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Approximation Theory and Functional Analysis

J.B. PROLLA Editor

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Edited by

João B. PROLLA
Universidade Estadual de Campinas, Brazil



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FOREWORD

This book contains the Proceedings of the International Symposium on Approximation Theory held at the Universidade Estadual de Campinas (UNICAMP), Brazil, during August 1 - 5, 1977. Besides the texts of lectures delivered at the Symposium, it contains some papers by invited lecturers who were unable to attend the meeting.

The Symposium was supported by the International Mathematical Union, by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), by German and Spanish government agencies, and by UNICAMP itself.

The organizing committee was constituted by Professors Silvio Machado, Leopoldo Nachbin, João B. Prolla (chairman), and Guido Zapata.

We would like to thank Professor Ubiratan D'Ambrosio, director of the Institute of Mathematics of UNICAMP, whose support made the meeting possible. Our special thanks are extended to Miss Elda Mortari who typed this volume.

João B. Prolla

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POLYNOMIAL APPROXIMATION AND A QUESTION OF G. E. SHILOV

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ABSTRACT Let E be an infinite dimensional real or complex Banach space. For $n = 0,1,2,\ldots,\infty$, let $a^n(E)$ be the algebra generated by all continuous polynomials on E which are homogeneous of degree $\leq n$. We discuss the completion of $a^n(E)$ with respect to several natural topologies, in the real and complex case. In particular, we prove that when E is a complex Banach space whose dual has the approximation property, then the τ_ω-completion of $a^1(E)$ can be identified with those holomorphic functions $f : E \to \mathbb{C}$ whose derivative $\hat{d}f : E \to E'$ is compact.

Let E be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with closed unit ball B_1. For each $n \in \mathbb{N}$, let $\mathcal{P}(^nE,F)$ be the space of n-homogeneous continuous polynomials $P : E \to F$, normed by $P \in \mathcal{P}(^nE,F) \longmapsto \|P\| \equiv \sup\{\|P(x)\| : x \in B_1\}$ $(\mathcal{P}(^0E,F) \equiv F)$. $C^m(E,F)$ is the space of m-continuously Fréchet differentiable functions from E to F and $H(E,F)$ is the space of holomorphic mappings from E to F, where E and F are complex Banach spaces. Throughout, if the range space F is

suppressed, then $F = \mathbb{K}$ is understood; thus for example $H(E) = H(E,\mathbb{C})$.

In this paper, we consider variations on the following problem posed by G. E. Shilov [8]. For each $n = 0,1,2,\ldots,\infty$, let $a^n(E)$ be the algebra generated by the collection of functions $P(^jE)$, $j \in \mathbb{N}$, $j \leq n$; thus $a^\infty(E) = \hat{\bigoplus}_{n=0}^{\infty} P(^nE) \equiv P(E)$. Then, what is the completion $(a^n(E),\tau)\hat{}$ of $a^n(E)$ with respect to some specified locally convex topology τ on $a^n(E)$? In the real case, this problem has been considered by many authors. In Section 1, we briefly outline some recent results in this case. When E is a complex Banach space, the above problem has apparently not been studied. In Section 2, we discuss the completion of $a^1(E)$ and $a^\infty(E)$ for several common topologies on the space $H(E)$. (Related results will also appear in [1].) In particular, we characterize the τ_ω-completion of $a^1(E)$ as a space of analytic functions having weakly uniformly continuous derivatives, and in terms of compact holomorphic mappings.

Some of the results in this paper were obtained while the author was a visitor at the Instituto de Matemática, Universidade Federal do Rio de Janeiro, supported in part by the CNPq and FINEP, to which the author expresses his gratitude.

SECTION 1. Among the most natural, and so far unsolved, versions of the question of Shilov is the following. Given E, a real infinite dimensional Banach space, let τ_b^o denote the topology on $a^\infty(E) = P(E)$ generated by the family of norms

$$P \in P(E) \to \sup\{|P(x)| : x \in B_m\}$$

where $B_m = \{x \in E : ||x|| \leq m\}$. Then, characterize the completion $a^\infty(E)\hat{}_{\tau_b^o}$ of $a^\infty(E)$. We recall that to each polynomial $P \in P(^nE)$ corresponds a unique symmetric continuous n-linear mapping $A : E \times E \times \ldots \times E \to \mathbb{K}$, via the transformation $P(x) = A(x, \ldots, x) \equiv Ax^n$. Thus, since

$$|P(x) - P(y)| = |A(x,\dots,x) - A(y,\dots,y)|$$

$$\leq |A(x,\dots,x,x) - A(x,\dots,x,y)|$$

$$+ |A(x,\dots,x,y) - A(x,\dots,y,y)| + \dots$$

$$\dots + |A(x,y,\dots,y) - A(y,\dots,y)|$$

$$\leq C_m \|x - y\|$$

for $P \in \mathcal{P}({}^n E)$, $x, y \in B_m$, and a constant C_m, we conclude that every element in $\mathcal{P}(E)$, and hence every element in $(\mathcal{P}(E),\tau_b^o)\hat{}$ is uniformly continuous on bounded subsets of E. However, it has been shown by Nemirovskiĭ and Semenov [6] that for any infinite dimensional Banach space E, there always exists a uniformly continuous function on B_1 which cannot be approximated uniformly on B_1 by polynomials. In connection with this, we remark that in many Banach spaces E, the norm function (which is obviously uniformly continuous on bounded sets) is not the uniform limit of polynomials on bounded sets. This was observed by Kurzweil [4] who showed that, for example in $E = C[0,1]$ (resp. ℓ_p, $1 \leq p$, p not even), the norm is not the uniform limit of 1 - (resp. $[p]$ -) differentiable functions. In particular, as Kurzweil noted, if $\inf\{|P(x)| : \|x\| = 1\} = 0$ for every $P \in \mathcal{P}({}^n E)$ and $n \in \mathbb{N}$, then, the norm cannot be uniformly approximated by polynomials on balls; this condition is closely connected with the uniform convexity of the space [5].

For arbitrary real Banach spaces E, the τ_b^o - completion of $a^1(E)$ was discussed in [2]. We briefly sketch the proof of a generalization of this result. Given a family $\mathcal{P} \subset \mathcal{P}(E)$, we say that a function $f : E \to F$ is $\mathcal{P}$ - uniformly continuous on bounded subsets of E

(abbreviated "P-continuous") if for each $\varepsilon > 0$ and bounded set $B \subset E$, there is $\delta > 0$ and a finite subcollection $\{P_1,\ldots,P_k\} \subset \mathcal{P}$ such that if $x, y \in B$ satisfy $|P_i(x) - P_i(y)| < \delta$ $(i = 1,\ldots,k)$, then $\|f(x) - f(y)\| < \varepsilon$. It is interesting to note that if $\mathcal{P}(^nE) \subset \mathcal{P}$ for some n, then $P \in \mathcal{P}(^nE,F)$ is $\mathcal{P}$-continuous if and only if P is compact. Indeed, if P is compact and $\varepsilon > 0$, then since $K \equiv \overline{P(B_1) - P(B_1)}$ is compact in F, there are unit vectors $\varphi_1,\ldots,\varphi_k$ in F' such that for any $y \in K$, $\|y\| < \varepsilon + \sup_{1\le i\le k} |\varphi_i(y)|$. Therefore, for any $x,y \in B_1$ which satisfy $|\varphi_i \circ P(x) - \varphi_1 \circ P(y)| < \varepsilon$ $(i = 1,\ldots,k)$, we have $\|P(x) - P(y)\| < 2\varepsilon$. The converse implication follows because P is then totally bounded. The case in which $\mathcal{P} = E'$ was discussed in [2].

PROPOSITION 1: *The space of $\mathcal{P}$-continuous functions $f : E \to F$ is the τ_b^o-completion of $A \otimes E$, where A is the algebra generated by $\mathcal{P}$ and $\{1\}$. (The τ_b^o topology is defined in a manner completely analogous to the case $F = \mathbb{R}$).*

PROOF: It is easy to see that every element of $A \otimes F$ is $\mathcal{P}$-continuous, and that the space of $\mathcal{P}$-continuous functions is τ_b^o-complete. To show density, let $f : E \to F$ be $\mathcal{P}$-continuous, $\varepsilon > 0$ and $B \subset E$ bounded, and let $\delta > 0$ and $\{P_1,\ldots,P_k\} \subset \mathcal{P}$ be selected as in the above definition. Define $\phi : E \to \mathbb{R}^k$ by $\phi(x) = (P_1(x),\ldots,P_k(x))$, and choose $x_1,\ldots,x_m \in B$ such that for any $x \in B$, $\|\phi(x) - \phi(x_i)\| < \delta/2$ for some $i = 1,\ldots,m$ (where $\mathbb{R}^k$ is given the sup-norm). There exist non-negative continuous functions $h_1,\ldots,h_m : \mathbb{R}^k \to \mathbb{R}$ such that

$$\sum_{i=1}^{m} h_i(y) \le 1 \quad \text{for all} \quad y \in \mathbb{R}^k,$$

$$\sum_{i=1}^{m} h_i(y) = 1 \quad \text{for} \quad y \in \bigcup_{j=1}^{m} B_{\mathbb{R}^k}(\phi(x_j), \delta/2), \text{and}$$

$$\operatorname{spt} h_i \subset B_{\mathbb{R}^k}(\phi(x_i),\delta) \quad \text{for} \quad i = 1,\ldots,m.$$

Choose polynomials $q_1,\ldots,q_m : \mathbb{R}^k \to \mathbb{R}$ such that for $i = 1,\ldots,m$,

$$\sup \{ |q_i(y) - h_i(y)| : y \in \bigcup_{j=1}^{m} B_{\mathbb{R}^k}(\phi(x_j),\delta) < \varepsilon/(m \cdot \max (1, \|f(x_i)\|)).$$

Then, the function $g : E \to F$ defined by $g(x) = \sum_{i=1}^{m} q_i \circ \phi(x) \cdot f(x_i) \in A \otimes F$ and, for all $x \in B$,

$$\|g(x) - f(x)\| \leq \sum_{i=1}^{m} |q_i(\phi(x)) - h_i(\phi(x))| \, \|f(x_i)\| + \left\| \sum_{i=1}^{m} h_i(\phi(x))(f(x_i) - f(x)) \right\| < 2\varepsilon ,$$

since $\sum_{i=1}^{m} h_i(\phi(x)) = 1$ for $x \in B$. Q.E.D.

When E is a separable Hilbert space with orthonormal basis $\{e_i\}$, Nemirovskiĭ and Semenov [6] have proved that the τ_b^o-completion of $\mathcal{P}(E)$ contains the regular functions on E, $\mathcal{R}(E)$. A mapping $f : E \to F$, where F is a Banach space, is said to be *regular* if for any bounded set $B \subset E$ and $\varepsilon > 0$, there is a finite set $\{A_1,\ldots,A_k\} \subset \mathcal{P}(^1E,E) \equiv \mathcal{L}(E,E)$ and $\delta > 0$ such that if $x,y \in B$ satisfy $|(A_j(x - y),e_i)| < \delta$ $(j=1,\ldots,k,\ i \in \mathbb{N})$, then $\|f(x) - f(y)\| < \varepsilon$. $\mathcal{R}(E,F)$ denotes the space of regular mappings from E to F, which is a Fréchet space with respect to the topology of uniform convergence on bounded subsets of E. Nemirovskiĭ and Semenov found that every $f \in \mathcal{R}(E)$ is contained in the τ_b^o - completion of $\mathcal{P}(E)$.

Since regular functions are bounded on bounded sets, $\mathcal{R}(E)$ is an algebra. It is not difficult to show that $a^1(E) \subset \mathcal{R}(E)$ and that

all functions f of the form $f(x) = \sum_{i=1}^{\infty} a_i \langle x, e_i \rangle^{n_i}$ are in $\mathcal{R}(E)$ provided $\sum_{i=1}^{\infty} |a_i|^2 < \infty$ and $n_i \geq 3$ for all i. On the other hand, $a^2(E) \not\subset \mathcal{R}(E)$ since $f(x) = \langle x,x \rangle \notin \mathcal{R}(E)$. Indeed, suppose $f \in \mathcal{R}(E)$, let $\gamma > 0$, and let $g : E \to \mathbb{R}$ be any function which is uniformly continuous on bounded subsets of E. For appropriate $\varepsilon > 0$, there are $A_1, \ldots, A_k \in \mathcal{L}(E,E)$ and $\delta > 0$ such that if $x, y \in B_1$ satisfy $|\langle A_j(x - y), e_i \rangle| < \delta$, then $\|x - y\| < \varepsilon$, and so $\|g(x) - g(y)\| < \gamma$. Thus any such g, being regular, is a τ_b^o-limit of polynomials, which contradicts the previously mentioned result of Nemirovskiĭ and Semenov.

It is trivial that $\mathcal{R}(E,E)$ is closed under composition on the left by elements of $\mathcal{L}(E,E)$. It is also closed under composition on the right by elements of $\mathcal{L}(E,E)$ since, if $A_1, \ldots, A_k \in \mathcal{L}(E,E)$ correspond to a given $f \in \mathcal{R}(E,E)$ and $\varepsilon > 0$, then $A_1 \circ T, \ldots, A_k \circ T \in \mathcal{L}(E,E)$ will correspond to $f \circ T$, for $T \in \mathcal{L}(E,E)$. In particular, the space $\mathcal{R}(E,E) \cap \mathcal{L}(E,E)$ is a closed 2-sided ideal in $\mathcal{L}(E,E)$ which contains the finite rank operators. Hence, either $\mathcal{R}(E,E) \cap \mathcal{L}(E,E) = \mathcal{L}_K(E,E)$, the compact operators, or $\mathrm{id} \in \mathcal{R}(E,E)$. In the second case, it follows that for $\varepsilon > 0$, there are $A_1, \ldots, A_k \in \mathcal{L}(E,E)$ and $\delta > 0$ such that if $x, y \in B_1$ satisfy $|\langle A_j(x - y), e_i \rangle| < \delta$ $(j = 1, \ldots, k,\ i \in \mathbb{N})$, then $\|x - y\| < \varepsilon$. But then

$$|\|x\|^2 - \|y\|^2| = |\sum_{i=1}^{\infty} (x_i^2 - y_i^2)| \leq \sum_{i=1}^{\infty} |x_i - y_i||x_i + y_i| \leq$$

$$\leq \Big(\sum_{i=1}^{\infty} (x_i - y_i)^2\Big)^{1/2} \|x + y\| < 2\varepsilon .$$

However, since $\|\ \|^2$ is not regular, $\mathrm{id} \notin \mathcal{R}(E,E)$.

Finally, we briefly review the case of differentiable approximation by polynomials. In [7], the authors examined $a^1(E)$ with respect to the topologies τ_c^n for $n = 0, 1, \ldots, \infty$. Here, τ_c^n is the locally convex topology generated by all seminorms of the form

$$f \in a^1(E) \to \sup \{ |\hat{d}^j f(x)(y)| : x \in K , y \in L\},$$

where K and L are compact subsets of E, $j \in \mathbb{N}$, $j \leq n$, and $\hat{d}^j f(x)$ denotes the jth-Taylor coefficient of f at x, an element of $P(^jE)$. It was found that if E has the approximation property, then $(a^1(E),\tau_c^n)\hat{} = C^n(E)$, for $n = 1,2,\ldots,\infty$. One notes that in this case, $(a^1(E),\tau_c^n)\hat{} = (P(E),\tau_c^n)\hat{}$. The completion of $a^1(E)$ with respect to τ_u^n (resp. τ_b^n) was studied in [2], where these locally convex topologies are defined by the generating family of seminorms of the form

$$f \in a^1(E) \to \sup \{ |\hat{d}^j f(x)(y)| : x \in X, y \in E, \|y\| \leq 1\} ,$$

where $X \subset E$ varies among the compact (resp. bounded) subsets of E and $j \in \mathbb{N}$, $j \leq n$. (Of course, since E is real, $(a^1(E),\tau)\hat{} = C(E)$ by the Stone-Weierstrass theorem for $\tau = \tau_u^o = \tau_c^o$). For $n \geq 1$, let $\overline{P_f(^nE)} \equiv$ the closure in $P(^nE)$ of span $\{\varphi^n : \varphi \in E'\} = (a^1(E),\tau_b^o)\hat{} \cap P(^nE)$. In the case of τ_u^n, it was shown that if E' has the bounded approximation property, then $(a^1(E),\tau_u^n)\hat{} = \{f \in C^n(E) : \hat{d}^j f(x) \in \overline{P_f(^jE)}$ for all $x \in E$, $j \in \mathbb{N}$, $j \leq n\}$. Let $C_{wu}(E,F)$ be the space of functions $f : E \to F$ which are weakly uniformly continuous on bounded subsets of E. That is, $C_{wu}(E,F) = \{f : E \to F :$ for all bounded sets $B \subset E$ and all $\varepsilon > 0$, there is a finite set $\{\varphi_1,\ldots,\varphi_k\} \subset E'$ and $\delta > 0$ such that if $x,y \in B$ satisfy $|\varphi_i(x - y)| < \delta$ $(i = 1,\ldots,k)$, then $\|f(x) - f(y)\| < \varepsilon\}$. Then, if E' has the bounded approximation property, $(a^1(E),\tau_b^n)\hat{} = \{f \in C^n(E) :$ for all $x \in E$, $j \in \mathbb{N}$, $j \leq n$, (i) $\hat{d}^j f(x) \in \overline{P_f(^jE)}$ and (ii) $\hat{d}^j f \in C_{wu}(E,P(^jE))\}$.

SECTION 2. In this section, E and F are complex Banach spaces. We first recall some of the usual topologies on $H(E,F)$. τ_o is the compact-open topology on $H(E,F)$. τ_∞ is the compact-open topology of

infinite order, generated by seminorms of the form

$$f \in H(E,F) \to \sup \{ \| \hat{d}^j f(x)(y) \| : x \in K , \ y \in E, \| y \| \leq 1 \},$$

where $K \subset E$ is compact and $j \in \mathbb{N}$. We will consider locally convex topologies τ between τ_∞ and τ_δ, where τ_δ is the bornological togolopgy associated to τ_o; in particular, our results are valid for the Nachbin ported topology τ_ω. In this section, we study the completion of $a^1(E)$ and $a^\infty(E)$ with respect to these topologies. In [1], this study is continued for the topology τ_b^o.

Of course, $H(E) = P(E)\hat{}$ for τ_δ, and hence for all weaker topologies, via the Taylor series expansion. Also, if E has the approximation property, then given a compact set $K \subset E$, $\varepsilon > 0$, and $f \in H(E)$, we can select $T \in E \otimes E$ such that $\| f - f \circ T \|_K \equiv \sup \{ |f(x) - f \circ T(x)| : x \in K \} < \varepsilon$. Then, since $T(E)$ is finite dimensional, we can find a complex polynomial $P : T(E) \to \mathbb{C}$ such that $\| f|_{T(E)} - P \|_{T(K)} < \varepsilon$. Thus, $\| f - P \circ T \|_K < 2\varepsilon$. Since $P \circ T \in a^1(E)$, we have shown that $(a^1(E), \tau_o)\hat{} = H(E)$. The converse implication, if $(a^1(E), \tau_o)\hat{} = H(E)$ then E has the approximation property, is apparently unknown.

To study $(a^1(E), \tau)\hat{}$, we first remark that $(a^1(E), \tau)\hat{} = \{ f \in H(E) : \hat{d}^n f(x) \in \overline{P_f(^nE)}$ for all $x \in E$ (equivalently, for $x = 0$) and all $n \in \mathbb{N} \}$. Now we consider holomorphic mappings $f : E \to F$ which are also in the space $C_{wu}(E,F)$ defined at the end of Section 1. We denote by $P_{wu}(^nE,F)$ the intersection of $P(^nE,F)$ and $C_{wu}(E,F)$; $P_{wu}(^1E,F)$ is also denoted $\mathcal{L}_{wu}(E,F)$. In [2], the following properties of spaces of functions which are weakly uniformly continuous on bounded sets were proved.

PROPOSITION 2: (a) *If* $f \in C_{wu}(E,F)$, *then* $\overline{f(B)}$ *is compact in* F *for all bounded sets* $B \subset E$.

(b) $P_f(^nE) \otimes F \subset P_{wu}(^nE,F)$ *for all* $n \in \mathbb{N}$.

(c) $P_{wu}(^nE,F)$ *is a closed subspace of* $P(^nE,F)$.

(d) *Let* $P \in P(^nE,F)$ *with associated symmetric* n*-linear mapping* A. *Then* $P \in P_{wu}(^nE,F)$ *if and only if the mapping* $C \in \mathcal{L}_{wu}(E,P(^{n-1}E,F))$, *where* $C(x)(y) \equiv Axy^{n-1}$. *In this case, the range of* C *is in fact contained in* $P_{wu}(^{n-1}E,F)$.

(e) E' *has the approximation property if and only if for every Banach space* F *and every* n, $\overline{P_f(^nE) \otimes F} = P_{wu}(^nE,F)$.

Let $P_K(^nE,F)$ denote the closed subspace of $P(^nE,F)$ consisting of all those polynomials P such that $\overline{P(B_1)}$ is compact in F. The space of compact holomorphic mappings from E to F, $H_K(E,F)$, is the set of holomorphic mappings $f : E \to F$ such that for each $x \in E$ (equivalently, for $x = 0$) there is a neighborhood U of x such that $\overline{f(U)}$ is compact in F. In [3], the following is proved.

PROPOSITION 3: *Let* $f \in H(E,F)$. *Then* $f \in H_K(E,F)$ *if and only if for every* $x \in E$ *(equivalently, for* $x=0$*) and* $n \in \mathbb{N}$, $\hat{d}^n f(x) \in P_K(^nE,F)$.

PROPOSITION 4: *Let* $P \in P(^nE)$. *Then the following conditions are equivalent.*

(a) $P \in P_{wu}(^nE)$,

(b) $\hat{d}P \in P_{wu}(^{n-1}E,E')$,

(c) $\hat{d}P \in P_K(^{n-1}E,E')$.

PROOF: (a) $\Rightarrow$ (b): Let $\varepsilon > 0$ and (without loss of generality) let $B = B_1$, and choose $\delta > 0$ and $\{\varphi_1,\ldots,\varphi_k\} \subset E'$ as in the definition of $C_{wu}(E)$. By the polarization formula, if $x, y \in B_1$ satisfy $|\varphi_i(x - y)| < \delta$ $(i = 1,\ldots,k)$ and if A is the symmetric n-linear mapping corresponding to P,

$$\|Ax^{n-1} - Ay^{n-1}\| = \sup \{ |Ax^{n-1}z - Ay^{n-1}z| : z \in B_1 \}$$

$$= \sup\{|\frac{1}{2^n n!} \sum_{\substack{\varepsilon_i = \pm 1 \\ 1 \le i \le n}} P(\varepsilon_1 x + \ldots + \varepsilon_{n-1} x + \varepsilon_n z) - P(\varepsilon_1 y + \ldots + \varepsilon_{n-1} y + \varepsilon_n z| : z \in B_1\}$$

$$\le \frac{n^n}{n!} \sup\{ \left| \frac{P(\varepsilon_1 x + \ldots + \varepsilon_{n-1} x + \varepsilon_n z)}{n} - \frac{P(\varepsilon_1 y + \ldots + \varepsilon_{n-1} y + \varepsilon_n z)}{n} \right| :$$

$$z \in B_1 , \varepsilon_i = \pm 1, \ 1 \le i \le n\}.$$

Since $\tilde{x} = \frac{1}{n}(\varepsilon_1 x + \ldots + \varepsilon_{n-1} x + \varepsilon_n z)$ and $\tilde{y} = \frac{1}{n}(\varepsilon_1 y + \ldots + \varepsilon_{n-1} y + \varepsilon_n z)$ are in B_1 and $|\varphi_i(\tilde{x} - \tilde{y})| < \delta$ $(i = 1,\ldots,k)$, we conclude that $\|Ax^{n-1} - Ay^{n-1}\| < \varepsilon \frac{n^n}{n!}$. Since $\hat{d}P(x) = nAx^{n-1}$, we conclude that $\hat{d}P \in P_{wu}({}^{n-1}E,E')$.

(b) $\Rightarrow$ (c): This implication follows from Proposition 2(a).

(c) $\Rightarrow$ (a): Assume $\hat{d}P \in P_K({}^{n-1}E,E')$, so that the mapping $x \in E \to \frac{1}{n} \hat{d}P(x) = Ax^{n-1} \in E'$ is compact. By the polarization formula, it follows that the $n-1$ linear mapping

$$(x_1,\ldots,x_{n-1}) \in E^{n-1} \to A(x_1,\ldots,x_{n-1}) \in E'$$

is compact. Hence, given $\varepsilon > 0$, there is a finite set $\{\varphi_1,\ldots,\varphi_k\} \subset E'$ such that for any $(x_1,\ldots,x_{n-1}) \in B_1^{n-1}$, $\|A(x_1,\ldots,x_{n-1}) - \varphi_i\| < \varepsilon$ for some $i = 1,\ldots,k$. Assume now that $x, y \in B_1$ satisfy $|\varphi_i(x - y)| < \varepsilon$ $(i = 1,\ldots,k)$. Then by the symmetry of A,

$$|P(x) - P(y)| = |Ax^{n-1}x - Ay^{n-1}y| \le$$

$$\le |Ax^{n-1}x - Ax^{n-1}y| + |Ayx^{n-2}x - Ayx^{n-2}y| + \ldots$$

$$\ldots + |Ay^{n-1}x - Ay^{n-1}y|.$$

For an arbitrary term above,

$$|Ay^j x^{n-j-1} x - Ay^j x^{n-j-1} y| \leq$$

$$\leq |(Ay^j x^{n-j-1} - \varphi_i)(x)| + |\varphi_i(x-y)| + |(\varphi_i - Ay^j x^{n-j-1})(y)| <$$

$$< 3\varepsilon ,$$

so that $|P(x) - P(y)| < 3\varepsilon n$. Q.E.D.

Note that in (c) $\Rightarrow$ (b) above, it is essential that the polynomial be a derivative since, in general, $P_{wu}({}^n E,F)$ is properly contained in $P_K({}^n E,F)$. For example, if E' has the approximation property and $F = \mathbb{C}$, then $P_{wu}({}^n E) = \overline{P_f({}^n E)} \subset P_K({}^n E) = P({}^n E)$, and the inclusion is proper in general.

Combining the above results, we get the following.

THEOREM 5: *Let* $f \in H(E)$. *Then* $\hat{d}f \in H_K(E,E')$ *if and only if for all* $n \in \mathbb{N}$ *and all* $x \in E$, $\hat{d}^n f(x) \in P_{wu}({}^n E)$. *Furthermore, if* E' *has the approximation property, then* $f \in (a^1(E),\tau)\hat{}$ *if and only if* $\hat{d}f \in H_K(E,E')$.

PROOF: We only prove the second part of the theorem. $f \in (a^1(E),\tau)\hat{}$ if and only if for any $n \in \mathbb{N}$ and $x \in E$, $\hat{d}^n f(x) \in \overline{P_f({}^n E)} = P_{wu}({}^n E)$ by Proposition 2(e). By Proposition 4, $\hat{d}^n f(x) \in P_{wu}({}^n E)$ if and only if $\hat{d}(\hat{d}^n f(x)) \in P_K({}^{n-1}E,E')$. Since $\hat{d}f = \sum_{n=1}^{\infty} \hat{d}(\frac{\hat{d}^n f(x)}{n!})$, the result follows by Proposition 3. Q.E.D.

Finally, we remark that in [1], we show that for any complex Banach space E, given $f \in H(E)$, $\hat{d}f \in H_K(E,E')$ if and only if f is locally weakly uniformly continuous.

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ANALYTIC HYPOELLIPTICITY OF OPERATORS OF PRINCIPAL TYPE

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Let

$$P(x,D) = P_m(x,D) + P_{m-1}(x,D) + \ldots \tag{1}$$

be a differential operator with analytic coefficients in an open set Ω of $\mathbb{R}^N$. Suppose that P is of principal type and, in addition, satisfies the hypoellipticity condition: along the null bicharacteristic strip of $\mathrm{Re}(P_m)$, the function $\mathrm{Im}(P_m)$ has only zeros of even order. It follows from [2] that the differential operator P is analytic hypoelliptic. Indeed, in his paper [2], Treves proves that, for differential operators of principal type, the following properties are equivalent: hypoellipticity; analytic-hypoellipticity; sub-ellipticity and the above condition on the zeros of the imaginary part of P_m.

Our aim is to present another proof of the fact that, for operators of principal type, the hypoellipticity condition above implies analytic-hypoellipticity. By using the factorization formula for pseudodifferential operators, we can replace, modulo analytic regularizing operators, the differential operator P by an analytic pseudodifferential operator $L = D_t - A(x,t,D_x)$ of order 1, where $A(x,t,D_x)$ is an analytic pseudodifferential operator of order 1, with respect to the variable x, only, and analytic coefficients depending on

(x,t).

Let, then,

$$L = D_t - A(x,t,D_x) \tag{2}$$

be a first order analytic pseudodifferential operator defined in Ω an open set in $\mathbb{R}^{n+1}$ which we can assume, without loss of generality, to contain the origin. Let

$$\sigma(L) = (\tau - \lambda(x,t,\xi)) + \lambda_0(x,t,\xi) + \lambda_{-1}(x,t,\xi) + \ldots \tag{3}$$

be the symbol of L, where the principal symbol $\tau - \lambda(x,t,\xi)$ is an analytic function of all its variables on $\Omega \times \mathbb{R}_{n+1} \setminus \{0\}$, homogeneous of degree 1 with respect to (ξ,τ), while each $\lambda_{-j}(x,t,\xi)$ is an analytic function in $\Omega \times \mathbb{R}_n \setminus \{0\}$, homogeneous of degree $-j$ with respect to ξ. We shall reason in a conic neighborhood of the point $(0,0,\xi^0,\tau^0)$ such that $\tau^0 = \lambda(0,0,\xi^0)$ with $(\xi^0,\tau^0) \neq (0,0)$. If we write $\lambda(x,t,\xi) = a(x,t,\xi) + ib(x,t,\xi)$ our basic assumption will be the following one:

(4) *the function* $b(x,t,\xi)$ *has only zeros of finite even order* $\leq 2k$ *along the integral curves of the differential system*

$$\frac{dx}{dt} = -\operatorname{grad}_\xi a(x,t,\xi), \quad \frac{d\tau}{dt} = \operatorname{grad}_x a(x,t,\xi)$$

contained in $W \times \Gamma'$, *where* Γ' *is the projection of a suitable cone in* $\mathbb{R}_{n+1}$ *containing* (ξ^0,τ^0).

THEOREM: *Under the above assumptions, the operator* L *has a local parametrix. More precisely, there is a continuous linear operator*

$K : C_c^\infty(U) \to \mathcal{D}'(U)$, *where* U *is a suitable neighborhood of the origin in* $\mathbf{R}^{n+1}$, *whose distribution kernel* $\mathbb{K}(x,t,y,s)$, *defined in* $U \times U$, *satisfies*

$$L\mathbb{K}(x,t,y,s) = \delta(x-y,t-s) + \mathbb{R}(x,t,y,s) \tag{5}$$

with the following additional properties:

a) $\mathbb{K}$ *is a regular kernel with respect to the variables* (x,t) *and* (y,s), *i.e., the mappings*

$$u(y,s) \to \langle \mathbb{K}(x,t,y,s), u(y,s)\rangle \quad \textit{and} \quad v(x,t) \to \langle \mathbb{K}(x,t,y,s), v(x,t)\rangle$$

transform $C_c^\infty(U)$ *into* $C^\infty(U)$;

b) *the kernel* $\mathbb{K}(x,t,y,s)$ *is an analytic function whenever* $(x,t) \neq (y,s)$;

c) *the operator* K *and its transpose* ${}^tK$ *are analytic-pseudo local in the following sense: let* ω *be any relatively compact open subset of* U *and let* $u \in E'(U)$; *if* u *is analytic in* ω, *then* Ku *and* ${}^tKu$ *are also analytic in* ω;

d) $\mathbb{R}(x,t,y,s)$ *is an analytic function in* $U \times U$.

PROOF: Assuming that the conic neighborhood $W \times \Gamma'$ is connected (which is always possible), then condition (4) implies that $b(x,t,\xi)$ never changes sign in $W \times \Gamma'$. Moreover, in what follows we are going to assume that $b \geq 0$. The case $b \leq 0$ is treated in a similar way.

Define the Fourier integral operator

$$Ku(x,t) = (2\pi)^{-n-1} \int\int e^{i(\langle x,\xi\rangle + t\tau)} K(x,t,\xi,\tau)\hat{u}(\xi,\tau)\,d\xi\,d\tau \tag{6}$$

with

$$(7) \qquad K(x,t,\xi,\tau) = \int_{-T}^{t} e^{i\phi(x,t,t',\xi)-i(t-t')\tau} k(x,t,t',\xi)\,dt'$$

where T is a small number greater than 0 to be chosen later and where the *phase function* ϕ and the *amplitude function* k are to be determined in such a way that

$$(8) \qquad LKu = u + Ru, \quad \text{for all} \quad u \in C_c^\infty(U),$$

with R an analytic regularizing operator.

We choose the phase function $\phi(x,t,t',\xi)$ as the solution of

$$(9) \qquad \begin{cases} \phi_t - \lambda(x,t,\xi + \phi_x) = 0 \\ \phi_{|t=t'} = 0, \end{cases}$$

with t and t' belonging to the interval $[-T,T]$. Since $\lambda(x,t,\xi)$ is an analytic function of all its variables, there is a unique solution ϕ of (9), analytic with respect to all its variables and homogeneous of degree 1 with respect to ξ.

As for the amplitude function, write

$$(10) \qquad k(x,t,t',\xi) = \sum_{\nu=0}^{\infty} k_\nu(x,t,t',\xi)$$

as a formal sum where each term k_ν is homogeneous of degree $-\nu$ with respect to ξ. The functions k_ν, $\nu = 0,1,2,\ldots$, are obtained as solutions of the following transport equations:

$$(11) \qquad \begin{cases} D_t k_0 - \sum_{j=1}^{n} \lambda_{\xi_j}(x,t,\xi + \phi_x)\, D_{x_j} k_0 + Ck_0 = 0 \\ k_{0|t=t'} = i \end{cases}$$

and

$$(12)\qquad \begin{cases} D_t k_\nu - \sum_{j=1}^{n} \lambda_{\xi_j}(x,t,\xi+\phi_x) D_{x_j} k_\nu + C k_\nu + \sum_{\nu'=0}^{\nu-1} Q_{\nu-\nu'} k_{\nu'} = 0 \\ k_\nu\big|_{t=t'} = 0 , \end{cases}$$

with $\nu = 1,2,\ldots$.

Setting

$$(13)\qquad K_j(x,t,\xi,\tau) = \int_{-T}^{t} e^{i\phi(x,t,t',\xi)-i(t-t')\tau} k_j(x,t,t',\xi)\,dt'$$

it can be proved, using suitable estimates for ϕ and k_j, that

$$(14)\qquad k = \sum_{j=0}^{\infty} k_j(x,t,t',\xi)$$

is an analytic symbol [1] and that K defined by (6) is a pseudodifferential operator. Finally, one can show that its distribution kernel $\mathbb{K}$ satisfies (5) and properties a), b), c) and d) of the theorem.

The existence of such a kernel implies then the analytic hypoellipticity of P.

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KOROVKIN APPROXIMATION IN FUNCTION SPACES

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INTRODUCTION

The starting point of this survey lecture is Korovkin approximation for a linear space $\mathcal{H}$ of continuous real-valued functions on a compact metrizable space X where the approximating operators are defined on the total space $C(X)$ of continuous real-valued functions on X. This type of setting is called here absolute Korovkin approximation. Chapter I recalls the main results, in particular the characterization of the Korovkin closure of the given function space $\mathcal{H}$. Motivations, details and references to the relevant literature can be found in the author's survey article [3].

Chapter II is devoted to the problem of determining the Korovkin closure in cases where it is not all of $C(X)$. The main tools are the introduction of the state space of $\mathcal{H}$ and the use of convexity arguments. The results of this Chapter arose from discussions with G. Leha. Details will be published elsewhere.

Chapter III studies problems of the so-called theory of relative Korovkin approximation. Here the approximating operators are no longer defined on all of $C(X)$ but rather on a fixed closed linear subspace $\mathcal{L}$ of $C(X)$ containing $\mathcal{H}$. Most of the results of this Chapter are due to Leha [7].

I. ABSOLUTE KOROVKIN APPROXIMATION

We shall treat here absolute Korovkin approximation only for spaces of continuous functions on a compact, even metrizable space though the main results essentially remain true for locally compact spaces [4].

Consequently, let X be a *compact metrizable* space, denote by $C(X)$ the linear space of all continuous real-valued functions on X and by $\mathcal{H}$ a *function space* (on X), i.e. a point separating, linear subspace of $C(X)$ containing the constant function 1. The space $C(X)$ will be considered as a normed space equipped with the usual norm $\|\cdot\cdot\|$ of uniform convergence (sup-norm).

A sequence $(T_n)_{n \in \mathbb{N}}$ of positive linear operators

$$T_n : C(X) \to C(X)$$

will be called ($\mathcal{H}$-) admissible if

$$\lim_{n\to\infty} \| T_n h - h \| = 0 \qquad \text{for all } h \in \mathcal{H}.$$

A function $f \in C(X)$ satisfying

$$\lim_{n\to\infty} \| T_n f - f \| = 0$$

for all admissible sequences will be called a *Korovkin function* (with respect to $\mathcal{H}$). The set of all these functions is the *Korovkin closure* $\mathrm{Kor}(\mathcal{H})$ of $\mathcal{H}$. Obviously, it is a linear space satisfying

$$\mathcal{H} \subset \mathrm{Kor}(\mathcal{H}) \subset C(X).$$

$\mathcal{H}$ is called a *Korovkin space* if $\mathrm{Kor}(\mathcal{H}) = C(X)$.

The Korovkin closure can be characterized by means of the following envelope technique: For an arbitrary $f \in C(X)$ two envelopes are defined:

$$\hat{f} = \inf \{h \in \mathcal{K} : h \geq f\}$$

and

$$\underset{\vee}{f} = \sup \{h \in \mathcal{K} : h \leq f\}.$$

Functions $f \in C(X)$ for which $\hat{f} = \underset{\vee}{f} (= f)$ are called $\mathcal{K}$-*affine*. The set $\hat{\mathcal{K}}$ of these functions is a linear subspace of $C(X)$ containing $\mathcal{K}$. It turns out to coincide with the Korovkin closure:

THEOREM 1: $$\mathrm{Kor}(\mathcal{K}) = \hat{\mathcal{K}}.$$

Another characterization of the Korovkin closure is obtained by means of the representing measures. A positive Radon measure μ on X is a *representing measure* for a point $x \in X$ (with respect to $\mathcal{K}$) if

$$\int h d\mu = h(x) \qquad \text{for all} \quad h \in \mathcal{K}.$$

The set of these measures will be denoted by $M_x(\mathcal{K})$. It always contains the Dirac measure ε_x defined by the unit mass in x.

The set $M_x(\mathcal{K})$ is then described by the following key lemma:

LEMMA 1: *For* $f \in C(X)$ *and* $x \in X$, *we have*

$$\left\{ \int f d\mu : \mu \in M_x(\mathcal{K}) \right\} = \left[\underset{\vee}{f}(x), \hat{f}(x) \right].$$

This leads to a new description of the functions in $\hat{\mathcal{K}}$, and hence of

the Korovkin closure: A function $f \in C(X)$ is $\mathcal{H}$-affine if and only if

$$\int f d\mu = f(x)$$

for all $x \in X$ and all representing measures $\mu \in M_x(\mathcal{H})$.

As a consequence of this and Theorem 1, we obtain a characterization of Korovkin spaces. It uses the notion of the *Choquet boundary* $\partial_{\mathcal{H}}X$ of X with respect to $\mathcal{H}$ which by definition is the set of points $x \in X$ permitting only ε_x as a representing measure:

$$\partial_{\mathcal{H}}X = \{x \in X : M_x(\mathcal{H}) = \{ \varepsilon_x \}\}.$$

THEOREM 2: *The given function space $\mathcal{H}$ is a Korovkin space if and only if* $\partial_{\mathcal{H}}X = X$.

It is this result which allows in many concrete examples a quick proof of a Korovkin-type theorem. In particular, Korovkin's classical result follows almost immediately. It states that, for a compact interval $X = [a,b]$ on the real line $\mathbb{R}$, the linear hull of the three functions 1, id, id^2 (id = identity map $x \to x$) is a Korovkin space.

II. DETERMINATION AND GEOMETRICAL INTERPRETATION OF THE KOROVKIN CLOSURE

In the existing literature few attention has been given to the determination of $\mathrm{Kor}(\mathcal{H})$ for the case where $\mathcal{H}$ is not a Korovkin space, hence where $\partial_{\mathcal{H}}X \neq X$. We shall have a closer look at this problem in particular for the case $X = [a,b]$ where $\mathcal{H}$ is the linear hull of 1, id and a third function $u \in C(X)$. A direct application of Theorem 1 in connection with the characterization of $\mathcal{H}$-affine functions by means of representing measures turns out to be difficult, in

general. However, ideas from the theory of integral representation in convex compact sets lead to a satisfactory method.

Continuing in the general situation of Chapter I, we denote by $S = S(\mathcal{H})$ the *state space* of $\mathcal{H}$ consisting of all positive linear forms $\varphi : \mathcal{H} \to \mathbb{R}$ which are normalized, i.e. $\varphi(1) = 1$. S is a convex subset of the topological dual $\mathcal{H}'$; it is *compact metrizable* in the space $E = \mathcal{H}'$ equipped with the weak topology $\sigma(\mathcal{H}', \mathcal{H})$. There is a canonical continuous embedding $j : X \to S$, namely $j(x) = \delta_x$ where δ_x is the evaluation functional for $x \in X$. The following properties are well-known ([1], pp. 79 - 82; [2], pp. 121 - 125).

$$j(\partial_{\mathcal{H}} X) = \text{ex } S \quad \text{(the set of extreme points)};$$

$$\mathcal{H} = \{l \circ j : l \in E'\};$$

in particular,

$$S = \overline{\text{conv}}\, j(X),$$

and

$$h = l \circ j \to l \mid S$$

is an order and norm preserving bijection of $\mathcal{H}$ onto a dense subspace of $A(S)$, the space of all continuous affine functions $a : S \to \mathbb{R}$. For the case $\dim \mathcal{H} < +\infty$ or, more generally, for $\mathcal{H}$ closed in $C(X)$ this is a bijection of $\mathcal{H}$ onto $A(S)$.

Let us consider now a compact set Y such that $\text{ex } S \subset Y \subset S$ (like $Y = j(X)$) and let us define for $g \in C(Y)$ the "geometrical" envelopes

$$\bar{g}_Y = \inf \{a \in A(S) : a \geq g \text{ on } Y\}$$

and

$$\underline{g}_Y = \sup \{a \in A(S) : a \leq g \quad \text{on} \quad Y\}.$$

By $A(Y,S)$ we denote the space

$$A(Y,S) = \{g \in C(Y) : \underline{g}_Y = \bar{g}_Y \quad \text{on} \quad Y\}.$$

As a consequence of the above properties of $j : X \to S(\mathcal{H})$ we obtain the canonical isomorphism

$$\phi : A(j(X),S) \to \hat{\mathcal{H}}$$

$$g \to g \circ j$$

of $A(j(X),S)$ onto $\hat{\mathcal{H}}$.

Example: Consider $X = [a,b]$ a compact interval in $\mathbb{R}$ and the function space $\mathcal{H}_u = \text{lin}\{1, \text{id}, u\}$ with a given function $u \in C([a,b])$. Since $j(X) = \{1\} \times G_u$ where G_u is the graph of u, the state space S can be identified with the closed convex hull of G_u.

Suppose that u is *neither convex nor concave*. Then there exists a point $y \in G_u$ such that S is the *face* F_y of S generated by $y \in S$. Consequently, for every $g \in A(G_u,S)$ the concave function $\bar{g}_{G_u} - \underline{g}_{G_u}$, defined on S, vanishes at y and hence at every point of $F_y = S$. This proves that $\bar{g}_{G_u} = \underline{g}_{G_u}$ is affine on S. Hence the restriction of functions of $A(S)$ to G_u defines a canonical isomorphism between $A(S)$ and $A(G_u,S)$. Consequently, $\hat{\mathcal{H}}_u = \mathcal{H}_u$.

Therefore, we can only expect to have $\hat{\mathcal{H}}_u = \text{Kor}(\mathcal{H}_u) \neq \mathcal{H}_u$ if u is concave or convex. If follows from Theorem 2 that $\hat{\mathcal{H}}_u = C(X)$ is

equivalent to the strict concavity or convexity of u. Since $id \in \mathcal{H}_u$ one only has to observe that every representing measure $\mu \in M_x(\mathcal{H}_u)$ has x as barycenter.

So let us assume that u is an element of the set K of all concave functions in $C([a,b])$. In what follows we shall get additional information about the behaviour of the map

$$u \to \mathrm{Kor}(\mathcal{H}_u) = \hat{\mathcal{H}}_u$$

defined on K. We can introduce a pre-order relation on K by defining $u \prec v$ $(u,v \in K)$ if $M_x(\mathcal{H}_v) \subset M_x(\mathcal{H}_u)$ holds for all $x \in [a,b]$. Then the relation $u \prec v$ expresses that v is more concave than u. As consequence of the characterization of $\mathcal{H}$-affine functions by means of representing measures and of Theorem 1 we obtain the implication

$$u \prec v \Rightarrow \mathrm{Kor}(\mathcal{H}_u) \subset \mathrm{Kor}(\mathcal{H}_v) \qquad (u,v \in K)$$

There are two extreme cases: the affine functions on [a,b] are the minimal, the strictly convex continuous functions on [a,b] are the maximal elements of K. The corresponding Korovkin closures are lin {1, id} = $A([a,b])$ and $C([a,b])$, respectively.

Much better results can be obtained by making use of Alfsens's notion of boundary (affine) dependencies [1]. We return to the general framework of this Chapter. For a point $y \in S$ the set B_y of all *boundary dependencies* is, by definition, the linear space of all signed Radon measures ν on S which are supported by the G_δ-set ex S and which annihilate all affine continuous functions on S:

$$\int a\,d\nu = 0 \qquad \text{for all} \quad a \in A(S).$$

As a consequence of the minimum principle for lower semicontinuous concave functions, a function $g \in A(Y,S)$ is uniquely determined by its restriction to $\operatorname{ex} S$, hence in particular to $\overline{\operatorname{ex} S}$. Therefore, $g \to g \mid \overline{\operatorname{ex} S}$ defines an order and norm preserving isomorphism of $A(Y,S)$ onto a certain linear subspace of $C(\overline{\operatorname{ex} S})$. This subspace and, consequently, $A(Y,S)$ can be described as follows:

PROPOSITION 1: *A function* $q \in C(\overline{\operatorname{ex} S})$ *is the restriction of a function in* $A(Y,S)$ *if and only if the following two conditions are satisfied:*

(a) $q \in A(\overline{\operatorname{ex} S}, S)$;

(b) $\int q\,d\nu = 0$ *for all* $\nu \in B_y$ *and all* $y \in Y \setminus \overline{\operatorname{ex} S}$.

Condition (b) is still redundant. By using the face F_y of S, generated by a point $y \in S$, it can be improved:

PROPOSITION 2: *For every function* $q \in C(\overline{\operatorname{ex} S})$ *and every point* $y \in S$ *the following two conditions are equivalent:*

(a) $\int q\,d\nu = 0$ *for all* $\nu \in B_y$;

(b) $\int q\,d\nu = 0$ *for all* $\nu \in \bigcup_{z \in F_y} B_z$.

We return now to the discussion of the

E x a m p l e : We choose for u a concave polygon on $[a,b]$ with n proper vertices. This means that u is of the form

$$u = \inf (a_1, \ldots, a_{n+1})$$

where $a_1, \ldots, a_{n+1}$ are affine functions on $[a,b]$ such that $a_j \leq a_k$

holds only in the trivial case $j = k$ $(n = 1,2,\ldots)$. For an arbitrary point y in the interior of $S = \overline{\text{conv}}\, G_u = \text{conv}\, G_u$ we have $F_y = S$. Furthermore $\text{ex}\, S$ is the set of the $n+2$ vertices of S. Therefore it follows from the two preceding propositions that $A(S) = A(S,S)$ and hence $\mathcal{K}_u$ is canonically isomorphic to the linear space of all $q \in A(\text{ex}\, S,S)$ satisfying

$$\int q\,d\nu = 0 \qquad \text{for all} \qquad \nu \in B_y .$$

Furthermore we know that $\hat{\mathcal{K}}_u$ is (by means of ϕ) canonically isomorphic to $A(G_u,S)$ and hence to the linear space of all $q \in A(\text{ex}\, S,S)$ satisfying

$$\int q\,d\nu = 0 \quad \text{for all} \quad \nu \in B_z \quad \text{and all} \quad z \in G_u \setminus \text{ex}\, S.$$

The latter condition is empty since $B_z = \{0\}$ for all $z \in G_u$. Indeed, every $z \in G_u$ has a unique representation as barycenter of a probability measure on $\text{ex}\, S$ since F_z is a segment or reduces to an extreme point. Consequently, $\hat{\mathcal{K}}_u$ is canonically isomorphic to $A(\text{ex}\, S,S)$. It is easy to check that y lies in exactly n triangles with vertices in $\text{ex}\, S$. These produce $n-1$ linear independent vectors of B_y. Since B_y is determined by a system of 3 linear equations in $n+2$ variables, we obtain $\dim B_y = n-1$. This proves our final result, namely

$$\dim \hat{\mathcal{K}}_u = \dim \mathcal{K}_u + n - 1 = n + 2.$$

Formally, this equality also holds for $n = 0$.

The canonical isomorphism between $\hat{\mathcal{K}}_u$ and $A(\text{ex}\, S,S)$ makes at the same time clear that the elements of $\hat{\mathcal{K}}_u$ are piecewise affine.

More precisely: Let $x_1 < \ldots < x_n$ be those points in $[a,b]$ where u has proper vertices. Then $\hat{\mathcal{H}}_u$ is the space of functions in $C([a,b])$ which are affine on every interval $[x_i, x_{i+1}]$, $i = 0,\ldots,n$, where $x_0 = a$ and $x_{n+1} = b$. This can be seen also directly by means of the representing measures. This description of $\hat{\mathcal{H}}_u$ makes the number $n+2$ of its dimension evident.

III. RELATIVE KOROVKIN APPROXIMATION

We return now to the situation studied in Chapter I. Hence $\mathcal{H}$ is a function space on a compact metrizable space X. We shall speak of *relative* Korovkin approximation if the role of $C(X)$ in absolute Korovkin approximation is taken over by a *closed function space* $\mathcal{L}$ containing $\mathcal{H}$ as linear subspace:

$$\mathcal{H} \subset \mathcal{L} = \bar{\mathcal{L}} \subset C(X).$$

Consequently, a sequence $(T_n)_{n \in \mathbb{N}}$ of positive linear maps is called $(\mathcal{H},\mathcal{L})$-admissible if

$$\lim_{n\to\infty} \| T_n h - h \| = 0 \qquad \text{for all} \qquad h \in \mathcal{H}.$$

A function $f \in \mathcal{L}$ is called a *relative Korovkin function* (with respect to $\mathcal{H}$ and $\mathcal{L}$) if

$$\lim_{n\to\infty} \| T_n f - f \| = 0$$

holds for all $(\mathcal{H},\mathcal{L})$-admissible sequences. The set of these functions is the relative Korovkin closure $\mathrm{Kor}(\mathcal{H},\mathcal{L})$ of $\mathcal{H}$ with respect to $\mathcal{L}$. $\mathrm{Kor}(\mathcal{H},\mathcal{L})$ is a function space satisfying

$$\mathcal{H} \subset \mathrm{Kor}(\mathcal{H},\mathcal{L}) \subset \mathcal{L} .$$

$\mathcal{H}$ is called a *Korovkin space with respect to* $\mathcal{L}$ if $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{L}$. As in the absolute case the main problems are to characterize $\mathrm{Kor}(\mathcal{H},\mathcal{L})$ and to decide whether $\mathcal{H}$ is a Korovkin space with respect to $\mathcal{L}$.

Let us consider first three

Examples: 1) $X = [-1,+1]$, $\mathcal{H} = \mathrm{lin}\{1, \mathrm{id}, \mathrm{id}^3\}$ and $\mathcal{L} = \mathrm{lin}\{1, \mathrm{id}, \mathrm{id}^2, \mathrm{id}^3\}$. It follows from the considerations in Chapter II that

$$\partial_{\mathcal{H}} X = [-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1] \quad \text{and} \quad \partial_{\mathcal{L}} X = X.$$

We shall see that $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{L}$.

2) $X = [0,1]$, $\mathcal{L} = \{f \in C(X) : \int_0^1 f(x)\,dx = f(0)\}$, $\mathcal{H} = \{f \in \mathcal{L} : \frac{1}{2}(f(\frac{1}{2}) + f(1)) = f(0)\}$. It is easy to check that

$$\partial_{\mathcal{H}} X = \partial_{\mathcal{L}} X =]0,1] .$$

We shall see that $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{H}$.

3) Let X be the closed unit disk in $\mathbb{R}^2$, $\mathcal{H}$ the space $A(X)$ of all affine functions on X, and let $\mathcal{L}$ be the space of all functions in $C(X)$ which are harmonic in the open disk. Then $\partial_{\mathcal{H}} X = \partial_{\mathcal{L}} X$ is the unit circle, i.e. the topological boundary of X. We shall see that $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{L}$.

Šaškin [9] announced a result that - at least for the case of a finite dimensional space $\mathcal{H}$ - the condition $\partial_{\mathcal{H}} X = \partial_{\mathcal{L}} X$ is necessary and sufficient for $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{L}$. The first two examples show that

this is not true. However, we shall see that the equality of the Choquet boundaries $\partial_{\mathcal{H}}X$ and $\partial_{\mathcal{L}}X$ is sufficient for $\mathcal{H}$ to be a Korovkin space with respect to $\mathcal{L}$ if in addition the common boundary $\partial_{\mathcal{H}}X = \partial_{\mathcal{L}}X$ is closed. This additional condition is fulfilled in the third example.

Crucial for the relative theory is the notion of the *relative Choquet boundary* $\partial_{\mathcal{H}}^{\mathcal{L}}X$ which by definition is the set

$$\partial_{\mathcal{H}}^{\mathcal{L}}X = \{x \in X : M_x(\mathcal{H}) = M_x(\mathcal{L})\}.$$

For $\mathcal{L} = C(X)$ this is exactly the definition of the Choquet boundary $\partial_{\mathcal{H}}X$, hence $\partial_{\mathcal{H}}X = \partial_{\mathcal{H}}^{C(X)}X$. Immediate consequences of the definition are the following two remarks:

$$\partial_{\mathcal{H}}X = \partial_{\mathcal{L}}X \Leftrightarrow \partial_{\mathcal{L}}X \subset \partial_{\mathcal{H}}^{\mathcal{L}}X ;$$

$$\partial_{\mathcal{H}}^{\mathcal{L}}X \subset \partial_{\mathcal{L}}X \Leftrightarrow \partial_{\mathcal{H}}^{\mathcal{L}}X = \partial_{\mathcal{H}}X.$$

Accordingly, we have in the above Examples:

1) $\partial_{\mathcal{H}}^{\mathcal{L}}X = \partial_{\mathcal{H}}X = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$;

2) $\partial_{\mathcal{H}}^{\mathcal{L}}X = \partial_{\mathcal{H}}X = \partial_{\mathcal{L}}X =]0,1]$

(since $M_x(\mathcal{H}) \neq M_x(\mathcal{L})$ for the origin $x = 0$);

3) $\partial_{\mathcal{H}}^{\mathcal{L}}X = \partial_{\mathcal{H}}X = \partial_{\mathcal{L}}X =$ topological boundary

(since $M_x(\mathcal{H}) \neq M_x(\mathcal{L})$ for all interior points of X).

Also the notion of $\mathcal{H}$-affine functions will be generalized. For a function $f \in C(X)$ the definition of the envelopes $\hat{f}$ and $\check{f}$ is the one given in Chapter I. A function f will be called $(\mathcal{H}, \mathcal{L})$-*affine*

if $f \in \mathcal{L}$ and if

$$\hat{f}(x) = \underset{\vee}{f}(x) \qquad \text{for all} \qquad x \in \overline{\partial_{\mathcal{L}} X}.$$

The set $\hat{\mathcal{K}}^{\mathcal{L}}$ of these functions is a linear subspace of $\mathcal{L}$:

$$\mathcal{K} \subset \hat{\mathcal{K}}^{\mathcal{L}} \subset \mathcal{L}.$$

Obviously, for $\mathcal{L} = C(X)$ we have $\hat{\mathcal{K}}^{C(X)} = \hat{\mathcal{K}}$.

The $(\mathcal{K},\mathcal{L})$-affine functions do not play the same role as in the absolute theory. The following result generalizes only one part of Theorem 1.

PROPOSITION 3: *Every $(\mathcal{K},\mathcal{L})$-affine function is a relative Korovkin function:*

$$\hat{\mathcal{K}}^{\mathcal{L}} \subset \mathrm{Kor}(\mathcal{K},\mathcal{L}).$$

We sketch the proof: Let f be a function in $\hat{\mathcal{K}}^{\mathcal{L}}$. Then $\hat{f}(x) = \underset{\vee}{f}(x)$ on the set $S = \overline{\partial_{\mathcal{L}} X}$. Compactness of S then leads, for a given number $\varepsilon > 0$, to the existence of finitely many functions $h'_1,\ldots,h'_n$ and $h''_1,\ldots,h''_n$ in $\mathcal{K}$ such that the two functions $\underline{h} = \sup(h'_1,\ldots,h'_n)$ and $\overline{h} = \inf(h''_1,\ldots,h''_n)$ satisfy

$$\underline{h} \le f \le \overline{h}$$

and

$$\overline{h}(x) - \underline{h}(x) < \varepsilon \qquad \text{for all} \qquad x \in S.$$

This implies for an arbitrary $(\mathcal{K},\mathcal{L})$-admissible sequence (T_n) that

$(T_n f)$ converges uniformly on S to f. From this and the maximum principle follows that $(T_n f)$ converges uniformly to f even on X. Indeed, a function $g \in \mathcal{L}$ satisfies $\|g\| < \varepsilon$ if and only if $|g(x)| < \varepsilon$ holds for all $x \in \partial_{\mathcal{L}} X$.

We obtain two corollaries:

COROLLARY 1: $\mathrm{Kor}(\mathcal{K},\mathcal{L}) = \mathcal{L}$ *if* $\overline{\partial_{\mathcal{L}} X} \subset \partial_{\mathcal{K}}^{\mathcal{L}} X$.

This follows by observing Lemma 1 which implies a characterization of the relative Choquet boundary, namely

$$\partial_{\mathcal{K}}^{\mathcal{L}} X = \bigcap_{f \in \mathcal{L}} \{\underset{\vee}{f} = \hat{f}\},$$

where $\{\underset{\vee}{f} = \hat{f}\}$ stands for the set $\{x \in X : \underset{\vee}{f}(x) = \hat{f}(x)\}$.

COROLLARY 2: $\mathrm{Kor}(\mathcal{K},\mathcal{L}) = \mathcal{L}$ *if the two Choquet boundaries* $\partial_{\mathcal{K}} X$ *and* $\partial_{\mathcal{L}} X$ *are closed and coincide.*

This follows from the first remark following the definition of the relative boundary. Corollary 2 settles Example 3.

It can be seen from Example 1 that one cannot expect to have the equality $\hat{\mathcal{K}}^{\mathcal{L}} = \mathrm{Kor}(\mathcal{K},\mathcal{L})$ in Proposition 3 without additional assumptions. Indeed, since $\partial_{\mathcal{L}} X = X$ we have $\hat{\mathcal{K}}^{\mathcal{L}} = \hat{\mathcal{K}} \cap \mathcal{L}$ in this example. However, we know from Chapter II that $\hat{\mathcal{K}} = \mathcal{K}$ since the function id^3 is neither convex nor concave on $[-1, +1]$. Furthermore Proposition 4 will make clear why $\mathrm{Kor}(\mathcal{K},\mathcal{L}) = \mathcal{L}$ and hence $\hat{\mathcal{K}}^{\mathcal{L}} \neq \mathrm{Kor}(\mathcal{K},\mathcal{L})$.

The proof of Proposition 3 uses a property of the closure S of the Choquet boundary $\partial_{\mathcal{L}} X$ which holds for much smaller closed sets S in certain cases. It is this observation which leads from Proposition 3 to Theorem 3.

A set $S \subset X$ will be called *$\mathcal{L}$-determining* if it is closed and if a function in $\mathcal{L}$ vanishes identically provided that it vanishes at

all points of S. A closed set $S \subset X$ will be called *strongly $\mathcal{L}$-determining* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the implication

$$|f(x)| < \delta \quad \text{for all} \quad x \in S \Rightarrow \|f\| < \varepsilon$$

holds for all $f \in \mathcal{L}$. Obviously, strongly $\mathcal{L}$-determining implies $\mathcal{L}$-determining. A closed set $S \subset X$ is strongly $\mathcal{L}$-determining if and only if the map $p_S : \mathcal{L} \to \mathcal{L} \mid S$, defined by restricting a function $f \in \mathcal{L}$ to the set S, is bijective and open.

We have seen that the closure of $\partial_{\mathcal{L}} X$ is always strongly $\mathcal{L}$-determining. If S is $\mathcal{L}$-determining and if $\mathcal{L} \mid S$ is closed in $C(S)$ then, by the open mapping theorem, S is strongly $\mathcal{L}$-determining. If $\mathcal{L}$ has finite dimension n then there exists a strongly $\mathcal{L}$-determining set S of cardinality n. It suffices to choose a base $f_1, \ldots, f_n$ of $\mathcal{L}$. A simple induction argument then yields the existence of n points $x_1, \ldots x_n \in X$ such that

$$\det (f_i(x_j)) \neq 0.$$

Consequently, $S = \{x_1, \ldots, x_n\}$ is $\mathcal{L}$-determining and by the preceding argument strongly $\mathcal{L}$-determining. In particular, if $\mathcal{L}$ is the set of real polynomials of degree $\leq n$ restricted to a compact interval $[a,b] \subset \mathbb{R}$, $a \neq b$, every set of $n+1$ different points $x_1, \ldots, x_{n+1} \in [a,b]$ is strongly determining. Therefore, in Example 1 the set $S = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$ is strongly $\mathcal{L}$-determining and contained in $\partial^{\mathcal{L}}_{\mathcal{K}} X$.

A simple revision of the proof of Proposition 3 now leads to the announced improvement:

THEOREM 3: *Let S be a strongly $\mathcal{L}$-determining set. Then every function $f \in \mathcal{L}$ satisfying*

$$\hat{f}(x) = \underset{\vee}{f}(x) \qquad \textit{for all} \qquad x \in S$$

is in $\mathrm{Kor}(\mathcal{K},\mathcal{L})$.

Since $\partial_{\mathcal{K}}^{\mathcal{L}} X$ is the intersection of all sets $\{\hat{f} = \underset{\vee}{f}\}$ with $f \in \mathcal{L}$, we obtain

COROLLARY: $\mathrm{Kor}(\mathcal{K},\mathcal{L}) = \mathcal{L}$ *if* $\partial_{\mathcal{K}}^{\mathcal{L}} X$ *contains a strongly* $\mathcal{L}$*-determining set.*

This corollary settles Example 1. It contains the corollaries of Proposition 3 as special cases.

For the case of absolute Korovkin approximation, that is for $\mathcal{L} = C(X)$, Theorem 1 states that $\mathrm{Kor}(\mathcal{K},\mathcal{L})$ equals $\hat{\mathcal{K}}^{\mathcal{L}}$. We have seen that in the relative theory one cannot expect a similar result without an additional assumption on $\mathcal{L}$. For $\mathcal{L} = C(X)$ the state space $S(C(X))$, defined in Chapter II, is the convex compact set $M_+^1(X)$ of all (Radon) probability measures on X, hence a simplex (in the sense of Choquet). It has been proved recently by Leha and Papadopoulou [8] that the corresponding property for general $\mathcal{L}$ leads to the complete generalization of Theorem 1.

Continuing the discussion in the general case of the relative theory, $\mathcal{L}$ is called *simplicial* if the state space $S(\mathcal{L})$ is a simplex. The result then is:

THEOREM 4: *For simplicial* $\mathcal{L}$ *a function is a relative Korovkin function if and only if it is* $(\mathcal{K},\mathcal{L})$*-affine:*

$$\mathrm{Kor}(\mathcal{K},\mathcal{L}) = \hat{\mathcal{K}}^{\mathcal{L}}.$$

The proof given in [8] makes use of the selection theorem of Lazar [6] for (metrizable) simplexes. An immediate consequence is

then the following result which contains Theorem 2 as a special case:

THEOREM 5: *$\mathcal{H}$ is a Korovkin space with respect to a simplicial space $\mathcal{L}$ if and only if*

$$\overline{\partial_{\mathcal{L}} X} \subset \partial_{\mathcal{H}}^{\mathcal{L}} X.$$

For the remaining part of the proof we only have to observe that $\overline{\partial_{\mathcal{L}} X}$ is contained in the intersection of all the sets $\{\hat{f} = \underset{\vee}{f}\}$ with arbitrary $f \in \hat{\mathcal{H}}^{\mathcal{L}} = \mathcal{L}$.

Since $\partial_{\mathcal{L}} X \subset \partial_{\mathcal{H}}^{\mathcal{L}} X$ is equivalent to $\partial_{\mathcal{H}} X = \partial_{\mathcal{L}} X$ we obtain a partial converse to Corollary 2 of Proposition 3:

COROLLARY: *$\partial_{\mathcal{H}} X = \partial_{\mathcal{L}} X$ holds if $\mathcal{H}$ is a Korovkin space with respect to a simplicial space $\mathcal{L}$.*

We are now in the position to finish the discussion of Example 2. Here $\mathcal{L}$ is simplicial since every continuous real function on a compact subset of $\partial_{\mathcal{L}} X =]0,1]$ is the restriction of a function in $\mathcal{L}$ (cf. [5], p. 169). From $\overline{\partial_{\mathcal{L}} X} = X = [0,1]$ it follows that $\hat{\mathcal{H}}^{\mathcal{L}} = \hat{\mathcal{H}} \cap \mathcal{L}$. But a function $f \in \mathcal{L} \setminus \mathcal{H}$ cannot be $\mathcal{H}$-affine since $M_x(\hat{\mathcal{H}}) = M_x(\mathcal{H})$ for all $x \in X$ according to Lemma 1. $\mu = \frac{1}{2}(\varepsilon_{1/2} + \varepsilon_1)$ is an $\mathcal{H}$-representing measure for $x = 0$; however, $\int f \, d\mu \neq f(0)$ according to the definition of $\mathcal{H}$. We thus obtain $\mathrm{Kor}(\mathcal{H},\mathcal{L}) = \mathcal{H}$.

Example 3 can be generalized as follows. Let X be the closure of an open, convex, relatively compact set $U \subset \mathbb{R}^n$, $n \geq 2$. Define $\mathcal{H} = A(X)$ and $\mathcal{L}$ as the set of functions $f \in C(X)$ which are harmonic in U. Again $\partial_{\mathcal{H}}^{\mathcal{L}} X \subset U^*$ where U^* denotes the topological boundary of U (and X). Furthermore $\partial_{\mathcal{H}} X = \mathrm{ex}\, X$ and $\partial_{\mathcal{L}} X = U^*$ since all boundary points of the convex set U are regular (cf. [2], p. 127). $\mathcal{L}$ is simplicial since every function $f \in C(U^*)$ is the restriction

of a function in $\mathfrak{L}$. It follows from the preceding Corollary and Corollary 2 of Proposition 3, or from Theorem 5, that $\mathfrak{K}$ is a Korovkin space with respect to $\mathfrak{L}$ if and only if $\operatorname{ex} X = U^*$, i.e. if and only if U is *strictly convex*.

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A REMARK ON VECTOR-VALUED APPROXIMATION ON COMPACT SETS, APPROXIMATION ON PRODUCT SETS, AND THE APPROXIMATION PROPERTY

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INTRODUCTION

After Grothendieck [21], a locally convex (ℓ.c.) space E is said to have the *approximation property* (for short, a.p.) if and only if the identity id_E of E can be approximated uniformly on every precompact subset of E by continuous linear operators from E into E of finite rank (i.e. with finite dimensional range). Many "concrete" ℓ.c. spaces are known to have the a.p., but a *counterexample of Enflo* (1972), with subsequent refinements due to Figiel, Davie, and Szankowski, shows that there are even closed subspace of ℓ^p without a.p. for each $p \geq 1$, $p \neq 2$.

In connection with the a.p., a criterion due to L. Schwartz [26] is very useful: Schwartz introduces for two ℓ.c. spaces E and F their ε-*product* by

$$E \varepsilon F := \mathcal{L}_e(F'_c, E),$$

where F'_c is the dual of F with the topology of uniform convergence on precompact subsets of F and where the subscript e on the space

$\mathcal{L}(F'_c,E)$ of all continuous linear operators from F'_c into E indicates the topology of uniform convergence on the equicontinuous subsets of F'.

If E and F are quasi-complete, one can easily show $E\varepsilon F \cong F\varepsilon E$, and the ε-product $E\varepsilon F$ of two complete spaces E and F is complete (cf. [26]). Moreover, the *ε-tensor product* $E \otimes_\varepsilon F$ of Grothendieck [21] is a topological subspace of $E\varepsilon F$. We can now formulate *Schwartz's criterion for the a.p.* ([26], Proposition 11, cf. also [3], I, 3.9, and [8]):

THEOREM (L. Schwartz): *The quasi-complete l.c. space* E *has the a.p. if and only if* $E \otimes F$ *is dense in* $E\varepsilon F$ *for each (quasi-) complete l.c. space* F *(or, equivalently, for each Banach space* F*). So if* E *and* F *are complete l.c. spaces such that* E *or* F *has the a.p., we get:* $E\varepsilon F = E \check{\otimes}_\varepsilon F$, *the completion of the ε-tensor product* $E \otimes_\varepsilon F$ *(which we will also call, for short, complete ε-tensor product).*

In fact, the applications of this theorem, say, in the case of function spaces E derive from the remark that the "abstract" operator space $E\varepsilon F$ can usually be identified with a "concrete" space of F-valued functions "of type E". And $E \otimes_\varepsilon F$ is the space of "corresponding" functions with finite dimensional ranges in F. Hence a proof of the a.p. of E is then *equivalent* to the approximation of certain F-valued functions by functions with values in finite dimensional subspaces of F for every (quasi-) complete *l.c.* space F or only for every Banach space F, a result which is of interest in both directions.

In this article, we will give some (rather simple) *new examples* of how to apply Schwartz's theorem to function spaces more general than, but essentially similar to the well-known uniform algebras H(K) and A(K) on compact subsets K of $\mathbb{C}^N$ ($N \geq 1$). More precisely, we deal here with spaces of continuous functions on a compact set K which

either are uniformly approximable by functions belonging, on open sets U containing K, to a given *subsheaf* F of the sheaf C of all continuous functions or have restrictions belonging to F on the interior $\overset{\circ}{K}$ of K.

The *general situation* is the subject of sections 1 and 2. In section 1, the vector-valued case is considered, while section 2 deals with "slice product" - results (on product sets). Finally, in section 3, we look at some of the *motivating examples* and survey the known results (and their relations) in this case.

So, in a sense, this paper is based on a generalization of the author's old article [2] and motivated, among other things, by the more recent article [27] of N. Sibony: We show the connection of some of Sibony's results with topological tensor product theory and with the a.p. of the spaces of scalar functions in question. The results of this paper will be combined with the technique of "localization" of the a.p. for subspaces of weighted Nachbin spaces (cf.[5] and [10]) in a subsequent paper to yield new examples of function spaces "of mixed type" with a.p. and to demonstrate applications of the localization procedure in some concrete cases.

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1. THE GENERAL VECTOR-VALUED CASE

Let X be a completely regular (Hausdorff) topological space and F a *closed locally convex* (*l.c.*) *subsheaf* of the sheaf C_X of all continuous (real or complex valued) functions on X, i.e., for each open subset U of X, $F(U)$ is a closed topological linear subspace of $C(U)$ with the compact-open topology co. In fact, it would be sufficient to require F to be a *presheaf* only, and we prefer to use presheaf notation throughout this paper. (For some of our notation compare [9] and [10]. A sheaf F as above was called "*sheaf of F-morphic functions*" in [9].)

Let E always denote a quasi-complete locally convex (Hausdorff) space (over $\mathbb{R}$ or $\mathbb{C}$). We will always assume that X is a $k_{\mathbb{R}}$-space, i.e. that any function $f : X \to \mathbb{R}$ (or, equivalently, any function $f : X \to Y$, Y any completely regular space) is continuous if and only if the restriction of f to each compact subset of X is continuous. (Each locally compact or metrizable space, and, more generally, each k-space is also a $k_{\mathbb{R}}$-space.) Then each open $U \subset X$ is again a $\kappa_{\mathbb{R}}$-space, cf. Blasco [12], and hence the sheaves C_X and F are *complete*, i.e. the spaces $(C(U), co)$ and $F(U)$ are complete for each open $U \subset X$.

Under these assumptions, there exists (cf. [10], 1.5) the "*E-valued sheaf* F^E *of* F", namely, for any open U in X,

$$F^E(U) = E \varepsilon F(U)$$

= the space of all continuous E-valued functions f on U which satisfy $e' \circ f \in F(U)$ for each $e' \in E'$, endowed with the topology co of uniform convergence on compact subsets of U (cf. [3] and [5]),

and the cononical restriction mappings of the sheaf F^E are just the

ordinary restrictions of functions. $\mathcal{F}^E$ is a *closed* subsheaf of the sheaf C_X^E of all continuous E-valued functions on X.

In our definition and in some of our results below, it may be helpful to keep the following motivating examples of sheaves $\mathcal{F}$ of $\mathcal{F}$-morphic functions in mind (cf. also [9] and [10] for more examples):

1. EXAMPLES: (i) X = complex monifold or just $\mathbb{C}^N$ ($N \geq 1$), $\mathcal{F} = \mathcal{O}$ = sheaf of holomorphic functions on X,

(ii) X open in $\mathbb{R}^n$ ($n \geq 1$), $L = P(x,D)$ a (linear) hypoelliptic differential operator with C^∞-coefficients, and $\mathcal{F} = \mathcal{N}_L$ = sheaf of null solutions of L, i.e. $\mathcal{N}_L(U) = \{f \in C^\infty(U); (L|_U)f \equiv 0\}$ for any any open U in X. (The closed graph theorem for Fréchet spaces implies that, on $\mathcal{N}_L(U)$, the topologies induced by $C^\infty(U)$ and by co coincide and hence that $\mathcal{N}_L(U)$ is a closed topological linear subspace of $(C(U), co)$.)

Especially, the sheaf $\mathcal{H}$ of harmonic functions on $\mathbb{R}^n$ satisfies all assumptions of 1. (ii) above, and also the "harmonic sheaves" of abstract potential theory are sheaves of $\mathcal{F}$-morphic functions. All the sheaves of example 1. are (FN)-sheaves.

2. DEFINITION: For a compact subset K of X, we define:

(i) $C(K,E)$:= the space of all continuous E-valued functions on K with the topology of uniform convergence on K,

(ii) $A_{\mathcal{F}}(K,E) := \{f \in C(K,E);\ f|_K \in \mathcal{F}^E(\overset{\circ}{K}),$

i.e. $e' \circ f|_K \in \mathcal{F}(\overset{\circ}{K})$ for each $e' \in E'\}$, and

(iii) $H_{\mathcal{F}}(K,E)$:= the closure in $C(K,E)$ of

$\{f \in C(K,E)$; there exists an open neighbourhood U of K (depending on f) and a function $g \in \mathcal{F}^E(U)$ [i.e. $g: U \to E$ continuous and $e' \circ g \in \mathcal{F}(U)$ for any $e' \in E'$] such that $g|_K = f\}$.

$H_f(K,E) \subset A_f(K,E)$ holds, and both are *closed* subspaces of $C(K,E)$ which we endow with the topology of uniform convergence on K (induced by $C(K,E)$). If $E = \mathbb{R}$ or $\mathbb{C}$, we write $C(K)$, $A_f(K)$, and $H_f(K)$, respectively.

Now, of course, if E is complete, all the spaces $C(K,E)$, $A_f(K,E)$, and $H_f(K,E)$ are complete, too. The equation $C(K,E) = E \varepsilon C(K)$ for quasi-complete E is well-known (cf. [3]), and, once this equation is well-understood, the proof of the first part of the following result is clear (see e.g. [3] or [5] for a description of $E \varepsilon F$, F an arbitrary subspace of $C(K)$, from which our result below is easily derived, too):

3. THEOREM: (1) $A_f(K,E) = E \varepsilon A_f(K)$.

(2) *Hence* $A_f(K,E) = E \check{\otimes}_\varepsilon A_f(K)$ *holds for all complete* ℓ.c. *(or, equivalently, for all Banach) spaces* E *if and only if* $A_f(K)$ *has the a.p.*

For the second part of 3, Schwartz's criterion for the a.p. (in the introduction) is needed. In other words, $A_f(K)$ has the a.p. if and only if, for arbitrary Banach space E, each function $f \in C(K,E)$ with $e' \circ f \mid_{\overset{\circ}{K}} \in f(\overset{\circ}{K})$ for any $e' \in E'$ may be approximated, uniformly on K, by continuous functions g on K with values in finite dimensional subspaces of E that satisfy $e' \circ g \mid_{\overset{\circ}{K}} \in f(\overset{\circ}{K})$, too, and hence have the form

$$g(x) = \sum_{i=1}^{n} e_i g_i(x) \qquad \text{for all} \quad x \in K;$$

$n \in \mathbb{N}$ finite (depending on g), $e_i \in E$, and $g_i \in A_f(K)$, $i = 1,\ldots,n$. (Remark that such an approximation with g_i *only* $\in C(K)$ is *always* possible by the a.p. of $C(K)$ and by the equation $C(K,E) = E \check{\otimes}_\varepsilon C(K)$ for complete ℓ.c. E.)

As to the a.p. of the corresponding space $H_F(K)$, the situation there is, in some sense, just *the opposite*:

4. THEOREM: *We assume that, for some basis $\mathcal{U}$ of neighbourhoods of* K, $F(U)$ *has the a.p. for each* $U \in \mathcal{U}$. [*For* (1) *below, we could also assume instead that* E *has the a.p.*]

(1) *Then* $E \otimes_\varepsilon H_F(K)$ *is a dense topological linear subspace of* $H_F(K,E)$, *and hence*

$$H_F(K,E) = E \check{\otimes}_\varepsilon H_F(K)$$

holds whenever E *is complete.*

(2) *Consequently* $H_F(K)$ *has the a.p. if and only if, for each complete l.c. (or each Banach) space* E,

$H_F(K,E) = \{f \in C(K,E)$; *for each* $e' \in E'$ *and each* $\varepsilon > 0$ *there exists an open neighbourhood* $U = U(e',\varepsilon)$ *of* K *and a function* $g = g(e',\varepsilon) \in F(U)$ *such that*

$$\sup_{x\in K} |(e' \circ f)(x) - g(x)| < \varepsilon\}.$$

PROOF: As $E \otimes_\varepsilon C(K)$ is a topological linear subspace of C(K,E) and as the ε-tensor product preserves topological linear subspaces, only density of $E \otimes H_F(K)$ in $H_F(K,E)$ must be verified for the first assertion. So let p be a continuous seminorm on E, $\varepsilon > 0$ and $f \in H_F(K,E)$. By definition, there exists an open set $U \supset K$ and a function $g \in F^E(U)$ such that $\sup_{x\in K} p(f(x) - g(x)) < \frac{\varepsilon}{2}$. But, again by definition, $F^E(U) = E \varepsilon F(U)$ (with the topology of uniform convergence on compact subsets of U). Without loss of generality, we may assume $U \in \mathcal{U}$, and hence the a.p. of $F(U)$ or of E and one direction of Schwartz's theorem from the introduction imply that $E \otimes F(U)$ is dense

in $\mathcal{F}^E(U)$. Therefore we can find $h \in E \otimes \mathcal{F}(U)$ with $\sup_{x\in K} p(g(x)-h(x)) < \frac{\varepsilon}{2}$. Now $h|_K \in E \otimes H_{\mathcal{F}}(K)$ holds and $\sup_{x\in K} p(f(x) - h(x)) < \varepsilon$, which proves the required density of $E \otimes H_{\mathcal{F}}(K)$ in $H_{\mathcal{F}}(K,E)$.

(2) is then clear from Schwartz's criterion because the space on the right hand side of the equation is nothing but $E \varepsilon H_{\mathcal{F}}(K)$ - as a close look will immediately reveal. □

In other words, it is *always* true (under the assumption of 4) that a function $f \in C(K,E)$ which can be approximated uniformly on K by functions extending to elements of $\mathcal{F}^E$ on open neighbourhoods of K may also be approximated uniformly on K by functions of the form

$$h(x) = \sum_{i=1}^{n} e_i h_i(x) \quad \text{for all} \quad x \in K;$$

$n \in \mathbb{N}$ finite (depending on h), $e_i \in E$, and $g_i \in H_{\mathcal{F}}(k)$, $i = 1,\ldots,n$. But the a.p. of $H_{\mathcal{F}}(K)$ is equivalent to the fact that, for arbitrary Banach space E, each function $f \in C(K,E)$ with the property that, given any $e' \in E'$, $e' \circ f$ may be approximated uniformly on K by (scalar) functions belonging to $\mathcal{F}$ on open sets containing K is already an element of $H_{\mathcal{F}}(K,E)$, i.e. can be approximated uniformly on K by E-valued functions belonging to $\mathcal{F}^E$ on open sets containing K. Or, to put it this way, $H_{\mathcal{F}}(K)$ has the a.p. if and only if, given any Banach space E and an arbitrary function $f \in C(K,E)$ as above, there exists for any $\varepsilon > 0$, *uniformly* for all e' in the unit ball E'_1 of E', an open set $U_o \supset K$ and a function $g_o : U_o \to E$ continuous with $e' \circ g_o \in \mathcal{F}(U_o)$ for each $e' \in E'_1$ such that

$$|(e' \circ f)(x) - (e' \circ g_o)(x)| < \varepsilon \quad \text{for all} \quad x \in K \quad \text{and all} \quad e' \in E'_1.$$

REMARK: The description of $E \varepsilon H_{\mathcal{F}}(K)$ as the right side of the equation in 4.(2) is of course *independent* of the hypothesis on $\mathcal{F}$ in 4 ,

and so is the inclusion $H_F(K,E) \subset E \varepsilon H_F(K)$ which follows from this description. Hence, as obviously $E \otimes_\varepsilon H_F(K)$ is *always* a topological linear subspace of $H_F(K,E)$, we have

$$E \check{\otimes}_\varepsilon H_F(K) \subset H_F(K,E) \subset E \varepsilon H_F(K)$$

whenever E is complete. So, by Schwartz's theorem, the a.p. of $H_F(K)$ clearly *implies* the equality

$$H_F(K,E) = E \check{\otimes}_\varepsilon H_F(K)$$

for complete *l.c.* spaces E, even *without* the hypothesis of 4.

Let E be complete and let the assumption of 4.(1) be satisfied [or let $H_F(K)$ have the a.p.]. Then the preceding two theorems imply:

$$\begin{array}{ccccc} H_F(K,E) & \subset & E \varepsilon H_F(K) & \subset & E \varepsilon A_F(K) \\ \| & & & & \| \\ E \check{\otimes}_\varepsilon H_F(K) & \subset & E \check{\otimes}_\varepsilon A_F(K) & \subset & A_F(K,E). \end{array}$$

So we obtain from Schwartz's theorem:

5. COROLLARY: *Let us again assume that, for some basis $\mathcal{U}$ of neighbourhoods of K, $F(U)$ has the a.p. for each $U \in \mathcal{U}$, and let $A_F(K) = H_F(K)$ be valid.*

Then $A_F(K) = H_F(K)$ has the a.p. if and only if $A_F(K,E) = H_F(K,E)$ holds for all complete l.c. [or, equivalently, for all Banach] spaces E.

If, in concrete examples, one examines the methods that lead to a proof of $A_F(K) = H_F(K)$, it turns out very aften that these methods also prove $A_F(K,E) = H_F(K,E)$ for, at least, arbitrary Banach spaces E. Corollary 5 shows that it suffices to prove the equality $A_F(K,E) = H_F(K,E)$ for all Banach spaces E to obtain *both* the a.p. of

$A_{\mathcal{F}}(K) = H_{\mathcal{F}}(K)$ *and* $A_{\mathcal{F}}(K,E) = H_{\mathcal{F}}(K,E)$ even for arbitrary complete ℓ.c. spaces E. On the other hand, sometimes the methods used in proving $A_{\mathcal{F}}(K) = H_{\mathcal{F}}(K)$ may also be adapted to yield a *direct* proof of the a.p. of this space, and then $A_{\mathcal{F}}(K,E) = H_{\mathcal{F}}(K,E)$ holds for all complete ℓ.c. spaces by Corollary 5, too. In fact, Corollary 5 demonstrates that the two approaches which we have just outlined are *equivalent*.

REMARK: Similarly, if E is a complete ℓ.c. space and if $A_{\mathcal{F}}(K) = H_{\mathcal{F}}(K)$, then the a.p. of E or of $A_{\mathcal{F}}(K) = H_{\mathcal{F}}(K)$ also implies $A_{\mathcal{F}}(K,E) = H_{\mathcal{F}}(K,E)$ *in general*.

2. APPROXIMATION ON PRODUCT SETS

Let us now turn to a description of the ε-product resp. complete ε-tensor product of two (or more) spaces of type $A_{\mathcal{F}}(K)$ resp. $H_{\mathcal{F}}(K)$. Such a description follows easily from the (well-known) general"*slice product theorem*" for subspaces of, say, $C(K_1 \times K_2)$. (This slice product theorem was first stated in Eifler [17], but he points out that the result is already implicitly contained in Grothendieck [21]. For more general slice product theorems, for some ideas connected with the underlying method, and for more applications compare [4] and [5].)

So let X_1 and X_2 be two completely regular (Hausdorff) spaces such that $X_1 \times X_2$ *is a* $k_{\mathbb{R}}$*-space*. Then both X_1 and X_2 must be $k_{\mathbb{R}}$-spaces, and, on the other hand, $X_1 \times X_2$ is $k_{\mathbb{R}}$, if both X_1 and X_2 are and if at least one of the spaces X_1, X_2 is even locally compact (or if both X_1 and X_2 are *hemicompact* $k_{\mathbb{R}}$-spaces). Let $\mathcal{F}_1$ resp. $\mathcal{F}_2$ denote closed ℓ.c. subsheaves of C_{X_1} resp. C_{X_2}. Then we know (by applying Blasco's result on the $k_{\mathbb{R}}$-property of open subsets of completely regular $k_{\mathbb{R}}$-spaces, cf. [12], instead of Arhangel'skiĭ's proposition on k-spaces in the proof of [10], 1.10) that the "product sheaf $\mathcal{F}_1 \varepsilon \mathcal{F}_2$" on $X_1 \times X_2$ exists:

$F_1 \varepsilon F_2$ is *uniquely determined* by the following requirements: For all open subsets $U_i \subset X_i$ $(i = 1,2)$, $(F_1 \varepsilon F_2)(U_1 \times U_2) = F_1(U_1) \varepsilon F_2(U_2)$, and, for all open $U_i \supset V_i$ on X_i $(i = 1,2)$,

$$r^{F_1 \varepsilon F_2}_{U_1 \times U_2, V_1 \times V_2} = r^{F_1}_{U_1 V_1} \varepsilon r^{F_2}_{U_2 V_2},$$

where r^F_{UV} denotes the canonical restriction mapping $F(U) \to F(V)$ with respect to the sheaf F and where the ε-product of continuous linear mappings is defined in, say, [7].

Let us now introduce the following *notation*: π_i is the canonical projection of $X_1 \times X_2$ onto X_i $(i = 1,2)$, and, for arbitrary open subsets U of $X_1 \times X_2$,

$$U_t := \{x \in X_2; (t,x) \in U\} (t \in X_1), U^x := \{t \in X_1; (t,x) \in U\} (x \in X_2).$$

Then we get a *general* description of $F_1 \varepsilon F_2$ on open sets $U \subset X_1 \times X_2$ as follows:

$$(F_1 \varepsilon F_2)(U) := \{f \in C(U);\ f(t,\cdot) \in F_2(U_t) \text{ for each } t \in \pi_1(U) \text{ and } f(\cdot,x) \in F_1(U^x) \text{ for each } x \in \pi_2(U)\},$$

endowed with the topology co of uniform convergence on compact subsets of U, and the canonical restriction mappings of the sheaf $F_1 \varepsilon F_2$ are just the ordinary restrictions of functions. $F_1 \varepsilon F_2$ is a *closed* ℓ.c. subsheaf of $C_{X_1 \times X_2} = C_{X_1} \varepsilon C_{X_2}$. Nuclearity of F_1 and F_2 is inherited by $F_1 \varepsilon F_2$ ([10], 1.2 c).

6. THEOREM: *Let* K_i *be a compact subset of* X_i $(i = 1,2)$. *Then we have:*

(1) $A_{\mathcal{F}_1}(K_1) \varepsilon A_{\mathcal{F}_2}(K_2) = \{f \in C(K_1 \times K_2);\ f(t,\cdot) \in A_{\mathcal{F}_2}(K_2)$ *and* $f(\cdot,x) \in A_{\mathcal{F}_1}(K_1)$ *for all* $(t,x) \in K_1 \times K_2\}$

$= \{f \in C(K_1 \times K_2);\ f(t,\cdot)\big|_{\overset{\circ}{K}_2} \in \mathcal{F}_2(\overset{\circ}{K}_2)$ *and* $f(\cdot,x)\big|_{\overset{\circ}{K}_1} \in \mathcal{F}_1(\overset{\circ}{K}_1)$ *for all* $(t,x) \in K_1 \times K_2\}$,

with the sup-norm of $C(K_1 \times K_2)$,

(2) $H_{\mathcal{F}_1}(K_1) \varepsilon H_{\mathcal{F}_2}(K_2) = \{f \in C(K_1 \times K_2);\ f(t,\cdot) \in H_{\mathcal{F}_2}(K_2)$ *and* $f(\cdot,x) \in H_{\mathcal{F}_1}(K_1)$ *for all* $(t,x) \in K_1 \times K_2\}$

$= \{f \in C(K_1 \times K_2);\ f(t,\cdot)$ *may be approximated uniformly on* K_2 *by functions belonging to* $\mathcal{F}_2$ *on open sets containing* K_2, *and* $f(\cdot,x)$ *may be approximated uniformly on* K_1 *by functions belonging to* $\mathcal{F}_1$ *on open sets containing* K_1 *for each* $(t,x) \in K_1 \times K_2\}$,

again with the sup-norm of $C(K_1 \times K_2)$, *and:*

(3) $$H_{\mathcal{F}_1}(K_1) \check{\otimes}_\varepsilon H_{\mathcal{F}_2}(K_2) \subset H_{\mathcal{F}_1 \varepsilon \mathcal{F}_2}(K_1 \times K_2) \subset H_{\mathcal{F}_1}(K_1) \varepsilon H_{\mathcal{F}_2}(K_2)$$
$$\subset A_{\mathcal{F}_1}(K_1) \varepsilon A_{\mathcal{F}_2}(K_2) \subset A_{\mathcal{F}_1 \varepsilon \mathcal{F}_2}(K_1 \times K_2).$$

PROOF: Parts (1) and (2) follow immediately from the slice product theorem for subspaces of $C(K_1 \times K_2)$ quoted above. To prove

$$H_{\mathcal{F}_1}(K_1) \check{\otimes}_\varepsilon H_{\mathcal{F}_2}(K_2) \subset H_{\mathcal{F}_1 \varepsilon \mathcal{F}_2}(K_1 \times K_2),$$

it suffices to verify $H_{\mathcal{F}_1}(K_1) \otimes H_{\mathcal{F}_2}(K_2) \subset H_{\mathcal{F}_1 \varepsilon \mathcal{F}_2}(K_1 \times K_2)$, which is immediate, too. The inclusion $H_{\mathcal{F}_1 \varepsilon \mathcal{F}_2}(K_1 \times K_2) \subset H_{\mathcal{F}_1}(K_1) \varepsilon H_{\mathcal{F}_2}(K)$ follows readily from the previous description of the sheaf $\mathcal{F}_1 \varepsilon \mathcal{F}_2$ and from the description of the ε-product on the right hand side, cf. (2).

