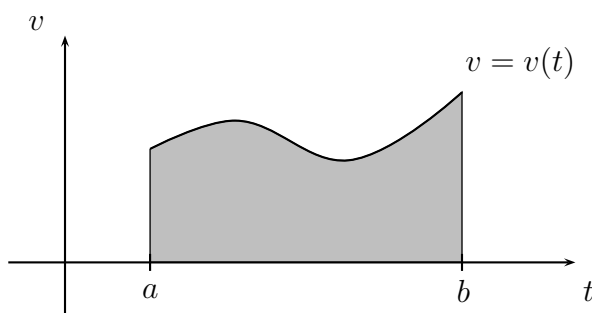

Notes for ‘Inferring displacement from velocity’

Important Ideas and Useful Facts:

- (i) **Area under the velocity curve:** Consider an object or particle moving along a straight line, where one direction is regarded as being positive, and the reverse direction is regarded as negative, and there is some reference point denoted by zero (as in the real line).

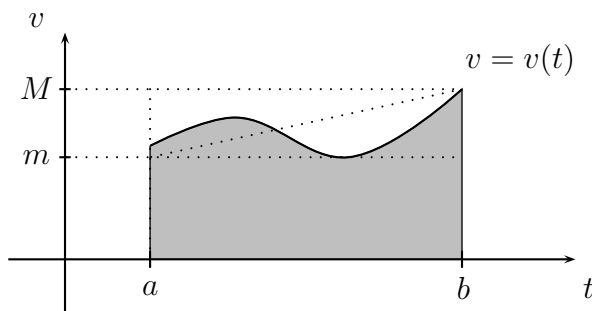
Denote displacement of the object or particle by $x = x(t)$ and its velocity by $v = v(t)$, functions of time t , using compatible units that are appropriate for the particular context.



The area under the velocity curve $v = v(t)$ for $a \leq t \leq b$, where $a < b$, represents the change in position $x(b) - x(a)$.

- (ii) **Upper and lower bounds for the area under the velocity curve over an interval:** As above, suppose m is the minimum and M is the maximum velocity for $a \leq t \leq b$. Then the *lower rectangle* of width $b - a$ and height m is a lower bound for the distance travelled. The *upper rectangle* of width $b - a$ and height M is an upper bound for the distance travelled over this interval. Hence

$$m(b - a) \leq \text{distance travelled} \leq M(b - a).$$



A useful estimate of the distance travelled is the average of the lower and upper bounds:

$$\frac{(m + M)(b - a)}{2},$$

that is, the area of the *trapezium* of base length $b - a$ and vertical side lengths m and M .

Examples:

1. A car travels along a straight road for 50 seconds at a uniform velocity of 95 kilometres per hour. Estimate, to the nearest 100 metres, how far the car traveled.

Solution: There are 1,000 metres in a kilometre and 3,600 seconds in an hour, so the distance travelled is

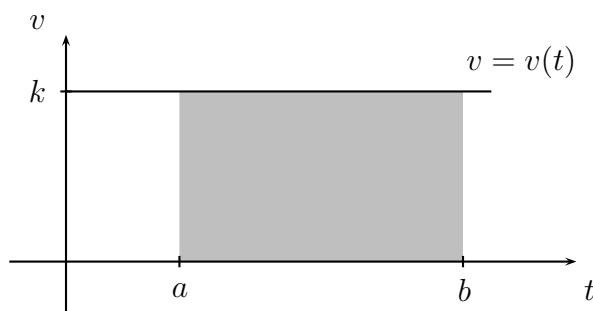
$$\frac{95 \times 1000}{3600} \times 50 = 1,319.4 \text{ m},$$

which we estimate to be 1,300 metres to the nearest 100 metres.

2. If the velocity curve $v = v(t) = k$ is constant then the distance travelled (in appropriate units) for $a \leq t \leq b$ is

$$k(b - a),$$

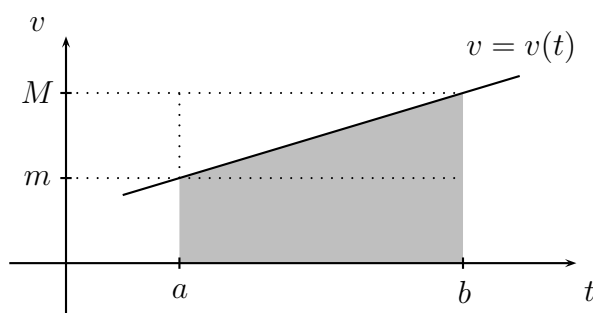
which is the area of the rectangular region under the curve, of width $b - a$ and height k units.



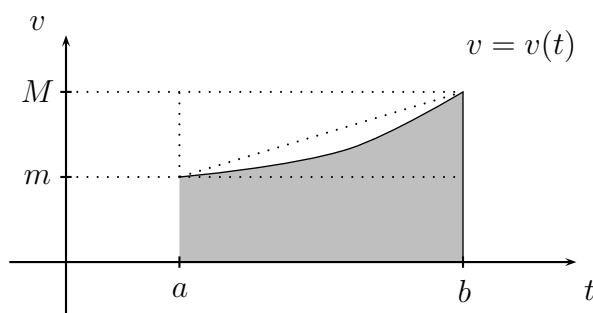
3. If the velocity curve $v = v(t)$ is linear and increasing (constant positive acceleration) then the distance travelled (in appropriate units) for $a \leq t \leq b$ is

$$\frac{(m + M)(b - a)}{2},$$

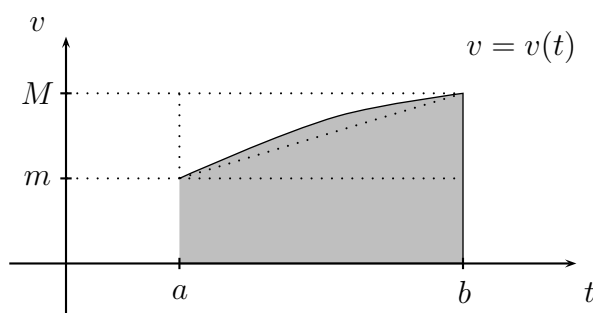
which is the area under the curve, and the average of the lower and upper rectangles of width $b - a$ and heights m and M respectively



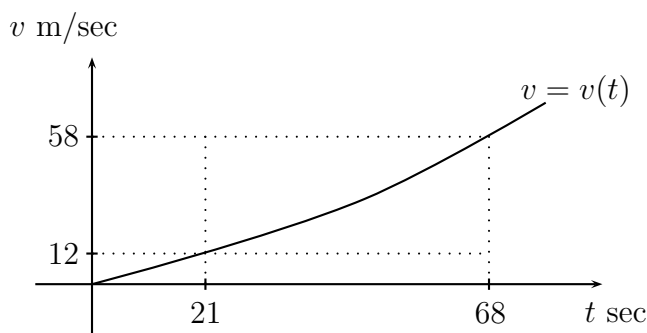
4. If the velocity curve $v = v(t)$ is increasing and concave up then the average of the lower and upper bounds is an overestimate of the distance travelled



5. If the velocity curve $v = v(t)$ is increasing and concave down then the average of the lower and upper bounds is an underestimate of the distance travelled



6. The velocity curve $v = v(t)$ m/sec of an accelerating object moving along a straight line is shown below, as a function of t sec, and is shaped slightly concave upwards.



Find lower and upper bounds for the distance travelled by the object over the time interval $21 \leq t \leq 68$ seconds. Use the average to estimate the distance travelled, and comment about whether this is likely to be an underestimate or overestimate.

Solution: The lower rectangle sitting just beneath the curve has area

$$(68 - 21) \times 12 = 564 ,$$

so a lower bound on the distance travelled is 564 metres. The upper rectangle sitting just above the curve has area

$$(68 - 21) \times 58 = 2726 ,$$

so an upper bound on the distance travelled is 2,726 metres.

The average is

$$\frac{(12 + 58)(68 - 21)}{2} = \frac{564 + 2726}{2} = 1645 ,$$

so a reasonable estimate of the distance travelled should be about 1,645 metres. We expect this to be a slight overestimate, as the curve appears to be slightly concave up.

