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INDEFINITE INTEGRAL

PINNACLE-CBSE

INDEFINITE INTEGRAL

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SYLLABUS

CBSE: Integration as inverse of differentiation. Properties of integrals. Integration by substitution, Partial fractions and their use in integrating rational functions, Integral of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{(px + q)dx}{\sqrt{ax^2 + bx + c}}, \text{ etc.}$$

Integration by parts, Integral of the types

$$\int e^{ax} \sin bx \, dx, \int \sqrt{x^2 \pm a^2} \, dx,$$

$$\int \frac{dx}{a + b \cos x}, \text{ etc.}$$

IITJEE: Integration as the inverse process of differentiation, indefinite integrals of standard functions, Integration by parts, integration by the methods of substitution and partial fractions.

Indefinite Integral

INDEFINITE INTEGRAL

In many practical problems, the derivative of a function is given and we are required to find the function which has the given derivative. The process of finding this function is called integration.

Definition: Basic Concept

Let $F(x)$ be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$. Then $F(x)$ is called the integral of $f(x)$. Symbolically, it is written as $\int f(x) dx = F(x)$.
 $f(x)$, the function to be integrated, is called the integrand.
 $F(x)$ is also called the anti-derivative (or primitive function) of $f(x)$.

Constant of Integration:

As the differential coefficient of a constant is zero, we have

$$\frac{d}{dx}(F(x)) = f(x) \Rightarrow \frac{d}{dx}[F(x) + c] = f(x).$$

Therefore, $\int f(x) dx = F(x) + c$.

The constant c is called the constant of integration and can take any real value. By assigning different value to c , we obtain different values of the integral.

Basic formulae:

Antiderivatives or integrals of some of the widely used functions (integrands) are given below:

- $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
- $\frac{d}{dx}(\ln |x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \ln |x| + c$
- $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c$
- $\frac{d}{dx}(a^x) = (a^x \ln a) \Rightarrow \int a^x dx = \frac{a^x}{\ln a} + c \quad (a > 0)$
- $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x dx = \sin x + c$
- $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x dx = -\cos x + c$
- $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x dx = \tan x + c$
- $\frac{d}{dx}(\operatorname{cosec} x) = (-\cot x \operatorname{cosec} x) \Rightarrow \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
- $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x dx = \sec x + c$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x dx = -\cot x + c$

- $$\frac{d}{dx} \left(\sin^{-1} \frac{x}{a} \right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \Rightarrow \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{or } -\cos^{-1} \left(\frac{x}{a} \right) + c_1$$
- $$\frac{d}{dx} \left(\tan^{-1} \frac{x}{a} \right) = \frac{a}{x^2 + a^2} \quad \Rightarrow \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$
- $$\frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}} \quad \Rightarrow \quad \int \frac{1}{|x| \sqrt{x^2 - 1}} dx = \sec^{-1}(x) + c$$
- $$\frac{d}{dx} (\operatorname{cosec}^{-1} x) = \frac{-1}{|x| \sqrt{x^2 - 1}} \quad \Rightarrow \quad \int \frac{dx}{|x| \sqrt{x^2 - 1}} = -\operatorname{cosec}^{-1} x + c$$
- $$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \ln |\sin x| + c$$
- $$\int \tan x \, dx = -\int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + c = \ln |\sec x| + c$$
- $$\int \sec x \, dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \ln |\sec x + \tan x| + c = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + c$$
- $$\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x (\cot x - \operatorname{cosec} x)}{\cot x - \operatorname{cosec} x} dx = \ln |\cot x - \operatorname{cosec} x| + c = \ln \left| \tan \frac{x}{2} \right| + c$$

Standard Formulae:

- $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right| + c$$
- $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$
- $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + c$$
- $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{x + a}{x - a} \right| + c$$
- $$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right| + c$$
- $$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + c$$
- $$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

Properties of indefinite integration:

Let $f(x)$ and $g(x)$ be two integrable functions. Then the following results hold:

i) $\int \alpha f(x) dx = \alpha \int f(x) dx$ where α is an arbitrary constant.

Proof: We know that $\frac{d}{dx} (\alpha p(x)) = \alpha \frac{d}{dx} (p(x))$.

$$\text{Let } f(x) = \frac{d}{dx} (p(x)) \quad \Rightarrow \quad p(x) = \int f(x) dx$$

$$\Rightarrow \alpha \int f(x) dx = \alpha p(x) = \int \frac{d}{dx}(\alpha p(x)) dx$$

$$= \int \alpha \frac{d}{dx}(p(x)) dx = \int \alpha f(x) dx$$

$$\text{ii)} \quad \int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx.$$

Proof: From the property of differentiation, we have

$$\frac{d}{dx}[p(x) \pm g(x)] = \frac{d}{dx}(p(x)) \pm \frac{d}{dx}(g(x))$$

If $\frac{d}{dx}(p(x)) = f(x)$ and $\frac{d}{dx}(g(x)) = g(x)$, then

$$\int f(x) dx = p(x) \text{ and } \int g(x) dx = q(x)$$

$$\text{so that } \int f(x) dx \pm \int g(x) dx = p(x) \pm q(x) = \int \frac{d}{dx}[p(x) \pm q(x)] dx$$

$$= \int \left[\frac{d}{dx} p(x) \pm \frac{d}{dx} (q(x)) \right] dx = \int [f(x) \pm g(x)] dx.$$

■ In general;

$$\int (\alpha_1 f_1(x) \pm \alpha_2 f_2(x) \pm \dots \pm \alpha_n f_n(x)) dx.$$

$$= \alpha_1 \int f_1(x) dx \pm \alpha_2 \int f_2(x) dx \pm \dots \pm \alpha_n \int f_n(x) dx$$

where $f_1, f_2, \dots, f_n$ are integrable function and $\alpha_1, \alpha_2, \dots, \alpha_n$ are arbitrary constants.

$$\text{iii)} \quad \int \frac{d}{dg}(f(g(x))) \frac{d}{dx}(g(x)) dx = f(g(x)).$$

Proof: We know that $\frac{d}{dx}(f(g(x))) = \frac{d}{dg} f(g(x)) g'(x)$ (Chain rule)

$$\Rightarrow \int \frac{d}{dg}(f(g(x))) \frac{d}{dx}(g(x)) dx = \int \frac{d}{dx}(f(g(x))) dx = f(g(x)).$$

$$\text{iv)} \quad \text{If } \int f(u) du = F(u) + c, \text{ then } \int f(ax + b) dx = \frac{1}{a} F(ax + b) + c, a \neq 0.$$

Proof: Write $u = ax + b$ so that $du = a dx$. Hence

$$\int f(u) = a \int f(ax + b) dx = F(ax + b) + c_1 \Rightarrow \int f(ax + b) dx = \frac{1}{a} F(ax + b) + c.$$

■ Trigonometric identities help in the simplification when the integral involves trigonometric function.

Illustration 1. Evaluate $I = \int \frac{(1 + \cos 4x)}{(\cot x - \tan x)} dx$.

Solution:

$$I = \int \frac{2 \cos^2 2x}{\cos^2 x - \sin^2 x} \cdot \sin x \cdot \cos x dx = \int 2 \cos 2x \frac{1}{2} \sin 2x dx$$

$$= \frac{1}{2} \int \sin 4x dx = -\frac{1}{8} \cos 4x + c.$$

Illustration 2. Evaluate $I = \int \frac{dx}{\sin^2 x \cdot \cos^2 x}$.

Solution: Transform the integrand in the following way.

$$\begin{aligned} \frac{1}{\sin^2 x \cos^2 x} &= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\ &= \sec^2 x + \csc^2 x = \frac{d}{dx} [\tan x - \cot x] \end{aligned}$$

$$\text{Hence } I = \int (\sec^2 x + \csc^2 x) dx = \tan x - \cot x + c.$$

Illustration 3. Evaluate $\int e^x (\cos x - \sin x) dx$.

Solution: Here $e^x (\cos x - \sin x)$ is the derivative of $e^x \cos x$.
 $\Rightarrow I = e^x \cos x + c.$

Illustration 4. Evaluate $\int 3x^{1/2} (1 + x^{3/2}) dx$.

Solution: Here $3x^{1/2} (1 + x^{3/2})$ is the derivative of $(1 + x^{3/2})^2$
 $\Rightarrow I = (1 + x^{3/2})^2 + c.$

Illustration 5. Evaluate $\int \cos 4x \cos 7x dx$.

Solution: When solving such problems it is expedient to use the following trigonometric identities:

$$\begin{aligned} \sin(mx) \cos(nx) &= (1/2) [\sin(m-n)x + \sin(m+n)x], \\ \sin(mx) \sin(nx) &= (1/2) [\cos(m-n)x - \cos(m+n)x], \\ \cos(mx) \cos(nx) &= (1/2) [\cos(m-n)x + \cos(m+n)x]. \end{aligned}$$

$$\text{Here } \cos 4x \cos 7x = (1/2) [\cos(3x) + \cos 11x]$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{2} \int (\cos 3x + \cos 11x) dx \\ &= \frac{1}{2} \left[\frac{\sin 3x}{3} + \frac{\sin 11x}{11} \right] + c. \end{aligned}$$

Illustration 6. Evaluate (i) $\int \frac{dx}{1 + \sin x}$, (ii) $\int \frac{\sin x}{1 + \sin x} dx$.

Solution: (i) $\int \frac{dx}{1 + \sin x} = \int \frac{1 - \sin x}{1 - \sin^2 x} dx = \int \frac{1 - \sin x}{\cos^2 x} dx$
 $\int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + c.$

(ii) Here $\int \frac{\sin x}{1 + \sin x} dx = \int \frac{\sin x + 1 - 1}{1 + \sin x} dx$
 $= \int \left(1 - \frac{1}{1 + \sin x} \right) dx$
 $= x - (\tan x - \sec x) + c.$

Exercise 1.**Integrate the following functions**

i) $\frac{4 - 5 \sin x}{\cos^2 x}$

ii) $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

iii) $\sqrt{x} + \frac{1}{\sqrt{x}}$

iv) $\sec x (\sec x + \tan x)$

v) $2x - 3 \cos x + e^x$

vi) $\left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2$

Methods of Integration

If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integrals of complex problems, generally four methods are used so that the integral is reduced to the standard form.

- Integration by substitution or by change of the independent variable.
- Integration by parts.
- Integration by using trigonometric identities.
- Integration by partial fractions.

Integration By Substitution**Direct Substitutions:**

If the integral is of the form $\int f(g(x)) g'(x) dx$, then put $g(x) = t$, provided $\int f(t) dt$ exists.

$$g(x) = t \Rightarrow \frac{dg(x)}{dx} = \frac{dt}{dx} \text{ so that the integral } \int f(g(x)) g'(x) dx$$

$$\text{become } \int f(t) \frac{dt}{dx} dx = \int f(t) dt.$$

The problem of integrating with respect to x is transformed to integrating $f(t)$ with respect to t and $\int f(t) dt = k(t) + c = k(g(x)) + c$.

Illustration 7. Evaluate $\int \frac{(1 + \ln x)^2}{x} dx$.

Solution: Let $1 + \ln x = t$ so that $\frac{1}{x} dx = dt$

$$\Rightarrow I = \int t^2 dt = \frac{1}{3} t^3 + c = \frac{(1 + \ln x)^3}{3} + c.$$

Illustration 8. Evaluate $\int \frac{\sin(\ln x)}{x} dx$.

Solution: Let $\ln x = t$. Then $dt = \frac{1}{x} dx$.

$$\text{Hence } I = \int \sin t dt = -\cos t + c = -\cos(\ln x) + c.$$

Illustration 9. Evaluate $\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \frac{1}{x}\right)}$.

Solution: The given integral can be written as

$$I = \int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left[\left(x + \frac{1}{x}\right)^2 + 1\right]\tan^{-1}\left(x + \frac{1}{x}\right)}$$

Let $\left(x + \frac{1}{x}\right) = t$. Differentiating we get $\left(1 - \frac{1}{x^2}\right)dx = dt$.

$$\text{Hence } I = \int \frac{dt}{(t^2 + 1)\tan^{-1}t}$$

Now make one more substitution $\tan^{-1}t = u$. Then

$$\frac{dt}{t^2 + 1} = du \text{ and } I = \int \frac{du}{u} = \ln|u| + c$$

Returning to t , and then to x , we have

$$I = \ln|\tan^{-1}t| + c = \ln\left|\tan^{-1}\left(x + \frac{1}{x}\right)\right| + c.$$

Illustration 10. Evaluate

$$(i) \int \frac{dx}{x(\log x)^m}, x > 0; \quad (ii) \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$$

Solution: (i) Let $\log x = t \Rightarrow \frac{1}{x} dx = dt$.

Substituting in the given integral, we have

$$\begin{aligned} \int \frac{dx}{x(\log x)^m} &= \int \frac{dt}{t^m} = \int t^{-m} dt \\ &= \frac{t^{-m+1}}{-m+1} + c = \frac{1}{1-m}(\log x)^{1-m} + c. \end{aligned}$$

■ For $m = 1$, $\int \frac{dx}{x \log x} = \int \frac{dt}{t} = \log|t| + c = \log|\log x| + c$

$$\begin{aligned} (ii) \text{ Let } I &= \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \\ &= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \left[\frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \right] dx \\ &= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(a-b)} = \frac{1}{(a-b)} \int (\sqrt{x+a} - \sqrt{x+b}) dx \\ &= \frac{1}{a-b} \left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2} \right] + c \\ &= \frac{2}{3(a-b)} [(x+a)^{3/2} - (x+b)^{3/2}] + c. \end{aligned}$$

Exercise 2.**Integrate the following:**

i) $\frac{\cos x + \sin x}{\cos x - \sin x},$

ii) $\frac{1}{\sin^{-1} x \cdot \sqrt{1-x^2}},$

iii) $x^x \log_e(ex),$

iv) $\left(\frac{4x+6}{x^2+3x+100} \right),$

v) $\frac{\cos \sqrt{x}}{\sqrt{x}},$

vi) $\frac{\sec^2(\log(x))}{x}.$

Standard Substitutions:

- For terms of the form $x^2 + a^2$ or $\sqrt{x^2 + a^2}$, put $x = a \tan \theta$ or $a \cot \theta$.
- For terms of the form $x^2 - a^2$ or $\sqrt{x^2 - a^2}$, put $x = a \sec \theta$ or $a \operatorname{cosec} \theta$.
- For terms of the form $a^2 - x^2$ or $\sqrt{a^2 - x^2}$, put $x = a \sin \theta$ or $a \cos \theta$.
- If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x = a \cos \theta$.
- For the type $\sqrt{(x-a)(b-x)}$, put $x = a \cos^2 \theta + b \sin^2 \theta$.
- For the type $(\sqrt{x^2 + a^2} \pm x)^n$ or $(x \pm \sqrt{x^2 - a^2})^n$, put the expression within the bracket = t.
- For the type $(x+a)^{-1-\frac{1}{n}}(x+b)^{-1+\frac{1}{n}}$ or $\left(\frac{x+b}{x+a}\right)^{\frac{1}{n}-1} \frac{1}{(x+a)^2}$ ($n \in \mathbb{N}, n > 1$),
put $\frac{x+b}{x+a} = t$.
- For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1, n_2 \in \mathbb{N}$ (and > 1), again put $(x+a) = t(x+b)$.

Illustration 11. Evaluate $\int \frac{dx}{(x+1)^{6/5}(x-3)^{4/5}}.$

Solution: $I = \int \frac{dx}{(x+1)^{6/5}(x-3)^{4/5}} = \int \frac{dx}{(x+1)^2 \left(\frac{x-3}{x+1} \right)^{4/5}}.$

Put $\left(\frac{x-3}{x+1} \right) = t \Rightarrow dt = \frac{4}{(x+1)^2} dx.$

Hence $I = \frac{1}{4} \int \frac{dt}{t^{4/5}} = \frac{5}{4} t^{1/5} + c = \frac{5}{4} \left(\frac{x-3}{x+1} \right)^{1/5} + c.$

Illustration 12. Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}.$ **Solution:** Writing $x = a \cos^2 \theta + b \sin^2 \theta = a + (b-a) \sin^2 \theta$, the given integral becomes

$$I = \int \frac{2(b-a) \sin \theta \cos \theta d\theta}{\{(a \cos^2 \theta + b \sin^2 \theta - a)(a \cos^2 \theta + b \sin^2 \theta - b)\}^{1/2}}$$

$$\begin{aligned}
 &= \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = \left(\frac{b-a}{b-a}\right) \int 2 d\theta \\
 &= 2\theta + c = 2\sin^{-1} \sqrt{\frac{x-a}{b-a}} + c.
 \end{aligned}$$

Indirect Substitutions:

If the integrand is of the form $f(x).g(x)$, where $g(x)$ is a function of the integral of $f(x)$, then put integral of $f(x) = t$.

