)(x) = \dfrac{x-3}{x^2}, \{x \mid x \neq 0\}$ **(b)** 0; undefined

33. (a) $(fg)(x) = \dfrac{8}{x(x+4)}, \{x \mid x \neq 0, -4\}$; $(f/g)(x) = \dfrac{x+4}{2x}, \{x \mid x \neq 0, -4\}$ **(b)** $\frac{1}{4}$, 1

35. (a) $P(x) = -0.4x^2 + 300x - 9300$

(b) 32 cameras



37. $R(x) = 0.15x - 0.000002x^2$ **39. (a)** The monthly change in value from 2007 to 2008 **(b)** The percentage change in value from 2007 to 2008 **(c)** The percentage decrease in value levels off in the last months of the year. **41. (a)** The proportion of the population that smokes **(b)** The proportion of smokers has steadily decreased since 1970.

Algebra Checkpoint 6.2 ▪ page 500

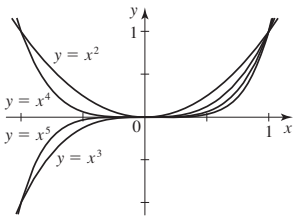
1. (a) 5^5 **(b)** $\dfrac{3^3}{5^3}$ **(c)** x^2 **(d)** $x^{3/2}$ **2. (a)** $5, -5$ **(b)** -3 **(c)** $\dfrac{\sqrt[3]{6}}{3}$ **(d)** 1.65669 **3. (a)** 25 **(b)** 27 **(c)** $2\sqrt[3]{2}$ **(d)** 7.70385 **4. (a)** $\pm\sqrt{\dfrac{x}{2}}$ **(b)** $\pm\dfrac{3}{2\sqrt{3A}}$ **(c)** $(9w)^5$ **(d)** $\left(\dfrac{2}{3}a_1\right)^{1/2.12}$

6.2 Exercises ▪ page 500

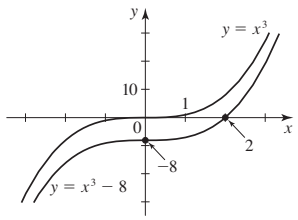
1. (a) x^2 **(b)** x^3 **2.** ① $y = x^4$; ② $y = x^2$; ③ $y = x^6$; ④ $y = x^3$; ⑤ $y = x^5$ **3.** directly; fifth; 7 **4.** $y = 4x^3$

5. (a) True **(b)** False

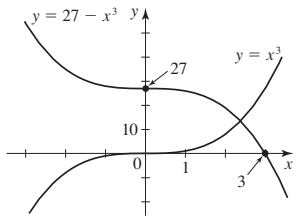
6.



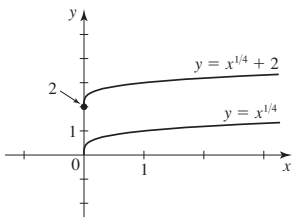
7. (a)



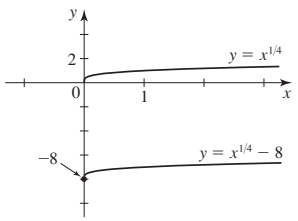
(b)



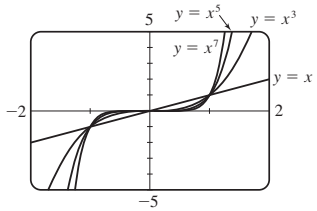
9. (a)



(b)

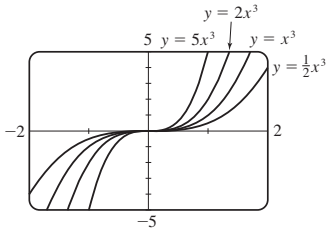


11.



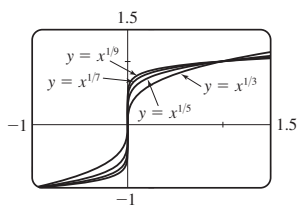
As c increases, the graph of $y = x^c$ becomes steeper when $|x| > 1$.

13.



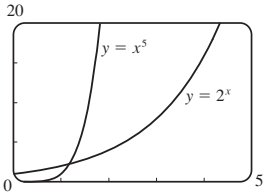
As c increases, the graph of $y = cx^3$ stretches vertically.

15.

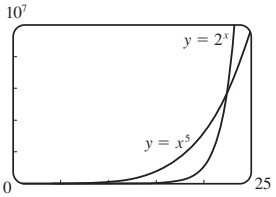


As c decreases, the graph of $y = x^c$ becomes less steep.

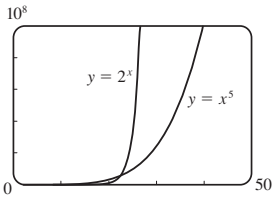
17. **(a)**



(b)



(c)



19. $V = kx^3$ **21.** $z = k\sqrt{y}$ **23.** $A = 5x^3$ **25.** $z = \frac{1}{4}\sqrt{t}$ **27.** 8 **29.** **(a)** $w = 2x^3$ **(b)** $w = 2.1x^3$ **(c)** No **31.** **(a)** $P = ks^3$ **(b)** $\frac{3}{250}$ **(c)** 324 W **33.** **(a)** $F = \frac{100}{9}s^2$ **(b)** No **35.** **(a)** $E = kT^4$ **(b)** 160,000

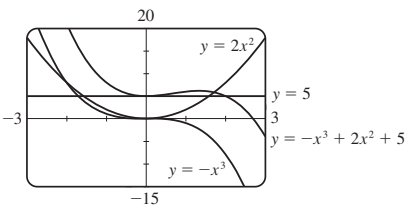
37. **(a)** $f^{-1}(D) = \sqrt{\dfrac{D}{0.033}}$ **(b)** The speed of a car with stopping distance D **(c)** 38.9 mi/h **39.** **(a)** 15.3 lb **(b)** 291.5 in.

Algebra Checkpoint 6.3 ▪ page 512

1. **(a)** $3(2x^2 + 1)$ **(b)** $r(r - 3)$ **(c)** $t^2(t^2 + 4)$ **(d)** $2y(y^2 - 3y + 2)$ **2.** **(a)** $(x + 9)(x - 9)$ **(b)** $(5m + 1)^2$ **(c)** $(t + 7)(t - 4)$ **(d)** $(3n + 2)(n - 1)$ **3.** **(a)** $3t^3(t + 2)(t - 2)$ **(b)** $(w^2 + 4)(w + 2)(w - 2)$ **(c)** $x^3(x + 7)(x - 4)$ **(d)** $(r^3 - 6)^2$ **4.** **(a)** $(x^2 + 1)(x + 3)$ **(b)** $(7z^3 + 5)(z^2 - 2)$

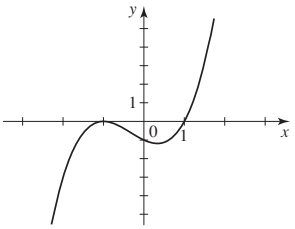
6.3 Exercises ▪ page 512

1. **(a)** power **(b)** $3x^4, -2x^3, 4; 3x^4; 4$ **2.** **(a)** factor; x **(b)** $2, -1, 3; 2, -1, 3$ **3.** positive, negative; test **4.** **(a)** ii **(b)** iii **(c)** i **(d)** iv **5.** No **7.**

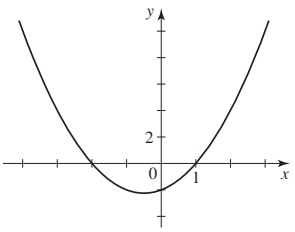


9. **(b)** $-1, 1$ **(c)** Negative on $(-\infty, -1)$ and $(-1, 1)$, positive on $(1, \infty)$

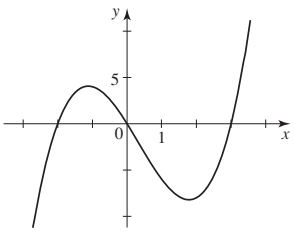
(d)



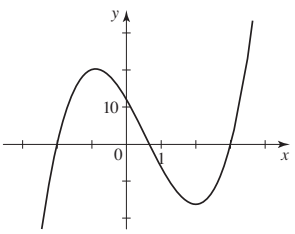
11.



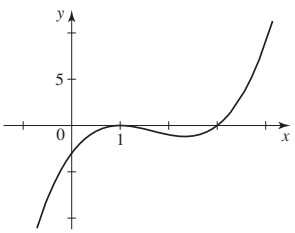
13.



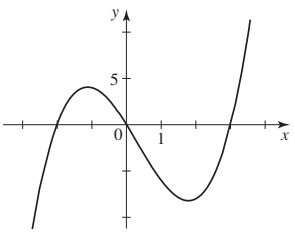
15.



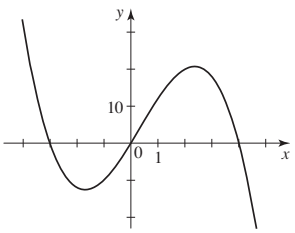
17.



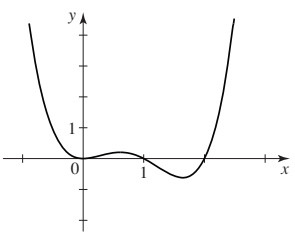
19. **(a)** $x(x - 3)(x + 2)$ **(b)**



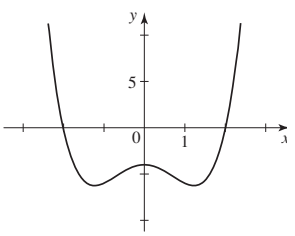
21. **(a)** $-x(x - 4)(x + 3)$ **(b)**



23. **(a)** $x^2(x - 2)(x - 1)$ **(b)**

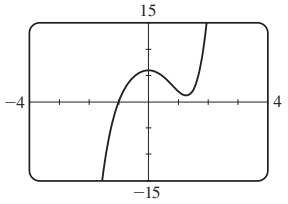


25. **(a)** $(x^2 + 1)(x - 2)(x + 2)$ **(b)**



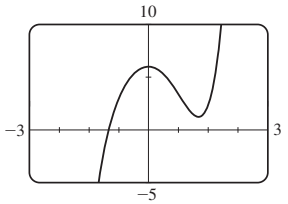
27. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$ **29.** $y \to \infty$ as $x \to \pm\infty$ **31.** $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$ **33.** III **35.** V **37.** VI

39. (a)



(b) Local maximum: $P(-0.53) = 13.13$; local minimum: $P(2.53) = -1.13$
(c) $-2, 2, 3$

41. (a)

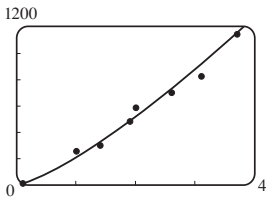


(b) Local maximum: $P(0) = 6$; local minimum: $P(1.26) = 1.24$ **(c)** -1

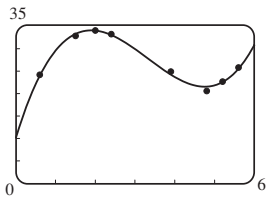
43. (a) 26 or 321 **(b)** No, \$5279 **45. (a)** It began snowing. **(b)** No **(c)** Just before midnight on Saturday night
47. (b) (0, 18) **(c)** 1728 in^3 **49.** 10.5 in. by 10.5 in. by 107.8 in. or 35.1 in. by 35.1 in. by 9.7 in.

6.4 Exercises ▪ page 522

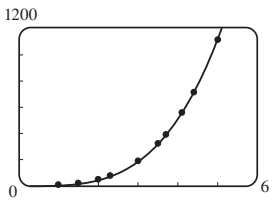
1. (a) scatter **(b)** power **2. (a)** exponential **(b)** power
5. (a) $y = 210.31x^{1.2994}$
(b)



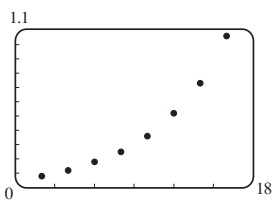
7. (a) $y = 1.07x^3 - 10.7x^2 + 29.0x + 10.2$
(b)



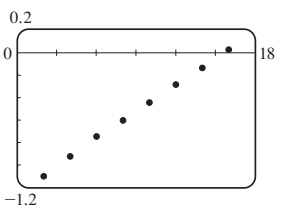
9. (a) Power
(b) $y = 4.00x^{3.50}$



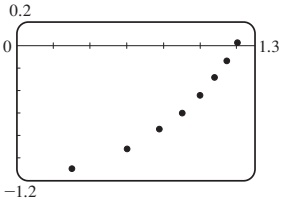
11. (a)



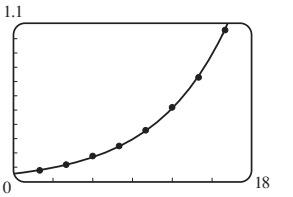
(b) Semi-log



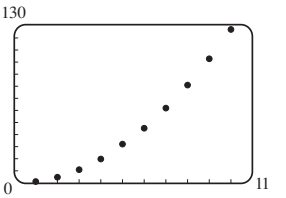
Log-log



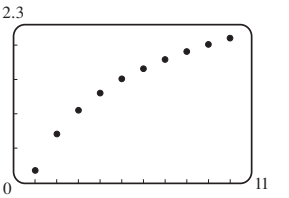
(c) Exponential
(d) $y = 0.0577(1.20023)^x$



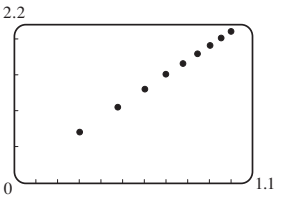
13. (a)



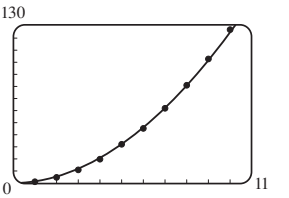
(b) Semi-log



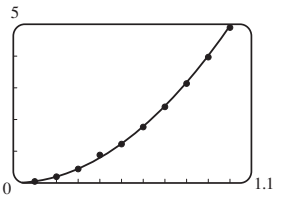
Log-log



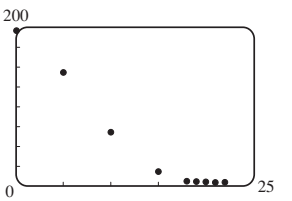
(c) Power
(d) $y = 1.40x^{1.9482}$



15. (a), (c)

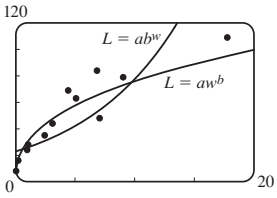


(b) $d = 4.96t^{2.0027}$
(d) 44.79 m
17. (a)

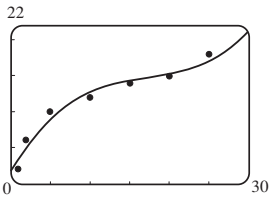


(b) $y = 302(0.8197)^x$
(c) $y = -0.00243x^4 + 0.135x^3 - 2.014x^2 - 4.06x + 199$
(d) Fourth-degree model

19. **(a)** $L = 30.6W^{0.395}$ **(b)** $L = 22.9(1.1263)^W$
(c) The power model is better. **(d)** 138 in.



21. **(a)** $y = 0.002x^3 - 0.105x^2 + 1.97x + 1.46$
(b) **(c)** 43 **(d)** 2.0 s



Algebra Checkpoint 6.5 ▪ page 532

1. **(a)** $\frac{1}{8}$ **(b)** 2 **(c)** 15,625 **(d)** $\frac{4}{25}$ **2.** **(a)** $\frac{1}{x}$ **(b)** $\frac{8}{y^6}$
(c) $-\frac{1}{b^3}$ **(d)** $\frac{1}{b^2}$ **3.** **(a)** $\frac{1}{8}$ **(b)** 2 **(c)** $\frac{1}{a^2}$ **(d)** a^2
4. **(a)** $\frac{1}{5}$ **(b)** $6, -6$ **(c)** $\frac{1}{8}$ **(d)** $\frac{1}{25}$

6.5 Exercises ▪ page 533

1. $y = x^{-1}$; $y = x^{-2}$ **2.** **(a)** inversely; fifth; 7 **(b)** $y = 4x^{-3}$
5. **(a)**

table **1**

x	$f(x)$
-0.1	-10^3
-0.01	-10^6
-0.001	-10^9
-0.00001	-10^{15}

table **2**

x	$f(x)$
0.1	10^3
0.01	10^6
0.001	10^9
0.00001	10^{15}

table **3**

x	$f(x)$
10	1.0×10^{-3}
50	8.0×10^{-6}
100	1.0×10^{-6}
100,000	1.0×10^{-15}

table **4**

x	$f(x)$
-10	-1.0×10^{-3}
-50	-8.0×10^{-6}
-100	-1.0×10^{-6}
$-100{,}000$	-1.0×10^{-15}

(b) As $x \to 0^-, f(x) \to -\infty$; as $x \to 0^+, f(x) \to \infty$
(c) As $x \to \infty, f(x) \to 0$; as $x \to -\infty, f(x) \to 0$

7. **(a)**

table **1**

x	$f(x)$
-0.1	10^4
-0.01	10^8
-0.001	10^{12}
-0.00001	10^{20}

table **2**

x	$f(x)$
0.1	10^4
0.01	10^8
0.001	10^{12}
0.00001	10^{20}

table **3**

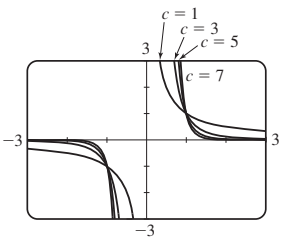
x	$f(x)$
10	1.0×10^{-4}
50	1.6×10^{-7}
100	1.0×10^{-8}
100,000	1.0×10^{-20}

table **4**

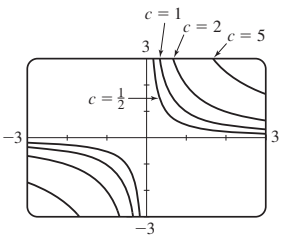
x	$f(x)$
-10	1.0×10^{-4}
-50	1.6×10^{-7}
-100	1.0×10^{-8}
$-100{,}000$	1.0×10^{-20}

(b) As $x \to 0^-, f(x) \to \infty$; as $x \to 0^+, f(x) \to \infty$
(c) As $x \to \infty, f(x) \to 0$; as $x \to -\infty, f(x) \to 0$

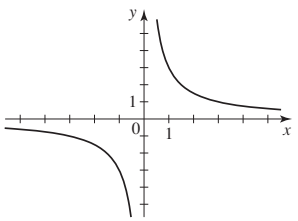
9.