7. A car accelerates from rest, reaching a speed of 24 m/sec after six seconds. The following velocities are recorded at the end of each second:

time from rest (sec)	0	1	2	3	4	5	6
velocity (m/sec)	0	8	13	17	20	23	24

Use the average of lower and upper rectangular approximation on each time interval, to estimate the distance travelled after the six seconds have elapsed. (You may assume the velocity curve is increasing throughout.)

Solution: Since the velocity of the car is increasing throughout, and the intervals are one second each, a lower bound L for the distance travelled will be obtained by adding up the velocities, in succession from 0 up to 23, since each of these will be a lower bound on the particular interval of one second:

$$L = 0 + 8 + 13 + 17 + 20 + 23 = 81 \text{ m} .$$

An upper bound U for the distance travelled will be obtained by adding up the velocities, in succession from 8 up to 24, since each of these will be an upper bound on the particular interval of one second:

$$U = 8 + 13 + 17 + 20 + 23 + 24 = 105 \text{ m} .$$

The average of these should be a reasonable estimate of the true distance travelled:

$$\frac{L + U}{2} = \frac{81 + 105}{2} = 93 \text{ m} .$$

Thus we expect the car travelled about 93 metres over these six seconds.

8. A car comes to a stop, ten seconds after the driver slams on the brakes. While the brakes are on, the following velocities are recorded:

time since brakes applied (sec)	0	2	4	6	8	10
velocity (m/sec)	34	24	16	9	4	0

Use the average of lower and upper rectangular approximations on each two second time interval to estimate the distance travelled over those ten seconds as the car comes to a halt. (You may assume the velocity curve is decreasing throughout.)

Solution: Since the velocity of the car is decreasing throughout, and the intervals are two seconds each, a lower bound L for the distance travelled will be obtained by adding up the velocities, in succession from 24 down to 0, since each of these will be a lower bound on the particular two second interval, and multiplying through by this common interval length of two seconds:

$$L = (24 + 16 + 9 + 4 + 0) \times 2 = 106 \text{ m} .$$

An upper bound U for the distance travelled will be obtained by adding up the velocities, in succession from 34 down to 4, since each of these will be an upper bound on the particular two second interval, and multiplying through by this common interval length of two seconds:

$$U = (34 + 24 + 16 + 9 + 4) \times 2 = 174 \text{ m} .$$

The average of these should be a reasonable estimate of the true distance travelled:

$$\frac{L + U}{2} = \frac{106 + 174}{2} = 140 \text{ m} .$$

Thus we expect the car travelled about 140 metres over these ten seconds, as it came to a halt.

Notes for ‘Areas bounded by curves’

Important Ideas and Useful Facts:

- (i) **Areas of rectangles and triangles:** The area of a rectangle of side lengths a and b is the product ab , and this can be taken as a geometric definition of multiplication of nonnegative real numbers. The area of a triangle of base length b and height h is $\frac{bh}{2}$.
- (ii) **Perimeter and area of a circle:** Consider a circle of radius r . The perimeter P is given by the formula $P = 2\pi r$ and the area A by the formula $A = \pi r^2$. Notice that $\frac{dA}{dr} = P$.
- (iii) **Sigma notation:** A sum of numbers, typically with many terms, is often abbreviated using *sigma notation*, employing the Greek capital $\sum$ for ‘sum’:

$$a_m + a_{m+1} + a_{m+2} + \dots + a_n = \sum_{i=m}^n a_i .$$

The general term or expression is indicated by a_i , say, where the subscript i ranges from m to n , say, in steps of one at a time, where $m \leq n$. The smallest value of the subscript m appears below the $\sum$ symbol and the largest value n appears above the symbol.

For example, if $a_i = i$, $m = 1$ and $n = 100$, then

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + \dots + 100 \quad \text{and} \quad \sum_{i=1}^{100} i^2 = 1^2 + 2^2 + 3^2 + \dots + 100^2 ,$$

the sums of the first 100 consecutive integers, and squares of consecutive integers, respectively. We have the following simple but useful properties, which allow one to break a sum into pieces and to bring constants outside the sum, where $b_m, \dots, b_n$ form any other list of numbers and c is a constant:

$$\sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i \quad \text{and} \quad \sum_{i=m}^n ca_i = c \sum_{i=m}^n a_i .$$

- (iv) **Trick of Gauss:** To add together a list of evenly spaced numbers (an *arithmetic sequence*), one can add them twice, but with one list backwards, and then divide by two. For example, consider the following sum, denoted by S :

$$\begin{array}{cccccccccccc} 1 & + & 2 & + & 3 & + & \dots & + & n-1 & + & n & = & S \\ n & + & n-1 & + & n-2 & + & \dots & + & 2 & + & 1 & = & S \\ \hline n+1 & + & n+1 & + & n+1 & + & \dots & + & n+1 & + & n+1 & = & 2S \end{array}$$

We therefore get $2S = n(n+1)$, so that, dividing by 2, and also using $\sum$ notation,

$$\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2} .$$

For example,

$$\sum_{i=1}^{100} i = 1 + 2 + \dots + 100 = \frac{100 \times 101}{2} = 5050 .$$

Examples:

1. Use the trick of Gauss to add up the first 50 consecutive integers:

$$1 + 2 + 3 + \dots + 49 + 50 .$$

Solution: Call this sum S . Then $2S = 50 \times 51$, so $S = \frac{50 \times 51}{2} = 1,275$.

2. Express the following sum using sigma notation, and evaluate it:

$$1 + 2 + 3 + \dots + 499 + 500 .$$

Solution: This sum becomes

$$\sum_{i=1}^{500} i = \frac{500 \times 501}{2} = 125,250 .$$

3. Use the trick of Gauss to add up the first 50 consecutive even integers:

$$2 + 4 + 6 + \dots + 98 + 100 .$$

Solution: Call this sum S . Then $2S = 50 \times 102$, since there are 50 terms and, when adding them twice, one can group each pair to add to 102, so $S = \frac{50 \times 102}{2} = 2,550$.

Alternatively, using sigma notation, and the formula for the sum of consecutive integers,

$$S = \sum_{i=1}^{50} 2i = 2 \sum_{i=1}^{50} i = \frac{2 \times 50 \times 51}{2} = 50 \times 51 = 2,550 .$$

4. Evaluate $\sum_{i=2}^6 2$.

Solution: This is an abbreviation for repeated addition of 2 using 5 occurrences (for $i = 2, 3, 4, 5, 6$), so the sum become $5 \times 2 = 10$.

5. Evaluate $\sum_{i=2}^6 (3i - 2)$.

Solution: This is an abbreviation for the following sum:

$$\begin{aligned} \sum_{i=2}^6 (3i - 2) &= (3 \times 2 - 2) + (3 \times 3 - 2) + (3 \times 4 - 2) + (3 \times 5 - 2) + (3 \times 6 - 2) \\ &= 4 + 7 + 10 + 13 + 16 = 50 . \end{aligned}$$

Alternatively, exploiting the formula for the sum of the first six consecutive integers,

$$\sum_{i=2}^6 (3i - 2) = \left(3 \sum_{i=2}^6 i \right) - \left(\sum_{i=2}^6 2 \right) = 3 \times \left(\frac{6 \times 7}{2} - 1 \right) - (5 \times 2) = 60 - 10 = 50 .$$

6. Evaluate $\sum_{i=1}^{100} (3i - 2)$.

Solution: Using properties of $\sum$ notation, we have

$$\sum_{i=1}^{100} (3i - 2) = \left(3 \sum_{i=1}^{100} i \right) - \left(\sum_{i=1}^{100} 2 \right) = \frac{3 \times 100 \times 101}{2} - 200 = 14,950 .$$

7. Find n such that $\sum_{i=1}^n i = 630$.

Solution: We want n such that

$$630 = \sum_{i=1}^n i = \frac{n(n+1)}{2} ,$$

so that $n^2 + n = 1260$, that is,

$$n^2 + n - 1260 = 0 .$$

By the quadratic formula,

$$n = \frac{-1 \pm \sqrt{1 + (4 \times 1260)}}{2} = \frac{-1 \pm \sqrt{5041}}{2} = \frac{-1 \pm 71}{2} .$$

Taking the positive square root, we get $n = 35$.