Illustration 13. Evaluate $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$.

Solution: Integral of the numerator $= \frac{x^{3/2}}{3/2}$.

Put $x^{3/2} = t$.

$$\text{We get } I = \frac{2}{3} \int \frac{dt}{\sqrt{a^3 - t^2}} = \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + c = \frac{2}{3} \sin^{-1} \left(\frac{x}{a} \right)^{\frac{3}{2}} + c.$$

Illustration 14. Evaluate $\int \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$.

Solution: $I = \int \frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2} dx$

Put $t = \sin x - \cos x \Rightarrow dt = (\cos x + \sin x)dx$

$$\begin{aligned}
 \Rightarrow I &= \int \frac{dt}{25 - 16t^2} = \frac{1}{40} \ln \left| \frac{5 - 4t}{5 + 4t} \right| + c \\
 &= \frac{1}{40} \ln \left| \frac{5 - 4(\sin x - \cos x)}{5 + 4(\sin x - \cos x)} \right| + c.
 \end{aligned}$$

Derived Substitutions:

Some times it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions.

Examples of such integrals are:

A. Algebraic Twins

$$\begin{aligned}
 \int \frac{2x^2}{x^4 + 1} dx &= \int \frac{x^2 + 1}{x^4 + 1} dx + \int \frac{x^2 - 1}{x^4 + 1} dx \\
 \int \frac{2}{x^4 + 1} dx &= \int \frac{x^2 + 1}{x^4 + 1} dx - \int \frac{x^2 - 1}{x^4 + 1} dx \\
 \int \frac{2x^2}{(x^4 + 1 + kx^2)} dx, \int \frac{2}{(x^4 + 1 + kx^2)} dx
 \end{aligned}$$

B. Trigonometric Twins

$$\int \sqrt{\tan x} \, dx, \int \sqrt{\cot x} \, dx, \quad \int \frac{1}{(\sin^4 x + \cos^4 x)} dx,$$

$$\int \frac{1}{(\sin^6 x + \cos^6 x)} dx, \quad \int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x} dx.$$

Method of evaluating these integrals are illustrated by mean of the following examples:

Illustration 15. Evaluate $\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$.

Solution: Let $I = \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 - 1 + \frac{1}{x^2}\right)} dx$.

Put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$\Rightarrow I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1} \left(\frac{x^2 - 1}{x}\right) + c$.

Illustration 16: Evaluate $\int \frac{1}{x^4 + 1 + 5x^2} dx$.

Solution: Let $I = \frac{1}{2} \int \frac{2}{x^4 + 1 + 5x^2} dx$

$$= \frac{1}{2} \int \frac{1+x^2}{x^4 + 1 + 5x^2} dx - \frac{1}{2} \int \frac{x^2 - 1}{x^4 + 1 + 5x^2} dx$$

$$= \frac{1}{2} \int \frac{1 + (1/x^2)}{x^2 + (1/x^2) + 5} dx - \frac{1}{2} \int \frac{1 - (1/x^2)}{x^2 + (1/x^2) + 5} dx = (I_1 - I_2)/2 \text{ (say).}$$

For I_1 , we write $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t^2 + 7} = \frac{1}{\sqrt{7}} \tan^{-1} \frac{t}{\sqrt{7}} = \frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{x - 1/x}{\sqrt{7}}\right).$$

For I_2 , we write $x + \frac{1}{x} = t$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$\Rightarrow I_2 = \int \frac{dt}{t^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x + 1/x}{\sqrt{3}}\right).$$

Combining the two results, we get $I = (I_1 - I_2)/2$

$$= \frac{1}{2\sqrt{7}} \tan^{-1} \left(\frac{x - 1/x}{\sqrt{7}}\right) - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x + 1/x}{\sqrt{3}}\right) + c.$$

Illustration 17. Evaluate $\int \sqrt{\tan x} \, dx$.

Solution: Put $\tan x = t^2 \Rightarrow \sec^2 x \, dx = 2t \, dt \Rightarrow dx = \frac{2t \, dt}{1+t^4}$

$$\Rightarrow I = \int \sqrt{\tan x} \, dx = \int t \frac{2t \, dt}{1+t^4}$$

$$= \int \frac{2t^2 \, dt}{1+t^4} = \int \frac{t^2+1}{t^4+1} \, dt + \int \frac{t^2-1}{t^4+1} \, dt.$$

This can be solved by the method used in Illustration -16.

Illustration 19. Evaluate $\int \frac{4}{\sin^4 x + \cos^4 x} \, dx$.

Solution: Let $I = 4 \int \frac{1}{\sin^4 x + \cos^4 x} \, dx = 4 \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x} \, dx$

$$= 4 \int \frac{(\tan^2 x + 1) \cos^2 x}{(\tan^4 x + 1) \cos^4 x} \, dx = 4 \int \frac{(\tan^2 x + 1) \sec^2 x}{(\tan^4 x + 1)} \, dx.$$

Now, put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow I = 4 \int \frac{1+t^2}{1+t^4} \, dt = 4 \int \frac{1/t^2 + 1}{t^2 + 1/t^2} \, dt.$$

Now, put $t - \frac{1}{t} = z \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dz$

$$\Rightarrow I = \int \frac{dz}{z^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t - 1/t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\tan x - 1/\tan x}{\sqrt{2}} + c.$$

Illustration 19. Evaluate $\int \frac{2 \cos x \, dx}{3 + 4 \sin 2x}$.

Solution: Here $\int \frac{2 \cos x \, dx}{3 + 4 \sin 2x} = \int \frac{\cos x + \sin x}{3 + 4 \sin 2x} \, dx + \int \frac{\cos x - \sin x}{3 + 4 \sin 2x} \, dx$

$$= \int \frac{(\cos x + \sin x) \, dx}{3 + 4(1 - (\sin x - \cos x)^2)} + \int \frac{(\cos x - \sin x) \, dx}{3 + 4((\sin x + \cos x)^2 - 1)}.$$

In the first integral put $\sin x - \cos x = t$, in second integral put $\sin x + \cos x = y$, and proceed.

- Similarly, we can solve integrals of the type:

$$\int \frac{1}{a \sin x + b \sec x} \, dx, \int \frac{1}{p \cos x + q \csc x} \, dx.$$

Illustration 20: Evaluate $\int \frac{dx}{2 \sin x + \sec x}$.

Solution: Let $I = \int \frac{dx}{2 \sin x + \sec x} = \int \frac{\cos x dx}{\sin 2x + 1} = \frac{1}{2} \int \frac{\cos x + \sin x + \cos x - \sin x}{\sin 2x + 1} dx$

$$= \frac{1}{2} \int \frac{\cos x + \sin x}{1 + \sin 2x} dx + \frac{1}{2} \int \frac{\cos x - \sin x}{1 + \sin 2x} dx$$

$$= \frac{1}{2} \int \frac{\cos x + \sin x}{1 + 1 - (\sin x - \cos x)^2} dx + \frac{1}{2} \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx.$$

Let $u = \sin x - \cos x$ and $v = \sin x + \cos x$.

Then $I = \frac{1}{2} \int \frac{du}{2 - u^2} + \frac{1}{2} \int \frac{dv}{v^2} = \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{2} - u}{\sqrt{2} + u} \right| - \frac{1}{2v} + c$

$$= \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{2} - \sin x + \cos x}{\sqrt{2} + \sin x - \cos x} \right| - \frac{1}{2(\sin x + \cos x)} + c.$$

Exercise 3.**Integrate the following functions:**

i) $\sqrt{\cot x}$	ii) $\frac{1}{x^4 + 3x^2 + 9}$	iii) $\frac{x^2 + 1}{(x^2 - 1)^2}$
iv) $\frac{\sin 2x}{p \cos^2 x + q \sin^2 x}$	v) $\frac{x}{\sqrt{x+4}}$	vi) $\frac{1}{1 + \cot x}$

Integrals of trigonometric functions:

(i). $\int \tan x dx = \log |\sec x| + c.$

Proof: Write $\cos x = t$ so that $-\sin x dx = dt$.

$$\text{Hence } \int \tan x dx = \int \frac{\sin x dx}{\cos x} = - \int \frac{dt}{t}$$

$$= -\log |t| + c = -\log |\cos x| + c$$

$$= \log |\sec x| + c.$$

(ii) $\int \cot x dx = \log |\sin x| + c$

Proof: Writing $\sin x = t$ and $\cos x dx = dt$, we get

$$\int \cot x dx = \int \frac{dt}{t} = \log |t| + c = \log |\sin x| + c$$

(iii) $\int \sec x dx = \log |\sec x + \tan x| + c.$

Proof: Here $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx.$$

Let $\sec x + \tan x = t \Rightarrow (\sec x \tan x + \sec^2 x) dx = dt$.

$$\text{Hence } \int \sec x dx = \int \frac{dt}{t} = \log |t| + c$$

$$= \log |\sec x + \tan x| + c.$$

$$(iv) \quad \int \cos \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c.$$

$$\text{Let } I = \int \cos \operatorname{cosec} x \, dx = \int \frac{\cos \operatorname{cosec} x (\cos \operatorname{cosec} x + \cot x)}{\cos \operatorname{cosec} x + \cot x} \, dx.$$

$$\text{Writing } \operatorname{cosec} x + \cot x = t \Rightarrow -(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x) \, dx = dt$$

$$\text{so that } I = -\int \frac{dt}{t} = -\log |t| + c$$

$$= \log \left| \frac{1}{\cos \operatorname{cosec} x + \cot x} \right| + c$$

$$= \log \left| \frac{(\cos \operatorname{cosec} x - \cot x)}{\cos \operatorname{cosec}^2 x - \cot^2 x} \right| + c$$

$$= \log |\operatorname{cosec} x - \cot x| + c.$$

Illustration 21. Evaluate the following:

$$(i) \quad \int \frac{\sin 2x}{\sin 4x} \, dx \quad (ii) \quad \int \frac{1 - \sin x}{1 - \cos x} \, dx$$

$$(iii) \quad \int \sin^7 x \, dx \quad (iii) \quad \int \frac{dx}{x \cos^2 (1 + \log x)}$$

Solution: (i) We have $\int \frac{\sin 2x}{\sin 4x} \, dx = \int \frac{\sin 2x}{2 \sin 2x \cos 2x} \, dx$

$$= \frac{1}{2} \int \sec 2x \, dx = \frac{1}{4} \int \sec t \, dt \quad \text{where } 2x = t$$

$$= \frac{1}{4} \log |\sec t + \tan t| + c = \frac{1}{4} \log |\sec 2x + \tan 2x| + c.$$

$$(ii) \quad \text{Here } \int \frac{1 - \sin x}{1 - \cos x} \, dx = \int \frac{1 - 2 \sin x / 2 \cos x / 2}{2 \sin^2 x / 2} \, dx$$

$$= \frac{1}{2} \int \operatorname{cosec}^2 \frac{x}{2} \, dx - \int \cot \frac{x}{2} \, dx = -\frac{1}{2} \frac{\cot x / 2}{1/2} - 2 \log \left| \sin \frac{x}{2} \right| + c$$

$$= -\cot \frac{x}{2} - 2 \log \left| \sin \frac{x}{2} \right| + c.$$

$$(iii) \quad \text{We write } \int \sin^7 x \, dx = \int \sin^6 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^3 \sin x \, dx.$$

$$\text{Let } \cos x = t \Rightarrow -\sin x \, dx = dt.$$

$$\text{Hence } \int \sin^7 x \, dx = -\int (1 - t^2)^3 \, dt$$

$$= -\int (1 - 3t^2 + 3t^4 - t^6) \, dt = -\left(t - t^3 + \frac{3}{5}t^5 - \frac{1}{7}t^7\right) + c$$

$$= -\left(\cos x - \cos^3 x + \frac{3}{5}\cos^5 x - \frac{1}{7}\cos^7 x\right) + c.$$

$$(iv) \quad \text{Let } 1 + \log x = t \Rightarrow \frac{1}{x} \, dx = dt.$$

$$\text{Hence } \int \frac{dx}{x \cos^2 (t + \log x)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt$$

$$= \tan t + c = \tan (1 + \log x) + c.$$

Exercise 4. Evaluate the integrals of the following functions with respect to x :

i) $\frac{\sin x}{\sin(x+a)}$

ii) $\sec^2 x \tan x$

iii) $\frac{1}{\sin x + \cos x}$

iv) $e^x \operatorname{cosec} x (1 - \cot x)$

v) $\frac{\sec x \operatorname{cosec} x}{\log \cot x}$

vi) $\sin^3 x \cos^2 x$.

Integrals of some particular functions:

Let us now prove some of the results given on page 2.

$$\begin{aligned} \blacksquare \quad I &= \int \frac{dx}{a^2 + x^2}; \text{ we write } x = a \tan \theta \text{ and get} \\ I &= \int \frac{a \sec^2 \theta d\theta}{a^2 \sec^2 \theta} = \frac{1}{a} \int d\theta = \frac{\theta}{a} + c = \frac{1}{a} \tan^{-1} \frac{x}{a} + c. \\ \blacksquare \quad I &= \int \frac{dx}{a^2 - x^2} = \int \frac{dx}{(a-x)(a+x)} = \frac{1}{2a} \int \left(\frac{1}{a-x} + \frac{1}{a+x} \right) dx \\ &= \frac{1}{2a} [-\log |a-x| + \log |a+x|] + c \\ &= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c. \end{aligned}$$

Alternatively, we may write $x = a \sin \theta$ and get

$$\begin{aligned} I &= \frac{1}{a} \int \sec \theta d\theta = \frac{1}{a} \log |\sec \theta + \tan \theta| + c \\ &= \frac{1}{2a} \log \left| \frac{1+\sin \theta}{1-\sin \theta} \right| + c = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c. \end{aligned}$$

$$\begin{aligned} \blacksquare \quad I &= \int \frac{dx}{\sqrt{x^2 + a^2}}; \text{ we write } x = a \tan \theta \text{ and get} \\ I &= \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + c = \log \left| \sqrt{\tan^2 \theta + 1} + \tan \theta \right| + c \\ &= \log \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| + c. \end{aligned}$$

$$\begin{aligned} \blacksquare \quad I &= \int \sqrt{a^2 - x^2} dx; \text{ we write } x = a \sin \theta, dx = a \cos \theta d\theta \text{ and get} \\ I &= a^2 \int \cos^2 \theta d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta) d\theta = \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + c \\ &= \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + c = \frac{a^2}{2} \left[\sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{a^2} \right] + c \\ &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2 - x^2}}{2} + c. \end{aligned}$$

$$\blacksquare \quad I = \int \frac{dx}{|x| \sqrt{x^2 - a^2}}; \text{ we write } x = a \sec \theta, dx = a \sec \theta \tan \theta d\theta \text{ and get}$$

$$I = \int \frac{a \sec \theta \cdot \tan \theta \cdot d\theta}{|a \sec \theta| \cdot |a \tan \theta|}.$$

Since $\theta = \sec^{-1} \frac{x}{a}$ and its range $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$,

$a \sec \theta$ and $a \tan \theta$ is positive. Hence

$$I = \frac{1}{a} \int \theta d\theta = \frac{1}{a} \sec^{-1} \frac{x}{a} + c.$$

Remarks: If $f(x)$ is a differentiable function of x , then

$$(i) \quad \int \frac{f'(x)}{(1+f^2(x))} dx = \tan^{-1}(f(x)) + c$$

$$(ii) \quad \int \frac{f'(x)}{\sqrt{1-f^2(x)}} dx = \sin^{-1}(f(x)) + c$$

$$(iii) \quad \int \frac{f'(x)}{f(x)} dx = \log(f(x)) + c$$

by writing $f(x) = t$.