Finally $A_{F_1}(K_1) \varepsilon A_{F_2}(K_2) \subset A_{F_1 \varepsilon F_2}(K_1 \times K_2)$, because $\overset{\circ}{K_1 \times K_2} = \overset{\circ}{K}_1 \times \overset{\circ}{K}_2$ and hence (by the description of $F_1 \varepsilon F_2$):

$$A_{F_1 \varepsilon F_2}(K_1 \times K_2) = \{f \in C(K_1 \times K_2);\ f\big|_{\overset{\circ}{K}_1 \times \overset{\circ}{K}_2} \in (F_1 \varepsilon F_2)(\overset{\circ}{K}_1 \times \overset{\circ}{K}_2) = F_1(\overset{\circ}{K}_1) \varepsilon F_2(\overset{\circ}{K}_2)\}$$

$$= \{f \in C(K_1 \times K_2);\ f(t,\cdot)\big|_{\overset{\circ}{K}_2} \in F_2(\overset{\circ}{K}_2) \text{ and } f(\cdot,x)\big|_{\overset{\circ}{K}_1} \in F_1(\overset{\circ}{K}_1) \text{ (only) for all } (t,x) \in \overset{\circ}{K}_1 \times \overset{\circ}{K}_2\}.$$

As 6.(1) (together with the description of $A_{F_1 \varepsilon F_2}(K_1 \times K_2)$ at the end of the preceding proof) shows, $A_{F_1}(K_1) \varepsilon A_{F_2}(K_2)$ will in general be *strictly* contained in $A_{F_1 \varepsilon F_2}(K_1 \times K_2)$, and it is easy to construct *examples* for this phenomenon. However, a simple topological assumption forces equality here, as part (2) of our next result demonstrates.

7. THEOREM: (1) *Let, for some basis* $\mathcal{U}_1$ *of neighbourhoods of* K_1, $F_1(U_1)$ *have the a.p. for each* $U_1 \in \mathcal{U}_1$ *or let, for some basis* $\mathcal{U}_2$ *of neighbourhoods of* K_2, $F_2(U_2)$ *have the a.p. for each* $U_2 \in \mathcal{U}_2$. [*Instead of this, we could also require* $H_{F_1}(K_1)$ *or* $H_{F_2}(K_2)$ *to have the a.p.*]. *Then*

$$H_{F_1 \varepsilon F_2}(K_1 \times K_2) = H_{F_1}(K_1) \check{\otimes}_\varepsilon H_{F_2}(K_2) \quad \textit{holds.}$$

(2) *If* K_1 *and* K_2 *are "fat", i.e. satisfy* $\overline{\overset{\circ}{K}_i} = K_i$ $(i = 1,2)$, *we get:*

$$A_{F_1}(K_1) \varepsilon A_{F_2}(K_2) = A_{F_1 \varepsilon F_2}(K_1 \times K_2).$$

PROOF: (1) The remark in brackets is obvious from 6.(3) and Schwartz's theorem. For the proof of (1) under the assumption on F_1 resp. F_2,

it suffices to show density of $H_{F_1}(K_1) \otimes H_{F_2}(K_2)$ in $H_{F_1 \varepsilon F_2}(K_1 \times K_2)$. So let $f \in H_{F_1 \varepsilon F_2}(K_1 \times K_2)$ and $\varepsilon > 0$ be given and find an open set U containing $K_1 \times K_2$ and a function $g \in (F_1 \varepsilon F_2)(U)$ such that

$$\sup_{x \in K_1 \times K_2} |f(x) - g(x)| < \frac{\varepsilon}{2} .$$

Without loss of generality we may assume $U = U_1 \times U_2$ with $U_i \in \mathcal{U}_i$ $(i = 1,2)$, and hence

$$(F_1 \varepsilon F_2)(U) = F_1(U_1) \varepsilon F_2(U_2) = F_1(U_1) \check{\otimes}_\varepsilon F_2(U_2)$$

by Schwartz's theorem, because $F_1(U_1)$ or $F_2(U_2)$ has the a.p. Then there exists $h \in F_1(U_1) \otimes F_2(U_2)$ such that

$$\sup_{x \in K_1 \times K_2} |g(x) - h(x)| < \frac{\varepsilon}{2} .$$

Now $h|_{K_1 \times K_2} \in H_{F_1}(K_1) \otimes H_{F_2}(K_2)$ and $\sup_{x \in K_1 \times K_2} |f(x) - h(x)| < \varepsilon$.

(2) Notice that, by the identity $C(K_1 \times K_2) = C(K_1, C(K_2)) = C(K_2, C(K_1))$, for arbitrary $f \in A_{F_1 \varepsilon F_2}(K_1 \times K_2)$, $I_1 : t \to f(t,\cdot)$ resp. $I_2 : x \to f(\cdot,x)$ yield continuous linear mappings of K_1 resp. K_2 into $C(K_2)$ resp. $C(K_1)$. The characterization of $A_{F_1 \varepsilon F_2}(K_1 \times K_2)$ at the end of the proof of 6 implies $I_1(\overset{\circ}{K}_1) \subset A_{F_2}(K_2)$ and $I_2(\overset{\circ}{K}_2) \subset A_{F_1}(K_1)$ and hence also $I_1(\overline{\overset{\circ}{K}}_1) \subset A_{F_2}(K_2)$ and $I_2(\overline{\overset{\circ}{K}}_2) \subset A_{F_1}(K_1)$. So, for fat sets K_1 and K_2, the assertion follows immediately from 6.(1).□

8. COROLLARY: (1) *If* $A_{F_i}(K_i) = H_{F_i}(K_i)$ $(i = 1,2)$ *holds and if one of these spaces has the a.p., then we obtain*

$$H_{F_1 \varepsilon F_2}(K_1 \times K_2) = A_{F_1}(K_1) \varepsilon A_{F_2}(K_2) .$$

So we get even $H_{F_1 \varepsilon F_2}(K_1 \times K_2) = A_{F_1 \varepsilon F_2}(K_1 \times K_2)$ *for fat sets* K_1 *and* K_2.

(2) $H_{F_1 \varepsilon F_2}(K_1 \times K_2)$ *has the a.p. whenever both* $H_{F_1}(K_1)$ *and* $H_{F_2}(K_2)$ *have the a.p.*

(3) *If* K_1 *and* K_2 *are fat and if both* $A_{F_1}(K_1)$ *and* $A_{F_2}(K_2)$ *have the a.p.,* $A_{F_1 \varepsilon F_2}(K_1 \times K_2)$ *has the a.p., too.*

PROOF: (1) is clear from Schwartz's theorem, 6.(3), and 7.(2). (2) and (3) follow from 7 by aid of the result (Schwartz [26], Proposition 11, Corollaire 2) that the ε-product of two compete l.c. spaces with a.p. also enjoys the a.p. □

Induction on 7 and 8.(1) using, among other (obvious) things, that finite ε-products are *associative* and that ε-products of complete spaces with a.p. are again spaces with a.p. yields now:

9. COROLLARY: *Let* $X_1, \ldots, X_n$ *be completely regular (Hausdorff) spaces with* $X_1 \times \ldots \times X_n$ *a* $k_{\mathbb{R}}$*-space, let* $F_1, \ldots, F_n$ *be closed l.c. subsheaves of* $C_{X_1}, \ldots, C_{X_n}$, *and* $K_1, \ldots, K_n$ *compact subsets of* $X_1, \ldots, X_n$, *respectively.*

(1) *Let, for some basis* $\mathcal{U}_i$ *of neighbourhoods of* K_i, $F_i(U_i)$ *have the a.p. for each* $U_i \in \mathcal{U}_i$ $(i = 1, \ldots, n$ *except for at most one* $i)$ *or let all but one* $H_{F_i}(K_i)$ $(i = 1, \ldots, n)$ *have the a.p. Then*

$$H_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n) = H_{F_1}(K_1) \check{\otimes}_\varepsilon \ldots \check{\otimes}_\varepsilon H_{F_n}(K_n)$$

is true, and if all $H_{F_i}(K_i)$ $(i = 1, \ldots, n)$ *have the a.p., the same holds for* $H_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n)$.

(2) *Let all the sets* $K_1, \ldots, K_n$ *be fat. Then*

$$A_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n) = A_{F_1}(K_1) \varepsilon \ldots \varepsilon A_{F_n}(K_n)$$

holds true, and if all $A_{F_i}(K_i)$ $(i = 1,\ldots,n)$ *have the a.p., the same holds for* $A_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n)$.

(3) *Let, for each* $i = 1,\ldots,n$, K_i *be fat and* $A_{F_i}(K_i) = H_{F_i}(K_i)$. *If then all these spaces (except for at most one) have the a.p.,*

$$A_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n) = H_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n)$$

is valid, too.

For the corresponding spaces of functions with values in a quasi-complete *l.c.* space E (see section 1), we get e.g.:

10. COROLLARY: (1) *Let* $K_1,\ldots,K_n$ *be fat. Then*

$$A_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n, E) = E \varepsilon A_{F_1}(K_1) \varepsilon \ldots \varepsilon A_{F_n}(K_n)$$

is true.

(2) *Let* E *be complete and let, for some basis* $\mathcal{U}_i$ *of neighbourhoods of* K_i, $F_i(U_i)$ *have the a.p. for each* $U_i \in \mathcal{U}_i$ $(i = 1,\ldots,n)$. *Then*

$$H_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n, E) = E \check{\otimes}_\varepsilon H_{F_1}(K_1) \check{\otimes}_\varepsilon \ldots \check{\otimes}_\varepsilon H_{F_n}(K_n)$$

is valid.

(3) *Let* E *be complete, let* K_i *be fat and* $A_{F_i}(K_i) = H_{F_i}(K_i)$ *for each* $i = 1,\ldots,n$. *Then if all the spaces* $A_{F_i}(K_i) = H_{F_i}(K_i)$ *have the a.p.* $(i = 1,\ldots,n)$,

$$A_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n, E) = H_{F_1 \varepsilon \ldots \varepsilon F_n}(K_1 \times \ldots \times K_n, E)$$

holds, too.

PROOF: (1) is a consequence of 3.(1) and 9.(2). Let us remark that, under the hypothesis of (2), $(F_1 \varepsilon \ldots \varepsilon F_n)(U)$ (as ε-product of complete

spaces with a.p.) satisfies the a.p. for each U in the basis

$$\mathcal{U}^p := \{U_1 \times \ldots \times U_n ;\ U_i \in \mathcal{U}_i \quad (i = 1,\ldots,n)\}$$

of neighbourhoods of $K_1 \times \ldots \times K_n$. Hence (2) follows from 4.(1) and 9.(1). Finally (3) is implied by 9.(2), (3) and by the remark at the very end of section 1. □

3. DISCUSSION OF THE MOTIVATING EXAMPLES

In this final section, we will look at some of the known results in the case of our motivating examples of sheaves F (cf. 1 above) and will point out that, between some theorems in the literature, strong relations follow from our previous discussion. It is not intended here to survey *all* the relevant articles, but we will rather illustrate some of the ideas which might play a rôle, when one tries to apply the results of sections 1 and 2, by specific examples.

Perhaps the case most people have been interested in is $F = \mathcal{O}$, the *nuclear* Fréchet sheaf of holomorphic functions on a complex manifold X. For simplicity, however, we will only deal with holomorphic functions on $X = \mathbb{C}^N$ ($N \geq 1$) here. It is clear that finite ε-products of sheaves $\mathcal{O}$ are nothing but the corresponding sheaf $\mathcal{O}$ on the product and that, for any quasi-complete l.c. space E, $\mathcal{O}^E$ is just the sheaf of E-valued holomorphic functions. When $F = \mathcal{O}$, we will write, for short, $A(K,E)$, $H(K,E)$ instead of $A_F(K,E)$, $H_F(K,E)$, respectively.

For $F = \mathcal{O}$, some of the results in sections 1 and 2 are apparently part of the "folklore" of the subject, but usually not easily accessible in the literature: We have already pointed out in the introduction that this paper is based on a generalization of the "old" article [2]. Later on (in [1], section 1), O. B. Bekken looked at closed subspaces of $C(K)$, K compact, with the so-called "*slice property*"

and showed that this property implies the a.p. After the proper change of notation and some identifications (using the fact that each Banach space is a closed subspace of $C(K')$ for some compact K') his results there are quite similar to our theorem 3 (for Banach spaces E). In section 3 of [1], (making use of the nuclearity of $\mathcal{O}$) Bekken obtains a proposition related to (but somewhat weaker than) our theorem 4. For a detailed account of the relation of the slice property with the a.p. and the consequences of a theorem of Milne in this connection, see also [6].

As usual with spaces of holomorphic functions, we must now split up our discussion for the cases $N = 1$ and $N \geq 2$. If $N = 1$ i.e. K is a compact subset of the complex plane, the problem is completely solved: $A(K)$ and $H(K)$ have then *always* the a.p. (whereas it remains *an open problem* whether even the Banach algebra $H^\infty(D)$ of all bounded holomorphic functions on the open unit disk D enjoys the a.p. Remark that the a.p. of the disk algebra $A(\bar{D}) = H(\bar{D})$ is really quite trivial !). This interesting result is due to the joint effort of several people (and also, unfortunately, not easily accessible in the literature in its full generality): Eifler [17], Gamelin-Garnett [19], section 6 for $H(K)$, and Davie [15] for $A(K)$ (they all use *vector - valued* results). More generally, Gamelin [18], section 12 has pointed out that the constructive techniques (and the approximation scheme) of Mergelyan and Vitushkin show that the so-called "T-*invariant*" algebras have the a.p. As to $A(K) = H(K)$ in the case of one variable, a *necessary and sufficient* condition (involving *continuous analytic capacity*) was given by Vitushkin, see e.g. [19] and [29].

For $N \geq 2$, there are only *partial* results. Remark first that, by an example of Diederich and Fornaess, there exists a relatively compact domain G of holomorphy in $\mathbb{C}^N$ with C^∞-boundary such that $A(K) \neq H(K)$ for $K = \bar{G}$. For a survey of some related recent work on

the question when A(K) = H(K) in several complex variables, we refer to Birtel [11], and for results in "*finite S. Ps. C. manifolds*" to Rossi-Taylor [25].

It is known now that A(K) (or H(K)) has the a.p. for the following types of compact sets $K \subset \mathbb{C}^N$:

(i) K is the closure of a *strictly pseudoconvex region* with sufficiently smooth (say, C^3 -) boundary, or:

(ii) K is the closure of a *regular Weil polyeder*.

Both conditions imply K fact (trivially), and A(K) = H(K) (in case (i), this approximation theorem is due to Henkin-Lieb - Kerzman, in case (ii), it is a result of Petrosjan). (i) was proved e.g. in Bekken [1], section 2, applying a vector-valued version of Henkin's separation of singularities result. It also follows from Sibony [27], Proposition 4 (in view of our Corollary 5). Sibony [27], p. 173 has also remarked that Petrosjan's arguments may be modified to yield A(K,E) = H(K,E) for each Fréchet space E if K is the closure of a regular Weil polyeder, and hence (ii) follows again from our Corollary 5.

REMARK: The method of "*localization of the a.p.*" for certain function spaces (cf. [5] and [10]) may be used to show that A(K') has the a.p. for compact sets K' that are "sufficiently well" *disjoint unions* of sets K as above and that some related function spaces have the a.p., too (cf. [5], Corollary 15), but we will not go into details here.

Let us now explicitly state what we get from the preceding results by applying our Corollaries 9 and 10:

14. THEOREM: (1) H(K) *has the a.p. if* $K = K_1 \times \ldots \times K_n$ *with* K_i (i = 1,...,n) *either*

(i) *any compact subset of* $\mathbb{C}$ *or*

(ii) *the closure of a strictly pseudoconvex region with sufficiently smooth boundary or of a regular Weil polyeder.*

(2) $A(K)$ *has the a.p. under the same conditions if one requires in* (1)(*i*), *additionally,* K_i *to be fat. And* $A(K) = A(K_1) \check{\otimes}_\varepsilon \ldots \check{\otimes}_\varepsilon A(K_n)$ *is then true.*

(3) $H(K) = A(K)$ *holds for* $K = K_1 \times \ldots \times K_n$ *with* K_i $(i = 1, \ldots, n)$ *either* (*i*) *a fat compact set in* $\mathbb{C}$ *with* $H(K_i) = A(K_i)$ *or as in* (1)(*ii*) *above.*

Let then E *be an arbitrary complete l.c. space.*

(4) *Under the assumptions of* (2),

$$A(K,E) = E \check{\otimes}_\varepsilon A(K_1) \check{\otimes}_\varepsilon \ldots \check{\otimes}_\varepsilon A(K_n) \quad \textit{holds.}$$

(5) *Under the assumptions of* (3), *we have* $A(K,E) = H(K,E)$, *too.*

11.(3) is related to a result of Weinstock [30], p. 812, where, instead of the assumption of a smooth boundary in 11.(1)(ii), he needs only the so-called "*segment property*" of K. (Weinstock's methods are quite different, however.) At this point, a few remarks on Sibony's paper [27] are also in order (in connection with our preceding results):

Proposition 1 of [27] is, in some sense, easy, if not trivial, as our theorem 4.(1) (and its simple proof) demonstrates: It is *not* necessary to invoke Gleason's theorem at this point; the well-known nuclearity (or even the a.p.) of $\mathcal{O}$ and simple tensor product arguments suffice! Corollaire 3 of [27] corresponds with 7.(1) and 10.(2) in this paper. As we have already noted above, however, Sibony's proposition 4 is really a *non-trivial* result based on Henkin's method and implies the a.p. of $A(K)$ in case (i). Hence, by our Corollary 5, it is (essentially) *equivalent* to theorem 2.4 of Bekken [1]. Finally, Corollaire 8 of [27] corresponds with our theorem 11.(5). It should perhaps be pointed out that, whereas part of Sibony's proofs looks as

though they are based on theorems and methods which are just true in his given *special* situation, it turns out from our discussion above that what is really needed is only a proof of the a.p. of A(K) (=H(K)) to make everything work, even in many *other* cases.

We turn to sheaves $\mathcal{F}$ of harmonic functions or, more generally, of null-solutions of hypoelliptic differential operators with C^∞-coefficients now. These are again nuclear Fréchet sheaves, and hence our assumption that $\mathcal{F}$ has the a.p. is certainly satisfied. For nuclearity of the sheaves in axiomatic potential theory, cf. Constantinescu-Cornea [14], § 11.

In this case, we will assume for the moment that the compact set K is the closure of some open subset U of X (and hence fat). A very nice phenomenon may occur here which yields a *completely trivial* solution to the question of the a.p. for $A_{\mathcal{F}}(K)$: If we suppose that U is a *regular* set for the *Dirichlet problem* with respect to the sheaf $\mathcal{F}$, i.e. for each $g \in C(\partial U)$ there exists a unique function $f \in A_{\mathcal{F}}(\bar{U})$ with $f|_{\partial U} = g$, the continuous linear restriction mapping $L : f \to f|_{\partial U}$ is bijective from $A_{\mathcal{F}}(K)$ onto $C(\partial U)$ and hence yields a topological isomorphism of these Banach spaces. (A maximum principle for functions in $A_{\mathcal{F}}(K)$ will imply that L is even an *isometry*.) Then $A_{\mathcal{F}}(K)$ certainly has the a.p. In fact, it would be enough for such a result that $L : f \to f|_{K'}$ is bijective from $A_{\mathcal{F}}(K)$ onto *a closed subspace with a.p.* of $C(K')$ for some closed subset K' of K (say, a closed set $K' \subset \partial U$).

Let for instance $\mathcal{F}$ be the sheaf $\mathcal{H}$ of (real) *harmonic* functions on $\mathbb{R}^n$ ($n \geq 2$). We refer e.g. to Ho-Van-Thi-Si [22], p. 617/8, 621/2, 626, 637 for conditions concerning, say, the equalities

(i) $A_{\mathcal{H}}(K) = H_{\mathcal{H}}(K)$, and

(ii) $A_{\mathcal{H}}(K)|_{\partial K} = C(\partial K)$ (or, equivalently, by the maximum principle, $L : A_{\mathcal{H}}(K) \to C(\partial K)$ bijective and isometric).

Let us only note that in general a suitable (outside) *cone condition* implies both (i) and (ii) and that, in the case $n = 2$, (i) and (ii) are valid for a compact set K such that each point $x \in \partial K$ is a boundary point of a connected component of the complement of K. So then $A_{\mathcal{H}}(K) = H_{\mathcal{H}}(K)$ has the a.p.

We also refer to Weinstock [31] for results on $A_F(K) = H_F(K)$ for sheaves $F = N_L$ (on $\mathbb{R}^n$) of null solutions of (linear) elliptic partical differential operators L of order m with constant coefficients in this connection and to Vincent-Smith [28] for $A_F(K) = H_F(K)$ in the setting of harmonic sheaves F of axiomatic potential theory. It would lead us too far afield even to give only *complete references* for all interesting relevant results in this direction.

Another argument then yields the a.p. of $A_F(K)$ and $H_F(K)$ even in a much more general setting:

12. THEOREM: *Let again* $\mathcal{H}$ *be the sheaf of harmonic functions on* $\mathbb{R}^n$ $(n \geq 2)$ *and* K *an arbitrary compact subset of* $\mathbb{R}^n$.

(1) *Then both* $A_{\mathcal{H}}(K)$ *and* $H_{\mathcal{H}}(K)$ *always have the a.p.*

(2) *Hence* $A_{\mathcal{H}}(K,E) = E \check{\otimes}_\varepsilon A_{\mathcal{H}}(K)$ *holds for each complete l.c. space* E, *and, for such an* E, $A_{\mathcal{H}}(K) = H_{\mathcal{H}}(K)$ *always implies* $A_{\mathcal{H}}(K,E) = H_{\mathcal{H}}(K,E)$.

PROOF: As Ho-Van-Thi-Si [22], p. 621, 634 shows, both $A = H_{\mathcal{H}}(K)$ and $A = A_{\mathcal{H}}(K)$ are *simplicial* spaces, i.e. the null measure is the only A-*maximal* measure (or, equivalently, measure concentrated in the Choquet boundary of A) orthogonal to A. This means (cf. Effros-Kazdan [16], p. 99) that the state space $S = S(A)$ is a *simplex* and that A is order isometric to the Banach subspace A(S) of all *affine* functions in C(S). However, it is well-known that each such *simplex space* A(S) has the a.p.: In fact, A(S)' is *an abstract* (L) - *space*. (This

argument can be found e.g. in the proof of Corollary 2.6, p. 477 of Namioka-Phelps [23].) (2) follows from (1) and 3.(2), 5 above. □

For the connection between simplicial spaces and the solution of "*weak Dirichlet problems*" see Effros-Kazdan [16]: $A_{\mathcal{H}}(K)$ (say) is simplicial if and only if each continuous function defined on a compact subset of the *Choquet boundary* of $A_{\mathcal{H}}(K)$ may be extended to an element of $A_{\mathcal{H}}(K)$ of the same norm.

But now we get the a.p. of $A_F(K)$ and $H_F(K)$ for *many* sheaves F of *axiomatic potential theory* and *all* sets K = closure of a relatively compact open set U: In fact, under certain axioms on the underlying *harmonic space* (X,F), it is known that $A_F(K)$ resp. $H_F(K)$ is *again simplicial*, and then we may proceed as in the proof of theorem 12 to carry the corresponding results over to this (much more general) setting. For the relevant axioms needed here and the p $A_F(K)$ resp. $H_F(K)$ is a simplicial space, we refer to Effros - Kazd [16], Cor. 4.3, p. 108 and Cor. 4.2, p. 112. (For a necessary and sufficient condition for $A_F(K) = H_F(K)$ in this setting see [16], theorem 4.4). In [16], the axioms still excluded *general* open sets U for *degenerate* elliptic equations, but the corresponding problem was solved affirmatively by Bliedtner-Hansen [13], and we refer to [13] for the most general results on simplicial spaces $A_F(K)$.

In concluding, we should point out that the ε-product $\mathcal{H}_1 \varepsilon \mathcal{H}_2$ of two sheaves of harmonic functions in axiomatic potential theory yields nothing but the (*multiply* resp.) *separately harmonic* functions of Gowrisankaran [20] resp. Reay [24]. We leave it to the reader to combine our preceding remark on the a.p. of $A_F(K)$ resp. $H_F(K)$ in axiomatic potential theory with the results in section 2 above to obtain, say, theorem 11 and lemma 23 of [24] without any effort. Of course, we could also immediately state results for "mixed" (say) holomorphic - harmonic sheaves $\mathcal{O} \varepsilon \mathcal{H}$ etc., but the preceding examples

and applications may suffice.

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THE COMPLETION OF PARTIALLY ORDERED VECTOR SPACES AND KOROVKIN'S THEOREM

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In the present paper we will give a new proof of a generalization of Korovkin's theorem using the completion of a partially ordered vector space by Dedekind-cuts. The given proof works not only in the case of $C[0,1]$ but also for certain partially ordered real vector spaces where a mode of convergence is defined, which is compatible with the linear structure and the partial ordering of the considered linear space.

Let X be a real vector space with a partial ordering defined by a cone K, the set of all positive elements of X including 0. The space X is called Dedekind-complete if every non-empty subset which is bounded from above has a supremum and if every non-empty subset which is bounded from below has an infimum. In the following we assume that the partial ordering is Archimedean which is defined by

$$\bigwedge_{n \in \mathbb{N}} n \cdot v \leqq u \Rightarrow v \leqq \Theta$$

for all elements $u, v \in X$.

Then we have the following

THEOREM: *Let X be a partially ordered real vector space, which is*

Archimedean. Then we have:

There is a unique determined Dedekind-complete partially ordered real vector space δX *with the following properties:*

(i) *There exists a subspace* $\hat{X}$ *of* δX *such that* X *and* $\hat{X}$ *are isomorphic.*

(ii) *Every element* $x^{\#} \in \delta X$ *satisfies*

$$x^{\#} = \sup \{ l \in X \mid l \leqq x^{\#} \}$$

$$= \inf \{ u \in X \mid x^{\#} \leqq u \}$$

δX *is called the Dedekind-completion of* X.

If in addition the ordering in X is directed i.e.

$$\underset{a,b \in X}{\forall} \quad \underset{c,d \in X}{\exists} \quad c \leqq a \leqq d \;\&\; c \leqq b \leqq d,$$

then δX is also a lattice.

For a proof of the theorem compare LUXEMBURG, ZAANEN [2].

DEFINITION: A subspace X of a partially ordered $\mathbb{R}$-vector space is called *Dedekind-dense in* Y iff $X \subset Y \subset \delta X$.

For stating the generalized Korovkin-theorem we have to define the mode of convergence in a partially ordered vector space. We use some results of BANASCHEWSKI [1]:

A subset $E \subset K \setminus \{0\}$ defines a convergence generating set in X if E satisfies the following conditions:

I. $$\underset{\varepsilon,\varepsilon' \in E}{\forall} \quad \underset{\delta \in E}{\exists} \quad \delta \leqq \varepsilon \;\&\; \delta \leqq \varepsilon'$$

II. $$\underset{\varepsilon \in E}{\forall} \quad \underset{\delta \in E}{\exists} \quad 2\delta \leqq \varepsilon$$

REMARK: Since we assume X to be Archimedean we have

III. $\inf E = 0$.

Now we define a mode of convergence as follows:

A sequence $(x_n) \subset X$ converges to an element $\bar{x} \in X$ iff

$$\underset{\varepsilon \in E}{\forall} \quad \underset{n_o \in \mathbb{N}}{\exists} \quad \underset{n \geqq n_o}{\forall} \qquad \bar{x} - \varepsilon \leqq x_n \leqq \bar{x} + \varepsilon$$

In this case we write $x_n \underset{E}{\to} \bar{x}$. This mode of convergence has the following properties:

(a) Constant sequences are convergent.

(b) If (x_n) converges to $\bar{x}$, then every subsequence of (x_n) converges also to $\bar{x}$.

(c) $x_n \underset{E}{\to} \bar{x} \;\&\; y_n \underset{E}{\to} \bar{y} \Rightarrow x_n + y_n \underset{E}{\to} \bar{x} + \bar{y}$

(d) $x_n \underset{E}{\to} \bar{x} \Rightarrow \underset{\alpha \in \mathbb{R}}{\forall} \quad \alpha x_n \underset{E}{\to} \alpha \bar{x}$

Further we assume

(e) Let (x_n) be a sequence such that

$$x_1 \geqq x_2 \geqq x_3 \geqq \cdots$$

and such that $\inf(x_n)$ exists, then

$$x_n \underset{E}{\to} \inf(x_n).$$

(f) Let (x_n) be a sequence such that

$$x_1 \leqq x_2 \leqq x_3 \leqq \cdots$$

and such that $\sup(x_n)$ exists, then

$$x_n \underset{E}{\to} \sup(x_n).$$

Now we can state the generalization of Korovkin's theorem:

THEOREM 1: *Let Y be a partially ordered $\mathbb{R}$-vector space and let E be a convergence generating set in Y. Further let X be an Archimedean*

partially ordered $\mathbb{R}$*-vector space, which is Dedekind-dense in* Y.

Let (L_n) *be a sequence of monotonic operators*

$$L_n : Y \to Y$$

such that

$$\underset{x \in X}{\forall} \; L_n(x) \underset{E}{\to} A(x)$$

where

$$A : Y \to Y$$

is a monotonic operator such that the restriction $A_{|X}$ *is a bijective map of* X *onto* X *and* $A_{|X}$ *is of monotonic type (i.e.* $A_{|X}(x_1) \leqq A_{|X}(x_2) \Rightarrow x_1 \leqq x_2$*). Then we have*

$$\underset{y \in Y}{\forall} \; L_n(y) \underset{E}{\to} A(y).$$

PROOF: For the proof let $y \in Y \subset \delta X$ be given. Then define the sets

$$U_y := \{u \in X \mid y \leqq u\},$$

$$L_y := \{l \in X \mid l \leqq y\}.$$

For each $u \in U_y$ and $l \in L_y$ we have

$$l \leqq y \leq u.$$

Since L_n and A are monotonic we have

$$L_n(l) \leqq L_n(y) \leqq L_n(u)$$

and

$$A(l) \leqq A(y) \leqq A(u).$$

For abbreviation we set

$$l_n := L_n(l), \quad y_n := L_n(y), \quad u_n := L_n(u).$$

We now prove: (y_n) converges to an element y_o. Since by assumption

$$L_n(l) \underset{E}{\to} A(l) \qquad \text{and} \qquad L_n(u) \underset{E}{\to} A(u)$$

we have

$$\underset{\varepsilon \in E}{\forall} \quad \underset{n_o \in \mathbb{N}}{\exists} \quad \underset{n \geqq n_o}{\forall} \quad A(l) - \varepsilon \leqq l_n \leqq y_n \leqq u_n \leqq A(u) + \varepsilon$$

or

$$A(l) - \varepsilon \leqq y_n \leqq A(u) + \varepsilon .$$

From this we conclude that the elements

$$s_n := \sup \{ y_k \in \delta X \mid k \geqq n \}$$

and

$$i_n := \inf \{ y_k \in \delta X \mid k \geqq n \}$$

exist and also satisfy the relation

$$\underset{n \geqq n_o}{\forall} \quad A(l) - \varepsilon \leqq i_n \leqq y_n \leqq s_n \leqq A(u) + \varepsilon .$$

Consequently we have

$$\underset{n \geqq n_o}{\forall} \quad A(l) - \varepsilon \leqq \bar{i} \leqq \bar{s} \leqq A(u) + \varepsilon$$

where

$$\bar{i} := \sup \{ i_n \} \qquad \text{and} \qquad \bar{s} := \inf \{ s_n \} .$$

This is true for every $\varepsilon \in E$; thus we have also

$$(*) \qquad \underset{l \in L_y}{\forall} \quad \underset{u \in U_y}{\forall} \quad A(l) \leqq \bar{i} \leqq \bar{s} \leqq A(u).$$

Now let $u \in U_{\bar{i}}$. Then we have $\bar{i} \leqq u$ and by (*)

$$\underset{l \in L_y}{\forall} \quad A(l) \leqq \bar{i} \leqq u.$$

Since A is of monotonic type and bijective we have

$$\underset{l \in L_y}{\forall} \quad l = A^{-1} A(l) \leqq A^{-1}(u)$$

and consequently $A^{-1}(u) \in U_y$. From this we conclude

$$A(y) \leqq AA^{-1}(u) = u$$

and hence

$$u \in U_{A(y)}, \quad \text{i.e.} \quad U_{\bar{i}} \subset U_{A(y)}.$$

Now let $u \in U_{A(y)}$. Then we have $A(y) \leqq u$ and

$$\underset{l \in L_y}{\forall} \quad l = A^{-1}A(l) \leqq A^{-1}(u)$$

and consequently $A^{-1}(u) \in U_y$. Using (*) we conclude

$$\tilde{i} \leqq AA^{-1}(u) = u$$

and hence $u \in U_{\tilde{i}}$, i.e. $U_{A(y)} \subset U_{\tilde{i}}$. Consequently we have

$$U_{\tilde{i}} = U_{A(y)}.$$

Similarly one can prove $L_{\tilde{i}} = L_{A(y)}$ and $U_{\bar{s}} = U_{A(y)}$, $L_{\bar{s}} = L_{A(y)}$. Using this we conclude: $\bar{i} = \bar{s} = A(y)$.

From the relations

$$\underset{\varepsilon \in E}{\forall} \quad \underset{n_o \in \mathbb{N}}{\exists} \quad \underset{n \geqq n_o}{\forall} \quad A(y) - \varepsilon \leqq i_n \leqq s_n \leqq A(y) + \varepsilon$$

and $i_n \leqq y_n \leqq s_n$ we conclude

$$A(y) - \varepsilon \leqq y_n \leqq A(y) + \varepsilon .$$

Since $\varepsilon \in E$ was arbitrary we have

$$y_n \xrightarrow[E]{} A(y).$$

REMARK: Let $C[a,b]$ be the vector lattice of all real-valued continuous functions on $[a,b]$ under the ordering defined by "$f \leqq g$ iff $f(t) \leqq g(t)$ for all $t \in [a,b]$". Then the set of all polynomials of degree $\leqq 2$ is Dedekind-dense in $C[a,b]$. Since the set $E:=\{\varepsilon \cdot 1 \in C[a,b] \mid \varepsilon > 0\}$ generates convergence in the sup-norm we can conclude the classical Korovkin-theorem from theorem 1.

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APPROXIMATION BY ALGEBRAIC CONVOLUTION INTEGRALS

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Dedicated to Professor Einar Hille on the occasion
of his eightyfifth birthday on 28 June 1979,
in friendship and high esteem.

1. INTRODUCTION AND PRELIMINARY RESULTS

In 1963 one of the authors [7] raised the question[*)] whether it is possible to construct a singular convolution integral of a function f defined on $[-1,1]$, say, which is an algebraic polynomial of degree n and which approximates f uniformly on $[-1,1]$ with order $O(n^{-\alpha})$, $0 < \alpha \leq 1$, provided f belongs to

$$(1.1) \qquad \mathrm{Lip}_1(\alpha;C) := \{f \in C;\ \omega_1(\eta;f;C) = O(\eta^{\alpha}),\ \eta \to 0+\},$$

where $\omega_1(\eta;f;C)$ is the modulus of continuity of f (see (7.3)) and $C \equiv C[-1,1]$ is the set of all real (or complex)-valued continuous functions f on $[-1,1]$ endowed with the usual sup-norm $\|\cdot\|_C$. The point is that this integral should not be derived under some (trigonometric)

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*) This problem has since been recalled in some language or other by, for example, G. Freud [28] and R. DeVore [21, 23].

substitution from a corresponding integral which is a trigonometric polynomial of degree n (such as the integral of Fejér-Korovkin; see e.g. [14], [9, p. 80]).

This problem was solved in some form or other by Freud [27] (1964), Sallay [49] (1964), Saxena [50] (1967), Vértesi [59] (1967), Vértesi-Kis [60] (1967), DeVore [20] (1968), Bojanic-DeVore [6] (1969), Bojanic [5] (1969), Chawla [18] (1970), Mathur [41] (1971), Rodina [46] (1973), Freud-Sharma [29] (1974), Szabados [56] (1976) and Butzer-Stens [14] (1976). Most of the polynomials constructed by these authors are of degree $4n - k$ with $k = 2$ or 4, some approximate f uniformly only on $[-1 + \varepsilon, 1 - \varepsilon]$ for each $\varepsilon > 0$.

The natural extension of this problem, posed in [8], is whether an algebraic polynomial of degree n can be constructed which gives uniform approximation to the associate f on the whole $[-1, 1]$ with order $\mathcal{O}(n^{-1-\alpha})$ provided the derivative f' belongs to $\mathrm{Lip}_1(\alpha; C)$, $0 < \alpha \leq 1$.

In this respect Bavinck [1, p. 69], Lupaş [38, 39] and Stens-Wehrens [55] considered the integral

$$(J_{2n}f)(x) := \frac{1}{2} \int_{-1}^{1} f(u)\chi_{2n}(x;u)\,du$$

(1.2)

$$\chi_{2n}(x;u) := \frac{3}{n^2 + 3n + 3} \sum_{k=0}^{2n} \frac{2k+1}{2} P_k(x)P_k(u) \cdot$$

$$\cdot \int_{-1}^{1} P_k(t)\,[\,P_n^{(2,0)}(t)\,]^2\,dt,$$

$P_n^{(\alpha,\beta)}$ being the Jacobi polynomial. Note that $\chi_{2n}(x;u) \geq 0$, $\int_{-1}^{1} \chi_{2n}(x;u)du = 2$ and, as will be seen below, the kernel can more simply be rewritten as

$$\chi_{2n}(x;u) = \frac{3}{n^2 + 3n + 3} \tau_u [P_n^{(2,0)}]^2(x).$$

Lupaş, for example, showed that

$$\|J_{2n}f - f\|_C \leq (1 + 2\sqrt{3})\omega_1(1/2n;f;C). \tag{1.3}$$

However, $(J_{2n}f)(x)$ is a polynomial of degree $2n$, and not n.

One purpose of this paper is to present a systematic approach to these two problems, thus to study direct approximation theorems for algebraic approximation processes that are built up analogously to well-known trigonometric convolution integrals. Normally one would expect to examine the convergence of convolution integrals like

$$\frac{1}{2} \int_{-1}^{1} \chi_n(x - u) f(u) du \tag{1.4}$$

to $f(x)$ for $n \to \infty$. The best known example of such an integral is that of Landau-Stieltjes (see e.g. [37, p. 9], [48, p. 147], [22, p. 26], [40]), the kernel of which is given by

$$\chi_{2n}(x) := \frac{1}{2\lambda_{2n}} [1 - (\frac{x}{2})^2]^n , \quad \lambda_{2n} := \frac{1}{2} \int_0^1 t^{-1/2} (1 - t)^n dt .$$

Here it is known (see e.g. [22, p. 221]) that $f \in \mathrm{Lip}_1(\alpha;C[-1+\varepsilon, 1 - \varepsilon])$, $0 < \alpha \leq 1$, $0 < \varepsilon < 1$, implies that

$$\|\frac{1}{2} \int_{-1}^{1} \chi_{2n}(x - u) f(u) du - f(x)\|_{C[-1+\varepsilon, 1-\varepsilon]} = O(n^{-\alpha/2}). \tag{1.5}$$

The integral is again an algebraic polynomial of degree $2n$. Here, however, difficulties occur at the end points ± 1 of the interval $[-1, 1]$ since the classical translation operator τ_u' used, namely $(\tau_u' g)(x) := g(x - u)$, leads one outside of the interval $[-1, 1]$.

The question now is whether it is possible to employ an algebraic convolution concept (which depends on an associated translation concept) for which these difficulties do not occur and for which there holds some "convolution theorem" in the form that if T is a suitable transform such as

$$(1.6) \qquad T[f](k) = \frac{1}{2}\int_{-1}^{1} f(u)\Phi_k(u)w(u)du \qquad (k \in \mathbb{P} = \{0,1,2,\ldots\})$$

for suitable functions Φ_k, orthogonal with respect to the weight $w(x)$, then

$$(1.7) \qquad T[f * g](k) = T[f](k)\, T[g](k) \qquad (k \in \mathbb{P}),$$

$f * g$ being the suitable convolution of f and g. This would enable one to use integral transform methods and, as is well-known, such methods usually enable one to solve a variety of problems by a reduction to a standard procedure (recall the Fourier transform method in the solution of partial differential equations; see e.g. [9, Chap.VII]). Hereby the aim is to employ purely algebraic means in the proofs, the only connection with the periodic theory being of structural nature, namely an approach via convolution integrals together with transform methods. Therefore in none of the proofs results of Fourier analysis will be used, as was the case in a few instances in the Chebyshev transform approach of Butzer-Stens [12,13,14,15,16], Stens [54].

The transform we shall apply is the quite well developed Legendre transform. Although Fourier-Legendre series have been known for at least a century, the product formula leading to the translation operator being already known to Gegenbauer [30] in 1875, the Legendre transform point of view seems to have been first emphasized by Tranter [58] only in 1950 (see also [51, p. 423, 454] and the literature cited there). The main results needed here are built up in Stens-

Wehrens [55], but let us recall some of the basic concepts.

Letting X stand either for the space $C[-1,1]$ or $L^p(-1,1) \equiv L^p$, $1 \le p < \infty$, of all real (or complex)-valued measurable functions f defined on $[-1,1]$ for which the norm

$$\|f\|_p := \{\frac{1}{2} \int_{-1}^{1} |f(u)|^p \, du\}^{1/p}$$

is finite, the Legendre transform of $f \in X$ is defined by

$$(1.8) \qquad L[f](k) \equiv f\hat{}(k) := \frac{1}{2} \int_{-1}^{1} f(u) P_k(u) \, du \qquad (k \in \mathbb{P}).$$

Here $P_k(x)$ is the Legendre polynomial of degree k, namely

$$P_k(x) = \frac{(-1)^k}{2^k k!} \frac{d^k}{dx^k}(1 - x^2)^k \qquad (x \in [-1,1];\ k \in \mathbb{P}).$$

In view of the fact that

$$(1.9) \qquad |P_k(x)| \le P_k(1) = 1 \qquad (x \in [-1,1];\ k \in \mathbb{P}),$$

one has

$$(1.10) \qquad |L[f](k)| \le \|f\|_X \qquad (f \in X;\ k \in \mathbb{P}),$$

so that (1.8) defines a bounded linear operator mapping X into (c_o), the space of all real (or complex)-valued sequences $\{a_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} a_k = 0$.

The classical translation operator τ_h' is here replaced by

$$(1.11) \qquad (\tau_h f)(x) := (1/\pi) \int_{-1}^{1} f(xh + u\sqrt{1-x^2}\sqrt{1-h^2}) \sqrt{1-u^2}^{-1} \, du \qquad (x, h \in [-1,1]).$$

In contrast to τ_h', τ_h defines for each $h \in [-1,1]$ a positive linear operator from X into itself with $\|\tau_h\|_{[X,X]} = 1$ and the usual

$\lim_{h\to 1-} \|\tau_h f - f\|_X = 0$. The associated convolution product $f * g$ is defined as

$$(1.12) \qquad (f * g)(x) := \frac{1}{2}\int_{-1}^{1} (\tau_u f)(x) g(u)\,du \qquad (x \in [-1,1]).$$

If $f \in X$, $g \in L^1$, the convolution $f * g$ exists as an element of X together with

$$(1.13) \qquad \|f * g\|_X \leq \|f\|_X \|g\|_1$$

$$(1.14) \qquad \mathcal{L}[f * g](k) = \mathcal{L}[f](k)\,\mathcal{L}[g](k) \qquad (k \in \mathbb{P})$$

which is the form taken on by the convolution theorem (1.7) in the Legendre case.

The derivative also being defined via translation, it is to be expected that the derivative in the Legendre frame will be unusual. The strong (Legendre) derivative of $f \in X$ is the function $g \in X$ for which

$$\lim_{h\to 1-} \left\| \frac{f - \tau_h f}{1 - h} - g \right\|_X = 0;$$

we write $D^1 f = g$. Derivatives D^r of higher order $r = 2,3,\ldots$ are defined iteratively. The set of all $f \in X$ for which $D^r f$ exists is denoted by W_X^r. Note that the strong derivative $D^1 f$, $f \in W_X^1$, coincides with the pointwise derivative

$$\frac{d}{dx}\left(\frac{x^2 - 1}{2}\,\frac{d}{dx} f(x)\right).$$

The counterparts of the modulus of continuity and Lipschitz class here take on the form

$$(1.15) \qquad \omega_1^L(\delta; f; X) \equiv \omega_1^L(\delta; f) := \sup_{\delta \leq h \leq 1} \|\tau_h f - f\|_X \qquad (-1 < \delta < 1)$$

(1.16) $\quad \mathrm{Lip}_1^L(\alpha;X) := \{f \in X;\ \omega_1^L(\delta;f) = O((1-\delta)^\alpha), \delta \to 1-\}\ (0 < \alpha \leq 1).$

The main purpose of this paper is to give a unified treatment of algebraic approximation theory via the Legendre transform method, in particular, to study conditions upon the sequence of functions $\{\chi_n\}_{n\in\mathbb{P}} \subset L^1(-1,1)$ such that

$$\lim_{n\to\infty} \| f * \chi_n - f \|_X = 0 \qquad (f \in X), \tag{1.17}$$

and to investigate the rate of convergence in (1.17), expressing it in terms of the modulus of continuity (1.15). In addition to such direct approximation theorems, the matching inverse theorems will also be considered, emphasis being placed upon the so-called case of non-optimal approximation. The case of optimal or saturated approximation is dealt with briefly. The aim will be to employ elementary means in establishing the direct theorems (thus without appealing to the general theorems based upon intermediate space methods of Berens [3] and Butzer-Scherer [10, 11], as was carried out by Bavinck [1, 2]). Concerning the inverse theorems, they will either be deduced via the associated theorems of best algebraic approximation (as developed in Stens-Wehrens [55]) or from a general result based upon a Bernstein-type inequality. This material is considered in Sec. 2.

One major aspect is to study various examples of suitable kernels that can be classified under the Legendre transform approach. These are given by various summability methods of the Fourier-Legendre series of $f \in X$:

$$f \sim \Sigma_{k=0}^{\infty} (2k+1) \hat{f}(k) P_k(x), \tag{1.18}$$

namely by the Fejér means of $f \in X$ (Sec. 5), by the Fejér-Korovkin means (Sec. 3), the Rogosinski means (Sec. 6), certain de La Vallée

Poussin sums (Sec. 5), by the Weierstrass and integral means, as well as by the Landau-Stieltjes integral in the Legendre frame, all three treated in Sec. 4.

The Fejér means are defined by

$$(1.19)\quad (\sigma_n f)(x) := \Sigma_{k=0}^{n} (1 - \frac{k}{n+1})(2k+1) f\hat{}(k) P_k(x) \qquad (x \in [-1,1]\ ;\ n \in \mathbb{P})$$

which may be rewritten in the form of an algebraic convolution integral

$$(1.20)\qquad (\sigma_n f)(x) := \frac{1}{2} \int_{-1}^{1} F_n(u)(\tau_u f)(x)\,du$$

where

$$(1.21)\quad F_n(x) := \Sigma_{k=0}^{n} (1 - \frac{k}{n+1})(2k+1) P_k(x) \qquad (x \in [-1,1];\ n \in \mathbb{P}).$$

A particular case of the results to be established asserts that

$$f \in \mathrm{Lip}_1(\alpha;C) \Rightarrow f \in \mathrm{Lip}_1^{L}(\alpha/2;C) \Rightarrow \|\sigma_n f - f\|_C = O(n^{-\alpha}). \quad (0 < \alpha < 1).$$

This solves the stated problem in its original form on the whole $[-1,1]$ for $0 < \alpha < 1$, $\sigma_n f$ being an algebraic polynomial precisely of degree n.

For the more difficult case $\alpha = 1$ we proceed as follows. In Fourier analysis the Fejér-Korovkin kernel may be defined as that even non-negative trigonometric polynomial t_n of the form

$$t_n(\theta) = 1 + 2\,\Sigma_{k=1}^{n} a_k \cos k\theta \qquad (\theta \in \mathbb{R};\ n \in \mathbb{P})$$

for which the coefficient a_1 takes on its largest possible value (given by $\cos(\pi/(n+2))$). In the corresponding algebraic case this amounts to finding that algebraic polynomial

$$p_n(x) = 1 + \Sigma_{k=1}^{n} (2k+1)b_k P_k(x) \qquad (x \in [-1,1];\ n \in \mathbb{P}),$$

which is non-negative on $[-1, 1]$ and for which b_1 attains its maximum. The solution of this extremal problem is exactly the Fejér – Korovkin kernel for the Legendre case, defined by

$$(1.22) \qquad K_n(x) := (1-\bar{x}_N^2) \begin{cases} \left(\dfrac{P_N(x)}{x-\bar{x}_N}\right)^2, & x \neq \bar{x}_N \\ \left(P_N'(\bar{x}_X)\right)^2, & x = \bar{x}_N, \end{cases}$$

$\bar{x}_N$ being the largest root of $P_N(x)$, and $N = [\![n/2]\!] + 1$ ($[\![x]\!]$ denotes the largest integer $\leq x$).

Again a special case of our results states that

$$f' \in \mathrm{Lip}_1(\alpha;C) \Rightarrow \| f * K_n - f \|_C = O(n^{-1-\alpha}) \qquad (0 < \alpha \leq 1).$$

This shows that the associated Fejér-Korovkin means $f \bullet K_n$ solve our problem even in its extended form on the whole $[-1, 1]$, $f \bullet K_n$ actually being a (pure) algebraic polynomial of degree n. Most of the above results are not only valid in $C[-1, 1]$ but also in $L^p(-1,1)$.

As mentioned, two of the authors set up a Chebyshev transform method with essentially the same aim in mind, namely to give a unified approach to as many problems as possible on the approximation of functions f belonging to $C[-1, 1]$ or $L_w^p(-1,1)$, $1 \leq p < \infty$, by *algebraic* polynomials. The Chebyshev method has the advantage that a greater variety of problems can be considered, such as all those connected with moduli of continuity of higher order, including the fractional case. The disadvantage, however, is that it is not as "purely" algebraic as is the Legendre transform approach. Although the latter is

more intricate as it is not connected with the periodic Fourier theory, it has the advantage that no "bad-looking" weight factors enter into the picture as is the case with the weight $w(x) = (1 - x^2)^{-1/2}$ in the Chebyshev theory.

The question of course arises why not treat the matter by the more general Jacobi transform approach. The reason is that we first wanted to present an approach that is not only as uncomplicated but also as complete as possible. However, much of the material presented can readily be carried over to the Jacobi frame. As indicated Bavinck [1, 2] considered more or less some of our results in the latter frame. But it can perhaps be said that his aim was to generalize *trigonometric* approximation theory to the Jacobi frame without being concerned with the connections to the problems of *algebraic* approximation in the classical sense.

For a basic unsolved problem in the Legendre approach see [17].

2. GENERAL THEOREMS ON CONVOLUTION INTEGRALS

This section is concerned with theorems on the convergence of general convolution integrals

$$(2.1) \qquad (I_\rho f)(x) := (f * \chi_\rho)(x) = \frac{1}{2}\int_{-1}^{1} (\tau_u f)(x) g(u)\,du$$

$$(f \in X;\ x \in [-1, 1];\ \rho \in \mathbf{A})$$

where $\{\chi_\rho\}_{\rho \in \mathbf{A}}$ is a kernel, i.e. $\chi_\rho \in L^1(-1, 1)$ with

$$(2.2) \qquad \frac{1}{2}\int_{-1}^{1} \chi_\rho(u)\,du = 1 \qquad (\rho \in \mathbf{A}),$$

ρ being a parameter ranging over some set $\mathbf{A}$ which is either an interval (a,b) with $0 \le a < b \le \infty$, or the set $\mathbf{P}$. Let ρ_0 be one of

the points a,b or $+\infty$.

In the following M denotes a positive constant, the value of which may be different at each occurrence. M is always independent of the quantities at the right margin.

PROPOSITION 1: *Let* $\{\chi_\rho\}_{\rho \in \mathbb{A}}$ *be a kernel such that*

$$\|\chi_\rho\|_1 \leq M \qquad (\rho \in \mathbb{A}). \tag{2.3}$$

The integrals $\{I_\rho\}_{\rho \in \mathbb{A}}$ *form an approximation process on* X, *i.e.*

$$\lim_{\rho \to \rho_0} \|I_\rho f - f\|_X = 0 \tag{2.4}$$

for each $f \in X$, *if and only if*

$$\lim_{\rho \to \rho_0} \hat{\chi}_\rho(k) = 1 \qquad (k \in \mathbb{N} = \{1,2,3,\ldots\}). \tag{2.5}$$

In this proposition, the proof of which follows by the Banach-Steinhaus theorem, it may be difficult to verify condition (2.3) in the applications. If the kernel is however positive, i.e. $\chi_\rho(u) \geq 0$ for almost all $u \in (-1,1)$, $\rho \in \mathbb{A}$, then (2.3) is always satisfied with $M = 1$ in view of (2.2). This leads to the following Bohman-Korovkin-type result:

PROPOSITION 2: *If the kernel* $\{\chi_\rho\}_{\rho \in \mathbb{A}}$ *is positive, each of the following assertions is equivalent to* (2.4) *and* (2.5):

$$\lim_{\rho \to \rho_0} \hat{\chi}_\rho(1) = 1, \tag{2.6}$$

$$\lim_{\rho \to \rho_0} \int_{-1}^{\delta} \chi_\rho(u)\,du = 0 \qquad (-1 < \delta < 1). \tag{2.7}$$

PROOF: That (2.5) implies (2.6) is obvious. On the other hand, if (2.6) holds, then by (2.2) and the positivity of the kernel,

$$0 \le \int_{-1}^{\delta} \chi_\rho(u)\,du \le \frac{1}{1-\delta}\int_{-1}^{\delta}(1-u)\chi_\rho(u)\,du$$

$$\le \frac{1}{1-\delta}\int_{-1}^{1}(1-u)\chi_\rho(u)\,du = \frac{2}{1-\delta}\,[1-\hat{\chi}_\rho(1)] = o(1) \qquad (\rho\to\rho_0).$$

This gives (2.7). Concerning (2.7) $\Rightarrow$ (2.4), one uses the fact that for $f \in X$

$$\|I_\rho f - f\|_X \le \frac{1}{2}\int_{-1}^{1}\|(\tau_u f)(\cdot) - f(\cdot)\|_X \chi_\rho(u)\,du \tag{2.8}$$

$$\le \|f\|_X \int_{-1}^{\delta}\chi_\rho(u)\,du + \omega_1^L(\delta;f)$$

which implies (2.4) by standard arguments. □

When dealing with positive kernels, there holds the following general result on the rate of convergence in (2.4):

PROPOSITION 3: *Let $\{\chi_\rho\}_{\rho\in\mathbb{A}}$ be a positive kernel, and let φ be a strictly positive function defined on $\mathbb{A}$ such that $\lim_{\rho\to\rho_0}\varphi(\rho) = 0$. The following assertions are equivalent:*

$$|1-\hat{\chi}_\rho(1)| = O(\varphi(\rho)) \qquad (\rho\to\rho_0), \tag{2.9}$$

$$\|I_\rho f - f\|_X \le M\omega_1^L(1-\varphi(\rho);f;X) \qquad (f\in X;\ \rho\in\mathbb{A}). \tag{2.10}$$

PROOF: Let us first mention the following inequalities needed (see [55; Sec. 5]),

$$\omega_1^L(\delta;g;X) \le 6(1-\delta)\|D^1 g\|_X \qquad (g\in W_X^1;\ -1<\delta<1) \tag{2.11}$$

$$(2.12)\quad \omega_1^L(\delta_1;g;X) \le 12\left(1 + \frac{1-\delta_1}{1-\delta_2}\right)\omega_1^L(\delta_2;g;X) \quad (g \in X;\ -1 < \delta_1, \delta_2 < 1).$$

Now if (2.9) holds, then (2.10) follows as in (2.8) by (2.12) and (2.2) since

$$\|I_\rho f - f\|_X \le 12\,\omega_1^L(1-\varphi(\rho);f)\frac{1}{2}\int_{-1}^{1}\left[1 + \frac{1-u}{\varphi(\rho)}\right]\chi_\rho(u)\,du$$

$$= 12\,\omega_1^L(1-\varphi(\rho);f)\left\{1 + \frac{1-\hat{\chi}(1)}{\varphi(\rho)}\right\} \le M\omega_1^L(1-\varphi(\rho);f) \quad (\rho \in \mathbb{A}).$$

To prove the converse, we apply

$$(2.13)\qquad (g \star P_k)(x) = \hat{g}(k)P_k(x) \qquad (g \in X;\ x \in [-1,1];\ k \in \mathbb{P})$$

for $k = 1$ and $g = \chi_\rho$ to deduce

$$|\hat{\chi}_\rho(1) - 1| = \|P_1\|_X^{-1}\,\|P_1 \star \chi_\rho - P_1\|_X \le M\omega_1^L(1-\varphi(\rho);P_1).$$

Since $P_1 \in W_X^1$, (2.9) follows by (2.11).□

The particular case $\varphi(\rho) = 1 - \hat{\chi}_\rho(1)$ gives, noting that

$$(2.14)\qquad 1 - \hat{\chi}_\rho(1) = \frac{1}{2}\int_{-1}^{1}(1-u)\chi_\rho(u)\,du > 0 \qquad (\rho \in \mathbb{A}),$$

COROLLARY 1: *For each positive kernel* $\{\chi_\rho\}_{\rho\in\mathbb{A}}$ *there holds*

$$(2.15)\qquad \|I_\rho f - f\|_X \le 24\,\omega_1^L(\hat{\chi}_\rho(1);f;X) \qquad (f \in X;\ \rho \in \mathbb{A}).$$

Inequality (2.15) shows that the approximation error depends on the smoothness properties of the functions involved. Therefore one might expect that the rate of convergence in (2.4) could be arbitrarily good if f is sufficiently smooth. Later on it will be seen that

this situation really takes place for at least one convolution integral. But for a majority of approximation processes there exists a critical order which cannot be transpassed unless f is a constant. This is the familiar saturation phenomenon.

A handy criterion deciding whether the saturation property holds for a convolution integral $\{I_\rho\}_{\rho \in \mathbb{A}}$ is given by

PROPOSITION 4: a) *If the kernel* $\{\chi_\rho\}_{\rho \in \mathbb{A}}$ *of the process* $\{I_\rho\}_{\rho \in \mathbb{A}}$ *satisfies condition*

$$\limsup_{\rho \to \rho_0} \frac{1 - \hat{\chi}_\rho(k)}{\varphi(\rho)} > 0 \qquad (k \in \mathbb{N}), \tag{2.16}$$

where φ *is a function as in Prop. 3, then*

$$\| I_\rho f - f \|_X = o(\varphi(\rho)) \qquad (\rho \to \rho_0) \tag{2.17}$$

implies that $f = \text{const.}$ (a.e.)*).

b) *If moreover there exists some* $k_0 \in \mathbb{N}$ *such that*

$$\limsup_{\rho \to \rho_0} \frac{1 - \hat{\chi}_\rho(k_0)}{\varphi(\rho)} < +\infty, \tag{2.18}$$

then there exists at least one non-constant (non-trivial) $f \in X$ *such that*

$$\| I_\rho f - f \|_X = O(\varphi(\rho)) \qquad (\rho \to \rho_0). \tag{2.19}$$

In other words, $\{I_\rho\}_{\rho \in \mathbb{A}}$ *is saturated in* X *with order* $O(\varphi(\rho))$, $\rho \to \rho_0$.

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*) "(a.e.)" means that an assertion holds for all $x \in [-1, 1]$ if $X = C[-1,1]$, and for almost all $x \in [-1, 1]$ if $X = L^p(-1,1)$, $1 \leq p < \infty$.

PROOF: By (2.17) one has on account of (1.10) and (1.14) that

$$|\chi_\rho^\wedge(k) - 1|\ |f^\wedge(k)| \le \|I_\rho f - f\|_X = o(\varphi(\rho)) \qquad (\rho \to \rho_o;\ k \in \mathbb{N}).$$

This means that $\lim_{\rho\to\rho_o} |1 - \chi_\rho^\wedge(k)|\,|f^\wedge(k)|\,/\varphi(\rho) = 0$, $k \in \mathbb{N}$. On the other hand, by (2.16) there exists a subsequence $\{\rho_j\}_{j=1}^\infty$, $\lim_{j\to\infty}\rho_j = \rho_o$, such that $\lim_{j\to\infty} |\chi_{\rho_j}^\wedge(k) - 1|\,/\varphi(\rho_j) = c_k > 0$, each $k \in \mathbb{N}$. By the uniqueness theorem (see [55, Sec. 2]) this gives $f = \text{const.}$ (a.e.).

Concerning part b), one has for $f = P_{k_o}$ by (2.13) and (2.18) that

$$\|I_\rho P_{k_o} - P_{k_o}\|_X = |\chi_\rho^\wedge(k_o) - 1|\ \|P_{k_o}\|_X = O(\varphi(\rho)) \qquad (\rho \to \rho_o),$$

which completes the proof.□

Whereas Cor. 1 gives a direct approximation result, an inverse one is given by

PROPOSITION 5: *Let $\{\chi_n\}_{n\in\mathbb{P}}$ be a kernel satisfying a Bernstein-type inequality of order $\gamma > 0$, namely*

$$(2.20)\qquad f * \chi_n \in W_X^1, \quad \|D^1(f * \chi_n)\|_X \le M\, n^\gamma \|f\|_X \qquad (f \in X;\ n \in \mathbb{P}).$$

If $f \in X$ can be approximated by the convolution integral $f * \chi_n$ such that

$$(2.21)\qquad \|f * \chi_n - f\|_X = O(n^{-\gamma\alpha}) \qquad (n \to \infty)$$

for some $0 < \alpha < 1$, then $f \in \mathrm{Lip}_1^L(\alpha;X)$.

PROOF: Let us follow the classical arguments using telescoping sums (cf. [9, p. 110]. Setting $U_2 = f * \chi_{2^2}$, $U_n = f * \chi_{2^n} - f * \chi_{2^{n-1}}$ for $n = 3,4,\ldots$, then by (2.21)

(2.22) $$\|U_n\|_X \le M 2^{-n\gamma\alpha} \qquad (n = 2,3,\ldots),$$

(2.23) $$\lim_{n\to\infty} \|\Sigma_{k=2}^{n} U_k - f\|_X = 0, \quad \|f\|_X \le \Sigma_{k=2}^{\infty} \|U_k\|_X .$$

Hence for arbitrary $h \in (-1,1)$ and integral $m \ge 2$

(2.24) $$\|\tau_h f - f\|_X \le \Sigma_{k=2}^{m} \|\tau_h U_k - U_k\|_X + 2 \Sigma_{k=m+1}^{\infty} \|U_k\|_X .$$

Since U_k belongs to W_X^1 for every $k \ge 2$, one has by (2.11)

(2.25) $$\|\tau_h U_k - U_k\|_X \le 6(1 - h)\|D^1 U_k\|_X .$$

Since the convolution product is commutative and associative, one can rewrite U_k as

$$U_k = (f - f \ast \chi_{2^{k-1}}) \ast \chi_{2^k} - (f - f \ast \chi_{2^k}) \ast \chi_{2^{k-1}} \qquad (k = 3,4,\ldots).$$

This implies by (2.25), (2.20) and (2.21) that

$$\|\tau_h U_k - U_k\|_X \le M(1 - h)\ \{2^{k\gamma} 2^{(1-k)\gamma\alpha} + 2^{(k-1)\gamma} 2^{-k\gamma\alpha}\}$$

$$\le M(1 - h) 2^{k\gamma(1-\alpha)},$$

which is also valid for $k = 2$. This yields by (2.24) and (2.22) that

(2.26) $$\omega_1^L(\delta; f) \le M\{(1 - \delta)\Sigma_{k=2}^{m} 2^{k\gamma(1-\alpha)} + \Sigma_{k=m+1}^{\infty} 2^{-k\gamma\alpha}\} \quad (-1 < \delta < 1).$$

If one now chooses m such that for $\delta > 1 - 2^{-\gamma}$, $2^{m-1} < (1-\delta)^{-1/\gamma} \le 2^m$, then (2.26) gives (as e.g. in [9, p. 101]) that $\omega_1^L(\delta; f) \le M(1 - \delta)^{\alpha}$. This is the desired assertion. □

Finally a result on best approximation by algebraic polynomials will be stated; it may be used for establishing direct approximation theorems for certain approximation processes. It also gives another way in proving inverse theorems for polynomials kernels leading to optimal approximation. As usual, let

$$E_n(f;X) = \inf_{p_n \in P_n} \|f - p_n\|_X \qquad (n \in \mathbb{P})$$

be the best approximation of $f \in X$ be algebraic polynomials of degree $\leq n$, P_n being the set of such polynomials. Concerning the rate of convergence in

$$\lim_{n\to\infty} E_n(f;X) = 0 \qquad (f \in X) \tag{2.27}$$

we have the following counterpart of the classical Jackson and Bernstein theorems (see [55, Sec. 5]).

PROPOSITION 6: a) *For each* $r \in \mathbb{P}$, *there holds*

$$E_n(f;X) \leq Mn^{-2r}\, \omega_1^L(1 - n^{-2};\, D^r f;X) \qquad (f \in W_X^r;\ n \in \mathbb{N}).$$

b) *The following statements are equivalent for* $0 < \alpha < 1$:

$$E_n(f;X) = O(n^{-2(r+\alpha)}) \qquad (n \to \infty), \tag{2.28}$$

$$D^r f \in \mathrm{Lip}_1^L(\alpha;X), \text{ i.e. } \omega_1^L(\delta;D^r f;X) = O((1-\delta)^\alpha) \quad (\delta \to 1-). \tag{2.29}$$

The purpose of the next sections is to apply the foregoing results to several concrete approximation processes.

3. THE ALGEBRAIC CONVOLUTION INTEGRAL OF FEJÉR-KOROVKIN

Concerning Cor. 1 it is of interest to search for positive kernels $\{\chi_n\}_{n=1}^{\infty}$ which belong to P_n such that the modulus $\omega_1^L(\chi_n\hat{}(1);f;X)$ is as small as possible. This amounts to determing such kernels for which $\chi_n\hat{}(1)$ is as close to 1 as possible. Note that $\chi_n\hat{}(1) < 1$ in view of (2.14). This will lead to the Fejér-Korovkin kernel of Sec. 1.

Therefore we will now discuss some extremal properties of the coefficients of the elements of the set

$$(3.1)\qquad NP_n^+ := \{p_n \in P_n,\ p_n(x) \geq 0 \ \text{ for } \ x \in [-1,1],\ p_n\hat{}(0)=1\} \quad (n \in \mathbb{P}).$$

THEOREM 1: a) *For all* $p_n \in NP_n^+$ *there holds*

$$(3.2)\qquad |p_n\hat{}(1)| \leq \bar{x}_{m+1},$$

where $m = [\![(n+1)/2]\!]$, *and* $\bar{x}_{m+1}$ *is the largest root of* P_{m+1}.

b) *For each even* $n \in \mathbb{P}$ *there exists a unique* $p_n \in NP_n^+$ *such that*

$$(3.3)\qquad p_n\hat{}(1) = \bar{x}_{m+1}.$$

c) *For* $n \in \mathbb{N}$ *odd there exists no* $p_n \in NP_n^+$ *such that* (3.3) *holds.*

d) *For each* $p_n \in NP_n^+$ *and each* $j \in \mathbb{N}$ *there holds*

$$(3.4)\qquad \frac{j(j+1)}{2(2n+1)^2} \leq 1 - p_n\hat{}(j) \leq 72j(j+1)(1-p_n\hat{}(1)),$$

the right hand inequality being valid for all $n \in \mathbb{P}$, *the left hand inequality only for* $n \geq N_o = N_o(j)$.

PROOF: First we need the Gauss-Jacobi mechanical quadrature formula

(see e.g. [19, p. 74], [57, p. 47;(15.3.2)]). It states that for all $q_{2k-1} \in P_{2k-1}$, $k \in \mathbb{N}$, there holds

$$\int_{-1}^{1} q_{2k-1}(u)\,du = \sum_{j=1}^{k} \lambda_{j,k} q_{2k-1}(x_{j,k}), \tag{3.5}$$

where $x_{j,k}$, $1 \le j \le k$, denote the roots of the Legendre polynomial P_k with $-1 < x_{j+1,k} < x_{j,k} < 1$, $j = 1,2,\ldots,k-1$, and $\lambda_{j,k}$ the "Legendre abscissas"

$$\lambda_{j,k} = 2(1 - x_{j,k}^2)^{-1}(P_k'(x_{j,k}))^{-2} \qquad (1 \le j \le k).$$

Now let $p_n \in NP_n^+$. Choosing $m = [\![(n+1)/2]\!]$ one has by (3.5)

$$1 = \hat{p_n}(0) = \frac{1}{2} \sum_{j=1}^{m+1} \lambda_{j,m+1} p_n(x_{j,m+1}) \tag{3.6}$$

$$\hat{p_n}(1) = \frac{1}{2} \int_{-1}^{1} u p_n(u)\,du = \frac{1}{2} \sum_{j=1}^{m+1} \lambda_{j,m+1} x_{j,m+1} p_n(x_{j,m+1}). \tag{3.7}$$

Since $p_n(x_{j,m+1}) \ge 0$, $\lambda_{j,m+1} \ge 0$, $1 \le j \le m+1$, part a) follows immediately by

$$|\hat{p_n}(1)| \le \frac{1}{2} \max_{1 \le j \le m+1} |x_{j,m+1}| \sum_{j=1}^{m+1} \lambda_{j,m+1} p_n(x_{j,m+1}) = \bar{x}_{m+1}.$$

To prove parts b) and c) we first assume the existence of a $p_n \in NP_n^+$ satisfying (3.3). By (3.6) and (3.7) one has

$$0 = \bar{x}_{m+1}\hat{p_n}(0) - \hat{p_n}(1) = \frac{1}{2} \sum_{j=2}^{m+1} \lambda_{j,m+1}(\bar{x}_{m+1} - x_{j,m+1}) p_n(x_{j,m+1}).$$

Since all terms in this sum are non-negative and $\lambda_{j,m+1}(\bar{x}_{m+1} - x_{j,m+1}) \neq 0$, one has by (3.6)

$$p_n(x_{j,m+1}) = 0 \qquad (2 \le j \le m+1) \tag{3.8}$$

$$p_n(x_{1,m+1}) = 2/\lambda_{1,m+1} \,. \tag{3.9}$$

Therefore p_n is uniquely determined at the points $x_{j,m+1}$, $1 \le j \le m+1$. Since the $x_{i,m+1}$, $i = 2,\dots,m+1$ are in $(-1,1)$ and p_n is to be positive, it must have m double roots. This implies that for even $n \in \mathbb{P}$

$$p_n(x) = (1-\bar{x}_{m+1}^2) \cdot \begin{cases} \left(\dfrac{P_{m+1}(x)}{x-\bar{x}_{m+1}}\right)^2 & , \quad x \neq \bar{x}_{m+1} \\ \left(P'_{m+1}(\bar{x}_{m+1})\right)^2 & , \quad x = \bar{x}_{m+1} \end{cases} \tag{3.10}$$

is the only polynomial in NP_n^+ satisfying (3.3).

For n odd, i.e. $m = (n+1)/2$, (3.8) states that p_n possesses $(n+1)/2$ double zeros, meaning that $p_n(x) \equiv 0$. But this is a contradiction to (3.9), proving c).

Concerning part d), the right side of (3.4) follows from

$$|\hat{p_n}(j) - 1| \le \|P_j\|_C^{-1} \, \|P_j * p_n - P_j\|_C \le 24\, \omega_1^L(\hat{p_n}(1); P_j)$$

$$\le 6 \cdot 24 \|D^1 P_j\|_C (1 - \hat{p_n}(1)) = 72 j(j+1)(1-\hat{p_n}(1)),$$

where (2.13), Cor. 1, (2.11) and the equality

$$(D^1 P_j)(x) = \frac{j(j+1)}{2} P_j(x) \qquad (x \in [-1,1];\ j \in \mathbb{P}) \tag{3.11}$$

(cf. [55, Sec. 3]) were applied. In order to verify the left side of (3.4) fix $j \in \mathbb{N}$. For $n \ge j+1$ one has by (3.5) that

$$|\hat{p_n}(j)| \le \frac{1}{2} \sum_{i=1}^{n} \lambda_{i,n} \, |P_j(x_{i,n})| \, p_n(x_{i,n}) \tag{3.12}$$

$$\leq \max_{x_{n,n} \leq x \leq x_{1,n}} |P_j(x)| = \max_{0 \leq x \leq x_{1,n}} |P_j(x)|,$$

the latter equality being valid as $|P_j(x)|$ is even, and $x_{1,n} = -x_{n,n}$. From [57, Sec. 7.21] it now follows that $|P_j(x)|$ attains its maximum value in $[0, 1-\eta]$ at $x = 1-\eta$ if $\eta > 0$ is small enough. This means that $\max_{0 \leq x \leq x_{1,n}} |P_j(x)| = |P_j(x_{1,n})| = P_j(x_{1,n})$ if n is chosen large enough. So (3.12) yields that

$$|\hat{p_n}(j)| \leq P_j(x_{1,n}) \qquad (n \geq N_o).$$

In view of Bruns' inequality (see e.g. [57. (6.21.5)])

$$\cos(2\pi/(2n+1)) \leq x_{1,n} \leq \cos(\pi/(2n+1)), \tag{3.13}$$

we find with a suitable $\xi_n \in (x_{1,n}, 1)$ that

$$1 = P_j(1) = P_j(x_{1,n}) + (1 - x_{1,n}) P_j'(\xi_n)$$

$$1 - P_j(x_{1,n}) \geq |P_j'(\xi_n)|(1 - \cos(\pi/(2n+1))) \geq |P_j'(\xi_n)| \frac{2}{(2n+1)^2}.$$

Since $\lim_{n\to\infty} P_j'(\xi_n) = P_j'(1) = j(j+1)/2$, the proof of d) is complete. □

The polynomial p_n of (3.10) satisfies for even $n \in \mathbb{N}$ the same extremal property as does the trigonometric Fejér-Korovkin kernel. Therefore one may justly call the kernel K_n of (1.22) the (Legendre-) Fejér-Korovkin kernel. If n is even, the K_n are precisely the polynomials p_n of (3.10); if n is odd then $K_n(x) = p_{n-1}(x)$.

Concerning the approximation behaviour of the associated Fejér-Korovkin convolution integral we have

THEOREM 2. *Let $\{K_n\}_{n \in \mathbb{P}}$ be the Legendre-Fejér-Korovkin kernel defined*

by (1.22). *One has for* $f \in X$

a) $\|f * K_n - f\|_X \le 24\, \omega_1^L(\bar{x}_N; f; X)$ $\qquad (n \in \mathbb{P})$.

b) $f \in \mathrm{Lip}_1^L(\alpha; X) \Leftrightarrow \|f * K_n - f\|_X = O(n^{-2\alpha})$ $\qquad (n \to \infty;\ 0 < \alpha < 1)$.

c) *The convolution integral of Fejér-Korovkin is saturated in* X *with order* $O(n^{-2})$.

The proof of part a) follows by Cor. 1. Concerning b), $f \in \mathrm{Lip}_1^L(\alpha; X)$ implies by (3.13) that $\omega_1^L(\bar{x}_N; f) = O((1 - \bar{x}_N)^\alpha) = O(n^{-2\alpha})$. The converse can be derived either from Prop. 5 since $f * K_n \in P_n \subset W_X^1$ and

$$\|D^1 p_n\|_X \le M n^2 \|p_n\|_X \qquad (p_n \in P_n;\ n \in \mathbb{P}) \tag{3.14}$$

(see [55, Sec. 5]), or from Prop. 6. Part c) follows readily by Prop. 4, Thm. 1d) and the fact that $1 - K_n^\wedge(1) = 1 - \bar{x}_N = O(n^{-2})$.

Let us remark that Bavinck [2] employs this kernel to establish a Jackson-type theorem (in the Jacobi frame). In [1] he also treats the extemal property but does not distinguish between the cases n being even or odd; for odd n his proof is not quite correct since the quadrature formula used is no more exact in this case. Newman-Shapiro [43] and Feinerman-Newman [25, p. 103] handle the extremal problem for even n in a somewhat more general frame but do not discuss results of the type of Thm. 2 b), c).

As an immediate consequence of Thm. 1d) one has that for every positive kernel $\{\chi_n\}_{n \in \mathbb{P}}$, $\chi_n \in P_n$, there holds

$$\limsup_{n \to \infty} n^2 (1 - \chi_n^\wedge(j)) > 0 \qquad (j \in \mathbb{N}).$$

An application of Prop. 4 a) then leads to the following algebraic counterpart of a result in [9, p. 88] which is related to a

result of Korovkin [36, p. 128].

COROLLARY 2: *If* $\chi_n \in NP_n^+$, *then*

$$\|f \star \chi_n - f\|_X = o(n^{-2}) \qquad (n \to \infty)$$

for some $f \in X$ *implies that* $f = \text{const.}$ (a.e.).

In other words, one cannot approximate a non-constant function $f \in X$ by positive polynomial convolution integrals of degree n with an order better than $O(n^{-2})$.

4. THREE FURTER POSITIVE ALGEBRAIC CONVOLUTION INTEGRALS

Whereas the first example below is an algebraic polynomial of degree 2n, the second and third are non-polynomial.

4.1. THE SINGULAR INTEGRAL OF LANDAU-STIELTJES

As remarked in the introduction, the classical Landau-Stieltjes integral approximates functions $f \in C[-1, 1]$ uniformly only on $[-1 + \varepsilon, 1 - \varepsilon]$ for $\varepsilon \in (0,1)$. Using the kernel of this integral we now construct a new singular integral by means of the Legendre convolution which forms an approximation process in $C[-1, 1]$ as well as in $L^p(-1,1)$. This kernel is defined by

$$\Lambda_{2n}(x) := (1/\lambda_{2n})[1 - (\frac{1-x}{2})^2]^n \qquad (x \in [-1,1];\ n \in \mathbb{P}),$$

(4.1)

$$\lambda_{2n} := \frac{1}{2}\int_{-1}^{1}[1 - (\frac{1-u}{2})^2]^n\,du \qquad (n \in \mathbb{P}).$$

The Λ_{2n} belong to NP_{2n}^+, and for the λ_{2n} one has $\lambda_o = 1$,

$$\lambda_{2n} = (1/2)\int_0^1 t^{-1/2}(1-t)^n dt = \frac{2n(2n-2).....2}{(2n+1)(2n-1)...3} \qquad (n \in \mathbb{N})$$

$$(4.2) \qquad (\pi/(4n+4))^{1/2} \leq \lambda_{2n} \leq (\pi/(4n+2))^{1/2} \qquad (n \in \mathbb{P}).$$

Let us list the properties of Λ_{2n} needed for the approximation theorem.

LEMMA 1: a) *For all* $n \in \mathbb{P}$ *there holds*

$$(4.3) \qquad \hat{\Lambda_{2n}}(1) = 1 - 1/(n+1)\lambda_{2n}.$$

b) *For each* $k \in \mathbb{N}$, *there exists a constant* $M > 0$ *such that*

$$(4.4) \qquad Mn^{-1/2} \leq 1 - \hat{\Lambda_{2n}}(k) \leq 72k(k+1)/(n+1)\lambda_{2n} \qquad (f \in X;\ n \in \mathbb{P}).$$

c) *The following Bernstein-type inequality holds:*

$$(4.5) \qquad f * \Lambda_{2n} \in W_X^1,\ \|D^1(f * \Lambda_{2n})\|_X \leq 6n^{1/2}\|f\|_X \qquad (f \in X;\ n \in \mathbb{P}).$$

PROOF: Part a) follows immediately by integrating the identity

$$(4.6) \qquad (1-x)\Lambda_{2n}(x) = (2/(n+1)\lambda_{2n})\ \frac{d}{dx}\left[1 - \left(\frac{1-x}{2}\right)^2\right]^{n+1}.$$

By applying a) and (3.4) we obtain the right-hand side of inequality (4.4). Concerning the left side, one has in view of the positivity of Λ_{2n} and (4.6) for each $\delta \in (0,2)$, $k \in \mathbb{N}$,

$$1 - \hat{\Lambda_{2n}}(k) \geq \frac{1}{2}\int_{1-\delta}^1 (1 - P_k(u)\Lambda_{2n}(u)\,du$$

$$= (1/(n+1)\lambda_{2n})\int_{1-\delta}^1 \frac{1-P_k(u)}{1-u}\left\{\frac{d}{du}\left[1 - \left(\frac{1-u}{2}\right)^2\right]^{n+1}\right\} du$$

$$\geq (1/(n+1)\lambda_{2n}) \inf_{1-\delta \leq u \leq 1} |P_k'(u)| \int_{1-\delta}^{1} \{\frac{d}{du}[1-(\frac{1-u}{2})^2]^{n+1}\}\, du.$$

Since $P_k'(1) = k(k+1)/2$, we may choose some $\delta \in (0,2)$ such that $\inf_{1-\delta \leq u \leq 1} |P_k'(u)| > 0$. Furthermore, the integral can be estimated from below by $(\delta/2)^2$. The desired inequality now follows by (4.2).

To prove part c) we obtain from [55, Sec. 3] and (1.13) that $f * \Lambda_{2n} \in P_{2n} \subset W_X^1$ and

$$\|D^1(f * \Lambda_{2n})\|_X = \|f * D^1\Lambda_{2n}\|_X \leq \|f\|_X \|D^1\Lambda_{2n}\|_1 \qquad (f \in X;\ n \in \mathbb{P}).$$

Therefore we have only to show that

$$(4.7) \qquad \|D^1\Lambda_{2n}(x)\|_1 \equiv \|\frac{d}{dx}[\frac{x^2-1}{2}\ \frac{d}{dx}\ \Lambda_{2n}(x)]\|_1 \leq 6n^{1/2} \qquad (n \in \mathbb{P}).$$

First let $n \geq 2$. By the estimate

$$\lambda_{2n}\,|\frac{d}{dx}[\frac{x^2-1}{2}\ \frac{d}{dx}\ \Lambda_{2n}(x)]|$$

$$= \frac{n}{n-1}\,|\{\frac{d}{dx}[1-(\frac{1-x}{2})^2]^{n-1}\}\{[1-(\frac{1-x}{2})^2]\frac{3x+1}{2} + \frac{n-1}{4}(1-x)(x^2-1)\}|$$

$$\leq \frac{n}{n-1}\{\frac{d}{dx}[1-(\frac{1-x}{2})^2]^{n-1}\}\ (2 + (n-1)(1-x)^2/2)$$

one finds that

$$\lambda_{2n}\|D^1\Lambda_{2n}\|_1 \leq \frac{n}{n-1}\int_{-1}^{1}\{\frac{d}{du}[1-(\frac{1-u}{2})^2]^{n-1}\}\, du$$

$$+ \frac{n}{4}\int_{-1}^{1}(1-u)^2\{\frac{d}{du}[1-(\frac{1-u}{2})^2]^{n-1}\}\, du$$

$$= n/(n-1) + 1 \leq 3,$$

which yields (4.5) by (4.2). A similar calculation shows that (4.7) also holds for $n = 1$; the case $n = 0$ is obvious. □

As an application of Cor. 1, Prop. 5 and Prop. 4 we obtain

THEOREM 3. *For the singular integral of Landau-Stieltjes* $f \star \Lambda_{2n}$, $f \in X$, *there holds*

a) $\|f \star \Lambda_{2n} - f\|_X \leq 24\, \omega_1^L(1 - [(n+1)\lambda_{2n}]^{-1}; f; X)$ $\qquad (n \in \mathbb{P})$.

b) $f \in \mathrm{Lip}_1^L(\alpha; X) \Leftrightarrow \|f \star \Lambda_{2n} - f\|_X = O(n^{-\alpha/2})$ $\quad (n \to \infty;\ 0 < \alpha < 1)$.

c) *The integral of Landau-Stieltjes is saturated in* X *with order* $O(n^{-1/2})$.

4.2. THE INTEGRAL OF WEIERSTRASS

The kernel of Gauss-Weierstrass with respect to the Legendre system is given by

$$(4.8) \qquad w_t(x) := \sum_{k=0}^{\infty} \exp\{-\frac{k(k+1)}{2} t\} (2k+1) P_k(x) \qquad (x \in [-1,1];\ t > 0).$$

LEMMA 2: *The kernel* $w_t(x)$ *is non-negative for all* $x \in [-1,1]$, $t > 0$, *and satisfies the Bernstein-type inequality*

$$(4.9) \qquad (f \star w_t) \in W_X^1,\ \|D^1(f \star w_t)\|_X \leq M t^{-1} \|f\|_X \qquad (f \in X;\ 0 < t \leq 1).$$

The proof of the first part can be found in Bochner [4, p. 1146] or Hille-Phillips [33, p. 612], and (4.9) can be deduced from Görlich-Nessel-Trebels [31, (3.9)] since the Legendre series of $f \in X$ is (C,1)-summable (see Sec. 5 below).

An application of Cor. 1, Prop. 5 with $\chi_n = w_{1/n}$, and Prop. 4

yields

THEOREM 4. *For the algebraic Gauss-Weierstrass convolution integral* $f \star w_t$, $f \in X$, *there holds*

a) $\|f \star w_t - f\|_X \leq 24\, \omega_1^L(e^{-t}; f; X)$ $\qquad (t > 0)$.

b) $f \in \mathrm{Lip}_1^L(\alpha; X) \Leftrightarrow \|f \star w_t - f\|_X = O(t^\alpha)$ $\qquad (t \to 0+;\ 0 < \alpha < 1)$.

c) *The integral* $f \star w_t$ *is saturated in* X *with order* $O(t)$, $t \to 0+$.

Concerning the infinitesimal generator of the Weierstrass as well as Abel method of summation of the Legendre series see e.g.[33,p.610].

4.3. THE INTEGRAL MEANS

In [55, Sec. 3] there had been introduced the (Legendre-) integral means $A_h f$, $f \in X$, $h \in (-1,1)$. They are defined as the convolution integral with kernel

$$\kappa(x;h) := \begin{cases} \log \dfrac{(1+x)(1-h)}{(1-x)(1+h)} \log^{-1}\left(\dfrac{2}{1+h}\right) & , \quad -1 < h \leq x < 1 \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{4.10}$$

the Legendre coefficients of which are given by

$$[\kappa(\cdot;h)]\hat{}(k) = \begin{cases} 1 & k = 0 \\ \dfrac{1 - P_k(h)}{k(k+1)\log(2/(1+h))} & , \quad k \in \mathbb{N}. \end{cases} \tag{4.11}$$

Note that

$$(4.12) \qquad \lim_{h\to 1-} \frac{1 - [\kappa(\cdot;h)]\hat{}(k)}{1-h} = \frac{k(k+1)}{8} \qquad (k \in \mathbb{N}).$$

Since $\kappa(x;h) \geq 0$, $x \in [-1,1]$, $h \in (-1,1)$ and $\lim_{h\to 1-}[\kappa(\cdot;h)]\hat{}(1) = 1$, the $A_h f$ define an approximation process on X for $h \to 1-$.

THEOREM 5. *For the integral means* $A_h f = f * \kappa(\cdot;h)$, $f \in X$, *one has:*

a) $\|A_h f - f\|_X \leq M\omega_1^L(h;f;X)$ $\qquad (h \in (-1,1))$,

b) $f \in \mathrm{Lip}_1^L(\alpha;X) \Leftrightarrow \|A_h f - f\|_X = O((1-h)^\alpha)$ $\qquad (h \to 1-;\ 0 < \alpha < 1)$,

c) *The integral means are saturated in* X *with order* $O(1-h)$, $h \to 1-$.

PROOF: Part a) follows by (4.12), Cor. 1 and (2.12), yielding also the direct part of b). Concerning the inverse part in b), we make use of the Bernstein inequality (cf. [55, Sec. 3])

$$A_h f \in W_X^1, \quad \|D^1 A_h f\|_X \leq M(1-h)^{-1}\|f\|_X \qquad (f \in X;\ h \in (-1,1)),$$

and apply Prop. 5, setting $\chi_n(x) = \kappa(x;1-1/n)$. Finally part c) follows from Prop. 4 by (4.12). □

5. THE INTEGRALS OF FEJÉR AND DE LA VALLÉE POUSSIN

In analogy to the trigonometric theory, the Fejér kernel with respect to the Legendre system is defined by

$$(5.1) \qquad F_n(x) := \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right)(2k+1)P_k(x) \equiv \frac{1}{n+1}\sum_{k=0}^n D_k(x)$$

$$(x \in [-1,1];\ n \in \mathbb{P}),$$

where D_n is the (Legendre-)Dirichlet kernel

$$(5.2) \qquad D_n(x) = \Sigma_{k=0}^{n} (2k+1)P_k(x) \qquad (x \in [-1,1]\ ;\ n \in \mathbb{P}).$$

The associated algebraic Fejér convolution integral $(\sigma_n f) := f \star F_n$ (recall (1.19), (1.20)) is actually the n-th (C,1)-mean of the partial sums of the Fourier-Legendre series (1.18) of $f \in X$. The L^1 - norms of the F_n are uniformly bounded (see [32]), i.e. there exists $1 \leq F < \infty$ such that $\|F_n\|_1 \leq F$, $n \in \mathbb{P}$. Since

$$\hat{F_n}(k) = \begin{cases} (1 - \frac{k}{n+1}) & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n, \end{cases}$$

with $\lim_{n\to\infty} \hat{F_n}(k) = 1$, $k \in \mathbb{N}$, one has by Prop. 1

$$(5.3) \qquad \lim_{n\to\infty} \|f \star F_n - f\|_X = 0 \qquad (f \in X).$$

Next consider the de La Vallée Poussin kernel

$$(5.4) \qquad v_{m,n}(x) := \frac{1}{m+1} \Sigma_{k=n-m}^{n} D_k(x) \qquad (x \in [-1,1]\ ;\ m,\ n \in \mathbb{P},\ m \leq n).$$

In view of the identity $(F_{-1} = 0)$

$$(5.5) \qquad v_{m,n}(x) = (1 - \frac{m-n}{m+1})F_n(x) + \frac{m-n}{m+1} F_{n-m-1}(x),$$

it follows that

$$\|v_{m,n}\|_1 \leq (1 + \frac{2(n-m)}{m+1})F \qquad (m,n \in \mathbb{P},\ m \leq n).$$

If m depends on n in such a way that

$$(5.6) \qquad \sup_{n\in\mathbb{N}} \frac{n}{m(n)+1} < \infty,$$

then $\|v_{m,n}\|_1 \leq M$, $n \in \mathbb{P}$. In this case Prop. 1 again implies that the associated integral $f * v_{m,n}$ forms an approximation process on X, noting that the Legendre coefficients of $v_{m,n}$ are given by

$$(5.7) \qquad \hat{v}_{m,n}(k) = \begin{cases} 1 & , \quad 0 \leq k \leq n - m \\ (n+1-k)/(m+1) & , \quad n - m + 1 \leq k \leq n \\ 0 & , \quad k > n. \end{cases}$$

In contrast to the trigonometric Fejér kernel and that of (Legendre-) Fejér-Korovkin treated in Sec. 3, the (Legendre-) Fejér kernel fails to be positive (e.g. $F_n(-1) = -1/2$ if n is odd; see also [26]). Using (5.5) this enables one to show that the (Legendre-) de La Vallée Poussin kernel is not a positive kernel, at least for certain $m,n \in \mathbb{P}$. Hence one cannot apply Cor. 1 in order to obtain direct approximation theorems. We proceed in another fashion; in the periodic case it is due to Stečkin [53].