8. Evaluate $\sum_{i=1}^{200} \left(\frac{1}{i} - \frac{1}{i+1} \right)$.

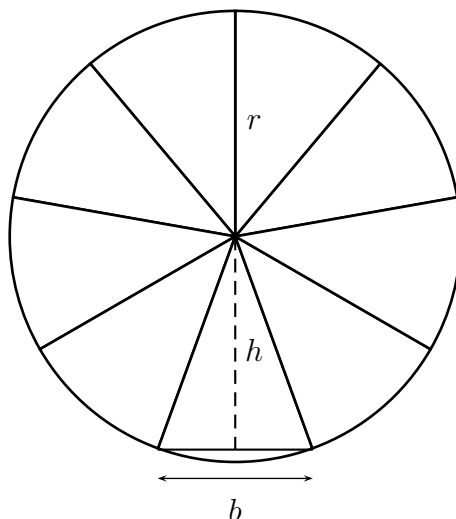
Solution: This is an example of a *collapsing sum*, since, if one writes it out in full, all of the intermediate terms cancel out leaving only the fraction $\frac{1}{1}$ at the beginning and the negative fraction $-\frac{1}{201}$ at the end:

$$\sum_{i=1}^{200} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{1} - \frac{1}{201} = \frac{200}{201} .$$

One can also see this more formally as follows, using properties of sigma notation, and replacing $i + 1$ by j in the second sum, and noting that j goes from 2 to 201 as i goes from 1 to 200:

$$\begin{aligned} \sum_{i=1}^{200} \left(\frac{1}{i} - \frac{1}{i+1} \right) &= \left(\sum_{i=1}^{200} \frac{1}{i} \right) - \left(\sum_{i=1}^{200} \frac{1}{i+1} \right) = \left(\sum_{i=1}^{200} \frac{1}{i} \right) - \left(\sum_{j=2}^{201} \frac{1}{j} \right) \\ &= \frac{1}{1} + \left(\sum_{i=2}^{200} \frac{1}{i} \right) - \left(\sum_{j=2}^{200} \frac{1}{j} \right) - \frac{1}{201} = \frac{1}{1} + 0 - \frac{1}{201} = \frac{200}{201} . \end{aligned}$$

9. We explain the Greek thought experiment that deduces the formula for the area A of a circle of radius r from the perimeter $P = 2\pi r$. We divide a circle of radius r into n sectors all of exactly the same size and shape.



The area of each sector is approximated by a triangle with base b and height h , say, and then

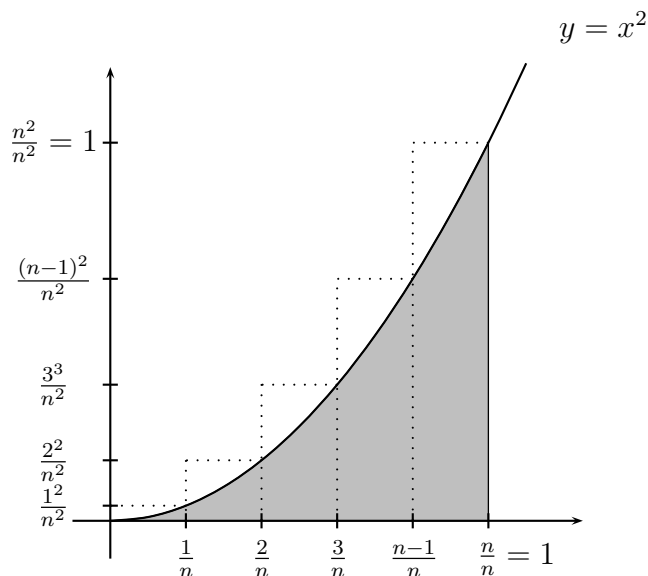
$$A \approx n \times (\text{area of one triangle}) = n \times \frac{bh}{2} = \frac{nbh}{2}.$$

As $n \rightarrow \infty$ the approximation improves, approaching equality, and in the limit we have,

$$A = \lim_{n \rightarrow \infty} \frac{nbh}{2} = \frac{1}{2} \left(\lim_{n \rightarrow \infty} nb \right) \left(\lim_{n \rightarrow \infty} h \right) = \frac{1}{2} (2\pi r)(r) = \pi r^2,$$

noting that nb approaches the perimeter $P = 2\pi r$, and h approaches the radius r .

10. We will use an approximation method involving rectangles to find the area A under the parabola $y = x^2$ for $0 \leq x \leq 1$, by which we mean the area shaded below, between the curve, the x -axis and the vertical line $x = 1$.



Upper rectangles that sit above the curve suffice for our argument. We divide the interval $[0, 1]$ on the x -axis into n equal subintervals, each of width $\frac{1}{n}$. We want to think of n as getting arbitrarily large (but, for the convenience of interpreting the diagram above, $n = 5$).

As i goes from 1 to n , the rectangle sitting above the curve, over the subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, has height $\frac{i^2}{n^2}$ and width $\frac{1}{n}$, so has area $\left(\frac{1}{n}\right)\left(\frac{i^2}{n^2}\right) = \frac{i^2}{n^3}$.

The area A under the curve is approximated by adding together the areas of all of these rectangles, which can be expressed succinctly using sigma notation, noting the common factor of $\frac{1}{n^3}$:

$$A \approx \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \dots + \frac{(n-1)^2}{n^3} + \frac{n^2}{n^3} = \frac{1}{n^3} \sum_{i=1}^n i^2.$$

As $n \rightarrow \infty$ this approximation improves toward equality and we get the following limit:

$$A = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n i^2.$$

It is far from obvious how to evaluate this limit. There is a trick involving looking at n^3 in a clever way, called a *telescoping sum*. For example, a ‘telescope’ for 5^3 looks like the following:

$$\begin{aligned} 5^3 &= 5^3 - 4^3 + 4^3 - 3^3 + 3^3 - 2^3 + 2^3 - 1^3 + 1^3 - 0^3 \\ &= (5^3 - 4^3) + (4^3 - 3^3) + (3^3 - 2^3) + (2^3 - 1^3) + (1^3 - 0^3). \end{aligned}$$

In general, for any large positive integer n , we have the following ‘telescope’, which can be manipulated using sigma notation, and making use of the formula for the sum of consecutive integers:

$$\begin{aligned} n^3 &= n^3 - (n-1)^3 + (n-1)^3 - (n-2)^3 + \dots - 2^3 + 2^3 - 1^3 + 1^3 - 0^3 \\ &= \sum_{i=1}^n (i^3 - (i-1)^3) = \sum_{i=1}^n (i^3 - (i^3 - 3i^2 + 3i - 1)) = \sum_{i=1}^n (3i^2 - 3i + 1) \\ &= 3\left(\sum_{i=1}^n i^2\right) - 3\left(\sum_{i=1}^n i\right) + \left(\sum_{i=1}^n 1\right) = 3\left(\sum_{i=1}^n i^2\right) - \frac{3n(n+1)}{2} + n. \end{aligned}$$

Rearranging this, we get the following elegant expression for the sum of squares of consecutive integers:

$$\begin{aligned} \sum_{i=1}^n i^2 &= \frac{1}{3} \left(n^3 + \frac{3n(n+1)}{2} - n \right) = \frac{1}{6} (2n^3 + 3n^2 + 3n - 2n) \\ &= \frac{n}{6} (2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

We can now return to finding the area A under the parabola, by evaluating the limit we found previously, using this formula and our usual tricks for manipulating limits:

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{n})(2 + \frac{1}{n})}{6} = \frac{(1+0)(2+0)}{6} = \frac{1}{3}. \end{aligned}$$

Important Ideas and Useful Facts:

- (i) Riemann sums, areas under curves and the definite integral:

Suppose that $a < b$ and consider a continuous function f defined for x along the interval $a \leq x \leq b$. Divide this interval into n subintervals (not necessarily of equal width) by choosing

$$t_0 = a < t_1 < t_2 < t_3 < \dots < t_{n-1} < t_n = b ,$$

so that $[t_{i-1}, t_i]$ denotes the i th subinterval for $i = 1$ to n . Choose points

$$x_1, x_2, \dots, x_n$$

within successive subintervals, that is, $x_i \in [t_{i-1}, t_i]$ for $i = 1$ to n . Denote by Δx_i the width of the i th subinterval, that is

$$\Delta x_i = t_i - t_{i-1} ,$$

for $i = 1$ to n . We think of Δx_i as a small change in x , as x moves across the interval from t_{i-1} to t_i . Then the (signed) area under the curve (that is, bounded by the curve, the x -axis and the vertical lines $x = a$ and $x = b$) is approximated by the *Riemann sum*

$$\sum_{i=1}^n f(x_i) \Delta x_i = f(x_1) \Delta x_1 + f(x_2) \Delta x_2 + \dots + f(x_n) \Delta x_n .$$

Each summand $f(x_i) \Delta x_i$ is the area of an approximating rectangle of width Δx_i and height $f(x_i)$. The exact value of the (signed) area under the curve is the limit of the Riemann sums

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i ,$$

and called the *definite integral of f from a to b* . This limit exists regardless of the choices made in partitioning the interval, or in choosing each x_i in the i th subinterval, for $i = 1$ to n , provided the widths of the subintervals approach zero as $n \rightarrow \infty$.

- (ii) **Upper and lower Riemann sums:** A Riemann sum is called *lower* if the rectangles always sit beneath the curve and it provides a lower bound for the area under the curve.

If the curve is increasing then the lower Riemann sum is formed always by choosing the left-hand endpoints of the subintervals.

If the curve is decreasing then the lower Riemann sum is formed always by choosing the right-hand endpoints of the subintervals.

A Riemann sum is called *upper* if the rectangles always sit above the curve, and it provides an upper bound for the area under the curve.

If the curve is increasing then the upper Riemann sum is formed always by choosing the right-hand endpoints of the subintervals.

If the curve is decreasing then the upper Riemann sum is formed always by choosing the left-hand endpoints of the subintervals.

(iii) Properties of the Definite Integral:

- (a) $\int_a^b c \, dx = c(b-a)$ and $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$ for any constant c .
- (b) $\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$.
- (c) $\int_a^a f(x) \, dx = 0$ and $\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$ (defined also when $a \geq b$).
- (d) $\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$.

Examples:

1. Evaluate $\int_{-2}^5 3 \, dx$, the area under the constant function 3 over the interval $[-2, 5]$.

Solution: The definite integral is the area of a rectangle of height 3 units and width 7, so

$$\int_{-2}^5 3 \, dx = 3 \times 7 = 21.$$

2. Evaluate and interpret the definite integral $\int_5^{-2} 3 \, dx$,

Solution: Moving backwards along the interval $[-2, 5]$ has the effect of measuring the base of a rectangle (appearing in the Riemann sum) as negative. Since the integrand is positive, one expects to produce a negative area. This is confirmed by the following formal manipulation, using the answer to the previous exercise, where we multiply the definite integral by negative one when we swap the terminals:

$$\int_5^{-2} 3 \, dx = - \int_{-2}^5 3 \, dx = -21.$$

3. Evaluate and interpret the definite integral

$$\int_2^2 (f(x) + 3g(x) - 7h(x)) \, dx,$$

where f , g and h are any continuous functions defined on an interval containing $x = 2$.

Solution: The definite integral evaluates to zero, as the interval $[2, 2] = \{2\}$ over which we are integrating is degenerate, consisting of a single point, so the area under the curve vanishes.

4. Evaluate

$$\int_0^1 (2 + 6x^2) \, dx,$$

given that $\int_0^1 x^2 \, dx = \frac{1}{3}$.

Solution: We have

$$\int_0^1 (2 + 6x^2) \, dx = \int_0^1 2 \, dx + 6 \int_0^1 x^2 \, dx = (2 \times 1) + 6 \times \left(\frac{1}{3}\right) = 4.$$

5. Evaluate

$$\int_1^5 (6 - f(x) + 2g(x)) dx ,$$

given that $\int_1^5 f(x) dx = 14$ and $\int_1^5 g(x) dx = -5$.

Solution: We have

$$\begin{aligned} \int_1^5 (6 - f(x) + 2g(x)) dx &= \int_1^5 6 dx - \int_1^5 f(x) dx + 2 \int_1^5 g(x) dx \\ &= (6 \times (5 - 1)) - 14 + 2(-5) = 24 - 14 - 10 = 0 . \end{aligned}$$

6. Find

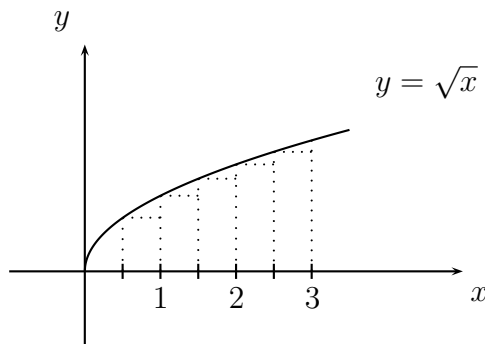
$$\int_1^7 f(x) dx ,$$

given that $\int_3^5 f(x) dx = 2$ and $\int_1^3 f(x) dx = 5$ and $\int_7^5 f(x) dx = -2$.

Solution: We have

$$\begin{aligned} \int_1^7 f(x) dx &= \int_1^3 f(x) dx + \int_3^5 f(x) dx + \int_5^7 f(x) dx \\ &= \int_1^3 f(x) dx + \int_3^5 f(x) dx - \int_7^5 f(x) dx = 5 + 2 - (-2) = 9 . \end{aligned}$$

7. Use a *lower* Riemann sum with 6 subintervals, by finding the areas of the lower rectangles beneath the curve, as indicated in the following diagram, to find a lower bound for the definite integral $\int_0^3 \sqrt{x} dx$, quoting your answer to two decimal places.



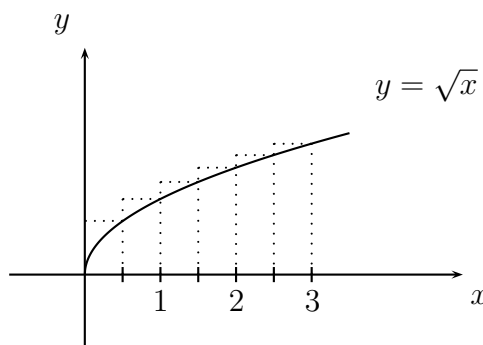
Solution: Each rectangle has width 0.5 and the heights of the lower rectangles are, in succession from left to right,

$$0, \sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2}, \sqrt{2.5} .$$

Hence the lower Riemann sum is

$$0.5(0 + \sqrt{0.5} + 1 + \sqrt{1.5} + \sqrt{2} + \sqrt{2.5}) \approx 2.96 .$$

8. Use an *upper* Riemann sum with 6 subintervals, by finding the areas of the upper rectangles, as indicated in the following diagram, to find an upper bound for the definite integral $\int_0^3 \sqrt{x} \, dx$, quoting your answer to two decimal places.



Solution: Each rectangle has width 0.5 and the heights of the rectangles are, in succession from left to right,

$$\sqrt{0.5}, 1, \sqrt{1.5}, \sqrt{2}, \sqrt{2.5}, \sqrt{3}.$$

Hence the upper Riemann sum is

$$0.5(\sqrt{0.5} + 1 + \sqrt{1.5} + \sqrt{2} + \sqrt{2.5} + \sqrt{3}) \approx 3.83.$$

9. Use the average of the lower and upper bounds found in the previous two exercises to estimate $\int_0^3 \sqrt{x} \, dx$, quoting your answer to one decimal place.

Comment on whether you think this estimate will be an underestimate or an overestimate of the true area under the curve.