Illustration 22. Evaluate the integral with respect to x of the function

$$(i) \quad \frac{1}{x^4 + a^2}, \quad (ii) \quad \frac{2x^2}{x^4 + a^2}, \quad (iii) \quad \frac{x}{(9-x^2)^{3/2}}.$$

Solution: (i) Let $I = \int \frac{dx}{x^4 + a^2} = \int \frac{x^2 + a - (x^2 - a)}{2a(x^4 + a^2)} dx.$

$$= \frac{1}{2a} \int \frac{x^2 + a}{x^4 + a^2} dx - \frac{1}{2a} \int \frac{x^2 - a}{x^4 + a^2} dx$$

$$= \frac{1}{2a} \int \frac{1 + a/x^2}{x^2 + a^2/x^2} dx - \frac{1}{2a} \int \frac{1 - a/x^2}{x^2 + a^2/x^2} dx$$

$$= \frac{1}{2a} \int \frac{1 + a/x^2}{\left(x - \frac{a}{x}\right)^2 + 2a} dx - \frac{1}{2a} \int \frac{1 - a/x^2}{\left(x + \frac{a}{x}\right)^2 - 2a} dx.$$

Writing $x - \frac{a}{x} = t$ and $x + \frac{a}{x} = z$, we get

$$I = \frac{1}{2a} \int \frac{dt}{t^2 + 2a} - \frac{1}{2a} \int \frac{dz}{z^2 - 2a}$$

$$= \frac{1}{2a} \cdot \frac{1}{\sqrt{2a}} \tan^{-1} \frac{t}{\sqrt{2a}} - \frac{1}{2a \cdot 2\sqrt{2a}} \log \left| \frac{z - \sqrt{2a}}{z + \sqrt{2a}} \right| + c$$

$$= \frac{1}{2a\sqrt{2a}} \tan^{-1} \left(\frac{x^2 - a}{x\sqrt{2a}} \right) - \frac{1}{4a\sqrt{2a}} \log \left| \frac{x^2 - \sqrt{2a}x + a}{x^2 + \sqrt{2a}x + a} \right| + c.$$

(ii) Let $I = \int \frac{2x^2}{x^4 + a^2} = \int \frac{x^2 + a + (x^2 - a)}{x^4 + a^2} dx$

and proceeding as in case (i), we get

$$I = \frac{1}{\sqrt{2a}} \tan^{-1} \left(\frac{x^2 - a}{x\sqrt{2a}} \right) + \frac{1}{2\sqrt{2a}} \log \left| \frac{x^2 - \sqrt{2a}x + a}{x^2 + \sqrt{2a}x + a} \right| + c.$$

(iii) In this case we write $x = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$ and get

$$\begin{aligned} I &= \int \frac{x}{(9-x^2)^{3/2}} dx = \int \frac{9 \sin \theta \cdot \cos \theta \cdot d\theta}{27 \cos^3 \theta} = \frac{1}{3} \int \sec \theta \cdot \tan \theta \cdot d\theta \\ &= \frac{1}{3} \sec \theta + c = \frac{1}{3\sqrt{1-\sin^2 \theta}} + c = \frac{1}{3\sqrt{1-x^2/9}} + c \\ &= \frac{1}{\sqrt{9-x^2}} + c. \end{aligned}$$

Alternatively, we write $9-x^2 = t \Rightarrow -2x dx = dt$ and get

$$I = -\frac{1}{2} \int \frac{dt}{t^{3/2}} = -\frac{1}{2} \int t^{-3/2} dt = t^{-1/2} + c = \frac{1}{\sqrt{9-x^2}} + c.$$

Exercise 5.

Evaluate the integrals, w.r.t. x , of the functions:

i) $\frac{1}{x + \sqrt{x}}$

ii) $\frac{\sin(2 \tan^{-1} x)}{1+x^2}$

iii) $\frac{\sqrt{a-x}}{\sqrt{a+x}}$

iv) $\frac{a^x}{\sqrt{1+a^{2x}}}, (a > 0)$

v) $\frac{x \tan^{-1}(x^2)}{1+x^4}$

vi) $\frac{x^3}{\sqrt{1+x^2}}$

Integration by Parts

If u and v be two functions of x , then integral of the product of these two functions is given by: $\int uv dx = u \int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx$.

This method is mainly used when the integrand is the product of two functions. Let $f(x)$ and $g(x)$ be two differentiable functions. Then

$$\begin{aligned} \frac{d}{dx} (f(x) g(x)) &= f'(x) g(x) + g'(x) f(x) \\ \Rightarrow f(x) g(x) &= \int f'(x) g(x) dx + \int g'(x) f(x) dx. \\ \text{or } \int f(x) g'(x) dx &= f(x) g(x) - \int f'(x) g(x) dx. \quad \dots(1) \end{aligned}$$

If we write $u(x) = f(x)$ and $v(x) = g'(x)$, then (1) gives

$$\int u(x) v(x) dx = u(x) \int v(x) dx - \int \left[\int v(x) dx \right] u'(x) dx.$$

= first function $\times$ integral of the second – integral of (product of derivative of the first function and integral of the second).

Note: In applying the above rule care has to be taken in the selection of the first function(u) and the second function (v). Normally we use the following methods:

(i) If in the product of the two functions, one of the functions is not directly integrable (e.g. $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) then we take it as the first function and the remaining function is taken as the second function. e.g. In the integration of $\int x \tan^{-1}x dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.

(ii) If there is no other function, then unity is taken as the second function e.g. In the integration of $\int \tan^{-1}x dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the first function.

(Inverse, Logarithmic, Algebraic, Trigonometric, Exponent)

In the above stated order, the function on the left is always chosen as the first function. This rule is called as **ILATE** e.g. In the integration of $\int x \sin x dx$, x is taken as the first function and $\sin x$ is taken as the second function.

(iv) Sometimes, it may be necessary to integrate by parts more than once to complete the integration process.

Illustration 23. Evaluate (i) $\int \sqrt{x^2 - a^2} dx$, (ii) $\int \sqrt{x^2 + a^2} dx$.

Solution: (i) Here $I = \int \sqrt{x^2 - a^2} dx = \sqrt{x^2 - a^2} \int 1 dx - \int \frac{x(2x)}{2\sqrt{x^2 - a^2}} dx$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$\Rightarrow 2I = x\sqrt{x^2 - a^2} - a^2 \ln(x + \sqrt{x^2 - a^2})$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2}) + c.$$

(ii) Here $I = \int \sqrt{x^2 + a^2} dx = \sqrt{x^2 + a^2} \int 1 dx - \int \frac{x(2x)}{2\sqrt{x^2 + a^2}} dx$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}} dx = x\sqrt{x^2 + a^2} - I + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow 2I = x\sqrt{x^2 + a^2} - a^2 \log|x + \sqrt{x^2 + a^2}| + c_1$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + c.$$

Illustration 24. Evaluate $\int \sec^3 \theta d\theta$.

Solution: Let $I = \int \sec^3 \theta d\theta = \sec \theta \int \sec^2 \theta d\theta - \int \tan \theta (\sec \theta \tan \theta) d\theta$

$$= \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta = \sec \theta \tan \theta - I + \ln |\sec \theta + \tan \theta|$$

$$\Rightarrow I = \frac{1}{2} [\sec \theta \tan \theta] + \frac{1}{2} \ln |\sec \theta + \tan \theta| + c.$$

Illustration 25. Evaluate $\int \ln(\sqrt{1-x} + \sqrt{1+x}) dx$.

Solution: Here we have only one function. If we take $u = \ln(\sqrt{1-x} + \sqrt{1+x})$ as the first function and $v = 1$ as the second function then

$$I = \ln(\sqrt{1-x} + \sqrt{1+x}) \int 1 dx - \int \left(\frac{d}{dx} (\ln(\sqrt{1-x} + \sqrt{1+x})) \right) \int 1 dx dx$$

$$\begin{aligned}
 &= x \ln(\sqrt{1-x} + \sqrt{1+x}) - \int \frac{1}{(\sqrt{1-x} + \sqrt{1+x})} \left(-\frac{1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}} \right) dx \\
 &= x \ln(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2} \int x \frac{\sqrt{1-x^2} - 1}{x\sqrt{1-x^2}} dx \\
 &= x \ln(\sqrt{1-x} + \sqrt{1+x}) - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx \\
 &= x \ln(\sqrt{1-x} + \sqrt{1+x}) - \frac{x}{2} + \frac{1}{2} \sin^{-1} x + c.
 \end{aligned}$$

Illustration 26. Evaluate $\int \frac{(\sqrt{x^2+1}) \{ \ln(x^2+1) - 2 \ln x \}}{x^4} dx$.

Solution:

First we make the substitution $1 + 1/x^2 = t$

$\Rightarrow -\frac{2}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$, so that we write the given integral as

$$I = \int \sqrt{1 + \frac{1}{x^2}} \ln \left(\frac{x^2+1}{x^2} \right) \frac{dx}{x^3} = -\frac{1}{2} \int \sqrt{t} \ln t dt.$$

Now, this integral can be evaluated by parts, taking $\ln t$ as the first function and $\sqrt{t}$ as the second function.

$$\text{Hence } I = -\frac{1}{2} \left[\frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \int t^{3/2} \cdot \frac{1}{t} dt \right] = \frac{t^{3/2}}{9} [2 - 3 \ln t] + c.$$

$$\text{Returning to } x, \text{ we obtain } I = \frac{(x^2+1) \sqrt{x^2+1}}{9x^3} \left[2 - 3 \ln \left(1 + \frac{1}{x^2} \right) \right] + c.$$

- **An important result:** In the integral $\int g(x)e^x dx$, if $g(x)$ can be expressed as $g(x) = f(x) + f'(x)$ then $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$.

Examples: (i) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \frac{e^x}{x} + c,$

(ii) $\int e^x (\sin x + \cos x) dx = e^x \sin x + c,$

(iii) $\int e^x (\cos x - \sin x) dx = e^x \cos x + c.$

Illustration 27. Evaluate $\int e^x \left(\frac{1-x}{1+x^2} \right)^2 dx$.

Solution: Here $\int e^x \left(\frac{1-x}{1+x^2} \right)^2 dx = \int e^x \frac{(1-2x+x^2)}{(1+x^2)^2} dx$

$$= \int e^x \left(\frac{1}{(1+x^2)} - \frac{2x}{(1+x^2)^2} \right) dx = \frac{e^x}{1+x^2} + c.$$

Note: Some times we write $\int \frac{f(x)}{\{p(x)\}^2} = \int \frac{f(x)}{p'(x)} \cdot \frac{p'(x)}{\{p(x)\}^2} dx$ and in order to evaluate the integral

by means of integration by parts, take $\frac{f(x)}{p'(x)}$ as the first function.

Illustration 28. Evaluate $\int \frac{x^2}{(x \sin x + \cos x)^2} dx$.

Solution: Let $I = \int \frac{x^2}{x \cos x} \cdot \frac{(x \cos x)}{(x \sin x + \cos x)^2} dx = \int (x \sec x) \frac{(x \cos x)}{(x \sin x + \cos x)^2} dx$

$$= x \sec x \int \frac{(x \cos x)}{(x \sin x + \cos x)^2} dx - \int \left[\left(\frac{d}{dx} (x \sec x) \right) \left(\int \frac{x \cos x}{(x \sin x + \cos x)^2} dx \right) \right] dx$$

$$= \frac{-x \sec x}{(x \sin x + \cos x)} - \int (\sec x + x \sec x \tan x) \left(\frac{-1}{x \sin x + \cos x} \right) dx$$

$$= \frac{-x \sec x}{(x \sin x + \cos x)} + \int \frac{\sec^2 x (x \sin x + \cos x)}{(x \sin x + \cos x)} dx$$

$$= \frac{-x \sec x}{(x \sin x + \cos x)} + \tan x + c.$$

Exercise 6.

Integrate the following functions:

i) $x^2 \cdot \sin x$,

ii) $x^n \log_e x$, $n \neq -1$,

iii) $e^x \frac{(x-1)}{(1+x)^3}$,

iv) $e^x \sec x (1 + \tan x)$,

v) $x (\log x)^2$,

vi) $\log \log x + \frac{1}{(\log x)^2}$.

Integration by Partial Fractions

A function of the form $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, is called a rational function. Consider the rational function $\frac{x+7}{(2x-3)(3x+4)} = \frac{1}{2x-3} - \frac{1}{3x+4}$.

The two fractions on the RHS are called partial fractions. To integrate the rational function on the LHS, it is enough to integrate the two fractions on the RHS, which are easily integrable. This is known as method of partial fractions. In case the degree of $P(x)$ (numerator) is not less than that of $Q(x)$ (denominator), we carry out the division of $P(x)$ by $Q(x)$ and reduce the degree of the numerator.

In order to write $P(x)/Q(x)$ in partial fractions, first of all we write

$Q(x) = (x-a)^k \cdots (x^2 + \alpha x + \beta)^r \cdots$ where binomials are different, and then set

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} + \frac{M_1 x + N_1}{x^2 + \alpha x + \beta} + \frac{M_2 x + N_2}{(x^2 + \alpha x + \beta)^2} + \cdots + \frac{M_r x + N_r}{(x^2 + \alpha x + \beta)^r} + \cdots$$

where $A_1, A_2, \dots, A_k, M_1, M_2, \dots, M_r, N_1, N_2, \dots, N_r$ are real constants to be determined. These are determined by reducing both sides of the above identity to integral form and equating the

coefficients of equal powers of x , which gives a system of linear equations in the coefficient. (This method is called the method of comparison of coefficients). The constants can also be obtained by substituting suitably chosen numerical values of x in both sides of the identity.

Note: Before proceeding to write a rational function as a sum of partial fractions, we should ascertain that it is either a proper rational fraction or is rewritten as one.

A rational function $\frac{P(x)}{Q(x)}$ is proper if the degree of polynomial $Q(x)$ is greater than the degree of

the polynomial $P(x)$. In case the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, we first write $\frac{P(x)}{Q(x)} = h(x) + \frac{p(x)}{Q(x)}$, where $h(x)$ is a polynomial and $p(x)$ is a polynomial of degree

less than the degree of polynomial $Q(x)$. Please go through the **Illustration -29** carefully.

Illustration 29. Evaluate $\int \frac{dx}{(x+2)(x^2+1)}$.

Solution: we write $\frac{1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{x^2+1}$ (ensure proper substitution)

or $1 = A(x^2+1) + (Bx+C)(x+2)$.

We now take $x = -2, x = 0, x = 1$.

This gives $A = \frac{1}{5}, B = \frac{-1}{5}, C = \frac{2}{5}$

$$\begin{aligned} \Rightarrow \int \frac{dx}{(x+2)(x^2+1)} &= \frac{1}{5} \int \frac{dx}{x+2} + \frac{1}{5} \int \frac{2-x}{x^2+1} dx \\ &= \frac{1}{5} \ln|(x+2)| - \frac{1}{10} \int \frac{2x}{x^2+1} dx + \frac{2}{5} \int \frac{dx}{x^2+1} \\ &= \frac{1}{5} \ln|(x+2)| - \frac{1}{10} \ln|(x^2+1)| + \frac{2}{5} \tan^{-1}(x) + c. \end{aligned}$$

Illustration 30. Evaluate $\int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x} dx$.

Solution: Since the power of the numerator is higher than that of the denominator, i.e., the fraction is improper, we have to single out the integral part. Dividing the numerator by the denominator we obtain:

$$\frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x} = x + 1 - \frac{x+2}{x(x^2-x-2)}.$$

$$\text{Hence } I = \int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x} dx = \int (x+1) dx - \int \frac{(x+2) dx}{x(x-2)(x+1)}.$$

Expand the remaining proper fraction into simple ones i.e.

$$\frac{x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

Hence, $x+2 = A(x-2)(x+1) + Bx(x+1) + C(x-2)$.

Substituting, in turn, the values 0, 2 and -1 of x

(the roots of the denominator) into both sides of the equality, we obtain
 $A = -1, B = 2/3, C = 1/3$

$$\Rightarrow I = \int (x+1) dx + \int \frac{dx}{x} - \frac{2}{3} \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{x+1}$$

$$= \frac{x^2}{2} + x + \ln|x| - \frac{2}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + c.$$

Illustration 31. Evaluate $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2(x+1)} dx$.