LEMMA 3. *For all* $f \in X$, $n,m \in \mathbb{P}$, $m \leq n$ *there holds*

$$(5.8) \qquad \|f * v_{m,n} - f\|_X \leq \frac{2(n+1)}{m+1} \, F E_{n-m}(f;X)$$

$$(5.9) \qquad \|f * F_n - f\|_X \leq \frac{8F}{n+1} \sum_{k=0}^{n+1} E_k(f;X).$$

PROOF: From (5.7) and the convolution theorem (1.14) it follows that for all $p_{n-m} \in \mathcal{P}_{n-m}$

$$(5.10) \qquad (p_{n-m} * v_{m,n})(x) = p_{n-m}(x) \qquad (x \in [-1,1],\ m,n \in \mathbb{P};\ m \leq n).$$

If $p_n^*(f)$ denotes a polynomial of best approximation in $\mathcal{P}_n$ to f

in X, then (5.10) yields

$$\|f \star v_{m,n} - f\|_X \leq \|f \star v_{m,n} - p^{\star}_{n-m} \star v_{m,n}\|_X + \|p^{\star}_{m-n} \star v_{m,n} - f\|_X$$

$$\leq \|v_{m,n}\|_1 \, \|f - p^{\star}_{m-n}\|_X + \|p^{\star}_{m-n} - f\|_X$$

$$\leq \frac{2(n+1)F}{m+1} \, E_{n-m}(f;X).$$

The proof of (5.9) being practically the same as in [53], we will only sketch it. One has by (5.1),(5.4) for all $k,n \in \mathbb{N}$, $2^{k-1} \leq n+1 < 2^k$

$$f - f \star F_n = \frac{1}{n+1} \{f - f \star v_{o,o} + \Sigma_{j=1}^{k-1} 2^{j-1}(f - f \star v_{2^{j-1}-1,2^j-1})$$

$$+ (n+1-2^{k-1})(f - f \star v_{n-2^{k-1},n})\} .$$

One deduces by (5.8)

$$\|f - f \star F_n\|_X \leq \frac{2F}{n+1} \{E_o(f;X) + \Sigma_{j=1}^{k-1} 2^j E_{2^{j-1}}(f;X) + (n+1)E_{2^{k-1}}(f;X)\}$$

$$\leq \frac{2F}{n+1} \{E_o(f;X) + 2E_1(f;X) + 4\Sigma_{j=2}^{k-1} \Sigma_{\ell=2^{j-2}+1}^{2^{j-1}} E_\ell(f;X)$$

$$+ \frac{n+1}{2^{k-2}} \Sigma_{\ell=2^{k-2}+1}^{n+1} E_\ell(f;X)\}$$

$$\leq \frac{8F}{n+1} \Sigma_{j=0}^{n+1} E_j(f;X),$$

which is the desired inequality for $n \geq 1$. If $n = 0$ it follows by (5.8)□.

Of course (5.8) gives a usable estimate only in the case when

$\lim_{n\to\infty}(n-m(n)) = +\infty$. Choosing e.g. $m = [\![\beta n]\!]$ for some $0 < \beta < 1$, one has in view of Prop. 6 and La. 3

THEOREM 6. *Let* $f * v_{m,n}$, $f \in X$, *be the algebraic de La Vallée Poussin convolution integral with integral* $m = [\![\beta n]\!]$, $0 < \beta < 1$, *defined via* (5.4). *One has for each* $r \in \mathbb{P}$

a) $\|f * v_{m,n} - f\|_X \le M n^{-2r} \omega_1^L(1-n^{-2}; D^r f; X)$ $\qquad (f \in W_X^r;\ n \in \mathbb{N})$.

b) $D^r f \in \mathrm{Lip}_1^L(\alpha; X) \Leftrightarrow \|f * v_{m,n} - f\|_X = O(n^{-2(r+\alpha)})$ $\qquad (n \to \infty;\ 0 < \alpha < 1)$.

c) *The integral* $f * v_{m,n}$ *is not saturated in* X, *i.e. for any arbitrary function* φ *as in Prop. 3 there exists an* $f \in X$, $f \neq$ const. (a.e.) (*e.g.* f *any polynomial*), *such that*

$$\|f * v_{m,n} - f\|_X = o(\varphi(n)) \qquad (n \to \infty).$$

In the case $n - m$ remains bounded, one can proceed similarly as for the Fejér means below, noting that $f * F_n = f * v_{n,n}$ (see [34] for the Gegenbauer case).

Concerning the Fejér means $f * F_n$ we have by (5.9), Prop. 6 and Prop. 4

THEOREM 7: *One has for* $f * F_n$, $f \in X$, $n \to \infty$,

a) $$f \in \mathrm{Lip}_1^L(\alpha; X) \Rightarrow \|f * F_n - f\|_X = \begin{cases} O(n^{-2\alpha}) & , \ 0 < \alpha < 1/2 \\ O(n^{-1}\log n) & , \ \alpha = 1/2 \\ O(n^{-1}) & , \ \alpha > 1/2. \end{cases}$$

b) $f \in \mathrm{Lip}_1^L(\alpha; X) \Leftrightarrow \|f * F_n - f\|_X = O(n^{-2\alpha})$ $\qquad (0 < \alpha < 1/2)$.

c) *The integral* $f \star F_n$ *is saturated in* X *with order* $O(n^{-1})$.

Note that Kallaev [35] not only showed that $f \star F_n$ is saturated in $C[-1,1]$ with order $O(n^{-1})$ but actually determined the saturation class, i. e. the class of functions f for which $\|f \star F_n - f\|_C = O(n^{-1})$. It consists of all $f \in C[-1,1]$ for which $G \in \mathrm{Lip}_1(1;C)$, where

$$G(x) := \int_{-1}^{x} \left(\frac{1-u^2}{x-u}\right)^{1/2} \psi'(u)\,du, \quad \psi(x) := \int_{x}^{1} f(u)(u-x)^{-1/2}\,du.$$

In other words, he studied part b) in the case $\alpha = 1/2$. For a pointwise analogue of part a) see Rafal'son [45].

6. THE ROGOSINSKI SUMMATION METHOD

It is well-known that the convolution integral generated by the Dirichlet kernel D_n of (5.2) fails to be an approximation process in X unless $X = L^p(-1,1)$, $4/3 < p < 4$ (see [44], [42]). Instead of D_n we now consider the "shifted" Dirichlet kernel

$$(6.1) \qquad R_n(x;x_n) \equiv R_n(x) := (\tau_{x_n} D_n)(x) \equiv \Sigma_{k=0}^{n} (2k+1) P_k(x_n) P_k(x)$$

$$(n \in \mathbb{P};\ x \in [-1,1]),$$

where $\{x_n\}$ is a suitable sequence of reals in $(-1,1)$. This kernel may by Christoffel's summation formula be rewritten in the closed form

$$R_n(x;x_n) = (n+1)\,\frac{P_n(x)P_{n+1}(x_n) - P_{n+1}(x)P_n(x_n)}{x_n - x} \quad (x \neq x_n).$$

The construction (6.1) corresponds to the trigonometric Rogosinski

kernel (see [47] and e.g. [9, p. 56, 106])

$$B_n(\zeta) := 1 + 2\Sigma_{k=1}^{n} \cos k\zeta_n \cos k\zeta \qquad (n \in \mathbb{P};\ \zeta \in \mathbb{R})$$

which forms an approximation process if e.g. $\zeta_n = \pi/(2n+1)$ or $\zeta_n = \pi/(2n)$. In view of Prop. 1 the x_n in (6.1) have to be chosen in such a way that

$$\|R_n\|_1 \leq M \qquad (n \in \mathbb{P}) \tag{6.2}$$

$$\lim_{n\to\infty} P_k(x_n) = 1 \quad (k \in \mathbb{N}) \quad \text{respect.} \quad \lim_{n\to\infty} x_n = 1, \tag{6.3}$$

for in this case

$$\lim_{n\to\infty} \hat{R_n}(k) = \lim_{n\to\infty} P_k(x_n) = P_k(1) = 1 \qquad (k \in \mathbb{N}). \tag{6.4}$$

The following lemma gives sufficient conditions for (6.2) and (6.3) to hold:

LEMMA 4: *Let* $\{x_n\}_{n\in\mathbb{P}} \subset (-1,1)$ *be such that*

$$|\bar{x}_n - x_n| = O(n^{-5/2}) \qquad (n \to \infty), \tag{6.5}$$

where $\bar{x}_n$ *is the largest zero of* $P_n(x)$, $\bar{x}_0$ *arbitrary. Then* (6.2) *and* (6.3) *are valid.*

PROOF: By Bruns' inequality (recall (3.13)) we have

$$|1 - x_n| \leq |1 - \bar{x}_n| + |\bar{x}_n - x_n| = O(n^{-2}) + O(n^{-5/2}) = O(n^{-2}) \tag{6.6}$$

for $n \to \infty$. This implies (6.3). Applying Abel's transformation twice we obtain

$$(6.7)\quad \Sigma_{k=0}^{n}(2k+1)P_k(x_n)P_k(x) = -\Sigma_{k=0}^{n-1} D_k(x)\bar{\Delta}P_k(x_n) + D_n(x)P_n(x_n)$$

$$= \Sigma_{k=0}^{n-2}(k+1)F_k(x)\bar{\Delta}^2P_k(x_n) - nF_{n-1}(x)\bar{\Delta}P_{n-1}(x_n) + D_n(x)P_n(x_n).$$

Here $\bar{\Delta}P_k(x_n) = P_{k+1}(x_n) - P_k(x_n)$ is the first and $\bar{\Delta}^2P_k(x_n) = P_{k+2}(x_n) - 2P_{k+1}(x_n) + P_k(x_n)$ the second difference of $\{P_k(x_n)\}_{k\in\mathbb{P}}$.

Making use of the well-known representation (see e.g. [57, p. 88])

$$P_k(x) = \frac{\sqrt{2}}{\pi}\int_x^1 \frac{\cos[(k+1/2)\arccos u]}{(u-x)^{1/2}(1-u^2)^{1/2}}\,du \qquad (x \in (-1,1))$$

and the generalized mean value theorem, we obtain with $\lambda(u) \in (0,1)$

$$\bar{\Delta}^2P_k(x_n) =$$

$$= \frac{\sqrt{2}}{\pi}\int_{x_n}^1 \frac{\cos[(k+5/2)\arccos u] - 2\cos[(k+3/2)\arccos u] + \cos[(k+1/2)\arccos u]}{(u-x_n)^{1/2}(1-u^2)^{1/2}}\,du$$

$$= \frac{\sqrt{2}}{\pi}\int_{x_n}^1 \frac{-\arccos^2 u\,\cos[(k+1/2+2\lambda(u))\arccos u]}{(u-x_n)^{1/2}(1-u^2)^{1/2}}\,du.$$

Since

$$0 \le \int_{x_n}^1 \frac{1}{(u-x_n)^{1/2}(1-u^2)^{1/2}}\,du \le \frac{1}{(1+x_n)^{1/2}}\int_{x_n}^1 \frac{1}{(u-x_n)^{1/2}(1-u)^{1/2}}\,du$$

$$= \frac{1}{(1+x_n)^{1/2}}\int_0^1 \frac{1}{(1-u)^{1/2}u^{1/2}}\,du = O(1) \qquad (n \to \infty),$$

one can estimate the second difference using (6.6) by

$$|\bar{\Delta}^2 P_k(x_n)| \le M \arccos^2 x_n = O(n^{-2}) \qquad (n \to \infty).$$

In a similar way one has for the first difference

$$|\bar{\Delta} P_{n-1}(x_n)| \le M \arccos x_n = O(n^{-1}) \qquad (n \to \infty),$$

and using Markov's inequality (e.g. [36, p. 99]) for $\lambda \in (0,1)$,

$$|P_n(x_n)| = |P_n(x_n) - P_n(\bar{x}_n)| = |x_n - \bar{x}_n| \, |P_n'(\bar{x}_n + \lambda(x_n - \bar{x}_n))|$$

$$\le M|x_n - \bar{x}_n| \, n^2 = O(n^{-1/2}) \qquad (n \to \infty),$$

Applying these three inequalities in (6.7) leads to

$$\|R_n\|_1 \le M\{n^{-2} \sum_{k=0}^{n-2} (k+1)\|F_k\|_1 + \|F_{n-1}\|_1 + \|D_n\|_1 n^{-1/2}\}$$

which yields (6.2) since $\|F_k\|_1 \le F$, $k \in \mathbb{P}$, and (see e.g. [32])

$$\lim_{n\to\infty} n^{-1/2} \|D_n\|_1 = 2\sqrt{2/\pi} \, . \, \square$$

In view of Prop. 1, La. 4 and that (6.3) implies (6.4), there follows

COROLLARY 3. *If* $\{x_n\}_{n\in\mathbb{P}} \subset (-1,1)$ *is such that* $|\bar{x}_n - x_n| = O(n^{-5/2})$, $n \to \infty$, *then one has for each* $f \in X$

$$\lim_{n\to\infty} \|f * R_n - f\|_X = 0. \tag{6.8}$$

REMARK: Note that the assumptions of La. 4 are fulfilled in the

particular case $x_n = \bar{x}_n$, so that one may use $\bar{x}_n$ instead of x_n in Cor. 3. The kernel $R_n(x;\bar{x}_n)$ may also be rewritten for $x \neq \bar{x}_n$ as

$$(6.9) \qquad R_n(x;\bar{x}_n) \equiv \Sigma_{k=0}^{n-1} (2k+1)P_k(\bar{x}_n)P_k(x) = (n+1)\frac{P_n(x)P_{n+1}(\bar{x}_n)}{\bar{x}_n - x}.$$

This form shows that the Fejér-Korovkin kernel K_{2n-2} of (1.22) is actually the square of $R_n(x;\bar{x}_n)$ apart from a normalization factor:

$$K_{2n-2}(x) = \frac{(1-\bar{x}_n^2)}{(n+1)^2 P_{n+1}^2(\bar{x}_n)} R_n^2(x;\bar{x}_n).$$

The analogous result in the trigonometric case, namely that this kernel is essentially the square of a particular Rogosinski kernel, was first noticed by Stark [52].

Let us finally consider the rate of approximation in (6.8). By the convolution theorem one has for each $p_n \in \mathcal{P}_n$

$$(6.10) \qquad (p_n \star R_n)(x) = (\tau_{x_n} p_n)(x) \qquad (x \in [-1,1]).$$

THEOREM 8. *Let $R_n(x;x_n)$ be defined as in (6.1) where $\{x_n\}_{n\in\mathbb{P}} \subset (-1,1)$ satisfies $|x_n - \bar{x}_n| = O(n^{-5/2})$ for $n \to \infty$. One has for $f \in X$*

a) $\|f \star R_n - f\|_X \leq M E_n(f;X) + \omega_1^L(x_n;f;X) \leq M\omega_1^L(1-n^{-2};f;X) \quad (n \in \mathbb{P})$

b) $f \in \mathrm{Lip}_1^L(\alpha;X) \Leftrightarrow \|f \star R_n - f\|_X = O(n^{-2\alpha}) \quad (n \to \infty;\ 0 < \alpha < 1).$

c) *The integral $f \star R_n$ of Legendre-Rogosinski is saturated in X with order $O(n^{-2})$.*

PROOF: Let $p_n^\star \in \mathcal{P}_n$ be a polynomial of best approximation to $f \in X$.

Using Minkowski's inequality and (6.10), one obtains

$$\|f * R_n - f\|_X \le \|f * R_n - p_n^* * R_n\|_X + \|p_n^* * R_n - f\|_X$$

$$\le \|R_n\|_1 \, E_n(f;X) + \|\tau_{x_n} p_n^* - \tau_{x_n} f\|_X + \|\tau_{x_n} f - f\|_X$$

$$\le (\|R_n\|_1 + 1) E_n(f;X) + \omega_1^L(x_n;f;X)$$

which is the first inequality of part a); the second follows by Prop. 6a), (6.6) and (2.12). Part b) follows by a) and Prop. 6b) or (3.14) and Prop. 5. Finally the saturation result can be derived from Prop. 4 since

$$\lim_{n\to\infty} \frac{1 - \hat{R_n}(k)}{1 - x_n} = \lim_{n\to\infty} \frac{1 - P_k(x_n)}{1 - x_n} = P_k'(1) = \frac{k(k+1)}{2} \qquad (k \in \mathbb{N}). \; \square$$

Note that Kallaev [35] states without proof the saturation class of a particular Rogosinski-type integral.

7. APPROXIMATION RATES EXPRESSED IN TERMS OF CLASSICAL MODULI OF CONTINUITY

The reader may object to the fact that the rates of convergence of the various convolution integrals given by parts b) and a) of Cor. 1 and Thms. 2-8 are expressed in terms of moduli of continuity or Lipschitz classes which are defined via the Legendre difference $(\tau_h f)(x) - f(x)$ and not the classical differences $(\Delta_h^1 f)(x) = f(x+h) - f(x)$, $(\Delta_h^2 f)(x) = f(x+h) + f(x-h) - 2f(x)$. However, the "Legendre" Lipschitz classes are actually equivalent to certain pointwise Lipschitz classes

defined via $(\Delta_h^i f)(x)$, $i = 1,2$, at least for $X = C[-1,1]$, as was shown in [55, Sec. 6].

The pointwise moduli of continuity of $f \in C[-1,1]$ for $x \in [-1,1]$ with respect to the differences $\Delta_h^i f(x)$, $i = 1,2$, and the associated Lipschitz classes of order $\alpha > 0$ are defined by

$$\Omega_1(\eta;x;f;C) := \sup\{|\Delta_h^1 f(x)|;\ h \in [-\eta,\eta] \cap [-1-x,\ 1-x]\}$$

$$\Omega_2(\eta;x;f;C) := \sup\{|\Delta_h^2 f(x)|;\ h \in [-\eta,\eta] \cap [-1-x,1-x] \cap [-1+x,1+x]\} \tag{7.1}$$

$$P\text{-}Lip_i(\alpha;C) := \{f \in C[-1,\ 1];\ \Omega_i(\eta;x;f;C) = O((\frac{\eta^2}{1-x^2})^{\alpha}),\ \eta \to 0+\}$$

It was indeed shown that

$$Lip_1^L(\alpha;C) = \begin{cases} P\text{-}Lip_1(\alpha;C) & (0 < \alpha < 1/2) \\ P\text{-}Lip_2(\alpha;C) & (0 < \alpha < 1). \end{cases} \tag{7.2}$$

Using these results we have

THEOREM 9: *The following assertions are equivalent for* $f \in X$ *and* $0 < \alpha < 1$:

(i) $f \in Lip_1^L(\alpha;C)$,

(ii) $f \in P\text{-}Lip_2(\alpha;C)$,

(iii) $E_n(f;C) = O(n^{-2\alpha})$ $(n \to \infty)$,

(iv) $\|f * K_n - f\|_C = O(n^{-2\alpha})$ $(n \to \infty)$,

(v) $\|f * \Lambda_{2n} - f\|_C = O(n^{-\alpha/2})$ $(n \to \infty)$,

(vi) $\|f * w_t - f\|_C = O(t^{\alpha})$ $(t \to 0+)$,

(vii) $\|A_h f - f\|_C = O((1-h)^{\alpha})$ $(h \to 1-)$,

(viii) $\|f \star v_{[\![\beta n]\!],n} - f\|_C = O(n^{-2\alpha})$ $(n \to \infty;\ 0 < \beta < 1)$,

(ix) $\|f \star R_n - f\|_C = O(n^{-2\alpha})$ $(n \to \infty)$,

provided $\{x_n\}$ *satisfies* $|x_n - \bar{x}_n| = O(n^{-5/2})$.

Furthermore, if $0 < \alpha < 1/2$, *the assertions* (i)-(ix) *are also equivalent to*

(x) $f \in \text{P-Lip}_1(\alpha;C)$,

(xi) $\|f \star F_n - f\|_C = (n^{-2\alpha})$ $(n \to \infty)$.

If one is only interested in direct approximation theorems, it is also possible to connect our results on rates of convergence with the classical moduli of continuity and Lipschitz classes, even in L^p-spaces. They are given by

$$\omega_1(\eta;f;X) = \sup_{0\le h\le\eta} \|\Delta_h^1 f\|_{X[-1,1-h]}$$

$$\omega_2(\eta;f;X) = \sup_{0\le h\le\eta} \|\Delta_h^2 f\|_{X[-1+h,1-h]} \tag{7.3}$$

$$\text{Lip}_i(\alpha;X) = \{f \in X;\ \omega_i(\eta;f;X) = O(\eta^\alpha), \eta \to 0+\} \quad (i = 1,2;\ \alpha > 0),$$

where $\|\cdot\|_{X[a,b]}$ indicates the X-norm taken over the interval $[a,b]$. By a well known result on best approximation (cf. [24, p. 145]), namely

$$f^{(r)} \in \text{Lip}_i(\alpha;X) \Rightarrow E_n(f;X) = O(n^{-r-\alpha}) \qquad (n\to\infty; r\in\mathbb{P}, \alpha > 0), \tag{7.4}$$

we find in view of Prop. 6b)

$$f \in \text{Lip}_1(\alpha;X) \Rightarrow f \in \text{Lip}_1^L(\alpha/2;X) \qquad (0 < \alpha \le 1) \tag{7.5}$$

$$f \in \text{Lip}_2(\alpha;X) \Rightarrow f \in \text{Lip}_1^L(\alpha/2;X) \qquad (0 < \alpha < 2). \tag{7.6}$$

The latter three implications will now be used to derive the direct results. In the case of the integrals of de La Vallée Poussin and Fejér-Korovkin one obtains by (5.8), and Thm. 2b) together with the inequality

$$\omega_2(\eta;f;X) \leq M\eta\omega_1(\eta;f';X) \qquad (f' \in X;\ \eta > 0), \tag{7.7}$$

respectively, the following

COROLLARY 4: a) *For the singular integral* $f * v_{[\beta n],n}$, $f \in X$, $0 < \beta < 1$, *one has for any* $r \in \mathbb{P}$

$$f^{(r)} \in \mathrm{Lip}_1(\alpha;X) \Rightarrow \|f * v_{[\beta n],n} - f\|_X = O(n^{-r-\alpha}) \quad (n \to \infty;\ 0 < \alpha \leq 1)$$

$$f^{(r)} \in \mathrm{Lip}_2(\alpha;X) \Rightarrow \|f * v_{[\beta n],n} - f\|_X = O(n^{-r-\alpha}) \quad (n \to \infty;\ 0 < \alpha \leq 2).$$

b) *For the singular integral* $f * K_n$, $f \in X$, *one has*

$$f \in \mathrm{Lip}_1(\alpha;X) \Rightarrow \|f * K_n - f\|_X = O(n^{-\alpha}) \quad (n \to \infty;\ 0 < \alpha \leq 1)$$

$$f \in \mathrm{Lip}_2(\alpha;X) \Rightarrow \|f * K_n - f\|_X = O(n^{-\alpha}) \quad (n \to \infty;\ 0 < \alpha < 2)$$

$$f' \in \mathrm{Lip}_1(\alpha;X) \Rightarrow \|f * K_n - f\|_X = O(n^{-1-\alpha}) \quad (n \to \infty;\ 0 < \alpha < 1).$$

To extend the last implication to $\alpha = 1$ we have to restrict the assertion to the case $X = C[-1,1]$.

COROLLARY 5: *For* $f' \in \mathrm{Lip}_1(1;C)$ *there holds*

$$\|f * K_n - f\|_C = O(n^{-2}) \qquad (n \to \infty).$$

PROOF: We need the following implications

$$(7.8)\quad f' \in \mathrm{Lip}_1(\alpha;C) \Rightarrow f \in \mathrm{Lip}_2(1+\alpha;C) \Rightarrow f \in \text{P-Lip}_2((1+\alpha)/2;C)\ (0<\alpha\leq 1)$$

$$(7.9)\quad f \in \text{P-Lip}_2(1;C) \Rightarrow \omega_1^L(\delta;f;C) = \omega_1(1-\delta;f;C) + O(1-\delta) \quad (\delta \to 1-).$$

(7.8) follows by (7.7) and the definition of Lip_2, P-Lip_2; (7.9) can be found in [55, Sec. 6]. If $f' \in \mathrm{Lip}_1(1;C)$, then one has by (7.8) and (7.9) that $\omega_1^L(\delta;f;C) = O(1-\delta)$, $\delta \to 1-$, yielding the assertion by Thm. 2b). □

Cor. 4b) and Cor. 5 solve the problem posed in the introduction as well as its extension not only in $C[-1,1]$ - space but also for $L^p(-1,1)$, $1 \leq p < \infty$, apart form the case $f' \in \mathrm{Lip}_1(1;L^p)$. The algebraic de La Vallée Poussin sums have a much better approximation behaviour. According to Cor 4a) they actually approximate any given $f \in X$ with the same order as do the algebraic polynomials of best approximation.

Let us recall that the integrals considered in this paper are convolution integrals of the form (2.1) (the convolution being understood in the Legendre sense) and not of the form (1.4). These integrals may, however, readily be rewritten in the form

$$(7.10)\qquad \frac{1}{2}\int_{-1}^{1} f(u)\,\chi_\rho(x;u)\,du \qquad (\rho \in A)$$

with $\int_{-1}^{1} \chi_\rho(x;u)\,du = 2, x \in [-1,1]$. For example, the Rogosinski integral can be written as (7.10) with

$$\chi_n(x;u) = (\tau_u R_n(\cdot;x_n))(x)$$

$$\equiv \sum_{k=0}^{n} (2k+1) P_k(x_n) P_k(x) P_k(u) \qquad (x,u \in [-1,1];\ n \in \mathbb{P}).$$

Another example of a singular integral written in the form (7.10) was the integral $J_{2n}f$ of (1.2). Lupaş' result (1.3) for this integral can also be derived from Cor. 1. Indeed, since

$$\left[\frac{3}{n^2+3n+3}\,(P_n^{(2,0)})^2\right]\hat{}\,(1) = \frac{n^2+3n}{n^2+3n+3} \qquad (n \in \mathbb{P}),$$

one has that for each $f \in X$

$$(7.11) \qquad \|J_{2n}f - f\|_X \le 24\,\omega_1^L\left(1 - \frac{3}{n^2+3n+3}; f; X\right) \qquad (n \in \mathbb{P}).$$

If $X = C[-1, 1]$, an easy calculation (cf. [55, Sec. 6]) shows that

$$\omega_1^L(\delta; f; C) \le M\,\omega_1(\sqrt{1-\delta}; f; C) \qquad (f \in C[-1,1];\ -1 < \delta < 1)$$

which yields

$$\|J_{2n}f - f\|_C \le M\,\omega_1(1/n; f; C) \qquad (f \in C[-1,1];\ n \in \mathbb{N}).$$

This is (1.3) apart from a constant. Using our methods it can even be shown that $J_{2n}f$ approximates f in $C[-1, 1]$-space with order $O(n^{-1-\alpha})$ provided $f' \in \mathrm{Lip}_1(\alpha; C)$, $0 < \alpha \le 1$. Moreover, $J_{2n}f$ is saturated in X with order $O(n^{-2})$. A result of the type of Thm. 2b) also holds for $J_{2n}f$.

Let us finally conclude with an unsolved problem. Does there exist a triangular matrix of distinct nodes $\{x_{k,n}\}_{k=0}^n$, $n \in \mathbb{P}$, $-1 \le x_{k,n} \le +1$ and a triangular matrix of fundamental functions $\{\varphi_{k,n}(x)\}_{k=0}^n$ $n \in \mathbb{P}$, defined on $[-1, +1]$, such that the linear summator operators $(L_n f)(x) = \Sigma_{k=0}^n f(x_{k,n})\varphi_{k,n}(x)$, $f \in C[-1, 1]$, are positive algebraic polynomials of degree n and

$$\|L_n f - f\|_{C[-1,1]} = O(n^{-\alpha})$$

provided $f \in \mathrm{Lip}_2(\alpha; C[-1, 1])$, $0 < \alpha \leq 2$?

The point is that the $\varphi_{k,n}$ should be defined as constructively as the corresponding functions in the case of the Bernstein polynomials defined over $[-1, 1]$, namely

$$\varphi_{k,n}(x) = 2^{-n}\binom{n}{k}(1 + x)^k(1 - x)^{n-k},$$

and these are known to approximate only with order $O(n^{-\alpha/2})$ under the same hypothesis.

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NON-ARCHIMEDEAN WEIGHTED APPROXIMATION

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1. INTRODUCTION

Let F be a non-archimedean, non-trivially valued field, which is locally compact for its natural topology, X a zero-dimensional Hausdorff space, E a non-archimedean locally convex Hausdorff space over F, and Γ the directed family of all non-archimedean continuous seminorms on E. (For the definition of these terms, see [1], [2] and [3].)

$C(X,E)$ denotes the vector space, over F, of all continuous functions from X to E, and L is a fixed vector subspace of $C(X,E)$. An L-weight is a function w from X to F such that $p \circ (wf)$ is uppersemicontinuous and null at infinity, for every $p \in \Gamma$ and $f \in L$. If W is a set of L-weights, then the non-archimedean seminorms $f \in L \to \sup_{x \in X} [w(x)f(x)]$, for $p \in \Gamma$, $w \in W$, endow L with a weighted topology τ_W, under which L is a non-archimedean locally convex space. In this case, (L, τ_W) is called a *non-archimedean Nachbin space*.

If S is a collection of functions from X to E, we set $S(x) = \{s(x) \in E;\ s \in S\}$, for each $x \in X$, and we say that S is *non-vanishing* on X if for every $x \in X$ there exists $s \in S$ such that $s(x) \neq 0$. W is called *directed* if, given $w_1, w_2 \in W$, there exists $\lambda > 0$ and $w \in W$ such that $|w_i| \leq \lambda |w|$, for $i = 1,2$. The following

facts are clear:

(i) If W is non-vanishing on X, then (L,τ_W) is Hausdorff.

(ii) If W is directed, then the sets $\{f \in L; \sup_{x\in X} p[w(x)f(x)] < \varepsilon\}$, for $p \in \Gamma$, $w \in W$, $\varepsilon > 0$, form a basis of 0-neighborhoods for (L,τ_W).

(iii) If W is non-vanishing at X, then, for each $x \in X$, $\delta_x : L \to E$, defined by $\delta_x(f) = f(x)$, is continuous and linear.

2. EXAMPLES

(i) If $L = C(X,E)$ and W is the set of F-characteristic functions of compact subsets of X then τ_W is the compact-open topology in $C(X,E)$.

(ii) If $L = C_o(X,E)$, the vector space of all continuous functions from X to E which vanish at infinity, and $W = \{1\}$, then τ_W is the uniform topology in $C_o(E,X)$.
In particular, if X is compact, we get $C_o(X,E) = C(X,E)$.

(iii) If X is locally compact, $L = C_b(X,E)$, the vector space of all bounded continuous functions from X to E, and $W = C_o(X,F)$, then τ_W is called the strict topology in $C_b(X,E)$.

We notice that in all these cases W is non-vanishing at X, and that, in the first two cases, W is directed.

3. THE BOUNDED CASE OF THE NON-ARCHIMEDEAN BERNSTEIN-NACHBIN PROBLEM

Given a subalgebra A of $C(X;F)$ and a Nachbin space (L,τ_W), the non-archimedean Bernstein-Nachbin problem consists in describing the closure of a vector subspace $M \subset L$ which is a module over A, that is, such that $AM \subset M$.

For $x, y \in X$, we say that $x \equiv y \pmod{A}$ if and only if $\varphi(x) = \varphi(y)$, for every $\varphi \in A$. We denote by X/A the quotient of X

by this equivalence relation. Each $Y \in X/A$ being a closed subset of X, one easily sees that $W|_Y$ is a set of $L|_Y$-weights, and then $(L|_Y, \tau_{W|_Y})$ is a non-archimedean Nachbin space. M is then said to be *localizable under* A in (L,τ_W) if the τ_W-closure of M in L coincides with (or, equivalently, contains) the set $\mathcal{L}_A(M) = \{f \in L$ such that $f|_Y$ belongs to the $\tau_{W|_Y}$ closure of $M|_Y$ in $L|_Y\}$.

The *restricted* non-archimedean Bernstein-Nachbin problem consists in asking for conditions under which localizability holds.

The *bounded case* of the non-archimedean Bernstein-Nachbin problem occurs when the functions in A are bounded on the support of every function in W.

We have then the following:

THEOREM 1: *In the bounded case, every* A*-module contained in* L *is localizable under* A *in* (L,τ_W).

PROOF: The proof is an adaptation of the one given by Nachbin ([4]) in the real case, and is based on the following

LEMMA: *Let* A *be a unitary closed subalgebra of* $C_b(X,F)$ *equipped with the supremum norm. If, for every* $Y \in X/A$, K_Y *is a compact subset of* X *which is disjoint from* Y, *then there exist* $Z_1,\ldots,Z_n \in X/A$ *and* $\varphi_1,\ldots,\varphi_n \in A$ *such that:*

(i) $\varphi_i|_{K_{Z_i}} = 0 \quad i = 1,\ldots,n$;

(ii) $\|\varphi_i\|_\infty = \sup_{x\in X} |\varphi_i(x)| \leq 1, \quad i * 1,\ldots,n$;

(iii) $\sum_{i=1}^{n} \varphi_i(x) = 1, \quad x \in X.$

PROOF OF THE LEMMA: Let βX be the Banaschewski compactification of X and $\beta : C_b(X,F) \to C(\beta X,F)$ the isometry such that $\beta(f) = \beta f$ is the

only continuous extension of f to βX (see [5]). Then βA, the image of A under β, is a unitary closed subalgebra of $C(\beta X,F)$. If G is the quotient space $\beta X/\beta A$ and $\pi : \beta X \to G$ the canonical projection, then G is compact, Hausdorff, and zero-dimensional (see [6]). Now, for $Y \in G$, we have that $Y \cap X \neq \phi$ if and only if $Y \in \pi(X)$ and, in this case, $Y \cap X \in X/A$. Then $K_{Y \cap X}$ is a compact subset of X, which is disjoint from $Y \cap X$. This implies that $\bigcap_{Y \in \pi(X)} \pi(K_{Y \cap X}) = \phi$. By the finite intersection property, there exist $Y_1,\ldots,Y_n \in \pi(X)$ such that $\bigcap_{i=1}^{n} \pi(K_{Y_i \cap X}) = \phi$. Putting $Z_i = Y_i \cap X$ and $V_i = G \setminus \pi(K_{Z_i})$, for $i = 1,\ldots,n$, then $(V_i)_{1 \leq i \leq n}$ is an open covering of G. By [6], Lemma 2.1, we can find $h_1,\ldots,h_n \in C(G,F)$ such that:

(i) $h_i|_{\pi(K_{Z_i})} = 0 \quad i = 1,\ldots,n$;

(ii) $\|h_i\|_\infty \leq 1 \quad i = 1,\ldots,n$

(iii) $\sum_{i=1}^{n} h_i(t) = 1 \quad t \in G$.

Letting $\psi_i = h_i \circ \pi \in C(\beta X,F)$, then each ψ_i is constant in each element of $\beta X/\beta A$, so that, by the non-archimedean Stone-Weierstrass Theorem for compact Hausdorff zero-dimensional spaces (see [6]), each ψ_i belongs to βA. Putting $\varphi_i = \psi_i|_X$, we get the desired result.

Now we go back to the proof of the Theorem. If B is the subalgebra of $C(X,F)$ generated by A and the constant function 1, then it is clear that M is an A-module if and only if it is a B-module, and that M is localizable under A in L if and only if it is localizable under B in L. Thus, we can assume A to be unitary.

Suppose first that $A \subset C_b(X,F)$ and let $\bar{A}$ be the closure of A in $C_b(X,F)$. Clearly, $X/A = X/\bar{A}$. Let then $f \in \mathcal{L}_A(M)$, $\varepsilon > 0$, $w_1,\ldots,w_m \in W$ and $p = \| \ \| \in \Gamma$ be given. For each $Y \in X/A$, there exists $g_Y \in M$ such that, for $x \in Y$ and $j = 1,\ldots,m$, $|w_j(x)|\|f(x) - g_Y(x)\| < \varepsilon$.

The set $K_Y = \bigcup_{j=1}^{m} \{x \in X;\ |w_j(x)|\,\|f(x) - g_Y(x)\| \geq \varepsilon\}$ is compact and disjoint from Y. By the Lemma, we can find $Z_1,\ldots,Z_n \in X/A$

and $\varphi_1, \ldots, \varphi_n \in \bar{A}$ such that $\varphi_i|_{K_{Z_i}} = 0$, $\|\varphi_i\|_\infty \leq 1$. Letting $g_i = g_{Z_i}$, we get, for $x \in X$, $i = 1,\ldots,n$, $j = 1,\ldots,m$:

$$|w_j(x)|\ |\varphi_i(x)|\ \|g_i(x) - f(x)\| \leq \varepsilon|\varphi_i(x)|,$$

as one can see by distinguishing the cases $x \in K_{Z_i}$ and $x \notin K_{Z_i}$. It follows that, for $x \in X$, $j = 1,\ldots,m$,

$$|w_j(x)|\ \Big\|\sum_{i=1}^n \varphi_i(x)g_i(x) - f(x)\Big\| = |w_j(x)|\ \Big\|\sum_{i=1}^n \varphi_i(x)[g_i(x) - f(x)]\Big\| \leq$$

$$\leq |w_j(x)|\ \max_i\ |\varphi_i(x)|\ \|g_i(x) - f(x)\| \leq \varepsilon\ \max_i\ |\varphi_i(x)| \leq \varepsilon\ .$$

Since each $\varphi_i \in \bar{A}$, there exist $a_i \in A$, for each i, such that $\|a_i - \varphi_i\|_\infty < \frac{\varepsilon}{k}$, where $k = 1 + \max\limits_{i,j} \sup\limits_{x\in X} |w_j(x)|\ \|g_i(x)\|$. Then

$$|w_j(x)|\ \Big\|\sum_{i=1}^n [a_i(x) - \varphi_i(x)]g_i(x)\Big\| \leq |w_j(x)|\max_i|a_i(x) - \varphi_i(x)\|g_i(x)\|$$

$$< \frac{\varepsilon}{k}\ \max_{i,j}\ |w_j(x)|\ \|g_i(x)\| < \varepsilon\ ,$$

so that

$$|w_j(x)|\ \Big\|\sum_{i=1}^n a_i(x)g_i(x) - f(x)\Big\| =$$

$$= |w_j(x)|\ \Big\|\sum_{i=1}^n [a_i(x) - \varphi_i(x)]g_i(x) + \sum_{i=1}^n \varphi_i(x)g_i(x) - f(x)\Big\| < \varepsilon.$$

As $a_ig_i \in AM \subset M$, this proves that f belongs to the closure of M in L.

Finally, in the general case when $A \subset C(X,F)$ and for each $\varphi \in A$, $w \in W$, φ is bounded on the support of w, given $f \in \mathcal{L}_A(M)$, $\varepsilon > 0$, $w_1,\ldots,w_m \in W$, $p \in \Gamma$, we take $Y = \bigcup\limits_{j=1}^m \operatorname{supp} w_j$, which is closed in X, and get $A|_Y \subset C_b(Y,F)$. Replacing X, W, L, A, M, τ_W by

Y, $W|_Y$, $L|_Y$, $A|_Y$, $M|_Y$, $\tau_{W|_Y}$ respectively, we apply the previous result, obtaining that $\mathcal{L}_{A|_Y}(M|_Y)$ is contained in the closure of $M|_Y$ in $L|_Y$. Since it is easily seen that $f|_Y \in \mathcal{L}_{A|_Y}(M|_Y)$, we can find $g|_Y \in M|_Y$ such that $p[w_j(x)(f(x) - g(x))] < \varepsilon$, for $x \in Y$, $j=1,\ldots,m$. But since $w_j(x) = 0$ if $x \in X \setminus Y$, this holds for every $x \in X$, which proves that f belongs to the τ_W-closure of M in L. So M is localizable under A in (L,τ_W).

COROLLARY: *If A is separating and we are in the bounded case, then:*

(i) *Given $f \in L$, then f belongs to the τ_W-closure of M in L if $f(x)$ belongs to the closure of $M(x)$ in E, for every $x \in X$.*

(ii) *M is dense in L, if $M(x)$ is dense in $L(x)$, for every $x \in X$.*

PROOF: (i) Since A is separating, the elements of X/A are singletons. Given then $\varepsilon > 0$, $p \in \Gamma$, $w_1,\ldots,w_m \in W$, $x \in X$, there exists, by hypothesis, $g \in M$ such that $p[f(x) - g(x)] < \frac{\varepsilon}{k}$, where $k = 1 + \max_j |w_j(x)|$. Then $p[w_j(x)(f(x) - g(x))] < \varepsilon$, for $j=1,\ldots,m$, which proves that $f \in \mathcal{L}_A(M)$. The result follows by Theorem 1.

(ii) Follows from (i).

REMARK: It W is non-vanishing on X, then the converses of (i) and (ii) in the Corollary hold, even without the hypotheses on A.

4. THE SCALAR CASE

In the scalar case, $E = F$ is (non-archimedean) normed by the absolute value. If the subspace L of $C(X,F)$ is fixed, then every subalgebra A of $C(X,F)$ contained in L is an A-module. We have then, as particular case of Theorem 1:

THEOREM 2: *In the bounded case, every subalgebra* A *of* C(X,F) *contained in* L *is localizable under itself in* (L,τ_W).

THEOREM 3 (Stone-Weierstrass): *Let* A *be a subalgebra of* C(X,F) *contained in the subspace* L *of* C(X,F), *and assume the bounded case. Then, in order that a given* $f \in L$ *belongs to the* τ_W*-closure of* A *in* L, *it is sufficient that both*

(i) $g(x) = 0$, *for every* $g \in A$, *implies* $f(x) = 0$, *and*

(ii) $g(x) = g(y)$, *for every* $g \in A$, *implies* $f(x) = f(y)$.

If W *is non-vanishing on* X, *then these conditions are also necessary.*

PROOF: For the sufficiency, it is enough, by Theorem 2, to show that $f \in \mathcal{L}_A(A)$. Given $Y \in X/A$, (ii) implies that f is constant in Y. Let λ be this constant value. Fixed $x_o \in Y$, if $\lambda \neq 0$, (i) implies that there exists $g \in A$, such that $g(x_o) \neq 0$. Since g is also constant in Y, then $h = \frac{\lambda}{g(x_o)}\, g \in A$ and $h|_Y = f|_Y$. (If $\lambda = 0$, $h = 0$ will do). Then, given $\varepsilon > 0$. $w_1,\ldots,w_m \in W$, $|w_j(x)|\,|h(x) - f(x)| = 0 < \varepsilon$, for $x \in Y$, $j = 1,\ldots,m$. So that $f \in \mathcal{L}_A(A)$.

As to the necessity, assume that W is non-vanishing on X, and that f belongs to $\bar{A}$, the τ_W-closure of A in L. Then if $x \in X$ is such that $g(x) = 0$, for every $g \in A$, take δ_x as in § 1, and we get:

$$f(x) = \delta_x(f) \in \delta_x(\bar{A}) \subset \overline{\delta_x(A)}^E = \overline{\{0\}}^E = \{0\},$$

since E is Hausdorff, so that $f(x) = 0$. Similarly, if $g(x) = g(y)$, for every $g \in A$, the same reasoning with $\delta_x - \delta_y$ shows that $f(x) = f(y)$.

COROLLARY: (a) *If* A *is a subalgebra of* C(X,F) *and* $f \in C(X,F)$,

then f *belongs to the compact-open closure of* A *if and only if*

(i) $g(x) = 0$, *for every* $g \in A$, *implies* $f(x) = 0$

(ii) $g(x) = g(y)$, *for every* $g \in A$, *implies* $f(x) = f(y)$.

(b) *If* A *is a subalgebra of* $C_o(X,F)$ *and* $f \in C_o(X,F)$, *then* f *belongs to the uniform closure of* A *if and only if*

(i) $g(x) = 0$, *for every* $g \in A$, *implies* $f(x) = 0$

(ii) $g(x) = g(y)$, *for every* $g \in A$, *implies* $f(x) = f(y)$.

(c) *If* X *is locally compact,* A *is a subalgebra of* $C_b(X,F)$ *and* $f \in C_b(X,F)$, *then* f *belongs to the strict closure of* A *if and only if*

(i) $g(x) = 0$, *for every* $g \in A$, *implies* $f(x) = 0$.

(ii) $g(x) = g(y)$, *for every* $g \in A$, *implies* $f(x) = f(y)$.

5. DENSITY IN TENSOR PRODUCTS

If S and T are, respectively, vector subspaces of C(X,F) and E, then $S \otimes T$ denotes the set of all finite sums of functions of the form $x \to s(x)t$, with $s \in S$, $t \in T$. Similarly, if X_1 and X_2 are zero-dimensional Hausdorff spaces, and S_1 and S_2 are, respectively, vector subspaces of $C(X_1,F)$ and $C(X_2,F)$, then $S_1 \otimes S_2$ denotes the set of all finite sums of the functions of the form $(x_1,x_2) \to s_1(x_1)s_2(x_2)$, with $s_1 \in S_1$, $s_2 \in S_2$.

THEOREM 4: *If* A *is separating and non-vanishing on* X, $A \otimes E \subset L$, *and if we are in the bounded case, then* $A \otimes E$ *is* τ_W*-dense in* L.

PROOF: $A \otimes E$ is an A-module, and $(A \otimes E)(x) = E$, for every $x \in X$, since A is non-vanishing at X. It suffices then to apply Theorem 1, Corollary.

COROLLARY 1:

(i) $C(X,F) \otimes E$ *is dense in* $C(X,F)$, *for the compact - open topology.*

(ii) *If* X *is locally compact,* $K(X,F) \otimes E$ *is dense in* $C_o(X,E)$, *for the uniform topology.* ($K(X,F)$ *is the set of all continuous scalar functions with compact support*).

(iii) *If* X *is locally compact,* $C_b(X,F) \otimes E$ *is dense in* $C_b(X,E)$, *for the strict topology.*

COROLLARY 2 (Dieudonnē):

(i) $(C(X_1,F) \otimes C(X_2,F)) \otimes E$ *is dense in* $C(X_1 \times X_2,E)$, *for the compact-open topology.*

(ii) $C(X_1,F) \otimes C(X_2,F)$ *is dense in* $C(X_1 \times X_2) \otimes F$.

6. EXTENSION THEOREMS

THEOREM 5: *If* E *is a non-archimedean Frēchet space over* F, *and* Y *is a non-empty compact subset of the zero-dimensional Hausdorff space* X, *then every* E*-valued continuous function on* Y *can be extended to a bounded continuous function on* X.

PROOF: We will employ a technique due to De La Fuente [7]. The linear mapping $T_Y : C(X,E) \to C(Y,E)$, defined by $T_Y(f) = f|_Y$, is clearly continuous for the compact-open topologies in both spaces. For $S \subset C(X,E)$, denote $T_Y(S)$ by $S|_Y$. Then $A = C_b(X,F)|_Y$ is a unitary subalgebra of $C(Y,F)$, and $M = C_b(X,E)|_Y$ is an A - module. Since the constant functions belong to M, $M(x) = E$, for each $x \in Y$. By Theorem 1, Corollary, $C_b(X,E)|_Y$ is dense in $C(Y,E)$.

Assume first that X is compact. Then $C(X,E)$ is a Frēchet space, and so is its quotient by the closed subspace $K = T_Y^{-1}(0)$. Now

we claim that $C(X,E)/K$ is linearly and topologically isomorphic to $C(X,E)|_Y$, for which it is enough to prove that T_Y is a topological homomorphism. Indeed, given U, a basic neighborhood of 0 in $C(X,E)$, then $U = \{g \in C(X,E);\ p[g(x)] < \varepsilon;\ x \in X\}$ for some $p \in \Gamma$, $\varepsilon > 0$. Then $V = \{h \in C(Y,E);\ p[h(x)] < \varepsilon;\ x \in Y\}$ is an open neighborhood of 0 in $C(Y,E)$. Since it is evident that $T_Y(U) \subset V \cap [C(X,E)|_Y]$, it is enough to prove the reverse inclusion. Let then $h = g|_Y \in V$, with $g \in C(X,E)$. Then $G = \{t \in X;\ p[g(t)] \geq \varepsilon\}$ is compact and disjoint from Y. By ultra-normality of X, there exists $\varphi \in C(X,F)$ such that φ is 0 on G, 1 on Y, and $|\varphi| \leq 1$ on X. Then $f = \varphi g \in C(X,E)$ is such that $f \in U$ and $T_Y(f) = h$, which proves that $h \in T_Y(U)$. Therefore, $C(X,E)|_Y = C_b(X,E)|_Y$ is complete, and thus closed in $C(Y,E)$. Since it is also dense, we get $C_b(X,E)|_Y = C(Y,E)$.

Now, in the general case, take $\beta_F X$ the Banaschewski compactification of X. By the previous result, $C(Y,E) = C(\beta_F X,E)|_Y$. Then, given $f \in C(Y,E)$, there exists $h \in C(\beta_F X,E)$ such that $f = h|_Y$. The function $g = h|_X$ is the required extension.

THEOREM 6: *If E is a non-archimedean Fréchet space over F, and Y is a closed subset of the zero-dimensional locally compact Hausdorff space X, then every function in $C_o(Y,E)$ can be extended to a function in $C_o(X,E)$.*

PROOF: We omit the proof, which is similar to that of Theorem 5.

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THÉORIE SPECTRALE EN UNE INFINITÉ DE VARIABLES

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1. L'utilité d'une théorie spectrale et d'un calcul fonctionnel holomorphe en une infinité de variables a été mise en lumière par la recherche de conditions d'unicité pour le calcul fonctionnel holomorphe d'un nombre fini de variables et des algèbres à spectres non compacts (cf [2]). Disons, de façon schématique, que l'unicité est établie pour un domaine spectral pseudoconvexe et en particulier polynomialement convexe et que, d'autre part, tout domaine de $\mathbb{C}^n$ peut s'interpréter comme la projection d'un domaine polynomialement convexe, mais d'un nombre infini de variables.

De façon classique, étant donnée une algèbre A, commutative et à élément unité (toutes les algèbres seront supposées désormais telles), on se donne des éléments $a_1,\ldots,a_n$ de A et on définit le spectre $\sigma(a)$ de $a = (a_1,\ldots,a_n)$ comme l'ensemble des points $s = (s_1,\ldots,s_n)$ de $\mathbb{C}^n$ tels que l'idéal engendré par $a_1 - s_1,\ldots,a_n - s_n$ soit propre, plus précisément comme le filtre des complémentaires des parties S, dites spectrales, sur lesquelles on peut trouver des fonctions $s \to u_i(s)$ bornées vérifiant $\Sigma(a_i - s_i)u_i(s) = 1$.

2. Pour décrire une situation semblable en dimension infinie, il est naturel de remplacer $\mathbb{C}^n$ par un espace localement convexe E et la donnée de $a_1,\ldots,a_n$ par celle d'une application linéaire bornée a

du dual E' de E dans A.

La notion de spectre correspond alors à ce qui suit: pour un système fini $\varphi = (\varphi_1,\dots,\varphi_n)$ d'éléments de E', c'est à dire une application linéaire continue φ de E dans $\mathbb{C}^n$, on peut considérer $\mathbf{a}_\varphi = (\mathbf{a}(\varphi_1),\dots,\mathbf{a}(\varphi_n))$ et son spectre $\sigma(\mathbf{a}_\varphi)$, plus précisément un ensemble S_φ spectral pour $\mathbf{a}_\varphi$. On s'intéresse à des familles (S_φ) vérifiant des conditions d'uniformité: n est fixe (ou majoré), et les fonctions u_i exprimant que $S_\varphi \in \sigma(\mathbf{a}_\varphi)$ sont bornées indépendamment de φ. Ainsi les ensembles spectraux sont-ils remplacés par des familles d'ouverts $\Omega_\varphi = \bar{\varphi}^{-1}(S_\varphi)$ qui s'ordonnent en un système projectif.

Avec les notations qui précèdent le calcul fonctionnel classique est un morphisme $f \to f[a]$ de l'algèbre $\mathcal{O}(\delta_S)$ des fonctions holomorphes à croissance polynomiale sur le domaine spectral S dans A, c'est à dire telles que $f\delta_S^N$ soit bornée pour un certain entier N, où δ_S est la distance dans $\mathbb{C}^N$ au complémentaire de S.

L'algèbre $\mathcal{A}$ qui intervient en dimension infinie a des éléments de la forme

$$f = \Sigma\, \lambda_\varphi\, f_\varphi \tag{1}$$

correspondant à une famille spectrale (S_φ), où $\Sigma\,|\lambda_\varphi| < \infty$, où $f_\varphi \in \mathcal{O}(\delta_{S\varphi})$ et vérifie dans cette algèbre des majorations indépendantes de φ. Le calcul fonctionnel s'obtient en posant

$$f[\mathbf{a}] = \Sigma\, \lambda_\varphi\, f_\varphi[\mathbf{a}_\varphi]. \tag{2}$$

Plus précisément on le définit d'abord pour des sommes telles que (1) n'ayant qu'un nombre fini de termes, et on le prolonge par passage au complété, l'algèbre $\mathcal{A}$ ayant été définie elle-même de cette façon.

3. Un problème, clef pour l'unicité du calcul fonctionnel en un nombre fini de variables, se pose: peut-on considérer l'algèbre $\mathcal{A}$ comme une algèbre de fonctions sur un domaine Σ de E? De façon évidente si Σ est la partie de la limite projective des Ω_φ définie par

$$\inf_\varphi \delta_{S_\varphi}(\varphi(s)) > 0,$$

on a un morphisme de $\mathcal{A}$ sur une algèbre $\mathcal{A}_\Sigma$ de fonctions sur Σ, dont l'injectivité n'est malheureusement pas claire. S'il n'y a pas de problème dans le cas d'un produit, la situation n'est pas élucidée dans le cas d'un produit fibré sur un domaine de $\mathbb{C}^n$, sauf si ce dernier est pseudoconvexe (cf [1], [2]).

4. Délaissant ici le problème de savoir si les fonctions holomorphes du calcul fonctionnel sont des fonctions, concentrons-nous sur le spectre et cherchons si on peut remplacer dans certains cas le système projectif des Ω_φ par un domaine Ω de E. Pour cela il faut pouvoir connaître des familles (S_φ) à partir de la seule donnée de Ω.

Il est naturel de considérer, pour n donné, les familles équicontinues d'applications linéaires φ de E dans $\mathbb{C}^n$ et les familles des parties S_φ de $\mathbb{C}^n$ telles que S_φ contient l'image par φ de Ω, ce qui se traduit plus exactement par le fait que

$$\inf_\varphi (\delta_{S_\varphi} \circ \varphi)(s) > 0 \quad \text{pour tout} \quad s \in \Omega . \tag{3}$$

Il faut noter à ce sujet que la dernière condition rend en général impossible pour λ_{φ_0}, avec λ parcourant $]0, 1]$, le choix de λS_{φ_0}.

L'ouvert Ω sera spectral si pour tout choix (S_φ) conforme à ce qui précède on a $S_\varphi \in \sigma(a_\varphi)$ avec uniformité par rapport à φ.

Un cas particulièrement simple est celui d'une suite bornée (a_n) de $\mathbf{A}$ et d'une suite bornée (S_n) telle que $S_n \in \sigma(a_n)$ avec uniformité par rapport à n ; l'espace E est l'espace $\ell^\infty(\mathbb{C})$ et (a_n) s'identifie à une application $\mathbf{a}$ de $\ell^1(\mathbb{C})$ dans $\mathbf{A}$. Peut-on alors affirmer que la partie Σ du produit des S_n, définie par $\inf_n \delta_{S_n}(s) > 0$, est spectrale pour $\mathbf{a}$?

If faudrait pour cela que pour un élément φ de la sphère unité de E', c'est à dire une suite (λ_n) de $\ell^1(\mathbb{C})$ telle que $\Sigma|\lambda_n| = 1$ on ait $\varphi(\Omega) \in \sigma(\mathbf{a}_\varphi)$, c'est à dire

$$\sum_n \lambda_n S_n \in \sigma\left(\sum_n \lambda_n a_n\right), \tag{4}$$

et avec uniformité par rapport à (λ_n).

En effet, s'il existe $\varepsilon > 0$ tel que $\delta_{S_n}(z_n) \geq \varepsilon$, alors S_n contient la boule ouverte $B(z_n,\varepsilon)$ et $\varphi(\Omega) = \Sigma \lambda_n S_n$ contient la boule ouverte $B(\Sigma \lambda_n z_n,\varepsilon)$ de sorte que $\delta_{\varphi(\Omega)}(\Sigma \lambda_n z_n) \geq \varepsilon$.

5. On peut donc se poser de façon générale le problème suivant: étant donnée une suite bornée (a_n) de A^N et une suite (S_n) de $\mathbb{C}^N$ telle que $S_n \in \sigma(a_n)$ avec uniformité par rapport à n, est-ce que l'on a la relation (4) pour toute suite (λ_n) de $\ell^1(\mathbb{C})$ telle que $\Sigma|\lambda_n| = 1$, avec uniformité par rapport à (λ_n) ?

Considérons le cas particulier d'une algèbre de Banach. On vérifie tout d'abord, en prenant des caractères, l'inclusion suivante, dans laquelle S_n est remplacé par l'ensemble $sp(a_n)$ (qui est l'intersection du filtre $\sigma(a_n)$)

$$\sum_n \lambda_n \, sp(a_n) \supset sp\left(\sum_n \lambda_n a_n\right).$$

Cette même inclusion montre donc que pour tout choix de $S_n \in \sigma(a_n)$, on a la relation (4). Cependant il resterait à établir l'uniformité

par rapport au choix d'une suite (λ_n) de la sphère unite de $\ell^1(\mathbb{C})$.

Il n'y a pas de difficulté si on remplace la borne sur les coefficients u_i par le fait que S contienne un ε-voisinage du spectre avec $\varepsilon > 0$ fixe. En effet si A^ε désigne l'ensemble des points dont la distance à A est strictement inférieure à ε on a

$$\Sigma \lambda_n (sp(a_n))^\varepsilon = (\Sigma \lambda_n sp(a_n))^\varepsilon .$$

On est ainsi conduit à étudier la croissance des coefficients spectraux en fonction de la distance au spectre.

Dans un sens on a l'inégalité:

$$\sqrt{|u_1(s)|^2 + \ldots + |u_n(s)|^2} \geq d(s, sp(a))$$

qui s'établit facilement en prenant $t = \chi(a) \in sp(a)$ et en sachant que $|\chi(u_i)| \leq \|u_i\|$.

La question fondamentale concerne l'autre sens: peut-on pour tout $\varepsilon > 0$ trouver une borne des coefficients $u_i(s)$ avec $d(s,sp(a)) \geq \varepsilon$ qui soit indépendante de a, $\|a\| \leq 1$?

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MEROMORPHIC UNIFORM APPROXIMATION ON CLOSED SUBSETS OF OPEN RIEMANN SURFACES

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Dedicated in memory of Alice Roth

1. INTRODUCTION

Let F be a (relatively) closed subset of an open Riemann surface R. Denote by H(F) and M(F) respectively the holomorphic and meromorphic functions on (a neighbourhood of) F. Let A(F) denote the functions continuous on F and holomorphic on the interior F° of F. Recently, the problem of approximating functions in A(F) uniformly by functions in H(R) has been considered by Scheinberg [17]. In the present paper, we consider the problem of approximating a given function on F uniformly by functions in M(R) and obtain, as a corollary, a result related to Scheinberg's. Our method of approximation is based on the technique of the late Alice Roth [15].

We shall rely on Scheinberg [17] for some results on the topology of surfaces. Without loss of generality, we shall assume that every Riemann surface R is connected. A subset is *bounded* in R if its closure in R is compact. A Riemann surface R' is an *extension* of R if R is (conformally equivalent to) an open subset of R'. If

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furthermore $\bar{R} \neq R'$, R' is an *essential extension* of R. We shall say that a closed subset F of R is *essentially of finite genus* if F has a covering by a family of pairwise disjoint open sets, each of finite genus. Denote by $\bar{M}(F)$ the uniform limits on F of functions meromorphic on R with poles outside of F and by $\bar{H}(F)$ the uniform limits on F of functions holomorphic on R. R* will denote the one point compactification of R.

The central problem in the qualitative theory of approximation is that of approximating a given function on a given set. In this direction we state our principal theorem.

THEOREM 1: (*Localization*): *Let* F *be closed and essentially of finite genus in an open Riemann surface* R. *Then, a function* f *is in* $\bar{M}(F)$ *if and only if*

$$f \mid K \cap F \in \bar{M}(K \cap F) , \tag{1}$$

for every compact set K *in* R.

If we drop the condition that F be essentially of finite genus, then the theorem is no longer true [9]. However, we may drop the condition, for R planar, since it is trivially verified by all F. In this situation, Theorem 1 is due to Alice Roth [15].

An immediate consequence of Theorem 1 is the following Walsh-type theorem, which was first obtained for planar R by Nersesian [14].

THEOREM 2: *Let* F *be closed and essentially of finite genus in an open Riemann surface* R. *A sufficient condition for* $A(F) = \bar{M}(F)$ *is that*

$$A(F \cap \bar{V}) = \bar{M}(F \cap \bar{V}) ,$$

for every bounded open set V *in* R.

By the Bishop-Kodama Localization Theorem [12], we may replace the open sets V by parametric discs.

The following is a Runge-type theorem.

THEOREM 3: *Let* F *be closed and essentially of finite genus in an open Riemann surface* R. *Then* $H(F) \subset \bar{M}(F)$. *Moreover,* $H(F) \subset \bar{H}(F)$ *if and only if* $R^* \setminus F$ *is connected and locally connected.*

Recently, we proved Theorem 3 for more restricted pairs (F, R) [7].

From Theorem 2, we have a corollary on Walsh-type approximation by holomorphic functions.

THEOREM A: (*Scheinberg* [17]): *Let* F *be closed and essentially of finite genus in a open Riemann surface* R. *A sufficient condition for* $A(F) = \bar{H}(F)$ *is that* $R^* \setminus F$ *be connected and locally connected.*

Scheinberg actually obtained this result for somewhat more general pairs (F,R). For arbitrary pairs (F,R), the condition that $R^* \setminus F$ be connected and locally connected is also necessary but no longer sufficient [9]. In fact, Scheinberg has shown that there is no topological characterization of pairs (F,R) for which $A(F) = \bar{H}(F)$ [17].

PROOF OF THEOREM A: Since $R^* \setminus F$ is connected, it follows from the Bishop-Mergelyan Theorem [2] that F satisfies the hypotheses of Theorem 2, when the sets V are parametric discs. Thus, if $f \in A(F)$ and $\varepsilon > 0$, there is a $g_1 \in M(R)$ with

$$|f(z) - g_1(z)| < \varepsilon/2, \quad z \in F.$$

Now by Theorem 3, there is a $g \in H(R)$ such that

$$|g_1(z) - g(z)| < \varepsilon/2, \quad z \in F.$$

This completes the proof of the corollary.

A closed set F in R is called a set of Carleman approximation by meromorphic functions, if for each $f \in A(F)$ and each ε positive and continuous on F, there is a $g \in M(R)$ with

$$|f(z) - g(z)| < \varepsilon(z), \quad z \in F.$$

The next result characterizes such sets completely when $F^{\circ} = \emptyset$. This result is known for R planar [14].

THEOREM 4: *Let F be closed with empty interior in an open Riemann surface R. Then F is a set of Carleman approximation by meromorphic functions if and only if*

$$C(F \cap K) = \bar{M}(F \cap K),$$

for each compact set K.

2. FUSION LEMMA

Using Behnke-Stein techniques, Gunning and Narasimhan [11] have shown that every open Riemann surface R can be visualized in a very concrete way. Indeed, they showed that R can be spread (without ramification) above the finite plane $\mathbb{C}$. To be precise, they proved that R admits a locally injective holomorphic function ρ. Thus $R \xrightarrow{\rho} \mathbb{C}$ is the spread.

We wish to reconstruct the Cauchy kernel of Behnke-Stein on R, something resembling $(q - p)^{-1}$. Conceptually we prefer to think of p

and q as both lying on R, however, for proofs, it may be preferable to think of two copies R_p and R_q of R spread respectively above the z and ζ planes:

$$\rho \times \rho : R_p \times R_q \to \mathbb{C}_z \times \mathbb{C}_\zeta$$

$$(p , q) \longmapsto (z , \zeta).$$

We construct an open cover of $R \times R$. If $(p,q) \in R \times R$, let D_p and D_q be discs about p and q respectively which lie schlicht over $\mathbb{C}$. Set $U(p,q) = D_p \times D_q$. Consider the Cousin data which is $(\zeta - z)^{-1}$ on $U(p,q)$. Since $R \times R$ is Stein, the first Cousin problem can be solved. Hence there is a meromorphic function Φ on $R \times R$ whose singularities are on the diagonal. In the neighbourhood of a diagonal point, we have, in local coordinates (forever more given by $\rho \times \rho$), that

$$\Phi(\zeta , z) - \frac{1}{\zeta - z}$$

is holomorphic. $\Phi(\zeta , z)$ means $\Phi(p,q)$, where $\rho(p) = \zeta$ and $\rho(q) = z$. We shall persist in this abusive notation, since it is invariant under local change of charts within the atlas given by $\rho \times \rho$. We call the function Φ a Cauchy kernel on R since

$$\mathrm{Res}_z \Phi(\cdot , z) = 1 .$$

We shall now extend to surfaces the powerful Fusion Lemma of Alice Roth [15].

FUSION LEMMA: *Let K_1, K_2, and K be compact subsets of an open Riemann surface R, with K_1 and K_2 disjoint. There is a positive number*

a such that if m_1 *and* m_2 *are any two meromorphic functions on* R *satisfying, for some* $\varepsilon > 0$,

$$|m_1 - m_2|_K < \varepsilon , \tag{1}$$

then there is a function m, *meromorphic on* R *such that for* j = 1,2,

$$|m - m_j|_{K \cup K_j} < a\varepsilon . \tag{2}$$

PROOF: We may assume $K_2 \setminus K \neq \emptyset$. Thus, we can construct open neighbourhoods U_1 and U_2 of K_1 and K_2 respectively such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and $R \setminus U_2$ is precompact. Moreover, we may assume that the boundaries of U_1 and U_2 consist of finitely many disjoint smooth Jordan curves. Let E be the compliment of $U_1 \cup U_2$ in R. Let G be a precompact neighbourhood of $(R \setminus U_2) \cup K_2 \cup K$. then

$$I(z) = \int_E \int |\Phi(z , \zeta)| \, d\xi \, d\eta , \quad \zeta = \xi + i\eta , \tag{3}$$

is uniformly bounded for $z \in G$, where Φ is a Cauchy kernel for R.

We introduce now an auxiliary function $\emptyset \in C^1(R)$ with values in [0, 1] such that $\emptyset$ is 1 on U_1 and $\emptyset$ is 0 on U_2. Then, since (3) is uniformly bounded, there is a constant $a > 2$ such that

$$\int_E \int \left|\frac{\partial}{\partial \bar{\zeta}} \emptyset(\zeta)\right| |\Phi(z,\zeta)| \, d\xi \, d\eta < a - 2 , \tag{4}$$

for $z \in G$.

Returning now to our meromorphic m_1 and m_2, we put $q = m_1 - m_2$. By (1) we can find a precompact neighbourhood U of K such that $|q(z)| < \varepsilon$, $z \in \bar{U}$. We replace q by a function q_1 constructed as follows. First, set

(5) $$q_1 = q \quad \text{on} \quad \bar{U}_1 \cup \bar{U}_2 \cup \bar{U} .$$

Now set $q_1 = 0$ elsewhere. Thus,

(6) $$|q_1(z)| < \varepsilon , \quad z \in E.$$

Set

(7) $$g(z) = \frac{1}{\pi} \int_E \int q_1(\zeta)\ \Phi(z,\zeta)\ \frac{\partial \emptyset}{\partial \bar{\zeta}}\ d\xi\, d\eta .$$

From (6) and (4) we have

(8) $$|g(z)| < (a-2) \cdot \varepsilon , \quad z \in G .$$

Since g is a Cauchy integral, g is holomorphic outside of E. Consequently,

(9) $$f(z) = \emptyset(z) q_1(z) + g(z), \quad z \in R ,$$

is holomorphic in U_2 (for $q_1(z) = \infty$, set $\emptyset(z)q_1(z) = 0$). For $z \in U_1$,

$$f(z) = q_1(z) + g(z)$$

is meromorphic and has the same poles as q_1. To see that f is also holomorphic on U, we invoke the formula

$$\emptyset(z) = -\frac{1}{\pi} \int_E \int \frac{\partial}{\partial \bar{\zeta}}\ \Phi(z,\zeta)\, d\xi\, d\eta , \quad z \in R .$$

Hence

$$f(z) = \frac{1}{\pi} \int_E \int \frac{\partial \phi}{\partial \bar{\zeta}} [q_1(\zeta) - q_1(z)] \Phi(z,\zeta) d\xi \, d\eta \ , \quad z \in R \ ,$$

$$q_1(z) \neq \infty .$$

For $z \in U$, $q_1 = q$ and

$$[q_1(\zeta) - q_1(z)] \Phi(z,\zeta)$$

is holomorphic. Thus f is holomorphic in U, and hence f is meromorphic on $U_1 \cup U_2 \cup U$ with the same poles as q. By the Runge-Behnke-Stein Theorem [1] there is a meromorphic function m_3 on R for which

$$|m_3(z) - f(z)| < \varepsilon \ , \quad z \in K_1 \cup K_2 \cup K .$$

Finally we put $m = m_2 + m_3$, and we have the following estimates: on $K_1 \cup K$

$$|m - m_1| \leq |f - (m_1 - m_2)| + |m_3 - f| \leq$$

$$\leq |\phi - 1| \ |q| + |g| + |m_3 - f| <$$

$$< \varepsilon + (a - 2)\varepsilon + \varepsilon = a\varepsilon \ ;$$

on $K_2 \cup K$

$$|m - m_2| \leq |f| + |m_3 - f| \leq |\phi| \ |q| + |g| +$$

$$+ |m_3 - f| < \varepsilon + (a - 2)\varepsilon + \varepsilon = a\varepsilon .$$

This completes the proof of the fusion lemma.

In the fusion lemma, it is clear that if

$$A(K_j \cup K) = \bar{M}(K_j \cup K) , \quad j = 1,2,$$

we may take $m_j \in A(K_j \cup K)$, $j = 1,2$.

Another consequence of the fusion lemma is the following Bishop-Kodama Localization Lemma.

LOCALIZATION LEMMA (*Kodama* [12]): *Let* f *be given on a compact subset* K *of an open Riemann surface, and suppose that for each* $z \in K$ *there exists a closed parametric disc* D_z *with center* z *such that*

$$f \mid (K \cap D_z) \in \bar{M}(K \cap D_z) .$$

Then $f \in \bar{M}(K)$.

3. PROOFS OF THEOREMS

Consider first Theorem 1. The necessity is trivial. To prove the sufficiency, suppose first that R has an essential extension R' in which the R'-closure $\bar{F}$ of F is compact. We may assume that $\bar{R} \neq \bar{F}$ and that R' is open. Let $\{G_n\}$ be an exhaustion of R by domains with

$$\bar{G} \subset G_{n+1} \quad \text{and} \quad \cup G_n = R .$$

In the Fusion Lemma, let K_1, K, and K_2 be the sets $\bar{G}_n$, $F \cap \bar{G}_{n+1}$, $\bar{F} \setminus G_{n+1}$, and consider these as compact subsets of the Riemann surface R'. For each $n = 1,2,3,\ldots$, the Fusion Lemma gives us a positive number a_n, and we may assume that

$$1 < a_n < a_{n+1} .$$

If ε is a given positive number, we select the positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ so that

$$\varepsilon_{n+1} < \varepsilon_n \quad \text{and} \quad \sum_{n=1}^{\infty} \varepsilon_n < \frac{\varepsilon}{2} . \tag{10}$$

By the hypotheses there exist functions $q_n \in M(R)$ such that

$$|q_n(z) - f(z)| < \frac{\varepsilon_n}{2a_n} , \quad z \in F_n = F \cap \bar{G}_{n+1} , \tag{11}$$

and therefore

$$|q_{n+1}(z) - q_n(z)| < \frac{\varepsilon_n}{a_n} , \quad z \in F_n , \tag{12}$$

$$n = 1, 2, 3, \ldots .$$

By the Behnke-Stein Theorem [1], we may assume that $q_n \in M(R')$. Thus, by the Fusion Lemma, for each $n = 1, 2, 3, \ldots$, there exists a function $r_n \in M(R')$ such that, for $n = 1, 2, 3, \ldots$,

$$|r_n(z) - q_n(z)| < \varepsilon_n , \quad z \in \bar{G}_n \cup F_n , \tag{13}$$

$$|r_n(z) - q_{n+1}(z)| < \varepsilon_n , \quad z \in \bar{F} \setminus G_n . \tag{14}$$

The inequalities (13) yield

$$\sum_{n}^{\infty} |r_\nu(z) - q_\nu(z)| < \sum_{n}^{\infty} \varepsilon_\nu , \quad z \in \bar{G}_n .$$

Therefore

$$m(z) = q_1(z) + \sum_1^\infty (r_\nu(z) - q_\nu(z))$$

is meromorphic in

$$R = \bigcup_{n=1}^\infty G_n \ .$$

From (11), (13) and (10), there follows for $z \in F_1$,

$$|m(z) - f(z)| \leq |q_1(z) - f(z)| + \sum_1^\infty |r_\nu(z) - q_\nu(z)| <$$

$$< \frac{\varepsilon_1}{2a_1} + \sum_1^\infty \varepsilon_\nu < \varepsilon \ .$$

From (11), (13), (14) and (10), we also have

$$|m(z) - f(z)| \leq \sum_1^{n-1} |r_\nu(z) - q_{\nu+1}(z)| + |q_n(z) - f(z)| +$$

$$+ \sum_n^\infty |r_\nu(z) - q_\nu(z)| < \sum_1^{n-1} \varepsilon_\nu + \frac{\varepsilon_n}{2a_n} + \sum_n^\infty \varepsilon_\nu < \varepsilon \ ,$$

$$\text{for } z \in F_n \setminus F_{n-1} \ , \ n = 2,3,\ldots \ .$$

Thus f can be approximated uniformly on F by functions meromorphic on R, and the proof is complete for the special case that R has an essential extension in which F is bounded.

To prove Theorem 1 in general, we shall invoke the special case just proved to inductively construct a sequence of meromorphic functions which converges to an approximating function.

Suppose R is an arbitrary open Riemann surface and F is a closed subset of R for which F has a covering $\{ V_j \}$ by pairwise

disjoint open sets, each of finite genus. We may assume that each V_j meets F, from which it follows that the family $\{V_j\}$ is locally finite. For each j, we triangulate V_j and set

$$P_j = (\cup\{T : T \cap (F_j^{\circ})^{-} \neq \emptyset\})^{\circ},$$

where T represents an arbitrary 2-dimensional closed triangle of the triangulation, and $F_j = F \cap V_j$. We call $\{P_j\}$ a polygonal cover and P_j polygonal. It is clear that the segments which make up $\cup\, \partial P_j$ are locally finite. Repeating the same argument, we can find, for each j, a polygon Q_j with $P_j \subset Q_j \subset \bar{Q}_j \subset V_j$.

We may construct an exhaustion $\{G_j\}$ of R by polygonal bounded domains in such a way that

$$G_j \cap V_k = \emptyset, \quad k > j.$$

We may also assume that each ∂G_j is transversal to each ∂P_k and to each ∂Q_k. That is, $\partial G_j \cap \partial P_k$ and $\partial G_j \cap \partial Q_k$ are isolated sets.

By a result of Scheinberg [17, Theorem 3.2], each of the Riemann surfaces

$$G_j \cup Q_1 \cup Q_2 \cup \ldots \cup Q_k$$

admits a compact essential extension. Thus, by the special case of Theorem 1, there is a function $m_1 \in M(G_2 \cup Q_1)$ with

$$|f(z) - m_1(z)| < \frac{\varepsilon}{2}, \quad z \in F_1.$$

There exists a function $p_1 \in M(R)$ such that $m_1 - p_1$ is holomorphic on $\bar{G}_1 \cup \bar{P}_1$. Set

$$f_2 = \begin{cases} m_1 - p_1 & \text{on} \quad \bar{G}_1 \cup \bar{P}_1 , \\ f - p_1 & \text{on} \quad F_2 . \end{cases}$$

By the special case of Theorem 1, there is a function $g_2 \in M(G_3 \cup Q_1 \cup Q_2)$ such that

$$|f_2(z) - g_2(z)| < \frac{\varepsilon}{2^2} , \quad z \in \bar{G}_1 \cup \bar{P}_1 \cup F_2 .$$

Set $m_2 = g_2 + p_1$. Then,

$$|m_2(z) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} , \quad z \in F_1 \cup F_2 .$$

$$|m_2(z) - m_1(z)| < \frac{\varepsilon}{2^2} , \quad z \in \bar{G}_1 .$$

Set $f_1 = f$. Then, we may proceed inductively to construct a sequence

$$m_j \in M(G_{j+1} \cup Q_1 \cup \ldots \cup Q_j)$$

satisfying for $j = 2, 3, \ldots$,

$$|m_j(z) - f(z)| < \sum_{n=1}^{j} \frac{\varepsilon}{2^n} , \quad z \in \bigcup_{n=1}^{j} F_n$$

and

$$|m_j(z) - m_{j-1}(z)| < \frac{\varepsilon}{2^n} , \quad z \in \bar{G}_{j-1} .$$

It is clear that m_j converges to a function $m \in M(R)$ and

$$|m(z) - f(z)| < \varepsilon , \quad z \in F .$$

This completes the proof of Theorem 1.

Theorem 3 was proved in [7] for the special case that R has an essential extension in which F is bounded. Theorem 3 has two parts, one on meromorphic approximation and one on holomorphic approximation. The meromorphic approximation follows from the special case in exactly the same way as the general form of Theorem 1 followed from the special case of Theorem 1.

The proof of the holomorphic part of Theorem 3 also follows from the holomorphic special case, but we must define the sets P_j, Q_j and G_j more carefully so that

$$(G_{j+1} \cup Q_1 \cup \ldots \cup Q_j)^* \setminus \bar{G}_{j-1} \cup \bar{P}_1 \cup \ldots \cup \bar{P}_{j-1} \cup F_j$$

is connected and locally connected.

First of all the exhaustion $\{G_j\}$ can be (and usually is) constructed in such a way that $G^*_{j+1} \setminus G_j$ is connected, for each j.

For each j, let K_j be the set of bounded components of $R^* \setminus \bar{P}_j$, $R^* \setminus \bar{Q}_j$,

$$R^* \setminus (G_j \cup P_1 \cup \ldots \cup P_j)^-$$

and

$$R^* \setminus (G_j \cup Q_1 \cup \ldots \cup Q_j)^- .$$

These are finite in number. Connect each such component to the ideal boundary of R by a simple path which misses F. We may replace this path by a connected polygonal neighbourhood with the same property. Clearly we may assume that the family of all such path neighbourhoods over all j is locally finite and transversal to everything we have

constructed. Let P be the union of all these path neighbourhoods. Set

$$G_j' = G_j \setminus \bar{P} \ , \ P_j' = P_j \setminus \bar{P} \ , \ Q_j' = Q_j \setminus \bar{P} \ .$$

Then by construction, $R^* \setminus \bar{P}_j'$, $R^* \setminus \bar{Q}_j'$,

$$R^* \setminus (G_j' \cup P_1' \cup \ldots \cup P_j')^-$$

and

$$R^* \setminus (G_j' \cup Q_1' \cup \ldots \cup Q_j')^-$$

are all connected. It is easy to see that these sets are also locally connected since the boundaries are locally finite and polygonal. It follows that the covers G_j' , P_j' , Q_j' have the required properties. This completes the proof of Theorem 3.

We now prove Theorem 4. Suppose then, that $F^o = \emptyset$, and that

$$C(F \cap K) = \bar{M}(F \cap K) \ ,$$

for each compact set K. Let $f \in C(F)$ and let ε be a positive continuous function on F.

Let $\{G_n\}$ be an exhaustion of R by polygonal domains. Set

$$\varepsilon_n = \inf \{ \varepsilon(z) : z \in F \cap G_n \} \ .$$

By hypothesis, there is a $g_1 \in M(R)$ such that

$$|f(z) - g_1(z)| < \frac{\varepsilon_2}{2^2} \ , \quad z \in F \cap \bar{G}_1 \ .$$

Set $g_o = g_1$, $G_o = \emptyset$, and suppose, to obtain an induction, that $g_1, g_2, \ldots, g_{n-1}$ have been found in $M(R)$ with the following three properties:

$$|g_j(z) - f(z)| < \frac{\varepsilon_j}{2^j} + \frac{\varepsilon_{j+1}}{2^{j+1}}, \quad z \in F \cap (G_j \setminus G_{j-1}), \tag{15}$$

$$|g_j(z) - g_{j-1}(z)| < \frac{\varepsilon_{j+1}}{2^{j+1}}, \quad z \in \bar{G}_{j-1}. \tag{16}$$

$$|g_j(z) - f(z)| < \frac{\varepsilon_{j+1}}{2^{j+1}}, \quad z \in F \cap \partial G_j. \tag{17}$$

Let us construct g_n. First set $f_n = g_{n-1}$ on $\bar{G}_{n-1}$. Now extend f_n continuously to $\bar{G}_{n-1} \cup (F \cap \bar{G}_n)$ in such a way that $f_n = f$ on $F \cap \partial G_n$ and

$$|f_n(z) - f(z)| < \frac{\varepsilon_n}{2}, \quad z \in F \cap (G_n \setminus G_{n-1}).$$

Since, by assumption,

$$C(F \cap G_n) = \bar{M}(F \cap G_n),$$

and since G_n is a Lyapunov domain, it follows from Lemma 3 in [10] and from the Bishop-Kodama Localization Theorem [12] that

$$A(\bar{G}_{n-1} \cup (F \cap \bar{G}_n)) = \bar{M}(\bar{G}_{n-1} \cup (F \cap \bar{G}_n)).$$

Hence there is a function $h_n \in M(R)$ such that

$$|h_n(z) - (f_n(z) - g_{n-1}(z))| < \frac{\varepsilon_{n+1}}{2^{n+1}}, \quad z \in \bar{G}_{n-1} \cup (F \cap \bar{G}_n).$$

Set $g_n = g_{n-1} + h_n$. Then, for $z \in F \cap (G_n \setminus G_{n-1})$,

$$|g_n(z) - f(z)| \leq |g_n(z) - f_n(z)| + |f_n(z) - f(z)| \leq \frac{\varepsilon_{n+1}}{2} + \frac{\varepsilon_n}{2},$$

and so g_n satisfies (15). It is easily verified that g_n also satisfies (16) and (17). Thus, we have constructed inductively a sequence $g_n \in M(R)$ having the properties (15), (16), and (17).

From (16), we see that g_n converges to a function $g \in M(R)$. From (15) and (16), it follows that if $z \in F \cap (G_n \setminus G_{n-1})$, then for all $m > n$,

$$|f(z) - g_m(z)| \leq |f(z) - g_n(z)| + \sum_{j=n+1}^{m} |g_j(z) - g_{j-1}(z)| <$$

$$< \frac{\varepsilon_n}{2^n} + \frac{\varepsilon_{n+1}}{2^{n+1}} + \sum_{j=n+1}^{m} \frac{\varepsilon_{j+1}}{2^{j+1}} < \varepsilon_n .$$

As m tends to ∞, we have

$$|f(z) - g(z)| < \varepsilon_n \leq \varepsilon(z), \quad z \in F \cap (G_n \setminus G_{n-1}) .$$

This completes the proof.

If $F^{\circ} = \emptyset$ and $R^* \setminus F$ is connected and locally connected, Scheinberg [17] has shown that F is a set of Carleman approximation by functions holomorphic on R (see also [8]). This is implicit in the proof of Theorem 4. Indeed, we may construct an exhaustion compatible with F, that is, such that

$$R^* \setminus (F \cup \bar{G}_j)$$

is connected and locally connected for each j. Thus we may choose

the functions g_n from $H(R)$.

4. OPEN PROBLEMS

a) If R is planar and f is given on F, then it follows from Theorem 3 that approximation of f by functions holomorphic on F or by functions meromorphic on R are essentially equivalent. However, the example in [9] shows that for some closed sets in some Riemann surfaces, there are functions in $H(F)$ which cannot be approximated by functions from $M(R)$. The problem of approximation by function in $H(F)$ becomes, then, a separate question which has not been treated on arbitrary open Riemann surfaces.

b) If R is planar, the condition in Theorem 2 is not only sufficient but also necessary [14]. It would be of interest to know whether it is also necessary on an open Riemann surface.

c) There remains the problem of considering arbitrary pairs (F,R). Only Theorem 4 is complete in this respect. Scheinberg [17] has shown that there is no topological characterization of pairs (F,R) for which $A(F) = \bar{H}(F)$. This is not at all obvious, but it is easy to see that there is also no topological characterization of pairs (F,R) for which $A(F) = \bar{M}(F)$ (see [6]).

d) Scheinberg [17] has solved the problem of Carleman approximation by holomorphic functions for the case that $F^o = \emptyset$ (see also [8]). In the case where R is planar, necessary and sufficient conditions are known (necessity [5], sufficiency [13]) for Carleman approximation, even when $F^o \neq \emptyset$. What about Riemann surfaces ?

e) There is also the question of uniform approximation on unbounded sets in several complex variables. This is practically virgin territory. See, however, [4] and [16].

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WHITNEY'S SPECTRAL SYNTHESIS THEOREM IN INFINITE DIMENSIONS

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0. INTRODUCTION

In 1948 H. Whitney [13], based on a conjecture of L. Schwartz, proved that, given a non-empty open subset $U \subset \mathbb{R}^n$, the closure, with respect to the compact-open topology of order m, of an ideal $I \subset \mathcal{E}^m(U)$ is determined by its set of local ideals. The original proof was simplified in 1966 by B. Malgrange [5]. The main concern of this paper is to extend Whitney's theorem to open subsets of infinite dimensional spaces. In finite dimensions there are two equivalent formulations of this theorem:

THEOREM 1: *If* $I \subset \mathcal{E}^m(U)$ *is an ideal, then* $\bar{I} = \hat{I}$, *where*

$$\hat{I} = \cap \{I + I(a,k);\ a \in U,\ k \in \mathbb{N},\ k \leq m\} \qquad \textit{and}$$

$$I(a,k) = \{f \in \mathcal{E}^m(U);\ D^i f(a) = 0,\ 0 \leq i \leq k\}.$$

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THEOREM 2: *If* $I \subset \mathcal{E}^m(U)$ *is an ideal, then* $\bar{I} = \check{I}$, *where*

$$\check{I} = \cap \{I + I(a,k,\varepsilon);\ a \in U,\ k \in \mathbb{N},\ k \leq m,\ \varepsilon > 0\} \quad \textit{and}$$

$$I(a,k,\varepsilon) = \{f \in \mathcal{E}^m(U);\ \|D^i f(a)\| < \varepsilon,\ 0 \leq i \leq k\}.$$

In infinite dimensions, Whitney's theorem is false in formulation 1, even in the case $U = H$, a real separable Hilbert space, and $m = 1$. We present an example of this in section 2. In formulation 2 it is true, with respect to the usual compact-open topology, for the case $m = 1$ with some restrictions. The case $m \geq 2$ remains an open problem and our guess is that the theorem is false in this context.

Two other directions arise naturally in infinite dimensions: the first one is to consider subspaces of $\mathcal{E}^m(U)$ which coincide, in finite dimensions, with the whole space $\mathcal{E}^m(U)$; the second is to look for a new topology in $\mathcal{E}^m(U)$ which coincides, in finite dimensions, with the usual one.

In section 2 we consider the concept of differentiability type, which gives us a unified way to deal simultaneously with several subspaces of $\mathcal{E}^m(U)$.

In [12] Restrepo studied the closure of the algebra of polynomials of finite type in a Banach space of a certain kind, for the topology of the uniform convergence of the function and its derivative on bounded subsets. In [1] Aron and Prolla extended this result to a more general class of Banach spaces, considering the case $m \geq 2$ and polynomial algebras of vector functions weakly uniformly continuous on bounded subsets. In section 3 we study ideals of functions weakly uniformly continuous on bounded sets, with respect to the topology of the uniform convergence of order m on bounded sets.

In section 4, we consider the topology τ^c introduced by Prolla in [10].

Finally, in section 5, we use the results of section 4 to establish some facts about modules.

The results of this paper are taken from the author's Doctoral Dissertation at the Universidade Federal do Rio de Janeiro, written under the guidance of Professor J. B. Prolla.

1. PRELIMINARIES

In the sequel $\mathbb{N}$ stands for the set of natural integers $\{0,1,2,\ldots\}$, m stands for an element of $\mathbb{N} \cup \{\infty\}$ and i, j, k for elements of $\mathbb{N}$.

Let $E \neq 0$ and F be real normed linear spaces with topological duals E' and F' respectively, $U \subset E$ a non-empty open subset, $E' \otimes F$ the linear subspace of $\mathcal{L}(E;F)$ spanned by the applications $\varphi \otimes v : x \in E \to \varphi(x)v \in F$, $\varphi \in E'$, $v \in F$.

For X a real Hausdorff locally convex space, a function $f : U \to X$ is called b-*differentiable* if there is $Df : U \to \mathcal{L}(E;X)$ (necessarily unique) such that, for $x \in U$, $\lambda \in \mathbb{R}$,

$$Df(x)y = \lim_{\lambda \to 0} \frac{f(x + \lambda y) - f(x)}{\lambda}$$

uniformly with respect to y on each bounded subset of E.

In the same way, we define c-*differentiability* by replacing b by c and bounded by compact. We observe that if E is a normed space, b-differentiability is Fréchet differentiability and c-differentiability is Hadamard differentiability (Nashed [9]).

Let $\mathcal{L}^b(E;X)$ denote the space $\mathcal{L}(E;X)$ endowed with the topology τ^b of uniform convergence on bounded subsets of E and $\mathcal{L}^c(E;X)$ denote the space $\mathcal{L}(E;X)$ endowed with the topology τ^c of uniform convergence on compact subsets of E. By induction we may define $\mathcal{L}^b({}^0E|F) = F$ and, for $k \geq 1$, $\mathcal{L}^b({}^kE|F) = \mathcal{L}^b(E;\mathcal{L}^b({}^{k-1}E|F))$. In the same

way, replacing b by c, we have $\mathcal{L}^c({}^kE|F)$. Furthermore, let $C(U;X)$ denote the vector space of all continuous functions from U to X, endowed with the compact-open topology τ^o.

The space $\mathcal{E}^{bm}(U;F)$ and its topology τ^{bm} will be defined inductively as follows:

For $m=0$, $\mathcal{E}^{bo}(U;F) = C(U;F)$, $\tau^{bo} = \tau^o$ and we denote $D^o f = f$, if $f \in C(U;F)$.

For $m=1$, define $\mathcal{E}^{b1}(U;F)$ as the vector space of all $f \in C(U;F)$ which are b-differentiable and such that $Df \in C(U;\mathcal{L}^b(E;F))$. The topology τ^{b1} is defined as the topology for which the isomorphism

$$f \in \mathcal{E}^{b1}(U;F) \to (f,Df) \in C(U;F) \times C(U;\mathcal{L}^b(E;F))$$

is a homeomorphism. For uniformity of notation, $D^1 f = Df$.

Suppose we had already defined $\mathcal{E}^{b(k-1)}(U;F)$, $\tau^{b(k-1)}$ and $D^{k-1} : \mathcal{E}^{b(k-1)}(U;F) \to C(U;\mathcal{L}^b({}^{k-1}E|F))$, for some $k \geq 2$.

Define $\mathcal{E}^{bk}(U;F)$ as the vector space of all $f \in \mathcal{E}^{b(k-1)}(U;F)$ such that $D^{k-1}f$ is b-differentiable and $D(D^{k-1}f) \in C(U;\mathcal{L}^b({}^kE|F))$. Define $D^k : \mathcal{E}^{bk}(U;F) \to C(U;\mathcal{L}^b({}^kE|F))$ by $D^k f = D(D^{k-1}f)$ and the topology τ^{bk} as being the only one for which the isomorphism

$$f \in \mathcal{E}^{bk}(U;F) \to (D^i f)_{0 \leq i \leq k} \in \prod_{i=0}^{k} C(U;\mathcal{L}^b({}^iE|F))$$

is a homeomorphism.

Finally, define $\mathcal{E}^{b\infty}(U;F) = \bigcap_{k \in \mathbb{N}} \mathcal{E}^{bk}(U;F)$ and consider as $\tau^{b\infty}$ the topology for which the isomorphism

$$f \in \mathcal{E}^{b\infty}(U;F) \to (D^i f)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} C(U;\mathcal{L}^b({}^iE|F))$$

is a homeomorphism.

For the case $F = \mathbb{R}$, we will write $\mathcal{E}^{bm}(U;F) = \mathcal{E}^{bm}(U)$.

The space $\mathcal{E}^{cm}(U;F)$ and its topology τ^{cm} is defined inductively in the same way, by just replacing b by c in the above definition.

There is a natural identification between $\mathcal{L}({}^kE|F)$ and $\mathcal{L}({}^kE;F)$, the vector space of continuous k-linear maps from E^k to F. In fact, there is a homeomorphism between $\mathcal{L}^b({}^kE|F)$ (respectively $\mathcal{L}^c({}^kE|F)$) and $\mathcal{L}^b({}^kE;F)$ (respectively $\mathcal{L}^c({}^kE|F)$), the space $\mathcal{L}({}^kE;F)$ endowed with the topology τ^b (respectively τ^c).

On the other hand, the natural isomorphism between $\mathcal{L}_s({}^kE;F)$, the vector space of continuous symmetric k-linear maps from E^k to F, and $\mathcal{P}^k(E;F)$, the space of continuous k-homogeneous polynomials from E into F, is, actually, a homeomorphism, if we endow both spaces with the topology τ^b or both with the topology τ^c.