Solution: The average of the lower and upper bounds from the previous exercise is

$$\frac{2.96 + 3.83}{2} \approx 3.4,$$

to one decimal place.

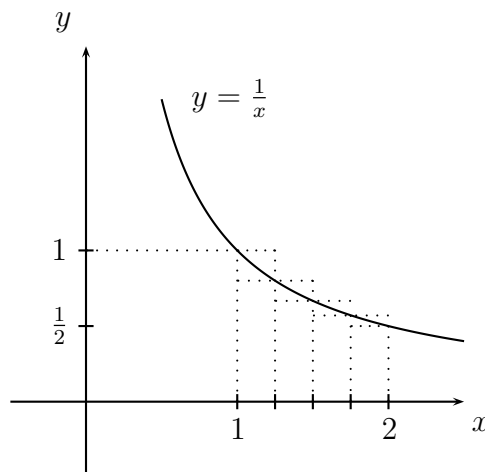
The average of the lower and upper Riemann sums for an increasing function, such as this one, uses areas of trapeziums, so should provide an underestimate of the true area under the curve, since the curve is concave down so sits just above the lines joining endpoints of the subintervals.

Soon you will be able to evaluate this area exactly using the Fundamental Theorem of Calculus. In fact,

$$\int_0^3 \sqrt{x} \, dx = 2\sqrt{3} \approx 3.5,$$

to one decimal place, which is slightly more than the estimate, confirming our expectations.

10. Use the average of the lower and upper Riemann sums, using 4 subintervals,



to estimate $\int_1^2 \frac{dx}{x}$, rounding off your final answer to three decimal places. Comment on whether you expect your estimate to be an underestimate or overestimate of the true area under the curve.

Solution: The partition of the interval uses the points

$$1, 1.25, 1.5, 1.75, 2.$$

Each subinterval has width 0.25. To form the lower Riemann sum L , we use the right-hand endpoints of the subintervals, and we get the lower bound

$$L = 0.25 \times \left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75} + \frac{1}{2} \right) \approx 0.6345,$$

to four decimal places. To form the upper Riemann sum U , we use the left-hand endpoints of the subintervals, and we get the upper bound

$$U = 0.25 \times \left(\frac{1}{1} + \frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75} \right) \approx 0.7595,$$

to four decimal places. The average of L and U is (keeping everything exact, this time),

$$\frac{L+U}{2} = 0.125 \times \left(\frac{1}{1} + \frac{2}{1.25} + \frac{2}{1.5} + \frac{2}{1.75} + \frac{1}{2} \right) \approx 0.697.$$

We estimate the area under the curve to be about 0.697, and expect this to be a slight overestimate, since the curve is concave up, so the curve sits slightly below lines joining endpoints of subintervals.

Soon you will be able to evaluate this area exactly using the Fundamental Theorem of Calculus. In fact,

$$\int_1^2 \frac{dx}{x} = \ln 2 \approx 0.693,$$

to three decimal places, which is slightly less than the estimate, confirming our expectations.

Notes for ‘The Fundamental Theorem of Calculus and indefinite integrals’

Important Ideas and Useful Facts:

- (i) **Antiderivatives and Uniqueness up to a Constant:** If f and g are functions such that f is differentiable and $f' = g$ then we call g the *derivative* of f and f an *antiderivative* of g .

If f_1 and f_2 are antiderivatives of g , and g is continuous on an interval, then the values of f_1 and f_2 differ by a constant on this interval, that is, there exists a constant C such that

$$f_1(x) = f_2(x) + C$$

for all x in the interval. We call C the *constant of integration*.

- (ii) **The Fundamental Theorem of Calculus:** If $a < b$ and f is continuous for $a \leq x \leq b$, and F is any antiderivative of f for $a \leq x \leq b$, then

$$\int_a^b f(x) dx = F(b) - F(a) .$$

Common notations for $F(b) - F(a)$ are $[F(x)]_a^b$, $F(x)]_a^b$ and $F(x)|_a^b$.

- (iii) **The Indefinite Integral:** If f is a continuous function then we write

$$\int f(x) dx$$

for any antiderivative of f , and call this the *indefinite integral* of f . Thus if we put $F(x) = \int f(x) dx$, for some choice of antiderivative, then $F'(x) = f(x)$. Choices of antiderivatives differ by a constant.

- (iv) **Some Properties and Standard Indefinite Integrals:**

$$(a) \quad \int k f(x) dx = k \int f(x) dx \quad \text{and} \quad \int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx .$$

$$(b) \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1 , \text{ and } \quad \int \frac{1}{x} dx = \ln |x| + C .$$

$$(c) \quad \int e^x dx = e^x + C , \quad \int \ln x dx = x \ln x - x + C .$$

$$(d) \quad \int \cos x dx = \sin x + C , \quad \int \sin x dx = -\cos x + C , \quad \int \sec^2 x dx = \tan x + C .$$

Examples and derivations:

1. Find the indefinite integral $\int (6x^2 + 4x - 1) dx$.

Solution: We can integrate each piece separately, and combine the constants as a single “plus C ” at the right, to get

$$\begin{aligned}\int (6x^2 + 4x - 1) dx &= 6 \int x^2 dx + 4 \int x dx - \int 1 dx \\ &= 6 \left(\frac{x^3}{3} \right) + 4 \left(\frac{x^2}{2} \right) - x + C \\ &= 2x^3 + 2x^2 - x + C.\end{aligned}$$

In practice, one usually goes immediately to the last step, adjusting the constants quickly as one goes.

2. Find the definite integral $\int_1^2 (6x^2 + 4x - 1) dx$.

Solution: Using the antiderivative from the previous exercise (without the “plus C ”), and the Fundamental Theorem of Calculus, we get

$$\int_1^2 (6x^2 + 4x - 1) dx = [2x^3 + 2x^2 - x]_1^2 = 16 + 8 - 2 - (2 + 2 - 1) = 19.$$

3. Find the indefinite integral $\int (\cos x - \sin x) dx$.

Solution: Again we can integrate each piece separately, and combine the constants as a single “plus C ” at the right, to get

$$\int (\cos x - \sin x) dx = \int \cos x dx - \int \sin x dx = \sin x + \cos x + C.$$

4. Find the definite integral $\int_{\pi/4}^{\pi/2} (\cos x - \sin x) dx$.

Solution: Using the result of the previous exercise, we have

$$\begin{aligned}\int_{\pi/4}^{\pi/2} (\cos x - \sin x) dx &= [\sin x + \cos x]_{\pi/4}^{\pi/2} = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) \\ &= 1 + 0 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = 1 - \sqrt{2}.\end{aligned}$$

5. Find the indefinite integral $\int 2x\sqrt{x} dx$.

Solution: We rewrite the integrand as a single fractional power of x , and as usual rewrite any constant to the right as a “plus C ”, to get

$$\int 2x\sqrt{x} dx = 2 \int x^{3/2} dx = 2 \left(\frac{x^{5/2}}{5/2} \right) + C = \frac{4x^{5/2}}{5} + C.$$

6. Find the definite integral $\int_1^4 2x\sqrt{x} \, dx$.

Solution: Using the result of the previous exercise, we have

$$\int_1^4 2x\sqrt{x} \, dx = \left[\frac{4x^{5/2}}{5} \right]_1^4 = \frac{4}{5} [x^{5/2}]_1^4 = \frac{4}{5} (32 - 1) = \frac{124}{5} .$$

7. Find the definite integral $\int_1^2 (x^{-2} - 2x^{-3}) \, dx$.

Solution: By the formula for integrating powers of x for each of the two pieces, we get

$$\int_1^2 (x^{-2} - 2x^{-3}) \, dx = [-x^{-1} + x^{-2}]_1^2 = \left(-\frac{1}{2} + \frac{1}{4} \right) - (-1 + 1) = -\frac{1}{4} .$$

8. Find the definite integral $\int_{-1}^1 \frac{e^x - e^{-x}}{2} \, dx$.

Solution: An obvious antiderivative of e^x is e^x . For e^{-x} , it is not so hard to guess an antiderivative, because its derivative is $-e^{-x}$, so multiplying through by negative one, gives e^{-x} . Hence, considering the indefinite integral first, and playing with minus signs for e^{-x} , we get

$$\int \frac{e^x - e^{-x}}{2} \, dx = \frac{1}{2} \left(\int e^x \, dx - \int e^{-x} \, dx \right) = \frac{1}{2} (e^x + e^{-x}) + C .$$

Hence, we get

$$\int_{-1}^1 \frac{e^x - e^{-x}}{2} \, dx = \frac{1}{2} [e^x + e^{-x}]_{-1}^1 = \frac{1}{2} (e + e^{-1} - e^{-1} - e) = 0 .$$

9. Suppose that F and G are antiderivatives of f on some interval. We sketch a proof why there exists a constant C such that

$$F(x) = G(x) + C$$

for all x on that interval.