Solution: Here $\frac{x^3 + 3x + 2}{(x^2 + 1)^2(x+1)} = \frac{x(x^2 + 1) + 2(x+1)}{(x^2 + 1)^2(x+1)}$

$$= \frac{x}{(x^2 + 1)(x+1)} + \frac{2}{(x^2 + 1)^2}.$$

Hence, the given integral

$$= \frac{1}{2} \int \frac{x+1}{x^2 + 1} dx - \frac{1}{2} \int \frac{1}{x+1} dx + 2 \int \frac{1}{(x^2 + 1)^2} dx$$

$$= \frac{1}{4} \int \frac{2x}{x^2 + 1} dx + \frac{1}{2} \int \frac{1}{x^2 + 1} dx - \frac{1}{2} \ln|x+1| + 2I \quad (\text{where } I = \int \frac{dx}{(x^2 + 1)^2}).$$

I can be found by putting $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$\Rightarrow I = \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int \cos^2 \theta d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{4} \frac{2x}{1+x^2}.$$

Hence, $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2(x+1)} dx$

$$= \frac{1}{4} \ln(x^2 + 1) + \frac{1}{2} \tan^{-1} x - \frac{1}{2} \ln|x+1| + \tan^{-1} x + \frac{x}{x^2 + 1} + c$$

$$= \frac{1}{4} \ln(x^2 + 1) + \frac{3}{2} \tan^{-1} x - \frac{1}{2} \ln|x+1| + \frac{x}{x^2 + 1} + c.$$

Note: Sometimes problems involving partial fractions can be easily solved by simple substitutions. See the following example.

Illustration 32. Evaluate $\int \frac{dx}{(x-b)^3(x-a)^2}$.

Solution: Put $x - b = t(x - a)$

$$\Rightarrow x = \frac{at - b}{t - 1} \Rightarrow dx = \frac{b - a}{(t - 1)^2} dt.$$

Also $x - a = \frac{a - b}{t - 1}$ and $x - b = \frac{(a - b)t}{t - 1}$.

Hence $I = \int \frac{(t - 1)^3}{(a - b)^3 t^3} \cdot \frac{(t - 1)^2}{(a - b)^2} \cdot \frac{b - a}{(t - 1)^2} dt$

$$= -\frac{1}{(a-b)^4} \int \frac{t^3 - 3t^2 + 3t - 1}{t^3} dt = -\frac{1}{(a-b)^4} \int \left(1 - \frac{3}{t} + \frac{3}{t^2} - \frac{1}{t^3}\right) dt$$

$$= -\frac{1}{(a-b)^4} \left[t - 3 \ln |t| - \frac{3}{t} + \frac{1}{2t^2} \right] \text{ where } t = \frac{x-b}{x-a}.$$

Exercise 7.**Integrate the following functions:**

i) $\frac{1}{3+2x-x^2}, x < 3$

ii) $\frac{1}{\sin x (3+2 \cos x)}$

iii) $\frac{1}{1+3e^x+2e^{2x}}$

iv) $\frac{\sin x}{\sin 4x}$

v) $\frac{x^2+1}{(x-1)^2(x+3)}$

vi) $\frac{(2x+1)}{(x+1)(x-2)}$

Algebraic Integrals

■ $I = \int \frac{dx}{ax^2 + bx + c}.$

Here $ax^2 + bx + c = a \left[x^2 + \frac{b}{a}x + \frac{c}{a} \right]$

$$= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right].$$

Substituting $x + \frac{b}{2a} = t$ and $\frac{4ac - b^2}{4a^2} = \pm p^2$, we have

$$I = \frac{1}{a} \int \frac{dt}{t^2 \pm p^2}.$$

■ $I = \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$ Using the substitution as above, we have

$$I = \frac{1}{\sqrt{|a|}} \int \frac{dt}{\sqrt{\pm t^2 \pm p^2}}$$

■ $I = \int \sqrt{ax^2 + bx + c} dx$ is reduced to

$$I = \sqrt{|a|} \int \sqrt{\pm t^2 \pm p^2} dt.$$

■ $\int \frac{px + q}{(ax^2 + bx + c)} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int (px + q) \sqrt{ax^2 + bx + c} dx$

In these type of integrals we write $px + q = \ell$ (diff. coefficient of $ax^2 + bx + c$) + m .Find ℓ and m by comparing the coefficient of x and the constant term on both sides of the identity.

In this way the question will reduce to the sum of two integrals which can be integrated easily (one of these integrals will be of the type given above).

Integrals of the type $\frac{ax^2 + bx + c}{px^2 + qx + r}$ or $\frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}}$

In this case substitute $ax^2 + bx + c = M(px^2 + qx + r) + N(2px + q) + R$.

Find M, N, & R. The integration reduces to integration of three independent functions.

In case of $\frac{p(x)}{ax^2 + bx + c}$, where $p(x)$ is a polynomial, we divide $p(x)$ by $ax^2 + bx + c$ and

obtain $\frac{p(x)}{ax^2 + bx + c} = q(x) + \frac{\text{linear polynomial}}{ax^2 + bx + c}$, where $q(x)$ is a polynomial in x .

Illustration 33. Evaluate $\int \frac{2 \sin 2\theta - \cos \theta}{6 - \cos^2 \theta - 4 \sin \theta} d\theta$.

Solution: Let $I = \int \frac{2 \sin 2\theta - \cos \theta}{6 - \cos^2 \theta - 4 \sin \theta} d\theta$
 $= \int \frac{(4 \sin \theta - 1) \cos \theta}{\sin^2 \theta - 4 \sin \theta + 5} d\theta$.
 Put $\sin \theta = t$
 $\Rightarrow I = \int \frac{4t - 1}{t^2 - 4t + 5} dt$
 $= 2 \int \frac{2t - 4}{t^2 - 4t + 5} dt + 7 \int \frac{dt}{t^2 - 4t + 5}$
 $= 2 \log |t^2 - 4t + 5| + 7 \tan^{-1}(t - 2) + c$ where $t = \sin \theta$.

Illustration 34. Evaluate $\int (2x - 4) \sqrt{4 + 3x - x^2} dx$.

Solution: Write $(2x - 4) = \ell$ (d.c. of $4 + 3x - x^2$) + m ,
 or $(2x - 4) = \ell(-2x + 3) + m$.
 Comparing the coefficients, we get
 $2\ell = 2, 3\ell + m = -4 \Rightarrow \ell = -1, m = -1$.
 Hence $I = -\int (-2x + 3) \sqrt{4 + 3x - x^2} dx - \int \sqrt{4 + 3x - x^2} dx$
 $= -\frac{2}{3} (4 + 3x - x^2)^{3/2} - \int \sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}$
 $= -\frac{2}{3} (4 + 3x - x^2)^{3/2} - \left[\frac{\left(x - \frac{3}{2}\right)}{2} \sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2} + \frac{25}{8} \sin^{-1} \left(\frac{2x - 3}{5} \right) \right] + c$.

Exercise 8.

Evaluate the following:

- i) $\int (x + 2) \sqrt{x^2 + 1} dx$, ii) $\int \frac{x + 1}{\sqrt{2x - x^2}} dx$, iii) $\int \left(\frac{2x + 1}{\sqrt{7 + 6x - 3x^2}} \right) dx$,
 iv) $\int \frac{dx}{9x^2 + 6x + 10}$, v) $\int \frac{dx}{\sqrt{16 - 6x - x^2}}$, vi) $\int \frac{(x + 2) dx}{\sqrt{3 + 2x + x^2}}$.

Integration of Irrational Algebraic Fractions:

(i). Irrational functions of $(ax+b)^{1/n}$ and x can be easily evaluated by the substitution

$$t^n = ax + b. \text{ Thus } \int f(x, (ax+b)^{1/n}) dx = \int f\left(\frac{t^n - b}{a}, t\right) \frac{nt^{n-1}}{a} dt.$$

(ii). For $\int \frac{dx}{(x-k)^f \sqrt{ax^2 + bx + c}}$, we substitute, $x - k = 1/t$.

This substitution will reduce the given integral to $-\int \frac{t^{f-1} dt}{\sqrt{At^2 + Bt + C}}$.

(iii). To integrate $\frac{1}{(Ax^2 + B)\sqrt{Cx^2 + D}}$, we first put $x = 1/t$, so that

$$\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}} = \int \frac{-1/t^2 dt}{(A/t^2 + B)\sqrt{C/t^2 + D}} = -\int \frac{t dt}{(A + Bt^2)\sqrt{C + Dt^2}}.$$

Now the substitution $C + Dt^2 = u^2$ reduces it to the form $\int \frac{du}{u^2 \pm a^2}$.

(iv). To evaluate $\int \frac{(ax^2 + bx + c) dx}{(dx + e)\sqrt{fx^2 + gx + h}}$

we write, $ax^2 + bx + c = A_1(dx + e)(2fx + g) + B_1(dx + e) + C_1$

where A_1 , B_1 and C_1 are constants which can be obtained by comparing the coefficients, of like terms on both sides. And given integral will reduce to the form

$$A_1 \int \frac{(2fx + g) dx}{\sqrt{fx^2 + gx + h}} + B_1 \int \frac{dx}{\sqrt{fx^2 + gx + h}} + C_1 \int \frac{dx}{(dx + e)\sqrt{fx^2 + gx + h}}.$$

Illustration 35. Evaluate $\int \frac{x}{(x-3)\sqrt{x+1}} dx$.

Solution: Put $x + 1 = t^2$. We get

$$\begin{aligned} I &= \int \frac{(t^2 - 1)2t dt}{(t^2 - 4)t} = 2 \int \frac{t^2 - 1}{t^2 - 4} dt \\ &= 2 \int \left\{ 1 + \frac{3}{t^2 - 4} \right\} dt = 2t + \frac{3}{2} \ln \left| \frac{t-2}{t+2} \right| + c \\ &= 2\sqrt{x+1} + \frac{3}{2} \ln \left| \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right| + c. \end{aligned}$$

Illustration 36. Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2 + x + 1}}$, $x > 1$.

Solution: Put $x - 1 = 1/t$ and $dx = -1/t^2 dt$.

$$\begin{aligned}
 \text{We get } I &= -\int \frac{dt}{\sqrt{3t^2 + 3t + 1}} = -\frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 + \frac{1}{12}}} \\
 &= -\frac{1}{\sqrt{3}} \ln \left| \left(t + \frac{1}{2}\right) + \sqrt{\left(t + \frac{1}{2}\right)^2 + \frac{1}{12}} \right| + c \\
 &= -\frac{1}{\sqrt{3}} \ln \left| \frac{1}{x-1} + \frac{1}{2} + \sqrt{\frac{12\left(\frac{1}{x-1} + \frac{1}{2}\right)^2 + 1}{12}} \right| + c
 \end{aligned}$$

Illustration 37. Evaluate $\int \frac{dx}{(x-3)^3 \sqrt{x^2 - 6x + 10}}$.

Solution:

Substituting $(x-3) = 1/t$

$$\Rightarrow dx = -1/t^2 dt$$

$$\text{we get, } \int \frac{dx}{(x-3)^3 \sqrt{x^2 - 6x + 10}}$$

$$= -\int \frac{t^2 dt}{\sqrt{1+t^2}} = \int \frac{dt}{\sqrt{1+t^2}} - \int \sqrt{1+t^2} dt$$

$$= \ln(t + \sqrt{1+t^2}) - \frac{t}{2} \sqrt{1+t^2} - \frac{1}{2} \ln(t + \sqrt{1+t^2}) + c$$

$$= \frac{1}{2} \ln(t + \sqrt{1+t^2}) - \frac{t}{2} \sqrt{1+t^2} + c$$

$$= \frac{1}{2} \left(\ln \left(\frac{1 + \sqrt{x^2 - 6x + 10}}{|x-3|} \right) \right) - \frac{\sqrt{x^2 - 6x + 10}}{2(x-3)^2} + c.$$

Illustration 38. Evaluate $\int \frac{x^3 + x^2 + x - 1}{(x^2 - 1)\sqrt{2 - x^2}} dx$.

Solution:

$$\text{Here } \int \frac{x^3 + x^2 + x - 1}{(x^2 - 1)\sqrt{2 - x^2}}$$

$$= \int \frac{x}{\sqrt{2-x^2}} dx + \int \frac{dx}{\sqrt{2-x^2}} + \int \frac{dx}{(x+1)\sqrt{2-x^2}} + \int \frac{dx}{(x-1)\sqrt{2-x^2}}$$

Put $x+1 = \frac{1}{t}$

Put $x-1 = \frac{1}{u}$

$$= -\sqrt{2-x^2} + \sin^{-1} \frac{x}{\sqrt{2}} + \log_e \left| \frac{x+1}{2+x+\sqrt{2-x^2}} \right| + \log_e \left| \frac{x-1}{2-x+\sqrt{2-x^2}} \right| + c.$$

Illustration 39. Evaluate $\int \frac{(2x^2 + 5x + 9)}{(x+1)\sqrt{x^2 + x + 1}} dx$.

Solution: Let $2x^2 + 5x + 9 = A(x+1)(2x+1) + B(x+1) + C$
 $= x^2 \cdot 2A + x(3A+B) + A+B+C$
 $\Rightarrow 2A = 2, 2A+B = 5, A+B+C = 9$
 $\Rightarrow A = 1, B = 2, C = 6.$

$$\begin{aligned} \text{Thus, } \int \frac{(2x^2 + 5x + 9)}{(x+1)\sqrt{x^2 + x + 1}} dx \\ = \int \frac{(2x+1)dx}{\sqrt{x^2 + x + 1}} + 2 \int \frac{dx}{\sqrt{x^2 + x + 1}} + 6 \int \frac{dx}{(x+1)\sqrt{x^2 + x + 1}} \\ = 4 \sqrt{x^2 + x + 1} + 2 \int \frac{dx}{\sqrt{(x+1/2)^2 + 3/4}} + I_1 \end{aligned}$$

$$\text{where } I_1 = 6 \int \frac{dx}{(x+1)\sqrt{x^2 + x + 1}}.$$

Put $x+1 = 1/t$

$$\Rightarrow dx = -1/t^2 dt$$

$$\Rightarrow I_1 = -6 \int \frac{dt}{\sqrt{t^2 - t + 1}} = -6 \int \frac{dt}{\sqrt{(t-1/2)^2 + 3/4}}$$

$$= -6 \ln \left(t - 1/2 + \sqrt{(t-1/2)^2 + 3/4} \right) + c$$

$$= -6 \ln \left(\frac{1-x+\sqrt{x^2+x+1}}{x+1} \right) + c$$

$$\text{Hence, } I = 4\sqrt{x^2 + x + 1} + 2 \ln \left(1+x+\sqrt{x^2 + x + 1} \right) - 6 \ln \left(\frac{1-x+\sqrt{x^2 + x + 1}}{x+1} \right) + c.$$

Trigonometric Integrals

Integrals of the form $\int R(\sin x, \cos x) dx$:

Here R is a rational function of $\sin x$ and $\cos x$. This can be translated into integrals of a rational function by the substitution: $\tan(x/2) = t$. This is the so called universal substitution. In this case

$$\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}, \quad x = 2 \tan^{-1} t; \quad dx = \frac{2dt}{1+t^2}.$$

Sometimes, instead of the substitution $\tan x/2 = t$, it is more advantageous to make the substitution $\cot x/2 = t$.

Universal substitution often leads to very cumbersome calculations. Indicated below are cases where the aim can be achieved with the aid of simpler substitutions.

- (a) If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, substitute $\cos x = t$.
- (b) If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, substitute $\sin x = t$.
- (c) If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, substitute $\tan x = t$.

Integrals of the form:

$$(i) \int \frac{p \cos x + q \sin x + r}{a \cos x + b \sin x + c} dx \quad (ii) \int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx$$

Rule for (i) : In this integral express numerator as l (denominator) + m (d.c. of denominator) + n . Find l , m , n by comparing the coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals

$$l \int \frac{dx}{a \cos x + b \sin x + c} + m \int \frac{\text{d.c. of (Denominator)}}{\text{Denominator}} dx + n \int \frac{dx}{a \cos x + b \sin x + c}.$$

Rule for (ii) : Express numerator as l (denominator) + m (d.c. of denominator) and find l and m as above.

For $I = \int \frac{dx}{a \cos x + b \sin x}$, we write $a = r \cos \theta$, $b = r \sin \theta$

and get $I = \int \frac{dx}{r \cos(x - \theta)} = \frac{1}{r} \int \sec(x - \theta) dx$

$$= \frac{1}{r} \log |\sec(x - \theta) + \tan(x - \theta)| + c$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \frac{b}{a}$.

Alternatively, we write $\tan \frac{x}{2} = t$ and get

$$I = \int \frac{2 dt}{a + 2bt - at^2} = \frac{-2}{a} \int \frac{dt}{\left(t - \frac{b}{a}\right)^2 - \left(\frac{a^2 + b^2}{a^2}\right)}$$

$$= \frac{-1}{\sqrt{a^2 + b^2}} \log \left| \frac{at - b - \sqrt{a^2 + b^2}}{at - b + \sqrt{a^2 + b^2}} \right| + c.$$

For $I = \int \frac{dx}{a \cos x + b \sin x + c}$, we write $\tan \frac{x}{2} = t$ and follow the procedure given above.