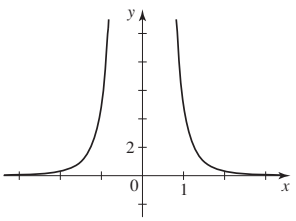
11.



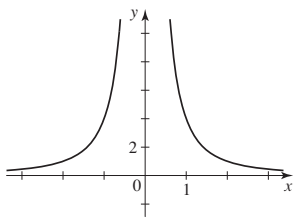
13.



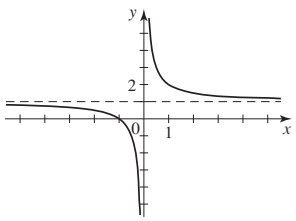
15.



17.



19.



21. $P = \dfrac{k}{T}$ **23.** $y = \dfrac{k}{\sqrt{t}}$ **25.** $z = \dfrac{6}{t}$

27. $s = \dfrac{45}{\sqrt{t}}$ **29.** 0.1875 **31.** **(a)** $P = \dfrac{26.169}{V}$

(b) 28.57 kPa **33.** **(a)** $L = \dfrac{7000}{d^2}$ **(b)** 0.7 dB

35. (a) $S = \dfrac{360}{n}$ (b) 72 lb **37.** (a) $H = \dfrac{k}{d^3}$
(b) 8 times the heat

Algebra Checkpoint 6.6 ▪ page 542

1. (a) $\dfrac{5}{x+1}$ (b) $\dfrac{1}{x+2}$ (c) $\dfrac{x+2}{x+1}$ (d) $\dfrac{x(2x+3)}{2x-3}$
2. (a) $\dfrac{2x}{x^2-1}$ (b) $\dfrac{3(x+2)}{x+3}$ (c) $\dfrac{1}{x^2+3x+2}$
(d) $\dfrac{x+2}{x(x+1)}$ **3.** (a) $3 - \dfrac{1}{x+2}$ (b) $1 - \dfrac{4}{x+1}$
(c) $8 + \dfrac{21}{x-2}$ (d) $2 - \dfrac{3}{x-1}$ **4.** (a) $x - 2 - \dfrac{16}{x-4}$
(b) $x^2 - 3x + 1 - \dfrac{1}{x+3}$

6.6 Exercises ▪ page 542

1. $\infty, -\infty$ **2.** a **3.** (a) factored (b) compound fraction
(c) standard **4.** (a) $-1, 2$ (b) $\frac{1}{3}$ (c) $-2, 3$ (d) 1
7. (a)

table **1**

x	$f(x)$
1.5	-3
1.9	-19
1.99	-199
1.999	-1999

table **2**

x	$f(x)$
2.5	5
2.1	21
2.01	201
2.001	2001

table **3**

x	$f(x)$
10	1.25
50	1.04
100	1.02
1000	1.002

table **4**

x	$f(x)$
-10	0.833
-50	0.962
-100	0.980
-1000	0.998

(b) As $x \to 2^-, f(x) \to -\infty$; as $x \to 2^+, f(x) \to \infty$
(c) As $x \to \pm\infty, f(x) \to 1$
9. (a)

table **1**

x	$f(x)$
1.5	-22
1.9	-430
1.99	$-40{,}300$
1.999	-4.003×10^6

table **2**

x	$f(x)$
2.5	-10
2.1	-370
2.01	$-39{,}700$
2.001	-3.997×10^6

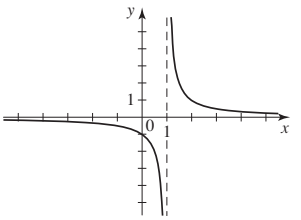
table **3**

x	$f(x)$
10	0.313
50	0.0608
100	0.0302
1000	0.00300

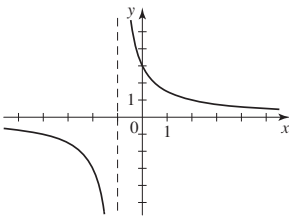
table **4**

x	$f(x)$
-10	-0.278
-50	-0.0592
-100	-0.0298
-1000	-0.00300

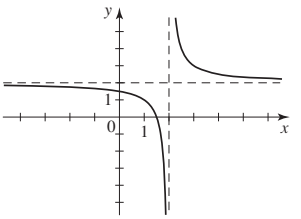
(b) As $x \to 2^-, f(x) \to \infty$; as $x \to 2^+, f(x) \to \infty$
(c) As $x \to \pm\infty, f(x) \to 0$ **11.** $1; -\frac{1}{4}$ **13.** $-1, 2; \frac{1}{3}$
15. $3; 3; x = 2; y = 2$ **17.** $-1, 1; \frac{1}{4}; x = -2, x = 2; y = 1$
19. $x = 2; y = 0$ **21.** No vertical asymptote; $y = 0$
23. $x = \frac{1}{2}, x = -1; y = 3$ **25.** $x = \frac{1}{3}, x = -2; y = \frac{5}{3}$
27.



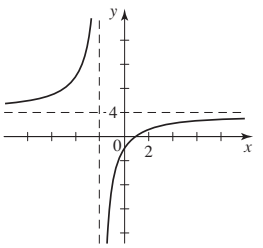
29.



31.

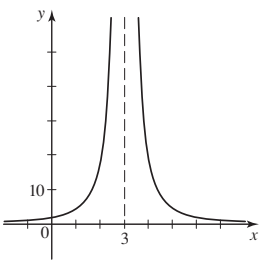


33.



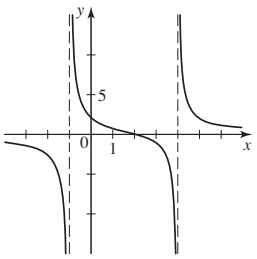
x-intercept 1
y-intercept -2
vertical $x = -2$
horizontal $y = 4$

35.



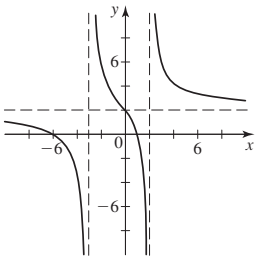
y-intercept 2
vertical $x = 3$
horizontal $y = 0$

37.



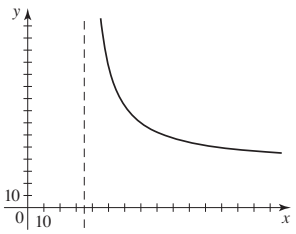
x-intercept 2
y-intercept 2
vertical $x = -1, x = 4$
horizontal $y = 0$

39.

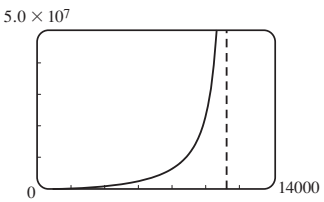


x-intercepts -6, 1
y-intercept 2
vertical $x = -3, x = 2$
horizontal $y = 2$

41. **(a)** $y = \dfrac{35x}{x - 35}$ **(b)** $y \to 35$ **(c)** $y \to \infty$



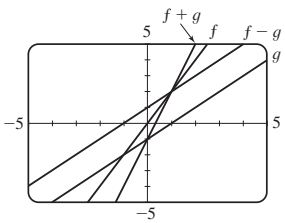
43. **(a)** 4.01×10^7 m
(b)



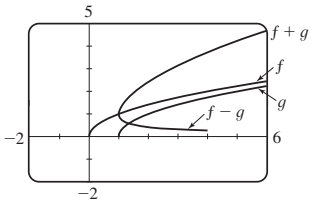
The escape velocity from the earth's gravitational pull

Chapter 6 Review Exercises ▪ page 548

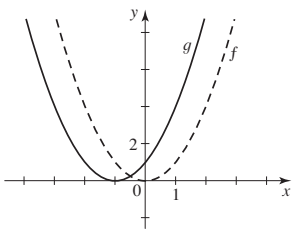
1. **(a)** $(f + g)(x) = 3x - 1, (-\infty, \infty); (f - g)(x) = x + 1, (-\infty, \infty); (fg)(x) = 2x^2 - 2x, (-\infty, \infty);$
$(f/g)(x) = \dfrac{2x}{x - 1}, (-\infty, 1) \cup (1, \infty)$ **(b)** 8; 4; 12; 3
(c)



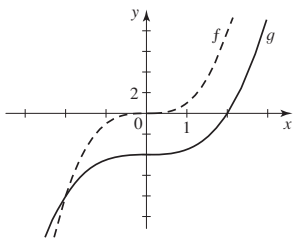
3. **(a)** $(f + g)(x) = \sqrt{x} + \sqrt{x - 1}, [1, \infty);$
$(f - g)(x) = \sqrt{x} - \sqrt{x - 1}, [1, \infty);$
$(fg)(x) = \sqrt{x}\sqrt{x - 1}, [1, \infty); (f/g)(x) = \dfrac{\sqrt{x}}{\sqrt{x - 1}}, (1, \infty)$
(b) 1; 1; 0; not defined
(c)



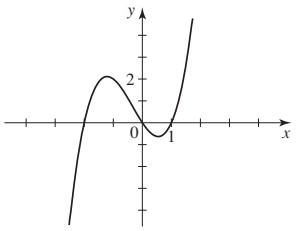
5.



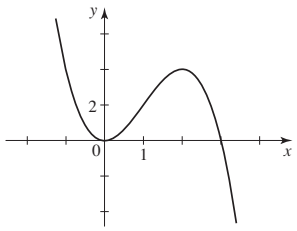
7.



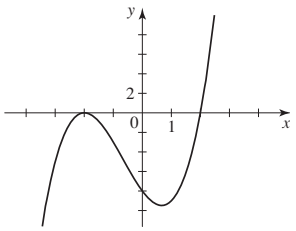
9. **(a)** $E = kv^2$ **(b)** $\frac{1}{4}$ **11.** **(a)** $L = kv^{1/3}$ **(b)** 18
13. 4
15.



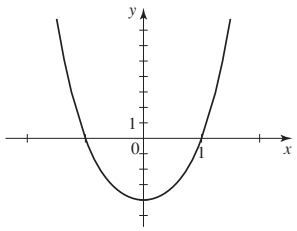
17.



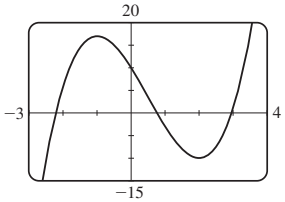
19.



21.

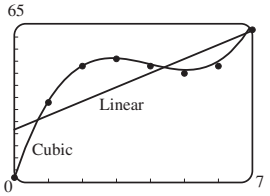


23. **(a)**



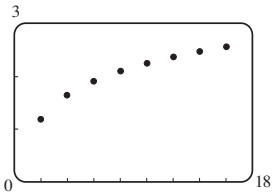
(b) Local maximum: $P(-1) = 17$; local minimum: $P(2) = -10$
(c) -2.22, 0.762, 2.96

25. **(a)** $y = 21.8 + 5.83x$;
$y = 0.965x^3 - 11.5x^2 + 42.3x + 1.79$
(b)

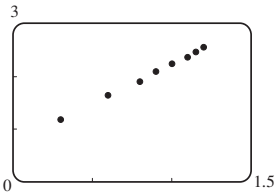


(c) Cubic

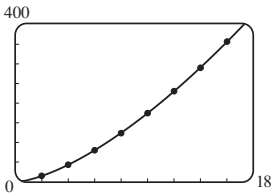
27. **(a)** Semi-log



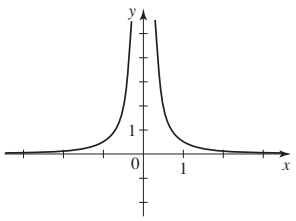
Log-log



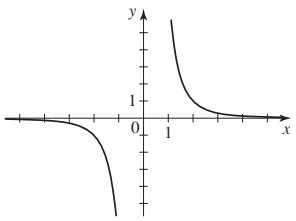
(b) Power model **(c)** $y = 5.3x^{1.5143}$



29.

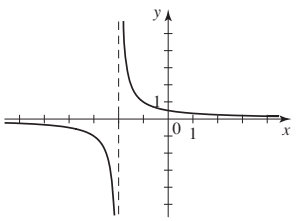


31.

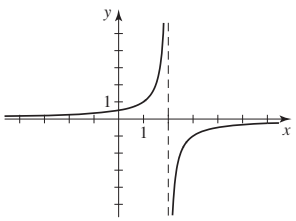


33. **(a)** $G = \dfrac{k}{r^3}$ **(b)** 300 **35.** 0.586

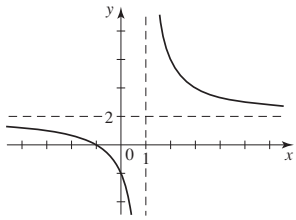
37.



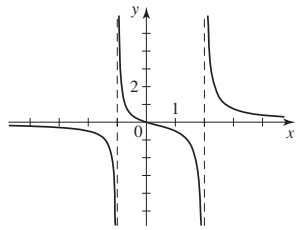
39.



41. **(a)** $-1; -2$
(b) $y = 2; x = 1$
(c)



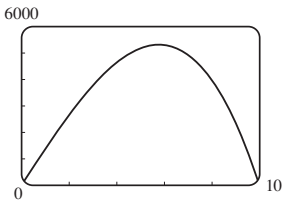
43. **(a)** 0; 0
(b) $y = 0; x = 2, x = -1$
(c)



45. **(a)** **(i)** Exponential **(ii)** Linear **(iii)** Power
(b) $g(x) = 4x$ **(c)** Linear; exponential; power
(d) $y = -1 + 3x$; $y = 0.800(2.5)^x$; $y = 2.00x^{1.32}$

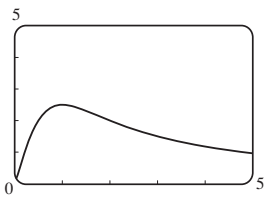
47. **(a)** Subtraction; \$390 **(b)** Average revenue per pen; average cost per pen

49. **(a)** $S = 1380x - 13.8x^3$ **(b)** [0, 10]
(c)



(d) 5.77 in.

51. **(a)** No; yes, $y = 0$ **(b)** $C(t) \to 0$ as $t \to \infty$
(c)



The maximum concentration was 2.5 mg/L at $t = 1$ h.

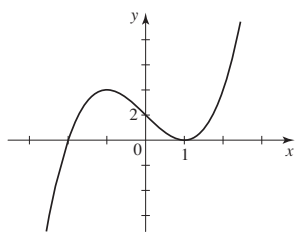
Chapter 6 Test ▪ page 553

1. $\sqrt{x} + x^2 - 1, [0, \infty)$; $\sqrt{x} - x^2 + 1, [0, \infty)$; $\sqrt{x}\,(x^2 - 1)$, $[0, \infty)$; $\dfrac{\sqrt{x}}{x^2 - 1}, [0, 1) \cup (1, \infty)$; 1; 1; 0; undefined

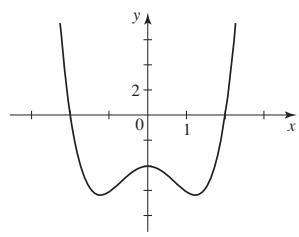
2. **(a)** I: h; II: f; III: g **(b)** (0, 0); (1, 1)

3. **(a)** $V = kx^3$ **(b)** 0.471 **(c)** 58.9 cm^3

4. **(a)**

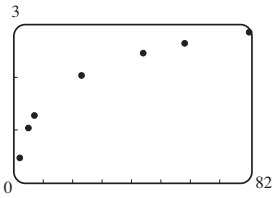


(b)

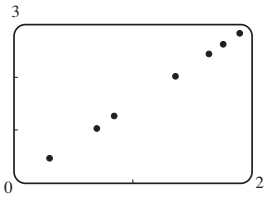


5. Local minima: $Q(-1.22) = -6.25$, $Q(1.22) = -6.25$; local maximum: $Q(0) = -4$

6. (a) Semi-log

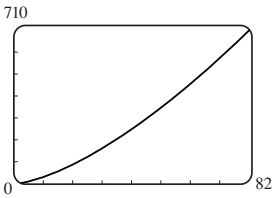


Log-log

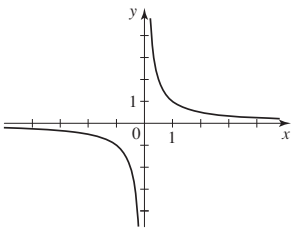


(b) Power model; $y = 1.02x^{1.49}$

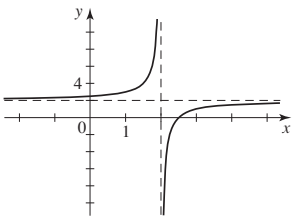
(c)



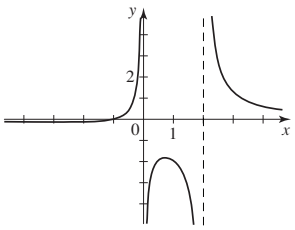
7. (a)



(b)