Sketch of proof: We have that

$$F'(x) = G'(x) = f(x)$$

for all x in the interval. The trick is to look at the function H defined by the rule

$$H(x) = F(x) - G(x) .$$

Then

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 ,$$

so that the tangent lines to the graph of H are all horizontal. Intuitively, it is clear that this forces the graph of H itself to be a horizontal line, for if the curve started to move away from the horizontal at some point, then it could not be approximated arbitrarily well by horizontal tangent lines. Hence $H(x)$ is constant for all x in the interval, so

$$F(x) - G(x) = H(x) = C ,$$

for some constant C , so that

$$F(x) = G(x) + C ,$$

completing the sketch of the proof.

10. We explain why there is a magnitude sign appearing in the standard integration formula

$$\int \frac{dx}{x} = \ln|x| + C .$$

For $x > 0$, certainly $\ln x$ is an antiderivative of $\frac{1}{x}$, so there is some constant C_1 such that

$$\int \frac{dx}{x} = \ln x + C_1 . \quad (1)$$

However, let us now consider $x < 0$. Then $-x > 0$ and the expression $\ln(-x)$ makes perfectly good sense. Put $u = -x$, so that $\frac{du}{dx} = -1$. By the Chain Rule,

$$\frac{d}{dx}(\ln(-x)) = \frac{d}{du}(\ln u) \frac{du}{dx} = \left(\frac{1}{u}\right)(-1) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x} .$$

So, in this case, $\ln(-x)$ is an antiderivative of $\frac{1}{x}$, so there must be some constant C_2 such that

$$\int \frac{dx}{x} = \ln(-x) + C_2 . \quad (2)$$

If $C_1 = C_2 = C$ then (1) and (2) can be combined in a single equation,

$$\int \frac{dx}{x} = \ln|x| + C , \quad (3)$$

exploiting the fact that $|x| = x$ for $x > 0$, and $|x| = -x$ for $x < 0$. This explains the usual formula.

In fact, (1) and (2) are more general than (3), because one could create a single antiderivative of $y = \frac{1}{x}$ using different constants in (1) and (2) for the different branches of the hyperbola, for example

$$F(x) = \begin{cases} \ln x + 3 & \text{if } x > 0 \\ \ln(-x) + 7 & \text{if } x < 0 \end{cases}$$

which would not be covered by (3). However, in practice, one is usually only concerned with one branch of the hyperbola, where the domain is a connected interval, so (3) suffices. Note that, because of this flexibility on the different branches of the hyperbola, the result guaranteeing uniqueness of the antiderivative up to addition by a single constant fails. This is not a contradiction, because the function $y = \frac{1}{x}$ is not continuous on the whole real line.

Examples and derivations:

1. Suppose that $y = f(x)$ is a continuous function defined over an interval $[a, b]$ where $a < b$. Recall that the Fundamental Theorem states that the area under the curve over this interval is

$$\int_a^b f(x) dx = F(b) - F(a)$$

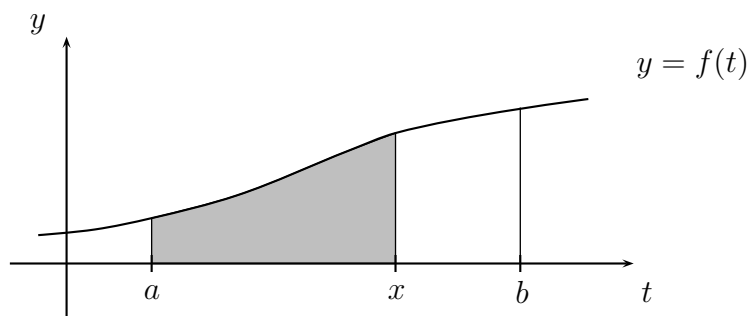
where $F(x)$ is any antiderivative of $f(x)$.

We provide a sketch of a proof of this fact. To simplify the argument, we assume that f is increasing over the interval $[a, b]$.

Sketch of proof: Consider the *area function* A given by the rule

$$A(x) = \int_a^x f(t) dt ,$$

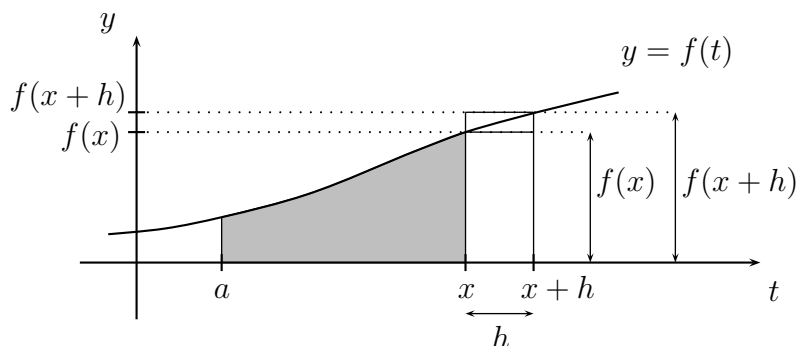
where we have used the variable x as the upper terminal in the definite integral, using instead t for the variable that appears in both the integrand and the differential.



Then $A(x)$ represents the area under the curve $y = f(t)$ over the interval $[a, x]$. In particular,

$$A(a) = \int_a^a f(t) dt = 0 \quad \text{and} \quad A(b) = \int_a^b f(t) dt .$$

Suppose that $a < x < b$ and choose a small positive real number h such that $x < x + h$.



We may draw lower and upper rectangles that enclose the curve $y = f(t)$ for $t \in [x, x+h]$.

On the one hand, the area of the curve over this interval is bounded below by the area of the lower rectangle, which is $hf(x)$. On the other hand, the area of the curve over this interval is bounded above by the area of the upper rectangle, which is $hf(x+h)$. This area is

$$\begin{aligned}\int_x^{x+h} f(t) dt &= \int_x^a f(t) dt + \int_a^{x+h} f(t) dt = \int_a^{x+h} f(t) dt + \int_x^a f(t) dt \\ &= \int_a^{x+h} f(t) dt - \int_a^x f(t) dt = A(x+h) - A(x) .\end{aligned}$$

We thus have the following cascade of inequalities:

$$hf(x) \leq A(x+h) - A(x) \leq hf(x+h) .$$

Dividing through by h , which is positive, retains the inequalities, and we get

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h) .$$

As $h \rightarrow 0^+$, of course $f(x) \rightarrow f(x)$. But also $f(x+h) \rightarrow f(x)$, since f is continuous. Thus the left and right-hand sides of this cascade of inequalities both tend to the same limit $f(x)$ as $h \rightarrow 0^+$. By the squeeze law,

$$\lim_{h \rightarrow 0^+} \frac{A(x+h) - A(x)}{h} = f(x) .$$

A similar argument, using small negative h and reversed inequalities above, proves that

$$\lim_{h \rightarrow 0^-} \frac{A(x+h) - A(x)}{h} = f(x) .$$

Thus we have the two-sided limit

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = f(x) .$$

This proves that $A(x)$ is an antiderivative of $f(x)$.

Suppose now that $F(x)$ is any antiderivative of $f(x)$. Then, there exists a constant C such that

$$F(x) = A(x) + C ,$$

for all $x \in [a, b]$.