Illustration 40. Evaluate $\int \frac{2 + 3 \cos \theta}{\sin \theta + 2 \cos \theta + 3} d\theta$.

Solution: Write the Numerator = l (denominator) + m (d.c. of denominator) + n

$$\Rightarrow 2 + 3 \cos \theta = l (\sin \theta + 2 \cos \theta + 3) + m (\cos \theta - 2 \sin \theta) + n.$$

Comparing the coefficients of $\sin \theta$, $\cos \theta$ and constant terms, we get

$$3l + n = 2 \quad 2l + m = 3 \quad l - 2m = 0$$

$$\Rightarrow l = 6/5, \quad m = 3/5 \quad \text{and} \quad n = -8/5.$$

$$\text{Hence } I = \int \frac{6}{5} d\theta + \frac{3}{5} \int \frac{\cos \theta - 2 \sin \theta}{\sin \theta + 2 \cos \theta + 3} d\theta - \frac{8}{5} \int \frac{d\theta}{\sin \theta + 2 \cos \theta + 3}$$

$$= \frac{6}{5} \theta + \frac{3}{5} \ln |\sin \theta + 2 \cos \theta + 3| - \frac{8}{5} I_3,$$

where $I_3 = \int \frac{d\theta}{\sin \theta + 2 \cos \theta + 3}.$

$$\text{Put } \tan \frac{\theta}{2} = t$$

$$\Rightarrow I_3 = 2 \int \frac{dt}{t^2 + 2t + 5} = 2 \int \frac{dt}{(t+1)^2 + 2^2}$$

$$= 2 \cdot \frac{1}{2} \cdot \tan^{-1} \left(\frac{t+1}{2} \right) = \tan^{-1} \left(\frac{\tan \theta / 2 + 1}{2} \right).$$

$$\text{Hence } I = \frac{6\theta}{5} + \frac{3}{5} \ln |\sin \theta + 2 \cos \theta + 3| - \frac{8}{5} \tan^{-1} \left(\frac{\tan \theta / 2 + 1}{2} \right) + C.$$

Illustration 41. Evaluate $\int \frac{3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$.

Solution:

Write the Numerator = l(denominator) + m(d.c. of denominator).

Therefore, $3 \sin x + 2 \cos x = l(3 \cos x + 2 \sin x) + m(-3 \sin x + 2 \cos x)$.

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$3 = 2l - 3m, \quad 2 = 3l + 2m \Rightarrow l = 12/13 \text{ and } m = -5/13.$$

$$\text{Hence, } I = l \int dx + m \int \frac{-3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx,$$

$$= \frac{12}{13} \int dx - \frac{5}{13} \int \frac{-3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$$

$$= \frac{12}{13} x - \frac{5}{13} \ln |3 \cos x + 2 \sin x| + c.$$

Illustration 42. Evaluate $\int \frac{dx}{\sin x (2 \cos^2 x - 1)}$.

Solution:

If in the expression $\frac{1}{\sin x (2 \cos^2 x - 1)}$ we substitute $-\sin x$ for $\sin x$, then the

integrand will change its sign. Hence, we take advantage of the substitution

$t = \cos x$; dt

$= -\sin x dx$. This gives

$$I = - \int \frac{dt}{(1-t^2)(2t^2-1)} \text{ and since } \frac{1}{(1-t^2)(2t^2-1)} = \frac{2}{(1-t^2)(2t^2-1)} - \frac{1}{1-t^2},$$

$$I = \frac{-1}{\sqrt{2}} \ln \left| \frac{1+\sqrt{2} \cos x}{1-\sqrt{2} \cos x} \right| + \frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right| + c.$$

Illustration 43. Evaluate $\int \frac{\sin^2 x \cos x}{\sin x + \cos x} dx$.

Solution:

Since the integrand doesn't change sign when $\sin x$ and $\cos x$ do change their

signs, we take advantage of the substitution $t = \tan x$,

$dt = dx/\cos^2 x$.

$$\text{Hence } I = \int \frac{t^2 dt}{(t+1)(t^2+1)^2}. \text{ Expand into partial fractions and solve.}$$

Integration of the type: $\int (\sin^M x \cos^N x) dx$

M & N $\in$ natural numbers.

- If one of them is odd, then substitute for term of even power.
- If both are odd, substitute either of the term.
- If both are even, use trigonometric identities only.

Illustration 44. Evaluate $\int \sin^3 x \cos^5 x \, dx$

Solution: Let $I = \int \sin^3 x \cos^5 x \, dx$
 $I = - \int p^5 (1-p^2) \, dp$
 (if $\cos x = p$, $-\sin x dx = dp$).
 $= \int p^7 \, dp - \int p^5 \, dp = \frac{p^8}{8} - \frac{p^6}{6} = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c.$

Alternatively:

$$I = \int R^3 (1-R^2)^2 \, dR \quad (\text{if } \sin x = R, \cos dx = dR)$$

$$= \int R^3 dR - \int 2R^5 dR + \int R^7 dR = \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + c.$$

Note: This problem can also be handled by successive reduction or by Trigonometric Identities.

Illustration 45. Evaluate $\int \sin^2 x \cos^4 x \, dx$.

Solution: We have $\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2} \right) \left(\frac{\cos 2x+1}{2} \right)^2 dx$
 $= \frac{1}{8} \int (1-\cos 2x)(\cos^2 2x + 2\cos 2x + 1) dx$
 $= \frac{1}{8} \int \left(\frac{\cos 4x+1}{2} \right) dx + \int \frac{2\cos 2x dx}{8} + \int \frac{dx}{8} - \frac{1}{8} \int \cos^3 2x dx$
 $- \frac{1}{4} \int \cos^2 2x dx - \frac{1}{8} \int \cos 2x dx$
 $I = \frac{1}{16} \frac{\sin 4x}{4} + \frac{x}{16} + \frac{\sin 2x}{8} + \frac{x}{8} - \frac{1}{8} \int \frac{\cos 6x + 3\cos 2x}{4} dx$
 $- \frac{1}{8} \left[(\cos 4x + 1) dx - \frac{1}{16} \sin 2x \right]$
 $= \frac{1}{64} \sin 4x + \frac{1}{16} \sin 2x + \frac{3x}{16} - \frac{1}{32} \left[\frac{\sin 6x}{6} + \frac{3\sin 2x}{2} \right] - \frac{1}{32} \sin 4x - \frac{x}{8}$
 $= -\frac{\sin 6x}{192} - \frac{\sin 4x}{64} + \frac{1}{64} \sin 2x + \frac{x}{16} + c.$

Exercise 9.

Find the integrals with respect to x , of the following functions:

$$\begin{array}{lll}
 \text{i)} & \frac{1}{3 \sin x + 4 \cos x}, & \text{ii)} \quad \frac{\sin x}{\sin x - \cos x}, \text{ iii)} \quad \frac{1}{3 + 2 \cos x}, \\
 \text{iv)} & \frac{\sin x}{\sin 3x}, & \text{v)} \quad \sec^4 x, \quad \text{vi)} \quad \frac{\sin x - \cos x}{3 \sin x + 4 \cos x}.
 \end{array}$$

Miscellaneous Exercise:

Find the integrals of the following functions:

$$\begin{array}{ll}
 \text{i)} & \frac{a + b \cos x}{\sin^2 x} \\
 \text{ii)} & \frac{1 + \cos x}{1 - \cos x} \\
 \text{iii)} & x \sqrt{1 + 2x^2} \\
 \text{iv)} & \sec x \log (\sec x + \tan x) \\
 \text{v)} & \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} \\
 \text{vi)} & \frac{1}{(1 + x^2) \sqrt{1 - x^2}} \\
 \text{vii)} & \frac{x \cos^{-1} x}{\sqrt{1 - x^2}} \\
 \text{viii)} & \sqrt{\sin x} \cos^3 x \\
 \text{ix)} & \frac{1}{x^2 (1 + x^4)^{3/4}} \\
 \text{x)} & \frac{x}{(x - 1)(x - 2)}
 \end{array}$$

ANSWERS TO EXERCISES

Exercise 1.

(i) $4 \tan x - 5 \sec x + c$

(iii) $\frac{2}{3} x^{3/2} + 2x^{1/2} + c$

(v) $x^2 - 3 \sin x + e^x + c$

(ii) $2 (\sin x + x \cos \alpha) + c$

(iv) $\tan x + \sec x + c$

(vi) $\frac{1}{2} x^2 - 2x + \log |x| + c.$

Exercise 2.

(i) $\log_e \left| \sec \left(x + \frac{\pi}{4} \right) \right| + c$

(iii) $x^x + c$

(v) $2 \sin (\sqrt{x}) + c$

(ii) $\log_e \sin^{-1} x + c$

(iv) $2 \ln (x^2 + 3x + 100) + c$

(vi) $\tan \log |x| + c$

Exercise 3.

(i) $-\frac{1}{\sqrt{2}} \tan^{-1} \left[\frac{\cot x - 1}{\sqrt{2} \cot x} \right] - \frac{1}{2\sqrt{2}} \ln \left[\frac{\sin x + \cos x - \sqrt{\sin 2x}}{\sin x + \cos x + \sqrt{\sin 2x}} \right] + c$

(ii) $\frac{1}{6} \left[\frac{1}{3} \tan^{-1} \left(\frac{x - \frac{3}{x}}{3} \right) - \frac{1}{2\sqrt{3}} \ln \left| \frac{x + \frac{3}{x} - \sqrt{3}}{x + \frac{3}{x} + \sqrt{3}} \right| \right] + c$

(iii) $\frac{x}{1-x^2} + c$

(v) $\frac{2}{3} (x-8)\sqrt{x+4} + c$

(iv) $\frac{1}{(q-p)} \log |p + (q-p) \sin^2 x| + c$

(vi) $\frac{x}{2} - \frac{1}{2} \log |\cos x + \sin x| + c.$

Exercise 4.

(i) $(x+a) \cos a - (\sin a) \log |\sin (x+a)| + c$

(iii) $\frac{1}{\sqrt{2}} \log \left| \sec \left(x - \frac{\pi}{4} \right) + \tan \left(x - \frac{\pi}{4} \right) \right| + c$

(iv) $e^x \operatorname{cosec} x + c$

(vi) $-\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + c$

(ii) $\frac{1}{2} \sec^2 x + c$

(v) $-\log |\log (\cot x)| + c$

Exercise 5.

(i) $2 \log (1 + \sqrt{x}) + c$

(iii) $a \sin^{-1} \left(\frac{x}{a} \right) + \sqrt{a^2 - x^2} + c$

(v) $\frac{1}{4} (\tan^{-1}(x^2))^2 + c$

(ii) $-\frac{1}{2} \cos (2 \tan^{-1} x) + c$

(iv) $\frac{\log |a^x + \sqrt{1+a^{2x}}|}{\log a} + c$

(vi) $\frac{1}{3} (1+x^2)^{3/2} - (1+x^2)^{1/2} + c$

Exercise 6.

- (i) $-x^2 \cos x + 2x \sin x + 2 \cos x + c$ (ii) $\frac{x^{n+1}}{n+1} \log_e x - \frac{x^{n+1}}{(n+1)^2} + c$
- (iii) $\frac{e^x}{(1+x)^2} + c$ (iv) $e^x \sec x + c$
- (v) $\frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{4} x^2 + c$ (v) $x \log (\log x) - \frac{x}{\log x} + c$

Exercise 7.

- (i) $\frac{1}{4} \log_e \left(\frac{x+1}{3-x} \right) + c$
- (ii) $\frac{1}{10} \log_e (1 - \cos x) - \frac{1}{2} \log_e (1 + \cos x) + \frac{2}{5} \log_e (3 + 2 \cos x) + c$
- (iii) $x + \log_e (1 + e^x) - 2 \log_e (1 + 2e^x) + c$
- (iv) $\frac{1}{8} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| - \frac{1}{4\sqrt{2}} \ln \left| \frac{\sqrt{2} \sin x - 1}{\sqrt{2} \sin x + 1} \right| + c$
- (v) $\frac{3}{8} \log |x-1| - \frac{1}{2(x-1)} + \frac{5}{8} \log |x+3| + c$
- (vi) $\frac{1}{3} \log |x+1| + \frac{5}{3} \log |x-2| + c$

Exercise 8.

- (i) $\frac{1}{3} (x^2 + 1)^{\frac{3}{2}} + \left[x\sqrt{x^2 + 1} + \ln |x + \sqrt{x^2 + 1}| \right] + c$ (ii) $-\sqrt{2x - x^2} + 4 \sin^{-1} \sqrt{\frac{x}{2}} + c$
- (iii) $-\frac{2}{3} \sqrt{7 + 6x - 3x^2} + \sqrt{3} \sin^{-1} \frac{\sqrt{3}(x-1)}{\sqrt{10}} + c$
- (iv) $\frac{1}{9} \tan^{-1} \left(\frac{3x+1}{3} \right) + c$ (v) $\sin^{-1} \left(\frac{x+3}{5} \right) + c$
- (vi) $\frac{1}{2} \log(x^2 + 2x + 3) + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + c.$

Exercise 9.

- (i) $\frac{1}{5} \log \left| \frac{2 \tan x / 2 - 1}{4 - 2 \tan x / 2} \right| + c$ (ii) $\frac{x}{2} + \frac{1}{2} \log |\sin x - \cos x| + c$
- (iii) $\frac{2}{5} \tan^{-1} \left(\frac{1}{\sqrt{5}} \tan \left(\frac{x}{2} \right) \right) + c$ (iv) $\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x} \right| + c$
- (v) $\tan \theta + \frac{1}{3} \tan^3 \theta + c$ (vi) $-\frac{1}{25} x - \frac{7}{25} \ln |3 \sin x + 4 \cos x| + c$

Miscellaneous Exercise:

(i) $-a \cot x - b \operatorname{cosec} x + c$

(ii) $-2 \cot x - 2 \operatorname{cosec} x - x + c$

(iii) $\frac{1}{6}(1+2x^2)^{3/2} + c$

(iv) $\frac{1}{2}[\log(\sec x + \tan x)]^2$

(v) $2 \sin(e^{\sqrt{x}}) + c$

(vi) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right) + c$

(vii) $-\sqrt{1-x^2} \cos^{-1} x - x + c$

(viii) $f23(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + c$

(ix) $-\frac{1}{x}(1+x^4)^{1/4} + c$

(x) $\log\left|\frac{(x-2)^2}{x-1}\right| + c.$

ASSIGNMENTS PROBLEMS

Level – 0

Find the integrals, with respect to x , of the following functions:

1. (i) $\frac{\cos x - \sin x}{\cos x + \sin x}$ (ii) $\frac{1}{\cos(x+a)\cos(x+b)}$
2. (i) $\frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}}$ (ii) $\frac{e^{2x} - 1}{e^{2x} + 1}$
3. (i) $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$ (ii) $\frac{\sec^2(2 \tan^{-1} x)}{1+x^2}$
4. (i) $\frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}$ (ii) $\frac{1}{x \cos^2(1 + \log x)}$
5. (i) $\frac{1}{\sqrt{x(1-2x)}}$ (ii) $\frac{x+2}{x^2 + 2x + 3}$
6. (i) $\frac{x}{\sqrt{8+x-x^2}}$ (ii) $\frac{x+2}{\sqrt{x^2 + 2x - 1}}$
7. (i) $\frac{x^2 - 1}{x^4 + 1}$ (ii) $\frac{x^2 + 1}{x^4 + x^2 + 1}$
8. (i) $x \sin^{-1} x$ (ii) $\frac{2 \cos 2x - \cos x}{6 - \cos^2 x - 4 \sin x}$
9. (i) $x\sqrt{1+x-x^2}$ (ii) $(x-5)\sqrt{x^2+x}$
10. (i) $x^3 \log 2x$ (ii) $\frac{e^x(1 - \sin x)}{1 - \cos x}$
11. (i) $\frac{e^x(2 + \sin 2x)}{1 + \cos 2x}$ (ii) $e^{-x} \sin^2 x$
12. (i) $\tan^{-1} x$ (ii) $\sec^{-1} x$
13. (i) $x^2 \tan^{-1} x$ (ii) $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$
14. (i) $e^{ax} \cos (bx + c)$ (ii) $(2x-1)(2x-3)^{5/2}$
15. (i) $\frac{x^2}{(x^2+2)(x^2+3)}$ (ii) $\frac{x^2}{(x-1)(x-2)(x-3)}$