Moreover, given f belonging to $\mathcal{E}^{bm}(U;F)$ or $\mathcal{E}^{cm}(U;F)$, $x \in U$, $k \leq m$, we may associate $D^k f(x)$ with an element $\hat{d}^k f(x)$ of $\mathcal{L}_s({}^kE;F)$ which may be identified with a polynomial $\hat{d}^k f(x)$ of $\mathcal{P}^k(E;F)$.

In that case, the τ^{bm} topology may be defined in $\mathcal{E}^{bm}(U;F)$ by the family of seminorms of the form

$$p_{K,k}(f) = \sup \{ \| \hat{d}^i f(x) \| ; \ x \in K, \ 0 \leq i \leq k \},$$

$K \subset U$ a compact subset, $k \leq m$.

The topology τ^{cm} may be defined in $\mathcal{E}^{cm}(U;F)$ by the family of seminorms:

$$p_{K,L,k}(f) = \sup \{ \| \hat{d}^i f(x)y \| ; \ (x,y) \in K \times L, \ 0 \leq i \leq k \},$$

$K \subset U$, $L \subset E$ compact subsets, $k \leq m$.

For details, see Nachbin [8].

2. IDEALS AND DIFFERENTIABILITY TYPES

The concept of holomorphy type for complex functions is already well known (Nachbin [7]). The same definition may be applied to real spaces (Aron and Prolla [1]).

DEFINITION 2.1: *A differentiability type from* E *to* F is a sequence of Banach spaces $P^{\theta k}(E;F)$, $k \in \mathbb{N}$, the norm on each being denoted by $P \to \| P \|_\theta$, which satisfies the following conditions:

i) $P^{\theta 0}(E;F)$ is the normed space of all constant functions from E to F, identified with F;

ii) each $P^{\theta k}(E;F)$ is a vector subspace of $P^k(E;F)$;

iii) there is a real number $\sigma \geq 1$ such that $j, k \in \mathbb{N}$, $j \leq k$, $x \in E$ and $P \in P^{\theta k}(E;F)$ imply $\hat{d}^j P(x) \in P^{\theta j}(E;F)$ and

$$\| \frac{1}{j!} \hat{d}^j P(x) \|_\theta \leq \sigma^k \| P \|_\theta \| x \|^{k-j} .$$

DEFINITION 2.2: Let θ be a differentiability type from E to F. We define the space $\&^{\theta m}(U;F)$ as the vector subspace of $\&^{bm}(U;F)$ of all functions f such that, for $x \in U$, $k \leq m$, we have $\hat{d}^k f(x) \in P^{\theta k}(E;F)$ and the mapping $x \in U \to \hat{d}^k f(x) \in P^{\theta k}(E;F)$ is continuous.

We endow $\&^{\theta m}(U;F)$ with the topology $\tau^{\theta m}$ defined by the family of seminorms;

$$p_{K,k}(f) = \sup \{ \| \hat{d}^i f(x) \|_\theta ; \ x \in K, \ 0 \leq i \leq k \},$$

where $K \subset U$ is a compact subset and $k \in \mathbb{N}$, $k \leq m$.

In the case $F = \mathbb{R}$ we will write $\&^{\theta m}(U;F) = \&^{\theta m}(U)$.

We remark that the space $\&^{bm}(U;F)$ is a particular case of this definition.

DEFINITION 2.3 (Aron and Prolla [1]): A differentiability type θ from E to F is called *compact* if it satisfies the following conditions for each $k \in \mathbb{N}$:

i) $P_f^k(E;F)$, the vector space of continuous k-homogeneous polynomials of finite type, is densely contained in $P^{\theta k}(E;F)$;

ii) for each $v \in F$ the map $\varphi \to \varphi^k \otimes v$ is continuous from $(E', \|\cdot\|)$ to $(P_f^k(E;F), \|\cdot\|_\theta)$;

iii) if $P \in E' \otimes E$, then $Q \circ P \in P^{\theta k}(E;F)$ for all $Q \in P^{\theta k}(E;F)$ and

$$\|Q \circ P\|_\theta \leq \|Q\|_\theta \, \|P\|^k.$$

EXAMPLES 2.4: For each $k \in \mathbb{N}$, let $P^{ck}(E;F)$ be the closure of $P_f^k(E;F)$ in $P^k(E;F)$ for the usual norm. Then $\theta = c$ is a compact differentiability type called *current compact type*.

If we consider, for each $k \in \mathbb{N}$, $P^{Nk}(E;F)$, the Banach space of all nuclear continuous polynomials from E to F, endowed with the nuclear norm $\|\cdot\|_N$, and if E has the approximation property, then $\theta = N$ is a compact differentiability type called *nuclear type* (see [2]).

PROPOSITION 2.5: *Let* F *be a Banach space and* θ *a differentiability type from* E *to* F.

If $P^{\theta k}(E;F) = P^k(E;F)$, $k \in \mathbb{N}$, $k \leq m$, *then* $\mathcal{E}^{bm}(U;F) = \mathcal{E}^{\theta m}(U;F)$ *topologically*.

PROOF: As we have $(P^{\theta k}(E;F), \|\cdot\|_\theta)$ a Banach space and the inclusion map $P^{\theta k}(E;F) \subset P^k(E;F)$ is continuous, then $\|\cdot\|$ and $\|\cdot\|_\theta$ are equivalent norms. □

COROLLARY 2.6: *Let* E *be a finite dimension normed space and* θ *a compact differentiability type from* E *to* F.

Then $\mathcal{E}^{bm}(U;F) = \mathcal{E}^{\theta m}(U;F)$ *topologically.*

PROOF: $P^k_f(E;F) = P^{\theta k}(E;F) = P^k(E;F)$, $k \in \mathbb{N}$. □

DEFINITION 2.7: Let θ be a differentiability type from E to F and $A \subset \mathcal{E}^{\theta m}(U;F)$ a non-empty subset.

We define:

$$\hat{A} = \cap \{A + I(a,k);\ a \in U,\ k \leq m;\ k \in \mathbb{N}\}$$

where $I(a,k) = \{f \in \mathcal{E}^{\theta m}(U;F);\ \hat{d}^i f(a) = 0,\ 0 \leq i \leq k\}$ and:

$$\check{A} = \cap \{A + I(a,k,\varepsilon);\ a \in U,\ k \leq m,\ k \in \mathbb{N},\ \varepsilon > 0\}$$

where $I(a,k,\varepsilon) = \{f \in \mathcal{E}^{\theta m}(U;F);\ \|\hat{d}^i f(a)\|_\theta < \varepsilon,\ 0 \leq i \leq k\}$.

PROPOSITION 2.8: *If* θ *is a differentiability type from* E *to* F *and* $A \subset \mathcal{E}^{\theta m}(U;F)$ *a non-empty subset, then* $\check{A}$ *is* $\tau^{\theta m}$*-closed.*

PROOF: Fix $a \in U$, $k \leq m$ and consider

$$B(a,k) = \cap \{A + I(a,k,\varepsilon);\ \varepsilon > 0\}.$$

If $f \notin B(a,k)$ there is $\varepsilon > 0$ such that $p(f - g) \geq \varepsilon$, for every $g \in A$, where

$$p(h) = \sup \{\|\hat{d}^i h(a)\|_\theta;\ 0 \leq i \leq k\},\quad h \in \mathcal{E}^{\theta m}(U;F).$$

Consider $V = \{h \in \mathcal{E}^{\theta m}(U;F);\ p(f - h) < \varepsilon/2\}$, which is a $\tau^{\theta m}$-neighborhood of f.

If there exists $h \in V \cap B(a,k)$, we have $p(h - g) < \varepsilon/2$ for

some $g \in A$. Then:

$$p(f - g) \leq p(f - h) + p(h - g) < \varepsilon,$$

which is a contradiction. So $V \cap B(a,k) = \phi$ which proves that $B(a,k)$ is closed. □

DEFINITION 2.9: Let θ be a differentiability type from E to F satisfying condition (iii) of Definition 2.3, $G \subset E' \otimes E$ and $A \subset \mathcal{E}^{\theta m}(U;F)$ non-empty subsets.

We say that (A,G) *satisfies condition* (L) if, given $g \in G$ and $V \subset U$ a non-empty open subset such that $g(V) \subset U$, we have $A \circ (g|V) \subset \overline{(A|V)}$, the closure being considered in $(\mathcal{E}^{\theta m}(V;F), \tau^{\theta m})$.

Similar conditions have been used by Lesmes [3], Prolla [10], Llavona [4].

EXAMPLE 2.10: Let E be a space with a Schauder basis $\{e_o, e_1, \ldots, e_n, \ldots\}$ and P_n the projection of E on the vector subspace spanned by $\{e_o, e_1, \ldots, e_n\}$.

Given $x \in E$, $x = \sum_{n \in \mathbb{N}} \lambda_n e_n$, let $g_n(x) = \lambda_n$. Let θ be a compact differentiability type from E to $\mathbb{R}$ such that $\mathcal{E}^{\theta m}(E)$ is an algebra and consider $I \subset \mathcal{E}^{\theta m}(E)$, the ideal generated by a subsequence $\{g_{n_i} ; i \in \mathbb{N}\}$. If $G = \{P_n ; n \in \mathbb{N}\}$ then (I,G) satisfies condition (L).

This example may be extended to a space with a generalized basis.

We remark that for the differentiability types introduced in 2.4, $\mathcal{E}^{\theta m}(U)$ is an algebra. More generally, if θ is a differentiability type from E to $\mathbb{R}$ such that given $P \in P^{\theta i}(E)$ and $Q \in P^{\theta j}(E)$ we have $PQ \in P^{\theta(k+j)}(E)$ and the mapping $(P,Q) \to PQ$ is continuous from $P^{\theta i}(E) \times P^{\theta j}(E)$ to $P^{\theta(i+j)}(E)$, then $\mathcal{E}^{\theta m}(U)$ is an algebra.

DEFINITION 2.11: We say that E has *property* (B) if there is a sequence $\{P_n ; n \in \mathbb{N}\} \subset E' \otimes E$ such that

i) $P_n x \to x$, $x \in E$

ii) $\varphi \circ P_n \to \varphi$, $\varphi \in E'$.

This definition was used by Restrepo [12] with the additional condition that the P_n are projections.

THEOREM 2.12: *Let* θ *be a compact differentiability type from* E *to* $\mathbb{R}$ *such that* $\mathcal{E}^{\theta m}(U)$ *is an algebra and let* $I \subset \mathcal{E}^{\theta m}(U)$ *be an ideal.*

Suppose that there is a sequence $G = \{P_n ; n \in \mathbb{N}\} \subset E' \otimes E$ *such that:*

i) E *has property* (B) *with respect to* G;

ii) (I,G) *satisfies condition* (L).

Then $\check{I}$ *is the* $\tau^{\theta m}$*-closure of* I *in* $\mathcal{E}^{\theta m}(U)$.

For the proof we need several lemmas.

LEMMA 2.13: *Let* θ *be a differentiability type from* E *to* $\mathbb{R}$ *such that* $\mathcal{E}^{\theta m}(U)$ *is an algebra and* $(E'|U) \subset \mathcal{E}^{\theta m}(U)$.

Let $E_1 \subset E$ *be a finite dimensional vector subspace,* $U_1 \subset E_1 \cap U$ *a non-empty open subset and consider* $R : g \in \mathcal{E}^{\theta m}(U) \to g|U_1 \in \mathcal{E}^{bm}(U_1)$.

If $I \in \mathcal{E}^{\theta m}(U)$ *is an ideal then the* τ^{bm}*-closure of* R(I) *is an ideal of* $\mathcal{E}^{bm}(U_1)$. *Moreover, if* $f \in \mathcal{E}^{\theta m}(U)$, $f \in \check{I}$, *then* Rf *belongs to the* τ^{bm}*-closure of* R(I) *in* $\mathcal{E}^{bm}(U_1)$.

PROOF: Let $A = R(\mathcal{E}^{\theta m}(U))$, which is a subalgebra of $\mathcal{E}^{bm}(U_1)$ because R is an algebra homomorphism.

Now A satisfies the hypotheses of Nachbin's theorem [6] because $1 \in A$ and $(E'|U) \subset \mathcal{E}^{\theta m}(U)$. Therefore A is τ^{bm}-dense in $\mathcal{E}^{bm}(U_1)$.

It is clear that $\overline{R(I)}$ is a vector subspace of $\mathcal{E}^{bm}(U_1)$. On the

other hand, if $Rf \in R(I)$ and $Rg \in A$ we have $R(fg) \in R(I)$, which implies $R(I) \cdot A \subset R(I)$. By continuity of multiplication, $\overline{R(I)} \cdot \bar{A} \subset \overline{R(I) \cdot A}$ and we conclude that $\overline{R(I)} \cdot \bar{A} \subset \overline{R(I)}$, which completes the proof that $\overline{R(I)}$ is an ideal.

Let now $f \in \check{I}$, $a \in U_1$, $k \leq m$, $\varepsilon > 0$. If $\sigma \geq 1$ is given by Definition 2.1, there is $g \in I$ such that

$$\| \hat{d}^i f(a) - \hat{d}^i g(a) \|_\theta < \varepsilon/\sigma^k, \quad 0 \leq i \leq k.$$

Then

$$\| \hat{d}^i (Rf)(a) - \hat{d}^i (Rg)(a) \| \leq \| \hat{d}^i f(a) - \hat{d}^i g(a) \| \leq$$

$$\sigma^i \| \hat{d}^i f(a) - \hat{d}^i g(a) \|_\theta < \varepsilon, \quad 0 \leq i \leq k.$$

So, we have $Rf \in (\overline{R(I)})^\vee$ and, by the classical Whitney's theorem, $Rf \in \overline{R(I)}$. □

LEMMA 2.14: *Let* E *be a space satisfying property* (B) *with respect to* $\{P_n ; n \in \mathbb{N}\}$, *let* θ *be a compact differentiability type from* E *to* F, $\varepsilon > 0$, $k \in \mathbb{N}$, $K^i \subset P^{\theta i}(E;F)$ *compact subsets*, $1 \leq i \leq k$.

There is $n_o \in N$ *such that*

$$\| Q - Q \circ P_n \|_\theta < \varepsilon$$

for all $n \geq n_o$ *and for all* $Q \in K_i$, $1 \leq i \leq k$.

PROOF: See Aron and Prolla [1].

LEMMA 2.15: *Let* E *be a space satisfying property* (B) *with respect to* $\{P_n ; n \in \mathbb{N}\}$, *let* θ *be a compact differentiability type from* E

to $\mathbb{R}$, $f \in \mathcal{E}^{\theta m}(U;F)$, $k \in \mathbb{N}$, $k \leq m$, $K \subset U$ *a compact subset,* $\varepsilon > 0$.

There are $n_o \in \mathbb{N}$ *and* $V \subset U$, *a non-empty open subset, such that:*

i) $K \subset V$

ii) $P_n(V) \subset U$, $n \geq n_o$

iii) $p_{K,k}(f|V - f \circ (P_n|V)) < \varepsilon$, $n \geq n_o$.

PROOF: Let $M \geq 1$ be such that $\|P_n\| \leq M$, $n \in \mathbb{N}$, and let $0 < \delta < \operatorname{dist}(K, E \setminus U)$ be such that $x \in K$, $y \in U$, $\|x - y\| < \delta$ imply $\|\hat{d}^i f(x) - \hat{d}^i f(y)\|_\theta < \varepsilon/2M^k$, $0 \leq i \leq k$.

By (B) and Lemma 2.14, there is $n_o \in N$ such that, for $n \geq n_o$:

$$\begin{cases} \|x - P_n x\| < \delta/2, & x \in K \\ \|Q - Q \circ P_n\|_\theta < \varepsilon/2, & Q \in \hat{d}^i f(K),\ 1 \leq i \leq k. \end{cases}$$

Let $r = \delta/2M$ and, for each $x \in K$, $B(x,r) = \{t \in U; \|t - x\| < r\}$. By compactness, there are $x_1, \ldots, x_s \in K$ such that

$$K \subset \cup \{B(x_i, r);\ 1 \leq i \leq s\}.$$

Consider the non-empty open subset $V = \cup \{B(x_i, r);\ 1 \leq i \leq s\}$. If $a \in V$, $\|a - x_i\| < r$ for some i. Then, if $n \geq n_o$, $\|x_i - P_n a\| \leq$

$$\leq \|P_n a - P_n x_i\| + \|P_n x_i - x_i\| < \|P_n\| \ \|a - x_i\| + \frac{\delta}{2} < \delta,$$

and we have $P_n(V) \subset U$.

Finally observe that for $x \in K$, $1 \leq i \leq k$, $n \geq n_o$:

$$\|\hat{d}^i(f|V)(x) - \hat{d}^i(f \circ (P_n|V))\ (x)\|_\theta \leq$$

$$\|\hat{d}^i f(x) - \hat{d}^i f(x) \circ P_n\|_\theta + \|\hat{d}^i f(x) \circ P_n - \hat{d}^i(f \circ (P_n|V))(x)\|_\theta$$

$$< (\varepsilon/2) + \|\hat{d}^i f(x) - \hat{d}^i f(P_n x)\|_\theta \ \|P_n\|^i < \varepsilon. \ \square$$

LEMMA 2.16: *Let* θ *be a compact differentiability type from* E *to* F, $P \in E' \otimes E$, $E_1 = P(E)$.

The space $\mathcal{P}^k(E_1;F)$ *endowed with the norm* $\|Q\|_P = \|Q \circ P\|_\theta$ *is a Banach space for each* $k \in \mathbb{N}$.

PROOF: See Aron and Prolla [1].

PROOF OF THEOREM 2.12: It is clear that $I \subset \check{I}$. By Proposition 2.8, $\bar{I} \subset \check{I}$.

Let $f \in \check{I}$, $K \subset U$ a compact subset, $k \in \mathbb{N}$, $k \leq m$, $\varepsilon > 0$. By Lemma 2.15, there are $n_o \in \mathbb{N}$ and $V \subset U$ a non-empty open subset such that $K \subset V$, $P_n(V) \subset U$ and

$$p_{K,k}((f|V) - f \circ (P_n|V)) < \varepsilon/3, \quad n \geq n_o .$$

Fix $n \geq n_o$ and consider $P = P_n$, $E_1 = P(E)$, $U_1 = U \cap E_1$, $K_1 = P(K) \subset U_1$.

If we define in $\mathcal{P}^i(E_1)$, $1 \leq i \leq k$ then norm $\|Q\|_P = \|Q \circ P\|_\theta$, then $(\mathcal{P}^i(E_1), \|\cdot\|_P)$ is a Banach space by Lemma 2.16, and $\|\cdot\|_P$ is equivalent to the usual norm $\|\cdot\|$ in $\mathcal{P}^i(E_1)$.

Hence the topology τ^{bm} may be defined in $\mathcal{E}^{bm}(U_1)$ by the family of seminorms:

$$p_{P,L,j}(h) = \sup \{\|\hat{d}^i h(x)\|_P;\ x \in L, \quad 0 \leq i \leq j\},$$

$L \subset U_1$ a compact subset, $j \in \mathbb{N}$, $j \leq m$.

By using notation and results from Lemma 2.13, there is $g \in I$ such that

$$p_{P,K_1,k}(Rf - Rg) < \varepsilon/3.$$

Then:

$$\|\hat{d}^i f(x) - \hat{d}^i (g \circ P)(x)\|_\theta \leq$$

$$\|\hat{d}^i f(x) - \hat{d}^i (f \circ P)(x)\|_\theta + \|\hat{d}^i (f \circ P)(x) - \hat{d}^i (g \circ P)(x)\|_\theta$$

$$\leq p_{K,k}((f|V) - f \circ (P|V)) + \|\hat{d}^i f(Px) \circ P \sim \hat{d}^i g(Px) \circ P\|_\theta$$

$$< (\varepsilon/3) + \|\hat{d}^i (Rf)(Px) \circ P - \hat{d}^i (Rg)(Px) \circ P\|_\theta =$$

$$(\varepsilon/3) + \|\hat{d}^i (Rf)(Px) - \hat{d}^i (Rg)(Px)\|_P < 2\varepsilon/3 ,$$

for all $x \in K$, $0 \leq i \leq k$.

So we have $p_{K,k}((f|V) - g \circ (P|V)) < 2\varepsilon/3$ and as (I,G) satisfies condition (L), it is possible to find $h \in I$ such that $p_{K,k}(g \circ (P|V) - (h|V)) < \varepsilon/3$. We conclude that $p_{K,k}(f - h) < \varepsilon$ and the proof is complete. □

COROLLARY 2.17: *In the same conditions of Theorem 2.12, if* I *is* $\tau^{\theta m}$*-closed, then* $I = \hat{I} = \check{I}$.

COROLLARY 2.18: *Let* $I \subset \mathcal{E}^{bl}(U)$ *be an ideal and suppose there is a sequence* $G = \{P_n ; n \in \mathbb{N}\} \subset E' \otimes E$ *such that:*

i) E *has property* (B) *with respect to* G.

ii) (I,G) *satisfies condition* (L).

Then $\check{I}$ *is the* τ^{bl}*-closure of* I *in* $\mathcal{E}^{bl}(U)$.

PROOF: If θ is any compact differentiability type from E to $\mathbb{R}$ we have $E' = P_f^l(E) = P^{\theta l}(E) = P^l(E)$.

By Proposition 2.5, $\mathcal{E}^{\theta l}(U) = \mathcal{E}^{bl}(U)$ topologically. □

EXAMPLE 2.19: If $I \subset \mathcal{E}^{\theta m}(U)$ is an ideal and E is infinite dimensional, it is not always true that $\hat{I}$ is closed. Hence Whitney's

theorem is false in the formulation $\bar{I} = \hat{I}$. We will give a counter-example for $m = 1$, remarking once more that $\mathcal{E}^{\theta 1}(U) = \mathcal{E}^{bl}(U)$ topologically for any compact differentiability type θ.

Let H be a real separable Hilbert space of infinite dimension and let $\{e_i ; i \in \mathbb{N}\}$ be an orthonormal basis for H.

Denote by $S \subset H$ the vector subspace algebraically spanned by $\{e_i ; i \in \mathbb{N}\}$. Then $\bar{S} = H$.

Consider $I \subset \mathcal{E}^{bl}(H)$ the ideal generated by $\{g_i ; i \in \mathbb{N}\}$ where $g_i(x) = \langle x, e_i \rangle$, $x \in H$. Explicitly:

$$I = \{ \sum_{i=0}^{n} f_i g_i ; f_i \in \mathcal{E}^{bl}(H), \quad n \in \mathbb{N} \}.$$

If $P_n : H \to H$ denotes the projection of H on the vector subspace spanned by $\{e_i ; 0 \leq i \leq n\}$ then H has property (B) with respect to $G = \{P_n ; n \in \mathbb{N}\}$ and (Example 2.10) (I,G) satisfies condition (L).

Consider the closed ideal $I_o = \{f \in \mathcal{E}^{bl}(H); f(0) = 0\}$ and let us characterize $\hat{I}$.

For a fixed $a \in H$,

$$I + I(a,1) = \{f \in \mathcal{E}^{bl}(H); \exists\, g \in I,\ g(a) = f(a),\ \hat{d}g(a) = \hat{d}f(a)\}.$$

If $g \in I$, then $g(0) = 0$ and $\hat{d}g(0) \in S$ and, on the other hand, as $g_i \in I$, $i \in \mathbb{N}$, $\hat{d}g_i(0) = e_i$, given $u \in S$ there is $g \in I$ such that $\hat{d}g(0) = u$. Then, if $a = 0$, $I + I(a,1) = \{f \in \mathcal{E}^{bl}(H); f(0) = 0, \hat{d}f(0) \in S\}$. If $a \neq 0$, we claim that $I + I(a,1) = \mathcal{E}^{bl}(H)$. In fact, let $f \in \mathcal{E}^{bl}(H)$, $f(a) = \lambda$, $\hat{d}f(a) = v$. There is $i \in \mathbb{N}$ such that $g_i(a) \neq 0$, because $a \neq 0$. Consider $h \in \mathcal{E}^{bl}(H)$, $h(a) = \frac{\lambda}{g_i(a)}$ and $\hat{d}h(a) = \frac{1}{g_i(a)}(v - \frac{\lambda e_i}{g_i(a)})$. Then $t = g_i h \in I$, $t(a) = \lambda$ and $\hat{d}t(a) = v$, which proves that $f \in I + I(a,1)$.

We conclude that $\hat{I} = I + I(0,1)$ and we claim that it is not τ^{bl}-closed. In fact:

$$I + I\ (0,1) \subsetneq I_o\ ,\ \text{because for}\ \ v \in H,\ v \notin S,$$

the function $f(x) = \langle x,y \rangle$ belongs to I_o but does not belong to $I + I(0,1)$.

On the other hand, if $g \in I_o$, $\hat{d}g(0) = v$, there is a sequence $\{s_i\ ;\ i \in \mathbb{N}\} \subset S$, such that $s_i \to v$ as $i \to \infty$. Consider $f_i(x) = g(x) - \langle x, v - s_i \rangle$, $x \in H$. Each f_i belongs to $I + I(0,1)$ and $f_i \to g$, for the topology τ^{bl} as $i \to \infty$. Then $\overline{I + I(0,1)} = I_o$, which establishes the claim.

Furthermore $I \neq \hat{I}$. In fact: $f(x) = \langle x,x \rangle$ belongs to $\hat{I}$, but does not belong to I.

By Corollary 2.18, $\bar{I} = \check{I}$. Moreover: $\hat{I} \subset \check{I} = \bar{I} \subset I_o = (\hat{\bar{I}})$ implies $\bar{I} = I_o$.

Then:

$$I \subsetneq \hat{I} \subsetneq \check{I} = \bar{I} = I_o \subsetneq \mathcal{E}^{bl}(H).$$

3. IDEALS OF FUNCTIONS WEAKLY UNIFORMLY m-CONTINUOUSLY DIFFERENTIABLE ON BOUNDED SETS

DEFINITION 3.1: A function $f : E \to F$ is called *weakly uniformly continuous on bounded sets* (*wucbs*) if given $B \subset E$ a bounded subset and $\varepsilon > 0$, there are $\delta > 0$ and $\varphi_1, \ldots, \varphi_k \in E'$ such that $x,y \in B$, $|\varphi_i(x) - \varphi_i(y)| < \delta$, $1 \leq i \leq k$, imply $\|f(x) - f(y)\| < \varepsilon$.

DEFINITION 3.2: Let θ be a differentiability type from E to F. We define:

$$\mathcal{E}_w^{\theta m}(E;F) = \{f \in \mathcal{E}^{\theta m}(E;F);\ \hat{d}^k f : E \to P^{\theta k}(E;F)$$

$$\text{is wucbs},\ k \in \mathbb{N},\ k \leq m\}.$$

We endow $\mathcal{E}_w^{\theta m}(E;F)$ with the $\tau_w^{\theta m}$-topology, defined by the family of seminorms:

$$p_{B,k}(f) = \sup \{ \| \hat{d}^i f(x) \|_\theta ; x \in B, 0 \leq i \leq k \},$$

$B \subset E$ a bounded subset, $k \in \mathbb{N}$, $k \leq m$.

We remark that $\mathcal{E}_w^{\theta m}(E;F) = \mathcal{E}^{bm}(E;F)$ for any compact type of differentiability θ, whenever E is a finite dimensional space. Notice also that for θ a compact differentiability type from E to F, $P_f(E;F) \subset \mathcal{E}_w^{\theta m}(E;F)$ (see Aron and Prolla [1]).

DEFINITION 3.3: Let θ be a differentiability type from E to F and $A \subset \mathcal{E}_w^{\theta m}(E;F)$ a non-empty subset. We define:

$$\check{A} = \cap \{A + I(a,k,\varepsilon); a \in U, k \leq m, \varepsilon > 0\}$$

where

$$I(a,k,\varepsilon) = \{f \in \mathcal{E}_w^{\theta m}(E;F); \| \hat{d}^i f(a) \|_\theta < \varepsilon, 0 \leq i \leq k\}.$$

We may define $\hat{A}$ in a similar way and, by introducting the natural modifications in Example 2.19, we see that $\hat{A}$ is not always a $\tau_w^{\theta m}$-closed subset of $\mathcal{E}_w^{\theta m}(E;F)$. On the other hand, $\check{A}$ is $\tau_w^{\theta m}$-closed for any $A \subset \mathcal{E}_w^{\theta m}(E;F)$ a non-empty subset. The proof of this fact is similar to 2.8.

PROPOSITION 3.4: *If θ is a differentiability type from E to F satisfying (iii) of Definition 2.3, then $\mathcal{E}_w^{\theta m}(E;F) \circ P \subset \mathcal{E}_w^{\theta m}(E;F)$, for all $P \in E' \otimes E$.*

PROOF: Let $f \in \mathcal{E}_w^{\theta m}(E;F)$ and $P \in E' \otimes E$. If $k \in \mathbb{N}$, $k \leq m$, $x \in E$, then $\hat{d}^k(f \circ P)(x) = \hat{d}^k f(Px) \circ P$.

Let $B \subset E$ be a bounded subset and $\varepsilon > 0$. As $P(B) \subset E$ is a bounded subset, there are $\delta > 0$ and $\varphi_1,\dots,\varphi_s \in E'$ such that $x,y \in B$, $|\varphi_i(Px) - \varphi_i(Py)| < \delta$, $1 \leq i \leq s$, imply

$$\|\hat{d}^k f(Px) - \hat{d}^k f(Py)\|_\theta < \varepsilon/\|P\|^k.$$

Then:

$$\|\hat{d}^k(f \circ P)(x) - \hat{d}^k(f \circ P)(y)\|_\theta \leq \|\hat{d}^k f(Px) - \hat{d}^k f(Py)\| \ \|P\|^k < \varepsilon$$

which proves that $x \in E \to \hat{d}^k(f \circ P)(x) \in P^{\theta k}(E;F)$ is wucbs. □

DEFINITION 3.5: Let θ be a differentiability type from E to F satisfying (iii) of Definition 2.3, and let $G \subset E' \otimes E$ and $A \subset \&_w^{\theta m}(E;F)$ be non-empty subsets.

We say that (A,G) *satisfies condition* (L) if given $g \in G$ we have $A \circ g \subset \bar{A}$, the closure being considered with respect to the topology $\tau_w^{\theta m}$.

DEFINITION 3.6: We say that E *has property* (B*) if there is a sequence $\{P_n ; n \in \mathbb{N}\} \subset E' \otimes E$ such that $\varphi \circ P_n \to \varphi$, for all $\varphi \in E'$.

THEOREM 3.7: *Let θ be a compact differentiability type from E to $\mathbb{R}$ such that $\&_w^{\theta m}(E)$ is an algebra and let $I \subset \&_w^{\theta m}(E)$ be an ideal.*

Suppose there is a sequence $G = \{P_n ; n \in \mathbb{N}\} \subset E' \otimes E$ such that:

i) *E has property (B*) with respect to G;*

ii) *(I,G) satisfies condition (L).*

Then $\check{I}$ is the $\tau_w^{\theta m}$-closure of I in $\&_w^{\theta m}(E)$.

For the proof we need the following lemmas:

LEMMA 3.8: *Let* θ *be a differentiability type from* E *to* $\mathbb{R}$ *such that* $\mathcal{E}_w^{\theta m}(E)$ *is an algebra and* $E' \subset \mathcal{E}_w^{\theta m}(E)$.

Let $E_1 \subset E$ *be a finite dimensional subspace and consider* $R : g \in \mathcal{E}_w^{\theta m}(E) \to g|E_1 \in \mathcal{E}^{bm}(E_1)$.

If $I \subset \mathcal{E}_w^{bm}(E)$ *is an ideal then the* τ^{bm}*-closure of* $R(I)$ *is an ideal of* $\mathcal{E}^{bm}(E_1)$. *Moreover, if* $f \in \mathcal{E}_w^{\theta m}(E)$, *and* $f \in \check{I}$, *then* Rf *belongs to the* τ^{bm}*-closure of* $R(I)$ *in* $\mathcal{E}^{bm}(E_1)$.

PROOF: Analogous to 2.13.

LEMMA 3.9: *Let* E *be a space satisfying property* (B*) *with respect to* $\{P_n ; n \in \mathbb{N}\}$; θ *a compact differentiability type from* E *to* F, $f \in \mathcal{E}_w^{\theta m}(E;F)$, $k \in \mathbb{N}$, $k \leq m$, $B \subset E$ *a bounded subset,* $\varepsilon > 0$.

There is $n_o \in \mathbb{N}$ *such that*

$$p_{B,k}(f - f \circ P_n) < \varepsilon, \quad n \geq n_o .$$

PROOF: See Aron and Prolla [1].

PROOF OF THEOREM 3.7: It is clear that $\bar{I} \subset \check{I}$.

Conversely, let $f \in \check{I}$, $B \subset E$ a bounded subset, $k \leq m$, and $\varepsilon > 0$ be given.

By Lemma 3.9, there is $n_o \in N$ such that

$$p_{B,k}(f - f \circ P_n) < \varepsilon/3, \quad n \geq n_o .$$

Fix $n \geq n_o$ and let $P = P_n$. If $E_1 = P(E)$, by using notation and results from Lemma 3.8, Rf belongs to the τ^{bm}-closure of $R(I)$ in $\mathcal{E}^{bm}(E_1)$. Furthermore, $P(B) \subset E_1$ is a bounded subset, then a relatively compact subset, and the topology τ^{bm} may be defined in $\mathcal{E}^{bm}(E_1)$ by the family of seminorms:

$$p_{P,L,j}(h) = \sup \{\| \hat{d}^i h(x) \circ P\|_\theta ;\ x \in L,\ 0 \leq i \leq j\},$$

$L \subset E_1$ a compact subset, $j \in \mathbb{N}$, $j \leq m$.

So, there is $g \in I$ such that:

$$\| \hat{d}^i (Rf)(Px) \circ P - \hat{d}^i (Rg)(Px) \circ P\|_\theta < \varepsilon/3,$$

$x \in B$, $0 \leq i \leq k$,

and using the fact that (I,G) satisfies condition (L), there is $h \in I$ such that

$$p_{B,k}(g \circ P - h) < \varepsilon/3.$$

Then:

$$\| \hat{d}^i f(x) - \hat{d}^i h(x)\|_\theta \leq$$

$$\| \hat{d}^i f(x) - \hat{d}^i (f \circ P)(x)\|_\theta + \| \hat{d}^i (f \circ P)(x) - \hat{d}^i (g \circ P)(x)\|_\theta +$$

$$\| \hat{d}^i (g \circ P)(x) - \hat{d}^i h(x)\|_\theta < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

$x \in B$, $0 \leq i \leq k$, which concludes the proof. □

4. IDEALS OF $\mathcal{E}^{cm}(U)$

DEFINITION 4.1: For $A \subset \mathcal{E}^{cm}(U;F)$ a non-empty subset we define

$$\check{A} = \cap \{A + I(a,k,L,\varepsilon);\ a \in U,\ k \leq m,\ L \subset E \text{ compact},\ \varepsilon > 0\}$$

where

$$I(a,k,L,\varepsilon) = \{f \in \mathcal{E}^{cm}(U;F);\ \| \hat{d}^i f(a)v\| < \varepsilon,\ v \in L,\ 0 \leq i \leq k\}.$$

The definition of $\hat{A}$ extends naturally and obvious modifications in 2.19 show that $\hat{A}$ may be fail to be τ^{cm}-closed. By contrast, $\check{A}$ is always τ^{cm}-closed.

The definition of condition (L) for a pair (A,G), $G \subset E' \otimes E$ a non-empty subset, is naturally extended too.

THEOREM 4.2: *Let* $I \subset \mathcal{E}^{cm}(U)$ *be an ideal and suppose there is* $G \subset E' \otimes E$ *such that*

i) i_E, *the identity of* E, *belongs to the closure of* G *in* $\mathcal{L}^c(E;E)$;

ii) (I,G) *satisfies condition* (L).

Then $\check{I}$ *is the* τ^{cm}*-closure of* I *in* $\mathcal{E}^{cm}(U)$.

LEMMA 4.3: *Let* $I \subset \mathcal{E}^{cm}(U)$ *be an ideal,* $E_1 \subset E$ *a finite dimensional vector subspace,* $U_1 \subset U \cap E_1$ *a non-empty open subset.*

If we consider $R : g \in \mathcal{E}^{cm}(U) \to g|U_1 \in \mathcal{E}^{cm}(U_1)$ *then the* τ^{cm}*-closure of* $R(I)$ *is an ideal of* $\mathcal{E}^{cm}(U_1)$. *Moreover, if* $f \in \mathcal{E}^{cm}(U)$, $f \in \check{I}$, *then* Rf *belongs to the* τ^{cm}*-closure of* $R(I)$ *in* $\mathcal{E}^{cm}(U_1)$.

PROOF: Analogous to 2.13. We just remark that $\mathcal{E}^{bm}(U_1) = \mathcal{E}^{cm}(U_1)$ topologically because E_1 is a finite dimension vector space.

PROOF OF THEOREM 4.2: It is clear that $\bar{I} \subset \check{I}$.

Let $f \in \check{I}$, $K \subset U$ and $L \subset E$ compact subsets, $k \leq m$, $\varepsilon > 0$. By Lemma 3.1, Prolla and Guerreiro [11], there are $u \in G$ and $V \subset U$ a non-empty open subset such that

$$p_{K,L,k}((f|V) - f \circ (u|V)) < \varepsilon/3.$$

Consider $E_1 = u(E)$, $U_1 = E_1 \cap U$, $K_1 = u(K)$ and $L_1 = u(L)$. By

using notation and results from Lemma 4.3, there is $g \in I$ such that

$$p_{K_1,L_1,k}(Rf - Rg) < \varepsilon/3.$$

On the other hand (I,G) satisfies condition (L) so there is $h \in I$ such that

$$p_{K,L,k}(g \circ (u|V) - (h|V)) < \varepsilon/3.$$

Then:

$$|\hat{d}^i f(x)v - \hat{d}^i h(x)v| \leq$$

$$|\hat{d}^i f(x)v - \hat{d}^i f(u(x))u(v)| + |\hat{d}^i f(u(x))u(v) - \hat{d}^i g(u(x))u(v)| +$$

$$|\hat{d}^i g(u(x))u(v) - \hat{d}^i h(x)v| \leq$$

$$p_{K,L,k}((f|V) - f \circ (u|V)) + p_{K_1,L_1,k}(Rf - Rg) +$$

$$p_{K,L,k}(g \circ (u|V) - (h|V)) < \varepsilon,$$

$(x,v) \in K \times L$, $0 \leq i \leq k$.
This shows that $f \in \bar{I}$. □

5. $\mathcal{E}^{cm}(U)$ - SUBMODULES OF $\mathcal{E}^{cm}(U;F)$

THEOREM 5.1: *Let* F *be a space with the approximation property and* $W \subset \mathcal{E}^{cm}(U;F)$ *an* $\mathcal{E}^{cm}(U)$*-submodule satisfying:* $(\varphi \circ W) \otimes v \subset \bar{W}$, *for all* $\varphi \in F'$, $v \in F$.

Suppose there is $G \subset E' \otimes E$ *such that:*

i) i_E *belongs to the closure of* G *in* $\mathcal{L}^c(E;E)$;

ii) (W,G) *satisfies condition* (L).

Then $\check{W}$ *is the* τ^{cm}*-closure of* W *in* $\mathcal{E}^{cm}(U;F)$.

The proof of 5.1 uses the following two lemmas. In both, $W \subset \mathcal{E}^{cm}(U;F)$ is an $\mathcal{E}^{cm}(U)$-submodule and $\varphi \in F'$.

LEMMA 5.2: *The vector subspace* $\varphi \circ W = \{\varphi \circ g;\ g \in W\}$ *is an ideal of* $\mathcal{E}^{cm}(U)$. *Moreover,* $(\varphi \circ W, G)$ *satisfies condition* (L), *if* (W,G) *satisfies condition* (L).

PROOF: If $h \in \mathcal{E}^{cm}(U)$ and $g \in W$, then $hg \in W$ and, so $\varphi \circ (gh) = h(\varphi \circ g) \in \varphi \circ W$. Therefore $\varphi \circ W$ is an ideal.

Suppose now that (W,G) satisfies (L) and let $g \in G$ and $V \subset U$ be a non-empty open subset such that $g(V) \subset U$. If we consider $K \subset V$ and $L \subset E$ compact subsets, $f \in W$, $k \leq m$, $\varepsilon > 0$, there is $h \in W$ such that

$$\|\hat{d}^i(f \circ g)(x)y - \hat{d}^i h(x)y\| < \varepsilon/(1 + \|\varphi\|),$$

$(x,y) \in K \times L$, $0 \leq i \leq k$.

Then:

$$|\hat{d}^i(\varphi \circ f \circ g)(x)y - \hat{d}^i(\varphi \circ h)(x)y| \leq \|\varphi\| \, \|\hat{d}^i(f \circ g)(x)y - \hat{d}^i h(x)y\| < \varepsilon,$$

$$(x,y) \in K \times L, \quad 0 \leq i \leq k.$$

This proves that $(\varphi \circ W) \circ (g|V) \subset \overline{(\varphi \circ W|V)}$. □

LEMMA 5.3: *Suppose that* i_E *belongs to the closure of* G *in* $\mathcal{L}^c(E;E)$, *for some* $G \subset E' \otimes E$, *and that* (W,G) *satisfies condition* (L).

If $f \in \check{W}$, *then* $\varphi \circ f$ *belongs to the* τ^{cm}*-closure of* $\varphi \circ W$ *in* $\mathcal{E}^{cm}(U)$.

PROOF: Consider $f \in \check{W}$, $a \in U$, $k \leq m$, $\varepsilon > 0$ and $L \subset E$ a compact

subset. There is $g \in W$ such that

$$\|\hat{d}^i f(a)y - \hat{d}^i g(a)y\| < \varepsilon/(1 + \|\varphi\|),$$

$y \in L$, $0 \leq i \leq k$.

Then:

$$|\hat{d}^i(\varphi \circ f)(a)y - \hat{d}^i(\varphi \circ g)(a)y| \leq \|\varphi\| \, \|\hat{d}^i f(a)y - \hat{d}^i g(a)y\| < \varepsilon,$$

$y \in L$, $0 \leq i \leq k$, which proves that $\varphi \circ f \in (\varphi \circ W)^{\vee}$. Since Lemma 5.2 enables us to apply Theorem 4.2, we conclude that $\varphi \circ f$ belongs to the τ^{cm}-closure of $\varphi \circ W$ in $\mathcal{E}^{cm}(U)$. □

PROOF OF THEOREM 5.1: Let $f \in \overset{\vee}{W}$, let $K \subset U$, $L \subset E$ be compact subsets, $k \leq m$, $\varepsilon > 0$ and define for $0 \leq i \leq k$ the set

$$A_i = \{\hat{d}^i f(x)v;\ (x,v) \in K \times L\}.$$

Then $A = \cup \{A_i ; 0 \leq i \leq k\}$ is a compact subset of F. By the approximation property, there are $n \in N$, $\varphi_j \in F'$, $v_j \in F$ such that:

$$\|y - \sum_{j=1}^{n} \varphi_j(y)v_j\| < \varepsilon/3, \quad y \in A.$$

By Lemma 5.3, each $\varphi_j \circ f$ belongs to $\overline{\varphi_j \circ W}$, so there are $g_j \in W$ such that

$$p_{K,L,k}(\varphi_j \circ f - \varphi_j \circ g_j) < \varepsilon_1$$

where $\varepsilon_1 = \varepsilon/3\ (1 + \sum_{j=1}^{n} \|v_j\|)$.

Let $h = \sum_{j=1}^{n} (\varphi_j \circ g_j) \otimes v_j$. By hypothesis, $h \in \bar{W}$. Consider $t \in W$ such that

$$p_{K,L,k}(h - t) < \varepsilon/3.$$

Then:

$$\|\hat{d}^i f(x)v - \hat{d}^i t(x)v\| \leq \|\hat{d}^i f(x)v - \hat{d}^i(\sum_{j=1}^{n}(\varphi_j \circ f) \otimes v_j)(x)v\| +$$

$$\|\hat{d}^i(\sum_{j=1}^{n}(\varphi_j \circ f) \otimes v_j)(x)v - \hat{d}^i h(x)v\| + \|\hat{d}^i h(x)v - \hat{d}^i t(x)v\| <$$

$$\varepsilon/3 + \sum_{j=1}^{n} |\hat{d}^i(\varphi_j \circ f - \varphi_j \circ g_j)(x)v| \, \|v_j\| + p_{K,L,k}(h - t) <$$

$$\varepsilon/3 + \sum_{j=1}^{n} \varepsilon_1 \|v_j\| + \varepsilon/3 < \varepsilon,$$

$$(x,v) \in K \times L, \quad 0 \leq i \leq k.$$

This proves that $p_{K,L,k}(f - t) < \varepsilon$, and so $f \in \bar{W}$, as desired.□

In order to drop the approximation property of the space F, we will introduce a new topology.

DEFINITION 5.4: We will denote by τ^{cm}-$*$ the topology defined in $\mathcal{E}^{cm}(U;F)$ by the family of seminorms:

$$f \to p_{K,L,k}(\varphi \circ f),$$

where $K \subset U$, $L \subset E$ are compact subsets, and $\varphi \in F'$, $k \leq m$, $k \in \mathbb{N}$.

Notice that for any subset $A \subset \mathcal{E}^{cm}(U;F)$ it follows from the above definition that $f \in \mathcal{E}^{cm}(U;F)$ belongs to the τ^{cm}-$*$ closure of A if, and only if, $\varphi \circ f$ belongs to the τ^{cm}-closure of $\varphi \circ A$ in $\mathcal{E}^{cm}(U)$, for each $\varphi \in F'$.

DEFINITION 5.5: Let $A \subset \mathcal{E}^{cm}(U;F)$ be a non-empty subset.

We define:

$$A^* = \cap \{A + I(a,k,L,\varphi,\varepsilon);\ a \in U,\ k \leq m,\ L \subset E \text{ compact},\ \varphi \in F',\ \varepsilon > 0\},$$

where

$$I(a,k,L,\varphi,\varepsilon) = \{f \in \mathcal{E}^{cm}(U;F);\ |\hat{d}^i(\varphi \circ f)(a)v| < \varepsilon,\ 0 \leq i \leq k,\ v \in L\}.$$

If $A \subset \mathcal{E}^{cm}(U;F)$ is a non-empty subset, an argument similar to 2.8 shows that A^* is τ^{cm}-$*$ closed.

DEFINITION 5.6: Let $A \subset \mathcal{E}^{cm}(U;F)$ and $G \subset E' \otimes E$ be non-empty subsets.

We say that (A,G) *satisfies condition* (L^*) if given $g \in G$ and $V \subset U$ a non-empty open subset such that $g(V) \subset U$, we have $Ao(g|V) \subset \overline{(A|V)}$, the closure considered in $(\mathcal{E}^{cm}(V;F), \tau^{cm}\text{-}*)$.

THEOREM 5.7: *Let* $W \subset \mathcal{E}^{cm}(U;F)$ *be an* $\mathcal{E}^{cm}(U)$*-submodule. Suppose there is* $G \subset E' \otimes E$ *such that* i_E *belongs to the closure of* G *in* $\mathcal{L}^c(E;E)$ *and* (W,G) *satisfies condition* (L^*).

Then W^* *is the* τ^{cm}-$*$ *closure of* W.

PROOF: Apply Lemma 5.2 and Theorem 4.2.

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RECENT PROGRESS IN BIRKHOFF INTERPOLATION

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§1. INTRODUCTION

The first paper [3] on Birkhoff interpolation is due to G. D. Birkhoff himself, which he presented to the American Mathematical Society when he was only 19 years old. Its style is old-fashioned; the main interest is in identities, remainder formulas, and mean value theorems. Birkhoff was interested in the sign of the kernels which appear in these formulas, and proved the important and deep theorem about their number of zeros.

In 1955 - 58, Tűran and his pupils studied the "0 - 2 lacunary interpolation", which prescribes the values of P_n and P_n'' at the knots. They studied a very special selection of knots - zeros of derivatives of Lagrange polynomials - and obtained many beautiful results

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(see [2], [45] and [41]).

In 1966, I. J. Schoenberg [39] asked when the interpolation problem with a given structure is solvable for all possible sets of knots. This is the problem of regularity or poisedness of the interpolation matrix, which has proved to be exceedingly difficult. K. Atkinson and A. Sharma [1] and D. Ferguson [7] gave the basic theorems of regularity, Karlin and Karon [13] contributed the theorem about coalescence, and Lorentz ([18], [19], [22]), theorems of singularity. Among the applications of Birkhoff interpolation, we mention the uniqueness problem for monotone approximation (Lorentz - Zeller [29], R. A. Lorentz [30]), and the Birkhoff quadrature formulas (Lorentz and Riemenschneider [24]). In recent years several papers have dealt with the Birkhoff interpolation problem for spline functions (Karlin-Karon [13] and others).

The present report attempts to give an exposition of this theory for polynomial interpolation. For the sake of brevity, we omit spline interpolation, and "lacunary interpolation" with special knots. This paper is based on the 1975 report [20] of one of us to the Center of Numerical Analysis, University of Texas in Austin.

A last remark: the name "Birkhoff" interpolation problem (rather than "Hermite-Birkhoff") seems to be completely justified from all possible mathematical points of view; both historical as well as those of substance.

§2. BASIC DEFINITIONS AND THEOREMS

2.1. DEFINITIONS Let $S = \{g_0, g_1, \ldots, g_N\}$ be a system of n times continuously differentiable functions on a set A which is either an interval $[a,b]$ or the circle $\mathbf{T}$. A linear combination $P = \sum_{j=0}^{N} a_j g_j$ will be called a *polynomial* in the system S.

A matrix

(2.1.1) $$E = (e_{ik})_{i=1,\ k=0}^{m\quad n},\ m \geq 1,\ n \geq 0$$

is an *interpolation matrix for* S if its elements e_{ik} are zero or one and if the number of ones in E is equal to $N+1$, $|E| = N + 1$. In general we do not allow empty rows, that is an i for which $e_{ik} = 0$, $k = 0,\ldots,n$. A *set of knots* $X = \{x_1,\ldots,x_m\}$ consists of m distinct points of the set A. The elements E, X, S and the data c_{ik} (defined for $e_{ik} = 1$) determine a *Birkhoff interpolation* problem which consists in finding a polynomial P satisfying

(2.1.2) $$P^{(k)}(x_i) = c_{ik} \quad (e_{ik} = 1).$$

The system (2.1.2) consists of $N + 1$ linear equations with $N + 1$ unknowns a_j. The pair E, X is called *regular* if equations (2.1.2) have a (unique) solution for each given set of c_{ik}; otherwise the pair E, X is *singular*. A pair E, X is regular if and only if the determinant of the system

(2.1.3) $$D(E,X) = \{g_o^{(k)}(x_i),\ldots,g_N^{(k)}(x_i);\ e_{ik} = 1\}$$

is different from zero. Formula (2.1.3) gives a row of the determinant corresponding to the entry $e_{ik} = 1$ in E; the order of the rows in (2.1.3) is taken as the lexicographical ordering of (i,k), $e_{ik} = 1$. By A(E;X) we denote the $(N + 1) \times (N + 1)$ matrix given by (2.1.3).

The basic notion of this report is that of a *poised* or *regular* matrix E. An interpolation matrix E has this property if the pair E, X is regular for each set of knots X in a given class. One can consider several types of regularity; *order regularity*, if $A = [a,b]$ and the knots must satisfy $a \leq x_1 < \ldots < x_m \leq b$; *real regularity* (*complex regularity*) when the knots are arbitrary distinct real

(complex) numbers; and, *trigonometric regularity* which is order regularity on the circle, $-\pi \le x_1 < \ldots < x_m < \pi$. The regularity of E means that the determinant $D(E,X) \neq 0$ for <u>all</u> (admissible) sets of knots, while singularity means that $D(E,X)$ vanishes for <u>some</u> X. We shall distinguish between *strong singularity* [6], when $D(E,X)$ takes values of different sign, and *weak singularity*, when $D(E,X)$ vanishes without a change of sign.

A matrix is singular if and only if some non-trivial polynomial P is *annihilated by* E, X for some admissible X; this means that P satisfies the homogeneous equations $P^{(k)}(x_i) = 0$ for $e_{ik} = 1$. For a singular matrix we consider $r(E)$, the lowest possible rank of the matrix $A(E,X)$. Then

$$d = d(E) = N + 1 - r(E) \tag{2.1.4}$$

the defect of E, is the largest possible dimension of the subspace of polynomials P, annihilated by E, X for some X.

EXAMPLES: A *Lagrange interpolation matrix* has $m = N + 1$ and ones in the column 0. A *Taylor interpolation matrix* is an $1 \times (N+1)$ matrix consisting of a single row of ones. A *Hermite matrix* has blocks $e_{i0} = \ldots = e_{i,k_i} = 1$ of ones in each row while the remaining entries are all zeros. An *Abel matrix* prescribes each derivative $P^{(k)}$ at exactly one point and thus has only a single one in each column.

2.2. THE ALGEBRAIC CASE Here $A = [a,b]$, S consists of the functions

$$g_0(x) = \frac{x^N}{N!}, \ldots, g_{N-1}(x) = \frac{x}{1}, g_N(x) = 1, \tag{2.2.1}$$

and P are the algebraic polynomials of degree N. Since $P^{(k)} \equiv 0$ for $k > N$, we may assume that $n \le N$, and by adding columns of zeros to

E, we may set $n = N$; such matrices are called *normal*. We shall assume that $N = n$ in what follows.

Now formula (2.1.3) becomes

$$D(E,X) = \left\{ \frac{x_i^{n-k}}{(n-k)!}, \frac{x_i^{n-k-1}}{(n-k-1)!}, \ldots, \frac{x_i^{-k}}{(-k)!}; e_{ik} = 1 \right\}, \tag{2.2.2}$$

if we agree to replace $1/p!$ by zero if $p < 0$.

PROPOSITION 2.1: *The determinant* $D(E,X)$ *is a homogeneous polynomial in* $x_1, \ldots, x_m$ *of joint degree* $1 + 2 + \ldots + n - \Sigma_{e_{ik}=1} k = \rho$ *which satisfies*

$$D(E,aX) = a^{\rho} D(E,X), \quad D(E,X + \alpha) = D(E,X), \tag{2.2.3}$$

$$D(E,X) = \left\{ \frac{(x_i+\alpha_0)^{n-k}}{(n-k)!}, \ldots, \frac{(x_i+\alpha_n)^{-k}}{(-k)!}; e_{ik} = 1 \right\}, \tag{2.2.4}$$

for arbitrary $\alpha_0, \ldots, \alpha_n$. *If* $X = \{x_1, \ldots, x_m\}$, *we write*

$$aX = \{ax_1, \ldots, ax_m\}, \quad X + \alpha = \{x_1 + \alpha, \ldots, x_m + \alpha\}.$$

In particular, it follows that the regularity (singularity) of E does not depend on the choice of the interval $[a,b]$.

For a normal matrix E, let $m(k)$ be the number of ones in column k and let $M(k) = \sum_{r=0}^{k} m(r)$ be the number of ones in columns $0,\ldots,k$. For example, $M(n) = n + 1$, while $M(0) > 0$ means that there are ones in column 0. The functions $m(k)$, $M(k)$ are called *the Pólya functions of* E. The condition

$$M(k) \geq k + 1, \quad k = 0, \ldots, n \tag{2.2.5}$$

is called the *Pólya condition*, and the corresponding matrix is called a *Pólya matrix*. Similarly, a *Birkhoff matrix* is a matrix E whose Pólya function satisfies the *Birkhoff condition*

$$M(k) \geq k + 2, \quad k = 0,1,\ldots,n-1. \tag{2.2.6}$$

These conditions play the following role. It is difficult to decide when E is singular, that is when $D(E,X) = 0$ for some X, but it is easy to see when $D(E,X)$ is zero for all X:

THEOREM 2.2: (G. D. Birkhoff, D. Ferguson, and B. Nemeth) *The determinant* $D(E,X)$ *is not identically zero if and only if* E *satisfies the Pólya condition.*

If $M(k) < k + 1$ for some k, then there is a non-trivial polynomial P of degree k which is annihilated by E. This proves the necessity of the condition. The sufficiency was proved incorrectly by G. D. Birkhoff [3] and later correctly proved independently by Nemeth [33] and D. Ferguson [7].

For normal matrices, condition (2.2.5) is equivalent to the assumption that any s last columns contain at most s ones:

$$\sum_{k=n_1}^{n} m(k) \leq n - n_1 + 1, \quad 0 \leq n_1 \leq n. \tag{2.2.7}$$

It is (2.2.7) that we call the *Pólya condition* for arbitrary (not normal) matrices. Condition (2.2.7) holds if and only if E can be made into a normal Pólya matrix by the addition of one or more supplementary rows.

A normal matrix E is *decomposable*, [1], $E = E_1 \oplus E_2$, if it can be split vertically into two normal matrices E_1, E_2. A matrix is indecomposable if and only if either it satisfies the Birkhoff condition,

or it consists of a single column. For each Pólya matrix E, there is the maximal *canonical decomposition*

$$E = E_1 \oplus \ldots \oplus E_\mu , \tag{2.2.8}$$

where each matrix E_j is either a one column or a Birkhoff matrix.

If $E = E_1 \oplus E_2$, then the matrix $A(E,X)$ which appears in (2.2.2) can be written, after a proper rearrangement of rows, as

$$A(E,X) = \left(\begin{array}{c|c} \overbrace{\qquad}^{n-r} & \overbrace{\qquad}^{r+1} \\ & A(E_1,X) \\ \hline A(E_2,X) & 0 \end{array}\right) \begin{array}{l} \} \; r+1 \\ \} \; n-r \end{array} . \tag{2.2.9}$$

As a corollary, we obtain

$$D(E,X) = \pm\, D(E_1,X)D(E_2,X). \tag{2.2.10}$$

Hence,

THEOREM 2.3: (Atkinson and Sharma [1]) *A decomposable matrix* $E = E_1 \oplus E_2$ *is regular if and only if both of its components are regular.*

2.3. REGULAR MATRICES By Theorem 2.2, the Pólya condition (2.2.5) is necessary for regularity. To obtain a workable sufficient condition, we need the following notion. By a *sequence* in a row i of the matrix E, we mean a continuous block of ones

$$e_{ik} = \ldots = e_{i\ell} = 1 \tag{2.3.1}$$

which is maximal. Therefore, for a sequence either $k = 0$ or else $e_{i,k-1} = 0$, and either $\ell = n$ or $e_{i,\ell+1} = 0$. A sequence is *odd*

(or *even*) if it has an odd (even) number of ones.

A sequence (2.3.1) is *supported* if there exist two ones in E to the NW and SW of $e_{ik} = 1$, in other words, if there are ones in positions (i_1,k_1), (i_2,k_2) in E with $i_1 < i$, $k_1 < k$; $i_2 > i$, $k_2 < k$. Already G. D. Birkhoff [3] noticed that odd supported sequences are not good for regularity; he called a matrix E *conservative*, if it has no odd supported sequences.

THEOREM 2.4: (Atkinson and Sharma [1]) *A normal Pólya matrix* E *is order regular if it has no odd supported sequences.*

The simplest proof of this theorem depends on an extended form of Rolle's theorem; another proof is based on the properties of Birkhoff's kernel, (see Section 7.3).

If f is continuously differentiable on [a,b], then between any two consecutive zeros α, β of f is an *odd* (or infinite) number of zeros of the derivative f'. Hence, if between α and β some number of zeros of f' of *even multiplicity* are known to exist, then f' must have an additional zero (with a new value or as an additional multiplicity of a known zero). Using this and induction, one proves

THEOREM 2.5: *If* f *is an* n*-times continuously differentiable function annihilated by a Pólya interpolation matrix* E *containing no odd supported sequences and a set of knots* $X = \{x_1,\ldots,x_m\}$, *then*

$$f^{(n)}(\xi) = 0 \quad \text{for some} \quad x_1 \le \xi \le x_m . \tag{2.3.2}$$

Applying this theorem to a polynomial P_n annihilated by E, X, we obtain $P_n \equiv 0$ and thus prove Theorem 2.4.

Since 2-row matrices have no odd supported sequences, we obtain

THEOREM 2.6 (Pólya [36], Whittaker [46]): *A* $2 \times (n+1)$ *interpolation*

matrix is regular if and only if it satisfies the Pólya condition.

The problem of order-regularity is much more complicated if the number of rows is more than two (even if it is only three, see §6). For *complex-regularity*, when the knots x_i may be arbitrary complex numbers, the situation is different. D. Ferguson [7] characterized complex-regular Birkhoff matrices (for another proof see [13]). Combining this with the decomposition formula (2.2.8) one obtains

THEOREM 2.7: *A Pólya matrix* E *is complex regular if and only if its canonical decomposition* (2.2.8) *consists only of Hermite matrices and of matrices with at most two non-zero rows.*

Unfortunately, the known proofs are not simple.

If E is not regular, its "distance from regularity" can be measured by its defect, given by formula (2.1.4). Using an argument from [5] we can prove

THEOREM 2.8: *For a normal Pólya matrix with exactly* p *odd supported sequences,*

$$d \leq \left[\frac{p+1}{2}\right]. \tag{2.3.3}$$

This inequality cannot be improved.

2.4. EXAMPLES, SYMMETRY, AND TRIGONOMETRIC INTERPOLATION Applying Theorem 2.4, we see that all Hermitian (hence all Lagrange and all Taylor) matrices are regular: they do not have odd supported sequences. Abel matrices are regular by Theorem 2.3 since they decompose into one column matrices, each with a single entry equal to one, and these are regular.

Computing the determinant D(E,X), one sees that in

$$(2.4.1)\quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the matrix E_1 is strongly singular, the matrix E_2 is weakly singular, while the matrix E_3 is regular in spite of the fact that it has (two) odd supported sequences. Thus, Theorem 2.4 of Atkinson - Sharma cannot be inverted. Nevertheless, this inversion is "usually" true. Matrices which have exactly one odd supported sequence in one of the rows (with other sequences of this row being even or not supported) are necessarily singular (see Theorem 5.1). There are also other results in this direction.

In his "lacunary interpolation", P. Túran has studied symmetric matrices. A *symmetric matrix* E should have an odd number, $2m+1$, of rows (to assure generality, we allow here a central row consisting only of zeros). The matrix E is symmetric if $e_{-i,k} = e_{i,k}$, $i = 1,\ldots,m$ and all k.

A symmetric matrix E is *symmetrically regular* if the pair E,X is regular for each symmetric set of knots $X = \{x_{-m},\ldots,x_0,\ldots,x_m\}$ with $x_0 = 0$, $x_{-i} = -x_i$. However, this notion can be reduced to regularity [22]: E is symmetrically regular if and only if both E_1 and E_2 are regular, where E_1 (or E_2) consists of row 0 of E in which all elements in odd (or even) positions have been replaced by ones while the other elements are left unchanged, and of rows $1,\ldots,m$ of E.

If the matrix E has some measure of symmetry, one can find some simple necessary conditions for regularity which complement the Pólya condition. For example, again let the rows of E be numbered $-m,\ldots,-1,0,\ldots,m$, and let q_j, $j = 1,\ldots,m$, be the number of k's for which $e_{-j,k} = e_{jk} = 1$. Also, let p_e, p_o be the number of zeros in row 0 of E in even or odd positions.

PROPOSITION 2.9 [23]: *Under the above assumptions, the following inequality is a necessary condition for the regularity of* E

$$q_1 + \ldots + q_m \leq \min(p_e, p_o). \tag{2.4.2}$$

Only a little is known about Birkhoff trigonometric interpolation. Here the system of functions S is

$$S = \{1, \cos x, \sin x, \ldots, \cos Nx, \sin Nx\}, \quad x \in [-\pi, \pi). \tag{2.4.3}$$

The polynomials in S are the trigonometric polynomials

$$T_N(x) = \frac{1}{2} a_o + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx).$$

An interpolation matrix is a $m \times (n+1)$ matrix $E = (e_{ik})_{i=1,k=0}^{m \quad n}$ with $2N+1$ ones. There is no *a priori* reason to assume that $n = 2N$ since derivatives $T_N^{(k)}$ of T_N of all orders are non-trivial. The determinant $D(E,X)$ for E is translation invariant, $D(E, X+\alpha) = D(E,X)$ for all α. The Pólya condition (2.2.5) is replaced here by the condition

$$M(0) > 0. \tag{2.4.4}$$

Also, an Atkinson-Sharma theorem holds for trigonometric interpolation. However, now one should consider cylindrical matrices (with row m of E proceeding row 1). In this case, when defining the support for a sequence of row i, one can take supporting ones from the same row.

THEOREM 2.10: *A matrix with* $m \geq 2$ *rows satisfying condition* (2.4.4) *is trigonometrically regular if it has no odd sequences except those*

beginning in column 0.

As further examples of results that hold for trigonometric interpolation, we mention:

PROPOSITION 2.11: (i) *Let* $i_1,\dots,i_p$ *be different rows of* E *and suppose* p *is prime. Let* R_1 (R_2) *be the set of even (odd)* $k > 0$ *for which there are ones in positions* (i_j,k), $j=1,\dots,p$. *Then the following inequalities are necessary for the regularity of* E

$$\max(|R_1|, |R_2|) \leq \frac{n}{p}. \tag{2.4.5}$$

(ii) (Johnson [12]). *A one row matrix* E *is regular if and only if it has* $N+1$ *ones in even positions and* N *ones in odd positions.*

§3. COALESCENCE OF MATRICES

3.1. LEVELING FUNCTIONS AND COALESCENCE The important concept of coalescence for two adjacent rows of a matrix was introduced by Karlin and Karon [13]. They also gave the Taylor formula, Theorem 3.3, although it was Lorentz and Zeller [29] who firmly established that $C \neq 0$ in that formula. Recently, Lorentz [22] put the method on a broader basis which allows multiple coalescence. Various applications of this method can be found in [13], [19] and [22] (see also §3.3, §5 and §6.2).

Let E be an $m \times (n+1)$ matrix, not necessarily normal, satisfying the Pólya condition (2.2.7) (see (3.1.1) below). We interpret E as a vertical grid of boxes. If $e_{ik} = 1$, then a ball occupies the i-th box in the k-th column. We place a tray of $n+1$ boxes under

the columns of the grid. Then the balls are permitted to fall from the grid into the boxes of the tray in such a way that if the box immediately below is occupied, then the ball rolls to the first available box on the right. The condition (2.2.7) assures us that no ball will roll out of the tray. The distribution of balls in the tray constitutes the one row matrix obtained by coalescence of the m rows of E. It is to be expected that the final arrangement of the balls in the tray is independent of manner in which the balls were allowed to fall. Here is an example of coelescence of a two row matrix:

1st row	1	0	1	1	1	0	0	1	0	0	0 . . .
2nd row	0	1	1	0	0	0	0	1	1	0	0 . . .
coalesced row	1	1	1	1	1	1	0	1	1	1	0 . . .
pre-coalesced 1st row	1	0	0	1	1	1	0	0	0	1	0 . . .

Fig. 1.

Let $m(k) = m_k$, $M(k) = M_k$ denote the Pólya functions of some interpolation matrix satisfying the Pólya condition

$$(3.1.1) \qquad \sum_{k=n_1}^{n} m(k) \leq n - n_1 + 1, \quad 0 \leq n_1 \leq n.$$

We shall use capital letters to denote the sum of a function, e.g. $G(k) = \sum_{r=0}^{k} g(r)$. The *level functions* m^0, M^0 of m and M are the largest functions g, G with integral values which satisfy

$$(3.1.2) \qquad 0 \leq g(k) \leq 1, \; G(k) = \sum_{r=0}^{k} g(r) \leq M(k), \; k = 0,\ldots,n.$$

This is equivalent to the following: if

$$(3.1.3) \qquad \mu_k = (\ldots((m_0 - 1)_+ + m_1 - 1)_+ + \ldots + m_{k-1} - 1)_+ + m_k, \; k = 0,\ldots,n,$$

then

$$(3.1.4) \qquad m^{O}(k) = \begin{cases} 1 & \text{if} \quad \mu_k \geq 1 \\ 0 & \text{if} \quad \mu_k = 0. \end{cases}$$

The *coefficient of collision* $\alpha(M) = \alpha(E)$ of M, or of the matrix E, measures the distance of M to the level function M^{O} and is defined by

$$(3.1.5) \qquad \alpha(M) = \alpha(E) = \sum_{k=0}^{n} (M(k) - M^{O}(k)).$$

(In the above interpretation of coalescence, this is the distance that the balls must roll.) A matrix E *has collisions* if $\alpha(E) > 0$. The basic properties of level functions are given in the following theorem.

THEOREM 3.1:

(i) $(M_1 + M_2)^{O} = (M_1^{O} + M_2)^{O} = (M_1^{O} + M_2^{O})^{O}$

(ii) $((M_1 + M_2)^{O} + M_3)^{O} = (M_1 + (M_2 + M_3)^{O})^{O} = (M_1 + M_2 + M_3)^{O}$

(iii) *If the function* $M_1 + M_2$ *satisfies* (3.1.1), *then so does* $M_1^{O} + M_2$.

(iv) $\alpha(M_1 + M_2) = \alpha(M_1) + \alpha(M_1^{O} + M_2)$

$= \alpha(M_1) + \alpha(M_2) + \alpha(M_1^{O} + M_2^{O})$.

The $1 \times (n+1)$ matrix E^{O} with Pólya functions m^{O}, M^{O} is called the *coalescence of the matrix* E *to one row*. More generally, if $E = E_1 \cup E_2$ is a decomposition of E into two disjoint sets of rows, then the *coalescence in* E *of the rows* E_1 *to one row* is the matrix $E_1^{O} \cup E_2$. From Theorem 3.1 we have

$$(3.1.6)\qquad (E_1 \cup E_2)^0 = (E_1^0 \cup E_2)^0 = (E_1^0 \cup E_2^0)^0,$$

$$((E_1 \cup E_2)^0 \cup E_3)^0 = (E_1 \cup (E_2 \cup E_3)^0)^0 = (E_1 \cup E_2 \cup E_3)^0,$$

and

$$(3.1.7)\qquad \alpha(E_1 \cup E_2) = \alpha(E_1) + \alpha(E_1^0 \cup E_2)$$

$$= \alpha(E_1) + \alpha(E_2) + \alpha(E_1^0 \cup E_2^0).$$

Moreover, the Pólya condition (3.1.1) is passed on to $E_1^0 \cup E_2$ from $E_1 \cup E_2$.

In particular, we can consider the coalescence of two rows in the matrix E, say E_i — the i-th row and E_j — the j-th row. The coalescence $(E_i \cup E_j)^0$ replaces these two rows by a single row, and has collision coefficient $\alpha_{ij} = \alpha(E_i \cup E_j)$. For a horizontal submatrix E_1 of E, we consider the coalescence $(E_1 \cup E_2^0)^0$ of E_1 with one row E_2^0 produced by coalescing a disjoint horizontal submatrix E_2 to one row. The coefficient of collision $\alpha(E_1 \cup E_2^0)$ can only increase with the number of ones in E_2^0 i.e. as E_2 increases in size. For $E_2 = E \setminus E_1$,

$$(3.1.8)\qquad \gamma(E_1) = \alpha(E_1 \cup E_2^0)$$

is called the *coefficient of maximal collision of* E_1 in the matrix E.

We have deliberately not mentioned any ordering of the rows in E. Indeed, the disjoint sets of rows, E_1 and E_2, could intertwine. This is good for considerations involving real (or complex) regularity, but we shall need to consider the order of the rows for applications to order regularity.

3.2. SHIFTS AND DIFFERENTIATION OF DETERMINANTS: For horizontal submatrices E_1 and E_2 of a matrix E, the row $(E_1 \cup E_2)^0$ can be obtained in the following way. Let $\tilde{E}_1$ be the $1 \times (n+1)$ matrix having the Pólya function $(M_1 + M_2)^0 - M_2^0$. Then the two rows $\tilde{E}_1$ and E_2^0 will be without collisions, i.e. $\alpha(\tilde{E}_1 \cup E_2^0) = 0$, and we obtain the row $(E_1 \cup E_2)^0$ from these two by simple addition (see Fig. 1 for an example). We call $\tilde{E}_1$ the *pre-coalesced* matrix with respect to E_2. If we denote the positions of the ones in E_1^0 and $\tilde{E}_1$ by $\ell_1 < \ldots < \ell_p$ and $\ell_1' < \ldots < \ell_p'$ respectively, then $\ell_j \leq \ell_j'$, $j = 1,\ldots,p$ and $\tilde{E}_1$ is formed from E_1^0 by shifting the ones from positions ℓ_j to ℓ_j'. The coefficient of collision for $E_1 \cup E_2^0$ is

$$(3.2.1) \qquad \alpha(E_1 \cup E_2^0) = \alpha(E_1) + \alpha(E_1^0 \cup E_2^0) = \alpha(E_1) + \sum_{j=1}^{p} (\ell_j' - \ell_j).$$

Since the order of the rows is important for studying the order regularity of matrices, we now consider the placement of the coalesced rows in E and its effect on the determinant. Let $E_i = (\ell_1,\ldots,\ell_p)$ and E_j be two rows of E, and let $\tilde{E}_i = (\ell_1',\ldots,\ell_p')$ be the pre-coalesced row E_i with respect to E_j. The new $(m-1) \times (n+1)$ matrix $\tilde{E}$ obtained from E by omitting row E_i and replacing row E_j by $(E_i \cup E_j)^0$ is the matrix of *coalescence of row* E_i *to row* E_j *in* E. If $\tilde{X}$ is the set of knots X with x_i omitted, we can obtain the determinant $D(\tilde{E},\tilde{X})$ as follows. The rows of the matrix $A(E,X)$ appearing in (2.1.3) which correspond to E_i are given by derivatives of orders $\ell_1,\ldots,\ell_p$; we replace these derivatives by derivatives of orders $\ell_1',\ldots,\ell_p'$ and replace x_i by x_j. The new matrix will have determinant $(-1)^\sigma D(\tilde{E},\tilde{X})$. The *interchange number* σ is the number of interchanges required to bring the rows of $A(\tilde{E},\tilde{X})$ into the lexicographic order of $\tilde{E}$ from the order inherited from E.

Similarly, if $\tilde{E}_i = (\ell_1^*,\ldots,\ell_p^*) = E_i^*$ is the pre-coalescence of

E_1 with respect to $E\backslash E_i$, then we obtain the *matrix* E^* *of maximal coalescence* by omitting row E_i, selecting some $j \neq i$, and replacing row E_j by $E_j \cup E_i^*$ (sometimes referred to as *coalescence of row* E_i *to row* E_j *at infinity*). For practical purposes concerning regularity, this matrix is essentially independent of j. Indeed, by the nature of the coalescence, the columns $\ell_1^*,\ldots,\ell_p^*$ in $E \setminus E_i$ consist only of zeros. Furthermore, if E satisfies the Pólya condition then each vertical submatrix of $E \setminus E_i$ of columns k, $\ell_q^* < k < \ell_{q+1}^*$, $(\ell_0^* = -1)$, is a Pólya matrix. Therefore, E^* has a decomposition into Pólya matrices having single column components in positions $\ell_1^*,\ldots,\ell_p^*$. The formula (2.2.10) shows that the determinant $D(E^*,X^*)$ is independent (up to sign) of the choice of j.

Motivated by the need to bring a row E_1^o to a pre-coalesced form $\tilde{E}_1$, we define a *shift* $\Lambda : k \to k+1$ of a submatrix E_1 of E as follows: a shift moves a one, $e_{ik} = 1$, of some row i in E_1 into the position $e_{i,k+1} = 1$. A shift is defined on $e_{ik} = 1$ only if $e_{i,k+1} = 0$. As an example, again let ℓ_q, ℓ_q' represent the positions of ones in E_1^o, $\tilde{E}_1$ respectively; if ℓ_{q0}' is the largest ℓ_q' with $\ell_q' \neq \ell_q$, and if ℓ_{q1} is the first one of the sequence in E_1^o ending in ℓ_{q0}, then there is a shift in some row of E_1 with k such that $\ell_{q1} \leq k \leq \ell_{q0}$ which either increases ℓ_{q0} by one unit without changing the remaining ℓ_q or decreases $\alpha(E_1)$ by one unit with $\ell_{q1},\ldots,\ell_{q0}$ unchanged. The new matrix ΛE_1 reduces the collision coefficient (3.2.1) when ΛE_1, $(\Lambda E_1)^o$ replaces E_1, E_1^o. Such a shift is called a *reducing shift* of E_1.

A *multiple shift* $\tilde{\Lambda} = \Lambda_1 \ldots \Lambda_\beta$ of order β is a product of β simple shifts. It transforms E_1 into a matrix $\tilde{\Lambda}E_1$.

PROPOSITION 3.2: *Let* $E_i = (\ell_1,\ldots,\ell_p)$ *and* E_j *be two rows of* E, *and* $\tilde{E}_i = (\ell_1',\ldots,\ell_p')$ *be the pre-coalescence of* E_i *with respect to* E_j.

(a) *For a multiple shift* $\tilde{\Lambda}$ *of order* $\beta < \alpha = \alpha(E_i \cup E_j)$, *the rows* $\tilde{\Lambda}E_i$ *and* E_j *are in collision (i.e.* $\alpha(\tilde{\Lambda}E_i \cup E_j) > 0$).

(b) *There exists only one multiple shift* $\tilde{\Lambda}$ *of order* α *for which* $\tilde{\Lambda}E_i$ *and* E_j *have no collisions. The shift* $\tilde{\Lambda}$ *carries* $\ell_1,\ldots,\ell_p$ *into* $\ell_1',\ldots,\ell_p'$. *Although* $\tilde{\Lambda}$ *is unique, it has* $C(\geq 1)$ *representations by simple (reducing) shifts:* $\tilde{\Lambda} = \Lambda_1 \ldots \Lambda_\alpha$.

(c) *If* E_1 *is a horizontal submatrix of* E, *then for any multiple shift* $\tilde{\Lambda}$ *of order* $\beta > \gamma = \gamma(E_1)$ *(see (3.1.8)), the matrix* $(\tilde{\Lambda}E_1) \cup (E \setminus E_1)$ *does not satisfy the Pólya condition.*

For a system S of differentiable functions and the matrix $A(E,X)$ associated with E, X, S, we want to find the partial derivatives of the determinant $D(E,X)$ of (2.1.3). To differentiate $D(E,X)$ with respect to one of the parameters x_i, we have to differentiate each row of (2.1.3) which contains x_i. This leads to

$$\frac{\partial^\beta D(E,X)}{\partial x_i^\beta} = \sum_{\Lambda_1 \ldots \Lambda_\beta} D(\Lambda_1 \ldots \Lambda_\beta E, X) \tag{3.2.2}$$

(and a similar formula for mixed derivatives), where the sum is taken over all representations of the multiple shifts of order β in the i-th row of E.

We would like to examine the behaviour of $D(E,X)$ as a function of x_i as x_i approaches another knot x_j. This behaviour can be determined through the relationships between shifts, collisions, and coalescence.

THEOREM 3.3: *For* $x_i \to x_j$, $D(E,X)$ *has the Taylor expansion*

$$D(E,X) = \frac{(x_i - x_j)^\alpha}{\alpha!} (-1)^\sigma C\, D(\tilde{E},\tilde{X}) + \ldots, \tag{3.2.3}$$

where C *is defined in Proposition* 3.2(b), $\alpha = \alpha_{ij} = \alpha(E_i \cup E_j)$, *and* σ *is the interchange number.*

For polynomial interpolation, when S is the system (2.2.1), we have

THEOREM 3.4: (i) *If* E_1 *is a horizontal submatrix of* E, *then* D(E,X) *is a polynomial in the variables* x_i *corresponding to* E_1 *of joint degree not greater than* $\gamma(E_1)$.

(ii) *In a single variable* x_i *corresponding to a row* E_i *in* E, D(E,X) *has the highest term*

$$D(E,X) = \frac{x_i^{\gamma}}{\gamma!}(-1)^{\sigma^*} C^* D(E^*,X^*) + \ldots \tag{3.2.4}$$

with $\gamma = \gamma(E_i)$ *and* E^* *defined by the maximal coalescence.*

3.3. APPLICATIONS OF COALESCENCE For order regularity or order singularity, we take rows E_i and E_j to be adjacent in (3.2.3) whereas for real or complex regularity this is not necessary. This remark applies throughout this section.

3.3.1. Suppose that E is a normal Pólya matrix. We can give a very simple proof [19] of Theorem 2.2. If $D(\tilde{E},\tilde{X})$ is not identically zero in (3.2.3), then the same is true of D(E,X). By repeated coalescences of two rows, we finally reduce E to the one row form $\{1,1,\ldots,1\}$, for which the determinant is the Vandermonde determinant of the system $S = \{g_0,\ldots,g_n\}$. Therefore,

THEOREM 3.5: *If* E *is a normal Pólya matrix and the Vandermonde determinant of the system* S *is not identically zero, then* D(E,X) *is not identically zero.*

3.3.2. If the determinant $D(\tilde{E},\tilde{X})$ or $D(E^*,X^*)$ in (3.2.3) or (3.2.4) changes sign, then so does $D(E,X)$. Hence [13], [19]

THEOREM 3.6: *If one of the coalesced matrices* $\tilde{E}$ *or* E^* *is strongly singular, then so is the original matrix* E.

3.3.3. We can exploit the interchange number o occuring in formulas (3.2.3) (and even in (3.2.4) [22]), by comparing them after coalescence of several rows in different ways.

THEOREM 3.7: *Let* $q \geq 3$ *rows* $F_1,\ldots,F_q$ *of* E *be coalesced* $q-1$ *times, in two different ways, to produce the same single row. If* $\sigma_1,\ldots,\sigma_{q-1}$; $\sigma'_1,\ldots,\sigma'_{q-1}$ *are the corresponding interchange numbers, and if*

$$\sigma_1 + \ldots + \sigma_{q-1} \not\equiv \sigma'_1 + \ldots + \sigma'_{q-1} \pmod 2 \tag{3.3.1}$$

then E *is strongly real singular for any system* S. *The same conclusion holds for strong order singularity if, in addition, the rows* $F_1,\ldots,F_q$ *are adjacent and all coalescences are in one direction (i.e. row* i *to row* $i+1$, *or row* $i+1$ *to row* i).

The last statement is required since the coalescence of row i to row $i+1$ contributes the sign from $(x_i - x_{i+1})^{\alpha}$. This contribution is the same on both sides if all the coalescences have this same direction. In general, a less simple statement, taking into account the collisions for coalescences i to $i+1$, is true when the directions of the $q-1$ coalescences are free (see §6.2).

We give a more explicit formulation of Theorem 3.7 for three rows of E; $F_1 = (\ell'_1,\ldots,\ell'_p)$, $F_2 = (\ell''_1,\ldots,\ell''_q)$ and F_3. By $(F)_s$, we mean the positions of the ones of row F pre-coalesced with respect

to row F_s, and by $(F)_{st}$ - the precoalescence with respect to $(F_s \cup F_t)^\circ$. Further, we adopt the convention that two sequences following each other mean that their elements should be written out in the given order. By considering the interchange numbers for the coalescences $((F_1 \cup F_2)^\circ \cup F_3)^\circ$ and $(F_1 \cup (F_2 \cup F_3)^\circ)^\circ$, we obtain [22]

PROPOSITION 3.8: *The matrix* E *is strongly real singular if it contains three rows for which the two sequences*

$$((\ell_1', \ldots, \ell_p')_2 , \ \ell_1'', \ldots, \ell_q'')_3$$

and

$$(\ell_1', \ldots, \ell_p')_{23} (\ell_1'', \ldots, \ell_q'')_3$$

belong to different permutation classes.

Thus, a matrix can have three rows that are so bad that it is singular for any arrangement of ones in the other rows, and for all systems S.

3.3.4. By properly selecting knots x_i, Theorem 3.3 and 3.4 give us

PROPOSITION 3.9: *If*

$$\gamma_i - \sum_{j \neq i} \alpha_{ij} \tag{3.3.2}$$

is odd, where $\gamma_i = \gamma(E_i)$ *and* $\alpha_{ij} = \alpha(E_i \cup E_j)$ *are the coefficients of collision for row* i *in* E, *then* E *is strongly real singular for polynomial interpolation.*

3.3.5. We have restricted ourselves in (3.2.3) to the expansion of $D(E,X)$ in one variable for sake of simplicity. If several knots x_i approach x_j, we obtain multiple coalescence. The expansion will then contain, as its main term, a form of order α in several variables. If this form changes sign, the matrix E must be strongly singular. This requirement is particularly meaningful for real singularity when the values of the variables of the form are unrestricted.

3.4. EXAMPLES: Let $\tilde{E}$ be obtained from E by coalescence. The matrices E and $\tilde{E}$ can be regular, weakly singular or strongly singular in logically nine possible combinations. Theorem 3.6 rules out two combinations: $\tilde{E}$ strongly singular with E being regular or weakly singular. All the other combinations can occur. Indeed, by coalescing the matrices E_1, E_2 and E_3 of (2.4.1) to two row form, we see that $\tilde{E}$ may be regular when E is any of the three types. The following examples of E. Kimchi and N. Richter-Dyn [16] are less trivial.

$$\tilde{E} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(3.4.1)

$$E_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrices $\tilde{E}$ and E_5 are weakly singular; the matrix E_4 is regular; and the matrix E_6 is strongly singular.

§4. INDEPENDENT KNOTS

The connection between the concept of an odd supported sequence and the extended form of Rolle's Theorem was exploited in §2.3 to obtain a simple proof of the Atkinson-Sharma theorem. This simple connection suggests that a more detailed study of the information gained from Rolle's theorem is warranted. The method of independent knots was formulated by Lorentz and Zeller [28] and developed further by Lorentz ([18], [20]), in order to study singular interpolation matrices.

Let E be an $m \times n+1$ interpolation matrix, and f be an n-times differentiable function on $[a,b]$ which is annihilated by E and $X = (x_1,\ldots,x_m) \subset [a,b]$, that is, let f satisfy

$$f^{(k)}(x_i) = 0, \quad e_{ik} = 1 \quad \text{in} \quad E. \tag{4.1}$$

From the zeros of f and its derivatives specified by (4.1), we can derive further zeros by means of Rolle's theorem. A selection of a complete set of such zeros is called a "Rolle set" of zeros.

A *Rolle set* R *for a function* f annihilated by E, X is a collection R_k, $k=0,1,\ldots,n$, of *Rolle sets of zeros* (with multiplicities specified) *for each of the derivatives* $f^{(k)}$ selected inductively as follows: The set R_0 consists of the zeros of f as specified in (4.1). If $R_0,\ldots,R_k$ have already been selected, then we select R_{k+1} according to the following rules:

1° A zero of $f^{(k)}$ in R_k of multiplicity greater than one is also a zero of $f^{(k+1)}$ with its multiplicity reduced by one.

2° All zeros of $f^{(k+1)}$ (including multiplicities) as specified by (4.1) are included in R_{k+1}.

3° For any adjacent zeros α, β of $f^{(k)}$ belonging to R_k, we select, if possible, a zero of $f^{(k+1)}$ between them subject

to the restrictions:

(a) If the new zero is one of the x_i, then it is not listed in (4.1), or

(b) there is an additional multiplicity of x_i as a zero of $f^{(k+1)}$ which is not acknowledged by (4.1).

(c) In the event of (b), if t is the multiplicity of x_i as a zero of $f^{(k+1)}$ given by (4.1), then the multiplicity of x_i in R_{k+1} is defined as follows. We add to (4.1) the equation $f^{(k+t)}(x_i) = 0$, and determine the multiplicity of x_i as a zero of $f^{(k+1)}$ from these equations. This may connect two sequences in E and prescribe a multiplicity larger than $t + 1$.

If a zero does not exist subject to the restrictions in 3℘, then we say that a *loss* occurs at step $k + 1$. A Rolle set constructed without losses in any of its steps is called *maximal*. The function f may have many Rolle sets, some of them may be maximal, while others are not maximal.

Some properties of Rolle sets are immediate consequences of the selection procedure. First of all, the only multiple zeros of $f^{(k+1)}$ in R_{k+1} are among the points x_i in X. Secondly, the extended form of Rolle's theorem shows that a loss will not occur if the rows of E corresponding to x_i between adjacent zeros of R_k contain no odd supported sequences. (This was the connection used in §2.3.) We have

LEMMA 4.1: *If the matrix* E *has no odd supported sequences, then all Rolle sets of a function* f, *annihilated by* E, X, *are maximal.*

The number of Rolle zeros in a maximal Rolle set can be determined by induction:

LEMMA 4.2: *If* f *is annihilated by* E,X, *then for each* k, $k = 0,1,\ldots,n$,

the number of Rolle zeros of $f^{(k)}$ *in a maximal Rolle set for the function* f *is at least* M(k) - k.

Let $S = \{g_0,\dots,g_n\}$ be a system of n-times continuously differentiable functions which are linearly independent on each open subinterval of [a,b]. A set of knots $X \subset [a,b]$ is called *independent* with respect to the system S if for each interpolation matrix E, every polynomial P in S annihilated by E,X has a maximal Rolle set of zeros.

Using a weak form of Markov's inequality, which is valid for each system S, it is possible to show that Rolle zeros for $P^{(k+1)}$ can be selected away from the zeros of $P^{(k)}$. More precisely, (see [37] for algebraic polynomials)

LEMMA 4.3: *There is a monotone increasing function* $\delta(\ell)$, $0 \le \delta(\ell) \le \frac{1}{2}\ell$ *such that if* $\beta - \alpha \ge \ell$, $a \le \alpha < \beta \le b$ *and* $P^{(k)}(\alpha) = P^{(k)}(\beta) = 0$ *for some polynomial* P *in* S *and* k, k = 0,...,n - 1, *then there exists* ξ, $\alpha + \delta(\ell) \le \xi \le \beta - \delta(\ell)$ *for which* $P^{(k+1)}(\xi) = 0$.

For simplicity and without loss of generality, we take [a,b] = [-1,1]. From Lemma 4.3, one derives

THEOREM 4.4: *For each* ρ, $0 < \rho < 1$, *there is a sequence* $\{\pm y_s\}_{s=1}^{\infty}$ *with* $\rho < y_1$ *having the following property. Let* X *be a subset of* $[-\rho,\rho] \cup \{\pm y_s\}$, *and* E *be an interpolation matrix which has no odd supported sequences in the rows corresponding to knots* x_i, $-\rho \le x_i \le \rho$. *Then each polynomial* P *in* S, *annihilated by* E, X, *has a maximal Rolle set.*

For the proof, the points y_s are chosen inductively very close to 1 so that the selection of Rolle zeros in step 3° is always possible. It is essential for the proof of Theorem 5.1 - indeed, it is the main idea - that the "harmless" knots x_i can be made variable in

an interval $(-\rho, \rho)$, arbitrarily close to $(-1, 1)$. Clearly, any knot set X contained in $\{\pm y_s\}$ is independent with respect to the system S.

Theorem 4.4 gives another simple proof of Theorem 2.2 (Windhauer [47] or [20]). Assuming that E is a normal Pólya matrix, we take $X \subset \{\pm y_s\}$ and show that the pair E, X is regular. Indeed, a polynomial P_n annihilated by E, X is identically zero by a standard application of Lemma 4.2.

As has been pointed out in [19], Theorems 2.2 and 2.4 extend to equations of the form

$$D_0 D_1 \dots D_k f(x_i) = c_{ik} \quad (e_{ik} = 1), \tag{4.2}$$

where D_j are certain differential operators of order 1, and S is the Chebyshev system connected with these operators (for a definition of S, see [15, p. 9, p. 378 - 379]).

§5. CLASSES OF SINGULAR MATRICES

The Atkinson-Sharma theorem provides only a sufficient condition for the regularity of matrices; the condition is not necessary. However, a good guiding idea is that this condition is "normally" necessary, or at least necessary under some simple additional conditions. All theorems of this section refer to *interpolation by algebraic polynomials and order singularity*.

THEOREM 5.1: *An $m \times (n+1)$ normal Birkhoff matrix is strongly singular if it contains a row with precisely one odd supported sequence (all other sequences of this row being even or not supported).*

The simplicity of the statement of Theorem 5.1 belies the

difficulties of its proof. This theorem was first proved in the special case when the interior row consists of a single supported one by Lorentz and Zeller [28], and in full generality by Lorentz in [18] and in [20]. The theorem also appears in Karlin and Karon [13] as a consequence of the method of coalescence, but the proof offered there contains an essential gap.

A proof may be based [20] on Theorem 4.4 of the method of independent knots. The row containing a single odd supported sequence is assigned a variable knot y, $-\rho < y < \rho$, while the other knots are taken among the $\pm y_s$ of Theorem 4.4. It is necessary to study the zeros of $P^{(k)}(x) = \frac{\partial^k}{\partial x^k} P(x,y)$, where P is some polynomial in x, depending on the parameter y, and to show that for some y it has the root $x = y$. This is done by examining all zeros of $P^{(k)}(x)$ in $(-1, 1)$.