(There is some subtlety about this equation at the endpoints of the interval, but this is taken care of by the fact that f is continuous.)

To find C , observe that

$$F(a) = A(a) + C = 0 + C = C .$$

so $C = F(a)$, and the rule for F becomes

$$F(x) = A(x) + F(a) .$$

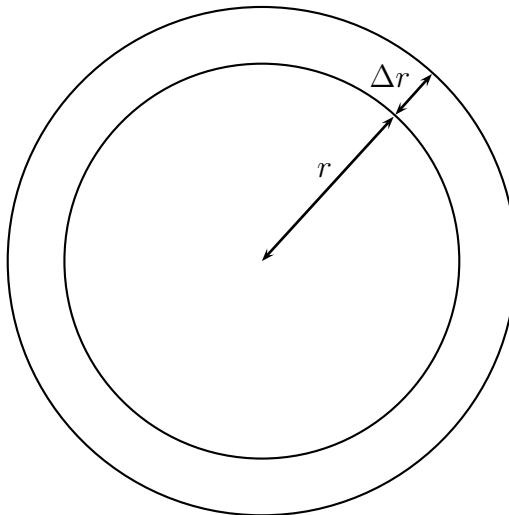
In particular, $F(b) = A(b) + F(a)$. Rearranging this, we get

$$\int_a^b f(t) dt = A(b) = F(b) - F(a) ,$$

completing the sketch of the proof of the Fundamental Theorem of Calculus.

2. We will use the idea of the previous proof to explain why the derivative of the formula for the area of a circle with respect to its radius should produce the formula for the perimeter (without knowing either explicitly).

Let $A = A(r)$ and $P = P(r)$ denote the area and perimeter respectively of a circle of radius r , and consider a small positive change Δr in the radius r , which propagates a small positive change ΔA in the area.



The annulus whose area is represented by ΔA has uniform width Δr throughout, a smaller perimeter $P(r)$ for the inner circle, and a larger perimeter $P(r + \Delta r)$ for the outer circle. Hence the area of the annulus must be bounded below by $P(r)\Delta r$ and bounded above by $P(r + \Delta r)\Delta r$, producing the following cascade of inequalities:

$$P(r)\Delta r \leq \Delta A \leq P(r + \Delta r)\Delta r .$$

Dividing through by Δr , which is positive, retains the inequalities, and we get

$$P(r) \leq \frac{\Delta A}{\Delta r} \leq P(r + \Delta r) .$$

As $\Delta r \rightarrow 0^+$, certainly $P(r) \rightarrow P(r)$, and also $P(r + \Delta r) \rightarrow P(r)$, by continuity of the perimeter function. By the squeeze law

$$\lim_{\Delta r \rightarrow 0^+} \frac{\Delta A}{\Delta r} = P(r) .$$

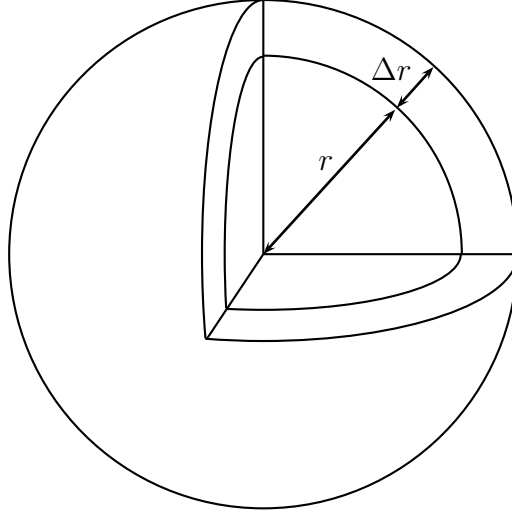
But this limit on the left-hand side is just the derivative (even though expressed here as a one-sided limit), so that

$$\frac{dA}{dr} = P ,$$

as expected. Of course, this matches behaviour we have already observed with regard to explicit formulae for the area and perimeter of a circle.

3. We can naturally extend the idea of the previous example to demonstrate that the derivative of a formula for the volume of a sphere with respect to its radius should produce a formula for the surface area (without knowing either explicitly).

Let $V = V(r)$ and $S = S(r)$ denote the volume and surface area respectively of a sphere of radius r , and consider a small positive change Δr in the radius r , which propagates a small positive change ΔV in the volume.



The “crust” whose volume is represented by ΔV has uniform thickness Δr throughout, all over the sphere. It interfaces with a smaller surface area $S(r)$ for the inner sphere, and a larger surface area $S(r + \Delta r)$ for the outer sphere. Hence the volume of the “crust” must be bounded below by $S(r)\Delta r$ and bounded above by $S(r + \Delta r)\Delta r$, producing the following cascade of inequalities:

$$S(r)\Delta r \leq \Delta V \leq S(r + \Delta r)\Delta r .$$

Dividing through by Δr , which is positive, retains the inequalities, and we get

$$S(r) \leq \frac{\Delta V}{\Delta r} \leq S(r + \Delta r) .$$

As $\Delta r \rightarrow 0^+$, certainly $S(r) \rightarrow S(r)$, and also $S(r + \Delta r) \rightarrow S(r)$, by continuity of the surface area function. By the squeeze law

$$\lim_{\Delta r \rightarrow 0^+} \frac{\Delta V}{\Delta r} = S(r) .$$

But this limit on the left-hand side is just the derivative (even though expressed here as a one-sided limit), so that

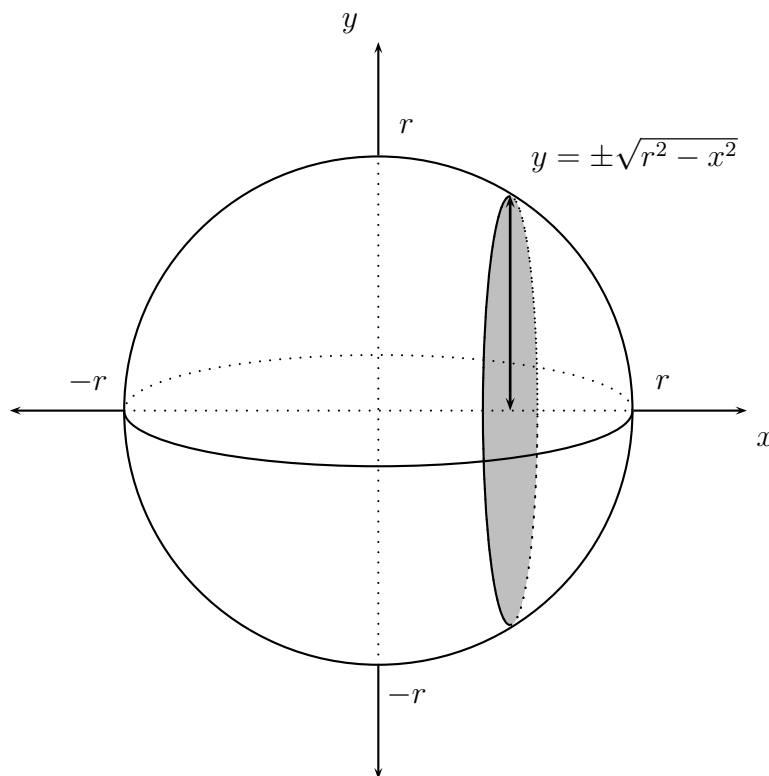
$$\frac{dV}{dr} = S ,$$

as expected.

The upshot of this is that we can deduce a formula for the surface area if we can find an explicit formula for the volume of a sphere, and then just differentiate it with respect to the radius.

4. We now find an explicit formula for the volume of a sphere by integrating its cross-sectional area. The method we are about to use is an instance of what is called the *disk method* in more advanced calculus courses, and we are calculating a *volume of revolution*.

Consider a sphere of radius r positioned in space with the xy -plane forming a vertical cross-section, with the centre of the sphere located at the origin.