ANSWERS TO ASSIGNMENT PROBLEMS

Level – 0

1. (i) $\log |\cos x + \sin x| + c$ (ii) $\frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + c$
2. (i) $\frac{1}{4} [\sin^{-1}(x^2)]^2 + c$ (ii) $\log |e^x - e^{-x}| + c$
3. (i) $\frac{1}{ab} \tan^{-1} \left(\frac{a}{b} \tan x \right) + c$ (ii) $\frac{1}{2} \tan(2 \tan^{-1} x) + c$
4. (i) $-\frac{1}{4} \sin^{-1} \left[\frac{1}{3} \cos^2 2x \right] + c$ (ii) $\tan \log(1+x)$
5. (i) $\frac{1}{\sqrt{2}} \sin^{-1}(4x-1) + c$ (ii) $\frac{1}{2} \log |x^2 + 2x + 3| + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + c$
6. (i) $\frac{1}{2} \sin^{-1} \left(\frac{2x-1}{\sqrt{33}} \right) - \sqrt{8-x+x^2} + c$ (ii) $\sqrt{x^2+2x-1} + \log \left| (x+1) + \sqrt{x^2+2x+1} \right| + c$
7. (i) $\frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + c$ (ii) $\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2-1}{\sqrt{3}x} \right) + c$
8. (i) $\frac{x^2}{2} \sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4} \sin^{-1} x + c$ (ii) $-\frac{5}{4} \log |1 + \sin x| + \frac{21}{5} \log |5 + \sin x| + c$
9. (i) $-\frac{1}{3} (1+x-x^2)^{1/2} + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + \frac{1}{8} (2x-1) \sqrt{1+x-x^2} + c$
 (ii) $\frac{1}{3} (x^2+x)^{3/2} - \frac{11}{8} \left[(2x+1) \sqrt{x^2+x} - \frac{1}{2} \log \left| (2x+1) + 2\sqrt{x^2+x} \right| \right] + c$
10. (i) $\frac{1}{4} x^4 \log 2x - \frac{1}{16} x^4 + c$ (ii) $-e^x \cot \left(\frac{x}{2} \right) + c$
11. (i) $e^x \tan x + c$ (ii) $-\frac{1}{2} e^{-x} - \frac{1}{10} e^{-x} (2 \sin 2x - \cos 2x) + c$
12. (i) $x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + c$ (ii) $x \sec^{-1} x - \log |x + \sqrt{x^2-1}| + c$
13. (i) $\frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log(1+x^2) + c$ (ii) $2 \left[x \tan^{-1} x - \log \sqrt{1+x^2} \right] + c$
14. (i) $\frac{e^{ax}}{a^2+b^2} [a \cos(bx+c) + b \sin(bx+c)] + k$
 (ii) $\frac{1}{9} (2x-3)^{9/2} + \frac{2}{7} (2x-3)^{7/2} + c$
15. (i) $-\sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + \sqrt{3} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + c$
 (ii) $1/2 \log |x-1| - 4 \log |x-2| + 9/2 \log |x-3| + c$

SOLVED PROBLEMS

Subjective:

Problem 1. Evaluate $\int \frac{\sin x}{\sqrt{3 \sin^2 x + 4 \cos^2 x}} dx$.

Solution:
$$\int \frac{\sin x}{\sqrt{3 \sin^2 x + 4 \cos^2 x}} dx$$

$$= \int \frac{\sin x}{\sqrt{3 + \cos^2 x}} dx \quad (\text{for } \cos x = p, -\sin x dx = dp)$$

$$= \int \frac{-dp}{\sqrt{3 + p^2}} = -\ln(p + \sqrt{p^2 + 3})$$

$$= -\ln(\cos x + \sqrt{\cos^2 x + 3}) + c.$$

Problem 2. Evaluate $\int \frac{dx}{(x - \beta)\sqrt{(x - \alpha)(\beta - x)}}$.

Solution: Put $x = \alpha \sin^2 \theta + \beta \cos^2 \theta$. Then $dx = 2(\alpha - \beta) \sin \theta \cos \theta d\theta$
and $(x - \alpha) = (\beta - \alpha) \cos^2 \theta$, and $(x - \beta) = (\alpha - \beta) \sin^2 \theta$

$$\Rightarrow I = \int \frac{4(\alpha - \beta) \sin \theta \cos \theta d\theta}{(\alpha - \beta) \sin 2\theta (\sin^2 \theta) (\beta - \alpha)}$$

$$= \frac{2}{\beta - \alpha} \int \operatorname{cosec}^2 \theta d\theta = -\frac{2}{\beta - \alpha} \cot \theta + c$$

$$= -\frac{2}{(\beta - \alpha)} \sqrt{\frac{(x - \alpha)}{(\beta - x)}} + c.$$

Problem 3. Prove that $\int \frac{x+2}{(x^2 + 3x + 3)\sqrt{x+1}} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}\sqrt{x+1}}{-x} \right) + c.$

Solution: Let $f(x) = \int \frac{x+2}{(x^2 + 3x + 3)\sqrt{x+1}} dx$.
Substituting $x + 1 = p^2$, $dx = 2p dp$, we get

$$f(x) = \int \frac{(p^2 + 1)2p dp}{p\{(p^2 - 1)^2 + 3(p^2)\}}$$

$$= 2 \int \frac{(p^2 + 1)}{p^4 + 1 - 2p^2 + 3p^2} dp = 2 \int \frac{(p^2 + 1)dp}{(p^4 + p^2 + 1)}$$

$$= 2 \int \frac{\left(1 + \frac{1}{p^2}\right) dp}{\left(p^2 + 1 + \frac{1}{p^2}\right)}$$

$$\begin{aligned}
 &= 2 \int \frac{\left(1 + \frac{1}{p^2}\right) dp}{\left(p - \frac{1}{p}\right)^2 + 3} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} \left(p - \frac{1}{p}\right) + c \\
 &= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{p^2 - 1}{\sqrt{3}p}\right) + c = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}\sqrt{x+1}}\right) + c \\
 &= \frac{2}{\sqrt{3}} \left[-\tan^{-1} \frac{-x}{\sqrt{3}\sqrt{x+1}}\right] + c = \frac{2}{\sqrt{3}} \left[\cot^{-1} \frac{-x}{\sqrt{3}\sqrt{x+1}} - \frac{\pi}{2}\right] + c \\
 &= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{-\sqrt{3}\sqrt{x+1}}{x}\right) + c_1.
 \end{aligned}$$

Problem 4. Evaluate $\int \frac{\sec x dx}{\sqrt{\sin(2x+A)+\sin A}}$.

Solution: Let $I = \int \frac{\sec x dx}{\sqrt{2\sin(x+A)\cos x}} = \int \frac{\sec^2 x dx}{\sqrt{\frac{2\sin(x+A)}{\cos x}}}$

$$\begin{aligned}
 &= \int \frac{1}{\sqrt{2}} \int \frac{\sec^2 x dx}{\sqrt{\tan x \cos A + \sin A}} \\
 &= \frac{\sec A}{\sqrt{2}} \int \frac{2p dp}{p} \\
 &\text{(If } \tan x \cos A + \sin A = p^2, \text{ then } \cos A \sec^2 x dx = 2p dp) \\
 &\text{i.e. } I = \sqrt{2} \sec A \int dp \\
 &= \sqrt{2} \sec A \sqrt{\tan x \cos A + \sin A} + c.
 \end{aligned}$$

Problem 5. Evaluate $\int \sin^4 x dx$.

Solution: Let $I = \int \sin^3 x \sin x dx = \sin^3 x (-\cos x) + \int 3\sin^2 x \cos^2 x dx$

$$\begin{aligned}
 &= -\sin^3 x \cos x + \int 3\sin^2 x (1 - \sin^2 x) dx \\
 &= -\sin^3 x \cos x + 3 \int \sin^2 x dx - 3 \int \sin^4 x dx \\
 4I &= -\sin^3 x \cos x + 3 \int \left(\frac{1 - \cos 2x}{2}\right) dx \\
 I &= \frac{1}{4} \left[-\sin^3 x \cos x + \frac{3}{2} \left[x - \frac{\sin 2x}{2}\right]\right] + c.
 \end{aligned}$$

Alternative:

$$\begin{aligned}
 I &= \int \sin^4 x dx = \frac{1}{4} \int (1 - \cos 2x)^2 dx \\
 &= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4}[x - \sin 2x] + \frac{1}{4} \int \left(\frac{\cos 4x + 1}{2} \right) dx \\
 &= -\frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + \frac{3x}{8} + c.
 \end{aligned}$$

Problem 6. Evaluate $\int (1 + (\tan x) \{ \tan (x + A) \}) dx$.

Solution: Let $I = \int (1 + (\tan x) \{ \tan (x + A) \}) dx$

$$\begin{aligned}
 &= \int \left[1 + \frac{\sin x \sin(x + A)}{\cos x \cos(x + A)} \right] dx = \int \frac{\cos(x + A - x)}{\cos x \cos(x + A)} dx \\
 &= \frac{\cos A}{\sin A} \int \frac{\sin(x + A - x)}{\cos x \cos(x + A)} dx \\
 &= \cot A \int \frac{\sin(x + A) \cos x - \sin x \cos(x + A)}{\cos x \cos(x + A)} dx \\
 &= \cot A \int \tan(x + A) dx - \cot A \int \tan x dx \\
 &= \cot A [\ln |\sec(x + A)| - \cot A \ln |\sec x|] \\
 &= \cot A \ln \left| \frac{\sec(x + A)}{\sec A} \right| + c.
 \end{aligned}$$

Problem 7. Evaluate $\int \cos 2\theta \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta$.

Solution: Let $I = \int \cos 2\theta \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta$

$$\begin{aligned}
 &= \int \cos 2\theta \ln \left(\frac{1 + \sin 2\theta}{\cos 2\theta} \right) d\theta \\
 &= \ln \frac{1 + \sin 2\theta}{\cos 2\theta} \cdot \frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2} \left\{ \frac{2 \cos 2\theta}{1 + \sin 2\theta} + \frac{2 \sin 2\theta}{\cos 2\theta} \right\} d\theta \\
 &= \frac{\sin 2\theta}{2} \ln \left(\frac{1 + \sin 2\theta}{\cos 2\theta} \right) - \int \tan 2\theta d\theta \\
 &= \frac{\sin 2\theta}{2} \ln \left(\frac{1 + \sin 2\theta}{\cos 2\theta} \right) - \frac{1}{2} \ln |\sec 2\theta| + c.
 \end{aligned}$$

Problem 8. Evaluate $I = \int \frac{(1+x)}{x(1+xe^x)^2} dx$.

Solution: Let $I = \int \frac{(1+x)}{x(1+xe^x)^2} dx$

$$\begin{aligned}
 &= \int \frac{(1+x)e^x}{(xe^x)(1+xe^x)^2} dx \quad (1+xe^x = p, e^x(1+x) dx = dp) \\
 &= \int \frac{dp}{(p-1)p^2} \\
 &\text{Let } \frac{1}{(p-1)p^2} = \frac{A}{(p-1)} + \frac{B}{p} + \frac{C}{p^2}
 \end{aligned}$$

$$\Rightarrow 1 = Ap^2 + B(p)(p-1) + C(p-1).$$

For $p = 1$, $p = 0$, and $p = -1$,

$A = 1$, $C = -1$ and $B = -1$.

$$\Rightarrow I = \int \frac{1}{(p-1)} dp - \int \frac{dp}{p} - \int \frac{dp}{p^2}$$

$$= \ln \frac{(p-1)}{p} + \frac{1}{p} + c = \ln \left(\frac{xe^x}{1+xe^x} \right) + \left(\frac{1}{1+xe^x} \right) + c.$$

Problem 9. Evaluate $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx$, $x > 0$.

Solution: Let $I = \int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx = \int \frac{(x-1)dx}{x\sqrt{x^2-1}}$

$$= \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{x\sqrt{x^2-1}}$$

$$= \ln \left(x + \sqrt{x^2-1} \right) - \sec^{-1} x + c.$$

Problem 10. Evaluate $\int \sqrt{x + \sqrt{x^2 + 2}} dx$.

Solution: Let $x + \sqrt{x^2 + 2} = p$ or $x^2 + 2 = p^2 + x^2 - 2px$

$$\frac{-2 + p^2}{2p} = x \text{ or } dx = \frac{p^2 + 2}{2p^2} dp$$

$$I = \int \frac{(p)^{1/2} (p^2 + 2)}{2p^2} dp = \frac{1}{2} \int p^{1/2} dp + \int p^{-3/2} dp$$

$$= \frac{1}{3} p^{3/2} + p^{-1/2} (-2) = \frac{1}{3} \left(x + \sqrt{x^2 + 2} \right)^{3/2} - 2 \frac{1}{\sqrt{x + \sqrt{x^2 + 2}}} + c.$$

Objective:

Problem 1. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

- (A) $\sin^{-1}(1-x) + c$ (B) $-\cos^{-1}(1-x) + c$
 (C) $\sin^{-1}(x-1) + c$ (D) $\cos^{-1}(x-1) + c$

Solution: Let $I = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}\left(\frac{x-1}{1}\right) + c$.

Hence (C) is the correct answer.

Problem 2. If $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = f(x) + c$ then $f(x)$ is equal to

- (A) $\sqrt{2} \sin^{-1}(\sin x - \cos x)$ (B) $\frac{\pi}{\sqrt{2}} - \sqrt{2} \cos^{-1}(\sin x - \cos x)$
 (C) $\sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right)$ (D) none of these

Solution: Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int \sqrt{2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$.

If $\sin x - \cos x = p$ then $(\cos x + \sin x)$

$dx = dp$ so that

$$I = \sqrt{2} \int \frac{dp}{\sqrt{1-p^2}} = \sqrt{2} \sin^{-1} p + c = \sqrt{2} \sin^{-1}(\sin x - \cos x) + c$$

$$= \frac{\pi}{\sqrt{2}} - \sqrt{2} \cos^{-1}(\sin x - \cos x)$$

$$= \sqrt{2} \tan^{-1} \frac{\sin x - \cos x}{\sqrt{1-(\sin x - \cos x)^2}} = \sqrt{2} \tan^{-1} \frac{\sin x - \cos x}{\sqrt{2 \sin x \cos x}}$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right).$$

Hence (A), (B), (C) are correct answers.

Problem 3. If $I = \int \frac{1}{2p} \sqrt{\frac{p-1}{p+1}} dp = f(p) + c$, then $f(p)$ is equal to

- (A) $\frac{1}{2} \ln(p - \sqrt{p^2 - 1})$ (B) $\left(\frac{1}{2} \cos^{-1} p + \frac{1}{2} \sec^{-1} p\right)$
 (C) $\ln \sqrt{p + \sqrt{p^2 - 1}} - \frac{1}{2} \sec^{-1} p$ (D) none of these

Solution: Let $I = \int \frac{1}{2p} \sqrt{\frac{p-1}{p+1}} dp$

$$\begin{aligned}
 &= \frac{1}{2} \int \frac{p-1}{p\sqrt{(p+1)(p-1)}} dp = \frac{1}{2} \int \frac{pdp}{p\sqrt{p^2-1}} - \frac{1}{2} \int \frac{dp}{p\sqrt{p^2-1}} \\
 &= \frac{1}{2} \log_e (p + \sqrt{p^2-1}) - \frac{1}{2} \sec^{-1} p.
 \end{aligned}$$

Hence (C) is the correct answer.

Problem 4.

If $\int \sqrt{\frac{\cos x - \cos^3 x}{(1 - \cos^3 x)}} dx = f(x) + c$, then $f(x)$ is equal to

- (A) $\frac{2}{3} \sin^{-1} \left(\cos^{\frac{3}{2}} x \right)$ (B) $\frac{3}{2} \sin^{-1} \left(\cos^{\frac{3}{2}} x \right)$
 (C) $\frac{2}{3} \cos^{-1} \left(\cos^{\frac{3}{2}} x \right)$ (D) $-\frac{2}{3} \sin^{-1} \left(\cos^{\frac{3}{2}} x \right)$

Solution:

$$\text{Let } I = \int \sqrt{\frac{\cos x - \cos^3 x}{(1 - \cos^3 x)}} dx = \int \frac{\sqrt{\cos x} (\sqrt{1 - \cos^2 x})}{\sqrt{1 - \left(\cos^{\frac{3}{2}} x \right)^2}} dx$$

$$\text{If } \cos^{\frac{3}{2}} x = p, \text{ then } \left(-\frac{3}{2} \cos^{\frac{1}{2}} x \sin x \right) dx = dp$$

$$I = -\frac{2}{3} \int \frac{dp}{\sqrt{1-p^2}} = -\frac{2}{3} \sin^{-1} \left(\cos^{\frac{3}{2}} x \right) + c = -\frac{2}{3} \cos^{-1} \left(\cos^{\frac{3}{2}} x \right) + c_1$$

Hence (C) and (D) are the correct answers.