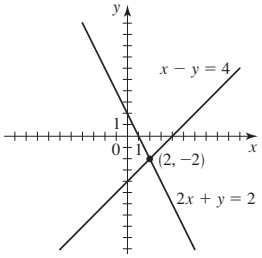
8. -1, none; $y = 0$, $x = 0$ and $x = 2$



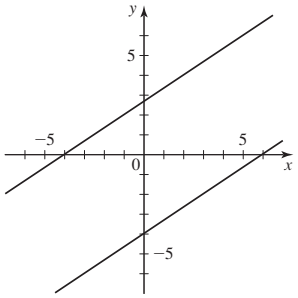
Chapter 7

7.1 Exercises ▪ page 576

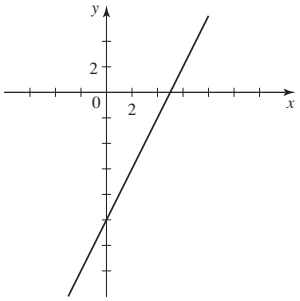
1. system **2.** x, y; equation **3.** (2, 1) **4.** substitution; elimination **5.** two, infinitely many **(a)** None **(b)** Infinitely many **6.** infinitely many; $1 - t$, $(1, 0)$, $(-3, 4)$, $(5, -4)$ **7. (a)** (3, 4) **8. (a)** $(-2, 2)$ and (4, 8) **9.** $(-2, 3)$ **11.** $(2, -2)$



13. No solution



15. Infinitely many solutions



17. (2, 2) **19.** (2, 1) **21.** $(3, -1)$ **23.** (2, 1) **25.** (3, 5) **27.** (1, 3) **29.** No solution **31.** $\left(x, \frac{1}{3}x - \frac{5}{3}\right)$ **33.** $\left(x, 3 - \frac{3}{2}x\right)$ **35.** $(-3, -7)$ **37.** (5, 10) **39.** No solution **41.** (3.87, 2.74) **43.** (61, 20) **45.** 12, 22 **47.** 5 dimes, 9 quarters **49.** 200 gallons regular, 80 gallons premium **51.** Plane's speed: 120 mi/h, wind speed: 30 mi/h **53.** 200 g Food A, 40 g Food B **55.** First: 25%, second: 10%

7.2 Exercises ▪ page 586

1. (a) 3 **(b)** 1 **(c)** 3, 1, 7 **(d)** 7, 1, 3 **2. (a)** $x + 3z = 1$ **(b)** -3; $4y - 5z = -4$ **5.** (1, 3, 2) **7.** (4, 0, 3)

9. $\begin{cases} x - 2y - z = 4 \\ -y - 4z = 4 \\ 2x + y + z = 0 \end{cases}$ **11.** $\begin{cases} 2x - y + 3z = 2 \\ x + 2y - z = 4 \\ 3y + 7z = 14 \end{cases}$

13. $(2, 1, -3)$ **15.** (1, 2, 1) **17.** (5, 0, 1) **19.** (0, 1, 2) **21.** $(1 - 3t, 2t, t)$ **23.** No solution **25.** No solution **27.** $(3 - t, -3 + 2t, t)$ **29.** $\left(2 - 2t, -\frac{2}{3} + \frac{4}{3}t, t\right)$ **31.** Short: \$30,000; intermediate: \$30,000; long: \$40,000 **33.** Corn: 250 acres; wheat: 500 acres; soybeans: 450 acres **35.** No solution **37.** Midnight Mango: 50; Tropical Torrent: 60; Pineapple Power: 30 **39.** A: 1500; B: 1200; C: 1000

7.3 Exercises ▪ page 599

1. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 2 & -3 \\ 0 & 2 & -1 & 3 \end{bmatrix}$ **2.** $\begin{cases} x + y + 2z = 4 \\ 3x + z = 2 \\ 5x + 2y - z = -2 \end{cases}$

3. (a) x, y **(b)** Dependent **(c)** $x = 3 + t$, $y = 5 - 2t$, $z = t$ **4. (a)** $x = 2$, $y = 1$, $z = 3$ **(b)** $x = 2 - t$, $y = 1 - t$, $z = t$ **(c)** No solution **5. (a)** 3×2 **(b)** 0 **7. (a)** 2×1 **(b)** 35 **9. (a)** 3×1 **(b)** 3 **11.** (1, 1, 2) **13.** (1, 0, 1) **15.** $(-1, 1, 2)$ **17.** $(0, 2, -1)$ **19.** $(-1, 0, 1)$ **21.** $(-1, 5, 0)$ **23.** $(10, 3, -2)$ **25.** Inconsistent **27.** Dependent, $(2 - 3t, 3 - 5t, t)$ **29.** Dependent, $(2 + 3t, t, 4)$ **31.** $(-2, 1, 3)$ **33.** No solution **35.** $(-9, 2, 0)$ **37.** $(5 - t, -3 + 5t, t)$ **39.** $(-2, 1, 0, 3)$ **41.** VitaMax: 2; Vitron: 1; VitaPlus: 2 **43.** Running: 5 mi; swimming: 2 mi; cycling: 30 mi **45.** Impossible

7.4 Exercises ■ page 607

1. Numerical **2.** Categorical **3.** Categorical **4.** Numerical

5. $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 10 & 3 \\ 0 & 4 & 2 \end{bmatrix}$ **7. (a)** 1 **(b)** 16 **(c)** 23 **(d)** 50 **9. (a)** 3 **(b)** 15 **(c)** 12 **(d)** 50

11. (a) $\begin{bmatrix} 0.40 & 0.13 & 0.11 \\ 0.32 & 0.48 & 0.15 \\ 0.28 & 0.39 & 0.74 \end{bmatrix}$ **(b)** The proportion of macaroni that is sold on Monday; the proportion of macaroni that is sold on Wednesday **13. (a)** $\begin{bmatrix} 19 & 23 & 17 \\ 13 & 17 & 12 \\ 6 & 5 & 12 \end{bmatrix}$ **(b)** 23

15. (a) $\begin{bmatrix} 200 & 80 & 60 \\ 160 & 300 & 80 \\ 140 & 240 & 400 \end{bmatrix}$ **(b)** 0.28 **17. (a)** $\begin{bmatrix} 32{,}000 \\ 42{,}000 \\ 44{,}000 \end{bmatrix}$ **(b)** \$42,000.00 **(c)** \$118,000

7.5 Exercises ■ page 616

1. dimension **2.** columns; rows **3.** (b), (c)

4. $\begin{bmatrix} 4 & \boxed{9} & -7 \\ 7 & -7 & \boxed{0} \\ \boxed{4} & -5 & -5 \end{bmatrix}$ **5.** (a), (b) **6.** A must be a square matrix.

7. $\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$ **9.** $\begin{bmatrix} 3 & 6 \\ 12 & -3 \\ 3 & 0 \end{bmatrix}$ **11.** Impossible

13. $\begin{bmatrix} 5 & 2 & 1 \\ 7 & 10 & -7 \end{bmatrix}$ **15. (a)** $\begin{bmatrix} 5 & -2 & 5 \\ 1 & 1 & 0 \end{bmatrix}$ **(b)** Impossible

17. (a) $\begin{bmatrix} 10 & -25 \\ 0 & 35 \end{bmatrix}$ **(b)** Impossible **19. (a)** Impossible **(b)** $\begin{bmatrix} 14 & -14 \end{bmatrix}$ **21. (a)** $\begin{bmatrix} -4 & 7 \\ 14 & -7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -8 \\ 4 & -17 \end{bmatrix}$

23. (a) $\begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 \\ 8 \\ -1 \end{bmatrix}$ **25. (a)** $\begin{bmatrix} 4 & -45 \\ 0 & 49 \end{bmatrix}$ **(b)** $\begin{bmatrix} 8 & -335 \\ 0 & 343 \end{bmatrix}$ **27. (a)** $\begin{bmatrix} 4690 & 1690 & 13{,}210 \end{bmatrix}$ **(b)** \$1690 **(c)** \$19,590 **29. (a)** $\begin{bmatrix} 105{,}000 & 58{,}000 \end{bmatrix}$ **(b)** Total amount (in ounces) of tomato sauce produced; total amount (in ounces) of tomato paste produced.

7.6 Exercises ■ page 625

1. (a) identity **(b)** A, A **(c)** inverse

2. (a) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ **(d)** $(-1, 3)$ **7.** $\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$ **9.** $\begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}$ **11.** No inverse

13. $\begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & -6 \end{bmatrix}$ **15.** No inverse

17. $\begin{bmatrix} -\frac{9}{2} & -1 & 4 \\ 3 & 1 & -3 \\ \frac{7}{2} & 1 & -3 \end{bmatrix}$ **19.** $(12, -8)$ **21.** $(126, -50)$

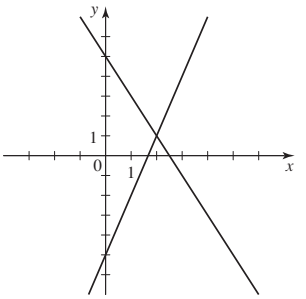
23. $(-38, 9, 47)$ **25.** $(-20, 10, 16)$

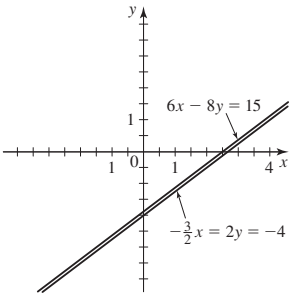
27. (a) $\begin{cases} x + y + 2z = 675 \\ 2x + y + z = 600 \\ x + 2y + z = 625 \end{cases}$

(b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 675 \\ 600 \\ 625 \end{bmatrix}$ **(c)** Standard: \$125; deluxe: \$150; leather: \$200

Chapter 7 Review Exercises ■ page 630

1. $(2, 1)$ **3.** No solution





5. $(2, 1)$ **7.** x = any number; $y = -12 + 3x$ **9.** $(1, 1, 2)$ **11.** $(0, 1, 2)$ **13.** $(4 - 2t, 2 - t, t)$ **15.** No solution **17.** $(1, -2, 3)$ **19.** Impossible **21.** $\begin{bmatrix} 4 & 18 \\ 4 & 0 \\ 2 & 2 \end{bmatrix}$ **23.** Impossible

25. $\begin{bmatrix} -3.5 & 10 \\ 1 & -4.5 \end{bmatrix}$ **27.** $\begin{bmatrix} 30 & 22 & 2 \\ -9 & 1 & -4 \end{bmatrix}$

29. $\begin{bmatrix} -0.5 & 5.5 \\ 3.75 & -1.5 \\ -0.5 & 1 \end{bmatrix}$ **33.** $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

35. $\begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$ **37.** $(65, 154)$ **39.** $\left(-\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$

41. (a)–(d), (f) $(8, 2)$ **(e)** $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix}$

43. (a) $\begin{cases} x = y + 4 \\ x + y = 22 \end{cases}$ **(b)** Kieran is 13; Siobhan is 9.

45. Haddock: 160 lb, sea bass: 340 lb, red snapper: 60 lb

47. **(a)** $\begin{cases} x + y = 18 \\ 20x + 50y = 600 \end{cases}$ **(b)** $\begin{bmatrix} 1 & 1 \\ 20 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 600 \end{bmatrix}$
(c)

$A^{-1} = \begin{bmatrix} \frac{5}{3} & -\frac{1}{30} \\ -\frac{2}{3} & \frac{1}{30} \end{bmatrix}$; $X = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$; ten \$20 bills and eight \$50 bills

Chapter 7 Test ▪ page 634

1. $(3, -1)$ **2.** Wind speed: 60 mph; airplane speed: 300 mph
3. **(a)** $(2, 1, -1)$ **(b)** Neither **4.** **(a)** No solution
(b) Inconsistent **5.** **(a)** $\left(\frac{1}{7} + \frac{1}{7}t, \frac{2}{7} + \frac{9}{7}t, t\right)$ **(b)** Dependent
6. Coffee: \$1.50; juice: \$1.75; doughnut: \$0.75
7. **(a)** Impossible **(b)** $\begin{bmatrix} 5 & -1.5 \\ -1.5 & 0 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 6 & 0 & -6 \end{bmatrix}$ **(d)** Impossible **(e)** Impossible

(f) $\begin{bmatrix} 5 \\ 0 \\ -6 \end{bmatrix}$ **8.** **(a)** $\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$; (70, 90)

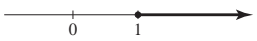
Algebra Toolkit **A**

A.1 Exercises ▪ page T5

2. **(a)** ba; Commutative **(b)** $(a + b) + c$; Associative
(c) $ab + ac$; Distributive **3.** **(a)** factor; factors **(b)** term; terms **4.** **(a)** Distributive Property; a; $ab + ac$
(b) Distributive Property; a; a; $a(b + c)$ **5.** **(a)** 50
(b) $0, -10, 50$ **(c)** $0, -10, 50, \frac{22}{7}, 0.583, 1.2\overline{3}, -\frac{1}{3}$
(d) $\sqrt{7}, \sqrt[3]{2}$ **6.** **(a)** $11, \sqrt{16}, \frac{15}{3}$ **(b)** $-11, 11, \sqrt{16}, \frac{15}{3}$
(c) $1.001, 0.\overline{3}, -11, 11, \frac{13}{15}, 3.14, \sqrt{16}, \frac{15}{3}$ **(d)** $-\pi$ **7.** 1
8. 5 **9.** -2 **10.** -301 **11.** Commutative Property
12. Commutative Property **13.** Associative Property
14. Distributive Property **15.** Distributive Property
16. Distributive Property **17.** Commutative Property
18. Distributive Property **19.** 2 **20.** -1 **21.** 2 **22.** 2
23. 36 **24.** 170 **25.** -30 **26.** -3 **27.** $3x + 21$
28. $8a - 16$ **29.** $6a - 18c$ **30.** $7x + 14y$
31. $-12ax + 8x$ **32.** $-18abc + 15bcd$
33. $8mnp + 12mnpq - 8mnq$ **34.** $-6pqs + 4pqrs + 10prs$
35. $3(x + 3)$ **36.** $6(y - 2)$ **37.** $-2(a + b)$
38. $-20(x - 2y)$ **39.** $b(a - 6)$ **40.** $2a(x + y)$
41. $-5b(a - 2c)$ **42.** $-30z(x + 3y)$
43. $2s(qr - 3qt + 6rt)$ **44.** $\frac{1}{8}ab(4x + 2y - z)$

A.2 Exercises ▪ page T12

2. a is to the left of b **3.** **(a)** $\{x \mid 2 < x < 7\}$ **(b)** $(2, 7)$
5. Absolute value; positive **6.** distance **7.** **(a)** $<$ **(b)** $>$
(c) $=$ **8.** **(a)** $<$ **(b)** $>$ **(c)** $=$ **9.** False **10.** True
11. True **12.** False **13.** False **14.** True **15.** False
16. True

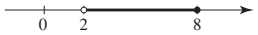
17.


18.


19.


20.

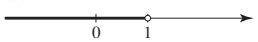

21. $x \le 1$ **22.** $x < -2$ **23.** $x > -\frac{3}{2}$ **24.** $x \ge 5$
25. **(a)** $x > 0$ **(b)** $t < 4$ **(c)** $a \ge -1$ **(d)** $-5 < x < \frac{1}{3}$
(e) $|p - 3| \le 5$ **26.** **(a)** $y < 0$ **(b)** $z > 1$ **(c)** $b \ge 8$
(d) $0 < w \le 17$ **(e)** $|y - 7| \ge 2$
27. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8\}$ **(b)** $\{2, 4, 6\}$
28. **(a)** $\{2, 4, 6, 7, 8, 9, 10\}$ **(b)** $\{8\}$
29. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **(b)** $\{7\}$
30. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **(b)** $\varnothing$
31. **(a)** $\{x \mid x \le 5\}$ **(b)** $\{x \mid -1 < x < 4\}$
32. **(a)** $\{x \mid x \ge -2\}$ **(b)** $\{x \mid -1 < x \le 5\}$
33. **(a)** All numbers greater than -3 and less than 0
(b) $\{x \mid -3 < x < 0\}$ **(c)**

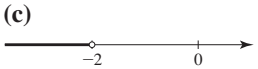

34. **(a)** All numbers greater than 2 and less than or equal to 8
(b) $\{x \mid 2 < x \le 8\}$ **(c)**


35. **(a)** All numbers greater than or equal to 2 and less than 8
(b) $\{x \mid 2 \le x < 8\}$ **(c)**


36. **(a)** All numbers greater than or equal to -6 and less than or equal to $-\frac{1}{2}$
(b) $\{x \mid -6 \le x \le -\frac{1}{2}\}$ **(c)**

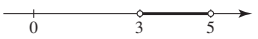

37. **(a)** All numbers greater than or equal to 2
(b) $\{x \mid x \ge 2\}$ **(c)**


38. **(a)** All numbers less than 1
(b) $\{x \mid x < 1\}$ **(c)**


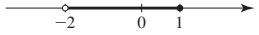
39. **(a)** All numbers less than -2
(b) $\{x \mid x < -2\}$ **(c)**


40. **(a)** All numbers greater than or equal to -3
(b) $\{x \mid x \ge -3\}$ **(c)**


41. $[1, 2]$


42. $(3, 5)$


43. $(-2, 1]$



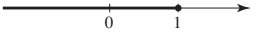
44. $(-5, 2)$



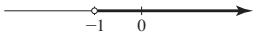
45. $[5, \infty)$



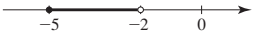
46. $(-\infty, 1]$



47. $(-1, \infty)$

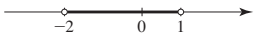


48. $[-5, -2)$

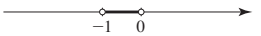


49. **(a)** $\{x \mid -3 \le x \le 5\}$ **(b)** $[-3, 5]$
50. **(a)** $\{x \mid -3 < x \le 5\}$ **(b)** $(-3, 5]$
51. **(a)** $\{x \mid 0 \le x < 2\}$ **(b)** $[0, 2)$
52. **(a)** $\{x \mid -2 < x \le 0\}$ **(b)** $(-2, 0]$

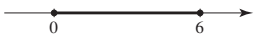
53.