The matrix E_3 in (2.4.1) shows that Theorem 5.1 fails to hold for matrices with two odd supported sequences in a row. For matrices with an odd number of odd supported sequences in one row some special cases are known (see Section 6.2). However, we have the example of the matrix

$$E_7 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

which is weakly singular. This example appears in [5] and has maximal defect, $d = 2$ (see (2.3.3)). It would be interesting to know whether a three row Birkhoff matrix with an odd number of odd supported sequences can be regular.

A second class of singular matrices is found by restricting all but one row. A row of the interpolation matrix E is *simple* if it contains a single entry one and remaining entries zero. The matrix E is

almost simple if all its rows but at most one are simple.

In the canonical decomposition (2.2.8) of an almost simple matrix, each component matrix is almost simple. More interestingly, the almost simple matrix E is strongly singular if (and only if) one of its component matrices is strongly singular. This observation is used in two ways. First of all it is enough to study the strong singularity of almost simple Birkhoff matrices. Secondly, one can assume the existence of a "smallest" regular or weakly singular almost simple Birkhoff matrix E_0 containing odd supported sequences. Theorem 5.1 and methods of coalescence are used to obtain a contradiction for the matrix E_0. In this way one obtains (Lorentz [19]):

THEOREM 5.2: *An almost simple Birkhoff matrix is regular if it has no odd supported sequences and is strongly singular otherwise.*

An interesting special case of this was established by K. Zeller in 1970. Let E be a normal interpolation matrix with at least three rows and which has non-zero entries only in an interior row i_o and in column zero of the other rows. Let $0 \leq k_o < \ldots < k_p \leq n$ be the positions of all zeros in row i_o. As a corollary to Theorem 5.3, we have

THEOREM 5.3: *A matrix of the type just described is regular if and only if all differences* $k_j - k_{j-1}$, $j = 1,\ldots,p$ *are odd.*

As long as the question of regularity of an interpolation matrix has not been solved, it is natural to ask whether it can be solved "most" of the time. That is, what is the probability that a normal interpolation matrix is regular? A first step in this direction is to count the number of Pólya matrices, or Birkhoff matrices, among all $m \times (n+1)$ normal interpolation matrices. Let $A(m,n)$, $P(m,n)$ and $B(m,n)$ denote the number of all $m \times (n+1)$ normal interpolation, Pólya

and Birkhoff matrices respectively. Here we allow some rows of the matrix to consist only of zeros, and assume $1 \le m \le n$. We shall outline the results of Lorentz and Riemenschneider [25].

THEOREM 5.4: *One has the equations*

$$A(m,n) = \binom{m(n+1)}{n+1}, \quad P(m,n) = \frac{1}{n+2}\binom{m(n+2)}{n+1}, \quad B(m,n) = \frac{1}{n}\binom{mn}{n+1} . \tag{5.2}$$

In particular, one has the strong asymptotic relations

$$P(m,n) \sim \frac{1}{n+2}\left[\frac{m}{m-1}\right]^m A(m,n), \quad n \to \infty$$

$$B(m,n) \sim \left[\frac{m-1}{m}\right]^{2m} P(m,n), \quad n \to \infty \tag{5.3}$$

The first relation (5.3) shows that (for large n) almost all normal matrices do not satisfy the Pólya condition; the second relation shows that there is a large proportion of Birkhoff matrices among the Pólya matrices. Theorem 2.2 implies now that almost all normal matrices are singular. However, whether a given matrix satisfies the Pólya or Birkhoff condition is easy to determine. Thus, the proper question should be: What is the probability that a normal $m \times (n+1)$ Pólya (or Birkhoff) matrix is regular ?

We assume that our $m \times n$ normal matrices satisfy the condition

$$(1+\delta)n / \log n \le m \le n, \tag{5.4}$$

where $\delta > 0$ is a constant.

THEOREM 5.5 [25]: *For each* $\varepsilon > 0$, *there is an* $n_0 = n_0(\varepsilon)$ *with the*

following property. Among all $B(m,n)$ *Birkhoff matrices satisfying* (5.4) *with* $n \geq n_0$, *at most* $\varepsilon B(m,n)$ *are regular. What is more, all but* $\varepsilon B(m,n)$ *of them have supported singletons.*

THEOREM 5.6 [25]: *For each* $\varepsilon > 0$, *there is an* $n_0 = n_0(\varepsilon)$ *so that for* $n \geq n_0$, *among all* $P(m,n)$ *Pólya matrices satisfying* (5.4), *at most* $\varepsilon P(m,n)$ *are regular.*

§6. THREE ROW MATRICES

6.1. ALMOST HERMITIAN MATRICES It is not clear in what respect the theory of regularity becomes simpler for three row matrices. The theorems on coalescence are not strong enough to reduce the general case to this one. Furthermore, we shall see that even very simple three row matrices present considerable difficulties. The results of §6 refer to order regularity.

We shall study $3 \times (n+1)$ normal Birkhoff matrices with the following placement of ones

$$e_{1k} = 1,\ 0 \leq k < p;\ e_{3k} = 1,\ 0 \leq k < q;$$

$$e_{2,k_1} = e_{2,k_2} = 1. \tag{6.1.1}$$

Then $p+q+1=n$ and $k_2 < n$; without loss of generality, we shall also assume that $k_1 < k_2 - 1$, $p \leq q$. For the knot set, we shall take $X = \{-1, x, 1\}$.

One of the smallest matrices of type (6.1.1), E_3 of (2.4.1), has served to show that regular matrices can have odd supported sequences. Generalizing this example, several authors (DeVore, Meir and Sharma [6], Lorentz and Zeller in [19], and Lorentz, Stangler, and Zeller

[26]) studied matrices of the form (6.1.1). It was hoped that in this way the problem of regularity could be completely solved for at least one nontrivial case. The incomplete success of this attempt leads one to believe that it is hardly possible to express the property of regularity in terms of simple properties of elements e_{ik} of a matrix E.

The method of the paper [6] was to apply known facts about the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In [19], the alternation properties of zeros of derivatives of the polynomial $(1+x)^p(1-x)^q$ were used. The first method gives more detailed information while the second method is applicable to wider classes of matrices.

THEOREM 6.1 [26]: *In order that the matrix* (6.1.1) *be regular, it is necessary that*

$$k_1 + k_2 = p + q + 1, \quad k_2 > q \tag{6.1.2}$$

or

$$p \le k_1 < k_2 \le q . \tag{6.1.3}$$

In the case (6.1.2), E *is regular if and only if* $p = q$. *In the case* (6.1.3), *the matrix can be either regular or singular, but the regularity of the matrix* (6.1.1) *implies the regularity of a similar matrix with parameters* k_1', k_2' *instead of* k_1, k_2 *if*

(a) $p \le k_1 \le k_1' < k_2' \le k_2 \le q$

or

(b) $k_1' = k_1 + 1, \quad k_2' = k_2 + 1 \le q.$

(Note that inequalities (a) have been stated incorrectly in the paper [26], namely with $k_1' \le k_1 < k_2 \le k_2'$. This error occured in the last lines of the proof in [26, p. 435]. The inequality (5.5) should

be replaced by the reverse one: "(5.5) $y'_{i+1}(\lambda) \leq y''_i(\lambda)$ for some i.")

The proof of this theorem is by the "chase method". As a candidate for the nontrivial polynomial annihilated by E, X, we take

$$P(x,\lambda) = (x - \lambda)(x + 1)^p(x - 1)^q.$$

We let λ change continuously from $-\infty$ to $+\infty$ and study what happens to those zeros of $P^{(k_1)}$ and $P^{(k_2)}$ whose existence is guaranteed by Rolle's theorem. The matrix is singular exactly when one of these zeros overtakes the other at some x_0, for then $P^{(k_1)}(x_0,\lambda) = P^{(k_2)}(x_0,\lambda) = 0$.

The second part of Theorem 6.1 means that in the triangle given by $p \leq x$, $y \leq q$, $x + 2 \leq y$, there exists a monotone increasing curve $y = \lambda(x)$, with slope at most one, so that E is singular above and on the curve, and regular below it. For $p = 1$, this curve was discovered in [6], and was shown to be the upper branch of the ellipse

$$(6.1.4) \qquad (q + 2)(x + y - 1)^2 - 4(q + 1)xy = 0;$$

moreover, E is weakly singular on this curve and strongly singular above.

For some values of the parameters, the statements of Theorem 6.1 were proved also in [6] and [19]; in addition, it was possible to distinguish between strong and weak singularity. One general case of weak singularity has been found to date, namely when $q = p + 1$, $k_1 + k_2 = p + q + 1$. For more details, consult the paper of DeVore, Meir and Sharma [6].

6.2. CRITERIA BASED ON COALESCENCE Let E be a $3 \times (n+1)$ normal Pólya matrix. For the knot set $X = \{0,x,1\}$, the determinant $D(E,X)$ is a polynomial in x. Clearly, E will be strongly singular if the sign of $D(E,X)$ is different in $(0,\varepsilon)$ and $(\eta,1)$ (ε, $1-\eta$ sufficiently small).

This simple observation is the essence of several criteria for the strong order singularity of E, although the statements themselves appear totally unrelated.

There are several equivalent forms in which this comparison of signs can be carried out. One of them is given by the special case of Proposition 3.8 when the matrix E consists just of the three ordered rows F_1, F_2, F_3 (of course, the interest of Proposition 3.8 is not limited to this case). Another form is one given by Karlin and Karon ([13, Theorem 2.3]): Let $(\ell_1,\ell_2,\ldots,\ell_q) = F_2$ be the positions of the ones in row 2, and let $(\ell_1^1,\ldots,\ell_q^1)$ and $(\ell_1^3,\ldots,\ell_q^3)$ be their positions in the pre-coalescence of row 2 with respect to row 1 and row 3 respectively.

PROPOSITION 6.2 (Karlin and Karon): *If* E *is a* $3 \times (n+1)$ *normal Pólya matrix, then* E *is strongly singular when*

$$(6.2.1)\qquad \sum_{j=1}^{q} \left\{ M(\ell_j - 1) + \ell_j + \ell_j^1 + \ell_j^3 + \sum_{k=\ell_j}^{\ell_j^1 - 1} e_{1,k} + \sum_{k=\ell_j}^{\ell_j^3 - 1} e_{3,k} \right\} \equiv 1 \pmod 2 .$$

In fact, this sum need only be taken over j *for which the* ℓ_j *are the first elements of odd supported sequences of row* 2.

The method of Karlin and Karon was to analyze the signs of the determinants involved by using arguments from the theory of total positivity due to S. Karlin. For the last statement of the theorem, one verifies that adjacent ones, $\ell_{j+1} = \ell_j + 1$, or unsupported ones, contribute 0 mod 2 in (6.2.1).

Both Proposition 6.2 and Proposition 3.8 are consequences of coalescence and the use of the Taylor's formula (3.2.3). This can be explained best of all if we define "directed coalescence" as follows. If F_1, F_2 are two adjacent rows of E, we define the directed coalescence $F_1 \Rightarrow F_2$ as the matrix derived from E by replacing row

F_1 by its pre-coalescence, $\tilde{F}_1$, with respect to F_2. The interchange number, $\sigma_{1,2}$, of the coalescence $F_1 \Rightarrow F_2$ is the number of interchanges needed to bring the sequence of integers $\tilde{F}_1$, F_2 into natural order. (Here a row F is represented by the positions of the ones as in §3.) In a similar way, $F_1 \Leftarrow F_2$ replaces row F_2 by its pre-coalescence with respect to F_1 and has the interchange number $\sigma_{2,1}$. Then (by (3.2.3))

$$\sigma_{2,1} \equiv \sigma_{1,2} + \alpha_{1,2} \pmod 2, \tag{6.2.2}$$

where $\alpha_{1,2} = \alpha(F_1 \cup F_2)$ is the coefficient of collision. For calculating the interchange numbers of further directed coalescences, the positions of the ones in $F_1 \Leftarrow F_2$ and $F_2 \Rightarrow F_1$ are assumed to be in their natural order; then, for example, $\sigma_{3,(2,1)} = \sigma_{3,(1,2)}$ for the directed coalescences $F_3 \Rightarrow (F_1 \Leftarrow F_2)$ and $F_3 \Rightarrow (F_1 \Rightarrow F_2)$ respectively.

Let δ be the sum of the exponents of powers of (-1) giving the signs mentioned at the beginning of this section. We can give several equivalent expressions for $\delta \pmod 2$ by means of directed coalescences and Theorem 3.3. For example, to obtain Proposition 3.8, we estimate the sign of $D(E,X)$ near 0 by means of the coalescences $(F_1 \Rightarrow F_2) \Rightarrow F_3$, and near 1 by means of $F_1 \Rightarrow (F_2 \Rightarrow F_3)$ and obtain in this way

$$\delta \equiv \sigma_{1,2} + \sigma_{(1,2),3} + \sigma_{2,3} + \sigma_{1,(2,3)} \pmod 2. \tag{6.2.3}$$

To obtain Proposition 6.2, we consider the coalescences $(F_1 \Leftarrow F_2) \Leftarrow F_3$ and $F_1 \Leftarrow (F_2 \Rightarrow F_3)$; this gives

$$\delta \equiv \sigma_{2,1} + \sigma_{3,(2,1)} + \sigma_{2,3} + \alpha_{2,3} + \sigma_{(2,3),1} \pmod 2. \tag{6.2.4}$$

(to see the equivalence of (6.2.3) and (6.2.4) directly, one uses (6.2.2) and an extended form of (3.1.7)). Equations (6.2.1) and $\delta \equiv 1 \pmod 2$ can be shown to be equivalent by the careful computation of the collision and interchange numbers of (6.2.4) by means of the quantities in (6.2.1).

Similar ideas give a special case of Propositions 3.8 and 6.2 found by Sharma and Tzimbalario [42]. Let E be a $3 \times (n+1)$ normal Birkhoff matrix with ones in positions $F_1 = (\ell_1', \ldots, \ell_p')$, $F_3 = (\ell_1''', \ldots, \ell_r''')$, and let $k_1 < \ldots < k_{p+r}$ be the positions of the zeros in row 2.

PROPOSITION 6.3: *If* $k_1 > \max(\ell_p' - p, \ell_r''' - r)$ *and if*

$$\sum_{j=1}^{p} (k_{r+j} - k_j) + pr \equiv 1 \pmod 2, \tag{6.2.5}$$

then E *is strongly singular.*

Here we use the coalescences $(F_1 \Rightarrow F_2) \Leftarrow F_3$ and $F_1 \Rightarrow (F_2 \Leftarrow F_3)$ to obtain

$$\sigma_{1,2} + \alpha_{1,2} + \sigma_{3,(1,2)} + \sigma_{3,2} + \sigma_{1,(3,2)} + \alpha_{1,(3,2)} \equiv \delta \pmod 2. \tag{6.2.6}$$

Under the assumptions of the theorem, it is easy to verify that (6.2.5) and $\delta \equiv 1 \pmod 2$ are equivalent by (6.2.6). Sharma and Tzimbalario proved Proposition 6.3 by using properties of special determinants.

An interesting special case of Proposition 6.3 is when the first and last rows of E are Hermitian. If, in addition, there is only one element in the Hermitian first row, the last row is Hermitian, and there is an odd number of odd supported sequences in E, then Proposition 6.3 implies that E is strongly singular (see Passow [35]).

§7. BIRKHOFF'S KERNEL

7.1. DEFINITION AND PROPERTIES Birkhoff's kernel, $K(t) = K_E(X,t)$ is a spline function (piecewise polynomial) intimately connected with Birkhoff interpolation by algebraic polynomials. It appears in the integral identities (7.1.4) and (7.1.6); the second of the these gives a remainder for the interpolation formula. The deepest theorem of Birkhoff in his famous paper [3] was an estimate of the number of zeros of his kernel. It is given here as a theorem about the number of changes of sign of splines with discontinuities linked to the matrix E.

As usual, let u_+^r for $r \geq 0$ mean u^r if $u \geq 0$, and 0 if $u \leq 0$, except that it is not defined if both $u = 0$, $r = 0$. We obtain the kernel $K_E(X,t)$ from the determinant $D(E,X)$ of (2.2.2) by replacing the elements of the first column in (2.2.2) by $(x_i-t)_+^{n-k-1}/(n-k-1)!$:

$$(7.1.1) \qquad K(t) = \left\{ \frac{(x_i - t)_+^{n-k-1}}{(n-k-1)!}, \frac{x_i^{n-k-1}}{(n-k-1)!}, \ldots, \frac{x_i^{-k}}{(-k)!} ; \ e_{ik} = 1 \right\} .$$

If $D_{ik}(X)$ are the algebraic components of the first column elements of the determinant (2.2.2), defined for $e_{ik} = 1$, then

$$(7.1.2) \qquad K_E(X,t) = \sum_{e_{ik}=1} D_{ik}(X) \frac{(x_i - t)_+^{n-k-1}}{(n-k-1)!} .$$

If the knots are ordered, $x_1 < \ldots < x_m$, then the determinant $K(t)$ is a polynomial in t in each of intervals $(-\infty,x_1), (x_1,x_2), \ldots, (x_m,+\infty)$, hence a spline. One sees that $K(t)$ is zero outside of $[x_1,x_m]$. {The same applies to the derivatives of the kernel. Thus, $K^{(j)}(t)$, $j = 0,\ldots,n-1$, is zero outside of $[A_j,B_j]$, where A_j (or B_j) is the smallest (or the largest) of the x_i with $e_{ik} = 1$ for some $k \leq n - j - 1$. } Integrating (7.1.2), we obtain

$$\int_{x_1}^{x_m} K(t)\,dt = D(E,X). \tag{7.1.3}$$

Let A_n denote the class of all $(n-1)$-times continuously differentiable functions f on $[a,b]$ for which $f^{(n-1)}$ is absolutely continuous.

THEOREM 7.1 (Birkhoff's Identity [3]): *Let E be a normal matrix with zeros in the last column. For each $f \in A_n$ and each set of knots X in $[a,b]$,*

$$\sum_{e_{ik}=1} D_{ik}(X) f^{(k)}(x_i) = \int_a^b f^{(n)}(t) K(t)\,dt. \tag{7.1.4}$$

The simplest special case of (7.1.4) is Taylor's formula with integral remainder.

From this theorem, we can obtain mean value formulas. Let $f \in C^n[a,b]$. If $K(t)$ does not change sign, then by using (7.1.4) and (7.1.3), we can obtain

$$\sum_{e_{ik}=1} D_{ik}(X) f^{(k)}(x_i) = f^{(n)}(\xi) D(E,X),\ x_1 < \xi < x_m\ (x_1 < x_i < x_m). \tag{7.1.5}$$

The same is true if K is of arbitrary sign, but f is a polynomial of degree not exceeding n. In both cases, the relations $f^{(k)}(x_i) = 0$, $e_{ik} = 1$ and $D(E,X) \neq 0$ imply $f^{(n)}(\xi) = 0$ for some ξ, $x_1 < \xi < x_m$.

Suppose now that E is a normal Pólya matrix without any restrictions and X is a set of knots for which $D(E,X) \neq 0$. If $f \in A_{n+1}$, there exists a polynomial P_n of degree at most n for which

$$P_n^{(k)}(x_i) = f^{(k)}(x_i),\quad e_{ik} = 1.$$

We would like to get a formula for the difference $f(x) - P_n(x)$. We extend E to $\tilde{E}$ by adding a 0-th row with only a single one, $e_{00} = 1$, and by adding an $(n+1)$-st column of zeros. Let x be different from the x_i, then $\tilde{X}$ is the set of knots obtained by adding x to X. The kernel $\tilde{K}(t) = K_{\tilde{E}}(\tilde{X},t)$ is *the Peano kernel* of the interpolation.

THEOREM 7.2 (Birkhoff): *One has for* $X \subset [a,b]$

$$(7.1.6) \qquad f(x) - P_n(x) = \frac{1}{D(E,X)} \int_a^b f^{(n+1)}(t)\tilde{K}(t)\,dt .$$

{A similar formula holds for the difference $f^{(k)}(x) - P_n^{(k)}(x)$, if we insert the one in the new 0-th row in position k, i.e. $e_{0,k} = 1$.}

7.2. NUMBER OF ZEROS OF SPLINES The deepest theorem of Birkhoff in [3] counts the number of *changes of sign* of a kernel $K_E(X,t)$. This estimate is also valid for other splines (D. Ferguson [8]). A generalization by Lorentz [21] concerns the *number of zeros* of splines. A function S on $(-\infty, +\infty)$ is a spline of finite support of degree n if there are points $x_1 < \ldots < x_m$ so that S is a polynomial of degree $\leq n$ on each interval (x_i, x_{i+1}), has exactly degree n for at least one i, and is zero outside of (x_1, x_m). Let $[a,b]$ be the smallest interval outside of which S vanishes.

THEOREM 7.3 [21]: *Let* S *be a spline of degree* $n-1$ *with knots* $x_1 < \ldots < x_m$. *Let* $E = (e_{ik})$ *be an* $m \times (n+1)$ *matrix of zeros and ones so that* $e_{i,n} = 0$, $i = 1,\ldots,m$, *and that* $e_{ik} = 1$ *whenever* $S^{(j)}$, $j = n-k-1$, *has a jump at* x_i. *If* p *is the number of odd supported sequences of* E, $N+1$ *is the number of ones in* E, *and* Z *is the number of zeros (counting their multiplicities) of* S *in* (a,b), *then*

$$(7.2.1) \qquad Z \leq N - n + p .$$

An essential part of this theorem is the definition of the multiplicity α of a zero z. A zero z may be a maximal interval (x_i,x_j) on which S vanishes, or a point, which may or may not be one of the knots. The definition of α in [21] is a follows. If S is continuous on z, then $\alpha = \min(\beta,\gamma)$, where β and γ are the multiplicities of S to the right and to the left of z. (In [21], max stands erroneously for min, but this was corrected in [20].) If S is discontinuous on z, then $\alpha = 1$ if S changes sign on z, $\alpha = 0$ otherwise. In [11], Jetter gives another apparently larger count for α. A different version of Theorem 7.3 is given in Schumaker [40], with still another definition of α. But also his p in (7.2.1) is different, so that his formulation does not contain Theorem 7.4.

Of course, for point zeros z of S that are not also knots, α is the ordinary multiplicity; and each change of sign of S is a zero. The proof of Theorem 7.3 uses a type of Rolle's theorem for splines.

7.3. APPLICATIONS OF THE BIRKHOFF KERNEL Let E again be a normal Birkhoff matrix. From Theorem 7.3 we derive

THEOREM 7.4 (Birkhoff): *The number of changes of sign in* $[a,b]$ *of the kernel* $K_E(X,t)$ *does not exceed the number* p *of odd supported sequences of* E.

Let $p=0$ and X be given. If $K(X,t)$ is not identically zero, from (7.1.3) we see that $D(E,X) \neq 0$. Thus, the pair E,X is regular. Since a simple additional argument shows that the case $K(t) \equiv 0$ is impossible, this gives a new proof of the Atkinson-Sharma Theorem, Theorem 2.4. Thus, this theorem is essentially contained in Birkhoff's results of 1906.

An interpolation matrix E is called *strongly regular* if for each X, the kernel $K(X,t)$ is of constant sign and does not vanish identically. We have the implications:

E conservative ⇒ E strongly regular ⇒ E regular,

and they cannot be inverted. For the last two statements, an example is given by the matrix E_3 of (2.4.1), which is regular but not strongly regular. Another example (Jaffe [10]) is the strongly regular matrix

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.3.1}$$

which is not conservative (contains odd supported sequences). See [10] for other examples.

It should be noted that the strong regularity of E is equivalent to the validity of the extended Rolle's theorem (Theorem 2.5) for E, for all X, all $f \in C^{(n)}[a,b]$, and some $a < \xi < b$.

§8. APPLICATIONS OF BIRKHOFF INTERPOLATION

8.1. RESTRICTED RANGE APPROXIMATION In the uniform approximation of a function $f \in C[a,b]$ by algebraic polynomials, we may want to restrict the approximating polynomials in some way. For example, we may assume the approximating polynomials P_n are increasing. More generally, we may restrict the P_n to satisfy [27]

$$\varepsilon_j P_n^{(k_j)}(x) \geq 0, \quad a \leq x \leq b, \quad j = 1,\ldots,p, \tag{8.1.1}$$

where $\varepsilon_j = \pm 1$ are given signs, and $1 \leq k_1 < \ldots < k_p \leq n$ are given integers. In analogy with the case $k_p = k_1 = 1$, this is still called the problem of monotone approximation.

Even more generally, one can restrict the ranges of the derivatives

by [38]

$$(8.1.2) \qquad \ell_j(x) \le P_n^{(k_j)}(x) \le u_j(x), \quad j = 1,\dots,p, \quad a \le x \le b.$$

For the bounding functions ℓ_j, u_j, it is necessary [38] to assume that either $u_j \equiv +\infty$, or that u_j is differentiable for $a < x < b$, and that either $\ell_j \equiv -\infty$, or that ℓ_j is differentiable. Problems concerning the uniqueness of a best restricted approximating polynomial are intimately related to Birkhoff interpolation.

THEOREM 8.1: *For each* $f \in C[a,b]$ *there exists a unique polynomial of the class* (8.1.1) *of best uniform approximation to* f.

This was proved by Lorentz and Zeller [27] for $p = 1$, and by R. A. Lorentz [30] for $p > 1$. For the proof, one first finds a Kolmogorov-type theorem, characterizing polynomials of best approximation of the family (8.1.1). These polynomials possess certain extremal properties; among them there exist *minimal polynomials* of best approximation, with the smallest number of extremal properties. For the proof of Theorem 8.1 it is sufficient to show that the difference, Q, between any best approximation polynomial and minimal polynomial is zero. This is achieved by means of the Atkinson-Sharma Theorem if $p = 1$, but requires a more sophisticated proof by induction for $p > 1$ [30].

A similar proof can be given [38] to establish the uniqueness in case of the more general problem (8.1.2). In [30], some additional restrictions are required for ℓ_j, u_j if $k_{j+1} = k_j + 1$. There is an indication in [17] that these restrictions can be avoided.

8.2. SIMULTANEOUS APPROXIMATION The simultaneous approximation of a function $f \in C^{k+1}[a,b]$ and of its derivatives of order k_j by a

polynomial P_n and its respective derivatives means approximation in the metric

$$(8.2.1) \qquad \| f \| = \max_{1 \le j \le p} \max_{a \le x \le b} |f^{(k_j)}(x)| + \max_{a \le x \le b} |f(x)|.$$

We assume $1 \le k_1 < \ldots < k_p \le k$. This problem is again related to Birkhoff interpolation. The set $B(f)$ of the polynomials of best approximation does not necessarily consist of one point. Using the minimal polynomials of (8.1.1), one can find [31] the dimension of $B(f)$ (see Chalmers [4], R. A. Lorentz [31], and also [17]).

8.3. CHEBYSHEV SYSTEMS ON $[-1, +1]$ When is the system $S = \{x^{k_1}, x^{k_2}, \ldots, x^{k_p}\}$, a Chebyshev system on $[-1, +1]$? Using Theorem 5.3, we obtain almost immediately [18], (see also Passow [34]):

THEOREM 8.2: *For integers* $0 \le k_1 < \ldots < k_p$, *the system* S *is a Chebyshev system on* $[-1, +1]$ *if and only if* $k_1 = 0$, *and if all differences* $k_{j+1} - k_j$ *are odd.*

8.4. OTHER TYPES OF INTERPOLATION Among other types of interpolation which are of considerable interest and should be investigated, we mention Birkhoff interpolation by polynomials of several variables, and Birkhoff interpolation by integral functions. For the last problem, there is the dissertation [9].

8.5. BIRKHOFF QUADRATURE MATRICES Corresponding to every Birkhoff interpolation problem, there is a quadrature problem formulated as follows:

Given the $m \times (n+1)$ interpolation matrix E with $N+1$ ones, a set of knots $X = \{x_1 < \ldots < x_m\} \subset [0,1]$, and a measure dg on $[0,1]$, when does there exist a quadrature formula of

(8.5.1) the form

$$\int_0^1 f(x)\,dg = \sum_{e_{ik}=1} c_{ik} f^{(k)}(x_i)$$

which is exact for polynomials of degree n?

The literature on this subject is meagre; besides the interesting paper of Micchelli and Rivlin [32], which treats some special cases, we have an attempt of a general theory in Lorentz and Riemenschneider [24] and Stieglitz [42], [43]. What follows is mainly based on [24].

A pair E, X is called q-*regular with respect to* dg if the problem (8.5.1) is solvable. Let $\mu_j = \int_0^1 x^j/j!\, dg(x)$, $j = 0,1,\ldots,n$, be the moments of dg, and let $B(E,X,dg)$ be the $(N+2) \times (n+1)$ matrix obtained from $A(E,X)$ of (2.2.2) by adding a last row consisting of the moments $(\mu_n,\ldots,\mu_0)$. The following result compares, in particular, regularity and q-regularity:

PROPOSITION 8.3: *A pair* E, X *is* q-*regular with respect to* dg *if and only if*

$$\text{rank } A(E,X) = \text{rank } B(E,X,dg); \tag{8.5.2}$$

in particular, if E, X *is regular, then* $N = n$ *and* E, X *is* q-*regular.*

A matrix E is q-*regular with respect to* dg if the pair E, X is q-regular for any set of knots X with $0 = x_1 < \ldots < x_m = 1$. The restriction $x_1 = 0$, $x_m = 1$ is necessary to obtain a new concept;

without this restriction, q-regularity (at least when $N = n$) would be identical with regularity. The regular matrices are strictly contained in the class of q-regular matrices. Indeed, the following singular matrices (Theorem 5.1) are q-regular with respect to dx on $[0,1]$

$$(8.5.3) \qquad E_8 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad E_9 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} .$$

The emphasis on the measure dg in the definition of a q-regular normal matrix is essential ([24], [43]). The matrix E is q-regular with respect to all non-zero measures $dg = w(x)dx \geq 0$, $w(x) \in L^1[0,1]$ if and only if E is a regular interpolation matrix.

A pair E, X can be q-regular even if the matrix E contains less than $n+1$ ones - it is sufficient to remember Gaussian quadrature formulas. Thus, a pair E, X can be q-regular without the matrix E satisfying the Pólya condition - Theorem 2.2 is not valid for q-regularity. We have, however,

THEOREM 8.4: *If* dg $(\not\equiv 0)$ *is a non-negative measure on* $[0,1]$ *which is not solely supported on the two point set* $\{0,1\}$, *then any matrix* E, *q-regular with respect to* dg, *must satisfy the Pólya condition.*

This was first proved by Stieglitz [42] for $dg = dx$. The general case needs a different proof, which is based, among other things, on differentiation of determinants similar to $D(E,X)$.

A normal Pólya matrix may be so badly singular that it is not q-regular for any reasonable measures. For simplicity, we assume that the measures are of the form

$$(8.5.4) \qquad dg = w(x)dx, \quad w(x) > 0, \quad w(x) \in L^1[0,1] .$$

THEOREM 8.5 [24]: *Let* E *be a normal Pólya matrix with exactly* p *odd sequences in the rows corresponding to* x_i, $0 < x_i < 1$. *If there is an* X *such that*

$$\text{rank } A(E,X) = n + 1 - \left[\frac{p+1}{2}\right], \tag{8.5.5}$$

then E, X *is not* q*-regular with respect to any measure* (8.5.4).

If E contains exactly one odd sequence in its interior rows which is also supported, then Theorem 8.5 and 5.1 imply that E is not q-regular for any measure (8.5.4). The examples in (8.5.3) show that the theorem does not hold if E contains other (non-supported) odd sequences in its interior rows.

A matrix E is called *Gaussian* if $N < n$ and yet the pair E,X is q-regular for some set X of knots and some measure dg.

PROPOSITION 8.6 [8], [24]: *Let* E *be Gaussian with knot set* X *for some measure* (8.5.4). *If* p *additional ones are required so that* E *would not have odd sequences in rows corresponding to* $0 < x_i < 1$, *then*

$$N \geq n - p . \tag{8.5.6}$$

In particular, $N \geq \frac{1}{2}(n-1)$.

A deeper and harder problem is to determine classes of Gaussian matrices. The known cases involve Hermitian and quasi-Hermitian matrices (see [32], [14]), when Proposition 8.6 is essentially invertible. The following interesting theorem is due to Micchelli and Rivlin [32].

THEOREM 8.7: *Let* I, J *be subsets of* $\{0, 1, \ldots, n\}$ *with* $|I| = k$, $|J| = \ell$, *and* $k + \ell - 2s = n + 1$. *If* w(x) *is as in* (8.5.4), *then*

there exists a unique selection of knots $0 < x_1 < \ldots < x_s < 1$ *for which the formula*

$$(8.5.7) \qquad \int_0^1 f(x)w(x)dx = \sum_{i\in I} a_i f^{(i)}(0) + \sum_{j\in J} b_j f^{(j)}(1) + \sum_{r=1}^{s} c_r f(x_r)$$

is (uniquely) solvable to be exact for all polynomials of degree at most n .

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APPROXIMATION POLYNOMIALE PONDÉRÉE ET PRODUITS CANONIQUES

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Soint E une partie fermée de $\mathbb{R}^+$, $p(x)$ une fonction définie sur E. On dit que $p(x)$ est un *poids de Bernstein* si notant par $C_o(E)$ les fonctions continues sur E et tendant vers zero à l'infini alors

$$x^n p(x) \in C_o(E) \quad \text{pour tout entier} \quad n > 0 \quad \text{et}$$

$$\{x^n p(x)\} \quad \text{est totale dans} \quad C_o(E).$$

On note par $H(q,E)$ la classe des fonctions réelles h harmoniques dans le complémentaire de E et verifiant

$$h(z) \geq 0 \quad \text{pour} \quad z \in \mathbb{C} \setminus E$$

$$h(x) \geq q(x) \quad \text{pour} \quad x \in E.$$

On sait que [3] $H(q,E) \neq \emptyset$ avec $q = -\log p$ entraine que $\{x^{n+2} p(x)\}_{n>0}$ est non totale dans $C_o(E)$. On se propose d'étudier le lien entre les poids de Bernstein sur E et la croissance de produits canoniques de la forme

$$W^{\mu}(z) = \int_E \log \left| 1 - \frac{z}{t} \right| d\mu(t)$$

où $d\mu$ est une mesure de Radon sur E. (Cf. [3] pour une première étude dans cette direction.)

Les deux problèmes paraissent differents, ils sont toutefois assez liés. Il est d'abord évident que $W^{\mu}(z)$ est une fonction harmonique dans $\mathbb{C} \setminus E$ et par suite qu'elle fournit un candidat pour etudier si $H(q,E) \neq \emptyset$. Les résultats intéressants vont dans l'autre direction. Les Théorèmes principaux sont énoncés ci dessous.

On dira qu'un point x_o de E est *régulier* si étant donné une fonction continue $h(x)$ quelconque définie sur E la solution du problème de Dirichlet $H(z)$ pour cette fonction satisfait à

$$\lim_{z=x_o} H(z) = h(x_o).$$

L'ensemble des points irréguliers de E est de capacité nulle. On a alors l'énoncé:

1.1 THÉORÈME: *Supposons que* E *est une partie de l'axe réel positif, que l'origine appartienne à* E *et soit un point régulier de* E, *et que l'intersection de* E *avec un ouvert soit ou bien de capacité positive ou bien vide. Supposons que* $q(e^{\sigma})$ *soit une fonction convexe de* σ. *Alors* $H(q,E)$ *est non vide si et seulement si il existe une mesure* μ *positive supportée par* E, *telle que*

$$W^{\mu}(x) \geq q(x) \quad \textit{pour tout} \quad x \in E,$$

le signe d'égalité ayant lieu quasi partout sur E.

En approchant μ par une mesure somme de masses de Dirac on obtient

1.2 PROPOSITION: *Soint* E *une partie de* $\mathbb{R}^+$ *et supposons que* $H(-\log p,E)$

soit non vide ($-\log p(e^{\sigma})$ *étant convexe*). *Alors il existe une partie discrète* $E_1 \subset E$ *telle que* p *ne soit pas un poids sur* E_1.

On considère l'ensemble des fonctions croissantes localement constantes sur le complémentaire de E ; on dira qu'une telle fonction $m \in H^+(E)$ [resp $m \in H^-(E)$] si $\hat{m}(x) \geq 0$ pour tout $x \in E$ [resp. $\hat{m}(x) \leq 0$ pour tout $x \in E$].

On note d'autre part par φ_E la fonction caracteristique de E, par

$$a(x) = \frac{1}{2 \log A} \int_{xA^{-1}}^{xA} \varphi_E(t) \frac{dt}{t} ,$$

où A dèsigne une constante fixée. D'autre part on aura à supposer dans certains cas que

1.3. $$\frac{\log p(x)}{\log p(tx)} = 0(1) \quad (x, tx \in E); \quad |\log t| = 0(1).$$

On a les énoncés suivants donnant des conditions nécessaires et des conditions suffisantes dépendant de fonctions arbitraires de H^+ et H^- .

1.4. *Supposons que* p *satisfasse à* (1.3) *et qu'on puisse trouver* $m \in H^+(E)$ *tel que*

$$\int_E \frac{\log p(x)}{a(x) m(x)} \frac{dx}{x} > -\infty ,$$

alors on peut trouver une patie discrète $E_1 \subset E$, *telle que* $W(E_1, p, 1)$ *soit non vide.*

1.5. *Supposons qu'on poisse trouver* $m \in H^-(E)$ *tel que*

I.5.1. $$\liminf \frac{-\tilde{m}(-x)}{\log p(x)} < +\infty$$

alors $W(E,p,0)$ *est vide.*

REMARQUE: On peut remplacer 1.5.1 par la condition

$$\liminf \frac{-1}{\log p(x)} \left[2 \int_1^x m(t)\frac{dt}{t} - \hat{m}(x) \right] < \infty.$$

1.6. Les énoncés 1.4 et 1.5 ramènent le problème à la détermination de fonctions d'épreuve des classes H^+ et H^-. Ces fonctions pourront être choisies adéquates au cas particulier considéré. Donnons quelques exemples:

D'abord un résultat général.

1.6.1. *Posons* $\lambda(x) = 1/2 \int_1^x \varphi_E(t)(dt|t)$, *alors* $e^{\lambda(x)} \in H^+$.

Ce résultat peut être précisé moyennant des hypothèses supplémentaires.

1.6.2. *Supposons que* $\lim a(x) = 0$, *posons*

$$\lambda^*(x) = \int_1^x \frac{1}{a(t)\,|\log a(t)|}\, \varphi_E(t)\frac{dt}{t}$$

alors quelle que soit la constante k, $e^{k\lambda^*(x)} \in H^+$.

1.6.3. *Supposons que* $\limsup a(x) < 1$, *alors on peut trouver* $\alpha > 1$ *tel que* $e^{\alpha\lambda(x)} \in H^+$.

Ceci montre qu'un résultat de Katznelson [2] ne peut pas être sensiblement amélioré.

1.6.4. *Supposons que* E *soit la réunion d'intervalles* $[a_k, b_k]$, *tels que*

$$\log b_k - \log a_k \to \infty, \text{ alors } e^{\beta\lambda(x)} \in H^-(E) \text{ pour tout } \beta > 1.$$

Ce résultat est à rapprocher de (4.1).

2. BALAYAGE AVEC LE NOYAU DE WEIERSTRASS

La théorie de balayage en potentiel logarithmique est développée soit pour des compacts, soit en utilisant les fonctions de Green par exemple celle du demi-plan. On se propose d'utiliser ici la théorie classique pour un cas non compact le noyau étant $\log |1 - zt^{-1}|$. Si μ est une mesure, on considérera l'intégrale (que l'on supposera toujours absolument convergente)

$$W^\mu(z) = \int \log |1 - zt^{-1}| \, d\mu(t).$$

D'autre part si μ est à support compact on peut considérer également

$$U^\mu(z) = \int \log \left|\frac{1}{z - t}\right| d\mu(t).$$

On a alors

$$W^\mu(z) = U^\mu(0) - U^\mu(z).$$

Nous allons commencer par écrire la fonction q comme un potentiel.

2.1. LEMME de factorisation: *Soit* $q(x)$ *une fonction positive, telle que* $q(e^\sigma)$ *soit une fonction convexe, à croissance linéaire pour* x *assez grand, et telle que*

2.1.1. $$\int_0^1 q(t)\, t^{-3/2}\, dt < \infty$$

alors on peut trouver une fonction $\hat{q}$ *positive vérifiant*

2.1.2. $$\int \log|1 - xt^{-1}|\ \hat{q}(t)\frac{dt}{t} = q(x), \quad x > 0.$$

PREUVE: L'hypothèse de convexité faite sur q implique que

2.1.3 $$q(x) = \int_0^x m(t)\frac{dt}{t} \quad \text{où} \quad m(t) \text{ est croissante.}$$

2.1.1. implique que

$$\int \frac{dm(t)}{t^{1/2}} < \infty .$$

Soit $Q(z)$ la transformée de Mellin de q. Transformons formellement les deux membres de 2.1.2. par Mellin, remarquant que

$$\int_0^\infty \log|1 - u|\ u^{z-1}du = -\frac{\operatorname{cotg} \pi z}{z} \qquad -1 < \operatorname{Re} z < 0$$

on obtient

$$\hat{Q}(z) = z\ \operatorname{tg}(\pi z)Q(z).$$

2.1.3. donne, notant par M la transformée de Mellin-Stieljès de dm

$$Q(z) = \frac{M(z)}{z^2} \qquad \text{d'où}$$

$$\hat{Q}(z) = -\frac{\operatorname{tg} \pi z}{z} \cdot Q(z).$$

Notons par $h(x)$ la fonction ayant pour transformée de Mellin $\frac{-\operatorname{tg} \pi z}{\pi z}$, $-\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}$. $h(x)$ se calcule par résidus et on trouve

$$h(x) = \log|1 + x^{1/2}| - \log|1 - x^{1/2}|$$

sur cette expression il est évident que

2.1.4. $h(x) > 0$ (Inégalité fondamentale)

2.1.5. $h \in L_{1,\alpha}$ $\quad -\frac{1}{2} < \alpha < \frac{1}{2}$

2.1.6. $h \in L_{\infty,\alpha}$ $\quad -\frac{1}{2} \leq \alpha \leq \frac{1}{2}$

où $L_{1,\alpha}$ désigne l'espace des fonctions sommables par rapport à la mesure $x^{\alpha-1}\,dx$; $L_{\infty,\alpha}$ l'espace des fonctions bornées par $x^{-\alpha}$. Ceci étant, justifions les opérations formelles effetuées ci-dessus.

Posons:

2.1.7. $$\hat{q} = h * dm$$

Alors $dm \in M_\alpha$ espace des mesures sommables pour x^α si $\alpha > -\frac{1}{2}$; d'où d'après 2.1.5. $\hat{q}$ est bien défini et $\hat{q} \in L_{1,\alpha}$, $-\frac{1}{2} < \alpha < \frac{1}{2}$.

2.1.8. Comme d'autre part $\log|1 - x| \in L_{1,\alpha}$, $-1 < \alpha < 0$, le produit de composition $k = \log|1 - x| * \hat{q}$ est bien défini et $\in L_{1,\beta}$ $-\frac{1}{2} < \beta < 0$. On a enfin

$$K(z) = \frac{-\operatorname{cotg} \pi z}{z} \cdot \hat{Q}(z) \quad \text{si} \quad \operatorname{Re} z = \beta,\ -\tfrac{1}{2} < \beta < 0,$$

et

$$\hat{Q}(z) = H(z)M(z).$$

D'où

$$K(z) = \frac{M(z)}{z^2} \qquad \text{et}$$

$$k(t) = \int_0^t \frac{M(x)}{x}\,dx \qquad \text{presque partout}$$

les deux membres étant continus ceci vaut partout d'où 2.1.2.

Posons

$$r(x) = \int_0^x \hat{q}(t)\frac{dt}{t}$$

il résulte de 2.1.8. que

2.1.9. $\lim_{x=\infty} r(x)$ existe

Nous allons monter un lemme élémentaire sur l'allure d'un potentiel d'une mesure portée pour l'axe réel.

2.2. LEMME: *Soit* $d\mu$ *une mesure positive à support compact portée par l'axe réel,* $\int \log|z - t|\, d\mu(t) = -U^{\mu}(z)$. *Supposons que*

$$\lim_{y=o} U^{\mu}(x_o + iy) \quad \textit{existe et soit finie .}$$

Alors on a

$$\lim_{y=o} U^{\mu}(x_o + iy) = U^{\mu}(x_o).$$

PREUVE: $U^{\mu}(x)$ est semi-continue inférieurement donc

$$U^{\mu}(x_o) < \lim U^{\mu}(x_o + iy).$$

D'où l'intégrale $-\int \log|1 - x_o t^{-1}|\, d\mu(t)$ est convergente. Remarquant que les points réguliers de E

$$\lim_{y=o} h(x + iy) = q(x)$$

d'où en utilisant 2.2.

$$\int \log|1 - xt^{-1}|\, d\mu(t) = q(x)$$

en tous les points réguliers de E, ceux-ci formant un ensemble partout dense sur E et $W^{\mu}(x)$ étant semi-continue supérieurement, q(x) continue, on obtieni

$$\int \log|1 - xt^{-1}|\, d\mu(t) \geq q(x) \quad \text{pout tout} \quad x \in E, \qquad \text{c.q.f.d.}$$

3. Nous nous proposons dans ce paragraphe de démontrer énoncés 1.2.

3.1. THÉORÈME: *Si* $H(E, -\log p)$ *est non vide, alors on peut trouver une mesure* $d\lambda$ *ayant pour support un ensemble discret* $E_1 \subset E$, *tel que*

$$\int x^n p(x)\, d\lambda(x) = 0\,, \qquad \int |d\lambda| < \infty\,.$$

DÉMONSTRATION: Soit $H(E, -\log p) \neq \emptyset$. Il existe d'après 1.1.

3.1.1. $$\int \log|1 - xt^{-1}|\, d\mu(t) \geq -\log p(x)\,, \quad x \in E\,.$$

Il résulte du fait que cette intégrale est $> -\infty$ que $\int_0^t d\mu = \mu(t)$ est une fonction continue. Soit

$$n(t) = \text{partie entière de } \mu(t)$$

et soit

$$\exp\left[-\int \log(1 - zt^{-1})\, dn(t)\right] = F(z).$$

F(z) est une fonction méromorphe n'admettant que des pôles simples.

D'autre part, posons

$$s(t) = \mu(t) - n(t) \qquad 0 \leq s(t) < 1$$

3.1.3. $$\int \log|1 - zt^{-1}|\, ds(t) = \mathrm{Re} \int \frac{z}{t - z}\, s(t) \frac{dt}{t} = \mathrm{Re} \int_0^{\infty} s(xu) \frac{1 + i\tau}{u - 1 - i\tau}\, \frac{du}{u}$$

où $\tau = yx^{-1}$; soit α tel que $p(\alpha) < 1$, $\alpha > 0$

$$= \int_0^{\alpha/x} + \int_{\alpha/x}^{1/2} + \int_{1/2}^{2} + \int_{2}^{+\infty} .$$

La première intégrale est inférieure à

$$\int_0^{\alpha} \mu(t) \frac{x}{x-t} \frac{dt}{t} = O(1).$$

La seconde à $\log x + O(1)$.

La dernière à $O(1)$.

Reste à évaluer la 3ème intégrale on le fera en posant $s = s_1 + \frac{1}{2}$, $|s_1| \leq \frac{1}{2}$, d'où

$$\mathrm{Re} \int_{1/2}^{2} s(xu) \frac{1+i\tau}{u-1-i\tau} \frac{du}{u} < \frac{1}{2} \int_{1/2}^{2} \left| \mathrm{Re} \frac{1+i\tau}{u-1-i\tau} \right| du + \frac{1}{2} \left| \int_{1/2}^{2} \mathrm{Re} \frac{1+i\tau}{u-1-i\tau} du \right|$$

$$+ \left| \mathrm{Re} \int \frac{1+i\tau}{u-1-i\tau} \left(1 - \frac{1}{u}\right) du \right| .$$

La première intégrale $< -\log \tau + O(1)$, la seconde et la troisième sont $O(1)$, d'où en tenant compte de 3.1.1.

$$|F(x+iy)| < Bp(x) \frac{x^A}{y}, \quad x \in E, \quad |y| < \frac{x}{2} .$$

Soit r un entier $> A + 2$, $b_1, \ldots, b_r$, r points de E distincts; alors on peut trouver une fraction rationnelle $H(z)$ ayant les b_k pour pôles simples et telle que

$$H(z) = o(z^{-r}), \quad z \to \infty$$

$$F_1(z) = F(z)\, H(z) \quad \text{vérifiera}$$

3.1.3. $$|F_1(z)| < Bp(x)|y^{-1}|x^{-2} \quad (z = x + iy).$$

On a

$$F_1(z) = \int \frac{d\rho(t)}{z-t}$$

où $\rho(t) = \Sigma$ Résidus de $F_1(x) < t$.

Le $m^{\underline{e}}$ résidu a vérifiant

$$|a_m| < B \frac{p(a_m)}{a_m^2} .$$

D'autre part on a $n(t) = 0(t^{1/2})$ en vertu de la formule de Carleman d'où $\int \frac{dn(t)}{t^2} < \infty$. Par suite si l'on pose

$$d\lambda(x) = (p(x))^{-1}d\rho(x) \quad \text{alors} \quad \int d\lambda < \infty \quad \text{et}$$

3.1.4. $$F_1(z) = \int \frac{p(t)}{z-t}\, d\lambda(t) = \gamma_n z^{-n} + \int -\frac{t^{n+1}}{z^{n+1}(z-t)}\, p(t)d\lambda(t)$$

où γ_n est le premier moment de $d\rho$ différent de zéro. Prenons $z = x+i$, $x \in E$; cette égalité contredirait 3.1.3. c.q.f.d.

4. FACTORISATION DU NOYAU $\log|1 - u|$

Nous allons décomposer dans l'algèbre de composition sur $(0,\infty)$ le noyau $\log|1 - u|$ dans le produit d'un *noyau positif* et d'un opérateur différentiel. C'est un fait bien connu que l'évaluation des produits de Weierstrass est compliqué par le fait que le noyau $\log|1-u|$ est $\langle 0$ si $u\rangle$ positif sinon la répartition des zéros intervenant localement et globalement : globalement par la fonction $\int^R \frac{n(t)}{t}\, dt$ lié par Nevanlinna à la moyenne m_R sur le cercle R. localement par les perturbations au voisinage des zéros à l'évaluation donnée par la moyenne m_R.

Nous allons donner une famille de telles factorisations du noyau $\log|1-u|$ dépendant d'une fonction arbitraire.

4.1. PROPOSITION: *Soit* $n(t)$ *une fonction à variation localement bornée, définie sur* $[0, +\infty[$, *telle que* $n(t) = O(t^{1/2})$, $t \to \infty$, *et soit* x_o *tel que*

4.1.1. $$n(x_o + h) - n(x_o) = O(\log^{-1} h).$$

Soit $s(t)$ *une fonction satisfaisant aux mêmes conditions. Supposons de plus que* $s(t) \to +\infty$, *que* $n(t)$ *soit nul au voisinage de zéro. Soient*

$$\varphi(x) = \int_o^\infty \log|1 - \frac{x}{t}|\, dn(t)$$

$$\gamma(x) = \int_o^\infty \log|1 - \frac{x}{t}|\, ds(t).$$

Posons

$$F(x,t) = x \int_o^t \frac{s(u)}{u-x} \frac{du}{u}, \quad si \quad 0 < t < x,$$

$$F(x,t) = x \int_t^\infty \frac{s(u)}{x-u} \frac{du}{u}, \quad si \quad x < t < +\infty,$$

$$\rho(x) = \frac{n(x)}{s(x)}.$$

Alors

$$\varphi(x_o) = \int F(x_o,t)\, d\rho(t) + \gamma(x_o)\rho(x_o).$$

PREUVE: L'hypothèse 4.1.1. permet d'écrire

$$\varphi(x_o) = \lim_{\varepsilon=o} \int_o^{x_o-\varepsilon} + \int_{x_o+\varepsilon}^{+\infty} \frac{xn(t)}{x-t} \frac{dt}{t}$$

$$x \int_{x+\varepsilon}^{+\infty} n(t)s^{-1}(t)\frac{s(t)}{x-t}\frac{dt}{t} = F(x,x+\varepsilon)\rho(x+\varepsilon) + \int_{x+\varepsilon}^{+\infty} F(x,t)d\rho(t)$$

d'où

$$\varphi(x) = \lim_{\varepsilon=o}[F(x,x+\varepsilon)\rho(x+\varepsilon) - F(x,x-\varepsilon)\rho(x-\varepsilon)] +$$

$$\lim_{\varepsilon=o} \int_o^{x-\varepsilon} + \int_{x+\varepsilon}^{+\infty} F(x,t)d\rho(t)$$

$\rho(x)$ satisfait en x_o la condition 4.1.1. ce qui permet d'écrire le premier crochet

$$\rho(x) \lim_{\varepsilon=o} [F(x,x+\varepsilon) - F(x,x-\varepsilon)]$$

$$= \rho(x)\lim [\int_o^{x-\varepsilon} + \int_{x+\varepsilon}^{+\infty} \frac{xs(u)}{x-u}\frac{du}{u}] = \rho(x)\gamma(x)$$

d'où 4.1.

4.2. COROLLAIRE: *On a*

$$\varphi(x) = x^{1/2} \int P(\frac{x}{t})d(\frac{n(t)}{t^{1/2}})$$

$$P(t) = \text{V.P.} \int_o^t \frac{u^{-1/2}}{u-1}du = \text{V.P.}\int_t^{+\infty} \frac{u^{-1/2}}{1-u}\,du = \log|\frac{1-t^{1/2}}{1+t^{1/2}}|.$$

PREUVE: Appliquons la proposition 4.1. avec $s(t) = t^{1/2}$, $0 < t < +\infty$. Alors une intégration par résidus donne

$$\gamma(x) = \text{V.P.}x \int_o^{+\infty} \frac{u^{1/2}}{x-u}\frac{du}{u} = 0, \quad x > 0.$$ D'autre part

$$F(x,t) = x \int_0^t \frac{u^{1/2}}{u-x} \frac{du}{u} = x^{1/2} \int_0^{x/t} \frac{v^{1/2}}{v-1} \frac{dv}{v}, \quad t < x$$

$$= x^{1/2} P\left(\frac{x}{t}\right)$$

en tenant compte de la deuxième expression donnée de P dans l'énoncé. Le fait que $\gamma(x) = 0$ montre que la deuxième expression proposée pour P est égale à la première. L'application de 4.1. établit alors 4.2.

Nous allons donner une première application du résultat de factorisation 4.1.

4.3. THÉORÈME: *Soit* p_1 *une fonction négative définie sur* E *et satisfaisant à* 1.9.1. *Posons*

4.3.1. $$\alpha(x) = \int_{E \cap [\frac{x}{2}, \frac{3x}{2}]} \frac{dt}{t} .$$

Soit $s(t)$ *une fonction croissante satisfaisant* 1.9.1, *telle que* $ds(t)$ *ait* E *pour support et que*

4.3.2. $$\int \log\left|1 - \frac{x}{t}\right| ds(t) > p_1(x) \qquad x \in E .$$

Supposons que

4.3.3. $$\int_E \frac{p_1(x)}{\alpha(x)\, s(x)} \frac{dx}{x} > -\infty .$$

Alors il existe une fonction $\varphi(z) < 0$, *harmonique dans le complémentaire de* E, *telle que*

$$\varphi(x) < p_1(x), \quad x \in E.$$

REMARQUE: Si E est composé d'intervalles de longueur logarithmique $> a > 0$,

alors on peut dans 4.3.3, prendre $\alpha(x) = 1$; de plus 4.3.2 a lieu si le produit canonique construit avec ds est simplement positif.

PREUVE: Posons

$$n(x) = s(x)\int_{E\cap[x;+\infty]} \frac{p_1(t)}{\alpha(t)s(t)}\ \frac{dt}{t}\ .$$

Alors dn a pour support E, et

$$d\rho = d(\frac{n}{s}) = -\ \frac{p_1(t)}{\alpha(t)s(t)}\ \frac{dt}{t} \qquad t \in E$$

$$= 0 \qquad t \notin E\ .$$

Escrivons 2.1, remarquant $\rho(x) = o(1)$ et utilisant 4.3.2,

$$\varphi(x) > \int F(x,t)d\rho(t) + o(p_1(x))\ .$$

On a si $\frac{x}{2} < t < x$

$$F(x,t) = \int \frac{s(x\xi)}{\xi - 1}\ \frac{d\xi}{\xi} < \theta\, s(x) \log|1 - \frac{t}{x}|$$

où $\theta > 0$, et une évaluation analogue pour $t \in [x, \frac{3}{2}\ x]$ d'où

$$\int F(x,t)d\rho(t) < \theta_1 \int_{E\cap[\frac{x}{2},\frac{3x}{2}]} \frac{p_1(t)s(x)}{\alpha(t)s(t)}\ \frac{dt}{t} < \theta_2 p_1(x)$$

où θ_1 et θ_2 sont deux constantes numériques positives d'où

$$\varphi(x) < \theta_3 p_1(x) \qquad x \geq x_o\ .$$

Prenons $n_1 = \theta_3^{-1}\, n$ et posons

$$\varphi_1(z) = \int \log|1 - \frac{z}{t}|\ dn_1(t) + A$$

où

$$- A = \sup [\, p_1(x) - \varphi(x)]^- , \quad 0 < x < x_o .$$

Le résultat suivant classique pour les fonctions entières d'ordre $\frac{1}{2}$ s'étend à $\varphi(z)$: il existe une suite infinie de cercles R_k tels que

$$\varphi(R_k e^{i\theta}) \to - \infty \quad \text{uniformément en} \quad \theta .$$

Dans $\{|z| < R_k\} \cap CE$, $\varphi(z)$ est harmonique négative sur la frontière, donc $\varphi(z) < 0$ quel que soit z.

5. Nous allons donner dans ce paragraphe des conditions suffisantes pour que la suite $\{x^n p(x)\}$ soit non totale dans l'espace $C_o(E)$ des fonctions continues sur E nulles à l'infini.

Etant donné $x \in E$ soit I_x le plus grand intervalle tel que $x \in I_x$, $I_x \subset E$. Posons

$$\alpha^*(x) = \inf \{ 1, \int_{I_x} \frac{dt}{t} \}$$

5.1. PROPOSITION: *Les notations étant celles de* 4.3, *supposons que les hypothèses de* 4.3, *sont satisfaites pour* $p_1(x) = \log p(x)$; *de plus supposons ou bien que*

5.1.1. $$\liminf(- p_1(x) \frac{\alpha^*(x)}{\alpha(x)}) > 0$$

ou bien que

$$\log \alpha^*(x) = 0(p_1(x)) \qquad \textit{et que}$$

5.1.2. $$\int_E \frac{- p_1(x)}{\alpha*(x)(1 - \log \alpha^*(x))\ s(x)} \frac{dx}{x} < \infty$$

ou bien que

5.1.3. $$\int_E \frac{-p_1(x)}{\alpha^*(x)s(x)} \frac{dx}{x} < \infty .$$

Alors la suite $\{x^n p(x)\}$ *est non totale dans* $C_o(E)$.

PREUVE: Nous allons construire un produit cononique

$$\varphi(z) = \int \log|1 - zt^{-1}| \, dn(t)$$

tel que la fonction conjuguée soit uniforme modulo 2π dans le complémentaire de E. Supposons par exemple que 5.1.3, est satisfait. Posons:

5.1.4. $$n(x) = s(x)\int_{E\cap[x,+\infty]} \frac{-p_1(t)}{\alpha^*(t)s(t)} \lambda(t)\frac{dt}{t}$$

où $\lambda(t)$ sera une fonction positive localement constante sur E c'est à dire constante sur chaque I_x. On a puisque p_1 et s satisfont à 1.9.2,

$$\int_{I_x} dn < \theta p_1(x)\lambda(x) \quad \text{où} \quad \theta > \theta_o > 0$$

cette quantité tendant vers $-\infty$ on peut déterminer la fonction λ telle que

$$\int_{I_x} dn = \text{entier}, \quad \lambda(x) \to 1, \quad x \to \infty.$$

Soit $g(z)$ la fonction holomorphe dans le complémentaire de E telle que $\log|g(z)| = \varphi(z)$.

Alors la formule 4.1, donne

$$\log |g(x)| < -\int s(x)\, d\left(\frac{n}{s}\right) < \theta_2 p_1(x)$$

d'où

$$|g^N(x)| < p(x), \quad \text{pour } N \text{ entier fixé assez grand, } x \in E .$$

Indiquons rapidement comment modifier un raisonnement classique pour conclure

$$g^N(z) = \int \frac{f(t)}{z - t}\, dt$$

où

$$f(t) = \lim_{y=o} \frac{1}{2i\pi} [g^N(x + iy) - g^N(x - iy)].$$

D'autre part la formule de Nevanlinna donne

$$\int \log |g(re^{i\theta})| d\theta = \int^2 \frac{n(t)}{t}\, dt .$$

Puisque $n(t) \to -\infty$ cette dernière quantité tend vers $-\infty$.

Soit q le premier entier tel que

$$a_q = \int f(t) t^q\, dt \neq 0;$$

alors

$$g^N(z) = z^{-q-1} a_q + z^{-q-2} \int \frac{f(t) t^{q+1}}{(z - t)}\, dt$$

$$\log |g^N(re^{i\theta})| < -(q - 1)\log r + \log(1 + r^{-1} y^{-1} b) + \log |a_q|$$

et d'autre part

$$\int_o^{2\pi} \log(1 + r^{-2} \sin^{-1}\theta\, b)\, d\theta = o(1)$$

ce qui contredirait la formule de Nevanlinna. $f(t)dt$ est ainsi orthogonale à $t^n (n \geq 0)$; d'autre part $f(t) < \frac{p(t)}{\pi}$ d'où $t^n \bar{p}(t)$ est non totale dans $C_o(E)$ où $\bar{p}(t) = t^2 p(t)$ ce qui entraine que $t^n p(t)$ est non totale dans $C_o(E)$.

Lorsque les hypothèses 5.1.1, et 5.1.2, sont satisfaites on définit encore $n(x)$ avec la même formule, les hypothèses 5.1.1, et 5.1.2, ayant pour effet que l'on peut determiner la fonction $\lambda(t)$ telle que $\int d n$ = entier et que $\lambda(t)$ soit bornée sur $(0, +\infty)$.

On peut d'autre part remarquer que les conditions 5.1.2 et 5.1.3 remplacent la condition 2.3 qui n'a plus besoin alors d'être vérifiée.

Nous allons maintenant donner diverses évaluations de la fonction $s(t)$ qui combinées avec 5.1, donneront des conditions néces — saires pour que $x^n p(x)$ soit totale dans $C_o(E)$

5.2. PROPOSITION: *Soit* $\mu(x) = \frac{1}{2} \int_{E \cap [0,x]} \frac{dt}{t}$, *alors on peut prendre dans* 5.1.

$$s(x) = e^{\mu(x)}.$$

PREUVE: Soit $\rho(x) = \frac{s(x)}{x^{1/2}}$ et appliquons 4.2. Alors

$$d\rho = 0 \quad \text{si} \quad x \in E$$

$$d\rho < 0 \qquad x \notin E$$

d'où, $P < 0$, on conclut

$$P * d\rho > 0$$

la fonction $e^{\mu(x)}$ est une fonction s.

COROLLAIRE: *Soit* E *composé d'intervalles de longueur logarithmique* $> h > 0$;

alors si

$$\int -\log p(x)\ e^{-\mu(x)}\ \frac{dx}{x} < \infty$$

la suite $x^n p(x)$ *est non totale dans* $C_o(E)$.

Nous allons donner sous des hypothèses supplémentaires de meilleures évaluations des $s(x)$ possibles.

Posons

$$\bar{\alpha}(x) = \int_{[\frac{x}{2},\ \frac{3x}{2}] \cap E} \frac{dt}{t}$$

5.3. PROPOSITION: *Si* $\lim \bar{\alpha}(x) = 0$, *posons*

$$v(x) = \int_{E \cap [0,x]} \frac{1}{\bar{\alpha}(t)\,|\log\ \bar{\alpha}(t)|}\ \frac{dt}{t}$$

alors quelle que soit la constante λ *on peut dans* 5.1 *prendre*

$$s(x) = e^{\lambda v(x)}.$$

PREUVE: Posons

$$\rho(x) = e^{\lambda v(x)} x^{-1/2} \qquad \text{alors}$$

$$d\rho = \eta\, \rho(x) \frac{dx}{x} \qquad \text{où}$$

$$\eta = -\frac{1}{2} \qquad x \notin E$$

$$\eta = \frac{\lambda}{\bar{\alpha}\,|\log\ \bar{\alpha}|} - \frac{1}{2} \qquad x \in E$$

on en déduit que

$$\rho(x\xi) = \xi^{-1/2}\rho(x)e^{b} \quad \text{où} \quad \frac{1}{2} < \xi < \frac{3}{2}$$

$$0 < b < \lambda\ (-\log\ \alpha)^{-1}$$

Remarquant que le noyau P introduit en 4.2 est croissant sur $[1,+\infty]$ et decroissant sur $[0,1]$, on a

$$\int_{x/2}^{3x/2} P(\frac{x}{t})d\rho(t) > \rho(x)\lambda \int_{xe^{-\frac{\alpha}{2}}}^{xe^{\frac{\alpha}{2}}} P(\frac{x}{t})e^{b}\frac{M}{t} - \rho(x)[\int_{e^{\frac{\alpha}{2}}}^{3/2} P(\xi^{-1})\xi^{-3/2}d\xi$$

$$+ \int_{\frac{1}{2}}^{e^{-\frac{\alpha}{2}}} P(\xi^{-1})\xi^{-3/2}d\xi\ ]\ (1+\alpha).$$

La première intégrale est $> -\rho(x)\lambda$ d'autre part si $x_n = 3^n x$ (n entier positif ou négatif)

$$\int_{\frac{1}{2}x_n}^{\frac{3}{2}x_n} P(\frac{x}{t})d\rho(t) = -\frac{\rho(x)}{2} \int_{\frac{3^n}{2}}^{\frac{3^{n+1}}{2}} P(\xi^{-1})\xi^{-3/2}d\xi(1+o(1))$$

et toutes ces intégrales étant positives d'où N désignant un entier fixé, on a $x \to \infty$

$$\int P(\frac{x}{t})d\rho(t) > \rho(x)(-\lambda + (1+o(1))\int_{3^{-N}}^{3^{N}} - P(\xi^{-1})\xi^{-3/2}d\xi)$$

cette intégrale se calcule et est $> N$ d'où $\int P(\frac{x}{t})d\rho(t) > 0$ si x assez grand et $N > \lambda$, d'où $e^{\lambda v(x)}$ est une fonction poids. c.q.f.d.

REMARQUE: Au lieu de fixer N et de faire tendre $x \to \infty$ on aurait pu prendre $N = N(x)$. Ce calcul ne peut être mené à bien que moyennant des hypothèses supplémentaires sur E. On obtient alors des fonctions poids de la forme $e^{\lambda(x)v(x)}$ où $\lambda(x) \to \infty$. Quel que soit E, ce procédé

de calcul ne permet pas d'obtenir $\lambda(x)$ croissant plus vite que $\log x$. On aurait pu d'autre part définir $\bar{\alpha}(x)$ en considérant l'intervalle $[xB^{-1}, xB]$ au lieu $[\frac{x}{2}, \frac{3x}{2}]$ où B désigne un nombre fixé $B > 1$.

Un cas intéressant est celui où

$$\lim v(x) < \infty , \quad x \to \infty$$

ceci est en particulier le cas où la série

$$\Sigma \frac{-1}{\log I_n} < \infty$$

où I_n désigne le n^e intervalle de l'ensemble E et

$$\log I_n = \log(\int_{I_n} \frac{dt}{t}) .$$

On pourrait comparer cette condition avec la condition de Wiener d'effilement à l'infini. On a l'amélioration suivante de 5.2.

5.4. PROPOSITION: *Si*

5.4.1. $$\limsup \bar{\alpha}(x) = \theta < 1$$

alors on peut trouver une constante λ, $\lambda > 1$ *telle que*

$$s(x) = e^{\lambda\mu(x)} \qquad \textit{vérifie} \quad 5.2.$$

PREUVE:

$$\rho(x) = s(x)x^{-1/2}$$

alors

$$d\rho = -\frac{1}{2}\rho(x)\frac{dx}{x} \qquad x \in E$$

$$= (\frac{\lambda - 1}{2})\rho(x)\frac{dx}{x} \qquad x \notin E$$

et d'autre part

$$\rho(x) = \xi^{\alpha}\rho(x), \quad -\frac{1}{2} < \alpha < \lambda - \frac{1}{2} .$$

Evaluons

$$\int_{\frac{x}{2}}^{\frac{3x}{2}} P(\frac{x}{t})d\rho(t) > \rho(x)(\frac{1-\lambda_1}{2} I_1 + \frac{1}{2} I_2) ,$$

choisissant γ vérifiant $\gamma < e^{\frac{\theta}{2}\log 3}$ où

$$0 < I_1 < \int_{\gamma^{-1}x}^{\gamma x} -P(\frac{x}{t})\frac{dt}{t} \quad \gamma^{1/2}$$

et

$$I_2 > \int_{\gamma x}^{3/2x} -P(\frac{x}{t})\frac{dt}{t} \; ; \quad \text{si } \lambda < \frac{I_1 + I_2}{I_1}$$

l'intégrale sur $[\frac{x}{2}, \frac{3x}{2}]$ sera positive; il en sera de même pour les intervalles $[\frac{3^n x}{2}, \frac{3^{n+1}x}{2}]$ d'où $e^{\lambda\mu(x)}$ est une fonction s .

5.5. REMARQUE: On dira que l'ensemble E est dense à l'infini dans F si posant

$$\alpha(x,B,E) = \int_{[xB^{-1}, xB]\cap E} \frac{dt}{t}$$

on a, B fixé,

5.5.1. $\liminf \frac{\alpha(x,B,E)}{\alpha(x,B,F)} > 0$ pour tout $B > 1$.

Posons

$$\mu^*(x) = \frac{1}{2}\int_{F\cap[0,x]} \frac{dt}{t} .$$

Alors si 5.5.1. vaut, $e^{\lambda\mu^*(x)}$ est une fonction poids pour E, quelle

que soit la constante λ , vérifiant $\lambda < 1$. La démonstration s'effectue comme en 5.4.

6.1. THÉORÈME: *Supposons que* $H(E,q)$ *soit non vide, alors quelle que soit la fonction* $r \in R^+(E)$, *on a*

6.1.1. $$\lim \frac{1}{q(x)} W^r(-x) = +\infty .$$

PREUVE: Supposons que 6.1.1. ne soit pas satisfaite la limite inférieure du premier membre de 6.1.1. sera égale à $b < \infty$. Nous allons construire une fonction q_1 telle que $q = o(q_1)$ et $H(E,q_1)$ soit non vide. Soit σ la mesure harmonique du complémentaire de E dans $\mathbb{C}$, alors

$$\int q'(x)(\sigma(\infty) - \sigma(x))\, dx < \infty.$$

Soit $\eta(x)$ une fonction croissante tendant vers l'infini telle que

$$\int \eta(x) q'(x)(\sigma(\infty) - \sigma(x))dx < \infty ,$$

on prendra q_1 tel que $q_1' = \eta q'$.

Alors

6.1.2. $$\liminf \frac{W^r(-x)}{q_1'(x)} = 0.$$

Soit $h(z)$ une fonction harmonique appartenant à $H(E,q_1)$, et μ la mesure associée par 1.1.

$$\liminf \frac{W^r(-x)}{W^\mu(x)} = 0$$

Remarquons que le maximum de W^r sur $|z| = R$ est atteint sur l'axe $x < 0$, son minimum sur $x > 0$, on obtient qu'il existe une suite

$R_k \to \infty$ telle que

$$|W^r(z)| < \frac{1}{k} W^\mu(z), \; |z| = R_k .$$

D'autre part on a sur E

$$|W^r(x)| < 0 < \frac{1}{k} W^\mu(x) , \quad x \in E$$

d'où en remarquant que W^r et W^μ sont harmoniques dans $\{z; |z| < R_k\} \cap \mathbb{C}\setminus E$. On obtient

$$|W^r(z)| < \frac{1}{k} W^\mu(z) , \quad |z| < R_k$$

k étant arbitrairement grand

$$W^r(z) \leq 0 .$$

Appliquons la formule de Nevanlinna

$$0 < \int_0^t r(u)\frac{du}{u} = \text{Moyenne de } W^r \text{ sur } |z| = t < 0$$

contradiction.

On peut remarquer que l'on aurait pu remplacer simplement l'hypothèse $H(E,q)$ non vide par celle qu'il existe une fonction $F(z)$, holomorphe dans le complémentaire de E, telle que $|F(x)| < W^{q(x)}$, $x \in E$. Il suffisait de remplacer dans le raisonnement ci-dessus $W^\mu(z)$ par la minorante harmonique de $-\log|F(z)|$.

6.2. PROPOSITION: *On a* $W^r(-x) < 2\int_0^x r(t)\frac{dt}{t} - W^r(x)$.

DÉMONSTRATION:

$$W^r(-x) = \int_0^{+\infty} \frac{x}{x+t}\, r(t)\frac{dt}{t}$$

$$W^r(-x) - \int_0^x \frac{r(t)}{t}\, dt = -\int_0^x \frac{1}{x+t} r(t)\, dt + \int_x^{+\infty} \frac{x}{x+t}\, r(t)\frac{dt}{t}$$

$$= \int_0^{\infty} \log\left(1 + \theta\left(\frac{x}{t}\right)\right) dr(t)$$

où $\theta(u) = u$, si $0 < u < 1$; et $\theta(u) = \frac{1}{u}$, si $1 < u < +\infty$.

De même on a

$$W^r(x) - \int_0^x r(t)\frac{dt}{t} = \int_0^{+\infty} \log\left[1 - \theta\left(\frac{x}{t}\right)\right] dr(t).$$

L'inégalité $-\log|1 - v| > \log|1 + v|$ donne

$$-\int_0^{\infty} \log\left|1 - \theta\left(\frac{x}{t}\right)\right| dr(t) > \int_0^{\infty} \log\left|1 + \theta\left(\frac{x}{t}\right)\right| dr(t), \quad \text{d'où 6.2.}$$

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SPACES OF DIFFERENTIABLE FUNCTIONS AND THE APPROXIMATION PROPERTY

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PREFACE When the author started to deal with the subject of the present article, he did not know too much of the various ways how one can do calculus in (real) topological vector spaces. He was mainly interested in a generalization of the theory of distributions to infinite dimensional locally convex (l.c) spaces, by means of duality theory and with relations to infinite dimensional holomorphy. Therefore he began studying spaces of differentiable functions on l. c. spaces by analyzing the definitions of C^n-functions used by Aron [3], Bombal Gordón and González Llavona [10] and Yamamuro [24]. Then he (re-) invented the notion of n times continuously γ-differentiable functions on an open subset Ω of a l.c. space E (2.4), where γ is an arbitrary covering of E by bounded subsets. As he realized later, this notion had been introduced with a slightly different definition by Keller [18] already. Further investigations showed that many results known to be true on open subsets of $\mathbb{R}^N$ carry over at least to Fréchet spaces or strong duals of Fréchet-Montel spaces if one takes as γ the system of all compact subsets of E. In order to be precise, we will now sketch the main results of the article.

In the first section most of the definitions are stated as well as some results which will be used in the sequel. In the second part

we introduce the l.c. space $C^n_\gamma(\Omega,F)$ of n times continuously γ-differentiable functions on an open subset Ω of the l.c. space E with values in the l.c. space F, where n is a natural number or ∞ and γ is a system as introduced above. For open subsets Ω in a quasi-complete l.c. space E, we show that $C^\infty_{co}(\Omega)$ is a Schwartz space if and only if E'_{co} is a Schwartz space.

In the third section we give a sufficient condition for the approximation property of $C^n_{co}(\Omega)$. The proof of the corresponding theorem 3.5 is a generalization of the proof given by Bombal Gordón and González Llavona [10] in the case of Banach spaces. (Actually an analysis of [10] led to the result presented here.) Since the proof uses a criterion for the a.p. due to Schwartz [22], we first characterize $C^n_\gamma(\Omega) \varepsilon F$ as a topological subspace of $C^n_\gamma(\Omega,F)$. The main lemma (3.3) for theorem 3.5 goes back to [10] as well as to Prolla and Guerreiro [20] (in the case of Banach spaces). It has also some further applications which generalize parts of the results of [20] and which may be of interest in connection with the theorem of Paley-Wiener - Schwartz for the elements of $C^\infty_{co}(E,\mathbb{C})'$.

In the last section we prove that, for open subsets Ω_1 and Ω_2 in certain l.c. spaces E_1 and E_2 respectively, there exists a natural topological isomorphism between $C^\infty_{co}(\Omega_1,C^\infty_{co}(\Omega_2))$ and $C^\infty_{co}(\Omega_1 \times \Omega_2)$. By the results on the a.p. of $C^\infty_{co}(\Omega)$ this also implies a representation of $C^\infty_{co}(\Omega_1 \times \Omega_2)$ as ε-tensor product $C^\infty_{co}(\Omega_1) \check{\otimes}_\varepsilon C^\infty_{co}(\Omega_2)$.

While finishing his investigations, the author received the preprint [12] of Colombeau, where spaces of C^∞-functions on Schwartz bornological vector spaces are studied. Colombeau has pointed out to the author that any function in $C^n_{co}(\Omega,F)$ is a C^n-function in Silva's large sense if the quasi-complete space E is given the compact bornology. For Fréchet spaces and for strong duals of Fréchet - Schwartz spaces E both notions coincide. This shows that the bornological setting is more general as far as the Schwartz property of $C^\infty_{co}(\Omega)$ is

concerned. However, it is not known how to prove the results of this paper with bornological methods if E is the strong dual of a Fréchet-Montel space only.

Let us come back to distributions again: It should be remarked that obviously some results in this article can also be regarded as propositions on the dual of $C^{\infty}_{co}(\Omega)$, a space which is a natural generalization of the distributions with compact support. The results mentioned in remark 4.8 will be the subject matter of a subsequent paper.

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1. PRELIMINARIES

In this section we shall fix the notation, recall some definitions and state some results which will be applied later. We shall use the theory of locally convex (l.c) spaces as it is presented e.g. in the books of Horváth [17], Köthe [19] and Schaefer [21].

Throughout this article, a l.c. space always means a *real* Hausdorff l.c. space, because we only want to deal with real differentiable functions.

1. Let E and F be l.c. spaces and let γ be a system of bounded subsets of E which cover E. Then, on the space L(E,F) of all continuous linear maps from E into F one can introduce the corresponding γ-topology of uniform convergence on the sets in γ; the resulting l.c. space is denoted by $L_{\gamma}(E,F)$. It is well known that this topology does not change if the system is enlarged in such a way that

it is directed under inclusion and that any subset of a set in γ belongs to γ. We shall always assume that γ has these properties.

By γ_σ, γ_{co}, γ_c and γ_b we denote the systems of all finite dimensional bounded, all compact, all precompact and all bounded subsets of E. The corresponding spaces $L_\gamma(E,F)$ are denoted by $L_\sigma(E,F)$, $L_{co}(E,F)$, $L_c(E,F)$ and $L_b(E,F)$. We write E'_γ instead of $L_\gamma(E, \mathbb{R})$ and $L_\gamma(E)$ instead of $L_\gamma(E,E)$.

2. DEFINITION: Let E and F be l.c. spaces, γ a system of bounded sets which covers E and $n \in \mathbb{N}$.

a) By $\mathcal{L}(E^n,F)$ we denote the linear space of all n-linear mappings from E^n into F.

b) Assume $n \geq 2$; $u \in \mathcal{L}(E^n,F)$ is called γ-*hypocontinuous*, if for any k with $1 \leq k \leq n$, any $S \in \gamma$ and any neighbourhood W of zero in F there is a neighbourhood V of zero in E such that $u(S^{k-1} \times V \times S^{n-k}) \subset W$.

c) We define the space $\mathcal{L}_\gamma(E^n,F)$ for $n=1$ by $L(E,F)$ and for $n \geq 2$ by

$$\mathcal{L}_\gamma(E^n,F) = \{u \in \mathcal{L}(E^n,F) \mid u \text{ is } \gamma\text{-hypocontinuous}\}.$$

Since, obviously, any $u \in \mathcal{L}_\gamma(E^n,F)$ is bounded on S^n for any $S \in \gamma$, we can endow $\mathcal{L}_\gamma(E^n,F)$ with the topology of uniform convergence on the system $\gamma^n = \{S^n \mid S \in \gamma\}$. As in the case of continuous linear mappings we write $\mathcal{L}_\sigma(E^n,F), \mathcal{L}_{co}(E^n,F), \ldots$ if $\gamma = \gamma_\sigma, \gamma_{co}, \ldots$. The elements of $\mathcal{L}_\sigma(E^n,F)$ are called *separately continuous n-linear mappings*. We write $\mathcal{L}_\gamma(E^n)$ instead of $\mathcal{L}_\gamma(E^n, \mathbb{R})$.

d) $u \in \mathcal{L}(E^n,F)$ is called *symmetric*, if for any permutation π of n elements and any $x = (x_1, \ldots, x_n) \in E^n$ we have

$u(x_{\pi(1)},\dots,x_{\pi(n)}) = u(x_1,\dots,x_n)$. We define

$$\mathcal{L}_\gamma^s(E^n,F) := \{u \in \mathcal{L}_\gamma(E^n,F) \mid u \text{ is symmetric}\}.$$

The closed linear subspace $\mathcal{L}_\gamma^s(E^n,F)$ of $\mathcal{L}_\gamma(E^n,F)$ is endowed with the induced topology.

The proof of the following lemma is an easy exercise.

3. LEMMA: *For any* $u \in \mathcal{L}_\gamma(E^n,F)$ *the following holds true:*

a) *For any* $S \in \gamma$ *the restriction* $u \mid S^n$ *is uniformly continuous.*

b) *For any* k, *with* $1 \le k \le n$, *and any* $S \in \gamma, u \mid S^{k-1} \times E \times S^{n-k}$ *is continuous.*

4. DEFINITION: Let E and F be l.c. spaces and γ a system of bounded subsets of E covering E.

a) The l.c. spaces $L_\gamma^n(E,F)$, $n \in \mathbb{N}$, are defined inductively by $L_\gamma^0(E,F) := F$ and $L_\gamma^{n+1}(E,F) := L_\gamma(E,L_\gamma^n(E,F))$.

b) There is a unique sequence $(\varphi^n)_{n \in \mathbb{N}}$ of linear mappings $\varphi^n : L_\gamma^n(E,F) \to \mathcal{L}(E^n,F)$ satisfying $\varphi^1 = \mathrm{id}_{L(E,F)}$ and $\varphi^{n+1}(u)[x_1,\dots,x_{n+1}] = \varphi^n(u(x_1))[x_2,\dots,x_{n+1}]$. Since φ^n is injective for any $n \in \mathbb{N}$ we may (and shall) regard $L_\gamma^n(E,F)$ as a linear space of n-linear mappings on E^n with values in F.

c) Let $n \in \mathbb{N}$ be fixed. An element $u \in L_\gamma^n(E,F)$ is called *symmetric*, if the corresponding n-linear mapping $\varphi^n(u)$ is symmetric. We define

$$L_\gamma^{ns}(E,F) := \{u \in L_\gamma^n(E,F) \mid u \text{ is symmetric}\}$$

and endow this linear space with the l.c. topology induced by $L^n_\gamma(E,F)$.

5. PROPOSITION: *Let* E *and* F *be l.c. spaces and let* γ *be a covering system of bounded subsets of* E. *Then the mapping* $\varphi^n : L^{ns}_\gamma(E,F) \to \mathcal{L}^s_\gamma(E^n,F)$ *is a topological isomorphism.*

PROOF: The proof is by induction on n. For $n=1$ the statement is obviously true. Hence, let us assume that φ^j is a topological isomorphism for $1 \leq j \leq n$. We shall prove that $\varphi^{n+1} : L^{n+1,s}_\gamma(E,F) \to \mathcal{L}^s_\gamma(E^{n+1},F)$ is a topological isomorphism. This will be done in several steps.

a) For any $u \in L^{n+1,s}_\gamma(E,F)$ $\varphi^{n+1}(u)$ is γ-hypocontinuous: Let $S \in \gamma$ and a neighbourhood W of zero in F be given. Then $U^n_{S,W} := \{m \in \mathcal{L}^s_\gamma(E^n,F) \mid m(S^n) \subset W\}$ is a neighbourhood of zero in $\mathcal{L}^s_\gamma(E^n,F)$. By induction hypothesis φ^n is a topological isomorphism. Since $u : E \to L^{ns}_\gamma(E,F)$ is continuous, there is a neighbourhood V of 0 in E, such that $\varphi^n \circ u(V) \subset U^n_{S,W}$, i.e. $u(V)[S^n] \subset W$. But this implies $\varphi^{n+1}(u)[V \times S^n] \subset W$. Since u is symmetric, this shows $\varphi^{n+1}(u) \in \mathcal{L}^s_\gamma(E^{n+1},F)$.

b) φ^{n+1} is bijective:

The injectivity of φ^{n+1} is clear. Let us show the surjectivity of φ^{n+1}. Take any $m \in \mathcal{L}^s_\gamma(E^{n+1},F)$ and define, for any $x_1 \in E$, $\hat{u}(x_1) : E^n \to F$ by $\hat{u}(x_1)[x_2,\ldots,x_{n+1}] := m(x_1,\ldots,x_{n+1})$. Then $\hat{u}(x_1)$ is in $\mathcal{L}^s_\gamma(E^n,F)$, since for any neighbourhood V of zero in E we have $\hat{u}(x_1)[V \times S^{n-1}] = m(\{x_1\} \times V \times S^{n-1})$. Thus we have defined a mapping $\hat{u} : E \to \mathcal{L}^s_\gamma(E^n,F)$, which is linear and continuous, because for any neighbourhood W of 0 in F there is a neighbourhood V in E such that $\hat{u}(V)[S^n] = m(V \times S^n) \subset W$, i.e. $\hat{u}(V) \subset U^n_{S,W}$. By induction

hypothesis $u := (\varphi^n)^{-1} \circ \hat{u}$ is in $L_\gamma^{n+1,s}(E,F)$ and $\varphi^{n+1}(u) = m$.

c) φ^{n+1} is a topological isomorphism:
This is easy to see, since $\{U_{S,W}^{n+1} \mid S \in \gamma, W \text{ neighbourhood of } 0 \text{ in } F\}$ is a fundamental system of neighbourhoods in $\mathcal{L}_\gamma^s(E^{n+1},F)$, while the sets $\tilde{U}_{S,W}^{n+1} = \{u \in L_\gamma^{n+1,s}(E,F) \mid u(S) \subset (\varphi^n)^{-1}(U_{S,W}^n)\}$ describe a fundamental system of neighbourhoods in $L_\gamma^{n+1,s}(E,F)$ if S runs through γ and W runs through the neighbourhoods of 0 in F.

6. DEFINITION: Let E and F be l.c. spaces.

a) The *ε-product of* E *and* F is defined as $E \varepsilon F := L_e(F_c', E)$, where e denotes the topology of uniform convergence on the equicontinuous subsets of F' (cf. Schwartz [22] or Bierstedt and Meise [5]).

b) The mapping $j : E \otimes F \to E \varepsilon F$, $j(\sum_{i=1}^n e_i \otimes f_i)[y'] := \sum_{i=1}^n \langle f_i, y' \rangle \cdot e_i$ is injective, hence $E \varepsilon F$ induces via j a l.c. topology on $E \otimes F$, called the *injective* or *ε-topology*. The completion $E \check{\otimes}_\varepsilon F$ of $E \otimes_\varepsilon F$ is called the *injective* or *ε-tensor product of* E *and* F.

c) E has the approximation property (a.p.) in the sense of Grothendieck, if $E' \otimes E$ is dense in $L_c(E)$ (cf. Schaefer [21], III, 9.1).

We shall use the following result of Schwartz [22], Ch. 1, §1, Prop. 11, in the form stated in Bierstedt and Meise [6].

7. THEOREM: *A quasi-complete l.c. space* E *has the a.p. iff for every Banach space* F *the algebraic tensor product* $E \otimes F$ *is dense in* $E \varepsilon F$.

8. k *and* $k_{\mathbb{R}}$*-spaces*: A completely regular topological space X is

a k-*space* ($k_{\mathbb{R}}$-*space*) if for any topological space Y (Y = $\mathbb{R}$ or equivalently Y a completely regular topological space) a function $f : X \to Y$ is continuous iff $f \mid K$ is continuous for any compact subset K of X. By Arhangel'skii [1] (Blasco [8]) open subsets of k-spaces ($k_{\mathbb{R}}$-spaces) are k-spaces ($k_{\mathbb{R}}$-spaces) again.

2. SPACES OF DIFFERENTIABLE FUNCTIONS

In this section we introduce l.c. spaces of n times continuously γ-differentiable functions and investigate their topological properties. Because of the applications in section 3, we are mainly interested in the completeness and the Schwartz property of such spaces. We begin by recalling some definitions.

1. DEFINITION: Let E and F be l.c. spaces, Ω an open subset of E, f a function on Ω with values in F and γ a system of bounded sets in E which covers E. f is called γ-*differentiable at a point* $a \in \Omega$ if there exists $u \in L(E,F)$ such that for every $S \in \gamma$ $\lim_{t \to 0} \frac{1}{t}(f(a+th) - f(a) - u(th)) = 0$ uniformly in $h \in S$ (i.e. for any $S \in \gamma$ and any continuous semi-norm q on F, there is $\delta > 0$ such that for any $t \in \mathbb{R}$, with $0 < |t| \leq \delta$,

$$\sup_{h \in S} q\left(\frac{1}{t}\,(f(a+th) - f(a) - u(th)) \leq 1\right).$$

Obviously u is uniquely determined by f and a; u is called the γ-*derivative of* f *in* a. We write f'(a) instead of u. If γ is the system of all bounded (finite) subsets of E, f is called *Fréchet-(Gâteaux-) differentiable at* a. f is called γ-*differentiable on* Ω, if f is γ-differentiable at any $a \in \Omega$.

For Gâteaux-differentiable functions there exist several generalizations of the classical mean value theorem (see e.g. Yamamuro

[24], 1.3). We shall use the following one, which is a consequence of the Hahn-Banach theorem and a result of classical calculus.

2. LEMMA: *Let* E *and* F *be l.c. spaces,* Ω *an open subset in* E *and let* $a,b \in \Omega$ *be such that* $S_{[a,b]} := \{a + t(b - a) \mid t \in [0,1]\}$ *is contained in* Ω. *Assume furthermore that* $f : \Omega \to F$ *is Gâteaux-differentiable at any* $x \in S_{[a,b]}$ *and that the mapping* $g : [0,1] \to L_\sigma(E,F)$, $g(t) := f'(a + t(b - a))$, *is continuous. Then the following holds true:*

$$f(b) - f(a) = \int_0^1 f'(a + t(b - a))[b - a]\,dt.$$

The following lemma indicates that γ-differentiability of a function f is already implied by Gâteaux differentiability and a continuity property of the derivative (see also Keller [18], 1.2.1 and Yamamuro [24], 1.4.4).

3. LEMMA: *Let* E *and* F *be l.c. spaces,* Ω *an open subset of* E *and* $f : \Omega \to F$ *Gâteaux differentiable on* Ω. *If* $f' : \Omega \to L_\gamma(E,F)$ *is continuous, then* f *is* γ*-differentiable on* Ω.

PROOF: Let a be any point in Ω, S any element of the system γ of bounded subsets of E and let q be any continuous semi-norm on F. By the continuity of f' in a, for $\varepsilon > 0$ there exists a convex balanced neighbourhood U of zero in E such that $a + U \subset \Omega$ and such that for any $x \in a + U$

$$\sup_{s \in S} q((f'(x) - f'(a))[s]) \leq \varepsilon .$$

Since S is bounded in E, we can find $\delta > 0$ with $\delta S \subset U$. By lemma 2 we have for any t with $0 < |t| \leq \delta$ and any $h \in S$:

$$\frac{1}{t}(f(a+th) - f(a) - f'(a)[th]) = \frac{1}{t}\left(\int_0^1 (f'(a+\tau th)[th] - f'(a)[th])\,d\tau\right.$$

$$= \int_0^1 (f'(a + \tau th) - f'(a))[h]\,d\tau.$$

This implies

$$\sup_{h\in S} q\left(\frac{1}{t}(f(a + th) - f(a) - f'(a)[th])\right) \leq \sup_{h\in S} \int_0^1 q((f'(a+\tau th) - f'(a))[h])\,d\tau \leq \varepsilon .$$

Hence f is γ-differentiable at a.