Problem 5.

$\int \frac{dx}{x(x^n + 1)}$ is equal to

- (A) $\frac{1}{n} \log_e \left(\frac{x^n}{x^n + 1} \right) + c$ (B) $-\frac{1}{n} \log_e \left(\frac{x^n + 1}{x^n} \right) + c$
 (C) $\log_e \left(\frac{x^n}{x^n + 1} \right) + c$ (D) none of these

Solution:

$$\text{Let } I = \int \frac{dx}{x(x^n + 1)} = \int \frac{dx}{x^{n+1} \left(1 + \frac{1}{x^n} \right)}.$$

$$\text{If } \left(1 + \frac{1}{x^n} \right) = p, \text{ then } -\frac{n}{x^{n+1}} dx = dp$$

$$\Rightarrow I = -\frac{1}{n} \int \frac{dp}{p} = -\frac{1}{n} \log_e p = -\frac{1}{n} \log_e \left(\frac{x^n + 1}{x^n} \right).$$

Hence (B) is the correct answer.

Problem 6. $\int \frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$ is equal to

(A) $\frac{1+\sqrt{x}}{(1-x)^2} + c$

(B) $\frac{1+\sqrt{x}}{(1+x)^2} + c$

(C) $\frac{1-\sqrt{x}}{(1-x)^2} + c$

(D) $\frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}} + c$

Solution:

Let $I = \int \frac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$

If $\sqrt{x} = \sin p$, then $\frac{1}{2\sqrt{x}} dx = \cos p dp$

$\Rightarrow I = \int \frac{2 \sin p \cos p dp}{(1+\sin p) \sin p \cos p}$

$= 2 \int \frac{dp}{(1+\sin p)} = 2 \int \frac{(1-\sin p) dp}{\cos^2 p}$

$= 2 \left\{ \int \sec^2 p dp - \int (\tan p \sec p) dp \right\}$

$= 2 (\tan p - \sec p) = 2 \left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}} \right) + c$

$= \frac{2(\sqrt{x}-1)}{\sqrt{(1-x)}} + c.$

Hence (D) is the correct answer.

Problem 7. $\int \frac{x^4-1}{x^2(x^4+x^2+1)^{1/2}} dx$ is equal to

(A) $\sqrt{\frac{x^4+x^2+1}{x}} + c$

(B) $\sqrt{x^2+1+\frac{1}{x^2}} + c$

(C) $\sqrt{\frac{x^4+x^2+1}{x^2}} + c$

(D) none of these

Solution:

Let $I = \int \frac{x^4-1}{x^2(x^4+x^2+1)^{1/2}} dx$

$= \int \frac{x^3 \left(x - \frac{1}{x^3} \right) dx}{x^3 \sqrt{x^2 + \frac{1}{x^2} + 1}}$

If $x^2 + \frac{1}{x^2} + 1 = p^2$, then $\left(2x - \frac{2}{x^3} \right) dx = 2p dp$

$\Rightarrow I = \int \frac{p dp}{p} = p + c = \sqrt{\frac{x^4+x^2+1}{x^2}} + c.$

Hence (B) and (C) are correct answers.

- Problem 8.** $\int \frac{dx}{\cos^6 x + \sin^6 x}$ is equal to
- (A) $\log_e (\tan x - \cot x) + c$ (B) $\log_e (\cot x - \tan x) + c$
 (C) $\tan^{-1} (\tan x - \cot x) + c$ (D) $\tan^{-1} (-2 \cot 2x) + c$

Solution: Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x}{1 + \tan^6 x} dx$
 $= \int \frac{(1 + \tan^2 x)^2 \sec^2 x dx}{1 + \tan^6 x}$
 If $\tan x = p$, then $\sec^2 x dx = dp$
 $\Rightarrow I = \int \frac{(1 + p^2)^2 dp}{1 + p^6} = \int \frac{(1 + p^2)}{p^4 - p^2 + 1} dp$
 $= \int \frac{p^2 \left(1 + \frac{1}{p^2}\right)}{p^2 \left(p^2 + \frac{1}{p^2} - 1\right)} dp$
 $= \int \frac{dk}{k^2 + 1} = \tan^{-1}(k) + c \quad \left(p - \frac{1}{p} = k, \quad \left(1 + \frac{1}{p^2}\right) dp = dk\right)$
 $= \tan^{-1} \left(p - \frac{1}{p}\right) + c = \tan^{-1} (\tan x - \cot x) + c$
 $= \tan^{-1} (-2 \cot 2x) + c$
 Hence (C) and (D) are correct answers.

- Problem 9.** $\int \frac{dx}{(x+a)^{\frac{8}{7}} (x-b)^{\frac{6}{7}}}$ is equal to
- (A) $\left(\frac{7}{a+b}\right) \left(\frac{x+a}{x-b}\right)^{\frac{1}{7}} + c$ (B) $\left(\frac{7}{a+b}\right) \left(\frac{x-b}{x+a}\right)^{\frac{1}{7}} + c$
 (C) $\frac{6}{a+b} \left(\frac{x-b}{x+a}\right)^{\frac{1}{6}} + c$ (D) $\frac{6}{a+b} \left(\frac{x+a}{x-b}\right)^{\frac{1}{6}} + c$

Solution: Let $I = \int \frac{dx}{(x+a)^{\frac{8}{7}} (x-b)^{\frac{6}{7}}} = \int \frac{dx}{(x+a)^2 \left(\frac{x-b}{x+a}\right)^{\frac{6}{7}}}$
 If $\left(\frac{x-b}{x+a}\right) = p$, then $\frac{a+b}{(x+a)^2} dx = dp \Rightarrow I = \frac{1}{a+b} \int \frac{dp}{p^{\frac{6}{7}}}$
 $= \frac{7}{a+b} \left(p^{\frac{1}{7}}\right) = \left(\frac{7}{a+b}\right) \left(\frac{x-b}{x+a}\right)^{\frac{1}{7}} + c$
 Hence (B) is the correct answer.

Problem 10. $\int \frac{3+2\cos x}{(2+3\cos x)^2} dx$ is equal to

(A) $\left(\frac{\sin x}{3\cos x + 2}\right) + c$

(B) $\left(\frac{2\cos x}{3\sin x + 2}\right) + c$

(C) $\left(\frac{2\cos x}{3\cos x + 2}\right) + c$

(D) $\left(\frac{2\sin x}{3\sin x + 2}\right) + c$

Solution:

Let $I = \int \frac{3+2\cos x}{(2+3\cos x)^2} dx$.

Multiplying Nr. & Dr. by $\operatorname{cosec}^2 x$

$$\Rightarrow I = \int \frac{(3\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x)}{(2\operatorname{cosec} x + 3\cot x)^2} dx$$

$$= - \int \frac{-3\operatorname{cosec}^2 x - 2\cot x \operatorname{cosec} x}{(2\operatorname{cosec} x + 3\cot x)^2} dx$$

$$= \frac{1}{2\operatorname{cosec} x + 3\cot x} = \left(\frac{\sin x}{2+3\cos x}\right) + c.$$

Hence (A) is the correct answer.

Alternative:

$$I = \int \frac{3\sin^2 x + 3\cos^2 x + 2\cos x}{(2+3\cos x)^2} dx$$

$$= \int \frac{\cos x}{(2+3\cos x)} dx + \int \frac{3\sin x \cdot \sin x}{(2+3\cos x)^2} dx$$

$$= \int \frac{\cos x}{2+3\cos x} dx + \frac{\sin x}{2+3\cos x} - \int \frac{\cos x}{2+3\cos x} dx = \frac{\sin x}{2+3\cos x} + c.$$

Problem 11. $\int \left(x + \frac{1}{x}\right)^{n+5} \left(\frac{x^2-1}{x^2}\right) dx$ is equal to

(A) $\frac{\left(x + \frac{1}{x}\right)^{n+6}}{n+6} + c$

(B) $\left(\frac{x^2+1}{x^2}\right)^{n+6} (n+6) + c$

(C) $\left(\frac{x}{x^2+1}\right)^{n+6} (n+6) + c$

(D) none of these.

Solution:

Here $I = \int p^{n+5} dp$.

If $x + \frac{1}{x} = p$ then, $\left(1 - \frac{1}{x^2}\right) dx = dp$,

$$\Rightarrow I = \int \left(x + \frac{1}{x}\right)^{n+5} \left(\frac{x^2-1}{x^2}\right) dx = \int p^{n+5} dp$$

$$= \frac{p^{n+6}}{n+6} + c = \frac{\left(x + \frac{1}{x}\right)^{n+6}}{n+6} + c. \text{ Hence (A) is the correct answer.}$$

- Problem 12.** $\int \frac{dx}{x + \sqrt{a^2 - x^2}}$ is equal to
- (A) $\frac{1}{2} \sin^{-1} \frac{x}{a} + \ln \sqrt{x + \sqrt{a^2 - x^2}} + c$ (B) $\frac{1}{2} \sin^{-1} \frac{x}{a} + \ln(x\sqrt{a^2 - x^2}) + c$
- (C) $\frac{1}{2} \sin^{-1} \frac{x}{a} + \ln(\sqrt{a + \sqrt{a^2 - x^2}}) + c$ (D) none of these.

Solution: Here $I = \int \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$ ($x = a \sin \theta$, $dx = a \cos \theta d\theta$)

$$= \frac{1}{2} \int d\theta + \frac{1}{2} \int \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} \theta + \frac{1}{2} \ln(\sin \theta + \cos \theta)$$

$$= \frac{1}{2} \sin^{-1} \frac{x}{a} + \ln \sqrt{x + \sqrt{a^2 - x^2}} + c.$$

Hence (A) is the correct answer.

- Problem 13.** $\int \frac{dx}{\sqrt{x+1} + (x+1)^{\frac{1}{3}}}$ is equal to
- (A) $2p^3 - 3p^2 + 6p - 6 \ln(1+p) + c$, where $p = (x+1)^{\frac{1}{6}}$
- (B) $2p^3 + 6p - 6 \ln(1+p) + c$, where $p = (x+1)^{\frac{1}{6}}$
- (C) $2p^3 + 3p^2 + 6p - 6 \ln(1+p) + c$, where $p = (x+1)^{\frac{1}{6}}$
- (D) none of these.

Solution: Here $I = \int \frac{6p^5 dp}{p^3 + p^2}$ ($(x+1) = p^6$, then $dx = 6p^5 dp$).

$$= 6 \int p^2 dp - 6 \int p dp + 6 \int dp - 6 \int \frac{p^2}{p^2(p+1)} dp$$

$$= \frac{6p^3}{3} - \frac{6p^2}{2} + 6p - 6 \ln(p+1) + c$$

$$= 2p^3 - 3p^2 + 6p - 6 \ln(1+p) + c, \text{ where } p = (x+1)^{\frac{1}{6}}.$$

Hence (A) is the correct answer.

- Problem 14** $\int \frac{(x^2 - 1)}{(x^2 + 1)\sqrt{x^4 + 1}} dx$ is equal to
- (A) $\sec^{-1} \left(\frac{x^2 + 1}{\sqrt{2} x} \right) + c$ (B) $\frac{1}{\sqrt{2}} \sec^{-1} \left(\frac{x^2 + 1}{\sqrt{2} x} \right) + c$
- (C) $\frac{1}{\sqrt{2}} \sec^{-1} \left(\frac{x^2 + 1}{\sqrt{2}} \right) + c$ (D) none of these.

Solution:

$$\text{Let } I = \int \frac{x^2 \left(1 - \frac{1}{x^2}\right) dx}{x^2 \left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2}\right)^{1/2}}.$$

$$\text{Let } x + \frac{1}{x} = p \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dp$$

$$\Rightarrow I = \int \frac{dp}{p\sqrt{p^2-2}} = \frac{1}{\sqrt{2}} \sec^{-1} \frac{p}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sec^{-1} \left(\frac{x^2+1}{\sqrt{2}x} \right) + c.$$

Hence (B) is the correct answer.

Problem 15. If $\int x e^x \cos x dx = f(x) + c$, then $f(x)$ is equal to

- (A) $\frac{e^x}{2} \{(1-x) \sin x - x \cos x\} + c$ (B) $\frac{e^x}{2} \{(x-1) \sin x + x \cos x\} + c$
 (C) $\frac{e^x}{2} \{(1+x) \sin x - x \cos x\} + c$ (D) none of these

Solution:Here $I = \text{Rear part of } \int x e^{(1+i)x} dx$

$$= \frac{x e^{(1+i)x}}{(1+i)} - \int \frac{e^{(1+i)x}}{(1+i)} dx = \frac{x e^{(1+i)x}}{(1+i)} - \frac{e^{(1+i)x}}{(1+i)^2}$$

$$= e^{(1+i)x} \left\{ \frac{x(1+i) - 1}{(1+i)^2} \right\}$$

$$= e^x [\cos x + i \sin x] \left[\frac{(x-1) + i x}{1 + 2i - 1} \right] = \frac{e^x}{-2} \{i \cos x - \sin x\} \{(x-1) + i x\}$$

$$= \frac{e^x}{-2} [(1-x) \sin x - x \cos x] + c = \frac{e^x}{2} [x \cos x + (x-1) \sin x] + c.$$

Hence (B) is the correct answer.

ASSIGNMENT PROBLEMS

Subjective:**Level – I**

Integrate the following w. r. t. x

- | | | | | |
|-----|-----|---|------|--|
| 1. | (i) | $\frac{x^2 + x}{\sqrt{2x+1}}$ | (ii) | $\frac{x + \sin x}{1 + \cos x}$ |
| 2. | (i) | $\cos \sqrt{x}$ | (ii) | $\cot^{-1} \sqrt{\frac{1-x}{1+x}}$ |
| 3. | (i) | $e^{-3x} \cos 2x$ | (ii) | $\sin^{-1} \sqrt{\frac{x}{a+x}}$ |
| 4. | (i) | $\frac{\sqrt{1+\ln x}}{x \ln x}$ | (ii) | $\frac{1}{\sqrt{1-\sin^4 x}}$ |
| 5. | (i) | $\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}$ | (ii) | $\frac{1}{(2x+1)(1+\sqrt{2x+1})}$ |
| 6. | (i) | $[(x+a)^{-3}(x+b)^{-5}]^{1/4}$ | (ii) | $\operatorname{cosec} x \sqrt{1+\cos x}$ |
| 7. | (i) | $\frac{e^{2x}}{\sqrt[4]{e^x - 1}}$ | (ii) | $\sqrt{\frac{a-x}{x-b}}$ |
| 8. | (i) | $\sqrt{\tan x} \sec x \operatorname{cosec} x$ | (ii) | $\frac{1}{1+3\sin^2 x}$ |
| 9. | (i) | $\frac{\cos x}{5-4\cos x}$ | (ii) | $\frac{x^3 \tan^{-1} x^4}{1+x^8}$ |
| 10. | (i) | $\frac{1}{(x^2-a^2)(x^2-b^2)}$ | (ii) | $\frac{\sqrt[3]{(1+x^3)}}{x^5}$ |

Level – II

1. Evaluate

$$(i) \int \frac{1}{x^3 \sqrt{x^4 + 1}} dx \quad (ii) \int \left(x + \frac{1}{x}\right)^{3/2} \left(\frac{x^2 - 1}{x^2}\right) dx$$