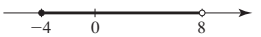
54.



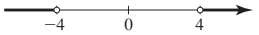
55.



56.



57.



58.



59. **(a)** 9 **(b)** 12 **(c)** **(d)** **60.** **(a)** 100 **(b)** 5 **(c)** 12 **(d)** **61.** 5 **62.** 4 **63.** **(a)** 15 **(b)** 24 **(c)** $\frac{7}{4}$ **64.** **(a)** $\frac{18}{35}$ **(b)** 19 **(c)** 0.8

A.3 Exercises ▪ page T18

1. 5^6 **2.** base; exponent **3.** add; 3^9 **4.** subtract; 3^3 **5.** multiply; 3^8 **6.** **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{8}$ **(c)** 2 **7.** **(a)** 81 **(b)** $\frac{1}{243}$ **8.** **(a)** 64 **(b)** $\frac{1}{16}$ **9.** **(a)** 25 **(b)** -25 **10.** **(a)** 10,000 **(b)** $-10{,}000$ **11.** **(a)** -216 **(b)** -216 **12.** **(a)** -2187 **(b)** -2187 **13.** **(a)** 1 **(b)** -1 **14.** **(a)** 1 **(b)** 1 **15.** **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{27}$ **16.** **(a)** $\frac{1}{10}$ **(b)** $\frac{1}{1{,}000{,}000}$ **17.** 125 **18.** 32 **19.** 64 **20.** 1 **21.** 1 **22.** -1 **23.** -9 **24.** 9 **25.** 1000 **26.** 27 **27.** $\frac{1}{9}$ **28.** 25 **29.** 1 **30.** $\frac{1}{2}$ **31.** $\frac{9}{16}$ **32.** $\frac{27}{8}$ **33.** $\frac{1}{100}$ **34.** $\frac{1}{2}$ **35.** x^{10} **36.** $\frac{1}{x^4}$ **37.** $8y^6$ **38.** $64x^2$ **39.** a^{18} **40.** $16a^{20}$ **41.** $-27a^3$ **42.** $16w^4$ **43.** y^3 **44.** $\frac{1}{x^4}$ **45.** a^6 **46.** z^4 **47.** $\frac{1}{64a^3}$ **48.** $\frac{3x^2}{4}$ **49.** $\frac{1}{24z^4}$ **50.** $\frac{1}{32}$ **51.** $\frac{81u^{20}}{16v^{12}}$ **52.** $-\frac{8x^6}{y^9}$

A.4 Exercises ▪ page T23

1. $5^{1/3}$ **2.** $\sqrt{5}$ **3.** Yes **4.** $4^{3/2} = (4^{1/2})^3 = 8$; $4^{3/2} = (4^3)^{1/2} = 8$ **5.** $\frac{2}{3}$ **6.** $-\frac{1}{3}$ **7.** $5^{-1/2}$ **8.** $7^{2/3}$ **9.** $\sqrt[3]{4^2}$ **10.** $\frac{1}{\sqrt{11^3}}$ **11.** $5^{3/5}$ **12.** $\frac{1}{\sqrt{2}}$ **13.** $\sqrt[5]{a^2}$ **14.** $x^{-5/2}$ **15.** **(a)** 4 **(b)** 2 **(c)** $\frac{1}{2}$ **16.** **(a)** 8 **(b)** -4 **(c)** 2 **17.** **(a)** $\frac{2}{3}$ **(b)** 4 **(c)** $\frac{1}{2}$ **18.** **(a)** 14 **(b)** 4 **(c)** 4 **19.** **(a)** 7 **(b)** -2 **(c)** $\frac{1}{5}$ **20.** **(a)** 2 **(b)** $-\frac{1}{3}$ **(c)** $\frac{3}{2}$ **21.** **(a)** $\frac{3}{2}$ **(b)** 4 **(c)** $\frac{125}{512}$ **22.** **(a)** $\frac{1}{4}$ **(b)** $\frac{1}{81}$ **(c)** 4 **23.** x^2 **24.** y^2 **25.** $2b^{1/2}$ **26.** $9a^{4/3}$ **27.** w **28.** 25 **29.** $\frac{1}{2}r^{1/8}$ **30.** $\frac{k}{3}$ **31.** $\frac{1}{x^3y^2}$ **32.** u^2v **33.** $\frac{r^6}{16q^3}$ **34.** $\frac{s^4}{64t}$ **35.** x^4 **36.** $5^{1/3}y^{4/3}$ **37.** $3st^2$ **38.** $2uv^2$ **39.** $y^{3/2}$ **40.** $b^{5/4}$ **41.** $10x^{7/12}$ **42.** $2a^{7/6}$ **43.** x **44.** $2x^{-2/3}$ **45.** $2x^{1/6}$ **46.** $\frac{1}{2}x^{1/4}y^{1/4}$ **47.** $y^{1/2}$ **48.** $s^{5/4}$

Algebra Toolkit **B**

B.1 Exercises ▪ page T31

1. like; $4a + b + 5$ **2.** like; $xy + 8b - 5c + 2$ **3.** **(a), (c)** **4.** $x^2 + 5x + 6$ **5.** $A^2 + 2AB + B^2$; $4x^2 + 12x + 9$ **6.** $A^2 - B^2$; $25 - x^2$ **7.** **(a)** A number times three, minus five; 4 **(b)** A number minus five, times three; -6 **8.** **(a)** A number minus two, divided by five; $\frac{1}{5}$ **(b)** A number minus two-fifths; $\frac{13}{5}$ **9.** **(a)** A number squared, plus one; 10 **(b)** A number plus one, squared; 16 **10.** **(a)** A number plus 5, times two; 16 **(b)** A number plus 2, times five; 25 **11.** **(a)** $2ax$, $-10a$, 1 **(b)** 15 **12.** **(a)** axy, b **(b)** 10 **13.** **(a)** $(xy)^2$, a^2 **(b)** 37 **14.** **(a)** $\frac{x}{2}$, b, 5 **(b)** 8 **15.** $7x + 5$ **16.** $-x - 3$ **17.** $7x^2 - 3x + 6$ **18.** $-2x^2 + 4x - 3$ **19.** $9aw + w$ **20.** $3by + b + 5$ **21.** $-2ax + 7ay - 6a$ **22.** $-bu + 2cv + 7$ **23.** $ax - ay - bx + by$ **24.** $ax - ay + bx - by$ **25.** $uw + au + bu + vw + av + bv$ **26.** $au + av + aw + bu + bv + bw$ **27.** $x^2 + x - 12$ **28.** $y^2 + 4y - 5$ **29.** $r^2 + 2r - 15$ **30.** $s^2 + 6s - 16$ **31.** $21t^2 - 26t + 8$ **32.** $8s^2 + 18s - 5$ **33.** $6x^2 + 7x - 5$ **34.** $14y^2 - 13y + 3$ **35.** $x^2 + 6x + 9$ **36.** $x^2 - 4x + 4$ **37.** $9x^2 + 24x + 16$ **38.** $1 - 4y + 4y^2$ **39.** $4u^2 + 4uv + v^2$ **40.** $x^2 - 6xy + 9y^2$ **41.** $4x^2 + 12xy + 9y^2$ **42.** $r^2 - 4rs + 4s^2$ **43.** $x^4 + 2x^2 + 1$ **44.** $4 + 4y^3 + y^6$ **45.** $x^2 - 25$ **46.** $y^2 - 9$ **47.** $9x^2 - 16$ **48.** $4y^2 - 25$ **49.** $x^2 - 9y^2$ **50.** $4u^2 - v^2$ **51.** $x - 4$ **52.** $a - b^2$ **53.** $-x^4 + x^2 - 2x + 1$ **54.** $-x^4 - 3x^2 - 4$ **55.** $4x^2 + 4xy + y^2 - 9$ **56.** $x^2 + 2xy + y^2 - z^2$ **57.** $6x^3 - 10x^2 + 4x + 2$ **58.** $6x^4 + 3x^3 + 3x^2 - 2x - 1$ **59.** $3x^6 + 2x - 1$ **60.** $4x^3 - 5x^2 + 2x - 4$ **61.** $x^3 + 4x^2 + 7x + 6$ **62.** $2x^3 + x^2 + 1$ **63.** $2x^3 - 7x^2 + 7x - 5$ **64.** $2x^3 - 5x^2 - x + 1$

B.2 Exercises ▪ page T37

1. **(a)** 3 **(b)** $2x^5$, $6x^4$, $4x^3$ **(c)** $2x^3$ **(d)** $2x^3(x^2 + 3x + 2)$ **2.** 10; 7; 2, 5; $(x + 2)(x + 5)$ **3.** $(A + B)(A - B)$; $(2x + 5)(2x - 5)$ **4.** $(A + B)^2$; $(x + 5)^2$ **5.** $5(a - 4)$ **6.** $-3(b - 4)$ **7.** $15x^3(2 + x)$ **8.** $6x(2x^2 + 3)$ **9.** $-2x(x^2 - 8)$ **10.** $3y^3(2y - 5)$ **11.** $ab(5 - 8c)$ **12.** $2x^2(x^2 + 2x - 7)$ **13.** $xy(2x - 6y + 3)$ **14.** $-7xy^2(x^3 - 2y - 3y^2)$ **15.** $(y - 6)(y + 9)$ **16.** $(z + 2)(z - 3)$ **17.** $(x - 1)(x + 3)$ **18.** $(x - 1)(x - 5)$ **19.** $(x - 3)(x + 5)$ **20.** $(x - 8)(x - 6)$ **21.** $(y - 5)(y - 3)$ **22.** $(z - 2)(z + 8)$ **23.** $(2x - 7)(x + 1)$ **24.** $(3x - 1)(x - 5)$ **25.** $(5x + 3)(x - 2)$ **26.** $(2x - 1)(x + 4)$ **27.** $9(x - 5)(x + 1)$

28. $(2x+1)(4x+3)$ **29. (a)** $(x+2)(x+6)$
(b) $(3x+4)(3x+8)$ **30. (a)** $(2x-1)(x+3)$
(b) $(2a+2b-1)(a+b+3)$ **31.** $(x+6)(x-6)$
32. $(y+10)(y-10)$ **33.** $(3a+4)(3a-4)$
34. $(2x+5)(2x-5)$ **35.** $(7+2y)(7-2y)$
36. $(2t+3s)(2t-3s)$ **37.** $(x+5)(x+1)$
38. $(4x+5)(4x-1)$ **39.** $4ab$ **40.** $\dfrac{4}{x}$ **41.** $(x+6)^2$
42. $(y+5)^2$ **43.** $(t-3)^2$ **44.** $(t+5)^2$ **45.** $(4z-3)^2$
46. $(5u-1)^2$ **47.** $(2w+y)^2$ **48.** $(r-3s)^2$
49. $3x(x+3)(x-3)$ **50.** $x(x+1)^2$ **51.** $x^2(x+3)(x-1)$
52. $2x(x+2)^2$ **53.** $x^2y^3(x+y)(x-y)$
54. $2x^2y(3x+y)(3x-y)$ **55.** $3(x-1)(x+2)$
56. $x(x+1)$ **57.** $(x+4)(x^2+1)$ **58.** $(3x-1)(x^2+2)$
59. $(2r+1)(r^2-3)$ **60.** $(3u+1)(1-3u^2)$
61. $(x+1)(x^2+1)$ **62.** $(x+1)(x^4+1)$
63. $(y-3)(y-2)(y+2)$ **64.** $(y-1)(y^2+1)$
65. $(t+2)(2t^2+1)$ **66.** $(3s+5)(s^2-2)$

B.3 Exercises ▪ page T43

1. (a), (c) **2.** numerator, denominator; $\dfrac{x+1}{x+3}$ **3. (a)** False **(b)** True **4. (a)** numerators, denominators; $\dfrac{2x}{(x+1)(x+3)}$ **(b)** invert; $\dfrac{3(x+2)}{x(x+5)}$ **5. (a)** 3 **(b)** $x(x+1)^2$ **(c)** $\dfrac{-x^2-x+1}{x(x+1)^2}$ **6. (a)** False **(b)** True **7.** 5 **8.** $-\frac{1}{3}$
9. $\dfrac{2}{x}$ **10.** $\dfrac{9x^2}{2}$ **11.** $\dfrac{5y}{10+y}$ **12.** $\dfrac{14t-1}{7}$ **13.** $\dfrac{x+2}{2(x-1)}$
14. $\dfrac{x+1}{3(x-2)}$ **15.** $\dfrac{1}{x+2}$ **16.** $\dfrac{x-2}{x-1}$ **17.** $\dfrac{x+2}{x+1}$
18. $\dfrac{x+4}{x-2}$ **19.** $\dfrac{y}{y-1}$ **20.** $\dfrac{y-6}{y+1}$ **21.** $\dfrac{1}{4(x-2)}$
22. $\dfrac{x-5}{x-4}$ **23.** $\dfrac{x+3}{x-3}$ **24.** $-\dfrac{x-1}{x+1}$ **25.** $\dfrac{1}{t^2+9}$ **26.** x
27. $\dfrac{x+5}{(x+4)(2x+3)}$ **28.** $\dfrac{1}{(2x-5)(3x-2)}$
29. $\dfrac{(2x+1)(2x-1)}{(x+5)^2}$ **30.** 1 **31.** $\dfrac{3(x+2)}{x+3}$ **32.** $\dfrac{x-5}{x+4}$
33. $\dfrac{3x+7}{(x-3)(x+5)}$ **34.** $\dfrac{2x}{(x+1)(x-1)}$
35. $\dfrac{1}{(x+1)(x+2)}$ **36.** $\dfrac{x^2+3x+12}{(x-4)(x+6)}$ **37.** $\dfrac{3x+2}{(x+1)^2}$
38. $\dfrac{2(5x-9)}{(2x-3)^2}$ **39.** $\dfrac{2x+1}{x^2(x+1)}$ **40.** $\dfrac{x^2+x+1}{x^3}$
41. $2+\sqrt{3}$ **42.** $\dfrac{3+\sqrt{5}}{2}$ **43.** $\dfrac{2(\sqrt{7}-\sqrt{2})}{5}$
44. $\dfrac{\sqrt{x}-1}{x-1}$ **45.** $\dfrac{y(\sqrt{3}-\sqrt{y})}{3-y}$ **46.** $2(\sqrt{x}+\sqrt{y})$
47. $x-2-\dfrac{16}{x-4}$ **48.** $x^2+x+\dfrac{6}{x-2}$
49. $2x^2-1-\dfrac{2}{2x+1}$ **50.** $\dfrac{1}{3}\left(x^2+x+2-\dfrac{1}{x+2}\right)$
51. x^4+1 **52.** $3x+1+\dfrac{5(x-1)}{2x^2+5}$

Algebra Toolkit C

C.1 Exercises ▪ page T54

1. (a), (c) **3. (a)** True **(b)** False **(c)** False **4.** cube; 5 **5. (a)** Five times a number minus three is twelve. **(b)** No **(c)** 3 **6. (a)** Four times a number plus nine is one. **(b)** No **(c)** -2 **7. (a)** Four times a number plus seven equals nine times that number minus three. **(b)** No **(c)** 2 **8. (a)** Two minus five times a number equals eight plus that number. **(b)** Yes **(c)** -1 **9.** -9 **10.** 4 **11.** -3 **12.** $\frac{5}{2}$ **13.** $-\frac{3}{4}$ **14.** -3 **15.** $\frac{32}{9}$ **16.** 3 **17.** 12 **18.** -70 **19.** $\frac{21}{11}$ **20.** 30 **21.** $\frac{1}{17}$ **22.** $\frac{5}{18}$ **23.** $\frac{10}{3}$ **24.** $\frac{35}{24}$ **25.** $\frac{29}{2}$ **26.** $\frac{13}{6}$ **27.** -20 **28.** -39 **29.** $-\frac{1}{3}$ **30.** $-\frac{4}{9}$ **31.** No solution **32.** No solution **33.** ± 7 **34.** $\pm 3\sqrt{2}$ **35.** $\pm 2\sqrt{6}$ **36.** $\pm\sqrt{7}$ **37.** ± 2 **38.** $\pm\sqrt{5}$ **39.** No real solution **40.** No real solution **41.** $-4, 0$ **42.** $5 \pm \sqrt{5}$ **43.** -3 **44.** -2 **45.** $\pm\sqrt[4]{21.3}$ **46.** $\pm\sqrt[6]{15}$ **47.** $\pm\sqrt[4]{8}$ **48.** $\sqrt[3]{5}$ **49.** $(2.6)^{2/3}$ **50.** $\left(\dfrac{5}{1.8}\right)^{8/7}$ **51.** $\left(\dfrac{2}{5}\right)^{1/4}$ **52.** $\pm\sqrt[6]{15}$ **53.** $-5, 1$ **54.** No real solution **55.** 8
56. $\dfrac{1}{\sqrt[5]{4}}-2$ **57.** $\pm\dfrac{\sqrt{35x}}{7}$ **58.** $\sqrt[3]{\dfrac{2t}{5}}$ **59.** $\dfrac{3y}{5y-4}$
60. $\dfrac{5l}{2(l+2)}$ **61.** $\dfrac{3y}{y+2}$ **62.** $\dfrac{5}{s-1}$ **63.** $\dfrac{P-2l}{2}$
64. $\sqrt[3]{\dfrac{3V}{4\pi}}$ **65.** $\dfrac{PV}{RT}$ **66.** $\dfrac{Fr^2}{GM}$ **67.** $\left(\dfrac{s}{1.2}\right)^{5/2}$ **68.** $\left(\dfrac{7A}{8}\right)^{5/9}$
69. $\left(\dfrac{1}{2A}\right)^{5/6}$ **70.** $\left(\dfrac{8}{5A}\right)^{10/7}$ **71.** $\dfrac{d}{s}$ **72.** $\dfrac{2L}{5}$ **73.** $\dfrac{1}{3\sqrt{5a_2}}$
74. $\sqrt[3]{\dfrac{5}{4x_2}}$