4. DEFINITION: Let E and F be l.c. spaces, $\Omega \neq \emptyset$ an open subset of E and γ a system of bounded subsets of E which covers E. For $n \in \mathbb{N}_\infty (:= \mathbb{N} \cup \{\infty\})$ we define *the space of n times continuously γ-differentiable functions on Ω with values in F* as

$$C^n_\gamma(\Omega,F) := \{f : \Omega \to F \mid \text{for any } j \in \mathbb{N}_0 \text{ with } 0 \leq j < n+1$$

$$f_j \in C(\Omega, L^j_\gamma(E,F)) \ (f_0 := f) \text{ and for any } j \in \mathbb{N}_0$$

$$\text{with } 0 \leq j < n \text{ is } f_j \text{ Gâteaux-differentiable}$$
$$\text{on } \Omega \text{ and } f_j' = f_{j+1}\} .$$

The vector space $C^n_\gamma(\Omega,F)$ is endowed with the *topology of uniform convergence of the derivatives up to the order n on the compact subsets of Ω*. This topology is given by the system $\{p_{\ell,K,S,q}\}$ of seminorms, where ℓ is any integer with $0 \leq \ell < n+1$, K is any compact subset of Ω, S is any element of γ and q is any continuous seminorm on F, and where $p_{\ell,K,S,q}$ is defined as

$$p_{\ell,K,S,q}(f) := \sup_{0\le j\le \ell} \sup_{x\in K} \sup_{y\in S^j} q(f_j(x)[y]).$$

In the sequel we shall write $f^{(j)}$ instead of f_j.

REMARKS: a) By lemma 3, f_j is really γ-differentiable on Ω for $0 \le j < n$.

b) A function f is in $C^n_\gamma(\Omega,F)$ iff it is of class C^n_γ on Ω in the sense of Keller [18], 2.5.0. The advantage of definition 4, however, will become clear pretty soon (see e.g. proposition 5).

c) Obviously we have for any semi-Montel space E:

$$C^n_{co}(\Omega,F) = C^n_c(\Omega,F) = C^n_b(\Omega,F).$$

5. PROPOSITION: *Let the open subset Ω of the l.c. space E be a $k_{\mathbb{R}}$-space and let $n \in \mathbb{N}_\infty$ be given. Assume that $L^j_\gamma(E,F)$ is complete for any $j \in \mathbb{N}_0$ with $0 \le j < n+1$. Then $C^n_\gamma(\Omega,F)$ is complete.*

PROOF: Let $(f_\alpha)_{\alpha\in A}$ be any Cauchy net in $C^n_\gamma(\Omega,F)$. The definition of the topology of $C^n_\gamma(\Omega,F)$ implies that for any compact subset K of Ω, and any j with $0 \le j < n+1$, $(f^{(j)}_\alpha | K)$ is a Cauchy net in $C(K,L^j_\gamma(E,F))$. Since $L^j_\gamma(E,F)$ is complete by hypothesis and since Ω is a $k_{\mathbb{R}}$-space, for any j with $0 \le j < n+1$ there is $g_j \in C(\Omega,L^j(E,F))$ such that $(f^{(j)}_\alpha)_{\alpha\in A}$ converges to g_j uniformly on every compact subset of Ω.

Now we shall show that for any j with $0 \le j < n$ the Gâteaux-derivative of g_j equals g_{j+1}: Let $a \in \Omega$ and $h \in E$ be given. Then there exists an open interval I in $\mathbb{R}$ on which the function $v_\alpha : t \to f^{(j)}_\alpha(a+th)$ is defined for any $\alpha \in A$. Obviously

$$v_\alpha \in C^1(I,L^j_\gamma(E,F)) \quad \text{and} \quad v'_\alpha(t) = f^{(j+1)}_\alpha(a+th)[h].$$

For any two l.c. spaces X and Y the evaluation map $\varepsilon: L_\gamma(X,Y) \times X \to Y$, $\varepsilon(u,x) = u(x)$ is separately continuous. Hence $(v'_\alpha)_{\alpha \in A}$ converges towards the function $w : t \to g_{j+1}(a + th)[h]$, uniformly on every compact subset of I. Thus, $v := \lim_{\alpha \to} v_\alpha$ is differentiable on I and $v' = w$. Because of $v(t) = g_j(a + th)$, this implies

$$\lim_{t \to o} \frac{1}{t}(g_j(a+th) - g_j(a)) = \lim_{t \to o} \frac{v(t)-v(o)}{t} = v'(0) = w(0) = g_j(a + th),$$

i.e. $$\lim_{t \to o} \frac{1}{t}(g_j(a + th) - g_j(a) - g_{j+1}(a)[th]) = 0.$$

This shows $g_o = \lim_{\alpha \to} f_\alpha \in C^n_\gamma(\Omega,F)$.

6. REMARKS: a) Let us recall that any open subset Ω of a metrizable l.c. space or a (DFM)-space is a $k_{\mathbb{R}}$-space.

b) Concerning the completeness of $L^j_\gamma(E,F)$ the following should be remarked: If F is complete and E bornological, then $L^j_b(E,F)$ is complete for any $j \in \mathbb{N}_o$. If F and E'_c are complete, and E''_{cc} equals E topologically, then $L^j_c(E,F)$ is complete for any $j \in \mathbb{N}_o$. Especially for any (F)-or(DFM)-space E and any complete l.c. space F, the space $L^j_c(E,F)$ is complete for every $j \in \mathbb{N}_o$.

7. COROLLARY: *Let E be any (DFM)-space, F any (F)-space and Ω an open subset of E. $C^n_b(\Omega,F) = C^n_{co}(\Omega,F) = C^n_c(\Omega,F)$ is an (F)-space for any $n \in \mathbb{N}_\infty$.*

PROOF: As it was shown by Dineen [13], Prop. 1 and Prop. 5, Ω is a hemicompact k-space. Hence $C^n_{co}(\Omega,F)$ is metrizable. By remark 6.b) and proposition 5, $C^n_{co}(\Omega,F)$ is complete.

The following lemma will be useful in the sequel.

8. LEMMA: *Let* E *be a l.c. space,* γ *a system of bounded subsets of* E *which contains the compact sets,* Ω *an open subset of* E *and* $n \in \mathbb{N}_\infty$. *Then* E'_γ *is a continuously projected topological subspace of* $C^n_\gamma(\Omega)$.

PROOF: Since $C^n_\gamma(\Omega)$ is defined only for $\Omega \neq \emptyset$, we can choose $a \in \Omega$. Then we define $A : C^n_\gamma(\Omega) \to C^n_\gamma(\Omega)$ by $A(f) : x \to \langle f'(a), x \rangle$. Since any continuous linear map y on E coincides with its own derivative and since $y^{(j)} = 0$ for $j \geq 2$, A projects $C^n_\gamma(\Omega)$ onto $E' \subset C^n_\gamma(\Omega)$. As $A(f)^{(j)} = 0$ for $j \geq 2$, the continuity of A follows from the estimates:

$$(1)\quad \sup_{x \in K} |A(f)(x)| = \sup_{x \in K} |\langle f'(a), x \rangle| \leq p_{1,\{a\},K}(f)$$

for any compact subset K of Ω.

$$(2)\quad \sup_{x \in K} \sup_{h \in S} |A(f)'(x)[h]| = \sup_{x \in K} \sup_{h \in S} |\langle f'(a), h \rangle| \leq p_{1,\{a\},S}(f)$$

for any compact subset K of Ω and any $S \in \gamma$.

Similar arguments show that E'_γ is a topological subspace of $C^n_\gamma(\Omega)$.

9. DEFINITION: Let E be a l.c. space. A subset K of E is called *very compact*, if there is a Banach disc B (i.e. a convex balanced bounded subset B of E for which E_B is a Banach space) in E such that K is contained in E_B and compact there.

10. REMARKS: a) The notion of very compact sets was introduced (with a different definition) by de Wilde [23], Chap. III, 4. Def. .

b) By a consequence of the Banach-Dieudonné theorem a subset K of E is very compact iff there exists a convex balanced

compact subset Q of E such that K is contained in E_Q and compact there.

11. PROPOSITION: *Let E be a l.c. space in which every compact subset is very compact. Then $C^\infty_{co}(\Omega)$ is a Schwartz space for any open subset Ω of E.*

PROOF: By a well-known characterization of Schwartz spaces, it suffices to show that for any compact subset K of Ω, any compact subset Q_o of E, and any $n \in \mathbb{N}$ there is a compact subset Q of E such that any sequence $(f_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty_{co}(\Omega)$ with $\sup_{\ell \in \mathbb{N}} p_{n+1,K,Q}(f_\ell) \leq 1$ contains a subsequence which is Cauchy with respect to the semi-norm p_{n,K,Q_o}.

Since the closed convex hull of a compact set in E is compact again, K can be covered by a finite number of compact convex sets. Hence, w.o.l.g., we may assume that K is convex. By hypothesis and by remark 10.b) there is a balanced convex compact subset Q of E, such that $K \cup Q_o \subset Q$ and K as well as Q_o are compact in E_Q.

Now take any sequence $(f_\ell)_{\ell \in \mathbb{N}}$ in $C^\infty_{co}(\Omega)$ with $\sup_{\ell \in \mathbb{N}} p_{n+1,K,Q}(f_\ell) \leq 1$. In order to show that $(f_\ell)_{\ell \in \mathbb{N}}$ contains a subsequence which is Cauchy with respect to the semi-norm p_{n,K,Q_o}, we proceed as follows: We fix j with $0 \leq j \leq n$ and define for any $\ell \in \mathbb{N}$, $g_{j,\ell} : K \times Q^j \to \mathbb{R}$ by $g_{j,\ell}(x,y_1,\ldots,y_j) := f_\ell^{(j)}(x)[y_1,\ldots,y_j]$. Then we show that $\{g_{j,\ell} | \ell \in \mathbb{N}\}$ is an equicontinuous subset of $C(K \times Q^j)$, where $K \times Q^j$ is given the topology induced by E_Q^{j+1}.

Let $(a,y), (b,z) \in K \times Q^j$ be given. Then multi - linearity of $f_\ell^{(j)}(x)$ implies

$$|g_{j,\ell}(b,z) - g_{j,\ell}(a,y)| \le |g_{j,\ell}(b,z) - g_{j,\ell}(a,z)| +$$

$$(1) \qquad + |g_{j,\ell}(a,z) - g_{j,\ell}(a,y)| \le |g_{j,\ell}(b,z) - g_{j,\ell}(a,z)| +$$

$$+ \sum_{k=1}^{j} |g_{j,\ell}(a,z_1,\dots,z_k, y_{k+1},\dots,y_j) - g_{j,\ell}(a,z_1,\dots,z_{k-1}, y_k,\dots,y_j)|.$$

The general mean value theorem (lemma 2) gives

$$(f_\ell^{(j)}(b) - f_\ell^{(j)}(a))[z] = \int_0^1 f_\ell^{(j+1)}(a + t(b-a))[z, b-a]\, dt.$$

By $\sup_{x\in K} \sup_{w\in Q^{j+1}} |f_\ell^{(j+1)}(x)[w]| \le p_{j+1,K,Q}(f_\ell) \le 1$, this implies (observe that only for $a \neq b$ there is something to prove):

$$|g_{j,\ell}(b,z) - g_{j,\ell}(a,z)| = |(f_\ell^{(j)}(b) - f_\ell^{(j)}(a))[z]| \le$$

$$\le \|b-a\|_{E_Q} \left| \int_0^1 f_\ell^{(j+1)}(a + t(b-a))\left[z, \frac{b-a}{\|b-a\|_{E_Q}}\right] dt \right| \le \|b-a\|_{E_Q}.$$

Concerning the other terms in (1), the following holds true for $1 \le k \le j$ (observe that only for $z_k \neq y_k$ there is something to prove):

$$\left| g_{j,\ell}(a,z_1,\dots,z_k,y_{k+1},\dots,y_j) - g_{j,\ell}(a,z_1,\dots,z_{k-1},y_k,\dots,y_j) \right|$$

$$= \|z_k - y_k\|_{E_Q} \left| g_{j,\ell}\left(a,z_1,\dots,z_{k-1},\frac{z_k - y_k}{\|z_k - y_k\|_{E_Q}},y_{k+1},\dots,y_j\right) \right| \le \|z_k - y_k\|_{E_Q}.$$

Thus we have shown

$$|g_{j,\ell}(b,z) - g_{j,\ell}(a,y)| \le \|b-a\|_{E_Q} + \sum_{k=1}^{j} \|z_k - y_k\|_{E_Q}.$$

Hence $\{g_{j,\ell} \mid \ell \in \mathbb{N}\}$ is equicontinuous on $K \times Q^j \subset E_Q^{j+1}$. Since $K \times Q_o^j$ is a compact subset of $K \times Q^j$ in E_Q^{j+1}, and since $\{g_{j,\ell} \mid \ell \in \mathbb{N}\}$ is uniformly bounded on $K \times Q_o^j$, $\{g_{j,\ell} \mid \ell \in \mathbb{N}\}$ is a relatively compact subset of $C(K \times Q_o^j)$ by the theorem of Arzela - Ascoli. But then it is possible to choose (inductively) a subsequence of $(f_\ell)_{\ell \in \mathbb{N}}$ which is a Cauchy - sequence with respect to the semi-norm p_{n,K,Q_o}. By the considerations at the beginning, the proof is now complete.

REMARK: A similar argument as in the proof of Proposition 11 was used in the article of Bierstedt and Meise [7], theorem 7.(a), where it was shown that the space $H(K)$ of holomorphic germs on a compact subset K of a metrizable Schwartz space is a Schwartz space again.

12. THEOREM: *Let E be a quasi-complete l.c. space. Then the following are equivalent:*

(1) E'_c *is a Schwartz space.*

(2) *Every compact subset of E is very compact.*

(3) *For any open subset Ω of E, $C^\infty_{co}(\Omega) = C^\infty_c(\Omega)$ is a Schwartz space.*

(4) *There exists an open subset $\Omega (\neq \emptyset)$ of E for which $C^\infty_{co}(\Omega)$ is a Schwartz space.*

PROOF: (1) $\Rightarrow$ (2). Let K be any compact subset of E. Then K° is a neighbourhood of zero in E'_c and hence $K^{\circ\circ}$ is equicontinuous in $(E'_c)'$. Since E is quasi-complete, the topology $\lambda(E',E)$ is compatible with the duality between E and E', thus, $K^{\circ\circ}$ is equicontinuous in $(E'_c)' = E$. By the dual characterization of Schwartz spaces (see e.g. Horváth [17] 3, §15, Prop. 5), there is a compact subset Q of E such that $K^{\circ\circ}$ is relatively compact in $(E'_c)'_{Q^{\circ\circ}}$. Since E is quasi-complete, $Q^{\circ\circ}$ is compact by the theorem of bipolars. But then K, being compact in

E, is compact in the Banach space $E_{Q^{\infty}}$.

(2) ⇒ (3) by proposition 11.

(3) ⇒ (4) trivial.

(4) ⇒ (1) by lemma 8.

REMARK: Using the concept of bornology and the notion of Silva differentiability, Colombeau [12] gives independent proofs of proposition 5 and theorem 12 in a more general setting.

13. COROLLARY: *Let* E *be any* (F)*-space or any* (DFS)*-space and let* Ω *be an arbitrary open subset of* E. *Then* $C^{\infty}_{co}(\Omega) = C^{\infty}_{c}(\Omega)$ *is a Schwartz space.*

PROOF: It is a consequence of the Banach-Dieudonné theorem (see e.g. Köthe [19], §21, 10.(3)) that any compact set in a Fréchet space is very compact. Since (DFS)-spaces can be represented as compact injective countable inductive limits of Banach spaces, by Floret - Wloka [14], §25, 2.2, every compact subset of a (DFS)-space is very compact.

14. REMARK: It would be interesting to know whether, concerning nuclearity, C^{∞}_{co}-functions behave similar as holomorphic functions on complex l.c. spaces (cf. Boland [9]). Since nothing in this direction seems to be known, let us remark that for any open subset Ω of $E = \bigoplus_{n \in \mathbb{N}} \mathbb{R}$ the space $C^{\infty}_{co}(\Omega)$ is nuclear.

But this result is essentially finite dimensional: For $n \in \mathbb{N}$ let $j_n : \mathbb{R}^n \to E$ denote the canonical embedding. Then $\Omega_n := j_n^{-1}(\Omega)$ is an open subset of $\mathbb{R}^n$, hence $C^{\infty}_{co}(\Omega_n)$ is nuclear for any $n \in \mathbb{N}$. Now it is a consequence of Yamamuro [24], (1.6.1), that $C^{\infty}_{co}(\Omega) = \operatorname{proj}_{\leftarrow n} C^{\infty}_{co}(\Omega_n)$. Since the projective limit of nuclear spaces is nuclear, this proves the nuclearity of $C^{\infty}_{co}(\Omega)$.

3. THE ROLE OF THE APPROXIMATION PROPERTY

The aim of this section is to derive a sufficient condition for the a.p. of $C^n_{co}(\Omega)$. This will be done by an application of theorem 1.7. Therefore, we first give (under appropriate hypotheses) a characterization of the ε-product of $C^n_\gamma(\Omega)$ and a quasi-complete l.c.space.

1. **THEOREM:** *Let* E *and* F *be l.c. spaces, let* γ *be a system of bounded subsets of* E *which contains the compact sets and let* $1 \le n \le \infty$. *Assume that* E^j *is a* $k_{\mathbb{R}}$*-space for* $1 \le j < n+2$ *and that* $C^n_\gamma(\Omega)$ *and* F *are quasi-complete. Then* $C^n_\gamma(\Omega)\,\varepsilon\,F$ *is topologically isomorphic to the topological linear subspace* $C^{np}_\gamma(\Omega,F)$ *of* $C^n_\gamma(\Omega,F)$ *where*

$$C^{np}_\gamma(\Omega,F) = \{f \in C^n_\gamma(\Omega,F) \mid \text{for any compact subset } K \text{ of } E, \text{ any } S \in \gamma, \text{ and } 1 \le j < n+1 \text{ the set } f^{(j)}(K)[S^j] \text{ is precompact in } F\}.$$

PROOF: The proof is similar as in the finite dimensional case, but becomes more involved, since we have to deal with total derivatives. The general idea is the following: Define $\Delta : \Omega \to C^n_\gamma(\Omega)'_c$ by $\Delta(x) := \delta_x$ and show that the mapping $f \to f \circ \Delta$ is a topological isomorphism between $C^n_\gamma(\Omega)\,\varepsilon\,F \approx L_e(C^n_\gamma(\Omega)'_c, F)$ and $C^{np}_\gamma(\Omega,F)$. This will be done in several steps.

a) For $0 \le j < n+1$ define the mapping $\Delta^j : \Omega \times E^j \to C^n_\gamma(\Omega)'_c$ by $\langle \Delta^j(x,y), f\rangle := f^{(j)}(x)[y]$. Then Δ^j is continuous.

By hypothesis E^{j+1} is a $k_{\mathbb{R}}$-space, hence by the result of Blasco [8], $\Omega \times E^j$ is a $k_{\mathbb{R}}$-space. Thus, it suffices to show that $\Delta \mid K \times Q^j$ is continuous for any compact set K in Ω and any compact subset Q of E. From the definition of the topology of $C^n_\gamma(\Omega)$ it is obvious that $\Delta^j(K \times Q^j)$ is an equicontinuous subset of $C^n_\gamma(\Omega)'$. On

equicontinuous subsets the topology of $C^n_\gamma(\Omega)'_c$ coincides with the weak topology, hence we only have to show that $\Delta^j|K\times Q^j : K \times Q^j \to C^n_\gamma(\Omega)'_\sigma$ is continuous. Let $f \in C^n_\gamma(\Omega)$ be arbitrary. Since the j-th derivative of f is symmetric, by proposition 1.5 we have the continuity of $f^{(j)} : K \to L^{ns}_\gamma(E,\mathbb{R}) \simeq \mathcal{L}^s_\gamma(E^j)$. For $S \in \gamma$, let $\rho^j_S : \mathcal{L}^s_\gamma(E^j) \to CB(S^j)$ denote the restriction. Then ρ^j_S is continuous, especially $\rho^j_Q : \mathcal{L}^s_\gamma(E^j) \to C(Q^j)$ is continuous. This shows that

$$\rho^j_Q \circ (f^{(j)} \mid K) \in C(K,C(Q^j)) \simeq C(K \times Q^j)$$

and hence the mapping $(x,y) \to f^{(j)}(x)[y] = \rho^j_Q \circ (f^{(j)} \mid K)(x,y)$ is continuous on $K \times Q^j$. Thus the continuity of Δ^j is proved.

b) For $1 \le j < n+1$ one can define a continuous linear mapping $\tilde{\Delta}^j : \Omega \to \mathcal{L}^s_\gamma(E^j, C^n_\gamma(\Omega))'_c)$ by $\tilde{\Delta}^j(x) := \Delta^j(x, \cdot\,)$.

By part a) and by the symmetry of the derivative it is obvious that $\Delta^j(x, \cdot\,)$ is a symmetric j-times linear mapping from E^j into $C^n_\gamma(\Omega)'_c$. Let us prove that $\Delta^j(x, \cdot\,)$ is γ-hypocontinuous: In part a) we have shown, that Δ^j is continuous in (x,o), hence for any neighbourhood W of zero in $C^n_\gamma(\Omega)'_c$ there is a neighbourhood U of zero in E such that

$$\Delta^j(x,u) = \Delta^j(x,u) - \Delta^j(x,o) \in W$$

for any $u \in U^j$. If $S \in \gamma$ is arbitrary, then there is $\lambda > 0$ such that $S \subset \lambda U$. Hence for $V := \lambda^{-j+1}U$ one has $\Delta^j(x, V \times S^{j-1}) \subset W$. By the symmetry of $\Delta^j(x, \cdot\,)$ this proves $\Delta^j(x, \cdot\,) \in \mathcal{L}^s_\gamma(E^j, C^n_\gamma(\Omega)'_c)$.

Now let us show the continuity of $\tilde{\Delta}^j$. First we observe that for any compact K in Ω and any $S \in \gamma$, by the definition of the topology of $C^n_\gamma(\Omega)$, the set $\Delta^j(K \times S^j) - \Delta^j(K \times S^j)$ is equicontinuous in $C^n_\gamma(\Omega)'$. By the coincidence of $\lambda(C^n_\gamma(\Omega)', C^n_\gamma(\Omega))$ and $\sigma(C^n_\gamma(\Omega)', C^n_\gamma(\Omega))$

on equicontinuous subsets of $C^n_\gamma(\Omega)'$, for any convex balanced neighbourhood W of zero in $C^n_\gamma(\Omega)'_c$ there are $f_1,\dots,f_m$ in $C^n_\gamma(\Omega)$ such that

$$(\Delta^j(K\times S^j) - \Delta^j(K\times S^j)) \cap W \supset (\Delta^j(K\times S^j) - \Delta^j(K\times S^j)) \cap \{f_1,\dots,f_m\}^\circ .$$

Since $f_k^{(j)} : \Omega \to \mathcal{L}^s_\gamma(E^j)$ is continuous for $1 \le k \le m$, for any $x \in K$ there exists a neighbourhood U of x such that for any $x' \in U$ and any k with $1 \le k \le m$

$$\sup_{y\in S^j} | f_k^{(j)}(x)[y] - f_k^{(j)}(x')[y] | \le 1.$$

Hence we have for any $x' \in U$ and any $y \in S^j$

$$\Delta^j(x,y) - \Delta^j(x',y) \in \{f_1,\dots,f_m\}^\circ .$$

By our first observation this shows for any $x' \in U \cap K$

$$\sup_{y\in S^j} q_W(\tilde{\Delta}^j(x)[y] - \tilde{\Delta}^j(x')[y]) \le 1,$$

where q_W denotes the gauge of W. Since $S \in \gamma$ was arbitrary, $\tilde{\Delta}^j|K$ is continuous for any compact subset K of Ω. But Ω is a $k_{\mathbb{R}}$-space, hence $\tilde{\Delta}^j$ is continuous.

c) For any $u \in F \varepsilon C^n_\gamma(\Omega) = L_e(C^n_\gamma(\Omega)'_c, F)$ the mapping $f_u := u\circ\Delta : \Omega \to F$ belongs to $C^{np}_\gamma(\Omega,F)$.

It is easy to see that for $0 \le j < n+1$ the mapping

$$u \circ \tilde{\Delta}^j : \Omega \to L^{js}_\gamma(E,F)$$

is continuous. Hence we have proved $f_u \in C^n_\gamma(\Omega,F)$, if we can show

$f_u^{(j+1)} = u \circ \tilde{\Delta}^{j+1}$ for $0 \leq j < n$. To do this, take j with $0 \leq j < n$, any $h \in E$, any $x \in \Omega$ and an arbitrary $S \in \gamma$ with $h \in S$. Choose $\delta > 0$ such that $x + th \in \Omega$ for t with $|t| \leq \delta$ and put $K := \{x + th \mid |t| \leq \delta\}$. Then remark that the set

$$\{\frac{1}{t}(\Delta^j(x + th,y) - \Delta^j(x,y) \mid y \in Q^j,\ 0 < |t| \leq \delta\}$$

is equicontinuous in $C_\gamma^n(\Omega)'$, since by 2.2 we have for any $f \in C_\gamma^n(\Omega)$ and any $y \in S^j$

$$|\langle \frac{1}{t}(\Delta^j(x+th,y) - \Delta^j(x,y),f\rangle| = |\frac{1}{t}(f^{(j)}(x+th)[y] - f^{(j)}(x)[y])|$$

$$= |\int_0^t f^{(j+1)}(x+\tau h)[h,y]d\tau| \leq |t|\, p_{j+1,K,S}(f) \leq \delta \cdot p_{j+1,K,S}(f).$$

Since the topology of $C_\gamma^n(\Omega)'_c$ coincides with the weak topology on equicontinuous subsets, we have

$$\lim_{t\to 0} \frac{1}{t}(\Delta^j(x + th,y) - \Delta^j(x,y) - \Delta^{j+1}(x,(th,y))) = 0$$

in $C_\gamma^n(\Omega)'_c$ uniformly in $y \in S^j$ if this holds in $C_\gamma^n(\Omega)'_\sigma$ uniformly in $y \in S^j$. But the latter is a consequence of the definition of $C_\gamma^n(\Omega)$, since for any $f \in C_\gamma^n(\Omega)$

$$\langle \frac{1}{t}(\Delta^j(x + th,y) - \Delta^j(x,y) - \Delta^{j+1}(x,(th,y))),f\rangle$$

$$= \frac{1}{t}(f^{(j)}(x + th) - f^{(j)}(x) - f^{(j+1)}(x)[th])[y]$$

tends to zero uniformly in $y \in S^j$ it t tends to zero. From this we get by induction

$$\lim_{t\to 0} \frac{1}{t}(f_u^{(j)}(x + th) - f_u^{(j)}(x) - u \circ \tilde{\Delta}^{j+1}(x)[th]) = 0$$

in $\mathcal{L}_\gamma^s(E^j,F)$. Hence $f_u^{(j)}$ is Gâteaux-differentiable in $x \in \Omega$ and $f_u^{(j+1)}(x) = u \circ \tilde{\Delta}^{j+1}(x)$. Since $u \circ \tilde{\Delta}^{j+1}$ is continuous, by 2.3 we have shown $f_u \in C_\gamma^n(\Omega,F)$.

In order to show that we even have $f_u \in C_\gamma^{np}(\Omega,F)$, take any compact K in Ω and any $S \in \gamma$. Then it follows from the theorem of Alaoglu-Bourbaki that for $0 \leq j < n + 1$ the set $\Delta^j(K \times S^j)$ is relatively compact in $C_\gamma^n(\Omega)'_c$, hence $u \circ \Delta^j(K \times S^j)$ is relatively compact in F. But we have shown above that

$$f_u^{(j)}(K)[S^j] = u \circ \Delta^j(K \times S^j).$$

d) The mapping $k : F \varepsilon C_\gamma^n(\Omega) \to C_\gamma^{np}(\Omega,F)$ defined by $k(u) := f_u$ is an injective topological homomorphism.

Let any compact K in Ω, any $S \in \gamma$ and $\ell < n + 1$ be given. For

$$V_{\ell,K,S,1} := \{f \in C_\gamma^n(\Omega) \mid p_{\ell,K,S}(f) \leq 1\}$$

we have

$$\begin{aligned}(\bigcup_{j=0}^{\ell} \Delta^j(K \times S^j))^\circ &= \bigcap_{j=0}^{\ell} (\Delta^j(K \times S^j))^\circ \\ &= \bigcap_{j=0}^{\ell} \{f \in C_\gamma^n(\Omega) \mid \sup_{x\in K} \sup_{y\in S^j} |f^{(j)}(x)[y]| \leq 1\} \\ &= V_{\ell,K,S,1}.\end{aligned}$$

By the theorem of bipolars this implies

$$V^{o}_{\ell,K,S,1} = \Big(\bigcup_{j=o}^{\ell} \Delta^j(K \times S^j)\Big)^{oo} = \overline{\Gamma \bigcup_{j=o}^{\ell} \Delta^j(K \times S^j)}^{\sigma}.$$

Since any $u \in F \varepsilon C^n_\gamma(\Omega)$ is weakly continuous on equicontinuous sub-sets, we have for any continuous semi-norm q on F and any

$$u \in F \varepsilon C^n_\gamma(\Omega) = L_e(C^n_\gamma(\Omega)'_c, F)$$

$$\sup_{v \in V^{o}_{\ell,K,S,1}} q(u(v)) = \sup_{v \in \bigcup_{j=o}^{\ell} \Delta^j(K \times S^j)} q(u(v))$$

$$= \sup_{o \le j \le \ell} \sup_{x \in K} \sup_{y \in S^j} q(u(\Delta^j(x,y)))$$

$$= \sup_{o \le j \le \ell} \sup_{x \in K} \sup_{y \in S^j} q(f_u^{(j)}(x)[y])$$

$$= p_{\ell,K,S,q}(f_u).$$

Hence the result claimed under d) is proved.

e) The mapping k defined in d) is surjective, i.e. a topological isomorphism.

The surjectivity of k will be proved by constructing a right-inverse j of k. Take any $f \in C^{np}_\gamma(\Omega,F)$ and any $y' \in F'$, then it is an easy exercise to show that $y' \circ f \in C^n_\gamma(\Omega)$. Hence we can define $u_f : F' \to C^n_\gamma(\Omega)$ by $u_f(y') := y' \circ f$. Let us assume for a moment that $u_f \in C^n_\gamma(\Omega) \varepsilon F$ holds. Then we can define $j : C^{np}_\gamma(\Omega,F) \to F \varepsilon C^n_\gamma(\Omega)$ by $j(f) = {}^t u_f$, since by our hypotheses transposition is a topological isomorphism between $C^n_\gamma(\Omega) \varepsilon F$ and $F \varepsilon C^n_\gamma(\Omega)$. The surjectivity of k is now proved, if we show $k \circ j = \mathrm{id}_{C^{np}_\gamma(\Omega,F)}$. But this is a consequence

of the following identity which holds for any $y' \in F$, any $x \in \Omega$, and any $f \in C^{np}_{\gamma}(\Omega,F)$:

$$\langle k \circ j(f)[x],y' \rangle = \langle {}^t u_f \circ \Delta^o(x),y' \rangle = \langle \Delta^o(x),u_f(y') \rangle$$

$$= \langle \Delta^o(x),y' \circ f \rangle = \langle f(x),y' \rangle .$$

Hence the proof of the theorem is complete, if we show $u_f \in C^n_{\gamma}(\Omega) \varepsilon F$. Let any compact subset K of Ω, any $S \in \gamma$, and $\ell < n+1$ be given. By hypothesis, the set $L_\ell := \bigcup_{j=o}^{\ell} f^{(j)}(K)[S^j]$ is precompact in F, hence L^o_ℓ is a neighbourhood of zero in F'_c. For any $y' \in L^o_\ell$ we have

$$p_{\ell,K,S}(u_f(y')) = \sup_{o \le j \le \ell} \sup_{x \in K} \sup_{y \in S^j} | u_f(y')^{(j)}(x)[y] |$$

$$= \sup_{o \le j \le \ell} \sup_{x \in K} \sup_{y \in S^j} | y'(f^{(j)}(x)[y]) | \le 1,$$

i.e. $u_f(L^o_\ell) \subset V_{\ell,K,S,1}$. Thus we have shown $u_f \in L(F'_c,C^n_{\gamma}(\Omega))$.

2. COROLLARY: *If the hypotheses of theorem 1 are satisfied for the system γ_{co} of all compact subsets of E, then we have for any open subset Ω of E:*

$$C^n_{co}(\Omega) \varepsilon F = C^n_{co}(\Omega,F).$$

This is the case for any (F)-space or any (DFM)-space E, any quasi-complete l.c. space F, and any $n \in \mathbb{N}_\infty$.

PROOF: The first part of the statement follows from theorem 1, and $C^{np}_{co}(\Omega,F) = C^n_{co}(\Omega,F)$. To prove this identity, let $f \in C^n_{co}(\Omega,F)$ be

given. Then for any $j \in \mathbb{N}$ with $1 \leq j < n+1$, any compact K in Ω, and any compact Q in E, by part a) of the proof of theorem 1, we have

$$\rho_Q^j \circ (f^{(j)} \mid K) \in C(K, C(Q^j, F)).$$

Now observe that the evaluation map $A : C(Q^j,F) \times Q^j \to F$, $A(g,x) := g(x)$, is continuous. Hence $B : K \times Q^j \to F$, $B(x,y) = A(f^{(j)}(x),y)$, is continuous and thus

$$f^{(j)}(K)[Q^j] = B(K \times Q^j)$$

is compact in F.

The second part of the statement follows from remark 2.6.

REMARK: Corollary 2 generalizes a result of Bombal Gordón and González Llavona [10], who characterized $C_{co}^n(E) \varepsilon F$ for Banach spaces E. Also for Banach spaces E, Aron [2] gave a somewhat different description of $C_b(E) \varepsilon F$.

Now we come to the main lemma for many of the results presented in the sequel. For Banach spaces E it goes back to Bombal Gordón and González Llavona [10] as well as to Prolla and Guerreiro [20].

3. LEMMA: *Let E be a quasi-complete barrelled l.c. space with the following property:*

(CFA): *For any compact subset K in E there exists an (F)-space E^K with a.p. and a continuous linear injective mapping $j_K : E^K \to E$ such that $K \subset j_K(E^K)$ and $j_K^{-1}(K)$ is compact in E^K.*

Furthermore let Ω be an open subset of E, let F be a normed space, and let $n \in \mathbb{N}_\infty$ and $f \in C_{co}^n(\Omega,F)$ be given. Then for any compact

subset K_o *of* Ω, *any compact* Q_o *in* E, *any* $\ell < n + 1$, *and any* $\varepsilon > 0$ *there is* $u \in E' \otimes E$ *and an open neighbourhood* ω *of* K, *such that* $f \circ u \in C^n_{co}(\omega,F)$ *and such that*

$$\sup_{x\in K} \sup_{o\le j\le \ell} \sup_{y\in Q_o^j} \| f^{(j)}(x)[y] - (f \circ u)^{(j)}(x)[y] \| \le \varepsilon .$$

PROOF: Put $K := K_o \cup Q_o$ and choose - according to (CFA)-an(F)-space E^K with a.p. for which there is a continuous embedding $j_K : E^K \to E$. Assume that the topology of E^K is given by the increasing system $(q_s)_{s\in\mathbb{N}}$ of semi-norms. Since E^K has the a.p., for any $m \in \mathbb{N}$ we can find $w_m \in (E^K)' \otimes E^K$ such that

$$\sup_{x\in K} q_m(w_m(x) - x) \le \frac{1}{m} .$$

By the quasi-completeness of E^K and E it follows from the injectivity of j_K that ${}^t j_K(E')$ is $\lambda((E^K)',E^K)$-dense in $(E^K)'$. Hence by approximating w_m appropriately we can find $u_m \in E' \otimes E^K \subset E' \otimes E$ such that

$$\sup_{x\in K} q_m(u_m(x) - (x)) \le \frac{2}{m} .$$

Consequently there is a sequence $(u_m)_{m\in\mathbb{N}}$ in $E' \otimes E^K$ such that for any $s \in \mathbb{N}$

$$\lim_{m\to\infty} \sup_{x\in K} q_s(u_m(x) - x) = 0,$$

i.e. u_m restricted to K tends to id_K in $C(K,E^K)$. Since the evaluation map $A : C(K,E^K) \times K \to E^K$, $A(f,x) := f(x)$, is continuous, the set

$$L := K \cup \bigcup_{m \in \mathbb{N}} u_m(K) = A((\{id_K\} \cup \{u_m \mid m \in \mathbb{N}\}) \times K)$$

is compact in E^K, hence also compact in E. Since K_o is a compact subset of K the same arguments show that for any $s \in \mathbb{N}$ the set

$$L_s := K_o \cup \bigcup_{m \geq s} u_m(K_o)$$

is compact in E^K and E.

Now let W_o be a convex balanced neighbourhood of zero in E for which $K_o + W_o \subset \Omega$. Since $\lim_{m \to \infty} u_m = id_K$ in $C(K,E)$ and since $K_o \subset K$, there exists $s_o \in \mathbb{N}$ such that for any $s \geq s_o$ and any $x \in K_o$

$$u_s(x) - x \in W_o .$$

Put $L_o := L_{s_o}$, then it follows

$$L_o \subset K_o + W_o \subset \Omega .$$

Since L_o is a compact subset of Ω and since for $0 \leq j \leq \ell$ the function $f^{(j)} : \Omega \to L^j_{co}(E,F)$ is continuous, there exists a convex balanced neighbourhood U of zero in E with $L_o + U \subset \Omega$ such that for any j with $0 \leq j \leq \ell$, any $x \in L_o$ and any $z \in E$ with $x - z \in U$ the following estimate holds

$$\sup_{y \in L^j} \| f^{(j)}(x)[y] - f^{(j)}(z)[y] \| < \frac{\varepsilon}{\ell + 1} \tag{1}$$

For $1 \leq j \leq \ell$ the set $f^{(j)}(L_o)$ is compact and hence bounded in $L^j_{co}(E,F) = L_{co}(E, L^{j-1}_{co}(E,F))$. E is barelled by hypothesis, hence $f^j(L_o)$ is equicontinuous in $L_{co}(E, L^{j-1}_{co}(E,F))$. This implies that there is a

neighbourhood W_j of zero in E such that for any $x \in L_o$, any $y_1 \in W_j$ and any $y' \in L^{j-1}$ we have

$$\|f^{(j)}(x)[y_1,y']\| \leq \frac{\varepsilon}{\ell+1} .$$

Since $f^{(j)}(x)$ is symmetric for any x, this means that we have

$$\|f^{(j)}(x)[y]\| \leq \frac{\varepsilon}{\ell+1} \tag{2}$$

for any $y = (y_1,\dots,y_j) \in E^j$, where $j-1$ of the y_k are in L and one of the y_k is in W_j.

The set $j_K^{-1}(\bigcap_{j=1}^{\ell}(W_j \cap U))$ is a neighbourhood of zero in E^K, hence there exists $s \in \mathbb{N}$ with $s \geq s_o$ such that

$$\{x \in E^K \mid q_s(x) \leq \frac{1}{s}\} \subset j_K^{-1}(\bigcap_{j=1}^{\ell}(W_j \cap U)).$$

Now we define $u := u_s$ and observe that by the choice of s and by our construction we get $u(x) \in K_o + U \subset \Omega$ for any $x \in K_o$ (from now on let us omit the map j_K, i.e. we regard u as mapping from E into E). Then the set $\omega := u^{-1}(\Omega)$ is an open neighbourhood of K_o and on ω we can define the mapping $f \circ u : \omega \to F$. By our definition 2.4 it is easy to see that $f \circ u \in C^n_{co}(\omega,F)$ and that for any $x \in \omega$, any j with $j < n+1$ and any $y \in E^j$ the following holds

$$(f \circ u)^{(j)}(x)[y] = f^{(j)}(u(x))[u(y)] . \tag{3}$$

In order to prove the desired estimate, we observe first that we have $u(K_o) \subset L_o$, and that for any $x \in K_o$, $u(x) - x \in U$. Hence we get from (1):

$$\|f(x) - f(z)\| \leq \frac{\varepsilon}{\ell+1} . \tag{4}$$

Then we observe that for any j with $1 \leq j \leq \ell$ we have $u(Q_o)^j \subset L_j$, hence for any $x \in K_o$ and any $y \in Q_o^j$ we get from (1), (2) and (3)

$$\|f^{(j)}(x)[y] - (f \circ u)^{(j)}(x)[y]\| \leq \|f^{(j)}(x)[y] - f^{(j)}(u(x))[u(y)]\|$$

$$\leq \|f^{(j)}(x)[y] - f^{(j)}(u(x))[y]\| + \|f^{(j)}(u(x))[y] - f^{(j)}(u(x))[u(y)]\|$$

(5)

$$\leq \frac{\varepsilon}{\ell+1} + \sum_{s=1}^{j} \|f^{(j)}(u(x))[(y_1,\ldots,y_{s-1},y_s - u(y_s),u(y_{s+1}),\ldots,u(y_j))]\|$$

$$\leq (\ell + 1) \cdot \frac{\varepsilon}{\ell + 1} = \varepsilon.$$

By (4) and (5), the proof of the lemma is complete.

4. REMARK: Let us recall from Bierstedt and Meise [6] that an inductive injective system $\{E_\alpha, j_{\alpha\beta}\}_{\alpha \in A}$ of l.c. spaces E_α is called *compactly regular*, if $E = \operatorname{ind}_{\alpha \to} E_\alpha$ is Hausdorff and if for any compact subset K of E there exists $\alpha \in A$ such that K is already a compact subset of E_α. Then it is obvious that any l.c. space E which can be represented as an inductive limit of a compactly regular inductive system $\{E_\alpha, j_{\alpha\beta}\}_{\alpha \in A}$ of (F)-spaces E_α with a.p. has the property (CFA). Hence in any of the following cases E has (CFA):

i) E is a (F)-space with a.p.

ii) $E = \operatorname{ind}_{n \to} E_n$, where $\{E_n, j_{nm}\}$ is a strict inductive system of (F)-spaces E_n with a.p.

iii) $E = \operatorname{ind}_{n \to} E_n$, where $\{E_n, j_{nm}\}$ is a compact injective inductive system of (F)-spaces E_n with a.p. For brevity we shall call any space of this type (DFSA)-space.

Using a trick which goes back to Aron and Schottenloher [4], we can now prove the desired result on the a.p. of $C^n_{co}(\Omega)$.

5. THEOREM: *Let* E *be as in lemma 3 and assume furthermore that for* $\gamma = \gamma_{co}$ *and* $n \in \mathbb{N}_\infty$ *all the hypotheses of theorem 1 on* E *and* $C^n_{co}(\Omega)$ *are satisfied. Then* $C^n_{co}(\Omega)$ *has the a.p. for any open subset* Ω *of* E.

PROOF: $C^n_{co}(\Omega)$ is quasi-complete by hypothesis. Hence, by theorem 1.7 the a.p. of $C^n_{co}(\Omega)$ can be proved by showing that $C^n_{co}(\Omega) \otimes F$ is dense in $C^n_{co}(\Omega) \varepsilon F$ for any Banach space F. By corollary 2 we may and shall identify $C^n_{co}(\Omega) \varepsilon F$ and $C^n_{co}(\Omega,F)$. Hence we only have to show that $C^n_{co}(\Omega) \otimes F$ is dense in $C^n_{co}(\Omega,F)$ for any Banach space F.

To do this, let any $f \in C^n_{co}(\Omega,F)$, any compact subset K_o of Ω, any compact Q_o in E, any $\ell < n+1$, and any $\varepsilon > 0$ be given. By lemma 3, there exists $u \in E' \otimes E$ and an open neighbourhood ω of K_o such that $f \circ u \in C^n_{co}(\omega,F)$ satisfies the estimates given in lemma 3. Let us define $E_o := \operatorname{Im} u$, $\Omega_o := \Omega \cap E_o$ and $f_o := f \mid \Omega_o$. Then $f_o \in C^n_{co}(\Omega_o,F)$, and by a classical (finite dimensional) result $C^n_{co}(\Omega_o,F) = C^n_{co}(\Omega) \overset{\vee}{\otimes}_\varepsilon F$. Since it was shown in the proof of lemma 3 that $u(K_o) \subset \Omega \cap E_o = \Omega_o$, and since $C^n_{co}(E_o)$ is dense in $C^n_{co}(\Omega)$, there exists $h \in C^n_{co}(E_o) \otimes F$ such that

$$p_{\ell,u(K_o),u(Q_o)}(f_o - h) \leq \varepsilon .$$

Then $g := h \circ u \in C^n_{co}(E) \otimes F$, and for any $x \in K_o$

$$\|f(x) - g(x)\| \leq \|f(x) - f \circ u(x)\| + \|f \circ u(x) - h \circ u(x)\| \leq$$

$$\leq \varepsilon + \|f_o(u(x)) - h(u(x))\| \leq 2\varepsilon .$$

Furthermore we have for any $x \in K_o$, any j with $1 \leq j \leq \ell$ and any $y \in Q_o^j$

$$\|f^{(j)}(x)[y]-g^{(j)}(x)[y]\| \leq \|f^{(j)}(x)[y]-(f\circ u)^{(j)}(x)[y]\| +$$

$$+ \|(f\circ u)^{(j)}(x)[y]-(h\circ u)^{(j)}(x)[y]\|$$

$$\leq \varepsilon + \|f_o^{(j)}(u(x))[u(y)]-h^{(j)}(u(x))[u(y)]\| \leq 2\varepsilon .$$

Hence we have shown $p_{\ell,K_o,Q_o}(f-g) \leq 2\varepsilon$, which proves the density of $C^n_{co}(E) \otimes F$ in $C^n_{co}(\Omega,F)$.

6. REMARK: All the hypotheses of theorem 6 are satisfied for any $n \in \mathbb{N}_\infty$ and any open subset Ω of E, if E is either an (F)-space with a.p. or a (DFSA)-space. This follows from 2.5, 2.6 and remark 4. We shall show now that for Fréchet spaces E with a.p. this result is optimal.

7. THEOREM: *For a Fréchet space* E *the following are equivalent:*

a) $C^n_{co}(\Omega)$ *has the a.p. for any* $n \in \mathbb{N}_\infty$ *and any open subset* $\Omega \neq \emptyset$ *of* E.

b) *There exist* $n \in \mathbb{N}_\infty$ *and an open subset* $\Omega \neq \emptyset$ *of* E *such that* $C^n_{co}(\Omega)$ *has the a.p.*

c) E *has the a.p.*

PROOF: (a) $\Rightarrow$ (b): trivial

(b) $\Rightarrow$ (c): By 2.8, $E'_{co} = E'_c$ is a continuously projected topological linear subspace of $C^n_{co}(\Omega)$, hence E'_c has the a.p. But for a Fréchet space E the a.p. of E'_c is equivalent to the a.p. of E, hence E has the a.p.

(c) $\Rightarrow$ (a): This is clear according to the remark 6.

REMARK: For Banach spaces E theorem 7 was shown by Bombal Gordón

and González Llavona [10] for $\Omega = E$. Again for Banach spaces E a slightly different version (using $C_b^n(\Omega)$ endowed with the topology induced by $C_{co}^n(\Omega)$) of theorem 7 was presented by Prolla and Guerreiro [20] and also by Aron [3].

8. COROLLARY: *There exists an* (FS)*-space* E *such that* $C_{co}^n(\Omega)$ *does not have the a.p. for any* $n \in \mathbb{N}_\infty$ *and any non-empty open subset* Ω *of* E.

PROOF: This is a consequence of theorem 7 and the existence of (FS)-spaces without a.p. The existence of such (FS)-space follows from Enflo's counterexample, as Hogbe-Nlend proved in [16].

Because of lemma 3, the method applied in the proof of theorem 5 can be used also to derive some further density results just by "lifting" density relations known in the finite dimensional case. Before stating them let us recall that a continuous n-homogeneous polynomial p on E is called *finite*, if there exist $y_1', \ldots, y_n' \in E'$ such that

$$p(x) = \prod_{j=1}^{n} \langle y_j', x \rangle \quad \text{for any} \quad x \in E.$$

By $P_f(E)$ we denote the linear hull of all continuous n-homogeneous polynomials on E, $n \in \mathbb{N}_0$. It is easy to see that $P_f(E) \subset C_{co}^\infty(E)$.

9. THEOREM: *Let* E *be a quasi-complete barrelled l.c. space with* (CFA). *Then for any* $n \in \mathbb{N}_\infty$ *and l.c. space* F *and any open subset* $\Omega (\neq \emptyset)$ *of* E *the space* $P_f(E) \otimes F$ *is dense in* $C_{co}^n(\Omega, F)$.

PROOF: Let any $f \in C_{co}^n(\Omega, F)$, any compact subset K of Ω, any compact subset Q of E, any $\ell < n+1$, any continuous semi-norm q on F,

and $\varepsilon > 0$ be given. We shall show that there exists $g \in P_f(E) \otimes E$ with

$$p_{\ell,K,Q,q}(f - g) \leq \varepsilon .$$

Let F_q denote the completion of the canonical normed space $F/\ker q$ and let $\pi : F \to F_q$ denote the canonical continuous linear map. Since $\pi \circ f \in C^n_{co}(\Omega,F_q)$, according to lemma 3 there exists $u \in E' \otimes E$ such that

$$\sup_{x\in K} \sup_{0\leq j\leq \ell} \sup_{y\in Q^j} \| (\pi \circ f)^{(j)}(x)[y] - (\pi \circ f \circ u)^{(j)}[y] \| \leq \varepsilon .$$

Now we proceed as in the proof of theorem 5 and define $E_o := \operatorname{Im} u$, $\Omega_o := \Omega \cap E_o$ and $f_o : (\pi \circ f)|\Omega_o$. Then $f_o \in C^n_{co}(\Omega_o,F_q) = C^n_{co}(\Omega_o) \check{\otimes}_\varepsilon F_q$. Since the polynomials on E_o are dense in $C^n_{co}(\Omega_o)$; since $\pi(F)$ is dense in F_q; and since $u(K)$ is contained in Ω_o (this was shown in the proof of lemma 3), there exists $h_o \in P(E_o) \otimes \pi(F) = P_f(E_o) \otimes \pi(F)$ such that

$$p_{\ell,u(K),u(Q),\|\cdot\|}(f_o - h_o) \leq \varepsilon .$$

Assume that $h_o = \sum_{i=1}^{m} p_i \otimes \pi(y_i)$, where $p_i \in P_f(E_o)$ and $y_i \in F$ for $i = 1,\ldots,m$. Then $h := \sum_{i=1}^{m} p_i \circ u \otimes y_i$ is in $P_f(E) \otimes F$ and as in the proof of theorem 5 it follows

$$p_{\ell,K,Q,q}(f - h) \leq 2\varepsilon .$$

Hence we have shown that $P_f(E) \otimes F$ is dense in $C^n_{co}(\Omega,F)$.

The following corollary is an immediate consequence of theorem 9.

10. COROLLARY: *Let* E *be a quasi-complete barrelled l.c. space with* (CFA). *Then for any* $n \in \mathbb{N}_\infty$, *any l.c. space* F, *and any open subset* $\Omega(\neq \emptyset)$ *of* E *the space* $C^\infty_{co}(E) \otimes F$ *is dense in* $C^n_{co}(\Omega,F)$.

Looking at theorem 5 and corollary 10 and their proof in the finite dimensional case one has the impression that condition (CFA) (or more or less the a.p.) together with finite dimensional results can be used instead of C^∞-functions with compact support. The following theorem is of the same nature. Before we state it, let us remark that an easy calculation shows the following: Let E be any l.c. space and let E' denote its (continuous) dual. For any system γ of bounded subsets of E (covering E) and any $y \in E'$, the function

$$e_y : x \to e^{\langle x,y \rangle}$$

belongs to $C^\infty_\gamma(E)$. Using this and the classical theorem of Paley – Wiener-Schwartz the proof of theorem 9 also gives

11. THEOREM: *Let* E *be a quasi-complete barrelled l.c. space with* (CFA). *Then for any* $n \in \mathbb{N}_\infty$, *any l.c. space* F, *and any open subset* $\Omega(\neq \emptyset)$ *of* E *the linear hull of the set* $\{e_y \cdot f \mid y \in E', f \in F\}$ *is dense in* $C^n_{co}(\Omega,F)$.

4. A KERNEL THEOREM FOR FUNCTIONS OF CLASS C^∞_{co}

In this section we shall show (under appropriate hypotheses) that any functions in $C^\infty_{co}(\Omega_1 \times \Omega_2)$ can be regarded as an element of $C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$ and vice versa. Using theorem 3.5 this also implies a tensor product representation for $C^\infty_{co}(\Omega_1 \times \Omega_2)$. Before we can prove our result we need several lemmas. The first lemma is an immediate consequence of definition 2.4.

1. LEMMA: *Let* E,F *and* G *be l.c. spaces, let* Ω *be an open subset of* E, *let* γ *be a covering system of bounded subset of* E, *and let* $u \in L(F,G)$ *be given.*

a) *For any* $f \in C^\infty_\gamma(\Omega,F)$ *the function* $u \circ f$ *belongs to* $C^\infty_\gamma(\Omega,G)$.

b) *For any* $f \in C^\infty_\gamma(\Omega,F)$ *and any* $j \in \mathbb{N}_0$ *the function* $f^{(j)}$ *belongs to* $C^\infty_\gamma(\Omega,L^j_\gamma(E,F))$.

c) *If* G *is a closed linear topological subspace of* F, *then any* $f \in C^\infty_{co}(\Omega,F)$ *with* $f(\Omega) \subset G$ *belongs to* $C^\infty_{co}(\Omega,G)$.

2. LEMMA: *Let* E_1, E_2 *and* F *be l.c. spaces and let* Ω_i *be an open subset of* E_i *for* $i = 1,2$. *Then for any* $f \in C^\infty_{co}(\Omega_1 \times \Omega_2, F)$, *by* $x_1 \to f(x_1, \cdot)$, *one defines a function* $g \in C(\Omega_1, C^\infty_{co}(\Omega_2,F))$.

PROOF: The mapping $i_2 : E_2 \to E_1 \times E_2$ defined by $i_2(x_2) = (0,x_2)$ is obviously linear and continuous, hence $i_2^j : E_2^j \to (E_1 \times E_2)^j$ is contonuous and j-linear, for any $j \in \mathbb{N}$. Thus i_2^j gives rise to a continuous linear map $(i_2^j)^* : \mathcal{L}^s_{co}((E_1 \times E_2)^j,F) \to \mathcal{L}^s_{co}(E_2^j,F)$, defined by $(i_2^j)^*(m)[x] = m([i_2^j(x)])$. If now f is any element of $C^\infty_{co}(\Omega_1 \times \Omega_2,F)$, then by lemma 1.b), 1.5, and lemma 1.c), $f^{(j)}$ is in $C^\infty_{co}(\Omega_1 \times \Omega_2, \mathcal{L}^s_{co}((E_1 \times E_2)^j,F)$. Then it follows from lemma 1.a) that $(i_2^j)^* \circ f^{(j)}$ is in $C^\infty_{co}(\Omega_1 \times \Omega_2, \mathcal{L}^s_{co}(E_2^j,F))$, hence $(i_2^j)^* \circ f^{(j)} \in C_{co}(\Omega_1 \times \Omega_2, L^j_{co}(E_2,F))$. Let us denote the function $(i_2^j)^* \circ f^{(j)}$ by g_j. Then it follows from $f \in C^\infty_{co}(\Omega_1 \times \Omega_2,F)$ that for any $j \in \mathbb{N}$ and any $x_1 \in \Omega_1$ the function $g_j(x_1, \cdot) : \Omega_2 \to L^j_{co}(E_2,F)$ is Gâteaux-differentiable and that its Gâteaux-derivative is $g_{j+1}(x_1, \cdot)$. This proves that for any $x_1 \in \Omega_1$ the function $f(x_1, \cdot)$ belongs to $C^\infty_{co}(\Omega_2,F)$, hence the function $g : \Omega_1 \to C^\infty_{co}(\Omega_2,F)$, $g(x_1) = f(x_1, \cdot)$ can be defined.

In order to show continuity of g on Ω_1, let any $x_1 \in \Omega$, any $\varepsilon > 0$, and any continuous semi-norm on $C^\infty_{co}(\Omega_2,F)$ of the form $p_{\ell,K_2,\Omega_2,q}$ be given. Since g_j is uniformly continuous on $\{x_1\} \times K_2$ for any j,

there exists a neighbourhood $V_1 \times V_2$ of zero in $E_1 \times E_2$ such that for any $(x_1,x_2) \in \{x_1\} \times K_2$ and any $(h_1,h_2) \in V_1 \times V_2$ we have for any j with $0 \leq j \leq \ell$

$$\tilde{p}_{j,Q_2,q}(g_j(x_1 + h_1, x_2 + h_2) - g_j(x_1,x_2)) \leq \varepsilon$$

(where $\tilde{p}_{j,Q_2,q}$ denotes the semi-norm $u \to \sup_{y \in Q_2^j} q(u(y))$ on $L^j_{co}(E,F)$). This implies for any $h_1 \in V_1$

$$p_{\ell,K_2,Q_2,q}(g(x_1 + h_1) - g(x_1)) =$$

$$= \sup_{x_2 \in K_2} \sup_{0 \leq j \leq \ell} \tilde{p}_{j,Q_2,q}((g(x_1 + h_1) - g(x_1))^{(j)}(x_2)) =$$

$$= \sup_{x_2 \in K_2} \sup_{0 \leq j \leq \ell} \tilde{p}_{j,Q_2,q}(g_j(x_1 + h, x_2) - g_j(x_1,x_2)) \leq \varepsilon,$$

hence g is continuous.

3. PROPOSITION: *For $i = 1,2$ let E_i be a quasi-complete l.c. space for which $(E_i)'_c$ is complete and which equals $(E_i)''_{cc}$ topologically, and let Ω_i be an open subset of E_i. Assume furthermore that E_2 is a $k_{\mathbb{R}}$-space. Then there exists a continuous linear and injective map $A : C^\infty_{co}(\Omega_1 \times \Omega_2) \to C^\infty_o(\Omega_1, C^\infty_{co}(\Omega_2))$, defined by $A(f) : x_1 \to f(x_1, \cdot)$ for any $f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$.*

PROOF: Let us show first that for any $f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$ the function $g : x_1 \to f(x_1, \cdot)$ belongs to $C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$.

Let i_1 denote the linear continuous mapping $i_1 : E_1 \to E_1 \times E_2$, $i_1(x_1) = (x_1,0)$ and let $f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$ be given. As in the proof of lemma 2 one shows that for any $j \in \mathbb{N}$ the mapping $\varphi_j := (i_1^j)^* \circ f^{(j)}$

belongs to $C^\infty_{co}(\Omega_1 \times \Omega_2, L^j_{co}(E_1, \mathbb{R}))$. Hence, by lemma 2 the mapping $g_j : x_1 \to \varphi_j(x_1, \cdot)$ is in $C(\Omega_1, C^\infty_{co}(\Omega_2, L^j_{co}(E_1, \mathbb{R})))$. Now observe that by our hypotheses it follows from 3.2 and general results on the ε-product that we have natural isomorphisms

$$C^\infty_{co}(\Omega_2, L^j_{co}(E_1, \mathbb{R})) \simeq C^\infty_{co}(\Omega_2) \varepsilon L^j_{co}(E_1, \mathbb{R}) \simeq L^j_{co}(E_1, C^\infty_{co}(\Omega_2)).$$

Using this isomorphism, we get from $\tilde{g}_j$ the mapping $g_j \in C(\Omega_1, L^j_{co}(E_1, C^\infty_{co}(\Omega_2))$ for any $j \in \mathbb{N}$. Obviously $g_o = A(f)$, and we shall prove now that g_{j+1} is the Gâteaux-derivative of g_j. In order to do this, let us first remark that for any $k \in \mathbb{N}$ we have $\varphi_j^{(k)} = (i_1^j)^* \circ f^{(j+k)}$ and that

$$(\tilde{g}_j(x_1))^{(k)} = (i_2^k)^* \circ \varphi_j^{(k)}(x_1, \cdot) = (i_2^k)^* \circ (i_1^j)^* \circ f^{(j+k)}(x_1, \cdot)$$

follows from the proof of lemma 2. Hence we get

$$(1) \qquad (g_j(x_1)[y_1])^{(k)}(x_2)[y_2] = f^{(j+k)}(x_1, x_2)[i_1^j(y_1), i_2^k(y_2)].$$

Now let $x_1 \in \Omega_1$, $h_1 \in E_1$, any $\ell \in \mathbb{N}$, any compact subset K_2 of Ω_2, any compact Q_2 of E_2, any compact subset Q_1 of E_1 and $\varepsilon > 0$ be given. We have to prove that there exists $\delta > 0$, such that for any t with $0 < |t| \leq \delta$

$$(2) \qquad \sup_{y_1 \in Q_1} p_{\ell, K_2, Q_2}\left(\frac{1}{t}(g_j(x_1 + th_1) - g_j(x_1) - g_{j+1}(x_1[th_1]))\right) \leq \varepsilon .$$

By (1) and 1.5 we have to estimate for $0 \leq k \leq \ell$

$$\frac{1}{t}(f^{(j+k)}(x_1 + th_1, x_2) - f^{(j+k)}(x_1, x_2) -$$

$$- f^{(j+k+1)}(x_1, x_2)[th_1])[i_1^j(y_1), i_2^k(y_2)] \tag{3}$$

$$= \int_0^1 (f^{(j+k+1)}(x_1 + \tau th_1, x_2) - f^{(j+k+1)}(x_1, x_2))[h_1, i_1^j(y_1), i_2^k(y_2)]\, d\tau.$$

Since $f^{(j+k+1)}$ is continuous on $\Omega_1 \times \Omega_2$, it is uniformly continuous on a suitable neighbourhood of the compact set $\{x_1\} \times K_2$. By uniform continuity of $f^{(j+k+1)}$ and (3) it is clear that there exists $\delta > 0$ satisfying (2). Consequently we have shown that $g = g_o$ is an element of $C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$.

Linearity and injectivity of A are obvious. Continuity of A follows immediately from (1) and the definition of the corresponding topologies.

Now we want to prove that A is surjective if we impose some further conditions.

4. LEMMA: *For* $i = 1,2$ *let* E_i *be a l.c. space and let* Ω_i *be an open subset of* E_i. *Assume that* $E_1^j \times E_2^k$ *is a* $k_{\mathbb{R}}$*-space for any* $(j,k) \in \mathbb{N}^2$. *Let* g *be a any function in* $C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$.

a) *For any* $(j,k) \in \mathbb{N}^2$ *the mapping* $f^{(j,k)}: \Omega_1 \times \Omega_2 \times E_1^j \times E_2^k \to \mathbb{R}$ *defined by* $f^{(j,k)}(x_1, x_2, y_1, y_2) := (g^{(j)}(x_1)[y_1])^{(k)}(x_2)[y_2]$ *is continuous.*

b) *For any* $(j,k) \in \mathbb{N}^2$, *any* $(h_1, h_2) \in E_1 \times E_2$ *one has*

$$\lim_{t \to o} \frac{1}{t}(f^{(j,k)}(x_1 + th_1, x_2 + th_2, y_1, y_2) - f^{(j,k)}(x_1, x_2, y_1, y_2) =$$

$$= f^{(j+1,k)}(x_1, x_2, (h_1, y_1), y_2) + f^{(j,k+1)}(x_1, x_2, y_1, (h_2, y_2))$$

uniformly in $y_1 \in Q_1^j$ *and* $y_2 \in Q_2^k$ *for any compact set* Q_i *in* E_i $(i = 1,2)$.

PROOF: a) Observe that for any open subset Ω of a l.c. space E, any l.c. space F and any $f \in C^\infty_{co}(\Omega,F)$ the function

$$(x,y) \to f^{(j)}(x)[y]$$

is continuous on $K \times Q^j$ for any compact subset K of Ω and any compact set Q in E and has values in F. Hence for any compact subset K_1 of Ω_1 and any compact subset Q_1 of E_1, the function

$$(x_1,y_1) \to g^{(j)}(x_1)[y_1]$$

belongs to $C(K_1 \times Q_1^j, C^\infty_{co}(\Omega_2))$. Since

$$\rho_k : C^\infty_{co}(\Omega_2, \mathcal{L}^s_{co}(E_2^k)) \to C(K_2 \times Q_2^k),\ \rho_k(f)(x_2,y_2) = f(x_2)[y_2]$$

is linear and continuous for any compact set K_2 in Ω_2 and any compact set Q_2 in E_2, we get from lemma 1.b) that $\rho_k((g^{(j)}(\cdot)[\cdot])^{(k)})$ belongs to $C(K_1 \times Q_1^j, C(K_2 \times Q_2^k)) = C(K_1 \times Q_1^j \times K_2 \times Q_2^k) \simeq C(K_1 \times K_2 \times Q_1^j \times Q_2^k)$. This proves the continuity of $f^{(j,k)}$, since $E_1^{j+1} \times E_2^{k+1}$ is a $k_{\mathbb{R}}$-space for any $(j,k) \in \mathbb{N}^2$.

b) The second assertion is a consequence of the following considerations:

$$\frac{1}{t}(f^{(j,k)}(x_1 + th_1, x_2 + th_2, y_1, y_2) - f^{(j,k)}(x_1,x_2,y_1,y_2))$$

$$= \frac{1}{t}\left[(g^{(j)}(x_1 + th_1)[y_1])^{(k)}(x_2 + th_2)[y_2] - (g^{(j)}(x_1 + th_1)[y_1])^{(k)}(x_1)[y_1]\right]$$

$$+ \frac{1}{t}\left[(g^{(j)}(x_1 + th_1)[y_1])^{(k)}(x_1)[y_1] - (g^{(j)}(x_1)[y_1])^{(k)}(x_2)[y_2]\right]$$

$$=: A(t) + B(t).$$

Since $g \in C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$, it is obvious that

$$\lim_{t\to 0} B(t) = (g^{(j+1)}(x_1)[(h_1,y_1)])^{(k)}(x_2)[y_2] = f^{(j+1,k)}(x_1,x_2,(h_1,y_1),y_2)$$

uniformly in $y_1 \in Q_1^j$ and $y_2 \in Q_2^k$.

By lemma 2.2 we get

$$A(t) = \int_0^1 (g^{(j)}(x_1 + th_1)[y_1])^{(k+1)}(x_2 + \tau th_2)[h_2,y_2]\,d\tau$$

$$= \int_0^1 f^{(j,k+1)}(x_1 + th_1, x_2 + \tau th_2, y_1, (h_2,y_2))\,d\tau .$$

It follows from a) that $f^{(j,k+1)}$ is uniformly continuous in a neighbourhood of the compact set $\{x_1\} \times \{x_2\} \times Q_1^j \times Q_2^k$, hence we also have

$$\lim_{t\to 0} A(t) = f^{(j,k+1)}(x_1,x_2,y_1,(h_2,y_2))$$

uniformly in $y_1 \in Q_1^j$ and $y_2 \in Q_2^k$.

5. THEOREM: *For $i = 1,2$ let E_i be a quasi-complete l.c.space which equals $(E_i)'_{cc}$ topologically and let Ω_i be an open subset of E_i. Assume furthermore that $E_1^j \times E_2^k$ is a $k_{\mathbb{R}}$-space for any $(j,k) \in \mathbb{N}^2$. Then the mapping $A : C^\infty_{co}(\Omega_1 \times \Omega_2) \to C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$, $A(f) : x_1 \to f(x_1,\cdot)$, is a topological isomorphism.*

PROOF: First let us show that A is surjective. Let g be any element of $C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$. By lemma 4 the function $f : (x_1,x_2) \to g(x_1)(x_2)$ is continuous on $\Omega_1 \times \Omega_2$. We shall prove $f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$. Then it is obvious that $A(f) = g$, hence A is surjective.

In order to prove $f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$ let us remark the following:

Let $(j,k) \in \mathbb{N}^2$ and a surjection $\sigma : \{1,\dots,j\} \cup \{1',\dots,k'\} \to \{1,\dots,j+k\}$ be given. Then we define a continuous linear map $\pi_\sigma : (E_1 \times E_2)^{j+k} \to E_1^j \times E_2^k$ by

$$\pi_\sigma((e_{11},e_{21}),\dots,(e_{1,j+k},e_{2,j+k}))$$

$$= ((e_{1,\sigma(1)},\dots,e_{1,\sigma(j)}),(e_{2\sigma(1')},\dots,e_{2\sigma(k')})).$$

By lemma 4.a) the function $\tilde{f}_\sigma^{(j,k)} := f^{(j,k)} \circ \pi_\sigma$ is continuous on $\Omega_1 \times \Omega_2 \times (E_1 \times E_2)^{j+k}$ and $\tilde{f}_\sigma^{(j,k)}(x_1,x_2,\cdot)$ is $(j+k)$-linear for any $(x_1,x_2) \in \Omega_1 \times \Omega_2$. Because of the continuity of $\tilde{f}_\sigma^{(j,k)}$ the mapping $f_\sigma^{(j,k)} : \Omega_1 \times \Omega_2 \to L_{co}^{j+k}(E_1 \times E_2, \mathbb{R})$, defined by

$$f_\sigma^{(j,k)}(x_1,x_2) : [y] \to \tilde{f}_\sigma^{(j,k)}(x_1,x_2,y)$$

is continuous. Using the mappings $f_\sigma^{(j,k)}$ it is easy to prove by induction that f belongs to $C_{co}^\infty(\Omega_1 \times \Omega_2)$. Let us show that f is Gâteaux-differentiable:

Define $\sigma_1 : \{1\} \cup \emptyset \to \{1\}$ by $\sigma_1(1) = 1$ and define $\sigma_2 : \emptyset \cup \{1'\} \to \{1\}$ by $\sigma_2(1') = 1$. Then we get from 4.b) that for $x = (x_1,x_2) \in \Omega_1 \times \Omega_2$ and $h = (h_1,h_2) \in E_1 \times E_2$

$$\lim_{t \to 0} \frac{1}{t}(f(x+th) - f(x) - f_{\sigma_1}^{(1,0)}(x)[th] - f_{\sigma_2}^{(0,1)}(x)[th]) = 0.$$

Hence $f'(x)[h] = (f_{\sigma_1}^{(1,0)}(x) + f_{\sigma_2}^{(0,1)}(x))[h]$, and $f \in C_{co}^1(\Omega_1 \times \Omega_2)$ by lemma 2.3. From this and lemmma 4.b) we get by induction that for any $\ell \in \mathbb{N}$ the function f is in $C_{co}^\ell(\Omega_1 \times \Omega_2)$ and that $f^{(\ell)}$ can be represented as a sum of $f_\sigma^{(j,k)}$ where the sum runs over all j and k in $\mathbb{N}_o$ with $j+k = \ell$ and over certain σ. This proves that

$f \in C^\infty_{co}(\Omega_1 \times \Omega_2)$.

Hence we have shown that $A : C^\infty_{co}(\Omega_1 \times \Omega_2) \to C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2))$ is bijective. From the representation of $(A^{-1}(g))^{(\ell)}$ indicated above it follows that A^{-1} is continuous. Then A is a topological isomorphism by proposition 3.

REMARK: Results of the same type as in theorem 5 are also given in the lecture notes of Fröhlicher and Bucher [15] (with a different definition of differentiability) and in Colombeau [11], [12]. It seems to be impossible to get the result on (DFM)-spaces given below by bornological methods.

Concluding this section, let us combine theorem 5 and some of the results in section 3. Then we get

6. THEOREM: *Let E_1 and E_2 be either (F)-spaces or (DFM)-spaces and let Ω_i be an open subset of E_i for $i = 1,2$. Then we have the following topological isomorphisms*

$$C^\infty_{co}(\Omega_1 \times \Omega_2) \cong C^\infty_{co}(\Omega_1, C^\infty_{co}(\Omega_2)) \cong C^\infty_{co}(\Omega_1) \varepsilon C^\infty_{co}(\Omega_2).$$

7. THEOREM: *For $i = 1,2$, let Ω_i be an open subset of the l.c. space E_i. Assume that either*

1) *E_1 and E_2 are (F)-spaces, one of which has a.p., or*

2) *E_1 and E_2 are (DFM)-spaces, one of which is a (DFSA)-space.*

Then the following holds

$$C^\infty_{co}(\Omega_1 \times \Omega_2) \cong C^\infty_{co}(\Omega_1) \check{\otimes}_\varepsilon C^\infty_{co}(\Omega_2).$$

8. REMARK: The dual of $C^\infty_{co}(\Omega)$ forms a natural generalization of the space of distributions with compact support to infinite dimensions.

It is obvious that many of the results of this article can also be regarded as results on the dual of $C^\infty_{co}(\Omega)$. E.g. theorem 3.10 is of importance in connection with the theorem of Paley - Wiener - Schwartz (in order to see this one has to extend several results to complex valued functions on Ω, then (for certain l.c. spaces E) one can define the Fourier-Laplace transform of any $T \in C^\infty_{co}(\Omega,\mathbb{C})'$ as a holomorphic function on the complexification of E'_c, satisfying a certain growth condition). Theorem 6 can be used to define a convolution $* : C^\infty_{co}(E)' \times C^\infty_{co}(E)' \to C^\infty_{co}(E)'$. The precise formulation of the results just mentioned will be contained in a subsequent paper.

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A LOOK AT APPROXIMATION THEORY

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1. INTRODUCTION

I would like to describe very briefly how I was led to become seriously interested in Approximation Theory, that is, to indicate the motivation that I had in my mind.

This field has developed in Brazil in the past ten years or so, thanks also to the work of Silvio Machado, João Bosco Prolla and Guido Zapata, as well as the research school that they formed.

If I had to reduce bibliographical references to a bare minimum, in what concerns the work of the Brazilian school in Approximation Theory and its relationship to the research of other groups, I would quote my monograph *Elements of Approximation Theory* (1967), as well as Prolla's monograph *Approximation of Vector Valued Functions* (1977) (see [34], [54]). However, the bibliography at the end is up-to-date and complete with respect to the work by Machado, Prolla, Zapata and myself. It is extremely incomplete otherwise. Let me emphasize the following aspects:

1) I shall restrict myself here to the real valued case. The vector valued case was treated in a desirable degree of generality through vector fibrations by Machado [16] and Prolla [40] (see also [35], [36]).

2) In the complex case, I point out the work by Machado on the Bishop and Weierstrass-Stone theorems [18]. (See also [54]).

3) We call attention to the work by Zapata on Mergelyan's theorem and quasi-analytic classes [65] (see also [54]).

4) Weighted approximation in the continuously differentiable case was studied by Zapata [63], [64].

5) A density theorem for polynomial algebras of continuously differentiable mappings in infinite dimensions and its relationship to the Banach-Grothendieck approximation property was investigated by Prolla and Guerreiro [53] (see also [38]).

6) Nonarchimedean Approximation Theory has been studied by Prolla [56], and Carneiro [7], [8].

2. APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE MAPPINGS

In 1947, Marshall Stone came from the University of Chicago to lecture at the Universidade Federal do Rio de Janeiro (known then as Universidade do Brasil) for three months. He offered a beautiful course on "Rings of Continuous Functions". Among other things, he talked about his celebrated paper A *Generalized Weierstrass Approximation Theorem* which he had just written. It was published next year in volume 21 (1948) of Mathematics Magazine. This is a good example of an article that became famous in spite of the fact that is was published in an obscure journal.

Stone's course dealt with continuous functions, and was going to have a lasting influence on me. It was during and shortly after it that, in 1948, I thought of and proved, but did not publish then,

what I called the Weierstrass-Stone theorem for modules [34]. I will come back to this aspect in a brief while. The reason I did not publish right away that result for modules was this. It took me years until 1960 - 1961, while I visited Brandeis University for four months, to realize the interest for Approximation Theory of modules in place of algebras, and to get started in weighted approximation proper for continuous functions.

In 1948, I went to the University of Chicago for a two year visit during 1948 - 1950, at the invitation of Stone. While there, I had an apportunity, in 1949, of presenting at André Weil's seminar the then recent article "On ideals of differentiable functions" by Hassler Whitney, just published in volume 70 (1948) of the American Journal of Mathematics. After my lecture, Irving Segal asked me: how about a similar result for algebras of continuously differentiable functions, along the lines of the Weierstrass-Stone theorem ? In other words, the problem was to describe the closure of a subalgebra of continuously differentiable functions, or equivalently, to describe the closed subalgebras of continuously differentiable functions, in the spirit of the Weierstrass-Stone theorem. To the best of my knowledge, this problem has remained open so far; see below for the conjecture that I have in mind in this respect.

Pressed by Segal's question, I studied immediately in 1949 [23] the noteworthy case of dense subalgebras, to obtain the following result. Let E be a real m-differentiable ($m = 0,1,\ldots,\infty$) manifold of finite dimension. Denote by $C^m(E)$ the algebra of all continuously m-differentiable real functions on E, endowed with the topology τ_m of uniform convergence on the compact subsets of E of such functions and all their differentials up to order m.

THEOREM 1: *If* $m \geq 1$, *a subalgebra* A *of* $C^m(E)$ *is dense in* $C^m(E)$ *for* τ_m *if and only if:*

(N1) *For every* $x \in E$, *there is* $f \in A$ *such that* $f(x) \neq 0$.

(N2) *For every* $x \in E, y \in E$, $x \neq y$, *there is* $f \in A$ *such that* $f(x) \neq f(y)$.

(N3) *For every* $x \in E$ *and every tangent vector* $t \neq 0$ *to* E *at* x, *there is* $f \in A$ *such that*

$$\frac{\partial f}{\partial t}(x) = df(x)(t) \neq 0.$$

These conditions do not depend on m. The case $m = 0$ excluded by the above result is covered by the Weierstrass-Stone theorem.

Coming back to the question Segal asked me in 1949, bit by bit I was led to formulate the following conjecture. If it is true, the Whitney ideal theorem and the above density theorem are subsumed by it. For the sake of simplicity of terminology and notation only, let us assume that $1 \leq m < \infty$ and that $E = U$ is a nonvoid open subset of $\mathbb{R}^n$ for some $n = 1, 2, \ldots$, extension to arbitrary E and to $m = \infty$ being easy. If A is a subalgebra of $C^m(U)$, consider the equivalence relation U/A defined by A on U, according to which $x, y \in U$ are equivalent when $f(x) = f(y)$ for all $f \in A$.

CONJECTURE 2: *If* $f \in C^m(U)$ *and* A *is a subalgebra of* $C^m(U)$ *then* f *belongs to the closure of* A *in* $C^m(U)$ *for* τ_m *if (and always only if), for every compact subset* K *of* U *contained in some equivalence class modulo* U/A *and every* $\varepsilon > 0$, *there is* $g \in A$ *such that* $|D^\alpha g(x) - D^\alpha f(x)| < \varepsilon$ *for any* $x \in K$ *and any partial derivative* D^α *of order at most equal to* m.

Notice that f belongs to the closure of A in $C^m(U)$ for τ_m when the above condition holds true for every compact subset K of U, by definition, not just for those K contained in some equivalence

class modulo U/A. The above conjecture is an aspect of what I called *localizability* (see below too). If true, the above conjecture has a natural extension to modules in place of algebras.

There is a more naive conjecture, which is easily seen to be false. We might indeed conjecture that every subalgebra A of $C^m(E)$ which is closed for τ_m is also closed for the topology τ_{ms} of simple convergence at points of E of functions and all their differentials up to order m. For $m=0$, this is indeed the case; as a matter of fact, the statement that τ_o and τ_{os} have to same closed subalgebras of $C^o(E) = C(E)$ is easily seen to be equivalent to the Weierstrass-Stone theorem.

EXAMPLE 3: *Let* A *be the subalgebra of* $C^1(\mathbb{R})$ *of all* $f \in C^1(\mathbb{R})$ *such that* $f(1/k) = f(0)$ *for all* $k=1,2,\ldots$ *and moreover* $\Sigma_{k=1}^{\infty} f'(1/n)/n^2 = 0$. *Then* A *is closed for* τ_1 *but it is not closed for* τ_{1s}.

A few years ago, I asked Jaime Lesmes the question of extending the above Theorem 1 to infinite dimensions. I also did raise that question during a lecture I gave at Madrid, where José Llavona got interested in it. Recent work along this line was done by Lesmes [13] and Prolla [49], [53] in Brazil, and by Llavona [14], [15] in Spain. We now summarize that aspect very succintly, along the lines of [38].

Let E, F be Hausdorff real locally convex spaces, $E \neq 0$, $F \neq 0$, U a nonvoid open subset of E and $m = 1,2,\ldots,\infty$. We denote by $C^m(U;F)$ the vector space of all mappings $f : U \to F$ that are continuously m-differentiable in the following sense:

1) f is finitely m-differentiable; that is, for every finite dimensional vector subspace S of E with $S \neq 0$ and $U \cap S$ nonvoid, we assume that the restriction $f|(U \cap S)$ is m-differentiable in the classical sense. Thus we have the differentials

$$d^k f : U \to \mathcal{L}_{as}({}^k E;F)$$

for $k = 0,1,\ldots, k \leq m$, with values in the vector space $\mathcal{L}_{as}({}^{k}E;F)$ of all symmetric k-linear mappings of E^k to F.

2) The mapping

$$(x,t) \in U \times E \to d^k f(x) t^k \in F$$

is continuous of every $k = 0,1,\ldots, k \leq m$. In particular, $d^k f(x)$ belongs to the vector subspace $\mathcal{L}_s({}^{k}E;F)$ of all continuous symmetric k-linear mappings of E^k to F.

We endow $C^m(U;F)$ with the topology τ_m defined by the family of seminorms depending on the parameters k, β, K, L

$$f \in C^m(U;F) \mapsto$$

$$p_{KL}^{k\beta}(f) = \sup \{\beta[\, d^k f(x) t^k] \; ; \; x \in K, \; t \in L\} \in \mathbb{R}$$

for $k = 0,1,\ldots, k \leq m$, β being a continuous seminorm on F and K,L being nonvoid compact subsets of U, E respectively.

We shall use the notion of polynomial algebra; see the convention on page 63, [54].

THEOREM 4: *Let* $m \geq 1$ *and* A *be a polynomial subalgebra of* $C^m(U;F)$. *Suppose that there is a subset* G *of the vector space* $E' \otimes E$ *of all continuous linear endomorphisms of* E *with finite dimensional images, such that:*

1) *The identity mapping* I_E *belongs to the closure of* G *for the compact-open topology on the vector space* $\mathcal{L}(E;E)$ *of all continuous linear endomorphisms of* E.

2) *For every* $J \in G$, *every nonvoid open subset* V *of* U *such that* $J(V) \subset U$ *and every* $f \in A$, *it follows that the restriction* $(f \circ J) \mid V = f \circ (J \mid V)$ *belongs to the closure in*

$C^m(U;F)$ *for* τ_m *of* $A \mid V$.

Then A *is dense in* $C^m(U;F)$ *for* τ_m *if and only if:*

(N1) *For every* $x \in U$, *there is* $f \in A$ *such that* $f(x) \neq 0$.

(N2) *For every* $x \in U$, $y \in U$, $x \neq y$, *there is* $f \in A$ *such that* $f(x) \neq f(y)$.

(N3) *For every* $x \in U$, $t \in E$, $t \neq 0$, *there is* $f \in A$ *such that*

$$\frac{\partial f}{\partial t}(x) = df(x)(t) \neq 0.$$

If E is finite dimensional, conditions 1) and 2) of Theorem 4 are satisfied by G reduced to I_E. Hence Theorem 4 implies Theorem 1.

Condition 1) of Theorem 4 implies that E has the Banach-Grothendieck approximation property, that is, I_E belongs to the closure of $E' \otimes E$ in $\mathcal{L}(E;E)$ for the compact-open topology. Thus Theorem 4 leads to the following conjecture:

CONJECTURE 5: *For every given* E, *the following conditions are equivalent:*

(C1) *For arbitrary* U, F, $m \geq 1$, *then every polynomial subalgebra* A *is dense in* $C^m(U;F)$ *for* τ_m *if (and always only if)* A *satisfies* (N1), (N2), (N3).

(C2) E *has the Banach-Grothendieck approximation property.*

It is known that (C1) implies (C2). The conjecture that (C2) implies (C1) is an attempt to improve Theorem 4.

In the direction of research that I just mentioned, there is more generally the question of studying Approximation Theory for algebras or modules of continuously differentiable vector valued mappings by using weights. This question however is still wide open, in spite of the available results. See the next section for the continuous case.

3. WEIGHTED APPROXIMATION FOR MODULES AND ALGEBRAS OF CONTINUOUS FUNCTIONS

Let me talk now about the Weiestrass-Stone theorem for modules, how it led me to the Bernstein approximation problem and what I then called the weighted approximation problem (or the Bernstein - Nachbin approximation problem, according to a more recent terminology by other authors).

Let E be a completely regular topological space, and $C(E)$ denote the algebra of all continuous real functions on E endowed with the compact-open topology.

There is the ideal theorem for $C(E)$ which reads as follows. If I is an ideal in $C(E)$ and $I^{-1}(0) = \cap_{f \in I} f^{-1}(0)$, then $f \in C(E)$ belongs to the closure of I in $C(E)$ if and only if f vanishes on $I^{-1}(0)$.

More generally, there is the Weiestrass-Stone theorem for a subalgebra A of $C(E)$ which reads as follows. Let E/A be the equivalence relation on E defined by $x_1 \sim x_2$ if $x_1, x_2 \in E$ and $f(x_1) = f(x_2)$ for every $f \in A$. Consider $A^{-1}(0) = \cap_{f \in A} f^{-1}(0)$ which either is one of equivalence classes modulo E/A or else is void. Then $f \in C(E)$ belongs to the closure of A in $C(E)$ if and only if f is constant on every equivalence class modulo E/A and f vanishes on $A^{-1}(0)$ if $A^{-1}(0)$ is nonvoid.

In the ideal theorem, we have a module I over the algebra $A = C(E)$.

In the Weierstrass-Stone theorem, we have a module A over the algebra $A + \mathbb{R}$ generated by A and all constant real functions on E. If we considered just a vector subspace W of $C(E)$, we would have a module W over the algebra A of all constant real functions on E. In the succession of these three cases, the algebra of multipliers varies from the largest to the smallest possibility containing the unit.

More generally, let A be a subalgebra of $C(E)$ which we may now assume to contain the unit without loss of generality, and let W be a vector subspace of $C(E)$ which is a module over A so that $AW \subset W$. The Weierstrass-Stone theorem for modules reads as follows. Introduce as before the equivalence relation E/A on E. Then $f \in C(E)$ belongs to the closure of W in $C(E)$ if, and only if, for every compact subset K of E contained in some equivalence class modulo E/A and every $\varepsilon > 0$, there is $g \in W$ such that $|g(x) - f(x)| < \varepsilon$ for every $x \in K$. This is an aspect of what later I called "localizability" (see below).

It is known that a topological vector space $\mathcal{W}$ has some representation by continuous real functions, that is $\mathcal{W}$ is isomorphic and homeomorphic to a topological vector subspace W of some $C(E)$, if and only if $\mathcal{W}$ is a Hausdorff locally convex space. Thus the following representation theory question arises naturally. Given a topological vector space $\mathcal{W}$ and an algebra $\mathcal{A}$ of linear operators of $\mathcal{W}$, where $\mathcal{A}$ contains the identity operator of $\mathcal{W}$, when does the pair $\mathcal{A}$, $\mathcal{W}$ have some representation by continuous real functions? More explictly, we want to know when we can find A, W in some $C(E)$ as above, a surjective vector space isomorphism and homeomorphism $i_m : \mathcal{W} \to W$ and a surjective algebra isomorphism $i_a : \mathcal{A} \to A$ so that $i_m[T(x)] = i_a(T) i_m(x)$ for every $T \in \mathcal{A}$ and $x \in \mathcal{W}$.

We have the following three results that I proved in 1956, but never published. Let $\mathcal{W}$ be a topological vector space, and $\mathcal{A}$ be an algebra of linear operators of $\mathcal{W}$. Assume that $\mathcal{A}$ contains the identity operator of $\mathcal{W}$ and is commutative (without commutativity of $\mathcal{A}$

we would below replace ideals by left ideals in A).

For every ideal J in A, let JW denote the A-invariant vector subspace of W spanned by the $T(x)$ with $T \in J$ and $x \in W$. We say that a subset X of W is A-convex if X is convex and $X = \cap_J (X + JW)$ for all ideals J in A of codimension 1 in A. We say that W is locally convex under A in case the A-convex neighborhoods of 0 in W form a basis of neighborhoods at 0. This implies local convexity in the usual sense, of course. If A is reduced to the scalar operators of W, then A-convexity and local convexity under A reduce to convexity and local convexity in the usual sense. The above definitions are subsumed by §3, [24].

A linear operator T on W is said to be "directed" when the neighborhoods V of 0 in W, for each of which there is $\lambda = \lambda(V) > 0$ such that $T(V) \subset \lambda V$, form a basis of neighborhoods at 0; in equivalent terms, when corresponding to every neighborhood U of 0 in W we may find another neighborhood V of 0 in W and $\varepsilon > 0$ such that $\cup_{k=0}^{\infty} T^k(\varepsilon^k V) \subset U$. More generally, the members of a collection C of linear operators on W are said to be "similarly directed" if the neighborhoods V of 0 in W, for each of which there is $\lambda = \lambda(V,T) > 0$ such that $T(V) \subset \lambda V$ for every $T \in C$, form a basis of neighborhoods at 0. Directedness of a linear operator implies its continuity. Both directedness and similar directedness reduce to continuity when W is a normed space. These concepts arise only in treating more general topological vector spaces. Thus the hypothesis in Theorem 6 below that the operators in A be similarly directed is automatically satisfied when W is a normed space.

THEOREM 6: *The pair A, W has some representation by continuous real functions if and only if W is a Hausdorff space which is locally convex under A, and the operators in A are similarly directed.*

THEOREM 7: *If the pair A, W has some representation by continuous real functions and S is a vector subspace of W which is invariant under A, then the quotient pair $A/S, W/S$ has some representation by continuous real functions if and only if S is closed in W.*

THEOREM 8: *If the pair A, W has some representation by continuous real functions, then spectral synthesis holds in the following sense. If S is a closed proper vector subspace of W which is invariant under A, then S is the intersection of all closed vector subspaces of W which are invariant under A, have codimension one in W and contain S.*

The passing to a quotient statement of Theorem 7 implies spectral synthesis in Theorem 8, which may be viewed as an abstract version of the Weierstrass-Stone theorem for modules. Let us also point out that, when A is reduced to the scalar operators of W, then Theorem 8 becomes the following statement. Every closed proper vector subspace S of a locally convex space W is the intersection of all closed vector subspaces of W which have codimension one in W and contain S. As it is classical, such a statement is equivalent to the Hahn-Banach theorem. Thus Theorem 8 may be looked upon as a generalization of both the Weierstrass-Stone theorem for modules and the Hahn-Banach theorem for locally convex spaces.

We may then ask the following natural question. To what extent the condition of the operators in A being similarly directed is crucial for the validity of Theorem 6, or Theorem 7, or Theorem 8? Local convexity under A is not superfluous. In fact, letting A be reduced to the scalars operators of W, then it may be false that every closed proper vector subspace S of W is the intersection of sll closed vector subspaces of W which have condimension one in W and contain S, in case W is not assumed to be locally convex. The

answer to the above natural question is no. The example that I found in 1957 led me to the classical Bernstein approximation problem, as I shall describe next.

EXAMPLE 9: Let W be the Fréchet space of all continuous real functions on $\mathbb{R}$ that are rapidly decreasing at infinity. Call $A = P(\mathbb{R})$ the algebra of all real polynomials on $\mathbb{R}$. Every $a \in C(\mathbb{R})$ that is slowly increasing at infinity gives rise to the continuous linear operator $T_a : f \in W \to af \in W$ which is directed if and only a is bounded. Thus A may be viewed as a commutative algebra of continuous linear operators of W containing the identity operator of W, but each such operator is directed if and only if the corresponding polynomial is constant. It is clear that W is locally convex under A. There is some $w \in W$ vanishing nowhere in $\mathbb{R}$ such that Aw is not dense in W (this is easily seen to be equivalent to existence of some $w \in W$ vanishing nowhere in $\mathbb{R}$ that is not a fundamental weight in the sense of BAP-2 or BAP-1 below). Then the closure $\overline{Aw}$ in W is a closed proper vector subspace of W which is invariant under A. Since w never vanishes in $\mathbb{R}$, it can be shown that $\overline{Aw}$ is not contained in any closed vector subspace of W which is invariant under A, having condimension one in W. Thus Theorem 8 does not hold in this case due to lack of directedness. A fortiori Theorem 7 and Theorem 6 do not hold in this case for the same reason.