2. For any natural number m , evaluate $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx, x > 0$.Integrate the following w. r. t. x :

$$3. \frac{\cos x - \cos 2x}{1 - \cos x}$$

$$4. \frac{(a^x + b^x)^2}{a^x b^x}$$

$$5. \frac{8x + 13}{\sqrt{4x + 7}}$$

$$6. \frac{x^3}{(x + 2)}$$

$$7 (i) \sin\left(m \ln \frac{x}{k}\right)$$

$$(ii) \frac{e^x (2 - x^2)}{(1 - x)\sqrt{1 - x^2}}$$

$$8 (i) \frac{x^3}{(2 + x^2)^{1/3}}$$

$$(ii) \sqrt{1 + \cos^2 x} \sin 2x \cos 2x$$

$$9 (i) \frac{\ln \frac{x}{e}}{(\ln x)^2}$$

$$(ii) \frac{x \ln x}{\sqrt{(x^2 - 1)^3}}$$

$$10 (i) \frac{(2 + x^2)^2 \cdot x^3}{(1 + x^2)}$$

$$(ii) \sin x \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

Objective:**Level – I**

- $I = \int e^{\tan^{-1}x} \left(\frac{1+x^2+x}{1+x^2} \right) dx$ is equal to
 (A) $x e^{\tan^{-1}x} + c$ (B) $x^2 e^{\tan^{-1}x} + c$
 (C) $\frac{1}{x} e^{\tan^{-1}x} + c$ (D) none of these
- $I = \int \frac{\cos x + x \sin x}{x(x + \cos x)} dx$ is equal to
 (A) $\log_e \left(\frac{x \sec x}{1 + x \sec x} \right) + c$ (B) $\log_e (1 + x \sec x) + c$
 (C) $\log_e (1 + x \cos x) + c$ (D) $\log_e (x \sec x) + c$
- $I = \int \sec^{2/3} x \operatorname{cosec}^{4/3} x dx$ is equal to
 (A) $3 \tan^{1/3} x + c$ (B) $-3 \tan^{-1/3} x + c$
 (C) $-3 \cot^{-1/3} x + c$ (D) $3 \cot^{-1/3} x + c$
- $I = \int \frac{1}{1 - \cos^4 x} dx$ is equal to
 (A) $2\sqrt{2} (\cot x + \sqrt{2} \tan^{-1} \sqrt{2} \cot x)$ (B) $-\frac{1}{2\sqrt{2}} (\sqrt{2} \cot x + \tan^{-1}(\sqrt{2} \cot x)) + c$
 (C) $2\sqrt{2} \{\cot x + \tan^{-1}(\cos x)\}$ (D) none of these
- $I = \int \frac{dx}{1+e^x}$ is equal to
 (A) $\log_e \left(\frac{1+e^x}{e^x} \right) + c$ (B) $\log_e \left(\frac{e^x}{1+e^x} \right) + c$
 (C) $\log_e (e^x) (e^x + 1) + c$ (D) $\log_e (e^{2x} + 1) + c$
- $I = \int e^{ax} \cos bx dx$ is equal to
 (A) real part of $\int e^{(a+bi)x} dx$ (B) imaginary part of $\int e^{(a+bi)x} dx$
 (C) neither real nor Imaginary part of $\int e^{(a+bi)x} dx$ (D) none of these
- $I = \int (\log_e x)^2 dx$ is equal to
 (A) $x \log_e x (\log_e x + 2) + c$ (B) $x \log_e x (2 \log_e x + 1) + c$
 (C) $x [(\ln x)^2 - 2(\ln x) + 2] + c$ (D) $x \{(\log_e x)^2 - 2(\log_e x - 2)\} + c$

8. $I = \int a^{px} b^{mx} dx$ is equal to
- (A) $(a^{px} b^{mx} \log_e a^p b^m) + c$ (B) $\left(\frac{\log_e a^p b^m}{a^{px} b^{mx}} \right) + c$
- (C) $\left(\frac{a^{px} b^{mx}}{\log_e a^p b^m} \right) + c$ (D) none of these
9. $I = \int \frac{dx}{x \log x \log(\log x)}$ is equal to
- (A) $\log [\log (\log x)] + c$ (B) $\log (\log x) + c$
- (C) $\log [\log (\log (1/x))] + c$ (D) none of these
10. $I = \int \frac{e^x}{(e^x + 1)^{1/4}} dx$ is equal to
- (A) $\frac{e^{2x} - e^{-2x}}{2} + c$ (B) $\frac{e^{2x} + e^{-2x}}{2} + c$
- (C) $\frac{e^x - e^{2x}}{2} + c$ (D) none of these .
11. Let $f(x) = \int e^x (x-1)(x-2) dx$. Then $f(x)$ decreases in the interval
- (A) $(-\infty, -2)$ (B) $(-2, -1)$
- (C) $(1, 2)$ (D) $(2, \infty)$
12. $\int e^x \sin x \cos x \cos 2x \cos 4x dx$.
- (A) $\frac{e^x}{4\sqrt{65}} \sin (8x - \tan^{-1} 8) + c$ (B) $\frac{e^x}{8\sqrt{65}} \sin (8x - \tan^{-1} 8) + c$
- (C) $e^x/8 \sin (8x - \tan^{-1} 8) + c$ (D) none of these
13. If $I = \int \frac{dx}{(2x+5)\sqrt{4x^2+20x+16}} = \frac{1}{m} \sec^{-1} \left(\frac{2x+5}{p} \right) + c$ then
- (A) $m = 1/6, p = 1/3$ (B) $m = -6, p = 3$
- (C) $m = 6, p = 3$ (D) $m = -6, p = -3$
14. $I = \int \frac{e^{x^2} (x^2 - 1)}{2x(x^2 + 1)^3} dx$ is equal to
- (A) $\frac{xe^{x^2}}{(x^2 + 1)^2} + c$ (B) $\frac{xe^{x^2}}{(x^2 + 1)^2} + c$
- (C) $\frac{e^{x^2}}{x(x^2 + 1)^2} + c$ (D) none of these
15. For $x > 1$, $\int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$ is equal to
- (A) $2x \tan^{-1}(x) - \ln(x^2 + 1) + c$ (B) $2x \cot^{-1} x + \log_e(x^2 + 1) + c$
- (C) $2x \tan^{-1}(x) + \log_e(x^2 + 1) + c$ (D) none of these

Level- II

1. $I = \int \frac{dx}{e^x + 4e^{-x}} = f(x) + c$ then $f(x)$ is equal to

(A) $2 \tan^{-1}(2e^x)$

(B) $\frac{1}{2} \tan^{-1}\left(\frac{e^x}{2}\right)$

(C) $2 \tan^{-1} \frac{e^x}{2}$

(D) $\frac{1}{2} \tan^{-1}(2e^{2x})$

2. $I = \int \frac{\log_e(\log_e x)}{x(\log_e x)} dx$ is equal to

(A) $\frac{1}{2} \sqrt{\log_e \log_e x} + c$

(B) $\frac{1}{2} (\log_e \log_e x)^2 + c$

(C) $\frac{1}{2} \log_e \log_e x$

(D) none of these

3. $I = \int \frac{(10x^9 + 10^x \log_e 10)}{(x^{10} + 10^x)} dx$ is equal to

(A) $10^x + x^{10} + c$

(B) $10^x - x^{10} + c$

(C) $10^x + x^{10}$

(D) $\log_e (10^x + x^{10}) + c$

4. $I = \int \left\{ \log_e \log_e x + \frac{1}{(\log_e x)^2} \right\} dx$ is equal to

(A) $x \log_e \log_e x + c$

(B) $x \log_e \log_e x - \frac{x}{\log_e x} + c$

(C) $x \log_e \log_e x + \frac{x}{\log_e x} + c$

(D) none of these

5. $I = \int \frac{\cos^4 x + 1}{\cot x - \tan x} dx$ is equal to

(A) $-\frac{1}{32} [\cos^2 2x + 4 \cos 2x + 10 \log_e \cos 2x] + c$

(B) $\frac{1}{32} [\cos^2 2x - 4 \cos 2x + 10 \log_e \cos 2x] + c$

(C) $\frac{1}{32} [\cos^2 2x - 4 \cos 2x - 10 \log_e \cos 2x] + c$

(D) none of these

6. $I = \int \sin^3 x \cos^3 x dx = f(x) + c$ then $f(x)$ is equal to

(A) $\frac{1}{24} (-4 \sin^6 x + 6 \sin^4 x)$

(B) $\frac{1}{24} (3 \sin 8x - 4 \sin 6x + 6 \sin 4x)$

(C) $\frac{1}{24} (4 \cos^6 x + 6 \cos^4 x)$

(D) $\frac{1}{24} (4 \cos^6 x - 3 \cos^8 x)$

7. $I = \int \sqrt{1 + 2 \tan x (\sec x + \tan x)} dx$ is equal to
 (A) $\log_e |\sec^2 x + \tan x \sec x| + c$ (B) $\log_e |1 + \tan x (\sec x + \tan x)| + c$
 (C) $\log_e |\sin x (\sec x - \tan x)| + c$ (D) none of these.
8. $I = \int \frac{dx}{(\sin x - 2 \cos x)(2 \cos x + \sin x)}$ is equal to
 (A) $\log_e \sqrt[4]{\frac{\tan x - 2}{2 \tan x + 1}} + c$ (B) $\frac{1}{4} \ln \left| \frac{\tan x - 2}{\tan x + 2} \right| + c$
 (C) $-\frac{1}{4} \log_e \left| \frac{2 \sin x + \cos x}{\sin x + 2 \cos x} \right| + c$ (D) none of these
9. $I = \int \frac{dx}{\sqrt{x}(x+9)}$ is equal to
 (A) $\frac{2}{3} \tan^{-1}(\sqrt{x}) + c$ (B) $\frac{2}{3} \tan^{-1}\left(\frac{\sqrt{x}}{3}\right) + c$
 (C) $\tan^{-1}\left(\frac{\sqrt{x}}{3}\right) + c$ (D) none of these
10. If $\int \operatorname{cosec} 2x dx = f(g(x)) + c$, then $g'(x)$ is equal to
 (A) $\frac{1}{2} \ln(\tan x)$ (B) $\tan \frac{1}{2x}$
 (C) $\ln |\sec x|$ (D) none of these
11. $I = \int (2x-3) \sqrt[6]{m+3nx-nx^2} dx$ is equal to
 (A) $\frac{-6}{7n} \sqrt[6]{(m+3nx-nx^2)^7} + c$ (B) $m \sqrt[6]{m+3nx-nx^2} + c$
 (C) $\frac{1}{n} \sqrt[6]{-(nx^2-3nx-m)^7} + c$ (D) none of these
12. If $A = \int e^x \sin x$ & $B = \int e^x \cos x dx$, then $A^2 + B^2$ is equal to
 (A) $\frac{e^{2x}}{2}$ (B) $\frac{1}{2} e$
 (C) $\frac{e^{3x}}{3}$ (D) none of these.
13. If $I = \int x \log_e \left(1 + \frac{1}{x}\right) dx = p(x) \log_e (x+1) + g(x) x^2 + Lx + c$, then
 (A) $p(x) = \frac{1}{2} x^2$ (B) $g(x) = \log_e x$
 (C) $L = 1$ (D) none of these

14. $I = \int \sqrt{1 + \sin 2x} dx$ is equal to

(A) $\cos x + \sin x + c \ln \left[-\frac{\pi}{4}, 0 \right]$

(B) $\cos x + \sin x + c \ln \left[\frac{3\pi}{4}, \pi \right]$

(C) $-\cos x - \sin x + c \ln \left[-\frac{\pi}{4}, 0 \right]$

(D) none of these

15. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log(9e^{2x} - 4) + c$, then

(A) $A = 3/2, B = 35/36, c \in \mathbb{R}$

(B) $A = -3/2, B = 35/36, c \in \mathbb{R}$

(C) $A = 1/2, B = 35/36, c = 3/2$

(D) none of these

ANSWERS TO ASSIGNMENT PROBLEMS

Subjective:**Level - I**

1. (i) $\frac{\sqrt{2x+1}}{5}(x^2 + x - 1) + c$ (ii) $x \tan \frac{x}{2} + c$
2. (i) $2[\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}] + c$ (ii) $\frac{1}{2} \left[\pi x - x \cos^{-1} x + \sqrt{1-x^2} \right] + c$
3. (i) $\frac{e^{-3x}}{13} (2 \sin 2x - 3 \cos 2x) + c$ (ii) $(x+a) \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{ax} + c$
4. (i) $2\sqrt{1+\ln x} + \ln \left| \frac{\sqrt{1+\ln x} - 1}{\sqrt{1+\ln x} + 1} \right| + c$ (ii) $\frac{1}{\sqrt{2}} \ln \left(\sqrt{2} \tan x + \sqrt{1+2\tan^2 x} \right) + c$
5. (i) $2/\pi[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2} - \frac{\pi}{2}x] + c$ (ii) $\ln \left| \frac{\sqrt{2x+1}}{\sqrt{2x+1}+1} \right| + c$
6. (i) $\frac{4}{(b-a)} \left[\frac{x+a}{x+b} \right]^{1/4} + c,$ (ii) $\sqrt{2} \ln \left(\tan \frac{x}{4} \right) + c$
7. (i) $\frac{4}{21} (e^x - 1)^{3/4} (3e^x + 4) + c$
(ii) $(a-b) \tan^{-1} \sqrt{\frac{x-b}{a-x}} + \sqrt{(a-x)(x-b)} + c$
8. (i) $2\sqrt{\tan x} + c$ (ii) $\frac{1}{2} \tan^{-1} (2 \tan x) + c$
9. (i) $-\frac{x}{4} + \frac{5}{6} \tan^{-1} \left[3 \tan \frac{x}{2} \right] + c$ (ii) $\frac{1}{8} (\tan^{-1} x^4)^2 + c$
10. (i) $\frac{1}{a^2 - b^2} \left[\frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) - \frac{1}{2b} \ln \left(\frac{x-b}{x+b} \right) \right] + c$ (ii) $\frac{-(1+x^3)^{4/3}}{4x^4} + c$

Level - II

1. (i) $-\frac{\sqrt{x^4+1}}{2x^2} + c$ (ii) $\frac{2}{5} \left(x + \frac{1}{x} \right)^{5/2} + c$
2. $\frac{1}{6(m+1)} \left[2x^{3m} + 3x^{2m} + x^m \right]^{\frac{m+1}{m}} + c$
3. $2 \sin x + x + c$
4. $\frac{\left(\frac{a}{b}\right)^x}{\log_e \left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\log_e \left(\frac{b}{a}\right)} + 2x + c$
5. $\frac{1}{3} (4x+7)^{\frac{3}{2}} - \frac{1}{2} (4x+7)^{\frac{1}{2}} + c$
6. $\frac{x^3}{3} - x^2 + 4x - 8 \ln |x+2| + c$
7. (i) $\frac{x}{(1+m^2)} \left[\sin \left(m \ln \frac{x}{k} \right) - m \cos \left(m \ln \frac{x}{k} \right) \right] + c$ (ii) $e^x \sqrt{\frac{1+x}{1-x}} + c$

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8. (i) $\frac{3}{10} (2 + x^2)^{2/3} (x^2 - 3) + c$

(ii) $\frac{-1}{5\sqrt{2}} [\cos 2x + 3]^{3/2} [\cos 2x - 2] + c$

9. (i) $\frac{x}{\ln x} + c$

(ii) $\sec^{-1} x - \frac{\ln x}{\sqrt{x^2 - 1}} + c$

10. (i) $\frac{x^6}{6} + \frac{3x^4}{4} + \frac{x^2}{2} - \frac{1}{2} \ln(1 + x^2) + c$

(ii) $x - \cos x \ln \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$

Subjective:**Level – I**

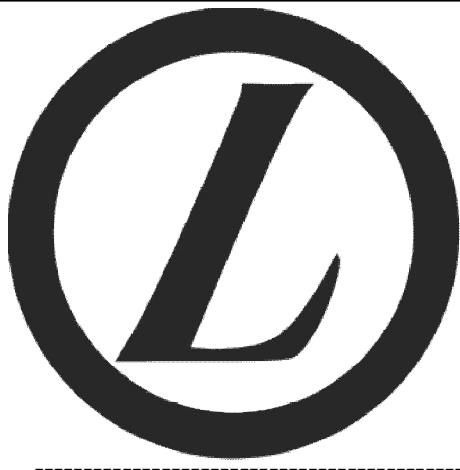
1. **A**
3. **B**
5. **B**
7. **C**
9. **A**
11. **C**
13. **C**
15. **B**

2. **A**
4. **B**
6. **A**
8. **C**
10. **D**
12. **B**
14. **B**

Level – II

1. **B**
3. **D**
5. **A**
7. **D**
9. **B**
11. **A**
13. **D**
15. **B**

2. **B**
4. **B**
6. **A**
8. **A**
10. **B**
12. **A**
14. **A**



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