C.2 Exercises ▪ page T61

1. (a) $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ **(b)** $\frac{1}{2}, -1, -4; -2, 4$
2. (a) $(x-5)(x+1)=0$ **(b)** $(x-2)^2=9$
(c) $x=\dfrac{-(-4) \pm \sqrt{(-4)^2-4(1)(-5)}}{2(1)}$ **3.** $-4, 3$ **4.** $-4, 1$
5. $3, 4$ **6.** $-6, -2$ **7.** $-\frac{1}{3}, 2$ **8.** $-\frac{3}{2}, \frac{5}{2}$ **9.** $-\frac{1}{2}, -3$
10. $-\frac{1}{2}, \frac{3}{2}$ **11.** $\frac{49}{4}; \frac{7}{2}$ **12.** $\frac{1}{4}; \frac{1}{2}$ **13.** $\frac{1}{16}; -\frac{1}{4}$ **14.** $\frac{1}{9}; -\frac{1}{3}$
15. $-1 \pm \sqrt{6}$ **16.** $2 \pm \sqrt{2}$ **17.** $3 \pm 2\sqrt{5}$
18. $-\frac{7}{2}, \frac{1}{2}$ **19.** $-2 \pm \sqrt{\dfrac{7}{2}}$ **20.** $1 \pm \dfrac{2}{\sqrt{3}}$ **21.** $\dfrac{5}{2} \pm \dfrac{3\sqrt{3}}{2}$
22. $-\dfrac{3}{2} \pm \dfrac{\sqrt{205}}{10}$ **23.** $-3, 5$ **24.** $-6, 1$ **25.** $2, 5$

26. $-10, -20$ **27.** $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **28.** $2 \pm \sqrt{2}$

29. $-1 \pm \frac{2\sqrt{6}}{3}$ **30.** $3 \pm 2\sqrt{2}$ **31.** $\frac{3}{4}$ **32.** $\frac{1}{4} \pm \frac{\sqrt{5}}{4}$

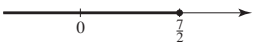
33. No real solution **34.** $-\frac{5}{2} \pm \frac{\sqrt{13}}{2}$ **35.** No real solution

36. No real solution

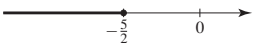
C.3 Exercises ▪ page T66

1. **(a)** $<$ **(b)** $\leq$ **(c)** $\leq$ **(d)** $>$ **2.** **(a)** True **(b)** False
3. 4 **4.** $-2, -1, 0, \frac{1}{2}$ **5.** $\sqrt{2}, 2, 4$ **6.** $1, \sqrt{2}, 2, 4$ **7.** 4
8. $\sqrt{2}, 2, 4$ **9.** $-2, -1, 2, 4$ **10.** $-1, 0, \frac{1}{2}, 1$

11. $(-\infty, \frac{7}{2}]$



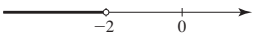
12. $(-\infty, -\frac{5}{2}]$



13. $(4, \infty)$



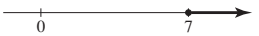
14. $(-\infty, -2)$



15. $(-\infty, 2]$



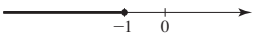
16. $[7, \infty)$



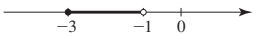
17. $[1, \infty)$



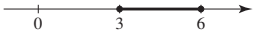
18. $(-\infty, -1]$



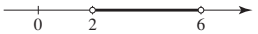
19. $[-3, -1)$



20. $[3, 6]$



21. $(2, 6)$



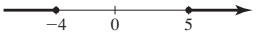
22. $(-1, 4]$



23. $(-2, 3)$



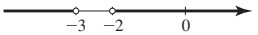
24. $(-\infty, -4] \cup [5, \infty)$



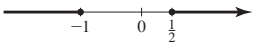
25. $[-3, 6]$



26. $(-\infty, -3) \cup (-2, \infty)$



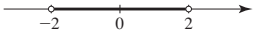
27. $(-\infty, -1] \cup [\frac{1}{2}, \infty)$



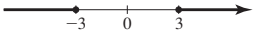
28. $(-1, 2)$



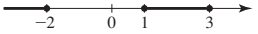
29. $(-2, 2)$



30. $(-\infty, -3] \cup [3, \infty)$



31. $(-\infty, -2] \cup [1, 3]$



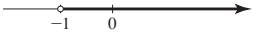
32. $(-1, 2) \cup (5, \infty)$



33. $(-\infty, -2) \cup (-2, 4)$



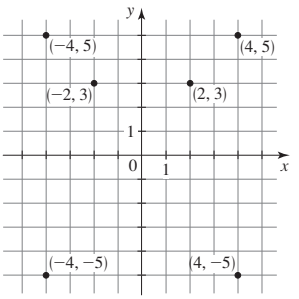
34. $(-1, \infty)$



Algebra Toolkit **D**

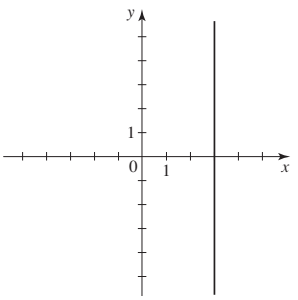
D.1 Exercises ▪ page T70

1. $(3, -5)$ **2.** II **3.** $\sqrt{(a-c)^2 + (b-d)^2}$; 10
4. $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$; $(4, 6)$
5.

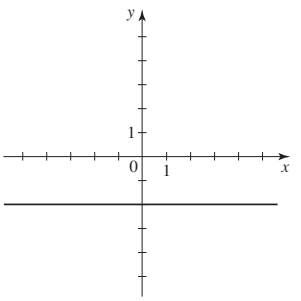


6. $A\,(5, 1)$, $B\,(1, 2)$, $C\,(-2, 6)$, $D\,(-6, 2)$, $E\,(-4, -1)$, $F\,(-2, 0)$, $G\,(-1, -3)$, $H\,(2, -2)$

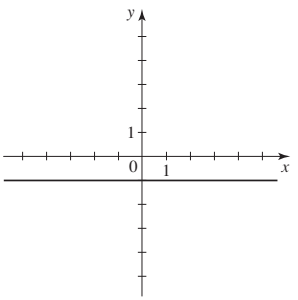
7.



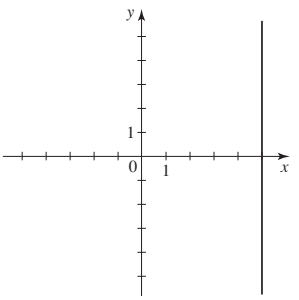
8.



9.

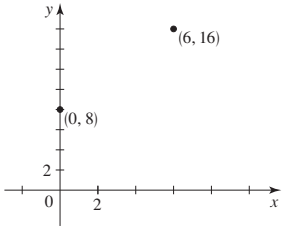


10.



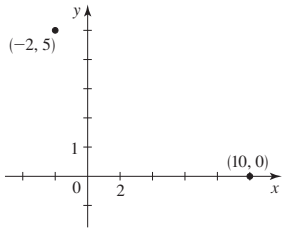
11. **(a)** $\sqrt{13}$ **(b)** $(1.5, 1)$ **12.** **(a)** 5 **(b)** $(0, \frac{1}{2})$ **13.** **(a)** 10 **(b)** $(1, 0)$ **14.** **(a)** $\sqrt{40}$ **(b)** $(1, -2)$

15.



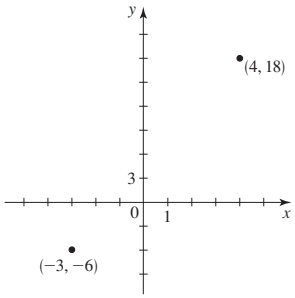
(b) 10 **(c)** (3, 12)

16.



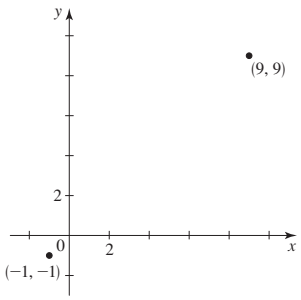
(b) 13 **(c)** (4, 2.5)

17.



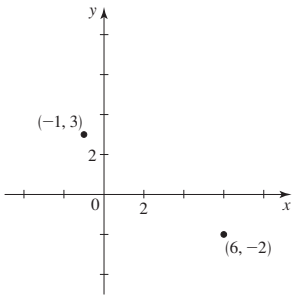
(b) 25 **(c)** $(\frac{1}{2}, 6)$

18.



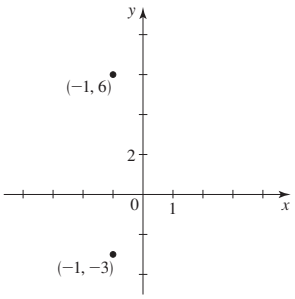
(b) $\sqrt{200}$ **(c)** (4, 4)

19.



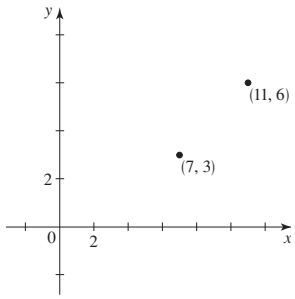
(b) $\sqrt{74}$ **(c)** (2.5, 0.5)

20.



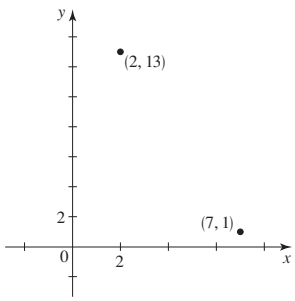
(b) 9 **(c)** $(-1, 1.5)$

21.



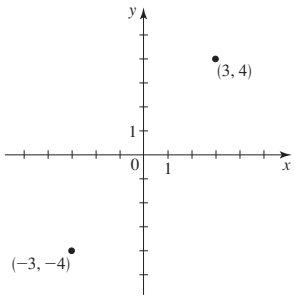
(b) 5 **(c)** (9, 4.5)

22.



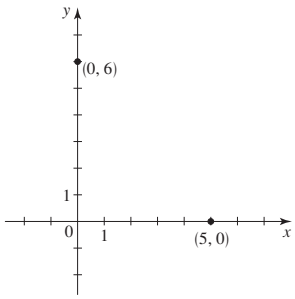
(b) 13 **(c)** (4.5, 7)

23.



(b) 10 **(c)** (0, 0)

24.



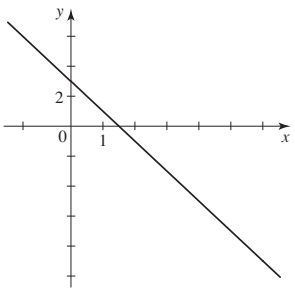
(b) $\sqrt{61}$ **(c)** (2.5, 3)

D.2 Exercises ▪ page T77

1. 3, 2; yes **2.** 1, 2; no **3. (a)** $y, x; -1$ **(b)** $x, y; \frac{1}{2}$
4. (1, 2); 3 **5. (a)** Subtracting y from x gives 7. **(b)** No
6. (a) Adding x to y^2 gives 6. **(b)** No **7. (a)** Subtracting $2x$ from y^3 gives 5. **(b)** Yes **8. (a)** Adding $3x$ to y^2, then subtracting $2y$, gives 0. **(b)** Yes

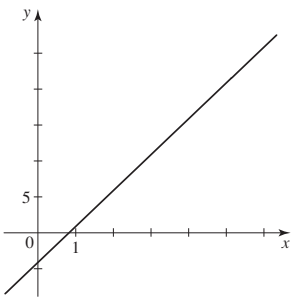
9.

x	y
0	3
1	1
2	−1
3	−3
4	−5
5	−7
6	−9



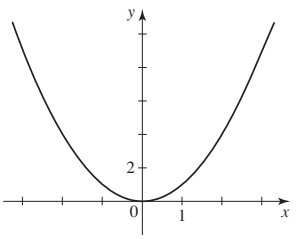
10.

x	y
0	−4
1	1
2	6
3	11
4	16
5	21
6	26



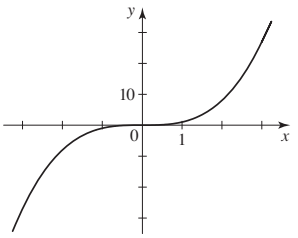
11.

x	y
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



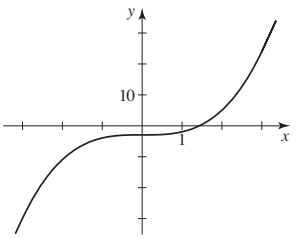
12.

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27



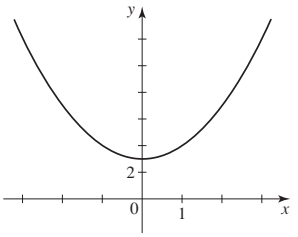
13.

x	y
-3	-30
-2	-11
-1	-4
0	-3
1	-2
2	5
3	24



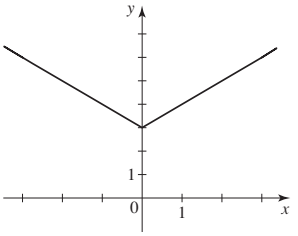
14.

x	y
-3	12
-2	7
-1	4
0	3
1	4
2	7
3	12



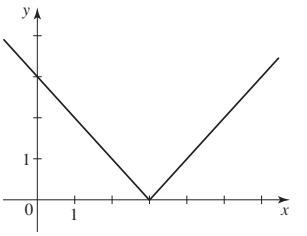
15.

x	y
-3	6
-2	5
-1	4
0	3
1	4
2	5
3	6

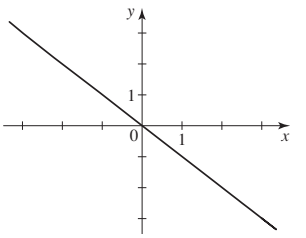


16.

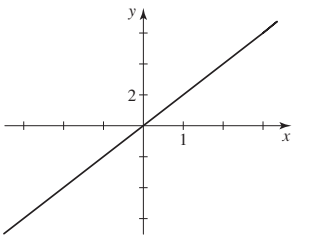
x	y
0	3
1	2
2	1
3	0
4	1
5	2
6	3



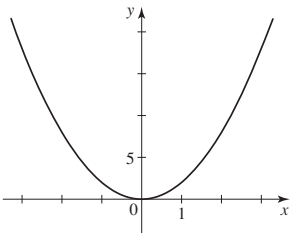
17.



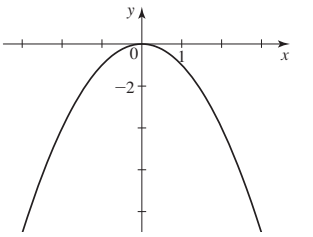
18.



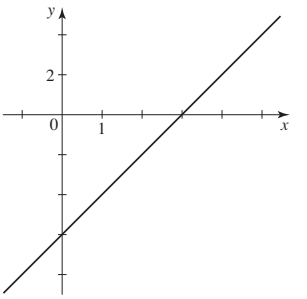
19.



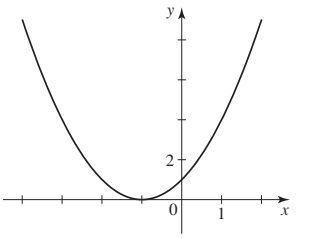
20.



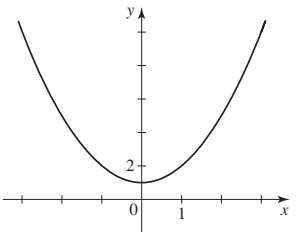
21.



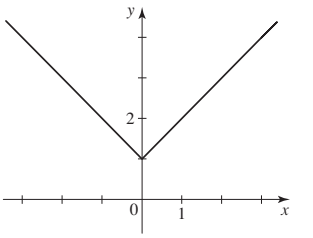
22.



23.

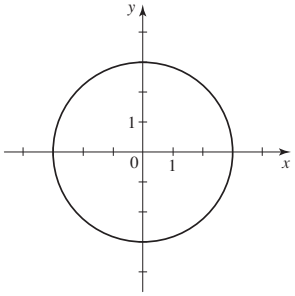


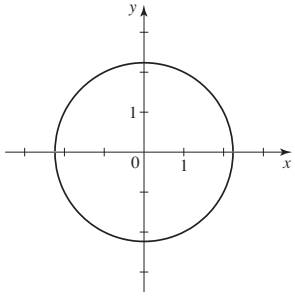
24.