This counterexample leads us to the *classical Bernstein approximation problem*, usually formulated in the following two forms, where $P(\mathbb{R}^n)$ is the algebra of all real polynomials on $\mathbb{R}^n$ for $n = 1, 2, \ldots$.

BAP-1. Let $v : \mathbb{R}^n \to \mathbb{R}_+$ be an upper semicontinuous "weight" and $Cv_\infty(\mathbb{R}^n)$ be the vector space of all $f \in C(\mathbb{R}^n)$ such that vf tends to 0 at infinity, seminormed by $\|f\|_v = \sup\{v(x) \cdot |f(x)| ; x \in \mathbb{R}^n\}$. Assume that v is rapidly decreasing at infinity, that is $P(\mathbb{R}^n) \subset Cv_\infty(\mathbb{R}^n)$.

When is $P(\mathbb{R}^n)$ dense in $Cv_\infty(\mathbb{R}^n)$? We then say that v is a *fundamental weight*. We shall denote by Ω_n the set of all such fundamental weights in the sense of Bernstein. For technical reasons we also introduce the set Γ_n of all such v such that $v^k \in \Omega_n$ for all $k > 0$. Clearly $\Gamma_n \subset \Omega_n$. This inclusion is proper.

B A P - 2. Let $C_\infty(\mathbb{R}^n)$ be the Banach space of all $f \in C(\mathbb{R}^n)$ tending to 0 at infinity, normed by $\|f\| = \sup\{|f(x)|\,;\, x \in \mathbb{R}^n\}$; it is the special case of $Cv_\infty(\mathbb{R}^n)$ when $v = 1$. Assume that $w \in C(\mathbb{R}^n)$ is rapidly decreasing at infinity, that is $P(\mathbb{R}^n)\, w \subset C_\infty(\mathbb{R}^n)$, and call w a *weight*. When is $P(\mathbb{R}^n)\, w$ dense in $C_\infty(\mathbb{R}^n)$? We then say that w is a *fundamental weight*.

If $w \in C(\mathbb{R}^n)$ is rapidly decreasing at infinity, then w is a fundamental weight in the sense of B A P - 2 if and only if w never vanishes on $\mathbb{R}^n$ and $|w|$ is a fundamental weight in the sense of B A P - 1. However a fundamental weight v in the sense of B A P - 1 may vanish on $\mathbb{R}^n$ and may fail to be continuous. It is in this sense that B A P - 1 is a better way of looking at the concept of fundamental weights in the sense of Bernstein than B A P - 2.

The following are the simplest criteria for an upper semicontinuous function $v : \mathbb{R} \to \mathbb{R}_+$ to belong to Γ_1, thus to Ω_1, by increasing degree of generality:

BOUNDED CASE: v *has a bounded support.*

ANALYTIC CASE: *There are* $C > 0$ *and* $c > 0$ *for which, for any* $x \in \mathbb{R}$, *we have*

$$v(x) \leq C \cdot e^{-c|x|}.$$

QUASI-ANALYTIC CASE: *We have* $\sum_{m=1}^{\infty} \frac{1}{\sqrt[m]{M_m}} = +\infty$ *where, for*

$m = 0,1,\ldots,$ *we set*

$$M_m = \sup \{ v(x) \cdot |x^m| ; \ x \in \mathbb{R} \} .$$

In BAP-1, the subalgebra $P(\mathbb{R}^n)$ of $C(\mathbb{R}^n)$ is contained in $Cv_\infty(\mathbb{R}^n)$, and we have the weight v in the definition of $Cv_\infty(\mathbb{R}^n)$. In BAP-2, the submodule $P(\mathbb{R}^n)w$ over the subalgebra $P(\mathbb{R}^n)$ of $C(\mathbb{R}^n)$ is contained in $C_\infty(\mathbb{R}^n)$, and we have the weight w in the definition of $P(\mathbb{R}^n)w$.

Thus I was led to the following general formulation of the *weighted approximation problem*. The viewpoint thus adopted embraces the Weierstrass-Stone theorem for modules, thus for algebras, and the Bernstein approximation problem. Actually, it is guided by the idea of extending the classical Bernstein approximation problem in the same style that the Weierstrass-Stone theorem generalizes the classical Weierstrass theorem (see [34] for details).

Let V be a set of upper semicontinuous positive real functions on a completely regular topological space E. We assume that V is *directed* in the sense that, if $v_1, v_2 \in V$, there are $\lambda > 0$ and $v \in V$ such that $v_1 \leq \lambda v$ and $v_2 \leq \lambda v$. Each element of V is called a *weight*. The vector subspace of $C(E)$ of all $f \in C(E)$ such that, for any $v \in V$ and any $\varepsilon > 0$, the closed subset $\{x \in E; \ v(x) \cdot |f(x)| \geq \varepsilon\}$ is compact, will be denoted by $CV_\infty(E)$. Each $v \in V$ determines a seminorm $f \to \|f\|_v = \sup \{ v(x) \cdot |f(x)| ; \ x \in E \}$ on $CV_\infty(E)$. the natural topology on the *weighted space* $CV_\infty(E)$ is defined by the family of all such seminorms.

Let $A \subset C(E)$ be a subalgebra containing the unit, and $W \subset CV_\infty(E)$ be a vector subspace. Assume that W is a module over A, that is $AW \subset W$. The *weighted approximation problem* consists of asking for a description of the closure of W in $CV_\infty(E)$ under such circumstances.

We say that W is *localizable under* A *in* $CV_\infty(E)$ when the following

condition holds true: if $f \in CV_\infty(E)$, then f belongs to the closure of W in $CV_\infty(E)$ if (and always only if), for any $v \in V$, any $\varepsilon > 0$ and any equivalence class X modulo E/A, there is $w \in W$ such that $v(x) \cdot |w(x) - f(x)| < \varepsilon$ for any $x \in X$. The *strict weighted approximation problem* consists of asking for necessary and sufficient conditions in order that W be localizable under A in $CV_\infty(E)$.

We denote by $G(A)$ a subset of A which topologically generates A as an algebra with unit, that is, such that the subalgebra of A generated by $G(A)$ and one is dense in A for the topology of $C(E)$. We also introduce a subset $G(W)$ of W which topologically generates W as a module of A, that is, the submodule over A of W generated by $G(W)$ is dense in W for the topology of $CV_\infty(E)$.

A basic result is then the following one.

THEOREM 10: *Assume that, for every* $v \in V$, *every* $a \in G(A)$ *and every* $w \in G(W)$, *there is* $\gamma \in \Gamma_1$ *such that*

$$v(x) \cdot |w(x)| \leq \gamma[a(x)]$$

for any $x \in E$. *Then* W *is localizable under* A *in* $CV_\infty(E)$.

We may combine Theorem 10 with the indicated criteria for membership of Γ_1. Let us consider explicitly the analytic case.

COROLLARY 11: *Assume that, for every* $v \in V$, *every* $a \in G(A)$ *and every* $w \in G(W)$, *there are* $C > 0$ *and* $c > 0$ *such that*

$$v(x) \cdot |w(x)| \leq C \cdot e^{-c \cdot |a(x)|}$$

for any $x \in E$. *Then* W *is localizable under* A *in* $CV_\infty(E)$.

As a particular case of the above results for modules, we have

the following ones for algebras. For simplicity sake, assume that V is strictly positive, that is, for every $x \in E$ there is $v \in V$ such that $v(x) > 0$. Let A be contained in $CV_\infty(E)$. We say that A is *localizable in* $CV_\infty(E)$ when the following condition holds true: if $f \in CV_\infty(E)$, then f belongs to the closure of A in $CV_\infty(E)$ if (and always only if) f is constant on every equivalence class modulo E/A.

We denote by $G(A)$ a subset of A which topologically generates A as an algebra with unit, that is such that the subalgebra of A generated by $G(A)$ and one is dense in A for the topology of $CV_\infty(E)$.

The particular case is then the following one.

THEOREM 12: *Assume that, for every* $v \in V$ *and every* $a \in G(A)$, *there is* $\gamma \in \Gamma_1$ *such that*

$$v(x) \leq \gamma[a(x)]$$

for any $x \in E$. *Then* A *is localizable in* $CV_\infty(E)$.

We may combine Theorem 12 with the indicated criteria for membership of Γ_1. Let us consider explictly the analytic case.

COROLLARY 13: *Assume that, for every* $v \in V$ *and every* $a \in G(A)$, *there are* $C > 0$ *and* $c > 0$ *such that*

$$v(x) \leq C \cdot e^{-c \cdot |a(x)|}$$

for any $x \in E$. *Then* A *is localizable in* $CV_\infty(E)$.

We quote [34], [37] for additional details.

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BANACH ALGEBRAS OVER VALUED FIELDS

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ABSTRACT

By "Gelfand theory" here is meant the study of the consequences of topologizing the maximal ideals of a Banach algebra. The theory is most rich when the underlying field is that of the complex numbers. If the underlying field is $\mathbb{R}$ or some other valued field, a theory can still be developed however and that is discussed here. First the Gelfand theory for complex Banach algebras is reviewed briefly; then the analogous theory for the case when the field carries a non-archimedean real-valued valuation is presented. In the course of the latter discussion, a Stone-Weierstrass theorem is needed. In the last part of the paper some versions of the Stone-Weierstrass theorem which hold in algebras of continuous functions over fields with nonarchimedean valuation are discussed.

1. CLASSICAL GELFAND THEORY. ([1], [9])

If G is an open subset of $\mathbb{C}$, the complex numbers, and X is a topological vector space, a map $x : G \to X$ is *analytic in* G if the

difference quotient has a limit at each point in G.

For the vector-valued version of Liouville's theorem to hold, the vector space must have a good supply of continuous linear functionals. The dual space X' must be *total* in the sense that if every $f \in X'$ vanishes at x, then x must be 0.

1.1. LIOUVILLE'S THEOREM: *If X is a* TVS *and X' is total then if $x: \mathbb{C} \to X$ is entire and bounded, then x must be constant.* ([1], p.211).

For the remainder of the results in this section we assume that X is a complex commutative Banach algebra with identity e ($\| e \| = 1$). A complex number λ is a *regular point of* $x \in X$ if $x - \lambda e$ is invertible. The set $\rho(x)$ of regular points of x is an open set. Moreover the resolvent map $r_x : \rho(x) \to X, \lambda \to (x - \lambda e)^{-1}$ is analytic, ([1], p. 208). An important consequence of these results is:

1.2. GELFAND - MAZUR THEOREM ([1], p. 212): (a) *For every $x \in X$, the spectrum $\sigma(x)$ of x, those complex numbers λ for which $x - \lambda e$ is not invertible, is not empty.*

(b) *If X is a division algebra (all nonzero elements have inverses) then X is isomorphic and isometric to $\mathbb{C}$.*

PROOF: (b) Since $\sigma(x) \neq \emptyset$, $x - \lambda e$ is not invertible for some $\lambda \in \mathbb{C}$. Since X is a division algebra, $x - \lambda e$ must be 0, i.e. $x = \lambda e$.

The proof of part (a) depends heavily on the Liouville theorem. Consequently one would suspect that this result would not transfer easily to Banach algebras over other fields, and indeed this is the case. Even in real Banach algebras there may be elements with empty spectrum. As long as the underlying field is $\mathbb{C}$, however, we can obtain versions of the above result for locally convex Hausdorff algebras and locally m-convex algebras ([1], p. 212 - 3). The only change that

occurs is that the "isometry" of part (b) is replaced by "homeomorphism". For a time it was wondered ([6]) if there were complex topological division algebras other than $\mathbb{C}$. Williamson ([12],[1, p.214]) showed that there were by providing an algebraically compatible topology for the field $\mathbb{C}(t)$ of rational functions in t with complex coefficients.

An important consequence of (b) above is that X/M is $\mathbb{C}$ for any maximal ideal M of X. We denote the coset (complex number) $x + M$ by $x(M)$. It now becomes possible to view X as a collection of functions on the space M of maximal ideals of X. We associate $x \in X$ with the function $\hat{x}$ which sends M into $x(M)$. Once M has been topologized, it becomes possible to view X as a collection of continuous functions mapping M into $\mathbb{C}$. Among other things, even without endowing M with a topology, it now follows that

$$\sigma(x) = \hat{x}(M). \tag{1.3}$$

In algebras of continuous or analytic functions ([1], p.202–3) characterizations such as 1.3 are the rule for describing spectra, i.e., the spectrum of a function x is its range.

We endow M with the weakest topology which will make each of the maps $\hat{x}$ continuous and call this the *Gelfand topology*. M then becomes a compact Hausdorff space.

A B^*-algebra is a Banach algebra with involution satisfying the condition $\|x^* x\| = \|x\|^2$. The celebrated representation theorem of Gelfand and Naimark states:

1.4. REPRESENTATION OF B^*-ALGEBRAS ([1], p. 259f.): *If X is a B^*-algebra, then X is isometrically isomorphic to the algebra $C(M, \mathbb{C})$ of continuous complex-valued functions on the compact space M of maximal ideals of X with sup norm (and pointwise operations).*

2. GELFAND THEORY OVER VALUED FIELDS

Here we assume that X is a commutative Banach algebra with identity over a field F where the norm on X and the valuation on F each satisfy the strong ("nonarchimedean") triangle inequality: $\|x + y\| \leq \max(\|x\|, \|y\|)$. Among the consequences of this inequality are that if $\|x\| \neq \|y\|$, then $\|x + y\| = \max(\|x\|, \|y\|)$ and that every point in a sphere is a center. All norms and valuations are assumed real-valued. A detailed discussion of such normed spaces and algebras can be found in [10], such spaces being called *nonarchimedean spaces*.

The critical result ((1.2)) that each element have nonempty spectrum fails to hold for nonarchimedean algebras. There may be elements with empty spectrum ([10], p. 105). The worst consequence of this is that we cannot say that X/M is F for each maximal ideal of X. X/M is merely a superfield of F. If we hypothesize separately that each element have nonempty spectrum then, exactly as in proof of (1,2)(b), division algebras are isometrically isomorphic to the underlying field. We define a *Gelfand algebra* to be a nonarchimedean commutative Banach algebra X with identity such that X/M = F for each maximal ideal M of X.

Although we cannot show that each element has nonempty spectrum in an arbitrary nonarchimedean Banach algebra, we can show for any x that $\sigma(x)$ is closed and bounded, the proof being about the same as for the complex case ([10], p. 114). Thus if F is locally compact, each element has compact spectrum. Also (cf. (1.3)) it is generally true that $\sigma(x) = \hat{x}(M) \cap F$.

In an attempt to duplicate the complex Gelfand theory, we wish to introduce a topology to the maximal ideals. Two main choices are available: Restrict consideration of what elements x are to be chosen or consider only certain maximal ideals. More specifically we consider ([10], p. 117f.):

2.1. THE GELFAND SUBALGEBRA X_g: Those $x \in X$ such that for every maximal ideal M, $x(M) \in F$ or

2.2. THE GELFAND IDEALS M_g: Those maximal ideals M such that $x(M) \in F$ for every x.

In the first case we retain all the M's; in the second, all the x's.

It now follows that (a) for each $M \in M$, $M \cap X_g$ is a Gelfand (maximal) ideal in X_g (i.e., $X_g / M \cap X_g = F$); (b) $X = X_g$ iff $M = M_g$ (X is a Gelfand algebra iff each maximal ideal is a Gelfand ideal or X coincides with its Gelfand subalgebra); (c) X_g is a closed subalgebra of X.

We may now consider the following topologies.

2.3. THE WEAK TOPOLOGY: Define the *weak Gelfand topology* to be the weakest topology for M such that each $x \in X_g$ is continuous. This induces the weak Gelfand topology on M_g.

2.4. THE STRONG TOPOLOGY: Define the *strong Gelfand topology* to be the weakest topology for M_g such that every $x \in X$ is continuous. This is clearly stronger than the weak Gelfand topology.

REMARKS: (a) Strong topologies yield spaces with more structure. (b) M_g is generally not big enough to yield information about X whereas the Gelfand ideals M_{gg} of X_g are rich enough to help describe X_g, e.g., if $x \in X_g$, then $\sigma(x) = \hat{x}(M_{gg})$. (c) These topologies are uniformities and M_g and M_{gg} are complete. Thus M_g or M_{gg} is compact iff they are totally bounded.

The last remark helps to obtain the following compactness result.

2.5. COMPACTNESS ([10], p. 124): *If* F *is locally compact then* M_g *and* M_{gg} *are strongly compact. Conversely if* M_{gg} *or* $M \cap X_g$ *is strongly compact, then either* F *is locally compact or the spectrum of any element in* X_g *is nonempty, compact, and nowhere dense.*

2.6. DISCONNECTEDNESS ([10], p. 125): *Each of the Gelfand topologies is 0-dimensional and each of the spaces* M_g, M_{gg}, $M \cap X_g$ *in the strong topology is totally disconnected and Hausdorff.*

2.7. SEPARATION ([10], p. 126): *The following statements are equivalent.*

(a) *The (weak) Gelfand topology on* M *is Hausdorff.*

(b) *The functions from* X_g *separate points.*

(c) *The functions from* X_g *separate points strongly.*

(d) *The map* $M \to M \cap X_g$ *is* 1 - 1.

Maximal ideals must always be of codimension 1. Conversely, in [5], Gleason proved that a linear subspace of codimension 1 in a complex commutative Banach algebra with identity is a maximal ideal iff it consists of singular elements. Hence, in a nonarchimedean Banach algebra, one might consider the question: If M is a linear subspace of codimension 1 consisting solely of singular elements, must M be a Gelfand ideal ? The fact that Gleason's argument uses deep theorems from complex variable theory gives warning that the nonarchimedean question could be difficult.

In [2] the authors considered Gleason's question in the topological algebra (endowed with the compact-open topology) C(T,F) of continuous function from a topological space T into a topological field F. It is shown there that Gleason's result is true if F is the field of complex numbers, false if F is the reals, and true if F is an ultraregular field containing at least three points under any of the following conditions.

1. F is not algebraically closed.
2. F possesses a sequence of distinct elements converging to 0.
3. F is discretely valued.
4. The topology of F is given by a valuation.
5. T is ultranormal.

We say that a Gelfand algebra is *regular* if the functions $\hat{x}$ separate points and closed subsets of M strongly.

2.8. REGULAR: X *is regular iff the (weak) Gelfand topology on* M *coincides with the hull-kernel topology on* M. ([10], p. 135).

In the complex case, X is regular iff the hull-kernel topology is Hausdorff and the proof relies heavily on the compactness of M in the Gelfand topology. By choosing nonarchimedean algebras in which M is not compact, one obtains counterexamples to "if the hull-kernel topology is Hausdorff, then X is regular".

Let U be the unit ball in X and let $W = \{x \mid \|x(M)\| \leq 1$ for each maximal ideal $M\}$. Since $\|x(M)\| \leq \|x\|$ for every M, it is clear that $U \subset W$. If $U = W$, we call X a V*-*algebra*. As will be seen shortly, the V*-algebras are the nonarchimedean analogs of B*-algebras (see (2.10)). It is easy to verify ([10], p. 148) that V*-algebras must be semisimple.

2.9. *If* T *is a 0-dimensional compact Hausdorff space and* F *is complete then* T *is homeomorphic to the space* M *of maximal ideals of* $C(T,F)$ *under the map* $t \to M_t = \{x \in C(T,F) \mid x(t) = 0\}$ *when* M *carries the Gelfand topology. Also,* $C(T,F)$ *is a* V*-*algebra* ([10], p. 154). *In addition, if* S *is 0-dimensional, compact and Hausdorff then* S *is homeomorphic to* T *iff* $C(S,F)$ *is isomorphic to* $C(T,F)$.

As a first representation theorem we have

2.10. ([10], p. 164) *If* X_g *is a* V*-*Gelfand algebra and* M_g *is compact*

then X_g *is isometrically isomorphic to* $C(M_g, F)$, *from which it follows that if* X *is a* V*-Gelfand algebra in which* M *is compact then* X *is isometrically isomorphic to* C(M,F).

For the proof of (2.10) one needs a version of a Stone-Weierstarss theorem for algebras of continuous functions which take values in a nonarchimedean valued field. Such theorems are the subject of the next and last part of the paper.

3. STONE-WEIERSTRASS THEOREMS

F denotes a field with nonarchimedean valuation.

Generalizing a result of Dieudonné ([4]), Kaplansky ([7]) obtained the following analog of the classical Stone-Weierstrass theorem.

3.1. KAPLANSKY-STONE-WEIERSTRASS THEOREM: ([7], [10, p. 162]): *If* T *is a compact Hausdorff space and* Y *a subalgebra of* C(T,F) *which separates points and contains constants then* Y *is dense in* C(T,F).

An immediate consequence of this is

3.2. ([7], [10, p. 163]): *If* T *is a locally compact Hausdorff space and* Y *a subalgebra of* $C_\infty(T,F)$*-continuous functions which vanish at infinity - which separates points and contains constants then* Y *in dense in* $C_\infty(T,F)$.

As has been observed by Nachbin ([8]), it is not really necessary to consider subalgebras Y for Stone-Weierstrass type theorems: sub-modules suffice. To quote just one of many possible illustrations of this viewpoint ([3], for example) we have the following result of Prolla's.

3.3. ([11], Cor. 2.5): *Let* T *be a compact Hausdorff space,* X *a*

nonarchimedean normed space over F *and* W *an* A*-submodule of* C(T,X), *where* A *is a separating subalgebra of* C(T,F).

Then W *is dense in* C(T,X) *iff for each* t *in* T, $W(t) = \{w(t) | w \in W\}$ *is dense in* X.

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APPROXIMATION OF PLURISUBHARMONIC FUNCTIONS

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If U is an open and connected subset of an infinite dimensional locally convex vector space E on $\mathbb{C}$, an application $f : U \to \mathbb{C}$ (resp. $[-\infty, +\infty[$) is said to be *holomorphic* (resp. *plurisubharmonic*) if

a) f is continuous (resp. upper semicontinuous)

b) the restriction of f to any finite dimensional subspace is holomorphic (resp. plurisubharmonic).

Let us denote by $H(U)$ (resp. $P(U)$, $P_C(U)$) the set of holomorphic (resp. plurisubharmonic, plurisubharmonic and continuous) functions on U. If K is a compact subset of U, let us denote by

$$\hat{K}_{H(U)} = \{x \in U,\ |f(x)| \leq |f|_K\ ,\ \forall f \in H(U)\}$$

$$\hat{K}_{P(U)} = \{x \in U,\ v(x) \leq \sup_K v,\ \forall v \in P(U)\}.$$

In $\mathbb{C}^n$, $n \geq 2$, the following results are well known (3):

1) Any v in $P(U)$ is the pointwise decreasing limit of C^∞ plurisubharmonic functions in a strictly smaller open set U' of U (ie $d(U', \complement U) > 0$).

2) If U is pseudo-convex (ie $\hat{K}_{P(U)}$ is compact in U for any compact K of U) then $\hat{K}_{P(U)} = \hat{K}_{P_C(U)}$.

3) If U is pseudo-convex, then for v in $P_c(U)$, $\varepsilon > 0$ and K compact subset of U there exist $f_1,\ldots,f_j$ in $H(U)$ and $a_1,\ldots,a_j$ positive numbers such that

$$|v - \sup_i a_i \log |f_i||_K < \varepsilon ,$$

4) If $K = \hat{K}_{H(U)}$ is compact in a pseudo-convex open set U, then any holomorphic function in a neighborhood of K can be approximated uniformly on K by elements of $H(U)$.

5) If U and U' are pseudo-convex, $U \subset U'$ then $\hat{K}_{H(U)} = \hat{K}_{H(U')}$ for any compact subset of U if and only if $H(U')$ is dense in $H(U)$ for the compact open topology.

Properties 3), 4) and 5) have been generalised to larger classes of locally convex spaces with Schauder basis including Banach spaces (6).

We shall investigate conditions 1 and 2. In $\mathbb{C}^n$, condition 1) is obtained by regularisation (ie convolution) of v by a Dirac sequence, so it is natural to consider some measure. For the sake of simplicity we shall consider here only (infinite dimensional) Banach spaces and Gaussian measures following Gross (5).

It is well known that in a Banach space E there are no substitute to the Lebesgue measure that means there does not exist a measure invariant by translations or rotations. A Gaussian measure μ on E can be characterized as follows: there exists an Hilbert space H_μ densely and continuously imbeded in E such that μ arises from the cylindrical Gauss measure on the cylindrical sets of H_μ. The triplet (H_μ, i, E) is called an abstract Wiener space. The following properties hold:

1) Let be T in $\mathcal{L}(E,E)$, if T restricted to H_μ is in $\mathcal{L}(H_\mu,H_\mu)$ and is unitary then μ is invariant by T (ie $\mu T^{-1} = \mu$).

2) Let be $\mu_x(A) = \mu(x + A)$, A Borel in E, then μ and μ_x are either equivalent or orthogonal, they are equivalent if and only if x belongs to H_μ.

We have the following Lemma:

LEMMA 1: *If μ is a Gaussian measure on E and if v is a plurisubharmonic function in an open subset U of E we have*

$$v(x) \leq \frac{1}{\mu[B(0,R)]} \int_{\|y\| \leq r} v(x+y)\ \mu(dy)$$

for r small enough.

PROOF: Suppose that v is bounded from above in the ball B(x,r), the mapping $x \to e^{i\theta}x$ induces a unitary mapping T_θ on H_μ, so μ is T_θ - invariant, and we have

$$\int_{\|y\| \leq r} v(x+ye^{i\theta})\ \mu(dy) = \int_{\|y\| \leq r} v(x+y)\ \mu(dy),$$

but $v(x) \leq \frac{1}{2\pi} \int_0^{2\pi} v(x + ye^{i\theta})d\theta$. The result follows from Fubini theorem.

Let us note $A(v,x,r) = \frac{1}{\mu(0,r)} \int_{\|y\| \leq r} v(x+y)\ \mu(dy)$, we have:

PROPOSITION 1: 1) *A(x,v,r) is a plurisubharmonic function of* x *and* $\mu(r)A$ *is a convex and increasing function of log* r.

2) $v(x) = \lim_{r=0} A(x,v,r)$.

3) *A(x,v,r) is infinitely* H_μ *- differentiable.*

Let us recall that a function φ is H_μ-differentiable if for any x in E the function $y \to \varphi(x+y)$, defined on H_μ, is differentiable at $y = 0$.

PROOF: 1) is a consequence of the fact that plurisubharmonic functions depending only from $\| x \|$ are logarithmically convex.

2) Since v is upper semicontinuous, for any $\varepsilon > 0$ we have $v(x + y) \leq v(x) + \varepsilon$ for $\| y \| \leq r_{x,\varepsilon}$ hence

$$v(x) \leq A(x,v,r) \leq v(x) + \varepsilon .$$

3) Is a consequence of a result of Gross (5).

Let us notice that, unless v is continuous, $A(v,x,r)$ is not in general a continuous function of x.

As a consequence of 2) and 3) we have:

PROPOSITION 2: *A plurisubharmonic function v is locally the pointwise limit of a sequence of infinitely H-differentiable plurisubharmonic functions.*

There is an other way to approximate bounded functions: let μ_t be a Gaussian measure of parameter $t > 0$, then $\mu_t(E) = 1$ and $\mu_t\{\| x \| \geq \alpha > 0\} \to 0$ if $t \to 0$ then Gross (5) has proved that the function $P_t f(x) = \int_E f(x + y)\mu_t(dy)$ is infinitely H_μ-differentiable if f is bounded and measurable. Moreover if f is uniformly continuous $P_t f$ tends to f uniformly on E.

PROOF: For $\varepsilon < 0$, there is η such that $|x - y| \leq \eta$ implies $|f(x) - f(y)| \leq \varepsilon$.

$$|P_t f(x) - f(x)| \leq \int_E |f(x + y) - f(x)| \, \mu_t(dy)$$

$$\leq \varepsilon \mu_t(\| y \| \leq \eta) + 2 \| f \|_\infty \, \mu_t(\| y \| \geq \eta)$$

$$\leq 2\varepsilon \quad \text{if} \quad t < t_\varepsilon .$$

If f is only continuous, then the convergence of $P_t f$ to f is uniform on every compact subset.

It is also well known (1) that there exist several separable Banach spaces such that the bounded and C^1 functions are not dense in the space of uniformly continuous and bounded functions. The preceding result gives a uniform approximation by H-infinitely differentiable function. For plurisubharmonic functions this kind of approximation gives more or less the same result as proposition 2.

Now we shall state the following proposition:

PROPOSITION 3: *Let* U *be a pseudo-convex open subset of a Banach space* E *and let* v *be a plurisubharmonic function on* U, *then* v *is the pointwise limit of a decreasing sequence of continuous (in fact Lipschitz) plurisubharmonic functions in* U.

COROLLARY: *Let* U *be a pseudo-convex open subset of a Banach space* E, *then*

$$\hat{K}_{P(U)} = \hat{K}_{P_c(U)}$$

for any compact subset K *of* U.

For the proof we shall follow an unpublised paper of C. Herves and M. Estevez (2). They first generalize in the Banach case an idea of (3): Let f be a lower semi-continuous function bounded from above, then for any integer k define

$$f_k(x) = \inf_y [k\|x - y\| + f(y)].$$

It is easy to show that $f_{k-1} \leq f_k \leq f$ and $|f_k(x) - f_k(y)| \leq k\|x - y\|$. Moreover f is the pointwise limit of the sequence f_k.

If U is pseudo-convex and v is a plurisubharmonic function in U we take this approximation sequence of the function f defined by $\exp(-v)$ in U and zero outside U. If we state

$$\tilde{U} = \{(z,w) \in U \times \mathbb{C}\ ,\ |w| < |f(z)|\}$$

and if we consider the norm $k\|z\| + |w|$ on $E \times \mathbb{C}$, we have

$$d((z,0),\ C\tilde{U}) = f_k(z).$$

Since $\tilde{U}$ is a pseudo-convex domain it follows that $-\log f_k$ is plurisubharmonic in U, moreover $v = \lim_k [-\log f_k]$. Proposition 3 is proved.

PROOF OF THE COROLLARY: It is sufficient to prove that $\hat{K}_{P_c(U)} \subset \hat{K}_{P(U)}$. If $x_o \notin \hat{K}_{P(U)}$ there is v in $P(U)$ such that $u(x_o) > \alpha > \sup_K v$. Proposition 3 implies that there is a decreasing sequence (v_n) in $P_c(U)$, hence:

$$K \subset \{x \in U,\ v(x) < \alpha\} = \bigcup_n \{x \in U,\ v_n(x) < \alpha\}.$$

Since K is compact and $v_{n+1} \leq v_n$ there is an index p such that

$$K \subset \{x \in U,\ v_p(x) < \alpha\}\ .$$

We have

$$v_p(x_o) \geq v(x_o) > \alpha \geq \sup_K v_p$$

hence x_o does not belong to $\hat{K}_{P_c(U)}$. The corollary is proved.

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THE APPROXIMATION PROPERTY FOR CERTAIN SPACES OF HOLOMORPHIC MAPPINGS

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§0. INTRODUCTION

If E and F are locally convex complex Hausdorff spaces, let $\mathcal{H}_S(E;F)$ be the vector space of all Silva-holomorphic mappings from E to F. (See Definition 1.13 below).

In section 1, after the preliminary definitions, we study the space $\mathcal{H}_S(E;F)$ endowed with the topology of uniform convergence on strict (see Definition 1.21) compact subsets of E.

In section 2, we prove that for a quasi-complete space E the following properties are equivalent:

(a) E has the S-approximation property (see Definition 1.31);

(b) $\mathcal{H}_S(E;\mathbb{C}) \otimes F$ is τ_S-dense in the space $\mathcal{H}_S(E;F)$, for every locally convex space F;

(c) $\mathcal{H}_S(E;\mathbb{C})$ with the topology τ_S, has the approximation property.

For Banach spaces, Aron and Schottenloher [2] and Aron [1] have some results about this for the space $(\mathcal{H}(E;\mathbb{C}), \tau_o)$, (where $(\mathcal{H}(E;\mathbb{C}),\tau_o)$ denotes the vector space of all holomorphic mappings from E to $\mathbb{C}$, endowed with the topology compact-open τ_o).

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§1. SILVA-HOLOMORPHIC MAPPINGS.

In this paper E and F are always locally convex complex Hausdorff spaces and U is a non-void open subset of E. $\mathcal{B}_E$ will denote the set of all closed absolutely convex bounded subsets of E. If $B \in \mathcal{B}_E$, E_B is the vector subspace of E generated by B and normed by the Minkowsky functional p_B determined by B. cs(E) is the set of non-trivial continuous seminorms on E.

1.1 DEFINITION: Let $n = 1,2,\ldots$; $\mathcal{L}_b({}^nE;F)$ will denote the vector space of all n-linear mappings from $E^n = E \times \ldots \times E$ (n times) to F, which are bounded on bounded subsets of E^n, endowed with the locally convex topology generated by all seminorms of the form:

$$\| T \|_{B_1,\ldots,B_n,\beta} = \sup \{ \beta(T(x_1,\ldots,x_n)); \; x_i \in B_i \, , \, i = 1,\ldots,n \}$$

where $T \in \mathcal{L}_b({}^nE;F)$, $B_1,\ldots,B_n \in \mathcal{B}_E$ and $\beta \in cs(F)$. If $B_1 = \ldots = B_n = B$, we denote $\| T \|_{B_1,\ldots,B_n,\beta} = \| T \|_{B,\beta}$. Notice that

$$\beta(T(x_1,\ldots,x_n)) \leq \| T \|_{B_1,\ldots,B_n,\beta} \cdot p_{B_1}(x_1) \ldots p_{B_n}(x_n),$$

for all $x_i \in E_{B_i}$, $i = 1,\ldots,n$.

The elements of $\mathcal{L}_b({}^nE;F)$ are called *Silva-bounded* (S-*bounded*) n-*linear mappings*. We will denote by $\mathcal{L}_{bs}({}^nE;F)$ the vector subspace of $\mathcal{L}_b({}^nE;F)$ of all such T that are symmetric. For $n = 0$, we define $\mathcal{L}_b({}^0E;F) = \mathcal{L}_{bs}({}^0E;F) = F$ and $\| T \|_{B,\beta} = \beta(T)$, for every $T \in \mathcal{L}_b({}^0E;F)$.

1.2 PROPOSITION: *If* F *is a complete locally convex space,* $\mathcal{L}_b({}^nE;F)$ *is complete. For every* F, $\mathcal{L}_{bs}({}^nE;F)$ *is a closed vector subspace of*

$\mathcal{L}_b({}^nE;F)$.

1.3 DEFINITION: For $T \in \mathcal{L}_b({}^nE;F)$, we define its symmetrization $T_s \in \mathcal{L}_{bs}({}^nE;F)$ by

$$T_s(x_1,\ldots,x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} T(x_{\sigma(1)},\ldots,x_{\sigma(n)}),$$

where S_n is the symmetric group of degree n.

1.4 PROPOSITION: *The mapping* $T : \mathcal{L}_b({}^nE;F) \to T_s \in \mathcal{L}_{bs}({}^nE;F)$ *is a continuous projection of* $\mathcal{L}_b({}^nE;F)$ *onto* $\mathcal{L}_{bs}({}^nE;F)$. *Furthermore* $T = T_s$ *if and only if* $T \in \mathcal{L}_{bs}({}^nE;F)$.

1.5 DEFINITION: Let $T \in \mathcal{L}_b({}^nE;F)$, $x \in E$ and $n \in \mathbb{N}$. If $n=1,2,\ldots$, we write Tx^n to denote $T(x,\ldots,x)$ where x is repeated n times. If $n = 0$, we define $Tx^0 = T$. A mapping $P : E \to F$ is a *Silva-bounded n-homogeneous polynomial* if there is $T \in \mathcal{L}_b({}^nE;F)$ such that $P(x) = Tx^n$, for every $x \in E$. To denote that T corresponds to P, we write $P = \hat{T}$. $\mathcal{P}_b({}^nE;F)$ denotes the vector space of all Silva-bounded n-homogeneous polynomials from E to F. On $\mathcal{P}_b({}^nE;F)$ we consider the locally convex topology τ_s generated by all seminorms of the form:

$$\| P \|_{B,\beta} = \sup \{\beta(P(x)); \quad x \in B\},$$

where $P \in \mathcal{P}_b({}^nE;F)$, $B \in \mathcal{B}_E$ and $\beta \in cs(F)$.

Notice that

$$\beta(P(x)) \leq \| P \|_{B,\beta} \cdot (p_B(x))^n, \quad \text{for all } x \in E_B .$$

1.6 PROPOSITION: *The mapping* $T \in \mathcal{L}_{bs}({}^nE;F) \to \hat{T} \in \mathcal{P}_b({}^nE;F)$, *is a vector space isomorphism and a homeomorphism of the first onto the second*

space. Moreover

$$\| \hat{T} \|_{B,\beta} \leq \| T \|_{B,\beta} \leq \frac{n^n}{n!} \| \hat{T} \|_{B,\beta} \qquad (1).$$

1.7 REMARK: $\frac{n^n}{n!}$ is the best universal constant occurring in (1). (Nachbin [9]).

1.8 PROPOSITION: *If* F *is a complete locally convex space,* $P_b({}^nE;F)$ *is complete for all* $n = 0,1,\ldots$.

1.9 DEFINITION: A *Silva-bounded polynomial* P *from* E *to* F is a mapping $P : E \to F$ for which there are $n = 0,1,\ldots,$ $P_k \in P_b({}^kE;F)$ $(k = 0,\ldots,n)$ such that $P = P_o + \ldots + P_n$. We will denote by $P_b(E;F)$ the vector space of all Silva-bounded polynomials from E to F.

1.10 PROPOSITION: *If* $P \in P_b(E;F)$, $P \neq 0$, *there is one and anly one way of writing* $P = P_o + \ldots + P_n$, *with* $n = 0,1,\ldots,$ $P_k \in P_b({}^kE;F)$ $(k = 0,\ldots,n)$ *and* $P_n \neq 0$.

1.11 DEFINITION: A *formal power series from* E *to* F *about* $\xi \in E$ is a series in $x \in E$ of the form

$$\sum_{n=o}^{\infty} A_n(x - \xi)^n,$$

where $A_n \in \mathcal{L}_{bs}({}^nE;F)$ $(n = 0,1,\ldots)$; or of the form

$$\sum_{n=o}^{\infty} P_n(x - \xi)$$

where $P_n = \hat{A}_n \in P_b({}^nE;F)$ $(n = 0,1,\ldots)$. A_n and P_n are called the coefficients of the power series.

1.12 LEMMA: *If the power series* $\sum_{n=0}^{\infty} P_n(x - \xi)$, *about* $\xi \in E$ *is such that for all* $B \in \mathcal{B}_E$ *and* $\beta \in cs(F)$, *there is some* $\rho_B > 0$, *such that*

$$\lim_{m\to\infty} \beta\Big(\sum_{n=0}^{m} P_n(x - \xi)\Big) = 0,$$

for all $x \in \xi + \rho_B B$, *then* $\beta(P_n(t)) = 0$, *for* $n = 0,1,\ldots$, *and* $t \in E$.

1.13 DEFINITION: A mapping $f : U \to F$ is said to be *Silva-holomorphic (S-holomorphic) on* U if, corresponding to every $\xi \in U$, there are $P_m \in P_b({}^mE;F)$ $(m = 0,1,\ldots)$ such that for all $\beta \in cs(F)$ and $B \in \mathcal{B}_E$ there is $\rho_B > 0$, satisfying $\xi + \rho_B B \subset U$, such that

$$f(x) = \sum_{n=0}^{\infty} P_n(x - \xi),$$

uniformly with respect to β on $\xi + \rho_B B$. The sequence $(P_n)_{n=0}^{\infty}$ is then unique at every point $\xi \in E$, by Lemma 1.12. We set the notations $\frac{1}{m!}\,\hat{\delta}^m f(\xi) = P_m$, $\frac{1}{m!}\,\delta^m f(\xi) = A_m$, if $\hat{A}_m = P_m$, for $m = 0,1,\ldots$. The notation

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!}\,\hat{\delta}^m f(\xi)(x - \xi),$$

indicates the Taylor series of f at ξ. $\mathcal{H}_S(U;F)$ denotes the vector space of all S-holomorphic mappings from U to F.

1.14 REMARK: The above concept of holomorphy was introduced by Sebastião e Silva [16]. We will denote by $\mathcal{H}(U;F)$ the vector space of all holomorphic mappings from U to F. For some basic properties of this space see Nachbin [8] and [9] and Noverraz [10]. Notice that a mapping $f \in \mathcal{H}_S(U;F)$ is holomorphic if and only if, f is continuous. $\mathcal{H}_S(U;F) = \mathcal{H}(U;F)$, for every open non-void subset U of a

seminormed or a Silva space E and for every locally convex Hausdorff space F. In general, $\mathcal{H}_S(U;F) = \mathcal{H}(U;F)$ if E is a holomorphically bornological space. (Matos [6]).

1.15 PROPOSITION: *A mapping* $f : U \to F$ *is S-holomorphic if and only if* $f|_{U \cap E_B}$ *is holomorphic on* $U \cap E_B$, *for every* $B \in \mathcal{B}_E$.

1.16 COROLLARY: *Let* $f \in \mathcal{H}_S(U;F)$, $\xi \in U$, $x \in U$ *and* $\rho > 1$ *be such that* $(1 - \lambda)\xi + \lambda x \in U$, *for every* $\lambda \in \mathbb{C}$, $|\lambda| \leq \rho$. *Then*

$$f(x) = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \frac{f((1-\lambda)\xi + \lambda x)}{\lambda - 1} \, d\lambda$$

1.17 COROLLARY: (Cauchy integral formula): *Let* $f \in \mathcal{H}_S(U;F)$, $\xi \in U$, $x \in E$ *and* $\rho > 0$ *be such that* $\xi + \lambda x \in U$, *for every* $\lambda \in \mathbb{C}$, $|\lambda| \leq \rho$. *Then*

$$\frac{1}{n!} \hat{\delta}^n f(\xi)(x) = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \frac{f(\xi + \lambda x)}{\lambda^{n+1}} \, d\lambda$$

for $n = 0, 1, \ldots$

1.18 COROLLARY: (Cauchy inequalities): *Let* $f \in \mathcal{H}_S(U;F)$, $\beta \in cs(F)$, $B \in \mathcal{B}_E$ $\xi \in U$ *and* $\rho > 0$ *be such that* $\xi + \rho B \subset U$. *Then*

$$\| \frac{1}{n!} \hat{\delta}^n f(\xi) \|_{B,\beta} \leq \frac{1}{\rho^n} \sup \{\beta(f(x)); \; x - \xi \in \rho B\}$$

for $n = 0, 1, \ldots$

1.19 DEFINITION: A mapping $f : U \to F$ is said to be *weakly Silva-holomorphic* if for every $\phi \in F'$ (where F' denotes the topological dual of F) the function $\phi \circ f$ is Silva-holomorphic.

1.20 PROPOSITION: *Let* F *be a space with the property that if* K *is a compact subset of* F, *then the closed absolutely convex hull of* K, $\Gamma(K)$, *is a compact subset of* F. *Then* $f : U \to F$ *is a weakly Silva-holomorphic mapping if and only if* f *is Silva-holomorphic.*

The proof of this proposition follows from Proposition 1.15 and Nachbin [8] .

1.21 DEFINITION: A subset K of E is said to be a *strict compact* set if there is $B \in \mathcal{B}_E$ such that K is contained in E_B and is compact in E_B. If E is normed, or Frēchet (or $\mathcal{L}F$), then $K \subset E$ is strict compact if and only if it is compact in E.

We will denote by τ_S the locally convex topology on $\mathcal{H}_S(U;F)$ of uniform convergence on the strict compact subsets of U.

1.22 PROPOSITION: *If* F *is a complete locally convex space, then* $(\mathcal{H}_S(U;F), \tau_S)$ *is complete.*

PROOF: Let $(f_\alpha)_{\alpha \in I}$ be a Cauchy net in $(\mathcal{H}_S(U;F), \tau_S)$ and $\beta \in cs(F)$. Then if $B \in \mathcal{B}_E$, $(f_\alpha|_{U \cap E_B})_{\alpha \in I}$ is a Cauchy net in $(\mathcal{H}(U \cap E_B;F), \tau_o)$ (τ_o is the compact - open topology). We know that $(\mathcal{H}(U \cap E_B;F), \tau_o)$ is complete, for F complete. Using this fact, it is easy to see that there is $f \in \mathcal{H}_S(U;F)$ such that $(f_\alpha)_{\alpha \in I}$ converges to f on $(\mathcal{H}_S(U;F), \tau_S)$.

We now define the notion of Silva-holomorphic mapping of compact type, which will be needed in the next section.

1.23 DEFINITION: For $\varphi \in E^*$, where E^* denotes the space of all linear mappings from E to $\mathbb{C}$, which are bounded on bounded subsets of E, and $b \in F$, we denote the S - bounded linear mapping $x \in E \to \varphi(x) \cdot b \in F$ by $\varphi \cdot b \in \mathcal{L}_b({}^nE;F)$. More generally, for $\varphi_i \in E^*$, $i = 1,\ldots,n$, $n \in \mathbb{N}$ and $b \in F$, we denote the S - bounded

n-linear mapping

$$(x_1,\dots,x_n) \in E^n \to \varphi_1(x_1)\dots\varphi_n(x_n)\cdot b \in F \quad \text{by} \quad \varphi_1 x \dots x\varphi_n \cdot b \in \mathcal{L}_b({}^nE;F).$$

The vector subspace of $\mathcal{L}_b({}^nE;F)$ generated by all elements of the form $\varphi_1 x \dots x\varphi_n \cdot b$, $\varphi_i \in E^*$, $i = 1,\dots,n$, and $b \in F$, is denoted by $\mathcal{L}_{bf}({}^nE;F)$. We define the vector subspace $\mathcal{L}_{bc}({}^nE;F)$ of $\mathcal{L}_b({}^nE;F)$, to be the closure of $\mathcal{L}_{bf}({}^nE;F)$ in $\mathcal{L}_b({}^nE;F)$. The topology on $\mathcal{L}_{bc}({}^nE;F)$ will always be the induced topology by $\mathcal{L}_b({}^nE;F)$. Hence, if F is a complete space then $\mathcal{L}_{bc}({}^nE;F)$ is a complete space. We define $\mathcal{L}_{bfs}({}^nE;F) = \mathcal{L}_{bf}({}^nE;F) \cap \mathcal{L}_{bs}({}^nE;F)$ and $\mathcal{L}_{bcs}({}^nE;F) = \mathcal{L}_{bc}({}^nE;F) \cap \mathcal{L}_{bs}({}^nE;F)$. For $n = 0$ we define all these spaces as F.

1.24 PROPOSITION: $\overline{\mathcal{L}_{bfs}({}^nE;F)} = \mathcal{L}_{bcs}({}^nE;F)$.

1.25 DEFINITION: $A \in \mathcal{L}_b({}^nE;F)$ is said to be a *Silva-bounded* n-*linear mapping of compact type* if and only if $A \in \mathcal{L}_{bc}({}^nE;F)$.

Analogously, for $\varphi \in E^*$, $b \in F$, we denote the Silva-bounded n-homogeneous polynomial given by $x \in E \to \varphi(x)^n \cdot b \in F$ by $\varphi^n \cdot b \in \mathcal{P}_b({}^nE;F)$. The vector subspace of $\mathcal{P}_b({}^nE;F)$ generated by all elements of the form $\varphi^n \cdot b$, $\varphi \in E^*$, $b \in F$ is denoted by $\mathcal{P}_{bf}({}^nE;F)$. We define the vector subspace $\mathcal{P}_{bc}({}^nE;F)$ of $\mathcal{P}_b({}^nE;F)$ to be the closure of $\mathcal{P}_{bf}({}^nE;F)$ in $\mathcal{P}_b({}^nE;F)$. The topology on $\mathcal{P}_{bc}({}^nE;F)$ will always be the induced topology by $\mathcal{P}_b({}^nE;F)$. Hence, if F is a complet space then $\mathcal{P}_{bc}({}^nE;F)$ is a complete space.

1.26 PROPOSITION: *The natural mapping* $T \in \mathcal{L}_{bs}({}^nE;F) \to \hat{T} \in \mathcal{P}_b({}^nE;F)$ *induces a topological and algebraic isomorphism between* $\mathcal{L}_{bcs}({}^nE;F)$ *and* $\mathcal{P}_{bc}({}^nE;F)$.

1.27 DEFINITION: $P \in \mathcal{P}_b({}^nE;F)$ is said to be a *Silva-bounded* n-*homogeneous*

polynomial of compact type if and only if $P \in \mathcal{P}_{bc}({}^{n}E;F)$.

1.28 DEFINITION: Let $\mathcal{H}_{Sc}(U;F)$ be the vector subspace of $\mathcal{H}_{S}(U;F)$ of all Silva-holomorphic mappings $f : U \to F$, such that for each $x \in U$ and $n \in \mathbb{N}$, $\frac{1}{n!}\hat{\delta}^{n}f(x) \in \mathcal{P}_{bc}({}^{n}E;F)$. An element $f \in \mathcal{H}_{Sc}(U;F)$ will be called a *Silva-holomorphic mapping of compact type of* U *into* F.

A main tool of this paper is the notion of ε-product introduced by Schwartz [14] which we want to review.

1.29 DEFINITION: Given two locally convex Hausdorff spaces E and F, we denote by F'_{c} the dual of F endowed with the topology of uniform convergence on all balanced convex compact subsets of F, and by $E \varepsilon F = \mathcal{L}_{\varepsilon}(F'_{c},E)$ the space of all linear continuous maps from F'_{c} to E, endowed with the topology of uniform convergence of all equicontinuous subsets of F'. The topology on $\mathcal{L}_{\varepsilon}(F'_{c},E)$ is generated by the seminorms $\beta \varepsilon \alpha$ defined by:

$$(\beta \varepsilon \alpha)(T) = \sup \{|\langle T(u),v\rangle|;\ u \in F',\ |u| \leq \beta,\ v \in E',\ |v| \leq \alpha\},$$

$T \in \mathcal{L}(F'_{c},E)$, $\beta \in cs(F)$ and $\alpha \in cs(E)$. We have that $E \varepsilon F \approx F \varepsilon E$.

1.30 DEFINITION: A locally convex Hausdorff space E is said to have the *approximation property*, if for every $\alpha \in cs(E)$, every $\varepsilon > 0$, and every balanced convex compact subset K of E, there is $T \in E' \otimes E$, such that $\alpha(T(x) - x) < \varepsilon$, for all $x \in K$.

1.31 DEFINITION: A locally convex Hausdorff space E is said to have the *S-approximation property* (S.a.p.), if given a strict compact subset K of E, there is $B \in \mathcal{B}_{E}$ such that $K \subset E_{B}$ and is compact in E_{B} and given $\varepsilon > 0$, there is $T \in E^{*} \otimes E$, such that $p_{B}(T(x) - x) < \varepsilon$, for all $x \in K$.

1.32 REMARK: If E has the S.a.p., E' = E*, and all compact subsets of E are strict, then E has the approximation property. Hence, if E is a normed space, or Frēchet, of $\mathcal{L}F$, which has the S.a.p., then E has the approximation property. If E is an inductive limit of a sequence $(E_n)_{n=0}^{\infty}$ of Banach spaces E_n, which have the approximation property, then E has the S.a.p. . Enflo in [3] gives an example of a Banach space which does not have the S.a.p. .

1.33 PROPOSITION: (Schwartz [14]:

a) *Let* E *and* F *be locally convex Hausdorff spaces. Then* $E \otimes_\varepsilon F$ *(tensor product of* E *and* F*, endowed with the* ε*-topology) is a topological vector subspace of* $E \varepsilon F$.

b) *A locally convex Hausdorff space* E *has the approximation property if and only if* $E \otimes F$ *is dense in* $E \varepsilon F$, *for all locally convex Hausdorff spaces* F.

c) *Let* E *be a quasi-complete space. Then* E *has the approximation property if and only if* $E \otimes F$ *is dense in* $E \varepsilon F$, *for all Banach spaces* F.

d) *If* E *and* F *are locally convex Hausdorff complete spaces, and* E *or* F *has the approximation property, then* $E \hat{\otimes}_\varepsilon F$ *is identical to* $E \varepsilon F$. ($E \hat{\otimes}_\varepsilon F$ *denotes the completion of* $E \otimes_\varepsilon F$).

§2. THE APPROXIMATION PROPERTY FOR $\mathcal{H}_S(U;\mathbb{C})$.

We begin our study with an examination of the closure of the tensor product $\mathcal{H}_S(U;\mathbb{C}) \otimes F$ in $(\mathcal{H}_S(U;F), \tau_S)$.

2.1 THEOREM: *Let* E *have the* S.a.p. *and let* U *be a balanced open subset of* E. *Then* $\mathcal{H}_{Sc}(U;\mathbb{C}) \otimes F$ *is* τ_S*-dense in* $\mathcal{H}_S(U;F)$, *for every locally convex space* F.

PROOF: Let $K \subset U$ be a stric compact set. By hypothesis there is

$B \in \mathcal{B}_E$ such that $K \subset U \cap E_B$ and is compact in E_B, so that for every $\varepsilon > 0$, there is $T \in E^* \otimes E$ such that $p_B(T(x) - x) < \varepsilon$, for all $x \in K$. Let $f \in \mathcal{H}_S(U;F)$, $\varepsilon > 0$ and $\beta \in cs(F)$. We first show that there is $\delta > 0$, $\delta \leq \operatorname{dist}_{E_B}(K, C_{E_B}(U \cap E_B))$ (where $C_{E_B}(U \cap E_B)$ is the complement of $U \cap E_B$ in E_B), such that $\beta(f(x) - f(y)) < \varepsilon$, whenever $x \in K$ and $p_B(x - y) < \delta$. Since $f|_{U \cap E_B}$ is continuous (Proposition 1.15), then for each $x \in K$, there is $\delta_x > 0$, $\delta_x \leq \operatorname{dist}_{E_B}(K, C_{E_B}(U \cap E_B))$, such that $\beta(f(x) - f(y)) < \varepsilon/2$, for $p_B(x - y) < \delta_x$. Since $K \subset U \cap E_B$ is compact is E_B, $K \subset \bigcup_{i=1}^{n} B(x_i, \delta_{x_i})$, for some set $\{x_1, \ldots, x_n\} \subset K$. ($B(a,r) = \{x \in E_B;\ p_B(x - a) < r$, when $a \in E_B$ and $r > 0\}$). Define

$$\gamma(x) = \sup \{ \delta_{x_i} - p_B(x - x_i);\ i = 1, \ldots, n\} \quad \text{for} \quad x \in K.$$

Then $\gamma : K \to R$ is continuous and $\gamma > 0$. Let $\delta = \inf \{\gamma(x);\ x \in K\}$. Now for any $x \in K$ and $y \in B(x, \delta)$, there is some i with $B(x,\delta) \subset B(x_i, \delta_{x_i})$, thus

$$\beta(f(x) - f(y)) < \beta(f(x) - f(x_i)) + \beta(f(x_i) - f(y)) < \varepsilon.$$

Since E has the S.a.p., there is $T \in E^* \otimes E$ such that $p_B(T(x) - x) < \delta$, for all $x \in K$. By the above, we get that

$$\beta(f(T(x) - f(x)) < \varepsilon,$$

for all $x \in K$. Let $\{g_1, \ldots, g_n\}$ be a basis in $T(E)$ and let

$$T(x) = \sum_{i=1}^{n} \phi_i(x) g_i, \quad \text{where} \quad x \in E \quad \text{and} \quad \{\phi_i\}_{i=1}^{n} \subset E^*.$$

Let $U_o = U \cap E_B \cap T(E)$. Since f is Silva-holomorphic, f can be considered as a holomorphic mapping from the finite dimensional balanced set U_o into F,

$$f(z) = f\Big(\sum_{i=1}^{n} z_i g_i\Big) = \sum_{\substack{|p| = 0 \\ p \in \mathbb{N}^n}}^{\infty} z^p f_p ,$$

where $(z_1,\dots,z_n) \in \mathbb{C}^n$, $f_p \in F$ and convergence is uniform on compact subsets of U_o. Since $T(K) \subset U \cap E_B$ and is compact in U_o, there is $M \in \mathbb{N}$, such that

$$\beta\Big(f(x) - \sum_{|p| \leq M} z^p f_p\Big) < \varepsilon, \text{ for all points } z = \sum_{i=1}^{n} \phi_i(x) g_i \in T(K).$$

Thus, if $x \in K$,

$$\beta\Big(f(x) - \sum_{|p| \leq M} \phi^p(x) f_p\Big) \leq \beta(f(x) - f(T(x))) + \beta\Big(f(T(x) - \sum_{|p| \leq M} \phi^p(x) f_p\Big) \leq 2\varepsilon.$$

Since

$$\sum_{|p| \leq M} \phi^p \cdot f_p \in \mathcal{H}_{Sc}(E;\mathbb{C}) \otimes F,$$

the proof is complete.

Now, we give an extension of the previous theorem to another class of subsets of E.

2.2 DEFINITION: Let U be a non-void open subset of E. U is said to be S-*Runge in* E if $P_b(E;\mathbb{C})$ is dense in $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$. U is said to be *finitely* S-*Runge in* E if for each finite dimensional subspace E_o of E, $U \cap E_o$ is S-Runge in E_o.

2.3 REMARK: If E is a Banach space, this definition coincides with the Definition 2.1 of Aron-Schottenloher [2]. If U is a balanced open subset of E, then U is finitely S-Runge and S-Runge in E. (Paques [11]).

2.4 THEOREM: *Let* E *have the* S.a.p. *and let* U *be an open subset*

of E *which is finitely* S-*Runge. Then* $\mathcal{H}_S(U;\mathbb{C}) \otimes F$ *is* τ_S - *dense in* $\mathcal{H}_S(U;F)$ *for every locally convex space* F.

For the proof of Theorem 2.4 it will be needed the following proposition, which has important corollaries.

2.5 PROPOSITION: *Let* F *be a space satisfying the following condition: If* K *is a compact subset of* F, *then the closed absolutely convex hull of* K, $\Gamma(K)$, *is a compact subset of* F. *If* U *is a non-void open subset of* E, *then*

$$(\mathcal{H}_S(U;\mathbb{C}),\ \tau_S) \in F \approx (\mathcal{H}_S(U;F),\ \tau_S).$$

(analogous formulas hold also for $(P_b(^nE;F),\ \tau_S)$, *for* $n \in \mathbb{N}$).

PROOF: Let $T : \mathcal{H}_S(U;F) \to \mathcal{H}_S(U;\mathbb{C}) \in F$ be defined by $(Tf)(\phi)(x) = (\phi \circ f)(x)$, for all $f \in \mathcal{H}_S(U;F)$, $\phi \in F'$ and $x \in U$. Clearly, $(Tf)(\phi)$ belongs to $\mathcal{H}_S(U;\mathbb{C})$. for each $f \in \mathcal{H}_S(U;F)$ and $\phi \in F'$.

We now show that the linear map $Tf : F'_c \to \mathcal{H}_S(U;\mathbb{C})$ is continuous. Indeed, let p be a τ_S-continuous seminorm on $\mathcal{H}_S(U;\mathbb{C})$ defined by $p(g) = \sup\{|g(x)|;\ x \in K\}$, where $K \subset U$ is a strict compact set. By hypothesis, the closed absolutely convex hull of $f(K)$ is a compact subset of F. Call it L. Let q be the seminorm on F' defined by $q(\phi) = \sup\{|\phi(t)|;\ t \in L\}$. It follows that

$$p((Tf)(\phi)) = \sup\{|(\phi \circ f)(x)|;\ x \in K\} \leq \sup\{|\phi(t)|;\ t \in L\} = q(\phi)$$

for all $\phi \in F'$. Hence $Tf \in \mathcal{L}(F'_c;\ \mathcal{H}_S(U;\mathbb{C}))$.

Let now $A \in \mathcal{H}_S(U;\mathbb{C}) \in F = \mathcal{L}(F'_c,\ \mathcal{H}_S(U;\mathbb{C}))$. For each $x \in U$, define $g(x) \in (F'_c)' = F$ by the formula $g(x)(\phi) = (A\phi)(x)$, for all $\phi \in F'$. Now g is weakly S-holomorphic, hence S - holomorphic by

Proposition 1.20. Clearly, $Tg = A$, and therefore T is onto $\mathcal{H}_S(U;\mathbb{C}) \varepsilon F$. On the other hand, T is injective by the Hahn-Banach Theorem. It remains to show that T is a homeomorphism.

Let $\beta \in cs(F)$ and $K \subset U$ be a strict compact subset. Let $\pi(g) = \sup \{|g(x)|;\ x \in K\}$, $g \in \mathcal{H}_S(U;\mathbb{C})$. then, for every $f \in \mathcal{H}_S(U;F)$, we have by the Hahn-Banach Theorem, that

$$\sup\{\beta(f(x));\ x \in K\} = \sup \{|\phi(f(x))|;\ \phi \in F',\ |\phi| \leq \beta,\ x \in K\}$$

$$= \sup \{|\langle \phi \circ f, \gamma \rangle|\ ;\ \gamma \in (\mathcal{H}_S(U;\mathbb{C}))',\ |\gamma| \leq \pi,\ \phi \in F',\ |\phi| \leq \beta\}$$

$$= (\beta \varepsilon \pi)(Tf).$$

This completes the proof.

2.6 COROLLARIES OF THE PROPOSITION 2.5: *If U is a non-void open subset of E, we have:*

a) *If F is a complete space and F or $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ has the approximation property, then*

$$(\mathcal{H}_S(U;\mathbb{C}), \tau_S) \hat{\otimes}_\varepsilon F \approx (\mathcal{H}_S(U;F), \tau_S).$$

In particular if E has finite dimension and F is a complete space, then

$$(\mathcal{H}_S(U;\mathbb{C}), \tau_S) \hat{\otimes}_\varepsilon F \approx (\mathcal{H}_S(U;F), \tau_S).$$

b) *If F has the approximation property and satisfies the condition of Proposition 2.5, then $\mathcal{H}_S(U;\mathbb{C}) \otimes F$ is τ_S-dense in $\mathcal{H}_S(U;F)$.*

c) *$(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ has the approximation property if and only if, $\mathcal{H}_S(U;\mathbb{C}) \otimes F$ is τ_S-dense in $\mathcal{H}_S(U;F)$, for all Banach spaces F.*

The proof of a) follows from Proposition 2.5 and Proposition 1.33 (d). The proof of b) follows from Proposition 2.5 and Proposition 1.33 (b); and c) follows from Proposition 2.5 and Proposition 1.33(c) and Proposition 1.22.

PROOF OF THEOREM 2.4: Let $K \subset U$ be a strict compact set, $\beta \in cs(F)$ and $f \in \mathcal{H}_S(U;F)$. By hypothesis, there is $B \in \mathcal{B}_E$ such that $K \subset U \cap E_B$ and is compact in E_B, so that given $\varepsilon > 0$, there is $T \in E^* \otimes E$ satisfying $p_B(T(x) - x) < \varepsilon$, for all $x \in K$.

As in the proof of Theorem 2.1, whenever $x \in K$, there is $\delta > 0$, $\delta \leq \operatorname{dist}_{E_B}(K, C_{E_B}(U \cap E_B))$, such that if $p_B(T(x) - x) < \delta$, then $\beta(f(x) - f(y)) < \varepsilon$. Let $U_o = U \cap U_B \cap T(E)$. Since $T(E)$ has finite dimension $(\mathcal{H}(U_o;\hat{F}), \tau_o) \approx (\mathcal{H}(U_o;\mathbb{C}), \tau_o) \hat{\otimes}_\varepsilon \hat{F}$ (by Corollary 2.6 a)), where $\hat{F}$ is the completion of F. Thus for $f|_{U_o}$ there is $f_1 \in \mathcal{H}(U_o;\mathbb{C}) \otimes \hat{F}$, with $\beta(f|_{U_o}(y) - f_1(y)) < \varepsilon$, for $y \in T(K)$.

Let $f_1 = \sum_{j=1}^{m} \varphi_j \otimes z_j$, where $\varphi_j \in \mathcal{H}(U_o;\mathbb{C})$ and $z_j \in \hat{F}$, $j = 1,\ldots,m$. There are $\bar{z}_j \in F$, $j = 1,\ldots,m$, with $\|\varphi_j\|_{T(K)} \cdot \beta(z_j - \bar{z}_j) < \varepsilon/2m$, and since U_o is S-Runge in $T(E)$, there are $\bar{\varphi}_j \in \mathcal{H}_S(T(E);\mathbb{C})$ with $\|\varphi_j - \bar{\varphi}_j\|_{T(K)} \cdot \beta(\bar{z}_j) < \varepsilon/2m$, $j = 1,\ldots,m$. Let $f_2 = \sum_{j=1}^{m} \bar{\varphi}_j \otimes \bar{z}_j \in \mathcal{H}_S(T(E);\mathbb{C}) \otimes F$, it follows that $\beta(f_1(y) - f_2(y)) < \varepsilon$, for all $y \in T(K)$. Thus letting $h = f_2 \circ T|_U \in \mathcal{H}_S(U;\mathbb{C}) \otimes F$, we get for all $x \in K$,

$$\beta(f(x) - h(x)) \leq \beta(f(x) - f|_{U_o}(T(x)) + \beta(f|_{U_o}(T(x) - f_1(T(x)) +$$

$$+ \beta(f_1(T(x)) - f_2(T(x))) < 3\varepsilon .$$

2.7 THEOREM: *Let E be a quasi-complete space and U be a non-void open subset of E. If $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ has the approximation property, then E has the* S.a.p. .

PROOF: We show that $E_S^* = (E^*, \tau_S)$ is a complemented subspace of

$(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$; hence E_S^* has the approximation property. From this, we have that, if E is a quasi-complete space, then E has the S.a.p.. For $a \in U$, the mapping $D_a : (\mathcal{H}_S(U;\mathbb{C}), \tau_S) \to E_S^*$, defined by $D_a(f) = \hat{\delta}^1 f(a)$, for $f \in \mathcal{H}_S(U;\mathbb{C})$, is a continuous projection onto E_S^*. Indeed, it is clear that $D_a^2 = D_a$. To show continuity, let K be a strict compact subset of E. Then there is $B \in \mathcal{B}_E$ such that $K \subset E_B$ and is compact in E_B. Let $\delta > 0$, be such that $a + \delta K \subset U \cap E_B$. From Cauchy inequalities, (Corollary 1.18) it follows that

$$\sup_{x\in K} |\hat{\delta}^1 f(a)(x)| \leq \frac{1}{\delta} \sup_{x\in K} |f(a + \delta x)|,$$

for all $f \in \mathcal{H}_S(U;\mathbb{C})$. Then D_a is continuous.

Now, we show that E has the S.a.p.. Since E_S^* has the approximation property, then for every balanced convex compact subset $\mathfrak{K}$ of E_S^*, for every strict compact subset K of E, and for every $\varepsilon > 0$, there is $g \in (E_S^*)' \otimes E^*$, such that $\|g(\varphi) - \varphi\|_K < \varepsilon$, for every $\varphi \in \mathfrak{K}$. Since $g \in (E_S^*)' \otimes E$,

$$g = \sum_{i=1}^{m} \varphi_i \otimes x_i, \quad \varphi_i \in (E_S^*)', \quad x_i \in E^*, \quad i = 1,\ldots,m.$$

Since, for each $i = 1,\ldots,m$, $\varphi_i : E_S^* \to \mathbb{C}$ is continuous $|\varphi_i(\varphi)| \leq c_i p_i(\varphi)$, for $\varphi \in E^*$, where $p_i(\varphi) = \|\varphi\|_{L_i}$, for some strict compact subset L_i of E. Let $B_i \in \mathcal{B}_E$ be such that $L_i \subset E_{B_i}$ and are compact in E_{B_i}, for each $i = 1,\ldots,m$. Since K is a strict compact subset of E, there is $B \in \mathcal{B}_E$ such that $K \subset E_B$ and is compact in E_B. Let $L = \Gamma(\bigcup_{i=1}^{m} L_i \cup K)$. Hence $|\varphi_i(\varphi)| \leq c\|\varphi\|_L$, where c is a constant. For $D \in \mathcal{B}_E$ so that $B \subset D$ and $B_i \subset D$, for $i = 1,\ldots,m$, L is a balanced convex compact set in the Banach space E_D. By the Hahn-Banach Theorem, there are, for each $i = 1,\ldots,m$, $\tilde{\varphi}_i : (E_D)' \to \mathbb{C}$ linear extensions of φ_i, so that $|\tilde{\varphi}_i(\varphi)| \leq \|\varphi\|_L$, for $\varphi \in (E_D)'$. Therefore, for $i = 1,\ldots,m$, $\tilde{\varphi}_i \in ((E_D)'_c)'$.

Let $\psi : E^*_S \to (E_D)'_c$, be defined by $\psi(T) = T|_{E_D}$, for $T \in E^*$. ψ is linear and continuous. Hence $\psi(\mathfrak{U}) = \mathfrak{U}_D$ is a balanced convex compact subset of $(E_D)'_c$, and then equicontinuous. Hence, we can write $\mathfrak{U}_D = V^o$, where V is a closed absolutely 0-neighborhood in E_D. ($V = \{v \in E_D,\ p_D(v) \leq \delta\}$, for some $\delta > 0$). Hence $\mathfrak{U}_D^o = V^{oo} = V$. Since we get $K \subset E_D$,

$$\sup_{t\in K} |g(\varphi)(t) - \varphi(t)| = \sup_{t\in K} |\tilde{g}(\varphi_D)(t) - \varphi_D(t)| < \varepsilon, \text{ if } \varphi_D \in \mathfrak{U}_D;$$

where $\tilde{g} = \sum_{i=1}^{m} \tilde{\varphi}_i \otimes x_i$. Hence,

$$\sup_{t\in K} \Big| \sum_{i=1}^{m} \tilde{\varphi}_i(\varphi_D) x_i(t) - \varphi_D(t) \Big| < \varepsilon, \quad \varphi_D \in \mathfrak{U}_D.$$

Therefore

$$\sup_{t\in K} \Big| \varphi_D\Big(\sum_{i=1}^{m} \tilde{\varphi}_i x_i(t) - t \Big) \Big| < \varepsilon, \quad \text{for} \quad \varphi_D \in V^o,$$

and then

$$1/\varepsilon \cdot \Big(\sum_{i=1}^{m} \tilde{\varphi}_i x_i(t) - t \Big) \in V,$$

that is,

$$\sup_{t\in K} p_D(g(t) - t) \leq \varepsilon\delta.$$

Since, $\tilde{g} \in E^* \otimes E$ and δ is independent of ε, it follows that E has the S.a.p..

2.8 DEFINITION: Let E be a locally convex complex Hausdorff space. E is said to have the S-*holomorphic approximation property* (S.H.a.p.) if given $K \subset E$, a strict compact set, there is $B \in \mathcal{B}_E$ such that $K \subset E_B$ and is compact in E_B and given $\varepsilon > 0$, there is $g \in \mathcal{H}_S(E;\mathbb{C}) \otimes E$ such that $p_B(g(x) - x) < \varepsilon$, for all $x \in K$.

It is clear that if E has the S.a.p., then E has the S.H.a.p.. For the converse it is needed that E be a quasi-complete space, that is, we have the following theorem, which contains the previous theorem for an open subset U of E, which is finitely S-Runge.

2.9 THEOREM: *Let* E *be a quasi-complete space and let* U *be an open subset of* E, *which is finitely* S-Runge. *Then the following conditions are equivalent:*

a) E *has the* S.H.a.p..

b) *For every locally convex space* F, $\mathcal{H}_S(U;\mathbb{C}) \otimes F$ *is* τ_S*-dense in* $\mathcal{H}_S(U;F)$.

c) $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ *has the approximation property.*

d) E *has the* S.a.p..

REMARK: The assumption of E to be a quasi-complete space is needed only in c) → d).

PROOF: b) → c) is part (c) of Corollary 2.6, which is true for any open subset of E. c) → d) is Theorem 2.7. d) → a) is obvious. It remains only to show that a) → b). This proof is analogous to the proof of Theorem 2.1, substituting $g \in \mathcal{H}_S(E;\mathbb{C}) \otimes E$ for $T \in E^* \otimes E$ (cf. Definition 2.8).

2.10 COROLLARY: *Let* E *be a quasi-complete space. Then* E *has the* S.a.p. *if and only if, for each* $n \in \mathbb{N}$, $(P_b({}^nE;\mathbb{C}), \tau_S)$ *has the approximation property.*

PROOF: If E has the S.a.p., it follows by Theorem 2.9, that for any open subset U of E, which is finitely S-Runge, $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ has the approximation property. Since for each $n \in \mathbb{N}$, $(P_b({}^nE;\mathbb{C}), \tau_S)$

is a complemented subspace of $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$, we have that $(P_b(^nE;\mathbb{C}), \tau_S)$ has the approximation property.

Conversely, in particular, E* having the approximation property, E has the S.a.p. (as in the proof of Theorem 2.7).

2.11 REMARK: By the previous Corollary, we have that if E is a quasi-complete space and U is an open subset of E, which is finitely S-Runge, then $(\mathcal{H}_S(U;\mathbb{C}), \tau_S)$ has the approximation property, if and only if, for each $n \in \mathbb{N}$, $(P_b(^nE;C), \tau_S)$ has the approximation property.

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THE APPROXIMATION PROPERTY FOR NACHBIN SPACES

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1. INTRODUCTION

Throughout this paper X is a Hausdorff space such that $C_b(X;\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) separates the points of X, and E is a non-zero locally convex space. Our aim is to prove that certain function spaces $L \subset C(X;E)$ have the approximation property as soon as E has the approximation property. We show this for the class of all Nachbin spaces $CV_\infty(X;E)$. Such spaces include $C(X;E)$ with the compact-open topology; $C_b(X;E)$ with the strict topology; $C_o(X;E)$ with the uniform topology. When $E = \mathbb{K}$, Bierstedt [1], using the technique of ε-products, had proved that $CV_\infty(X;\mathbb{K})$ has the approximation property, under the hypothesis that X is a completely regular $k_{\mathbb{R}}$-space, and that the family V of weights is such that given a compact subset $K \subset X$, one can find a weight $v \in V$ such that $v(x) \geq 1$ for all $x \in K$.

The technique we use here was suggested by the paper [5] of Gierz, who proved the analogue of Theorem 1 below for the case of X compact and bundles of Banach spaces. This technique of "localization" of the approximation property was used by Bierstedt, in the case of the partition by antisymmetric sets (Bierstedt [2]), but the main idea of representing the space of operators of L as another Nachbin space of cross sections is due to Gierz. However our presentation is

much simpler, in particular we do not use the concept of a locally $C(X)$-convex $C(X)$-module. In the Introduction to his paper, Gierz said that his method could be applied to the vector fibrations in the sense of [8], and this led to our effort at simplifying his proof and adapting it to our context.

2. THE APPROXIMATION PROPERTY FOR NACHBIN SPACES

A *vector fibration* over a Hausdorff topological space X is a pair $(X,(F_x)_{x\in X})$, where each F_x is a vector space over the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *cross-section* is then any element f of the Cartesian product of the spaces F_x, i.e., $f = (f(x))_{x\in X}$.

A *weight on* X is a function v on X such that $v(x)$ is a seminorm over F_x for each $x \in X$. A *Nachbin space* LV_∞ is a vector space L of cross-sections f such that the mapping

$$x \in X \to v(x)[f(x)]$$

is upper semicontinuous and null at infinity on X for each weight v belonging to a *directed set* V of weights (directed means that, given $v_1, v_2 \in V$, there is some $v \in V$ and $\lambda > 0$ such that $v_i(x) \leq \lambda v(x)$ $(i = 1,2)$ for all $x \in X$); the space L is then equipped with the topology defined by the directed set of seminorms

$$f \to \|f\|_v = \sup\{v(x)[f(x)];\ x \in X\},$$

and it is denoted by LV_∞.

Since only the subspace $L(x) = \{f(x);\ f \in L\} \subset F_x$ is relevant, we may assume that $L(x) = F_x$ for each $x \in X$.

The cartesian product of the spaces F_x has the structure of a $C(X;\mathbb{K})$-module, where $C(X;\mathbb{K})$ denotes the ring of all continuous

$\mathbb{K}$-valued functions on X, if we define the product ϕf for each $\phi \in C(X;\mathbb{K})$ and each cross-section f by

$$(\phi f)(x) = \phi(x) f(x)$$

for all $x \in X$. If $W \subset L$ is a vector subspace and $B \subset C(X;\mathbb{K})$ is a subalgebra, we say that W is a B-module, if $BW = \{\phi f; \phi \in B, f \in W\} \subset W$.

We recall that a locally convex space E has the *approximation property* if the identity map e on E can be approximated, uniformly on every totally bounded set in E, by continuous linear maps of finite rank. This is equivalent to say that $E' \otimes E$ is dense in $\mathcal{L}_c(E)$, the space $\mathcal{L}(E)$ with the topology of uniform convergence on totally bounded sets of E. Let $cs(E)$ be the set of all continuous seminorms on E. For each seminorm $p \in cs(E)$, let E_p denote the space E seminormed by p. If, for each $p \in cs(E)$, the space E_p has the approximation property, then E has the approximation property.

THEOREM 1: *Suppose that, for each $x \in X$, the space F_x equipped with the topology defined by the family of seminorms $\{v(x); v \in V\}$ has the approximation property. Let $B \subset C_b(X;\mathbb{K})$ be a self-adjoint and separating subalgebra. Then any Nachbin space LV_∞ which is a B-module has the approximation property.*

The idea of the proof is to represent the space $\mathcal{L}(W)$, where $W = LV_\infty$, as a Nachbin space of cross-sections over X, each fiber being $\mathcal{L}(W;F_x)$, and then apply the solution of the Bernstein-Nachbin approximation problem in the separating and self-adjoint bounded case. Before proving theorem 1 let us state some corollaries.

COROLLARY 1: *Let X be a Hausdorff space, and for each $x \in X$ let F_x be a normed space with the approximation property. Let $B \subset C_b(X;\mathbb{K})$ be a self-adjoint and separating subalgebra.*

Let L *be a vector space of cross-sections pertaining to* $(X;(F_x)_{x\in X})$ *such that*

(1) *for every* $f \in L$, *the map* $x \to \|f(x)\|$ *is upper semicontinuous and null at infinity;*

(2) L *is a* B*-module;*

(3) $L(x) = F_x$ *for each* $x \in X$.

Then L *equipped with norm* $\|f\| = \sup\{\|f(x)\|;\ x \in X\}$ *has the approximation property.*

PROOF: Consider the weight v on X defined by v(x) = norm of F_x, for each $x \in X$. Then LV_∞ is just L equipped with the norm $\|f\| = \sup\{\|f(x)\|;\ x \in X\}$.

REMARK: From Corollary 1 it follows that all "continuous sums", in the sense of Godement [6] or [7], of Banach spaces with the approximation property have the approximation property, if the "base space" X is compact and if such a "continuous sum" is a $C_b(X;\mathbb{K})$-module. In particular, all "continuous sums" of Hilbert spaces and of C*-algebras, in the sense of Dixmier and Douady [3] have the approximation property, if X is compact. Indeed, a "continuous sum" in the sense of [3] is a $C(X;\mathbb{K})$-module.

COROLLARY 2: *Let* X *be a Hausdorff space such that* $C_b(X;\mathbb{K})$ *is separating; let* V *be a directed set of real-valued, non-negative, upper semicontinuous functions on* X*; and let* E *be a locally convex space with the approximation property. Then* $CV_\infty(X;E)$ *has the approximation property.*

PROOF: By definition, $CV_\infty(X;E) = \{f \in C(X;E);\ vf$ vanishes at infinity, for all $v \in V\}$, equipped with the topology defined by the

family of seminorms

$$\| f \|_{v,p} = \sup \{ v(x)\ p(f(x)) ;\ x \in X \}$$

where $v \in V$ and $p \in cs(E)$.

Let L_v denote $CV_\infty(X;E)$ equipped with the topology defined by the above seminorms when $v \in V$ is kept fixed. Then, for each $x \in X$, either $L_v(x) = 0$ or $L_v(x) = E$ equipped with the topology defined by the seminorms $\{v(x)p;\ p \in cs(E)\}$. Hence in both cases, $L_v(x)$ has the approximation property. It remains to notice that all Nachbin spaces are $C_b(X;\mathbb{K})$-modules. Therefore L_v has the approximation property. Since $v \in V$ was arbitrary, $CV_\infty(X;E)$ has the approximation property.

COROLLARY 3: *Let* X *and* E *be as in Corollary 2. Then*

(a) $C(X;E)$ *with the compact-open topology has the approximation property.*

(b) $C_o(X;E)$ *with the uniform topology has the approximation property.*

REMARK: In (a) above, it is sufficient to assume that $C(X;\mathbb{K})$ is separating.

COROLLARY 4: (Fontenot [4]) *Let* X *be a locally compact Hausdorff space, and let* E *be a locally convex space with the approximation property. Then* $C_b(X;E)$ *with the strict topology* β *has the approximation property.*

PROOF: Apply Corollary 2, with $V = \{v \in C_o(X;\mathbb{R});\ v \geq 0\}$.

COROLLARY 5: *All Nachbin spaces of continuous scalar-valued functions*

have the approximation property.

PROOF: In Corollary 2, take $E = \mathbb{K}$.

3. PROOF OF THEOREM 1

Let $W = LV_\infty$ and let $A \subset W$ be a totally bounded set.

Let $v_o \in V$ and $\varepsilon > 0$ be given.

For each $T \in \mathcal{L}(W)$ consider the map

$$\varepsilon_x \circ T : W \to F_x$$

for $x \in X$, where $\varepsilon_x : W \to F_x$ is the evaluation map, i.e., $\varepsilon_x(f) = f(x)$, for all $f \in W$.

STEP 1: $\varepsilon_x \circ T \in \mathcal{L}(W;F_x)$.

PROOF: Just notice that $\varepsilon_x \in \mathcal{L}(W;F_x)$, since

$$v(x)[f(x)] \leq \|f\|_v, \quad \text{for any } v \in V.$$

For each $T \in \mathcal{L}(W)$, consider the cross-section $\hat{T} = (\varepsilon_x \circ T)_{x \in X}$; and for each $v \in V$ consider the weight $\hat{v}$ on X defined by

$$\hat{v}(x)[U(x)] = \sup \{ v(x)[(U(x))(f)]; \ f \in A\}$$

for every $U(x) \in \mathcal{L}(W;F_x)$. Then

$$\hat{v}(x)[\hat{T}(x)] = \hat{v}(x)[\varepsilon_x \circ T] = \sup \{ v(x)[(T f)(x)]; \ f \in A\}$$

for any $T \in \mathcal{L}(W)$.

STEP 2: *The map* $x \to \hat{v}(x)[\hat{T}(x)]$ *is upper semicontinuous and vanishes at infinity on* X, *for each* $T \in \mathcal{L}(W)$.

PROOF: Let $x_o \in X$ and assume

$$\hat{v}(x_o)[\hat{T}(x_o)] < h .$$

Choose h'' and h' such that

(1) $$\hat{v}(x_o)[\hat{T}(x_o)] < h' < h'' < h.$$

Let $\delta = 2(h'' - h')$. Then $\delta > 0$. Since $T(A)$ is totally bounded, there exist $f_1, f_2, \ldots, f_m \in A$ such that, given $f \in A$, there is $i \in \{1,2,\ldots,m\}$ such that

(2) $$\|T f - T f_i\|_v < \delta / 4.$$

Since $x \to v(x)[(T f_i)(x)]$ is upper semicontinuous, there are $V_1, V_2, \ldots, V_m$ neighborhoods of x_o such that

(3) $$v(x)[(T f_i)(x)] < v(x_o)[(T f_i)(x_o)] + \delta / 4$$

for all $x \in V_i$ $(i = 1,2,\ldots,m)$.

Let $U = V_1 \cap V_2 \cap \ldots \cap V_m$. Then U is a neighborhood of x_o in X. Let $x \in U$ and let $f \in A$. Choose $i \in \{1,2,\ldots,m\}$ such that (2) is true. Then

$$\begin{aligned} v(x)[(T f)(x)] &\leq v(x)[(T f)(x) - (T f_i)(x)] + v(x)[(T f_i)(x)] \\ &< \|T f - T f_i\|_v + v(x_o)[(T f_i)(x_o)] + \delta/4 \\ &< \delta/2 + v(x_o)[(T f_i)(x_o)] \\ &= h'' - h' + v(x_o)[(T f_i)(x_o)] . \end{aligned}$$

On the other hand, by (1), we have

$$v(x_o)[(T f_i)] \leq \hat{v}(x_o)[(\hat{T}(x_o)] < h' .$$

Hence $v(x)[(T f)(x)] < h''$ for all $f \in A$, and $x \in U$.
Therefore $\hat{v}(x)[\hat{T}(x)] \leq h'' < h$, for all $x \in U$.

Let us now prove that the mapping $x \to \hat{v}(x)[\hat{T}(x)]$ vanishes at infinity.

Let $\delta > 0$ be given and define

$$K_\delta = \{x \in X;\ \hat{v}(x)[\hat{T}(x)] \geq \delta\} .$$

Since $K_\delta = \emptyset$, if $\sup\{\|T f\|_v ;\ f \in A\} < \delta$, we may assume $\sup\{\|T f\|_v ;\ f \in A\} \geq \delta$.

Since $T(A)$ is totally bounded, there are $f_1,\ldots,f_m \in A$ such that, given $f \in A$, there is $i \in \{1,\ldots,m\}$ such that

$$\|T f - T f_i\|_v < \delta/4. \tag{4}$$

Let $K = \bigcup_{i=1}^{m} \{t \in X;\ v(t)[(T f_i)(t)] \geq \delta/2\}$.

Then K is compact, since each of the functions $x \to v(x)[(T f_i)(x)]$ vanishes at infinity. Let now $x \in K_\delta$ and choose $f \in A$ such that

$$v(x)[(T f)(x)] > \frac{3\delta}{4} . \tag{5}$$

Choose $f_i \in A$ satisfying (4). Then

$$v(x)[(T f)(x)] < v(x)[(T f_i)(x)] + \delta/4. \tag{6}$$

Therefore $\delta/2 < v(x)[(T f_i)(x)]$ and so $x \in K$, i.e., $K_\delta \subset K$. Since K_δ is closed, this ends the proof.

The above two steps show that the image $\mathcal{L} = \{\hat{T};\ T \in \mathcal{L}(W)\}$ of $\mathcal{L}(W)$ under the map $T \to \hat{T}$ is a Nachbin space $\mathcal{L}V_\infty$ of cross sections over X, pertaining to the vector fibration $(X;\ (\mathcal{L}(W;F_x))_{x \in X})$, if we take as family V of weights the family $V = \{\hat{v};\ v \in V\}$

STEP 3: *For every* $T \in \mathcal{L}(W)$, $v \in V$,

$$\sup_{f \in A} \|T f\|_v \leq \sup_{x \in X} \hat{v}(x)[\hat{T}(x)].$$

PROOF: Let $f \in A$. Then

$$\|T f\|_v = \sup_{x \in X} v(x)[(T f)(x)] =$$

$$= \sup_{x \in X} v(x)[(\varepsilon_x \circ T)(f)]$$

$$= \sup_{x \in X} \hat{v}(x)[\hat{T}(x)] = \|\hat{T}\|_{\hat{v}}.$$

Let now $F = \{\hat{T};\ T \in W' \otimes W\}$.

Our aim is to prove that we can find $T \in W' \otimes W$ such that

$$\sup_{f \in A} \|T f - f\|_{v_o} < \varepsilon.$$

Hence, by Step 3, it is enough to prove that $\|\hat{T} - \hat{I}\|_{\hat{v}_o} < \varepsilon$, where $\hat{I} = (\varepsilon_x)_{x \in X}$.

By the bounded case of the Bernstein-Nachbin approximation problem (Theorem 11, [8], pg. 314) it is enough to prove that

STEP 4: F *is a* B*-module.*

STEP 5: *For each* $x \in X$, $F(x)$ *is dense in* $\mathcal{L}(W;F_x)$, *equipped with the topology defined by the seminorms* $\{\hat{v}(x);\ v \in V\}$.

PROOF: To prove that $\hat{F}$ is a B-module, it is enough to prove that

$$(7) \qquad (M_\phi \circ T)\hat{} = \phi\, \hat{T}$$

for all $\hat{T} \in \hat{F}$, i.e., for all $T \in W' \otimes W$ and for all $\phi \in B$; and $M_\phi : W \to W$ is defined by $M_\phi(f) = \phi f$, for all $f \in W$.

Now to prove (7), one has to prove that

$$(8) \qquad (M_\phi \circ T)\hat{}(x) = (\phi\, \hat{T})(x)$$

for all $x \in X$. However,

$$(M_\phi \circ T)\hat{}(x) = \varepsilon_x \circ (M_\phi \circ T), \quad \text{and}$$

$$(\phi\, \hat{T})(x) = \phi(x)\hat{T}(x)$$

$$= \phi(x)(\varepsilon_x \circ T).$$

And, for all $f \in W$ one has

$$[\varepsilon_x \circ (M_\phi \circ T)](f) = \varepsilon_x((M_\phi \circ T)(f))$$

$$= \varepsilon_x(\phi(T f))$$

$$= \phi(x)(T f)(x)$$

$$= \phi(x)(\varepsilon_x \circ T)(f).$$

This ends the proof of step 4.

To prove step 5, we first notice that, since each F_x equipped with the topology defined by $\{v(x);\ v \in V\}$ has the approximation property, then $W' \otimes F_x$ is dense in $\mathcal{L}_c(W;F_x)$, a fortiori in $\mathcal{L}(W;F_x)$ with the topology of the seminorms $\{\hat{v}(x);\ v \in V\}$.

Hence, all we have to prove is that $F(x)$ contains $W' \otimes F_x$, for each $x \in X$.

Let then $T \in W' \otimes F_x$ be a continuous linear operator of finite rank, say

$$T = \sum_{i=1}^{n} \phi_i \otimes v_i$$

where $\phi_i \in W'$ and $v_i \in F_x$. Since $W(x) = F_x$, choose $f_i \in W$ such that

$$f_i(x) = v_i$$

for $i = 1,2,\ldots,n$.

Define $U = \sum_{i=1}^{n} \phi_i \otimes f_i$.

Then $U \in W' \otimes W$; so $\hat{U} \in F$. Now

$$\hat{U}(x) = \varepsilon_x \circ \left(\sum_{i=1}^{n} \phi_i \otimes f_i \right)$$

and therefore

$$\hat{U}(x)(f) = \sum_{i=1}^{n} \phi_i(f) f_i(x) = \sum_{i=1}^{n} \phi_i(f) v_i = T(f)$$

for all $f \in W$.

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ON CARDINAL SPLINE SMOOTHING

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INTRODUCTION

The present paper describes the methods, with some changes to be mentioned below, whereby I solved the numerical problem assigned to me at the Ballistics Research Laboratories in Aberdeen, Maryland, during the second World War (see [4]). The problem was to smooth very extended equidistant tables of drag functions (or drag coefficients) by approximating them by very smooth functions that were easily computable with their first and second derivatives.

The first question to be answered was this: *When may a "moving average" operation legitimately be called a smoothing formula ?* An answer is given in Section 1 of Part I. That it is a reasonable one is shown in Section 2 of Part I. These ideas were later greatly generalized by Fritz John in his important work on parabolic differential equations (see [3]). The connection is briefly mentioned in Section 2 of Part I.

The second essential ingredient of our results of Part II is the process of *cardinal spline interpolation* (see [9]). The required results are described in Section 3 of Part I.

The changes from my war-time approach are developed completely in Part II. This is the new contribution of the paper, and it arises

if we adapt the idea of E. T. Whittaker (see [10] and [7]) to the problem of smoothing a bi-infinite sequence of equidistant data. The result is the *cardinal spline smoothing process*.

Let me thank Professor Ilio Galligani for asking me to visit Rome in May of 1977, where I gave a short 10-hours course on Cardinal Spline Interpolation at his Istituto per le applicazioni del Calcolo "Mauro Picone".

PART I. PRELIMINARIES

1. **WHAT IS A SMOOTHING FORMULA ?** The several smoothing (or gradua — tion) methods discussed in [10] are all of the "moving average" type. By this we mean that we are given a sequence of real and symmetric weights

$$(1) \qquad (a_n),\ a_n = a_{-n} (-\infty < n < \infty),\ \sum_{-\infty}^{\infty} a_n = 1,$$

which we apply to the data (x_n) to produce their smoothed version (y_n), by the formula

$$(2) \qquad y_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} x_\nu\ ,\ (n = 0,\ \pm\ 1, \ldots).$$

In other words, we convolute the sequences (a_n) and (x_n), an opera — tion symbolically described by

$$(3) \qquad (y_n) = (a_n) * (x_n).$$

By deriving a formula (2) according to a definite idea, such as a least squares formula, or Whittaker's formulae (see [10, Chapter XI]), we may reasonably expect that it will transform a given se- quence (x_n) into a smoother sequence (y_n). This point becomes doubt- ful, however, in case the formula (2) was otherwise obtained, e.g.,

by some approximating procedure that is not strictly interpolatory. An example of such a procedure will be our cardinal spline smoothing of Part II. A criterion for (2) to be regarded as a *smoothing formula* was proposed by the author in [4, pp. 50 - 54] and proceeds as follows.