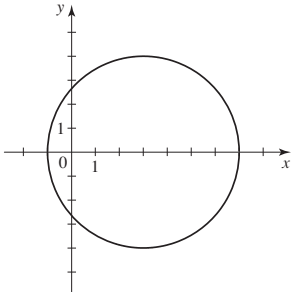
25. **(a)** 4 **(b)** 1, 3 **(c)** 0, 4; 0
26. **(a)** 4, -4 **(b)** -1, 7 **(c)** 3 **(d)** -2, 8; -4, 4
27. **(a), (b)** x-intercepts -3, 3; y-intercepts -2, 2

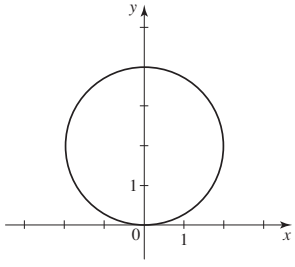
28. (a), (b) x-intercepts $-2, 2$; y-intercepts $-4, 4$
29. $3; -3$ **30.** $2, 3; 6$ **31.** $-3, 3; -9$
32. $\frac{1}{2}; 1$ **33.** $-2, 2; -2, 2$ **34.** $-1; 1$
35. $(0, 0); 3$ **36.** $(0, 0); \sqrt{5}$



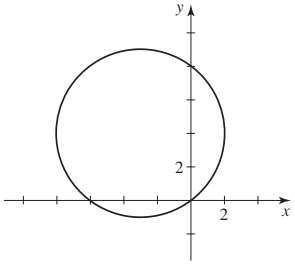


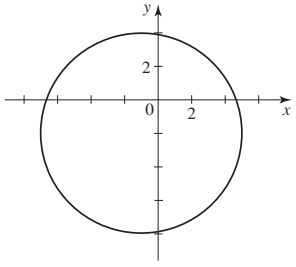
37. $(3, 0); 4$ **38.** $(0, 2); 2$





39. $(-3, 4); 5$ **40.** $(-1, -2); 6$



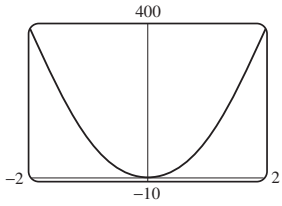


41. $(x - 2)^2 + (y + 1)^2 = 9$
42. $(x + 1)^2 + (y + 4)^2 = 64$
43. $(x + 2)^2 + (y - 2)^2 = 4$
44. $(x + 1)^2 + (y - 1)^2 = 9$
45. $(x - 1)^2 + (y + 2)^2 = 4; (1, -2); 2$
46. $(x - 1)^2 + (y - 1)^2 = 4; (1, 1); 2$
47. $(x - 2)^2 + (y + 5)^2 = 16; (2, -5); 4$
48. $x^2 + (y + 3)^2 = 7; (0, -3); \sqrt{7}$

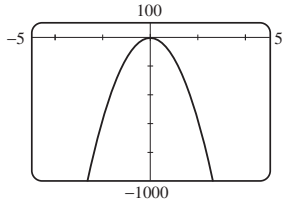
D.3 Exercises ▪ page T84

1. viewing rectangle **2.** zoom in **3.** (c)
4. (c) **5.** (c) **6.** (d)

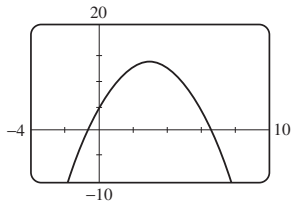
7.



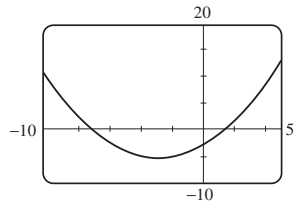
8.



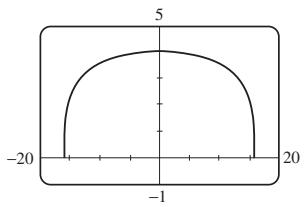
9.



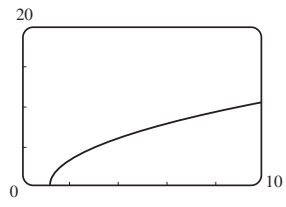
10.



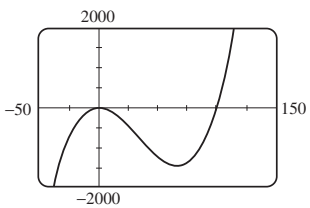
11.



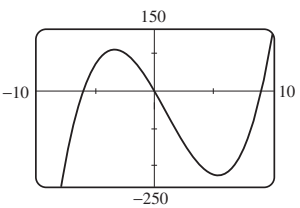
12.



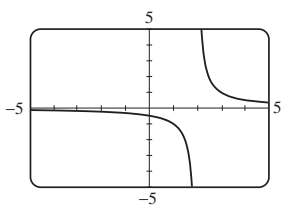
13.



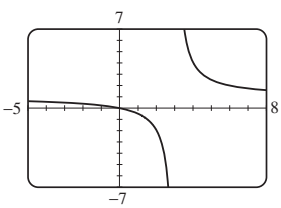
14.



15.

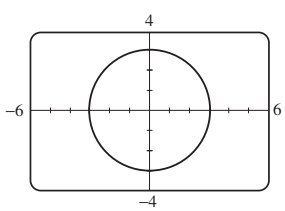


16.

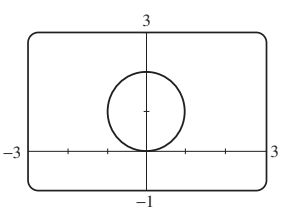


17. No **18.** No **19.** Yes; 2
20. Yes; 1

21.



22.



D.4 Exercises ▪ page T89

1. x **2.** above **3. (a)** $-1, 0, 1, 3$ **(b)** $[-1, 0] \cup [1, 3]$
4. (a) $1, 4$ **(b)** $(1, 4)$ **5.** -4 **6.** -6 **7.** $\sqrt{32}$ **8.** $\sqrt[3]{16}$
9. No real solution **10.** No real solution **11.** $\pm\frac{5}{2}$ **12.** 2.61
13. 3.00, 4.00 **14.** 0.25, 0.50 **15.** 1.00, 2.00, 3.00
16. $-1.00, -0.25, 0.25$ **17.** 1.62 **18.** 0, 2.31 **19.** 0.75
20. $-0.31, 0.81$ **21.** No solution **22.** No solution
23. $[-2.00, 5.00]$ **24.** $[-2.00, 0.25]$
25. $(-\infty, 1.00] \cup [2.00, 3.00]$
26. $(-1.00, -0.25) \cup (-0.25, \infty)$
27. $(-1.00, 0) \cup (1.00, \infty)$
28. $(-\infty, -0.535] \cup [0.535, \infty)$
29. $[-3.00, 6.00]$
30. $(-\infty, -3.00) \cup (-2.00, \infty)$
31. $(-1.00, 2.00) \cup (5.00, \infty)$
32. $(-\infty, -2.00) \cup (-2.00, 4.00)$

INDEX

Absolute value, T11
  distance between points on real line and, T12
Absolute value function, 70
Addition
  of algebraic expressions, T26–T27
  graphical, of functions, 485–86, 505
  of matrices, 605–6, 611–12
  of polynomials, T30
  of rational expressions, T41–T42
Aerodynamic lift, 503
Age and height, 5–6
Air pressure and elevation, 122–23
Alcohol, surge function and, 563, 565–66
Alcohol consumption, P1–P4, 11, 25, 52, 88
  absorption, P1, P3
  collecting data on, P1
  metabolism, P1, P3
  modeling absorption and metabolism, P3–P4, 261, 272, 295, 376
Algebraic expression(s)
  adding or subtracting, T26–T27
  average rate of change of function defined by, 147–48
  combining, T25–T32
  defined, T25
  factoring, T33–T38
  form and value of, T25–T26
  multiplying, T27–T29
  polynomials, combining, T29–T31
Algebraic method of solving equation, T85, T86
Ancient objects, finding the age of, 371–72
Anglerfish, depth and pressure experienced by, 35
Annual percentage yield (APY), 266
Archimedes, 209, 312, 568
  crown problem, 568, 573
Arctic sea ice, extent of, 197–98
Area(s)
  proportionality to square of "length" measurement, 557
  of similar objects, 555–56
Arithmetic expression, evaluating, T2
Arrow notation, 509, 528
Asbestos and cancer, links between, 194, 195
Aspirin metabolism, 308
Assets, fair division of, 242–45
Associative properties, T2
Astronaut, net change in weight of, 55–56
Asymptote(s)
  horizontal, 287, 288, 528, 540, 541
  of rational functions, 540, 541
  of reciprocal functions, 528
  vertical, 329, 528, 540, 541
Atlanta, population of, 150–51
Attenuated growth, 398
Augmented matrix of linear system, 590–91, 593, 594
Average age of preschoolers, 3
Average income, 3–4
Average (mean), 2–3
  definition of, 2
  formula for, 104–5
Average rate of change, 142–53, 272–75
  average speed, 145–46, 148
  definition of, 143
  for function defined by algebraic expression, 147–48
  of linear function, 155–56. *See also* Rate of change
  recognizing changes in, 229–33
Average speed of moving object, 145–46, 148

Bach, Johann Sebastian, 411
Back-substitution, solving triangular system using, 581, 582
Bacteria count, in colony-forming units per milliliter, 356
Bacterial growth, 248–49
  comparing growth rates, 264
  models for different time periods, 263
Bankruptcy, fair division of assets in, 242–45
Barometer, invention of, 476
Barton, Otis, 30
Base
  exponential function with base *a*, 286, 288–90
  exponential function with base *e*, 351
  exponential notation, T14
  of logarithm, 324, 326–27, 337–39
Bat cave, species-area relation in, 499, 517–18
Bathysphere, 30
Beebe, William, 30
Beer-Lambert Law, 363–64
Belgium, population of, 302–3
Bias in presenting data, 128–33, 130–33
Bicycle race, average speed in, 145–46
Billionaire ruler, 403
Binomials, T26
  multiplying, using FOIL, T28
Biodiesel fuels from seeds of *Jatropha curcsa*, 391–92
Biodiversity in Pasoh Forest Reserve of Malaysia, 526
Bird flight, weight and wingspan and, 525
Blood alcohol concentration (BAC), P1, P3, 11, 52, 88, 189, 376
  using surge functions to model data on, 563, 565–66
Body Mass Index (BMI), 51, 189–90
Boiling point, elevation and, 33–34
Bonds, coupon, 309
Boyle's Law, 111, 529, 534
Braking distance, finding maximum carrying capacity of road with, 562–63
Bridge science, 237–39
Bungee jumping, 148

California, population of, 282
Camera lenses, focusing distance for, 536, 537–39
Canceling, simplifying rational expressions by, T39
Cancer and asbestos, links between, 194, 195

Carbon dioxide levels in atmosphere, 197
Carrying capacity
in logistic growth model, 277, 298
of road, 80–81, 560–63
of road, safe following distance and, 560, 561–62
of road, using braking distance to find maximum, 562–63
Car sales, hybrid, 22, 302
Cartesian plane. *See* Coordinate plane
Catching up with leader on bike ride, 213–14
Categorical data, 602–11
collecting, 635
getting information from, 615–16
organizing, in matrix, 603–5, 636–37
proportions and, 604–5
tabulating, 604
Catfish, stocking pond with, 298
Causation
correlation and, 239–42
direction of, 242
Cell phone plan, 57
Cell phone usage in India, 363
Central tendency, 2
measures of, 2, 3, 4
Change, describing. *See* Function(s)
Change of base formula, 337–39
China, Internet usage in, 363
Chirping rate of crickets, temperature and, 198, 208
Chocolate-powered car, 34
Chord, 412
Christmas bird count, 22
Circle(s), T76–T77
equation of, identifying, T77
equation of, standard form of, T76–T77
graphing, T76
graphing, on graphing calculator, T83–T84
Closed interval, T9
Coefficient(s), 177–82
constant, 177–80
correlation, 195
heat transfer, 370
of x, 177, 180–82
Coefficient matrix, 621
Colony-forming units, 356
Columbia-Ecuador earthquake (1906), 346
Columns (matrix), 590
Combining "like terms," T26, T30
Combining logarithmic expressions, 336
Common factors
canceling, T39
factoring out, T33–T34
Common logarithms, 324–26
Common redpoll, wintering habits of, 25
Commutative properties, T2
Completing the square, T57–T59
defined, T58
to express quadratic function in standard form, 431
for general quadratic function, 440
solving quadratic equations by, T58–T59
Composite functions. *See* Composition of functions
Composition of functions, 377–79
Compound fraction form of rational function, 539
Compound interest, 265–66
calculating annual percentage yield, 266
comparing yields for different compounding periods, 266
continuously compounded, 354–55
formula, 265
investment growth and, 369
Concentration, mixtures and, 204–5
Conjugate radical, T42
Constant
of proportionality, 183, 496, 529, 557
spring, 188–89
universal gravitational, 104
Constant coefficient, 177
varying, 177–80
Constant function, 65
Constant rate of change, 153–65
linear model of, 156–57
Continuously compounded interest, 354–55
Cooling, Newton's Law of, 370–71, 375, 399, 427
Cooling coffee, 41
Coordinate, T7
Coordinate line. *See* Real number line (real line)
Coordinate plane, T67–T68, T71
distance between points on, T68–T69
graphing points and sets in, T68
graphing two-variable data in, 14–16
midpoint of line segment on, T69–T70
Correlation, causation and, 239–42
Correlation coefficient, 195
Cost
fixed, 26
model for, 89–90
unit, 26, 27
Cost comparison of gas-powered and hybrid-electric cars, 212–13
Coughing, mathematical model of, 88
Coupon bonds, 309
Crime scene investigation, 364
Newton's Law of Cooling and, 371
Crop yield
density (plants/acre) and, 465
rainfall and, 465
Cross-tab matrix, 603, 636
Crude oil imports in U.S., 493
Cube root function, 498

Data, 2–11. *See also* Modeling
analyzing, 136–37, 239
bias in presenting, 128–33
categorical. *See* Categorical data
collecting, 134–36, 238, 476–77
describing relationships in, 25–35
exponential, recognizing, 316–17
fitting exponential curves to, 274–75, 295–303
fitting polynomial curves to, 520–22
fitting power curves to, 517–18, 519–20
linear, recognizing, 233–37
linearizing, 408–9
log-log plot of, 518, 519
numerical, 602
one-variable, 2–5, 136–37
quadratic, recognizing, 478–79
scatter plot of, 14, 518, 519
semi-log plot of, 407–9, 518, 519
two-variable, 5–7, 13, 14–16, 37–38, 137
visualizing relationships in, 12–25
Data mining, 233
Dating, radiocarbon, 371–72
Dead Sea Scrolls, 376
Death in United States, leading causes of, 130–32
Decay factor, 253, 264
Decay rate, 253
instantaneous, 355–64
Decibel scale, 344–45
Decreasing functions, 78–79
Deforestation, paper usage and, 90–91
Degree of polynomial, 504
Demand, linear model for 170–71
Demand equation, 171, 214, 218
Demographics, life expectancy and, 23
Denominator
least common (LCD), T41
rationalizing the, T42
Dependent system, 571, 582–84, 596–98
Dependent variable, 37–38, 54
net change in, 38
Depreciation, straight-line, 176
Depth and pressure, 6–7, 14–15, 26
model for, 29–30

Difference(s)
first, 28–30
of functions, 484–87
second, 478–79, 480, 481
of squares, factoring, T35–T36
Dimension of matrix, 590
Dimensions of lot, quadratic function modeling, 453–54
Direct proportionality, 182–84
definition of, 183
to power function, 496–97
Discriminant, 452–53
Dissonance, 411–12
Distance, T11
maximum distance seen from a height, 61
between points on real line, T12
between two points in coordinate plane, T68–T69
Distance Formula, T68–T69
Distance function, inverse function of, 498
Distance-speed-time problem, 574–75
Distributive property, T3–T5
expanding using, T3–T4
factoring using, T4–T5
multiplying expressions using, T27–T28
multiplying polynomials using, T30
solving equations with, T48
Division. *See also* Quotient(s)
of assets, fair, 242–45
of exponents, T16
graphical, 536
long, T42–T43
of rational expressions, T40–T41
Domain
of function, 56–57, 77–78
of relation, 13, 14
Doppler effect, 111, 545
Drip irrigation system for garden, 91–92

Eames, Ray and Charles, 402
Earthquakes
intensity of, 346–47
magnitude of, 346, 349
Einstein, Albert, 101
Electrical capacity of solar panels, 182–84
Electrical resistance, 111
Elementary row operations, 591–92
Elements of set, T8
Elevation
air pressure and, 122–23
boiling point and, 33–34
temperature and, 28–29
Elimination method, 570–71
Gaussian, 581–82
Empty set, T8
End behavior
defined, 509
of polynomial function, 508–10
of rational function, 540
Energy, world consumption of, 48–49
Energy and mass formula, Einstein's, 101
Entries of matrix, 590
Environmental management, species survival and, 639
Enzymes in expectant mothers, levels of, 15–16
Equation(s), 25–35. *See also* System of equations
defined, T47
demand, 171, 214, 218
exponential, 365–66, 401–2
false, 582
in function form, 39
graphing, 27
of horizontal line, 171–72
inequality compared to, T62
of lines, 165–77. *See also* Linear equation(s)
logarithmic, 367–68
logistic growth, 277
matrix, 621–25
operations on, T47
power, T51–T53
quadratic. *See* Quadratic equation(s)
reading, T48
satisfying, T72
solutions (or roots) of, T47
solving the. *See* Solving the equation
that represent functions, 38–40, 42
two-variable, T71–T80
of vertical line, 171–72
Equilibrium point, 214
Equivalent system, operations leading to, 581
e (the number), 351
exponential models in terms of, 357–58
Ethiopia, population of, 374–75
Expanding expression
distributive property used in, T3–T4
logarithmic expressions, 335–36
Expectant mothers, levels of enzymes in, 15–16
Exponent(s), T14
integer, T14–T19
negative, T15–T16
rational, T20–T24
rules for working with, T16–T18, T21, T22
zero, T15–T16
Exponential decay, 252–55, 356
of "iceman," radiocarbon dating and, 372
modeling 252–55, 252–55, 264–65, 290, 356–57
Exponential equation(s), 365–66
logarithms used to solve, 401–2
steps in solving, 365
Exponential form, 327
Exponential function(s), 247–61
with base *a*, 286
with base *a*, effect of varying *a* or *C*, 288–90
with base *e*, 351
comparing linear functions and, 275–76
comparing power functions and, 494–95
for compound interest, 265–66
definition of, 249
finding, from a graph, 290–91
fitting exponential curves to data, 295–303
graphs of, 286–95, 406–9
inverse functions of, 385–86
modeling with, 249–55, 261–66, 274–75, 290, 295–98, 356–58
modeling with, appropriateness of, 518, 519
musical scale and, 409–12
natural, 351–52
rates of change of, 273–74
semi-log graph of, 407–9
transformations of, 420–21
Exponential growth, 248–85, 356
example of, 249–50
of investment, 265–66
linear growth versus, 275–76
modeling, 249–52, 249–52, 263, 265–66, 356, 357–58
of savings, 315–16
Exponential model(s), 276
appropriateness of, 296–97
changing time period in, 262–65, 266
for data, finding, 295–96
of decay, 252–55, 264–65, 290, 356–57
fitting model to data, 274–75, 295–97
getting information from, 368–71
of growth, 249–52, 263, 265–66, 356, 357–58
of investment growth, 265–66, 369
in terms of *e*, 357–58

Exponential model(s) (*continued*)
in terms of instantaneous growth rate, 358
of world population, 289, 296, 369–70
Exponential notation, T14–T15
for *n*th root of *a*, T20
for *n*th root of a^m, T21
Exponential patterns, 316–19
finding exponential functions fitting data, 317–19
recognizing exponential data, 316–17
Extraneous lines in graph, avoiding, T82–T83
Extraneous solutions, T50
Extrapolation, 191–92
Extreme numbers, 312–14

Factor(s), T1
canceling common, T39
factoring out common, T33–T34
Factored form of rational function, 539
Factoring
algebraic expressions, T33–T38
completely, T36
distributive property used in, T4–T5
factoring out common factors, T33–T34
graphing polynomial functions by, 505–8
by grouping, T37
solving quadratic equations by, 448–50, T56–T57
special factoring formulas, T35–T36
trinomials, T34–T35
Fair division of assets, 242–45
Falling sky diver, 61
False equation, 582
Family of linear equations, graphing, 177–78, 180–81
Family of logarithmic functions, graphing, 329–30, 338–39
Farms in United States, average rate of change in number of, 144–45
Fechner, Gustav, 344
Femur length, height and, 197
Fencing a garden, 94–96, 443
Financial problem, using linear system to model, 585–86
First differences, 28–30
Fish, length-at-age data for, 521–22
Fish population, limited, 277–78
Fitt's Law, 350
Fixed cost, 26
Focusing distance for camera lenses, 536, 537–39
FOIL, multiplying binomials using, T28
Force/mass/acceleration, Newton's formula of, 107
"Forgetting" curves, experimenting with, 525
Formula(s), 101–12
for average (mean), 104–5
compound interest, 265
definition of, 101
distance, T68–T69
equivalent, 105
finding, 102–3
midpoint, T69–T70
quadratic, 451, 454, 455, T59–T61
reading and using, 105–8
simple interest, 203
for surface area, 103
variables with subscripts in, 104–5
Fractional positive powers, power function with, 497–98
Fractions
properties of, T40
solving an equation involving, T50
Frequencies of musical notes, 410–12
Fudging the data, 128
Function(s), 35–52
absolute value, 70
average rate of change of, 142–53
composition of, 377–79
constant, 65
cube root, 498
decreasing, 78–79
definition of, 36–37
dependent variable in, 37–38, 54
difference of, 484–87
domain of, 56–57, 77–78
equations that represent, 38–40, 42
evaluating, 54–56
exponential. *See* Exponential function(s)
four ways to represent, 43
of functions, 377–79
graphical addition and subtraction of, 485–87, 505
graph of, 41–43, 64–88
graph of, finding values of its inverse from, 380
graph of, reading, 74–76
graph of, using graphing calculator, 67–68
identity, 66
increasing, 78–79
independent variable in, 37–38, 54
inverse, 379–86, 498
linear. *See* Linear function(s)
local maximum and minimum values of, 79–81
logarithmic. *See* Logarithmic function(s)
logistic, 297
modeling with, 89–100
natural exponential, 351–52
natural logarithm, 353
one-to-one, 383–85
piecewise defined, 57–58, 68–70
polynomial, 504–16, 520–22
power. *See* Power functions
product of, 488–89
quadratic. *See* Quadratic function(s)
quotient of, 488
range of, 77–78
rational, 536–45, 560–63
reciprocal, 527–30
as relation, 36–37
root, 497–98
as rule, 52–53
square root, 66–67, 497–98
squaring, 428–29, 432
sum of, 484–87
surge, 563–66
value of *f* at *x*, 53, 55–56, 75–76
vertical shifts, 414–15, 417–18
Function form, 39
Function notation, 52–54

Galileo Galilei, 153
Galileo's Law, 493, 498
Garden
fencing a, 94–96, 443
irrigating, 91–92
Gas mileage
formula for, 102
maximum, for car, 442
Gasoline price, year and, 38
Gas-powered and hybrid-electric cars, cost comparison of, 212–13
Gaussian elimination, 581–82
General form of equation of line, 172–73
General form of quadratic function, 429–30
completing the square for, 440
General Social Survey (GSS), 635
Geometry, constructing model involving, 205–6
Geometry of space, inverse square laws and, 531–32
Germany, population of, 375
Global warming, 141
Grade of road, 157, 158
Graph(s)

of absolute value function, 70
of circle, T76, T83–T84
of exponential functions, 286–95, 406–9
of family of linear equations, 177–78, 180–81
finding domain and range of function from, 77–78
finding exponential function from, 290–91
finding linear functions from, 160–61
finding local maximum and minimum values from, 79–81
finding values of function from, 75–76
of function, 41–43, 64–88
of function, finding values of its inverse from, 380
of function, reading, 74–76
of general linear equation, 172–73
horizontal shifts of, 416–18
of increasing and decreasing functions, 78–79
of inequalities, T8
intercepts, finding, T75. *See also* *x*-intercepts; *y*-intercepts
intersection points on, 68, 76
of intervals, T9, T10
of inverse square function, 530
of linear function, 155
of logarithmic functions, 328–30, 338–39
misleading, 128–30
of model, getting information from, 93–96
of natural exponential functions, 352
of parallel lines, 179, 180
of piecewise defined functions, 68–70
of polynomial function, 505–8
of power functions, 495
of quadratic functions, 432–34, 449–50
rates of change and shapes of, 230–33
of rational functions, 536–37, 539–41
reading, 16–17, 138–39, 473–75
of reciprocal function, 527–28
reflecting, 419–22, 495
of root functions, 497, 498
scatter plot, 14, 519
semi-log, 407–9, 518, 519
shifting, 414–18, 495
solving a polynomial equation graphically, 511
of squaring function, 432
stretching and shrinking, vertically, 418–19, 495
of system of two linear equations in two variables, 571–72
transformations of function and, 414–27, 495
two-variable data in coordinate plane, 14–16, T67–T68
of two-variable equations, T72–T75
verbal description from, 75
vertical shifts, 414–15, 417–18
Graphical addition of functions, 485–86, 505
Graphical division, 536
Graphical method of finding intersection points of linear functions, 211
Graphical method of solving equations, T85–T88
equation in an interval, T88
inequalities, T88–T89
quadratic equations, T87
Graphical subtraction of functions, 486–87
Graphing calculator, T80–T85
avoiding extraneous lines, T82–T83
choosing viewing rectangle, T80–T82
CubicReg command on, 521
exponential function, 296
family of exponential functions, 289
graphing circle on, T83–T84
graphing functions with, 67–68
logistic growth function, 278, 297–98
LOG key, 338
multiplying matrices on, 614–15
PwrReg command, 517, 519
QuadReg command on, 462, 463
two graphs on same screen, T82
Xmin and Xmax command, T81
Ymin and Ymax command, T81
Gravitation, Newton's Law of, 535
Gravitational force
between moon and astronaut in space ship, 73
Newton's formula for, 104, 107
Greater than symbol (>), T7
Grimshaw, John, 34
Grouping, factoring by, T37
Growth
attenuated, 398
exponential. *See* Exponential growth
linear model of, 156–57
logistic, 272, 276–78, 297–98
Growth factor, 249–51, 252
changing time period and, 262–65
Growth rate, 251–52
annual percentage yield, 266
changing time period and, 262–65
comparing, 264
instantaneous, 355–64
percentage rate of change, 272–75, 297

Half-life, 254, 264
Hanselman sextuplets, weights of, 8
Hare and tortoise race, modeling, 210–11
Health-care expenditures, U.S., 301
Heat transfer coefficient, 370
Height
age and, 5–6
of box, 106–7
femur length and, 197
Hidden variable, 239–42
Highway engineering, 80–81
Home sales in United States, 492
Hooker, Steven, 194
Hooke's Law, 188–89
Horizontal asymptote, 287, 288, 528, 540, 541
Horizontal lines, 171–72
Horizontal Line Test, 384–85
Horizontal shifts of graphs, 416–18
combining vertical shifts and, 417–18
House, median price of, 4–5
"Housing bubble," bursting of U.S., 259–60
Howler monkeys, modeling population of, 637–39
Hybrid car sales, 22, 302
Hydrogen ion concentration, pH scale and, 344

"Iceman" of Neolithic Age, radiocarbon dating of, 372
Identity function, 66
Identity matrix, 619
Image of *x* under *f*, 53
Income, average and median, 3–4
Inconsistent system, 571, 572, 582–84, 596–98
Increasing functions, 78–79
Independent variable, 37–38, 54
India
cell phone usage in, 363
population of, 270, 392
Inequalities, T62–T66
graphing, T8
linear, solving, T63
nonlinear, solving, T63–T65
operations on, T62
order on real line and, T7–T8
reversing direction of, T62
solving, graphically, T88–T89

Infant mortality, U.S., 190–91, 192, 195
Infinite intervals, T9
Initial population, 249
Initial value, 26, 156, 250, 356
Inner product, 612
Input(s), 12–14
  evenly spaced, 28
  in function, 36
Installation of flooring, average rate of, 143
Instantaneous rates of growth or decay, 355–64
  expressing exponential models in terms of, 358
Integer exponents, T14–T19
Integers, T1
Intelligibility, noise and, 199
Intensity
  of earthquakes, 346–47
  sound, decibel scale of, 344–45
Intercepts. *See* *x*-intercepts; *y*-intercepts
Interest
  compound, 265–66, 369
  continuously compounded, 354–55
  simple, 203–4
Internet usage in China, 363
Interpolation, 192
Intersection
  of intervals, T10–T11
  of sets, T8–T9
Intersection points, 68, 76
  equilibrium point, 214
  of linear functions, 211–15
Interval(s), T9–T11
  closed, T9
  frequency, on musical scale, 410–12
  graphing, T9, T10
  infinite, T9
  open, T9
  solving equation in, T88
  unions and intersections of, finding, T10–T11
  verbal description of, T10
Inverse function(s), 379–86
  definition of, 380
  of distance function, 498
  of exponential and logarithmic functions, 385–86
  finding values of, graphically, 380
  properties of, 382
  steps in finding, 380
Inverse of matrix, 620–25
Inverse proportionality, 528–30
Inverse square function, graph of, 530
Inverse square laws, 530–32
  laws of nature as, 531–32
Investment, growth of, 265–66, 369
Irrational numbers, T1
  e as, 351

Jet takeoff, sound intensity of, 345

Kepler's Law for Periods of Planets, 524
Kepler's Third Law, 493
King Hiero's crown, amount of gold in, 568, 573
Kirchhoff's Laws, 589

Labor force, U.S., 492
La Condamine, Charles-Marie de, 134
Lang Lang, 410
Law of Laminar Flow, 50
Law of the Lever, 209, 568
Law of the Pendulum, 503
Laws of Logarithms, 334–41
  defined, 335
  expanding and combining logarithmic expressions using, 335–36
Lead emissions, U.S., 524
Leading entry of matrix row, 592
Leading term of polynomial, 504
  end behavior of polynomial function and, 509–10
Leading variable in linear system, 595
Least common denominator (LCD), T41
Length-at-age data for fish, 521–22
Less than symbol (<), T7
Lever, Law of the, 209, 568
Life expectancy
  demographics and, 23
  in United States, 198
"Like terms," combining, T26, T30
Limited resources, growth with. *See* Logistic growth
Line(s)
  equations of, 165–77. *See also* Linear equation(s)
  extraneous, T82–T83
  horizontal, 171–72
  parallel, 178–80, 182
  perpendicular, 180–82
  regression (line of best fit), 189, 190–94, 195, 519
  slope of, 157, 158–59
  through two given points, finding equation for, 169
  vertical, 171–72
Linear data. *See* Linear patterns
Linear equation(s), 165–77, 201–19
  defined, T49
  general form of, 172–73
  graphing family of, 177–78, 180–81
  models that lead to, 202–9
  point-slope form of, 167–71
  slope-intercept form of, 166–67
  solving, T49–T51
  systems of. *See* Linear system(s)
  in two variables, 166
  varying coefficients in, 177–82
Linear function(s), 153–65
  average rate of change of, 155–56
  comparing exponential functions and, 275–76
  comparing squaring functions and, 428–29
  definition of, 154
  finding, from a graph, 160–61
  fitting patterns in data, finding, 234–37
  graphing, 155
  graphing quotients of, 536–39
  identifying, 154
  initial value of, 156
  intersection points of 211–15
  modeling supply and demand, 214–15
  modeling with, appropriateness of, 518, 519
  rate of change and, 155–57, 159–61
  slope and, 157–61
Linear growth versus exponential growth, 275–76
Linear inequality, T63
Linearizing data, 408–9
Linear model(s), 26–35, 275
  for cost, 89–90
  definition of, 26
  for demand, 170–71
  for depth-pressure data, 29–30
  getting information from, 30
  of growth, 156–57
  making, 165–77
  for radar, 166–67
  for temperature and elevation, 28–29
  for volume, 168–69
Linear patterns, 233–37
  finding functions to fit data, 234–37
  properties of linear data, 234–35
Linear regression, 189–201
  analyzing data using, 239
  correlation coefficient, 195
  line of best fit, 189, 190–94, 519
  line of best fit, used for prediction, 191–94
Linear scale, 342, 343
Linear system(s)
  augmented matrix of, 590–91, 593, 594
  elementary row operations to solve, 592

modeling with, 584–86
number of solutions of, 571
operations leading to equivalent, 581
in reduced row-echelon form, solutions of, 595
in several variables, 580–89
solving, using reduced row-echelon form, 594–95
solving, using row-echelon form, 593
in triangular form, 580, 581, 582
in two variables, 568–80
written as single matrix equation, 621–25
Line of best fit (regression line), 189, 190–91
correlation coefficient, 195
using, for prediction, 191–94
Line segment, midpoint of, T69–T70
Living alone, number of Americans, 200
Local maximum value, 79–81
Local minimum value, 79–81
Logarithm(s)
base *a*, 326–27
basic properties of, 328
change of base, 337–39
common (base 10), 324–26
comparing, 336–37
expanding and combining expressions, 335–37
exponential equations solved using, 401–2
laws of, 334–41
natural, 353–54
Logarithmic equations, 367–68
steps in solving, 367
Logarithmic form, 327
Logarithmic function(s), 328–34
with base *a*, 328
definition of, 328
graphing, 328–30
graphing, using change of base formula, 338–39
inverse functions of, 385–86
natural, 353
transformations of, 421–22
Logarithmic meter stick, 404
Logarithmic scales, 342–50, 407
decibel scale, 344–45
distance between semitones on, 410
linear scale compared to, 342–43
pH scale, 343–44
Richter scale, 345–47
Logarithm ruler, 403–4, 407
Logistic functions, 297
Logistic growth, 272, 276–78
modeling, 297–98
Logistic growth equation, 277
Log-log plot, 518, 519
Loma Prieta earthquake (1989), 43
Long division, T42–T43

Magnitude
of earthquakes, 346, 349
orders of, 405
Main diagonal, 619
Mass of earth, 108
Matrix, matrices
addition of, 605–6, 611–12
augmented, of linear system, 590–91, 593, 594
coefficient, 621
cross-tab, 603, 636
defined, 590
dimension of, 590
entries of, 590
identity, 619
inverse of, 620–25
multiplying, 612–16
multiplying, times a column matrix, 606–7
organizing categorical data in, 603–5, 636–37
proportion, 636
reduced row-echelon form of, 594–95
row-echelon form of, 592–94
rows and columns, 590
scalar multiplication of, 606, 611–12
singular, 623
solving systems of equations using, 590–602
subtraction of, 611–12
transition, 637–39
Matrix equations, 621–25
modeling with 624–25
solving, 622–25
Maximum area, modeling, 94–96
Maximum gas mileage for car, 442
Maximum height for model rocket, 442
Maximum revenue from ticket sales, 443–44
Maximum value of function, local, 79–81
Maximum value of quadratic function, 440–41
Maximum volume, finding, 510–11
Mean (average), 2–3
formula for, 104–5
Measurement
bias in, 128–33, 130–33
data collection, 134–36
Median, 3–5
Median income, 3–4
Median price of house, 4–5
Medication, exponential decay model for, 252–54
Meter stick, 404
Midpoint formula, T69–T70
Millionaire measuring tape, 403
Minimum value of function, local, 79–81
Minimum value of quadratic function, 440–41
Minneapolis, population of, 362
Misleading graphs, 128–30
Mixture problem, 575–76
Mixtures and concentration, model for, 204–5
Model, 26, 89. *See also* Linear model(s); Modeling
constructing, 202–6
Modeling, 89
absorption and metabolism of alcohol, P3–P4, 261, 272, 295, 376
definition of, 26
determining appropriate model for data, 518–20
direct proportionality, 182–84
with exponential functions, 249–55, 261–66, 274–75, 290, 295–98, 356–58
with functions, 89–100
getting information from graph of model, 93–96
with linear systems, 584–86
logistic growth, 297–98
with matrix equations, 624–25
models that lead to linear equations, 202–9
with polynomial functions, 510–11, 520–22
with power functions, 499, 517–20, 554–57
prediction based on model, P4
with quadratic functions, 442–44, 453–56, 461–66, 476–77
radioactive decay, 320–22
of real-world relationships, 88–100
species-area relation, 517–18
species survival, 637–39
supply and demand, 214–15, 218–19
with surge functions, 563, 565–66
with systems of equations, guidelines for, 574
using algebra to make model, P3
volume of box, 93–94
Model rocket
maximum height for, 442
quadratic function modeling height of, 455–56
Monomial, T26