To begin with, we assume the Laurant series

$$F(z) = \sum_{-\infty}^{\infty} a_n z^n, \quad (r < |z| < r^{-1}), \tag{4}$$

to converge in some ring containing the unit circle $|z| = 1$. Setting $z = e^{iu}$, we call the even function

$$\phi(u) = F(e^{iu}) = a_o + 2a_1 \cos u + 2a_2 \cos 2u + \ldots \tag{5}$$

the *characteristic function* of the formula (2). The regularity of $F(z)$ on $|z| = 1$, implies that $\phi(u)$ is regular in a certain strip $|\mathrm{Im}\ u| < \alpha$, whence the valid Taylor expansion

$$\phi(u) = \sum_{o}^{\infty} b_n u^n \quad (|u| < \alpha). \tag{6}$$

This expansion allows us to express easily an important property of (2): Its *reproductive power*, or *degree of exactness*.

We say that (2) has the degree of exactness $2m - 1$, where m is a natural number, provided that (2) reproduces exactly, i.e., $y_n = x_n$ for all n, a sequence (x_n), if this sequence represents the values $P(n)$ of a polynomial of degree $2m - 1$, but does not have this property for any higher degree. That the degree of exactness is always odd, follows from the symmetry condition (1). In terms of (6) we have the following easily established proposition:

The formula (2) *has the degree of exactness* $2m - 1$, *if and only if the expansion* (6) *is of the form*

$$\phi(u) = 1 - \lambda u^{2m} + \ldots \quad (\lambda \neq 0). \tag{7}$$

An example. Let (1) be the sequence

$$a_o = 5/8,\ a_1 = 1/4,\ a_2 = -1/16,\ a_n = 0 \quad \text{if} \quad n > 2,$$

when the convolution (2) assumes the form

$$y_n = (-x_{n-2} + 4x_{n-1} + 10x_n + 4x_{n+1} - x_{n+2})/16. \tag{8}$$

From (5) we obtain

$$\phi(u) = (5 + 4 \cos u - \cos 2u)/8 = 1 - \frac{1}{16} u^4 + \ldots \tag{9}$$

and (7) is seen to hold with $m = 2$, $\lambda = 1/16$. It follows that the formula (8) has the *degree of exactness* 3.

May (8) be regarded as a smoothing formula ? The general criterion is described by the following.

DEFINITION: *We say that* (2) *is a smoothing formula, provided that its characteristic function* $\phi(u)$ *satisfies*

$$-1 < \phi(u) < 1 \quad \text{if} \quad 0 < u < 2\pi . \tag{10}$$

This condition evidently implies that *the coefficient* λ, *appearing in* (7), *is positive*. The characteristic function (9) is easily seen to satisfy (10), and so (8) is a smoothing formula according to our definition of the term.

The criterion (10) raises the following kind of equation: Do the numerous smoothing formulae as given in [10] satisfy our criterion

(10)? See Greville's paper [1] where some of these questions are answered affirmatively.

In [4, pp. 50 - 54] good arguments in support of the criterion (10) are presented. Even more convincing reasons were given in [5, Part I]. In the next section we reproduce these arguments as presented in [6, pp. 200 - 204].

2. THE BEHAVIOR OF THE ITERATES OF A MOVING AVERAGE FORMULA : THE PROBLEM OF DE FOREST. A conclusive argument in support of the necessity of our condition (1.10) is furnished by the solution of the following problem first stated and attacked by Erasmus L. De Forest (1834-1888). If we subject the given sequence (x_ν) n times in succession to the same tranformation (1.2), we obtain a linear transformation

$$y_\nu^{(n)} = \sum_{k=-\infty}^{\infty} a_{\nu-k}^{(n)} x_k , \tag{1}$$

which is the n-fold iterate of (1.2). *What is the asymptotic behavior of the coefficients of* (1) *as* $n \to \infty$? This question was answered by De Forest and by G. B. Dantzig (for references see[5]) for the case when all coefficients of (1.2) are non - negative, hence necessarily that $m = 1$ in (1.7). A general solution is as follows.

Let (1.2) be such that (1.17), (1.10), and (1.7), are satisfied, hence that $\lambda > 0$. Let

$$G_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-v^{2m}} \cos vx \, dv, \tag{2}$$

which is the normal frequency function

$$G_1(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} \tag{3}$$

if $m = 1$, otherwise ($m = 2,3,\ldots$) $G_m(x)$ is an entire function having infinitely many zeros, all real.

The coefficients of (1) *satisfy the asymptotic relations*

$$a_\nu^{(n)} = (\lambda n)^{-\frac{1}{2m}} G_m(\nu(\lambda n)^{-\frac{1}{2m}}) + o(n^{-\frac{1}{2m}}) \quad \text{as } n \to \infty, \tag{4}$$

where the *"little o" symbol holds uniformly for all integers* ν.

For a proof see [5, Part I], where it is also shown by examples that (4) no longer holds if the equality sign is allowed in (1.10), and that the coefficient $a_o^{(n)}$ diverges exponentially to $+\infty$, as $n = 2k$ tends to infinity through even values, if the inequalities (1.10) are reversed anywhere in the interval $0 < u < 2\pi$.

The following discussion, while not directly related to our subject of smoothing, will show the connection of the asymptotic relation (4) with the wider field of parabolic differential equations.

Observe that (2) implies that

$$U(x,t) = t^{-\frac{1}{2m}} G_m(x\,t^{-\frac{1}{2m}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tv^{2m} + ixv}\, dv, \quad (t > 0). \tag{5}$$

The function under the integral sign is immediately seen to satisfy for all v, the differential equation

$$\frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m} u}{\partial x^{2m}} \quad (x \in \mathbf{R},\ t > 0), \tag{6}$$

which reduces to the familiar heat equation if $m = 1$. It follows that also $U(x,t)$, defined by (5), is a solution of (6) in the upper half-plane $t > 0$. On the other hand, applying to (2) Fourier's inversion formula and setting $v = 0$, we find that

$$\int_{-\infty}^{\infty} G_m(x)\,dx = 1 .$$

These remarks imply the following: *If* $f(x)$ *is continuous and* $o(|x|^{-2})$, say, as $|x| \to \infty$, *then*

$$(8) \qquad u(x,t) = t^{-\frac{1}{2m}} \int_{-\infty}^{\infty} G_m \{(x - v)t^{-\frac{1}{2m}}\} f(v)\,dv, \quad (t > 0),$$

is a solution of the differential equation (6) *satisfying the boundary condition*

$$(9) \qquad \lim_{t \to 0+} u(x,t) = f(x).$$

This particular solution $u(x,t)$ may now also be approximated by the following numerical procedure: Draw in the (x,t)-plane the rectangular lattice of points

$$(\nu \Delta x,\ n \Delta t) \quad (\nu = 0, \pm 1, \ldots;\ n = 0,1,2, \ldots).$$

Define on it a lattice function $u_{\nu,n}$ by starting with

$$u_{\nu,0} = f(\nu \Delta x),$$

and computing the values along each horizontal line from those on the line below it, by means of the transformation (1.2). This evidently amounts to iterating (1.2), and after n steps we obtain

$$(10) \qquad u_{\nu,n} = \sum_{\mu=-\infty}^{\infty} a_{\nu-\mu}^{(n)} f(\mu \Delta x).$$

For any given x and $t > 0$, (10) will go over into (8) if we do the following: *We first connect the mesh-sizes* Δx *and* Δt *by the relation*

(11) $$\Delta t = \lambda(\Delta x)^{2m}.$$

If the integers ν *and* n *are such that*

$$\nu\Delta x \to x, \qquad \textit{and} \qquad n\,\Delta t \to t \quad \textit{as} \quad \Delta x \to 0.$$

then

$$u_{\nu,n} \to u(x,t).$$

This follows readily from (10) and (8), in view of the asymptotic relation (4): (10) differs from a Cauchy-Riemann sum for the integral (8), by a quantity that tends to zero due to the uniformity in ν of the error term of (4).

It is interesting to note that it does not matter which formula (1.2) we use in this construction, as long as it is of the degree of exactness $2m-1$, i.e., it satisfies (1.7), and above all that it satisfies the stability condition (1.10), the term "stability"meaning here stability on iteration. For the general theory of F. John, of which the equation (6) is a special example, see [3].

In this section we dealt exclusively with formulae (1.2) which satisfy the symmetry relation. In [2] T. N. E. Greville dealt with the more difficult case of unsymmetric formulae.

3. CARDINAL SPLINE INTERPOLATION (see [9, Lectures 1 - 4]). The problem of *cardinal interpolation* is to find solutions f(x) of the interpolation problem

(1) $$f(\nu) = y_\nu, \qquad \text{for all integers } \nu,$$

where (y_ν) are the data. A formal solution is furnished by the series

$$f(x) = \sum_{-\infty}^{\infty} y_\nu \frac{\sin \pi(x-\nu)}{\pi(x-\nu)} \tag{2}$$

investigated in 1908 by de la Vallée Poussin, also later by E. T. Whittaker, who called it the *cardinal series*. The difficulty with (2) is the slow decay of the function $\frac{\sin \pi x}{(\pi x)}$ as $x \to \infty$. A much simpler solution of (1) is the *piecewise linear interpolant* $S_1(x)$ given by

$$S_1(x) = \sum_{-\infty}^{\infty} y_\nu M_2(x-\nu), \tag{3}$$

where $M_2(x)$ is the roof function defined by

$$M_2(x) = x+1 \text{ in } [-1,0],\ M_2(x) = 1-x \text{ in } [0,1] \text{ and } M_2(x) = 0 \text{ if } |x| > 1.$$

The purpose of *cardinal spline interpolation* is to bridge the gap between the piecewise linear $S_1(x)$ defined by (3), and the cardinal series (2). It aims at retaining some of the sturdiness and simplicity of (3), at the same time capturing some of the smoothness and sophistication of (2).

Let m *be a natural number, and let*

$$S_{2m-1} = \{S(x)\} \tag{4}$$

be the class of cardinal splines $S(x)$ *of degree* $2m-1$ *defined by the two conditions:*

$$S(x) \in C^{2m-2}(\mathbb{R}), \tag{5}$$

(6) *The restriction of* $S(x)$ *to every unit interval* $(\nu, \nu+1)$, *where* ν *is an integer, is a polynomial of degree* $\leq 2m-1$.

For $m = 1$ we find S_1 to be identical with the class (3) of continuous piecewise linear functions. Observe that S_{2m-1} contains the class of polynomials of degrees not exceeding $2m - 1$.

The role of the roof-function $M_2(x)$, of (3), is taken over by the so-called *central B-spline* $M_{2m}(x)$: *Writing* $x_+ = \max(x,0)$, *it may be defined by*

$$(7) \qquad M_{2m}(x) = \frac{1}{(2m-1)!}\,\delta^{2m}x_+^{2m-1} = \frac{1}{(2m-1)!}\sum_{\nu=0}^{2m}(-1)^{\nu}\binom{2m}{\nu}(x+m-\nu)_+^{2m-1}.$$

Clearly $M_{2m}(x) \in S_{2m-1}$; we also find that $M_{2m}(x) > 0$ in its support $-m < x < m$. The B-spline should be familiar in view of the fundamental identity

$$\delta^{2m}f(0) = \int_{-m}^{m} M_{2m}(x)\, f^{(2m)}(x)\,dx\,,$$

which also shows that $\int M_{2m}(x)\,dx = 1$ if we choose $f(x) = x^{2m}$.

The representation (3) also generalizes, and *every* $S(x) \in S_{2m-1}$ *admits a unique representation*

$$(8) \qquad S(x) = \sum_{-\infty}^{\infty} c_\nu\, M_{2m}(x - \nu),$$

where the c_ν *are constants.* This is the so-called *standard representation*. The converse is clear: Every series (8) furnishes an element of S_{2m-1}. We now try to solve the interpolation problem (1) by elements of S_{2m-1}. In this direction there are two different kinds of results.

A. *The data* (y_ν) *are of power growth* (See [8]). We say that the sequence (y_ν) is of *power growth*, and write

(9) $$(y_\nu) \in PG,$$

provided that

(10) $$y_\nu = O(|\nu|^\gamma) \quad \text{as} \quad \nu \to \pm\infty, \text{ for some } \gamma \geq 0.$$

Similarly, we write

(11) $$f(x) \in PG,$$

provided that $f(x) = O(|x|^{\gamma_1})$ as $x \to \pm\infty$, for some $\gamma_1 \geq 0$.

Below we exclude the trivial case when $m = 1$, since our problem is solved by (3) without any restriction on the (y_ν).

THEOREM 1: *If the sequence (y_ν) is of power growth, then the interpolation problem*

(12) $$S(\nu) = y_\nu \qquad \textit{for all} \quad \nu,$$

has a unique solution $S(x)$ such that

(13) $$S(x) \in S_{2m-1} \cap PG.$$

The assumption (9) of Theorem 1 is a rough one; it admits, e.g., all *bounded* sequences (y_ν), with $\gamma = 0$ in (10). The second assumption to which we now pass, is much more selective, and takes into account the finer structure of the sequence; in fact it admits only a narrow subclass of the sequences of PG. As usual, with stronger assumptions, stronger conclusions are possible: The interpolant $S(x)$ will exhibit an important extremum property.

B. *The case when* $\sum_{-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty$ (See [9, Lecture 6]). We introduce the classes of sequences and functions as follows:

$$\ell_2^m = \{(y_\nu);\ \sum_{-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty\}, \tag{14}$$

$$L_2^m = \{f(x);\ f,\ldots,f^{(m-1)} \text{ are absolutely continuous, } f^{(m)}(x) \in L_2(\mathbb{R})\}. \tag{15}$$

Of course ℓ_2^0 and L_2^0 are the familiar ℓ_2 and L_2, respectively. We may also describe ℓ_2^m as the class of sequences obtained from elements of ℓ_2 by n successive summation. Similarly the elements of L_2^m are obtained from those of L_2 by n successive integrations.

THEOREM 2. *If*

$$(y_\nu) \in \ell_2^m \tag{16}$$

then the interpolation problem

$$S(\nu) = y_\nu \qquad \textit{for all } \nu, \tag{17}$$

has a unique solution such that

$$S(x) \in S_{2m-1} \cap L_2^m. \tag{18}$$

This solution $S(x)$ *has the following extremum property: If* $f(x)$ *is an arbitrary function such that*

$$f(x) \in L_2^m \tag{19}$$

and

(20) $$f(\nu) = y_\nu \qquad \textit{for all} \quad \nu,$$

then

(21) $$\int_{-\infty}^{\infty} |f^{(m)}(x)|^2 \, dx > \int_{-\infty}^{\infty} |S^{(m)}(x)|^2 dx,$$

unless $f(x) = S(x)$ *for all real* x.

In words: If $(y_\nu) \in \ell_2^m$, then the spline interpolant $S(x)$ minimizes the integral

(22) $$I(f) = \int_{-\infty}^{\infty} |f^{(m)}(x)|^2 \, dx$$

among all sufficiently smooth interpolants of (y_ν).

If $y_\nu = P(\nu)$ for all ν, where $P(x) \in \pi_{m-1}$, then $P(x) \in S_{2m-1} \cap L_2^m$, and therefore $S(x) = P(x)$ by the unicity of the solution in Theorem 2. However, here $I(S) = 0$. In the general case of $(y_\nu) \in \ell_2^m$ we may therefore say that $S(x)$ is among all interpolants of (y_ν), the one that "is most nearly" a polynomial of degree $\leq m - 1$.

If $y_\nu = P(\nu)$, where $P(x) \in \pi_{2m-1}$, but $P(x) \notin \pi_{m-1}$, then again $P(x) \in S_{2m-1} \cap PG$, and so $S(x) = P(x)$ is the unique solution of Theorem 1. Theorem 2 does not apply here because $(y_\nu) \notin \ell_2^m$. There is no interpolant $f(x)$ such that $I(f) < \infty$.

How do we actually construct the solutions $S(x)$ *of these interpolation problems?* To answer this question let us for the moment assume that

(23) $$(y_\nu) \in \ell_1, \quad \text{hence a fortiori} \quad (y_\nu) \in \ell_2.$$

This insures the continuity of the periodic function

(24) $$T(u) = \sum_{-\infty}^{\infty} y_\nu e^{i\nu u}$$

which we call the *generating function* of the sequence (y_ν). Here and below we denote the relationship between a sequence and its generating function symbolically by writing

$$(y_\nu) \rightarrow T(u) . \tag{25}$$

We also require the generating function of the sequence $(M_{2m}(\nu))$, which is

$$\phi_{2m}(u) = \sum_{\nu=-(m-1)}^{m-1} M_{2m}(\nu) e^{i\nu u}. \tag{26}$$

This is a cosine polynomial of order $m-1$, because $M_{2m}(x) = 0$ if $|x| \geq m$. It is readily evaluated by (7), and we find that

$$\phi_2(u) = 1, \ \phi_4(u) = \frac{1}{3}(2 + \cos u), \ \phi_6(u) = \frac{1}{60}(33 + 26 \cos u + \cos 2u), \ldots .$$

It also has the property that

$$0 < \phi_{2m}(\pi) \leq \phi_{2m}(u) \leq \phi_{2m}(0) = 1 \quad \text{for all } u. \tag{27}$$

It follows that its reciprocal has an expansion

$$\frac{1}{\phi_{2m}(u)} = \sum_{-\infty}^{\infty} w_\nu e^{i\nu u} \tag{28}$$

with real coefficients w_ν, $w_{-\nu} = w_\nu$, that decay exponentially. Let us find the standard representation

$$S(x) = \sum_j c_j M_{2m}(x - j) \tag{29}$$

of the solution of the interpolation problem (17), which requires that

$$\sum_j c_j M_{2m}(\nu - j) = y_\nu \quad \text{for all} \quad \nu . \tag{30}$$

Furthermore let

$$C(u) = \sum_j c_j e^{iju}, \quad \text{or} \quad (c_j) \rightarrow C(u), \tag{31}$$

be the as yet unknown generating function of the (c_j). Since the convolution of two sequences has a generating function that is the product of the generating functions of the two sequences, we see by (24), (26), and (31), that the relations (30) are equivalent to the relation

$$C(u)\phi_{2m}(u) = T(u), \quad \text{or} \quad C(u) = \frac{T(u)}{\phi_{2m}(u)} . \tag{32}$$

Now (28) shows that $(c_\nu) = (y_\nu) * (w_\nu)$ and therefore

$$c_\nu = \sum_j y_j w_{\nu-j} \quad \text{for all} \quad \nu . \tag{33}$$

These are the coefficients of the interpolating spline (29).

EXAMPLES: 1. *If* $m = 1$, then $\phi_2(u) = 1$, hence $w_0 = 1$, $w_\nu = 0 (\nu \neq 0)$, and we obtain $c_\nu = y_\nu$ for all ν. *If* $m = 2$, we find (See [9, Lecture 4, Section 5]) that

$$w_\nu = \sqrt{3} \cdot \lambda^{|\nu|}, \quad \text{where} \quad \lambda = -2 + \sqrt{3} = -.26795.$$

2. If we choose $y_\nu = \delta_\nu$, where $\delta_0 = 1$, $\delta_\nu = 0 (\nu \neq 0)$, then (33) shows that $c_\nu = w_\nu$. Therefore the spline

$$L_{2m-1}(x) = \sum_\nu w_\nu M_{2m}(x - \nu) \tag{34}$$

is the solution of the interpolation problem

$$L_{2m-1}(\nu) = \delta_\nu , \quad \text{for all} \quad \nu. \tag{35}$$

The function (34) is the *fundamental function* of the process, and *the solution* $S(x)$ of *the general problem* (17) *is given by*

$$S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_{2m-1}(x - \nu). \tag{36}$$

This cardinal interpolation formula bridges the gap between the linear interpolant (3) and the cardinal series (2). In fact, notice that if $m = 1$ then (36) reduces to (3), while we have

$$\lim_{m\to\infty} S_{2m-1}(x) = \frac{\sin \pi x}{\pi x} . \tag{37}$$

Also every derivative $S^{(k)}_{2m-1}(x)$ converges to the corresponding derivative of the right side of (37), uniformly for all real x.

In our discussion we have assumed that (23) holds. However, *the ralations* (33), (29), *and* (36) *are valid for both Theorems* 1 *and* 2, *under their respective assumptions.*

PART II. THE CARDINAL SMOOTHING SPLINE

1. STATEMENT OF THE PROBLEM: We assume now that

$$(y_\nu) \in \ell_1 , \quad \text{hence} \quad \Sigma \, | y_\nu | < \infty , \tag{1}$$

and restrict ourselves to real-valued data and functions. We also recall the definitions (3.14) and (3.15) of Part I, of the classes ℓ_2^m and L_2^m. In view of the inclusion relations

(2) $$\ell_1 \subset \ell_2 \subset \ell_2^1 \subset \ldots \subset \ell_2^k \subset \ell_2^{k+1} \subset \ldots$$

(See [9, p. 104]), we observe that (1) implies that (y_ν) satisfies the assumptions of Theorem 2 for all m.

THE PROBLEM: *We are given* m *and a smoothing parameter* $\varepsilon > 0$. *Among all functions*

(3) $$f(x) \in L_2^m, \qquad \text{hence} \qquad f^{(m)}(x) \in L_2(\mathbb{R}),$$

we wish to find the solution of the problem

(4) $$J(f) = \varepsilon \cdot \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \sum_{-\infty}^{\infty} (f(\nu) - y_\nu)^2 = \text{minimum}.$$

LEMMA 1: *In solving the minimum problem* (4) *we may restrict the choice of admissible functions* $f(x)$ *to the elements of*

(5) $$S_{2m-1} \cap L_2^m .$$

PROOF: If $f(x)$ is such that $J(f) < \infty$, then $(f(\nu) - y_\nu) \in \ell_2$. We apply Theorem 2 to the sequence $(f(\nu))$, and let

(6) $$s(x) \in S_{2m-1} \cap L_2^m$$

be such that $s(\nu) = f(\nu)$ for all ν. But then

$$J(s) = \varepsilon \int (s^{(m)})^2 dx + \Sigma (s(\nu) - y_\nu)^2$$

$$= \varepsilon \int (s^{(m)})^2 dx + \Sigma (f(\nu) - y_\nu)^2,$$

and so

$$J(s) \leqq \varepsilon \int_{-\infty}^{\infty} (f^{(m)})^2 dx + \Sigma (f(\nu) - y_\nu)^2 = J(f),$$

in view of the extremum property of $s(x)$ as expressed by (3.21) of Theorem 2. Therefore, for any $f(x)$, the spline $s(x)$ that interpolates $f(x)$, produces a value $J(s) \leqq J(f)$.

Let us therefore find the solution

$$S(x) = \sum_j c_j M_{2m}(x - j) \tag{7}$$

of the minimum problem

$$J(S) = \varepsilon \int_{-\infty}^{\infty} (S^{(m)})^2 dx + \sum_{-\infty}^{\infty} (S(\nu) - y_\nu)^2 = \text{minimum}. \tag{8}$$

Here we need another

LEMMA 2: *If* (7) *satisfies* $S(x) \in L_2(R)$, *hence also* $(c_j) \in \ell_2$, *then*

$$\int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx = \sum_{j,\nu} \gamma_{j-\nu} c_j c_\nu , \tag{9}$$

where

$$(\gamma_\nu) \rightarrow (2 \sin \tfrac{u}{2})^2 \phi_{2m}(u). \tag{10}$$

PROOF: From (7) we find that

$$\int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx = \int \{ \sum_{j,k} c_j c_k M_{2m}^{(m)}(x - j) M_{2m}^{(m)}(x - k) \} dx \tag{11}$$

$$= \sum_{j,k} \gamma_{j-k} c_j c_k ,$$

where (γ_r) is the even sequence defined by

$$\gamma_r = \int_{-\infty}^{\infty} M^{(m)}(x)\, M^{(m)}(x - r)\,dx\,, \tag{12}$$

where, to simplify notations we dropped the subscript $2m$ of $M_{2m}(x)$. Integrations by parts show that

$$\gamma_r = (-1)^{m-1} \int_{-\infty}^{\infty} M'(x) M^{(2m-1)}(x - r)\,dx\,. \tag{13}$$

Observe that $M^{(2m-1)}(x)$ is a step function assuming in consecutive unit intervals the values

$$\ldots, 0, 0, 1, -\binom{2m-1}{1}, \binom{2m-1}{2}, \ldots, -1, 0, 0, \ldots\,. \tag{14}$$

This sequence has the generating function

$$\sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} e^{iju} = (1 - e^{iu})^{2m-1}, \tag{15}$$

except for a shift factor e^{iuk} which we disregard. Now (13) indicates that (γ_r) is the convolution of the sequence (14) with the sequence

$$\int_{\nu}^{\nu+1} M'(x)\,dx = M(\nu + 1) - M(\nu) \to (e^{-iu} - 1)\phi_{2m}(u).$$

However, in (13) the sequence (γ_r) appears as a sum of the form $\Sigma_\nu a_\nu b_{\nu-r}$. If we pass from (a_ν) to the reversed sequence $(a_{-\nu})$, we obtain a genuine convolution $\Sigma_\nu a_{-\nu} b_{\nu-r}$. Let us therefore reverse the first sequence (14). As we obtain the generating function of the reversed sequence by changing u into $-u$ in its original generating function, we find the generating function of (γ_r) to be (up to a shift factor e^{iuk}) the product

$$(-1)^{m-1}(1 - e^{-iu})^{2m-1} \cdot (e^{-iu} - 1)\, \phi_{2m}(u)$$

$$= (-1)^{m}(1 - e^{-iu})^{2m}\, \phi_{2m}(u)$$

$$= e^{-ium}(-1)^{m}(e^{iu/2} - e^{-iu/2})^{2m}\, \phi_{2m}(u)$$

$$= e^{-ium}(2 \sin \tfrac{u}{2})^{2m}\, \phi_{2m}(u).$$

Since (γ_r) is an *even* sequence, its generating function must be even, and therefore

$$(\gamma_r) \to (2 \sin \tfrac{u}{2})^{2m}\, \phi_{2m}(u)\ ,$$

establishing (10).

2. SOLUTION OF THE PROBLEM: From (8), (9), and (7) we find that

$$(1) \qquad J(S) = \varepsilon \sum_{j,\nu} \gamma_{j-\nu} c_j c_\nu + \sum_{\nu} \{ \sum_{j} c_j M(\nu - j) - y_\nu \}^2 .$$

Let us minimize this function of the (c_k). To obtain the normal equations, we differentiate $J(S)$ obtaining

$$\frac{1}{2} \frac{\partial}{\partial c_k} J(S) = \varepsilon \sum_{j} \gamma_{j-k} c_j + \sum_{\nu} \{ \sum_{j} c_j M(\nu - j) - y_\nu \} M(\nu - k) = 0 \quad (k \in \mathbb{Z}).$$

If we sum within the double-sum only with respect to ν, we obtain

$$(2) \qquad \sum_{\nu} M_{2m}(\nu - j) M_{2m}(\nu - k) = \alpha_{j-k}$$

where

$$(\alpha_\nu) \to (\phi_{2m}(u))^2. \tag{3}$$

The normal equations thus become

$$\varepsilon \sum_j \gamma_{j-k} c_j + \sum_j \alpha_{j-k} c_j = \sum y_\nu M_{2m}(\nu - k) \qquad (k \in \mathbb{Z}),$$

or

$$\sum_j (\alpha_{j-k} + \varepsilon \gamma_{j-k}) c_j = \sum_\nu y_\nu M_{2m}(\nu - k) \qquad (k \in \mathbb{Z}). \tag{4}$$

However, by (3) and (1.10) we find

$$(\alpha_\nu + \varepsilon \gamma_\nu) \to (\phi_{2m}(u))^2 + \varepsilon (2 \sin \tfrac{u}{2})^{2m} \phi_{2m}(u). \tag{5}$$

From

$$(M_{2m}(\nu)) \to \phi_{2m}(u),$$

and writing

$$(c_\nu) \to C(u), \quad (y_\nu) \to T(u), \tag{6}$$

we find the normal equations (4) to be equivalent to the relation

$$\{(\phi_{2m}(u))^2 + \varepsilon (2 \sin \tfrac{u}{2})^{2m} \phi_{2m}(u)\} C(u) = T(u) \phi_{2m}(u),$$

whence

$$(c_\nu) \to C(u) = \frac{T(u)}{\phi_{2m}(u) + \varepsilon (2 \sin \frac{u}{2})^{2m}}. \tag{7}$$

This establishes

THEOREM 3: *In terms of the expansion*

$$\frac{1}{\phi_{2m}(u) + \varepsilon(2 \sin \frac{u}{2})^{2m}} = \sum_{\nu=-\infty}^{\infty} w_\nu(\varepsilon) e^{i\nu u}, \tag{8}$$

where the coefficients $w_\nu(\varepsilon) = w_{-\nu}(\varepsilon)$ *decay exponentially, the coefficients* (c_j) *of the solution*

$$S(x) = \sum_j c_j M_{2m}(x - j) \tag{9}$$

of the minimum problem, are

$$c_\nu = \sum_{j=-\infty}^{\infty} y_j w_{\nu-j}(\varepsilon). \tag{10}$$

We call the solution (9) the *cardinal smoothing spline.*

3. *A few properties of the cardinal smoothing spline* $S(x) = S(x;\varepsilon)$.

A. We have assumed above that $\varepsilon > 0$. However, if we set $\varepsilon = 0$ in (2.8), it becomes

$$\frac{1}{\phi_{2m}(u)} = \sum_\nu w_\nu(0)\, e^{i\nu u}, \tag{1}$$

and a comparison with the expansion (3.28) of Part I, shows that $w_\nu(0) = w_\nu$ for all ν: *This shows that* $S(x;0) = S(x)$ *reduces to the interpolating cardinal spline* $S(x)$ *of Theorem* 2.

B. *What is the effect of the smoothing spline* $S(x) = S(x;\varepsilon)$ *on the original data* (y_ν)? This we answer by determining the "smoothed" sequence $(S(\nu))$, to compare it with (y_ν). By (2.9) and (2.10) we find

$$(S(\nu)) = (c_\nu) * (M_{2m}(\nu)),$$

and therefore, by (2.7),

$$(2)\qquad (S(\nu)) \to C(u)\ \phi_{2m}(u) = \frac{T(u)}{1 + \varepsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}} .$$

In terms of the expansion

$$(3)\qquad \frac{1}{1 + \varepsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}} = \sum_{\nu} \sigma_{\nu}(\varepsilon)\ e^{i\nu u}$$

(2) shows that the sequence $(S(\nu;\varepsilon))$ *arises from the data* (y_ν) *by the smoothing formula*

$$(4)\qquad S(\nu;\varepsilon) = \sum_{j} \sigma_{\nu-j}(\varepsilon)\, y_j .$$

Observe that by (2.8) and (3) the coefficients $\sigma_\nu(\varepsilon)$ are expressed in terms of $w_\nu(\varepsilon)$ by $\sigma_\nu(\varepsilon) = \sum_j M_{2m}(\nu - j)\ w_j(\varepsilon)$.

Is (4) a smoothing formula according to our definition of Part I, Section 1 ? That it is one we see if we inspect its characteristic function

$$(5)\qquad K(u;\varepsilon) = \frac{1}{1 + \varepsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}} ,$$

for it is evident that

$$(6)\qquad 0 < K(u;\varepsilon) < K(0;\varepsilon) = 1 \quad \text{for} \quad 0 < u < 2\pi .$$

C. *The smoothing power of the formula* (4) *increases with increasing* ε . In [4, Definition 2, p. 53] we gave good reasons for

the following definition: Of two different smoothing formulae having the characteristic functions $\phi(u)$ and $\tilde{\phi}(u)$, we say that the second has greater smoothing power, provided that

(7) $\quad |\tilde{\phi}(u)| \leqq |\phi(u)|$ for all u, excluding equality for all u.

However, if $0 < \varepsilon < \tilde{\varepsilon}$, it is clear by (5) that

$$0 < K(u;\tilde{\varepsilon}) < K(u;\varepsilon) \quad \text{if} \quad 0 < u < 2\pi ,$$

and the criterion (7) is satisfied.

D. *The degree of exactness of the smoothing formula* (4) *is* $= 2m-1$. This follows from (1.7) of Part I, because (5) shows that we have the expansion in powers of u

(8) $$K(u;\varepsilon) = 1 - \varepsilon u^{2m} + \ldots \quad .$$

E. If we drop our assumption (1.1), and assume only that (y_ν) *is of power growth, then our construction of the smoothing spline* $S(x) = S(x;\varepsilon)$ *by the formulae* (2.8), (2.10), *and* (2.9), *remains applicable*. Of course, its earlier connection with the funtional $J(S)$, of (1.8), no longer holds. In fact we will find that $J(S) = \infty$ for all splines S. Presumably, it is still true that our $S(x;\varepsilon)$ minimizes $J(S)$, provided that (y_ν) satisfies the condition

$$(y_\nu) \in \ell_2^m$$

of Theorem 2. However, this I was not able to establish.

In any case I recommend the cardinal smoothing spline $(S(x;\varepsilon))$, which represents the modification, found more than 30 years later, of may war-time approach to the problem of cardinal smoothing.

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A CHARACTERIZATION OF ECHELON KÖTHE-SCHWARTZ SPACES

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In [1], A. Grothendieck asks if each quasi-barrelled (DF)-space is bornological. We gave an answer to this question in [5] by constructing a class of quasi-barrelled (DF)-spaces which are neither bornological nor barrelled. In this paper, in the context of the Köthe's echelon spaces which are Montel, we characterize the spaces of Schwartz using certain non-bornological barrelled spaces. As a consequence, we prove the existence of non-bornological barrelled (DF)-spaces.

The vector spaces we use here are defined on the field K of the real or complex numbers. If $\langle E,F\rangle$ is a dual pair, we denote by $\mu(E,F)$ the Mackey topology on E. If E is a topological vector space, E' is its topological dual. In the sequel λ will be an echelon space and $\lambda^{\times}$ its α-dual. Let us suppose that the steps defining λ

$$\alpha^{(n)} = (a_1^{(n)}, a_2^{(n)},\ldots,a_p^{(n)},\ldots),\ n=1,2,\ldots$$

are all positive, they form an increasing sequence and, for each index p, there exists and index q such that $a_p^{(q)} \neq 0$. Let ε_n be the sequence such that all its terms vanish except n-th whose value is one. Generally, we follow the terminology of [2] for this kind of spaces. In particular, φ is the space generated by the vectors

ε_n, $n = 1, 2, \ldots$. Here we always consider φ as a subspace of $\lambda^{\mathbf{X}}[\mu(\lambda^{\mathbf{X}}, \lambda)]$.

Let $P = \{I_n : n = 1,2,\ldots\}$ be a partition of the set N of natural numbers, such that I_n is infinite, $n = 1,2,\ldots$. Let F be the filter of all the subsets of N such that, if $F \in F$ the complementary in I_n of $F \cap I_n$ is finite, $n = 1,2,\ldots$. Let $\{F_j : j \in J\}$ be the set of all the filters on N finer than F so that, if $M \in F_j$ for some $j \in J$, then $M \cap I_n \neq \emptyset$, $n = 1,2,\ldots$. It follows immediately that, with the relation of inclusion, this set is inductive ordered. Using Zorn's lemma, let $\mathcal{U}$ be a maximal element.

PROPOSITION 1: *For each* $n \in N$, *the restriction of* $\mathcal{U}$ *to* I_n *is an ultrafilter.*

PROOF: Let A_1 and A_2 be two non-empty subsets of I_n such that $A_1 \cup A_2 = I_n$, and $A_1 \cap A_2 = \emptyset$. Therefore, one of these sets, say A_1 intersects all the elements of $\mathcal{U}$ and then

$$A = A_1 \cup [\cup \{I_p : p \in N,\ p \neq n\}]$$

belongs to $\mathcal{U}$ and $A \cap I_n = A_1$. This completes the proof.

For each $U \subset N$, we denote by $\lambda^{\mathbf{X}}(U)$ the sectional subspace of $\lambda^{\mathbf{X}}[\mu(\lambda^{\mathbf{X}}, \lambda)]$ defined by $\lambda^{\mathbf{X}}(U) = \{\alpha = (a_1, a_2, \ldots, a_n, \ldots) : \alpha \in \lambda^{\mathbf{X}},\ a_n = 0, \forall n \in U\}$. If U_1 and U_2 belong to $\mathcal{U}$ it follows that $U_1 \cap U_2$ belongs to $\mathcal{U}$ and

$$\lambda^{\mathbf{X}}(U_1 \cap U_2) \supset \lambda^{\mathbf{X}}(U_1) \cup \lambda^{\mathbf{X}}(U_2)$$

and, therefore,

$$L = \cup \{\lambda^{\mathbf{X}}(U) : U \in \mathcal{U}\}$$

is a subspace of $\lambda^{\mathbf{x}}$ containing φ. Let us suppose that the topology of L is the one induced by $\mu(\lambda^{\mathbf{x}},\lambda)$.

PROPOSITION 2: *If λ is a Montel space and T is a barrel in L, it absorbs the bounded subsets of $\varphi \cap \lambda^{\mathbf{x}}(N \sim I_n)$, for each $n \in N$.*

PROOF: Let us suppose that there exists in $\varphi \cap \lambda^{\mathbf{x}}(N \sim I_n)$ a bounded normal subset B which is not absorbed by T. We now inductively construct a sequence (y_q) in B in the following way: Let us suppose that we have already obtained the elements $y_1, y_2, \ldots, y_q$ in B such that

$$y_p \notin pT,\ y_p = \sum_{r \in N(p)} a_r \varepsilon_r,\ a_r \in K,\ p = 1,2,\ldots,q\ ,$$

where $N(1), N(2), \ldots, N(q)$ are finite subsets of I_n, mutually disjoints, such that $N(1)$ contains the first element of I_n, and $N(I_p)$, $p > 1$, contains the first element of I_n which does not lie in $M(p-1)$, being $M(r) = N(1) \cup N(2) \cup \ldots \cup N(r)$.

The space $\varphi \cap \lambda^{\mathbf{x}}(N \sim I_n)$ is the topological direct sum of

$$E_1 = \varphi \cap \lambda^{\mathbf{x}}(N \sim M(q)) \quad \text{and} \quad E_2 = \varphi \cap \lambda^{\mathbf{x}}(N \sim (I_n \sim M(q)))\ .$$

Let B_1 be the projection of B onto E_1 according to E_2, and let B_2 be the projection of B onto E_2 according to E_1. Since B is a normal set it follows that $B_1 \cup B_2 \subset B$. Moreover, $B_1 + B_2 \supset B$. B_1 is a bounded subset of the finite-dimensional space E_1, hence T absorbs B_1. Since B is not absorbed by T, neither B_2. Therefore, we can find an element $y_{q+1} \in B_2 \subset B$ such that

$$y_{q+1} \notin (q+1)T.$$

The element y_{q+1} can be written in the form

$$y_{q+1} = \sum_{r \in N(q+1)} a_r \varepsilon_r , \quad a_r \in K ,$$

where $N(q+1)$ is a finite subset of I_n, disjoint from each set $N(1), N(2),\ldots,N(q)$ and that it contains the first element of I_n which is not contained in $M(q)$. The sets of the sequence $(N(q))$ define a partition of I_n. Let

$$P_1 = \cup \{N(2q-1) : q = 1,2,\ldots\}$$

$$P_2 = \cup \{N(2q) : q = 1,2,\ldots\} .$$

Since the restriction of $\mathcal{U}$ on I_n is an ultrafilter, there exists an $U \in \mathcal{U}$ such that $U \cap I_n$ coincides with P_1 or P_2, $U \cap I_n = P_1$, say. Therefore, $y_{2q} \in \lambda^{\times}(U)$, $q = 1,2,\ldots$. The space $\lambda^{\times}(U)$ is barrelled, because is a sectional subspace of $\lambda^{\times}[\mu(\lambda^{\times},\lambda)]$; hence T absorbs the set $\{y_2, y_4, \ldots, y_{2q}\ldots\}$ and it contradicts

$$y_{2q} \notin (2q)T, \quad q = 1,2,\ldots .$$

Since the normal hull of every bounded subset of $\varphi \cap \lambda^{\times}(N \sim I_n)$ is bounded, it follows that T absorbs every bounded subset of $\varphi \cap \lambda^{\times}(N \sim I_n)$.

PROPOSITION 3: *If λ is a Montel space and T is a barrel in L, it absorbs every bounded subset of φ.*

PROOF: Let us suppose that there is in φ a bounded and normal subset B not absorbed by T. Let us choose in B an element y_1 which is not in T. By a recurrent process, let us define a sequence (y_q)

in B. If $y_1, y_2, \ldots, y_q$ are already defined, such that

$$y_p \notin pT,\ y_p = \sum_{r \in N(p)} a_r \varepsilon_r,\ a_r \in K,\ p = 1,2,\ldots,q\ ,$$

where $N(1), N(2), \ldots N(q)$ are finite subsets of N so that

$$N(1) \subset I_1 \cup I_2 \cup \ldots \cup I_{n_1} = H_1\ ,$$

$$N(2) \subset I_{n_1+1} \cup I_{n_1+2} \cup \ldots \cup I_{n_2} = H_2\ ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$N(q) \subset I_{n_{q-1}+1} \cup I_{n_{q-1}+2} \cup \ldots \cup I_{n_q} = H_q\ .$$

Let $K_q = \cup \{H_p : p = 1,2,\ldots,q\}$. Then φ is the topological direct sum of

$$E_1 = \varphi \cap \lambda^{x}(N \sim K_q) \quad \text{and} \quad E_2 = \varphi \cap \lambda^{x}(K_q)$$

Let B_1 be the projection of B onto E_1 according to E_2 and B_2 the projection of B onto E_2 according to E_1. Since B is normal, $B_1 \cup B_2 \subset B$. Moreover, $B_1 + B_2 \supset B$.

From the previous proposition, it follows easily that T absorbs B_1. Since B is not absorbed by T, neither B_2, hence we can find an element $y_{q+1} \in B_2 \subset B$ such that

$$y_{q+1} \notin (q+1)T.$$

Then

$$y_{q+1} = \sum_{r \in N(q+1)} a_r \varepsilon_r,\ a_r \in K\ ,$$

being $N(q+1)$ a finite subset of N which not intersects K_q. Since

$$P = \{I_n : n = 1,2,\ldots\}$$

is a partition of N and $N(q+1)$ is finite, we can find a natural number n_{q+1}, larger than n_q, such that

$$N(q+1) \subset I_{n_q+1} \cup I_{n_q+2} \cup \ldots \cup I_{n_{q+1}} = H_{q+1}$$

Let $M = \cup \{N(q) : q = 1,2,\ldots\}$. Then $N \sim M$ intersects each I_n in a set with finite complement in I_n. Hence $N \sim M \in F \subset U$.

Since $\lambda^x(N \sim M)$ is barrelled and $y_q \in \lambda^x(N \sim M)$, $q = 1,2,\ldots$, it follows that T absorbs the set $\{y_1, y_2, \ldots, y_q, \ldots\}$ and it contradicts the fact of $y_q \notin qT$, $q = 1,2,\ldots$. Clearly, T absorbs each bounded subset of φ.

q.e.d.

Let $\lambda[\mu(\lambda,\lambda^x)]$ a Montel space which is not Schwartz. Therefore, there exists a positive integer k such that, if M is the ordered subset of all the natural numbers n so that $a_n^{(k)} \neq 0$, the sequence

$$\left(\frac{a_n^{(k)}}{a_n^{(p)}} \right)_{n \in M}$$

does not converges towards zero, $p = k+1, k+2, \ldots$, [2,p.422]. Let us define an increasing sequence in M

$$q_1 < q_2 < \ldots < q_i < \ldots$$

such that

$$\lim_{i \to \infty} \frac{a_{q_i}^{(k)}}{a_{q_i}^{(k+1)}} = c_1 \neq 0.$$

Since $\lambda[\mu(\lambda,\lambda^{x})]$ is a Montel space we can select a subsequence (m_i) of (q_i) so that, for a particular number $k_1 > k + 1$

$$\lim_{i\to\infty} \frac{a_{m_i}^{(k)}}{a_{m_i}^{(k_1)}} = 0 \qquad [2, \text{ p. } 421].$$

Let I_1 be the set $\{m_1, m_2, \ldots\}$. Obviously, $M \sim I_1$ is an infinite set. Let us suppose that we have constructed subsets of N, $I_1, I_2, \ldots, I_q$, so that $I_p \cap M$ is an infinite set and

$$I_p \cap I_r = \emptyset, \quad p \neq r, \quad p, r = 1,2,\ldots,q .$$

If $I_p = \{r_1, r_2, \ldots, r_i, \ldots\}$, suppose also that there are two natural numbers $k_p > k + p$, i_p, so that

$$\lim_{i\to\infty} \frac{a_{r_i}^{(k)}}{a_{r_i}^{(k+p)}} = c_p \neq 0, \quad \lim_{i\to\infty} \frac{a_{r_i}^{(k)}}{a_{r_i}^{(k_p)}} = 0, \quad i > i_p .$$

Let $H_q = \cup \{I_p : p = 1,2,\ldots,q\}$. If we arrange the terms of $H_q \cap M$ as a sequence

$$n_1 < n_2 < \ldots < n_i < \ldots$$

we obtain, for $u > k_p$, $p = 1,2,\ldots,q$, that

$$\lim_{i\to\infty} \frac{a_{n_i}^{(k)}}{a_{n_i}^{(u)}} = 0, \quad i > i_p, \quad (1).$$

From (1) and the condition of $\lambda[\mu(\lambda,\lambda^{x})]$ not being a Schwartz space, it follows that

$$M \sim H_q = \{s_1, s_2, \ldots, s_i, \ldots\}$$

is an infinite set and the sequence

$$\left(\frac{a_{s_i}^{(k)}}{a_{s_i}^{(k+q+1)}} \right)$$

does not converges to zero. Therefore, we can select a subsequence (t_i) of (s_i) and a positive integer $k_{q+1} > k + q + 1$ so that

$$\lim_{i\to\infty} \frac{a_{t_i}^{(k)}}{a_{t_i}^{(k+q+1)}} = c_{q+1} \neq 0, \quad \lim_{i\to\infty} \frac{a_{t_i}^{(k)}}{a_{t_i}^{(k_{q+1})}} = 0 .$$

Let I_{q+1} be the set $\{t_1, t_2, \ldots, t_i, \ldots\}$ together with the first element of N which does not lie in H_q. In this way we obtain a partition $P = \{I_n : n = 1,2,\ldots\}$ of N such that I_n is infinite, whose properties will be used in the sequel.

THEOREM 1: *If the Montel space* $\lambda[\mu(\lambda,\lambda^x)]$ *is not Schwartz, there is in* $\lambda^x[\mu(\lambda^x,\lambda)]$ *a dense subspace* G *which is barrelled and non bornological.*

PROOF: Using the number k and the partition P obtained above, we construct the space L as we did at the beginning of this paper and the subspace G of $\lambda^x[\mu(\lambda^x,\lambda)]$ which is the linear hull of L and the vector $\alpha^{(k)}$. We will prove that G is barrelled and non bornological. Let T be a barrel in G. By Proposition 3, T absorbs every bounded subset of φ. On the other hand, φ is a bornological space [3, p. 324], hence $T \cap \varphi$ is a neighbourhood of the origin in φ.

Since φ is dense in G, the closure of $T \cap \varphi$ in G is contained in T and it is a neighbourhood of the origin in G. Therefore, G is barrelled. Since L is a subspace of G whose codimension in G is not larger than one, to see that G is not bornological, according to a result of Mackey [4], it is suffices to prove that $\alpha^{(k)}$ is not the limit in the sense of Mackey, of a sequence lying in L.

Let us suppose that $(\beta^{(n)})$ is a sequence of L which congerges to $\alpha^{(k)}$ in the sense of Mackey. Consequently, there is a natural number p such that $(\beta^{(n)})$ converges to $\alpha^{(k)}$ in $\lambda_\alpha^x(k+p)$ for the topology of the norm $\|\cdot\|$ deduced from the unit ball

$$\{\beta = (b_1, b_2, \ldots, b_q, \ldots) : |b_q| \leq a_q^{(k+p)},\ q = 1,2,\ldots\}.$$

We can find a positive integer r so that

$$\frac{a_m^{(k)}}{a_m^{(k+p)}} > \frac{1}{2}\, c_p\,,\quad m > r,\ m \in I_p\,.$$

Given $\beta^{(n)} = (b_1^{(n)}, b_2^{(n)}, \ldots, b_q^{(n)}, \ldots)$, we have that

$$\|\alpha^{(k)} - \beta^{(n)}\| = \sup\left\{\frac{|a_q^{(k)} - b_q^{(n)}|}{a_q^{(k+p)}} : q \in N,\ a_q^{(k+p)} \neq 0\right\}.$$

We can find $U \in \mathcal{U}$ such that $\beta^{(n)} \in \lambda^x(U)$. Since $U \cap I_p$ is not finite, we can obtain a positive integer s in I_p, larger than r, such that $b_s^{(n)} = 0$. Then

$$\|\alpha^{(k)} - \beta^{(n)}\| \geq \frac{|a_s^{(k)} - b_s^{(n)}|}{a_s^{(k+p)}} = \frac{a_s^{(k)}}{a_s^{(k+p)}} > \frac{1}{2}\, c_p\,,$$

and we obtain a contradiction. Therefore, G is not bornological.

PROPOSITION 4: *Let* E *be a Fréchet-Schwartz space. If* F *is a barrelled subspace of* $E'[\mu(E',E)]$, *then* F *is bornological.*

PROOF: Let (A_n) be an increasing fundamental sequence of compact subsets of $E'[\mu(E',E)]$. Let us suppose first that F is dense in $E'[\mu(E',E)]$. Let B_n be the closure of $A_n \cap F$ in $E'[\mu(E',E)]$. Let F_n be the closure of the normed space $F_{A_n \cap F}$ in the Banach space E'_{A_n} with the topology induced by the one associated to E'_{A_n}. Since F is a (DF)-space, (B_n) is a fundamental sequence of compact sets in $E'[\mu(E',E)]$, [2, p. 402]. Given a positive integer p there are two positive integers q and r such that $A_p \subset B_q$ and B_q is a compact set in E'_{A_r}. Then A_p is a compact set in F_r and therefore we can afirm that $E'[\mu(E',E)]$ is the inductive limit of the sequence (F_n) of Banach spaces. Let u be a bounded linear form on F. Since $F \cap F_n$ is dense in F_n, the restriction u_n of u to $F \cap F_n$ can be extended to a continuous linear form v_n in F_n. Evidently, there exists a linear form v on E' which coincides with v_n in F_n, $n = 1,2,\ldots$. Then v is continuous on $E'[\mu(E',E)]$ and, consequently, its restriction u to F is continuous. Hence, F is bornological. The general case is proved obtaining the closure $\bar{F}$ of F in $E'[\mu(E',E)]$ and proving that $\bar{F}$ is the dual the Mackey of the Fréchet-Schwartz space $E/F^{\perp}$, with $F^{\perp}$ the orthogonal subspace of E to F.

THEOREM 2: *A Montel space* $\lambda[\mu(\lambda,\lambda^{x})]$ *is Schwartz if, and only if, each barrelled subspace of* $\lambda^{x}[\mu(\lambda^{x},\lambda)]$ *is bornological.*

PROOF: It follows from Theorem 1 and Proposition 4.

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THE RATIONAL APPROXIMATION OF REAL FUNCTIONS

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I. INTRODUCTION

This paper is closely related to the classical theory of best uniform approximation of continuous functions by quotients of polynomials. That is, let f be a continuous real function on $[0,1]$ and let

$$R_n^m = \{p/q : p \in P_m,\ q \in P_n,\ q \neq 0 \text{ on } [0,1],\ \text{and } p/q \text{ irreducible}\} \tag{1.1}$$

where P_n denotes the real polynomials of degree less than or equal n.

It is classically known that there is an $r \in R_n^m$ which is a best approximation to f [see for example Walsh, 1935]. That is,

$$\|f - r\| = \operatorname{dist}(f, R_n^m). \tag{1.2}$$

Furthermore the approximation is characterized by $f - \frac{p}{q}$ having the zero function as a best approximation from the linear space

$$qP_m + pP_n = P_N. \tag{1.3}$$

where $N = \max\{\partial q + m, \partial p + n\}$. Hence r is a best approximation to

f if and only if $f - r$ has an extremal alternation of length $N + 2$. (Achieser [1930]). It follows that best approximations are always unique. In this setting however the best approximation operator is not generally continuous. In fact, it is continuous at f if and only if f has a normal point p/q as a best approximation, that is, if either $\partial p = m$ or $\partial q = n$ (Werner [1965]). The complex rational functions, $R_n^m(\mathbb{C})$, are defined similarly with P_m and P_n replaced with $P_m(\mathbb{C})$ and $P_n(\mathbb{C})$, the polynomials with complex coefficients.

A complex function defined on [0,1] still has exactly one best approximation from $P_n(\mathbb{C})$. However approximation from $R_n^m(\mathbb{C})$ is not as well understood. It is still true that a best approximations exists in $R_n^m(\mathbb{C})$, but Walsh [1931] has constructed an example to show that if the domain of the functions is a particular "crescent moon" shaped region of the complex plane (instead of the interval [0,1] as in our setting), then there is a complex function with more than one best approximation in $R_n^m(\mathbb{C})$. More recently E. Saff and R. Varga made the surprising observation that in fact $(x - 1/2)^2$ has nonunique best approximations from $R_1^1(\mathbb{C})$ [1976].

If the function being approximated is the real function f, then its best approximation in $P_n(\mathbb{C})$ is also real, and the problem is reduced to the theory of approximation from P_n. But for the rational functions the Saff-Varga example shows that the analogous reduction is not valid. It appears natural to consider approximations to f from $\operatorname{Re} R_n^m(\mathbb{C})$ - the real parts of $R_n^m(\mathbb{C})$ functions.

This paper is an exposition of part of such a study. The detailed paper will appear elsewhere. As it turns out the theory of approximation from $\operatorname{Re} R_n^m$ is, an intriguing mix of the regularity of approximation from R_n^m with the pathology of that from $R_n^m(\mathbb{C})$. There are also some applications to approximation from $R_n^m(\mathbb{C})$.

II. EXISTENCE OF BEST APPROXIMATIONS

In the classical settings, the existence of a best approximation is easy to establish. The idea is that a minimizing sequence r_i for f (i.e. $\|r_i - f\| \to \text{dist}\,(f, R_m^n)$) has a subsequence with converging numerators and denominators. Cancelling common zeros of the limit numerator and denominator produces a best approximation to f. Here the problem is that the limit function may not be in $\text{Re}\, R_n^m(\mathbb{C})$.
For example, for each $\varepsilon > 0$

$$r_\varepsilon = \frac{(0)(1) + (\varepsilon x^n)(\frac{1}{\varepsilon} x^m)}{1^2 + (\varepsilon x^n)^2} \in \text{Re}\, R_n^m(\mathbb{C}). \tag{2.1}$$

So,

$$x^{m+n} \in c\ell\,\text{Re}\, R_n^m(\mathbb{C}). \tag{2.2}$$

But, one easily shows that x^{m+n} is not of the form

$$\frac{pq + st}{q^2 + s^2}, \quad p, t \in P_m, s, q \in P_n. \tag{2.3}$$

Hence the theory is actually about approximation from the closure of $\text{Re}\, R_n^m(\mathbb{C})$ (denoted herefter by $\mathcal{R}_n^m$). The first problem, then, is to characterize $\mathcal{R}_n^m$.

Clearly, $\mathcal{R}_n^m \subseteq \mathcal{Q}_n^{m+n}$, where

$$\mathcal{Q}_\mu^\nu = \{\frac{p}{q} : p \in P_\nu, q \in P_{2\mu}; q \geq 0 \text{ on } \mathbb{R}, \text{ and } q > 0 \text{ on } [0,1]\}. \tag{2.4}$$

In fact:

1. PROPOSITION: *If* $n \leq m + 1$, *then* $\mathcal{R}_n^m = \mathcal{Q}_n^{m+n}$.

2. PROPOSITION: *For all* m, *and* n, $\mathcal{R}_n^m$ *admits best approximations to all* $f \in C[0,1]$.

III. CHARACTERIZATION AND UNICITY OF APPROXIMATIONS FROM Q_n^m

As in the characterization of approximation from R_n^m, the idea is to change the problem to that of approximation from a more computable set. We will first state a special case so that the general case will appear less absurd. Suppose that $\frac{a}{b} \in Q_n^m$, that a and b have no common factors and that the degrees of a and b are such that $2n + \partial a \leq \partial b + m$.

Let

$$(3.1) \quad H(a,b) = \{h \in P_M : \operatorname{sgn} h(x) = -\operatorname{sgn} a(x) \text{ for } x \in Z(b)\}$$

where $M = \partial b + m$ and $Z(f)$ denotes the zero set of a function f.

3. PROPOSITION: *$\frac{a}{b}$ is a best approximation to f if and only if zero is a best approximation to $f - \frac{a}{b}$ from $H(a,b)$.*

Now in general suppose $\frac{a}{b} \in Q_n^m$. From the definition of Q_n^m we may assume that a and b have no common quadratic factors (i.e. $b \geq 0$). However it may be possible that a and b have some common real zeros. Let F be the greatest monic common divisor of a and b.

Put

$$(3.2) \qquad a_o = a/F \quad \text{and} \quad b_o = b/F .$$

Now put

$$(3.3) \qquad M = \max\{\partial b_o + m, \partial a_o + 2n\}$$

$$= \dim\{b_o P_m + a_o P_{2n}\} - 1 .$$

For $\frac{a}{b} \in Q_n^m$ we now define

$$(3.4) \qquad Z(a,b) = \begin{cases} Z(b_o) \cap \mathbb{R} & \text{if } 2n + \partial a \leq \partial b + m \\ [Z(b_o) \cap \mathbb{R}] \cup \{\infty\} \cup \{-\infty\} & \text{if } 2n + \partial a > \partial b + m \end{cases}$$

For convience we will write

(3.5) $f(\infty)$ for $\lim_{x\to\infty} f(x)$ and

$f(-\infty)$ for $\lim_{x\to-\infty} f(x)$,

when these limits exists.

Now define:

(3.6) $H(a,b) = \{Fh : h \in P_M : \operatorname{sgn} h(x) = -\operatorname{sgn} a_o(x)$

for $x \in Z(a,b)\}$.

4. COMMENT: With the above notation proposition 3 above is still valid.

Our interest in proposition 3 is that one can compute the number of possible sign changes of members of $H(a,b)$ and use this to derive an extremal alternation type of characterization for approximations from $\mathcal{Q}_n^m$. However the result separates into many cases depending on the number and parity of the points in $Z(a,b) \cap (-\infty, 0]$ and in $Z(a,b) \cap [1, \infty)$. Rather than presenting the complicated statement of the alternation theorem, we will give some of the consequences.

5. COROLLARY: *Best approximations from* $\mathcal{Q}_n^m$ *are unique.*

6. COROLLARY: *Suppose* a, *and* b *have no common factors,* $m + \partial b \geq 2n + \partial a$ *and* $Z(b) \cap \mathbb{R} = \phi$. *Then* $\frac{a}{b}$ *is a best approximation to* $f \in C[0,1]$ *from* $\mathcal{Q}_n^m$ *if and only if* $f - \frac{a}{b}$ *has an extremal alternation of length* $2 + \max\{m + \partial b, 2n + \partial a\}$.

7. COROLLARY: *A constant function is a best approximation, to a*

continuous function, from Q_n^m *if and only if the error function has an extremal alternation of length* $2 + \max\{m, 2n\}$.

8. COROLLARY: *If* $n \geq 1$ *there is a continuous function* f *and an* $r \in Q_n^m$ *such that*

(i) r *is a best approximation of* f *but*

(ii) $-r$ *is not a best approximation to* $f - 2r$.

IV. APPROXIMATION FROM $R_n^m(\mathbb{C})$:

In some special cases $\operatorname{dist}(f, R_n^m) = \operatorname{dist}(f, R_n^m(\mathbb{C}))$. Although this phenomenon only occurs in a restricted setting, there are some applications to the theory for approximation from $R_n^m(\mathbb{C})$. For example we can easily produce real functions which have nonunique best approximations from $R_n^m(\mathbb{C})$.

9. THEOREM: *Let* $f \in C[0,1]$ *and* $\frac{a}{b} \in R_n^m$. *If* $\frac{a}{b}$ *is a best approximation from* $R_n^m(\mathbb{C})$ *then*

$$\| f - \frac{a}{b} \| \geq \operatorname{dist}(f, R_n^m)$$

where $N = \min\{n - \partial b, m - \partial a\}$.

10. COROLLARY: *Let* $m \geq n$, $p \in P_{m-n}$, *and* $f \in C[0,1]$; *the following are equivalent:*

(i) p *is a best approximation to* f *from* $R_n^m(\mathbb{C})$,

(ii) p *is a best approximation to* f *from* R_n^m.

11. COROLLARY: *Let* $m \geq n$ *and* $f \in C[0,1]$. A *constant function* α *is a best approximation from* $R_n^m(\mathbb{C})$ *if and only if* $f - \alpha$ *has an*

extremal alternation of length $m + n + 2$.

12. EXAMPLE: (Saff - Varga) For every $m \geq n \geq 1$ there are continuous real functions which have nonunique best approximations from $R_n^m(\mathbb{C})$.

13. COROLLARY: (Saff - Varga) *Let* $n \leq m+1$. *Let* $\frac{a}{b} \in R_n^m(Z(a) \cap Z(b) = \phi)$ *be the best approximation to* f *from* $R_n^m(\mathbb{C})$. *Then* $f - \frac{a}{b}$ *must have an extremal alternation of length at least*

$$2 + m + \min\{n - \partial b, m - \partial a\}.$$

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FUNDAMENTAL SEMINORMS

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1. INTRODUCTION

Here we will consider a general problem of polynomial approximation in euclidean n-dimensional space. The subject of polynomial approximation was iniciated in 1885 with the first version of the Weierstrass theorem for uniform approximation on compact sets of euclidean space. The non-uniform approximation problem on the whole space was iniciated with the Bernstein paper of 1924 [2] and continued to be developed in the so called Bernstein problem. Classically this problem has been studied from the point of view of continuous and integrable functions in the natural context of weighted spaces. See, for instance [1], [4], [7], [9] and [12] for related developments and additional references. More recently, Bernstein's problem has also been considered in the context of weighted spaces of differentiable functions and distributions. See, for instance [10] and [13].

In this approach we use the unifying notion of fundamental seminorm in considering a polynomial approximation problem which contains all the above mentioned cases of Bernstein's problem. Further, this approach puts in focus the seminorm point of view in approximation

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theory which has been undertaken for instance in [3].

The main results are a characterization of fundamental seminorms on the real line (Theorem 1), a quasi-analytic criterion for such seminorms (Theorem 2) and a tensor product criterion for fundamental seminorms in the general case (Theorem 3).

We finish by listing some interesting open problems, some of them unsolved even in the classical case.

2. PRELIMINARIES

In the following, n, m and k will denote elements in $\mathbb{N}^*$, $\mathbb{N} \cup \{\infty\}$ and $\mathbb{N}^n$ respectively. We put $|k| = k_1 + \ldots + k_n$ and if $t \in \mathbb{R}^n$, $t^k = t_1^{k_1} \cdot \ldots \cdot t_n^{k_n}$.

$C^m(\mathbb{R}^n)$ is the algebra of all complex valued m-times continuously differentiable functions on $\mathbb{R}^n$. If $f \in C^{|k|}(\mathbb{R}^n)$, then $\partial^k f$ means $\dfrac{\partial^{|k|} f}{(\partial x_1)^{k_1} \ldots (\partial x_n)^{k_n}}$. We will consider also the following subalgebras of $C^m(\mathbb{R}^n)$:

$$P(\mathbb{R}^n) = \{p \in C^m(\mathbb{R}^n) \mid p \text{ is a polynomial}\}$$

$$C_c^m(\mathbb{R}^n) = \{f \in C^m(\mathbb{R}^n) \mid \text{support of } f \text{ is compact}\}$$

$$C_o^m(\mathbb{R}^n) = \{f \in C^m(\mathbb{R}^n) \mid \partial^k f \text{ vanishes at infinity, for all } k \in \mathbb{N}^n\}$$

For a bounded complex function f on $\mathbb{R}^n$ we let $\|f\|$ denote the number $\sup\{|f(x)|, x \in \mathbb{R}^n\}$. Also for $m \in \mathbb{N}$ we define the norm $\| \ \|_m$ on $C_o^\infty(\mathbb{R}^n)$ by

$$\|f\|_m = \sum_{|k| \leq m} \|\partial^k f\| .$$

The topology defined by the family of norms $\| \ \|_m$, $m \in \mathbb{N}$, will be denoted by τ. Unless explicitly stated, this is the topology to be

considered on $C_o^\infty(\mathbb{R}^n)$.

REMARK 1: The functions $t \to 1/(1 + t^2)$ and $t \to t/(1 + t^2)$ belong to $C_o^\infty(\mathbb{R})$. Hence for all $p \in P(\mathbb{R}^n)$ there exists $m_o \in \mathbb{N}$ such that $p/(1 + x_1^2)^m \ldots (1 + x_n^2)^m \in C_o^\infty(\mathbb{R}^n)$ for any $m \geq m_o$. From this follows that the set of all products gp, $g \in C_o^\infty(\mathbb{R}^n)$, $p \in P(\mathbb{R}^n)$ contains the sum of any two such products and also any polynomial.

DEFINITION 1: The *Bernstein space on* $\mathbb{R}^n$, denoted by B_n or simply B when $n = 1$, is the complex vector space of all products gp, $g \in C_o^\infty(\mathbb{R}^n)$, $p \in P(\mathbb{R}^n)$.

Let α be a seminorm on B_n, $A \subset B_n$. Then $\bar{A}^\alpha$ will denote the closure of A in the seminormed space $(B_n, \alpha) \equiv B_{n,\alpha}$.

DEFINITION 2: A seminorm α on B_n is *polynomially compatible* if the module operations

$$(g,f) \in C_o^\infty(\mathbb{R}^n) \times B_{n,\alpha} \to gf \in B_{n,\alpha}$$

$$f \in B_{n,\alpha} \to \bar{f} \in B_{n,\alpha}$$

are continuous. $SPC(\mathbb{R}^n)$ will denote the (directed) set of all polynomially compatible seminorms on B_n.

EXAMPLE 1: Let α be an *increasing seminorm* on B_n that is $\alpha(f) \leq \alpha(g)$ when $|f| \leq |g|$. Then $\alpha \in SPC(\mathbb{R})$. In fact, let $g \in C_o^\infty(\mathbb{R}^n)$, $f \in B_n$. Since $|gf| \leq \|g\| |f|$ then $\alpha(gf) \leq \|g\| \alpha(f)$. Also $\alpha(\bar{f}) = \alpha(f)$ since $|\bar{f}| = |f|$.

It is clear that finite positive linear combinations of increasing seminorms are also increasing seminorms.

EXAMPLE 2: Let $m \in \mathbb{N}^*$, α_k, $|k| \leq m$, be a family of increasing

seminorms on B_n such that $\alpha_{k'} \leq$ constant. α_k when $k \leq k'$, that is $k_i \leq k_i'$, $i = 1,\ldots,n$. Let

$$\alpha(f) = \sum_{|k| \leq m} \alpha_k(\partial^k f), \quad f \in B_n .$$

Then $\alpha \in SPC(\mathbb{R}^n)$. In fact, let $g \in C_o^\infty(\mathbb{R}^n)$, $f \in B_n$. Using Leibnitz's formula, the fact that α_k is increasing, and the condition on the family, it follows that

$$\alpha_k(\partial^k(gf)) \leq \sum_{k' \leq k} \binom{k}{k'} \|g\|_m \alpha_k(\partial^{k-k'} f) \leq \text{constant} \cdot \|g\|_m \alpha_{k-k'}(\partial^{k-k'} f).$$

Hence $\alpha(gf) \leq$ constant $\cdot \|g\|_m \alpha(f)$. Also, it is clear that $\alpha(\bar{f}) = \alpha(f)$.

REMARK 2: There exist seminorms $\alpha \in SPC(\mathbb{R}^n)$ which are not of the types described in Examples 1 or 2. For instance, this is the case for the seminorm defined by

$$\alpha(f) = |f(0)| + \sup\{|f'(x)|, x \in [0,1]\}, \quad f \in B.$$

Then for all $g \in C_o^\infty(\mathbb{R})$ and $f \in B$ we have

$$\alpha(gf) \leq \|g\|_1 \alpha(f), \quad \alpha(\bar{f}) = \alpha(f).$$

PROPOSITION 1: *Let* $\alpha \in SPC(\mathbb{R}^n)$. *Then* $C_c^\infty(\mathbb{R}^n)$ *is dense in* $B_{n,\alpha}$.

PROOF: Let $g_o \in C_o^\infty(\mathbb{R}^n)$, $p_o \in P(\mathbb{R}^n)$ be given. Then the mapping $g \in C_o^\infty(\mathbb{R}^n) \to gp_o \in B_{n,\alpha}$ is continuous. For $m = 1,2,\ldots$ let $\theta_m \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \theta_m \leq 1$, $\theta_m(x) = 1$ when $\|x\| \leq m$ and $\|\partial^k \theta_m\| \leq \frac{1}{m}$, $1 \leq |k| \leq m$. Then $\theta_m g_o \to g_o$ in $C_o^\infty(\mathbb{R}^n)$. Hence $\theta_m g_o p_o \in C_c^\infty(\mathbb{R}^n)$ for all m and $\theta_m g_o p_o \to g_o p_o$ in $B_{n,\alpha}$.

DEFINITION 3: $\alpha \in SPC(\mathbb{R}^n)$ is *fundamental* when $P(\mathbb{R}^n)$ is dense in

$B_{n,\alpha}$. We say also that α is a *Bernstein seminorm on* $\mathbb{R}^n$. *Bernstein's n-dimensional problem* consists in describing Bernstein seminorms on $\mathbb{R}^n$.

REMARK 3: All the cases of Bernstein approximation problem mentioned at the Instroduction consist in asking for necessary and sufficient conditions in order that some convenient seminorm $\alpha \in SPC(\mathbb{R}^n)$ be fundamental. Here is a useful result.

PROPOSITION 2: *Let* (E,β) *be a complex seminormed space of functions or distributions on* $\mathbb{R}^n$ *such that* $B_n \subset E$ *and* $\alpha = \beta \mid B_n \in SPC(\mathbb{R}^n)$. *Assume that there exists* $m \in \mathbb{N} \cup \{\infty\}$ *such that* $C_c^m(\mathbb{R}^n)$ *is dense in* E *and the induced topology on* $C_c^m(\mathbb{R}^n)$ *is weaker than the inductive limit topology.*

Then $P(\mathbb{R}^n)$ *is dense in* E *if and only if* α *is fundamental.*

PROOF: Necessity is obvious. Conversely, since $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c^m(\mathbb{R}^n)$ in the inductive limit topology [11], then it is also dense in E. From this it follows that $P(\mathbb{R}^n)$ is dense in E, since $\beta \mid B_n$ is fundamental.

EXAMPLE 3: Let u be m upper-semicontinuous nonnegative function on $\mathbb{R}^n$ such that $u(t)t^k$ vanishes at in finity for all $k \in \mathbb{N}^n$. (We say that u is a *weight on* $\mathbb{R}^n$). Let $E = Cu_\infty(\mathbb{R}^n)$ be the vector space of all complex continuous functions f on $\mathbb{R}^n$ such that uf vanishes at infinity, seminormed by $\beta(f) = \|uf\|$. Then $\beta \mid B_n$ is an increasing seminorm. For $m = 1,2,\ldots$ let $\theta_m \in C_c(\mathbb{R}^n)$ be such that $0 \leq \theta_m \leq 1$, $\theta_m(x) = 1$ if $\|x\| \leq m$. Then for any $f \in Cu_\infty(\mathbb{R}^n)$ $\beta(f - \theta_m f) \to 0$ when $m \to \infty$. Also $\beta(f) \leq \|u\| \|f\|$ for all $f \in C_c(\mathbb{R}^n)$. Thus the conditions of Proposition 2 are satisfied. When $P(\mathbb{R}^n)$ is dense in $Cu_\infty(\mathbb{R}^n)$, we say that u is a *fundamental weight.*

EXAMPLE 4: Let μ be a positive Borel measure on $\mathbb{R}^n$ such that t^k is μ-integrable for all $k \in \mathbb{N}^n$, $1 \leq p < +\infty$. Let $E = \mathcal{L}^p(\mu)$, $\beta = \mathcal{L}^p$-seminorm. Then $\beta|B_n$ is increasing. Also $C_c(\mathbb{R}^n)$ is dense in E and $\beta(f) \leq \mu(\mathbb{R}^n)^{1/p} \|f\|$ for all $f \in C_c(\mathbb{R}^n)$. Thus the conditions of Proposition 2 are satisfied.

EXAMPLE 5: Let $m \in \mathbb{N}^*$, u_k, $|k| \leq m$, be a family of weights on $\mathbb{R}^n$ such that $u_k \leq \text{constant} \cdot u_{k'}$ if $k' \leq k$. Let E be the vector space of all $f \in C^m(\mathbb{R}^n)$ such that $u_k \partial^k f$ vanishes at infinity for all k, $|k| \leq m$, $\beta(f) = \sum_{|k| \leq m} \|u_k \partial^k f\|$. Then $\beta|B_n$ is of the type described in Example 2. If θ_m, $m = 1,2,\ldots$ is as in the proof of Proposition 1, then $\beta(f - \theta_m f) \to 0$, when $m \to \infty$, for all $f \in E$. Also

$$\beta(f) \leq \left(\max_{|k| \leq m} \|u_k\| \right) \|f\|_m$$

for all $f \in C_c^m(\mathbb{R}^n)$. Thus the seminormed space (E,β) satisfies the hypothesis of Proposition 2.

EXAMPLE 6: Let m and u_k, $|k| \leq m$, be as in Example 5. If $d\lambda$ denotes Lebesgue measure on $\mathbb{R}^n$, let $d\mu_k = u_k \, d\lambda$ for all $|k| \leq m$. Given p, $1 \leq p < +\infty$, let E be the vector space of all complex distributions f on $\mathbb{R}^n$ such that $\partial^k f \in \mathcal{L}^p(\mu_k)$ for all k, $|k| \leq m$, $\beta(f) = \sum_{|k| \leq m} \left(\int |\partial^k f|^p d\mu_k \right)^{1/p}$. Then $\beta|B_n$ is also of the type described in Example 2. Furthermore $C_c^m(\mathbb{R}^n)$ is dense in E. The proof of this fact is similar to that used in proving density in Sobolev spaces [11]. Also $\beta(f) \leq \left(\max_{|k| \leq m} \mu_k(\mathbb{R}^n)^{1/p} \right) \|f\|_m$ for all $f \in C_c^m(\mathbb{R}^n)$. Once again the hypothesis of Proposition 2 are satisfied.

PROPOSITION 3: *Let α be a fundamental seminorm on $\mathbb{R}^n$.*

1) *If $\beta \in SPC(\mathbb{R}^n)$ is such that $\beta \leq \text{constant} \cdot \alpha$, then β is fundamental.*

2) *Let* $c \in \mathbb{R} \setminus \{0\}$, $x_o \in \mathbb{R}^n$ *and* $\varphi(x) = cx + x_o$, $x \in \mathbb{R}^n$. *If* $\beta(f) = \alpha(f \circ \varphi)$ *for all* $f \in B_n$, *then* β *is fundamental.*

PROOF: 1) is an immediate conseguence of Proposition 1. In the case of 2), observe that $P(\mathbb{R}^n) \circ \varphi = P(\mathbb{R}^n)$, $C_c^\infty(\mathbb{R}^n) \circ \varphi = C_c^\infty(\mathbb{R}^n)$.

3. MAIN RESULTS

In the characterization of dense subalgebras in spaces of differentiable functions the following is a crucial result.

LEMMA 1 (Nachbin's Lemma): *Let* $A \subset C^m(\mathbb{R}^n)$, $m \geq 1$, *be a set of real functions satisfying the following conditions:*

1) *For any* $x, y \in \mathbb{R}^n$ *such that* $x \neq y$, *there is* $g \in A$ *such that* $g(x) \neq g(y)$.

2) *For any* $x \in \mathbb{R}^n$ *there is* $g \in A$ *such that* $g(x) \neq 0$.

3) *For any* $x, v \in \mathbb{R}^n$, $v \neq 0$, *there is* $g \in A$ *such that* $\frac{\partial g}{\partial v}(x) \neq 0$.

Then given any $f \in C^m(\mathbb{R}^n)$ *and* $K \subset \mathbb{R}^n$ *compact there exist* $g_1, \ldots, g_\ell \in A$ *and* $h \in C^m(\mathbb{R}^\ell)$, $h(0) = 0$, *such that*

$$f = h(g_1, \ldots, g_\ell) \quad \text{on} \quad K.$$

PROOF: See [8].

DEFINITION 4: A set $A \subset C^m(\mathbb{R}^n)$, $m \geq 1$, satisfies conditions (N) if 1), 2) and 3) above are true.

LEMMA 2: *Let* $\alpha \in SPC(\mathbb{R}^n)$. *If* $A \subset C_o^\infty(\mathbb{R}^n)$ *is a self-adjoint subalgebra that satisfies conditions* (N), *then* A *is dense in* $B_{n,\alpha}$.

PROOF: From Proposition 1 it is enough to show that $C_c^\infty(\mathbb{R}^n) \subset \bar{A}^\alpha$.

Further, since the topology defined by α on $C_o^\infty(\mathbb{R}^n)$ is weaker than τ, we need only show that the closure of A in the latter topology contains $C_c^\infty(\mathbb{R}^n)$.

If C is a subset of $C_o^\infty(\mathbb{R}^n)$ we will denote by $\bar{C}$ its closure in the topology τ. Assume also that C is a subalgebra. In this case, if $g_1,\ldots,g_\ell \in C$ are real and $h \in C^\infty(\mathbb{R}^\ell)$ is such that $h(0) = 0$, then $h(g_1,\ldots,g_\ell) \in \bar{C}$. In fact, it is clear that $h(g_1,\ldots,g_\ell) \in C_o^\infty(\mathbb{R}^n)$. Since the set $G = \{g_1(x),\ldots,g_\ell(x),\ x \in \mathbb{R}^n\}$ is bounded in $\mathbb{R}^\ell$, using the Weierstrass approximation theorem for differentiable functions, we can approximate h on G by polynomials $p \in P(\mathbb{R}^\ell)$ without constant term, since $h(0) = 0$. Furthermore, $\partial^k g_i$ is bounded for all $k \in \mathbb{N}^n$, $i = 1,\ldots,\ell$. Hence $p(g_1,\ldots,g_\ell)$ approximates $h(g_1,\ldots,g_\ell)$ in the topology τ.

Let $f \in C_c^\infty(\mathbb{R}^n)$ be given. Assume $f \neq 0$ and let H be its support. Let A_1 be the set of real parts of functions in A. Since A is a self-adjoint algebra, then A_1 is a subalgebra of A and also satisfies conditions (N). In particular, for any $x \in H$ there exists $g \in A_1$ such that $g(x) \neq 0$. Choose $h \in C^\infty(\mathbb{R})$ such that $h \geq 0$, $h(g(x)) > 0$ and $h(0) = 0$. From the above remark, it follows that $h \circ g \in \bar{A}_1$. Also $h \circ g$ is positive on a neighbourhood of x. Hence by compactness, there exist $g_1 \in \bar{A}_1$ and $r > 0$ such that $g_1 \geq r$ on H, $g_1(0) = 0$. Let $h_1 \in C^\infty(\mathbb{R})$ be such that $h_1 = 1$ on $[r,+\infty)$, $h_1 = 0$ on a neighbourhood of 0. If $f_1 = h_1 \circ g_1$, then $f_1 = 1$ on H and $f_1 \in \bar{A}_1$ since this is a closed subalgebra. Also, f_1 has compact support, say K. Then from Nachbin's Lemma, there exist $g_1,\ldots,g_\ell \in A_1$, $h \in C^\infty(\mathbb{R}^\ell)$, $h(0) = 0$ such that $f = h(g_1,\ldots,g_\ell)$ on K. Hence $f = f_1 \cdot h(g_1,\ldots,g_\ell)$ on $\mathbb{R}^n$ and from the remark on subalgebras it follows that $f \in \bar{A}$, since $h(g_1,\ldots,g_\ell) \in \bar{A}$. Now the proof is complete.

DEFINITION 5: For $z \in \mathbb{C}\setminus\mathbb{R}$, let g_z be the complex function on $\mathbb{R}$ defined by

$$g_z(x) = \frac{1}{x - z}, \quad x \in \mathbb{R}.$$

It is clear that $g_z \in C_o^\infty(\mathbb{R})$.

LEMMA 3: *Let* $\alpha \in SPC(\mathbb{R})$. *If* $g_z \in \overline{P(\mathbb{R})}^\alpha$ *for some* $z \in \mathbb{C} \setminus \mathbb{R}$, *then* α *is fundamental.*

PROOF: Let $E = \overline{P(\mathbb{R})}^\alpha$. We claim that $g_z^m P(\mathbb{R}) \subset E$ for all $m \in \mathbb{N}$. In fact, for $m = 0$ this is evident. Assume that the proposition is true for some $m \in \mathbb{N}$. Let $p \in P(\mathbb{R})$. Since $q = g_z(p - p(z)) \in P(\mathbb{R})$, then from the assumption it follows that

$$g_z^{m+1}(p - p(z)) = g_z^m q \in E.$$

Now the mapping $f \in B_\alpha \to g_z^m f \in B_\alpha$ is continuous hence

$$g_z^{m+1} \in g_z^m \overline{P(\mathbb{R})}^\alpha \subset \overline{g_z^m P(\mathbb{R})}^\alpha \subset E.$$

Since E is a complex vector space we have

$$g_z^{m+1} p = g_z^{m+1}(p - p(z)) + p(z) g_z^{m+1} \in E.$$

So the claim is proved. Further, the mapping $f \in B_\alpha \to \bar{f} \in B_\alpha$ is continuous, hence E is self-adjoint since $P(\mathbb{R})$ is. So $(\bar{g}_z)^\ell = \overline{g_z^\ell} \in E$ for all $\ell \in \mathbb{N}$; whence

$$g_z^m (\bar{g}_z)^\ell \in g_z^m E \subset \overline{g_z^m P(\mathbb{R})}^\alpha \subset E \quad \text{for all} \quad m, n \in \mathbb{N}.$$

From this it follows that the complex algebra A generated by g_z and $\bar{g}_z$ is contained in E. Also A is self-adjoint and satisfies conditions (N) since $\{g_z\}$ satisfies conditions (N). From Lemma 2 it

follows that $P(\mathbb{R})$ is dense in B_α, that is, α is fundamental.

LEMMA 4: *Let* $\alpha \in SPC(\mathbb{R})$, $z, z' \in \mathbb{C} \setminus \mathbb{R}$. *Then there exists a positive constant* $C_{z,z'}$ *such that*

$$\alpha(g_z p) \leq C_{z,z'} \cdot \alpha(g_{z'}, p) \quad \textit{for all} \quad p \in P(\mathbb{R}).$$

PROOF: Let $p \in P(\mathbb{R})$. Since $g_z p = \frac{g_z}{g_{z'}} \cdot g_{z'} p = (1 + (z - z')g_z) g_{z'} p$ it follows

$$\alpha(g_z p) \leq \alpha(g_{z'}, p) + |z - z'| \alpha(g_z g_{z'}, p).$$

From the definition of α, there exists $C > 0$ and $m \in \mathbb{N}$ such that

$$\alpha(g f) \leq C \| g \|_m \alpha(f), \quad \text{for all} \quad g \in C_o^\infty(\mathbb{R}), \quad f \in B.$$

If $r = |y_{mz}|$, then $\| g_z \|_m = \sum_{k=0}^{m} \frac{k!}{r^{k+1}} = C_z$ and

$$\alpha(g_z p) \leq (1 + |z - z'| \, C \, C_z) \, \alpha(g_{z'}, p).$$

To finish, it is enough to observe that the number $C_{z,z'} = 1 + |z - z'| C C_z$ does not depend on p.

THEOREM 1: *Let* $\alpha \in SPC(\mathbb{R})$. *If there exists* $z \in \mathbb{C} \setminus \mathbb{R}$ *such that the set in complex plane*

$$P_\alpha(z) = \{p(z) \mid p \in P(\mathbb{R}), \ \alpha(g_i p) \leq 1\}$$

is unbounded, then α *is fundamental. Conversely if* α *is fundamental then* $P_\alpha(z)$ *is unbounded for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

PROOF: Assume that $P_\alpha(z)$ is unbounded. Let $p \in P(\mathbb{R})$ be such that

$\alpha(g_i p) \leq 1$ and $p(z) \neq 0$. If

$$q = - \frac{1}{p(z)} g_z(p - p(z))$$

then $q \in P(\mathbb{R})$ and $q - g_z = \frac{g_z p}{p(z)}$. By choosing a constant $C_{z,i} > 0$ as in Lemma 4 it follows that

$$\alpha(q - g_z) \leq \frac{C_{z,i}}{|p(z)|} .$$

Since $P_\alpha(z)$ is unbounded, then $g_z \in \overline{P(\mathbb{R})}^{\alpha}$ and from Lemma 3 α is fundamental.

Conversely assume that α is fundamental. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $n \in \mathbb{N}^*$ be given. Since $g_z \in \overline{P(\mathbb{R})}^{\alpha}$, there exists $p \in P(\mathbb{R})$ such that $\alpha(g_z - p) \leq \frac{1}{n}$. Let $q = n(1 - (x - z)p)$. Then $q \in P(\mathbb{R})$ and $\alpha(g_z q) = n\alpha(g_z - p) \leq 1$. If $C_{i,z}$ is a positive constant as is Lemma 4 it follows that $\alpha(g_i q) \leq C_{i,z}$. To finish we let $p_n = \frac{q}{C_{i,z}}$. Then $p_n \in P(\mathbb{R})$, $\alpha(g_i p_n) \leq 1$ and $p_n(z) = \frac{n}{C_{i,z}}$. Hence $P_\alpha(z)$ is unbounded.

THEOREM 2 (quasi-analytic criterion): *Let* $\alpha \in SPC(\mathbb{R})$. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\alpha(x^n)}} = + \infty$$

then α *is fundamental.*

PROOF: Let T be a continuous linear form on B_α such that T vanishes on $P(\mathbb{R})$. Let D denote the set of complex numbers z such that $\mathrm{Im} z < 1$. Define

$$h(z) = T(g_z), \quad z \in D.$$

It is enough to prove that $h = 0$ on D. In fact assuming this, from

Hahn-Banach theorem it follows that $g_z \in \overline{P(\mathbb{R})}^{\alpha}$ for all $z \in D$. Then α is fundamental from Lemma 3.

Let $z \in D$, $n \in \mathbb{N}$. If $n \geq 1$ then

$$g_z = g_z \frac{x^n}{z^n} - \frac{1}{z} \sim \frac{x}{z^2} - \ldots - \frac{x^{n-1}}{z^n}.$$

Since T vanishes on $P(\mathbb{R})$ it follows that $h(z) = \frac{1}{z^n} T(g_z x^n)$. This is also true for $n = 0$. Hence

$$|h(z)| \leq \frac{\|T\|}{|z|^n} \alpha(g_z x^n) \quad \text{for all} \quad z \in D, \quad n \in \mathbb{N}.$$

From the definition of α, there exist $C > 0$ and $m \in \mathbb{N}$ such that

$$\alpha(gf) \leq C \|g\|_m \alpha(f) \quad \text{for all} \quad g \in C_o^{\infty}(\mathbb{R}), \quad f \in B.$$

Since $\|g_z\|_m \leq (m+1)!$ for all $z \in D$ we have that

$$(*) \qquad |h(z)| \leq C\|T\|(m+1)! \frac{\alpha(x^n)}{|z|^n} \quad \text{for all} \quad z \in D, \ n \in \mathbb{N}.$$

Let $z, z_o \in D$, $z \neq z_o$. Then $g_z - g_{z_o} = (z - z_o) g_z g_{z_o}$. Hence

$$\left| \frac{h(z) - h(z_o)}{z - z_o} - T(g_{z_o}^2) \right| = |T((g_z - g_{z_o}) g_{z_o})| \leq$$

$$\leq |z - z_o| \|T\| \alpha(g_z g_{z_o}^2) \leq |z - z_o| \|T\| (C(m+1)!)^3 \alpha(1).$$

From this it follows that h is holomorphic on D. Since (*) is true, h is holomorphic on D and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\alpha(x^n)}} = +\infty,$$

then Denjoy conditions in Watson's problem are satisfied, hence h vanishes on D ([6]). Now the proof is complete.

COROLLARY 1: *Let* Δ *be the set of all seminorms* $\alpha \in SPC(\mathbb{R})$ *for which there are positive constants* C, c *and* N, $m \in \mathbb{N}$ *(all depending on* α*) such that*

$$\alpha(x^n) \leq C(c\, n \log n \cdot \ldots \cdot \log_m n)^n \quad \text{for all} \quad n \geq N$$

where $\log_m$ *is defined by* $\log_0 n = n$ *and* $\log_n n = \log(\log_{m-1} n)$ *if* $m \geq 1$.

Then Δ *is a directed set of fundamental seminorms.*

PROOF: This is a direct consequence of Theorem 2 observing that the "moments" of any two such seminorms have a common estimate of the same type.

THEOREM 3: *Let* $\alpha \in SPC(\mathbb{R}^n)$. *If there exist fundamental seminorms* $\alpha_1, \ldots, \alpha_n \in SPC(\mathbb{R})$ *such that*

$$\alpha(f_1 \otimes \ldots \otimes f_n) \leq \alpha_1(f_1) \cdot \ldots \cdot \alpha_n(f_n) \quad \text{for all} \quad f_1, \ldots, f_n \in B,$$

then α *is fundamental.*

PROOF: Let $\pi : B_{\alpha_1} \times \ldots \times B_{\alpha_n} \to B_{n,\alpha}$ be defined by $\pi(f_1, \ldots, f_n) = f_1 \otimes \ldots \otimes f_n$, for $f_1, \ldots, f_n \in B$. Then

$$\pi(C_c^\infty(\mathbb{R}) \times \ldots \times C_c^\infty(\mathbb{R})) \subset \pi(\overline{P(\mathbb{R})}^{\alpha_1} \times \ldots \times \overline{P(\mathbb{R})}^{\alpha_n}) \subset \overline{\pi(P(\mathbb{R}) \times \ldots \times P(\mathbb{R}))}^{\alpha}$$

since each α_i is fundamental and π is continuous. Hence if A is the complex subalgebra generated by

$$\pi(C_c^\infty(\mathbb{R}) \times \ldots \times C_c^\infty(\mathbb{R})) \quad \text{viz} \quad A = C_c^\infty(\mathbb{R}) \otimes \ldots \otimes C_c^\infty(\mathbb{R}),$$

then $A \subset \overline{P(\mathbb{R}^n)}^{\alpha}$. Since A is a self-adjoint subalgebra of $C_o^\infty(\mathbb{R}^n)$

and also satisfies conditions (N), from Lemma 2 it follows that A is dense in $B_{n,\alpha}$. Hence α is fundamental.

4. OPEN PROBLEMS

1. Give integral criteria like those in [7] for characterizing fundamental seminorms on $\mathbb{R}$.
2. Under what conditions on $\alpha \in SPC(\mathbb{R})$ is it true that α is fundamental if and only if $\sum_{i=1}^{\infty} \frac{1}{\sqrt[n]{\alpha(x^n)}} = +\infty$?
3. If $\alpha \in SPC(\mathbb{R})$ is not fundamental, describe $\overline{P(\mathbb{R})}^{\alpha}$.
4. If $\alpha \in SPC(\mathbb{R})$ is not fundamental, are there positive constants c, C such that for all $p \in P(\mathbb{R})$, $\alpha(p) \leq 1$ and $z \in \mathbb{C}$ we have $|p(z)| \leq C e^{c|z|}$?
5. Under what conditions on $\alpha \in SPC(\mathbb{R})$ is it true that α is fundamental if and only if the set $\{p \in P(\mathbb{R}),\ \alpha(p) \leq 1\}$ is unbounded in the space of entire functions on $\mathbb{C}$?
6. Give a characterization of fundamental seminorms on $\mathbb{R}^n$, $n \geq 2$.
7. Is the set of all fundamental seminorms on $\mathbb{R}$ directed? Same on $\mathbb{R}^n$?

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