Mosquito prevalence in Saga City, Japan, 199
Mount Kilimanjaro, temperatures on, 34
Moving object, average speed of, 145–46, 148
Multiplication. *See also* Product
  of algebraic expressions, T27–T29
  of exponents, T16
  of matrices, 612–16
  of matrices, getting information by, 615–16
  of matrix times a column matrix, 606–7
  of polynomials, T30
  of rational expressions, T40
Music, 409–12
  frequencies of notes in musical scale, 410–12
  intervals, frequencies, and dissonance, 411–12

National debt, U.S., 132–33, 270, 314
Natural exponential function, 351–52
Natural logarithm function, 353
Natural logarithms, 353–54
  properties of, 353
Natural numbers, T1
Negative exponents, T15–T16
Negative number, square root of, T20
Net change, 38, 42, 54–56
  in value of function, 55–56
Newton, Isaac, 104, 107
Newton's Law of Cooling, 370–71, 375, 399, 427
Newton's Law of Gravitation, 535
Nietzsche, Friedrich, 409
Noise and intelligibility, 199
Nonlinear inequalities, solving, T63–T65
  algebraically, T64–T65
  steps in, T64
Norman window, maximizing area admitting light in, 447
Northridge earthquake in Los Angeles (1994), 85
Notation
  arrow, 509, 528
  exponential, T14–T15, T20, T21
  function, 52–54
  scientific, 312–14
  set-builder, T8
Number(s)
  extreme, 312–14
  irrational, 351, T1
  natural, T1
  rational, T1
  real, T1–T7
  triangular, 479–80
  using letters to stand for, T3
Numerical data, 602
Numerical representation of function, 43
Nutritional analysis, 598

Olympic pole vault records, 193–94, 198–99
Olympic swimming records, 200
One-to-one function, 383–85
  test of, 384–85
One-variable data, 2–5
  analyzing, 136–37
Open interval, T9
Operations
  on equations, T47
  on inequalities, T62
  order of, T2
  on real numbers, T1–T2
Ordered pair, 12, T67
  as solution to equation, T72
Order of operations, T2
Order on real line, T7–T8
Orders of magnitude, 405
Origami, 401–2
Outliers, 4
Output, 12–14
  in function, 36

Pag, Andy, 34
Pan evaporation, 24
Paper consumption, function model for, 90–91
Parabola, 432. *See also* Quadratic function(s)
  opening downward, 434, 440
  opening upward, 433, 440
  vertex of, 432, 440
Parallel lines, 178–80
  finding equations of lines parallel to given line, 182
Parameter of system of solutions, 584
Patterns
  exponential, 316–19
  linear, 233–37
  quadratic, 478–81
Paulos, John Allen, 239
Pendulum, Law of the, 503
Percentage rate of change, 272–75, 297
  as constant, 273, 274
Perfect square, T57
  recognizing, T36
Perimeter, constructing model for, 205–6
Periods of Planets, Kepler's Law for, 524
Perpendicular lines, 180–82
  finding equations of lines perpendicular to given line, 182
Philadelphia, population of, 86
Phoenix Mars Lander, 152–53
pH scale, 343–44
Piecewise defined functions, 57–58
  graphing, 68–70
Pitch of roof, 157, 158
Pizza, cutting for maximum number of pieces, 481
Point-slope form of equation of line, 167–71
Pole vaulting, 192–94, 198–99
Polynomial(s)
  combining, T29–T31
  defined, T29–T30
  degree of, 504
  leading term of, 504, 509–10
Polynomial equation, solving graphically, 511
Polynomial function(s), 504–16
  defined, 504
  end behavior of 508–10
  graphing, 505–8
  modeling with 510–11, 520–22
  ratio of. *See* Rational functions
Population
  of Atlanta, 150–51
  bacterial growth, 248–49, 263, 264
  of Belgium, 302–3
  of California, 282
  carrying capacity and, 298
  changing time period for growth, 262–63
  comparing growth rates of, 264
  doomsday, 316
  of Ethiopia, 374–75
  of Germany, 375
  growth factor, 249–51, 252
  growth rate of, 251–52
  of India, 270, 392
  initial, 249
  limited fish, 277–78
  linear vs. exponential growth, 275–76
  logistic growth, 277–78, 297–98
  of Minneapolis, 362
  of Philadelphia, 86
  transition matrix to predict, 637–39
  U.S., census data on, 225, 300–301
  world, 282
  world, doubling of, 369–70
  world, exponential models for, 289, 296, 369–70
  world, limited, 285

Power equations, T51–T53
Power functions, 493–503
  combining. *See* Polynomial function(s)
  comparing exponential functions and, 494–95
  defined, 493
  direct proportionality to, 496–97
  with fractional positive powers, 497–98
  inverse proportionality to, 528–30
  inverse square laws, 530–32
  modeling with, 499, 517–20, 554–57
  with negative powers, 527–35
  with positive integer powers, 493
  reciprocal function, 527–30
  transformations of, 495
Powers of Ten (film), 402
Precipitation
  average annual, 16–17
  Olympic Peninsula in state of Washington, 10
Prediction
  based on model, P4
  using line of best fit for, 191–94
Preschoolers, average age of, 3
Pressure, depth and, 6–7, 14–15, 26
  model for, 29–30
Pressure, volume, and temperature formula, 101, 105
Principal square root, T20
Principle of Substitution, T27, T28
Product
  of functions, 488–89
  inner, 612
Product Formulas, T28–T29
Profit, finding, 487
Profit sharing, 245
Proportionality, 557–60
  analyzing problem of road capacity using, 560–63
  constant of, 183, 496, 529, 557
  direct, 182–84, 496–97
  inverse, 528–30
  in natural world, 559–60
  of surface area and volume, 558–59
Proportionality symbol, 557–60
Proportion matrix, 636
Proportions, categorical data and, 604–5

Quadrants, T67
Quadratic equation(s), 448–60, T56–T61
  defined, 448, T56
  discriminant of, 452–53
  solving, algebraic method, T86
  solving, by completing the square, T58–T59
  solving, by factoring, 448–50, T56–T57
  solving, graphical method, T87
  solving, using quadratic formula, 451, 454, 455, T59–T61
Quadratic formula, 451, 454, 455, T59–T61
  defined, T60
Quadratic function(s), 428, 429–47
  of best fit, 461–66, 476–77
  defined, 429
  finding, from its graph, 434, 450
  in general form, 429–30, 440
  graphs of, 432–34, 449–50
  identifying, 430
  maximum and minimum values of, 440–41
  modeling with, 442–44, 453–56, 461–66, 476–77
  in standard form, 431–34
Quadratic patterns, 478–81
Quotient(s). *See also* Division
  of functions, 488
  of linear functions, graphing, 536–39
  in long division, T43

Rabbit population, exponential growth model for, 252
Radar, linear model for, 166–67
Radical(s)
  conjugate, T42
  rational exponents and, T20–T24
Radioactive elements, decay of, 254–55
  of different amounts, 290
  half-lives, 254, 264
  modeling with coins, 320–21
  modeling with dice, 321–22
Radiocarbon dating, 371–72
Radium, exponential decay model for, 254–55, 264–65
Radius of ball, effect of doubling, 496–97
Rainfall and crop yield, 465
Range
  of function, 77–78
  of relation, 13, 14
Rate of change, 156
  average, 142–53, 155–56, 272–75
  changing, 229–33
  constant, 153–65
  finding linear model from, 157
  linear functions and, 155–57, 159–61
  percentage, 272–75, 297
  shapes of graphs and, 230–33
  slope and 159–61, 167
Rational exponents, T20–T24
  $a^{1/n}$, T20
  $a^{m/n}$, T21
  rules of exponents and, T21, T22
  simplifying by writing radicals as, T22–T23
Rational expressions, T39–T45
  adding, T41–T42
  defined, T39
  dividing, T40–T41
  long division of, T43
  multiplying, T40
  simplifying, T39, T42
  subtracting, T41–T42
Rational functions, 536–45
  analyzing problem of road capacity using, 560–63
  in compound fraction form, 539
  defined, 536
  in factored form, 539
  graph of, 536–37, 539–41
  quotients of two linear functions, 536–39
  in standard form, 539
Rationalizing the denominator, T42
Rational numbers, T1
Real number line (real line), T7
  distance between points on, T12
  graphing intervals on, T9, T10
  order on, T7–T8
Real numbers, T1–T7
  operations on, T1–T2
  properties of, T2–T5
Reciprocal function, 527–30
  graph of 527–28
Reduced row-echelon form of matrix, 594–95
Reflecting graphs, 419–22, 495
Regression, linear. *See* Linear regression
Regression line (line of best fit), 189, 190–94, 519
  correlation coefficient, 195
  using, for prediction, 191–94
Relation(s), 12–14
  definition of, 12
  domain of, 13, 14
  function as, 36–37
  range of, 13, 14
Relativity Theory, 50
Remainder in long division, T43
Resistors in parallel, 544–45
Richter, Charles, 345
Richter scale, 345–47
Rise, 158
Root(s). *See* Solution(s)

Root function, 497–98
Row(s)
  elementary row operations, 591–92
  of matrix, 590
Row-echelon form of matrix, 592–94
  reduced, 594–95
Run, 158

Safe following distance, 560
  carrying capacity of road and 561–62
San Francisco earthquake (1906), 346
SAT scores, change in rate of change of, 233
Savings, exponential growth of, 315–16
Scalar multiplication, 606
  of matrix, 606, 611–12
Scaling factor, 554–55
Scatter plot, 14, 519
Scientific notation, 312–14
Score(s)
  finding average, 104–5
  needed to get specific average, 106
Second differences, 478–79, 480, 481
Sedimentation rate, 165
Semi-log graphs (semi-log plot), 407–9, 518, 519
Semitone, 410
Set(s), T8
  in coordinate plane, graphing, T68
  empty, T8
  union and intersection of, T8–T9
Set-builder notation, T8
Shape and size
  power functions modeling relationships between, 554–57
  proportionality symbol used to relate, 557–60
Shifting graph, 414–18, 495
Shrinking graphs, 418–19
Sichuan, China, earthquake in (2008), 346–47
Similar objects, 554–57
  areas of, 555–56
  scaling factor of, 554–55
  volumes of, 556
Simon, Erika and Helmut, 372
Simple interest, 203–4
Simplifying rational expressions, T39, T42
Simpson, O.J., 370
Singular matrix, 623
Slope-intercept form of equation of line, 166–67
Slope of line, 157, 158–59
  definition of, 158
  linear functions and, 157–61
  parallel lines, 178
  perpendicular lines, 181
  point-slope form, 167–71
  rate of change and, 159–61, 167
  slope-intercept form, 166–67
Smoking in United States, 492
Snowfall, Sierra Nevada mountain range, 10
Solar cookers, 471
Solar panels, electrical capacity of, 182–84
Solution(s)
  of equation, T47
  equation with no, T50–T51
  extraneous, T50
  of linear system in reduced row-echelon form, 595
  of linear system in two variables, number of, 571
  of system of equations, 569
  of two-variable equation, T72
  of two-variable equation, table of, T73, T74
Solving the equation, T47–T56
  algebraic method, T85, T86
  with distributive property, T48
  graphical method, T85–T88
  linear equations, T49–T51
  for one variable in terms of others, T53–T54
  power equations, T51–T53
Sorensen, Søren Peter Lauritz, 343
Sound intensity, decibel scale of, 344–45
Special factoring formulas, T35–T36
Special Product Formulas, T28–T29
Species-area relation, 516
  in bat cave, 499, 517–18
  modeling, 517–18
Species survival, predicting, 637–39
Speed(s)
  average, 145–46, 148
  combining, 415
  doubling, 418
Sphere
  surface area of, 60
  volume of, 48, 496–97
Spring constant, 188–89
Square root
  of negative number, T20
  principal, T20
Square root function, 66–67, 497–98
Squaring function, 428–29
  graph of, 432
Staircase, slope of, 158
Standard form
  of equation of circle, T76
  of quadratic function, 431–34
  of rational function, 539
Starting time, horizontal shift of graph and changing, 416
Steepness. *See* Slope of line
Stopping distance of car, 503
Straight-line depreciation, 176
Streptococcus A bacterium, exponential growth of, 248–49
Stretching graphs, 418–19, 495
Subscripts, variables with, 104–5
Substitution, principle of, T27, T28
Substitution method, 569–70
Subtraction. *See also* Difference(s)
  of algebraic expressions, T26–T27
  graphical, of functions, 486–87
  of matrix, 611–12
  of polynomials, T30
  of rational expressions, T41–T42
Sumatra, earthquake in (2004), 346–47
Sum of functions, 484–87
Super-size portions, 226
Supply and demand, modeling, 214–15, 218–19
Supply equation, 214, 218
Surface area
  formula for, 103
  species-area relation in bat cave, 499, 517–18
  of sphere, 60
  volume and, 558–59
Surge functions, 563–66
  experimenting with, 563–65
  modeling alcohol data with, 563, 565–66
Survey
  analyzing, 136–37
  taking, 135–36, 635
Swimming pool fill time, 201–2
Symbolic representation of function, 43
System of equations, 567–602
  applications, 573–76
  defined, 568
  dependent, 571, 582–84, 596–98
  elimination method of solving, 570–71
  equivalent, operations leading to, 581
  graphical interpretation of, 571–72
  guidelines for modeling with, 574
  inconsistent, 571, 572, 582–84, 596–98
  linear, in several variables, 580–89
  linear, in two variables, 568–80
  linear, written as single matrix equation, 621–25

matrices to solve, using, 590–602
solution of, 569
substitution method of solving, 569–70

Table of solutions, T73, T74
Tacoma Narrows bridge collapse, 237
Temperature
average rates of change of, 230–33
chirping rate of crickets and, 198, 208
of cooling coffee, 41
elevation and, 28–29
time and, 6, 14–15
Term, T1
Ticket sales
maximum revenue from, 443–44
quadratic function modeling revenue from, 454–55
Time and temperature, 6, 14–15
Time periods in exponential models
changing, 262–65, 266
exponential decay models, 253
exponential growth model, 250
Time periods in logistic growth model, 277
Tire inflation, modeling proper, 461, 463
Tire Pressure Monitoring (TPM) system, 461
Tones in music, 410
Torricelli, Evangelista, 476
Torricelli's Law, 61, 392–93, 466, 476–77
testing theory, 477
Traffic management, 560–63
Trains traveling on same track, 179–80
Transformations of function, effects on graph, 414–27, 495
exponential functions, 420–21
horizontal shifts, 416–18
logarithmic functions, 421–22
power functions, 495
reading graph, 473–75
reflecting graphs, 419–22, 495
vertical shifts, 414–15, 417–18
vertical stretching and shrinking, 418–19, 495
Transition matrix, 637–39
Triangular form, linear system in, 580, 581, 582
Triangular numbers, 479–80
Trinomial, T26
factoring, T34–T35
Tuition fees, rate of change in, 229–30
Twain, Mark, 128
Two-variable data, 5–7
analyzing, 137
dependent and independent variables, 37–38, 54
graphing, in coordinate plane, 14–16
as relation, 13
Two-variable equations, T71–T80
of circle, T76–T77
defined, T71
graphs of, T72–T75
reading, T72
solution of, T72

Union
of intervals, T10–T11
of sets, T8–T9
Unit cost, 26, 27
U.S. Census Bureau, 134
U.S. national debt, 132–33, 270, 314
U.S. population, national census data on, 225, 300–301
U.S. Senate, women in, 22
Universal gravitational constant, 104

Value of function, 53
finding, from a graph, 75–76
local maximum and minimum, 79–81
net change in, 55–56
Variable(s), T25
dependent, 37–38, 54
hidden, 239–42
independent, 37–38, 54
leading, in linear system, 595
solving equation for one variable in terms of others, T53–T54
with subscripts, 104–5
Verbal description of interval, T10
Verbal representation of function, 43, 54
verbal description from graph, 75
Vertex of parabola, 432, 440
Vertical asymptote, 329, 528, 540, 541
Vertical lines, 171–72
Vertical line test, 41, 42–43
Vertical shifts of graphs, 414–15
combining horizontal shifts and, 417–18
Vertical stretching and shrinking of graphs, 418–19, 495
Viewing rectangle, choosing, T80–T82
Volume(s)
of box, modeling, 93–94
of can, 60
linear model for, 168–69
maximum, finding, 510–11
proportionality to cube of "length" measurement, 557
of similar objects, 556
of sphere, 48
of sphere, radius and, 496–97
surface area and 558–59
Voter survey, 42

Water rates, usage and, 58, 69
Weight
of astronaut, net change, 55–56
of whole world, 107–8
Well-Tempered Clavier, The (Bach), 411
Wheat, supply and demand for, 214–15
Windmill, watts of power generated by, 40
Wing loading, 112
Women
in science and engineering, 201
in U.S. Senate, 22
World population, 282
different models for, 289
doubling of, 369–70
exponential models for, 289, 296
limited, 285

x-axis, T67
x-coordinate, T67
Xeriscaping, 57
x-intercepts, T75
factored form of rational function to find, 539
graphing a polynomial function to find, 505–8
using discriminant to find number of, 452–53

y-axis, T67
y-coordinate, T67
y-intercepts, 158, 539, T75

Zero exponents, T15–T16
Zero-product property, 448, T56