The profile of the sphere in the xy -plane is a circle with equation $x^2 + y^2 = r^2$, so that

$$y = \pm\sqrt{r^2 - x^2}.$$

Consider x between $-r$ and r , and form a vertical cross-section of the sphere that is perpendicular to the xy -plane, passing through the x -axis at this particular value of x . The cross-section is a circle with radius $y = \sqrt{r^2 - x^2}$, so its area becomes

$$\pi y^2 = \pi(r^2 - x^2).$$

To get the volume of the sphere, we think of taking a “continuous sum” of these “infinitesimal” vertical cross-sections, as x ranges from $-r$ to r . This idea is intuitive, but it can be formalised using Riemann sums and circular disks of increasingly small widths. Using this idea, the volume of the sphere then becomes

$$\begin{aligned} \int_{-r}^r \pi y^2 dx &= \pi \int_{-r}^r (r^2 - x^2) dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \pi \left(r^3 - \frac{r^3}{3} - r^2(-r) + \frac{(-r)^3}{3} \right) \\ &= \pi \left(2r^3 - \frac{2r^3}{3} \right) = \frac{4\pi r^3}{3}, \end{aligned}$$

recovering (discovering!) the usual formula

$$V = \frac{4\pi r^3}{3},$$

for the volume of a sphere. We then recover (discover!) the usual formula for the surface area of a sphere:

$$S = \frac{dV}{dr} = \frac{d}{dr} \left(\frac{4\pi r^3}{3} \right) = 4\pi r^2.$$

Notes for ‘Integration by substitution’

Important Ideas and Useful Facts:

- (i) The word *integration* has two meanings: The word *integration* can refer to the process of forming an antiderivative of a function $f(x)$, expressed as the indefinite integral

$$\int f(x) \, dx .$$

The word *integration* can also refer to the process of finding or evaluating the area under the curve $y = f(x)$ over an interval $[a, b]$, represented by the definite integral

$$\int_a^b f(x) \, dx .$$

The terminology always has to be read in context. Of course, the two meanings are closely related, because if $F(x)$ is any antiderivative of $f(x)$ then the Fundamental Theorem of Calculus tells us that

$$\int_a^b f(x) \, dx = F(b) - F(a) .$$

- (ii) **Integration by substitution:** The method of *integration by substitution* typically refers to a method of antidifferentiation, and says that

$$\int \left(f(u) \frac{du}{dx} \right) dx = \int f(u) \, du ,$$

or, equivalently, in function notation, where $u = g(x)$ for some differentiable function g ,

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du .$$

The first form, expressed in Leibniz notation, is particularly easy to remember, because one can think of the differential dx , appearing at the end of the expression on the left-hand side, as cancelling with the differential dx appearing in $\frac{du}{dx}$, as if $\frac{du}{dx}$ were an ordinary fraction involving numbers. It is common to move freely backwards and forwards between the left and right-hand sides using equations involving differentials.

The method can also be applied using definite integrals, where we add appropriate terminals, a and b for the variable x , on the left-hand side, and $g(a)$ and $g(b)$ for the variable u , on the right-hand side:

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du .$$

Examples and derivations:

1. The rule for integration by substitution follows from the Chain Rule for differentiation. To see this, suppose that y is an antiderivative of $f(u)$ with respect to the variable u , so its derivative is $f(u)$, that is,

$$\frac{dy}{du} = f(u) .$$

We can express this relationship using an equation involving an indefinite integral:

$$\int f(u) \, du = y + C , \tag{1}$$

for some constant of integration C . But, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = f(u) \frac{du}{dx} ,$$

so that y is also an antiderivative of $f(u) \frac{du}{dx}$, but now with respect to x . Again, this can be expressed as an equation involving an indefinite integral:

$$\int \left(f(u) \frac{du}{dx} \right) dx = y + C , \tag{2}$$

for some constant of integration C , which may be taken to be the same constant appearing earlier. Hence, by (1) and (2), we have

$$\int \left(f(u) \frac{du}{dx} \right) dx = y + C = \int f(u) \, du ,$$

which finally gives us the integration by substitution formula.

2. Find the indefinite integral $\int 6x(x^2 + 1)^2 \, dx$ by making a substitution $u = x^2 + 1$.

Solution: Put $u = x^2 + 1$, so that $\frac{du}{dx} = 2x$, which we may write as $du = 2x dx$. Putting everything in terms of u , by reorganising the integrand and constants, we get

$$\int 6x(x^2 + 1)^2 \, dx = \int 3(x^2 + 1)^2 (2x dx) = \int 3u^2 \, du = u^3 + C = (x^2 + 1)^3 + C .$$

3. Find the indefinite integral $\int e^{-x} \, dx$.

Solution: Put $u = -x$. Hence $\frac{du}{dx} = -1$, so that $du = -dx$, which can be written as $dx = -du$. Expressing everything in terms of u , including the differential, we have

$$\int e^{-x} \, dx = \int e^u (-du) = - \int e^u \, du = -e^u + C = -e^{-x} + C .$$

4. Find the definite integral $\int_0^{\ln 2} e^{-x} \, dx$.

Solution: Using the indefinite integral from the previous exercise, we have

$$\int_0^{\ln 2} e^{-x} \, dx = [-e^{-x}]_0^{\ln 2} = -e^{-\ln 2} - (-e^{-0}) = -(e^{\ln 2})^{-1} - (-1) = -2^{-1} + 1 = \frac{1}{2} .$$

5. Find the indefinite integral $\int \left(\frac{x-1}{2}\right)^3 dx$.

Solution: Put $u = \frac{x-1}{2}$. Then $\frac{du}{dx} = \frac{1}{2}$, so that $du = \frac{1}{2}dx$, which can be rewritten as $dx = 2du$. Expressing everything in terms of u , including the differential, we have

$$\int \left(\frac{x-1}{2}\right)^3 dx = \int u^3 (2du) = 2 \int u^3 du = \frac{2u^4}{4} + C = \frac{u^4}{2} + C = \frac{(x-1)^4}{32} + C.$$

6. Find the definite integral $\int_1^2 \left(\frac{x-1}{2}\right)^3 dx$.

Solution: Using the indefinite integral from the previous exercise, we have

$$\int_1^2 \left(\frac{x-1}{2}\right)^3 dx = \left[\frac{(x-1)^4}{32}\right]_1^2 = \frac{1}{32}[(x-1)^4]_1^2 = \frac{1}{32}((2-1)^4 - (1-1)^4) = \frac{1}{32}.$$

7. Find the definite integral $\int_{\pi/4}^{\pi/2} \sin(2x) dx$.

Solution: Put $u = 2x$, so that $du = 2dx$, giving $dx = \frac{1}{2}du$. Putting everything in terms of u , we get the following indefinite integral:

$$\int \sin(2x) dx = \int \sin u \left(\frac{1}{2}du\right) = \frac{1}{2} \int \sin u du = \frac{-\cos u}{2} + C = \frac{-\cos(2x)}{2} + C.$$

Now we find the definite integral (keeping the terminals in terms of x):

$$\int_{\pi/4}^{\pi/2} \sin(2x) dx = \frac{1}{2}[-\cos(2x)]_{\pi/4}^{\pi/2} = \frac{1}{2}\left(-\cos(\pi) - \left(-\cos\left(\frac{\pi}{2}\right)\right)\right) = \frac{1}{2}(-(-1)+0) = \frac{1}{2}.$$

8. Find the definite integral $\int_0^{\pi/4} \cos\left(2x - \frac{\pi}{2}\right) dx$.

Solution: Put $u = 2x - \frac{\pi}{2}$, so that $du = 2dx$, giving $dx = \frac{1}{2}du$. Further, when $x = 0$ we have $u = -\frac{\pi}{2}$, and when $x = \pi/4$ we have $u = 0$. Hence, putting everything in terms of u , including converting the terminals, we get

$$\begin{aligned} \int_0^{\pi/4} \cos\left(2x - \frac{\pi}{2}\right) dx &= \int_{-\pi/2}^0 \cos u \left(\frac{1}{2}du\right) = \frac{1}{2} \int_{-\pi/2}^0 \cos u du = \frac{1}{2}[\sin u]_{-\pi/2}^0 \\ &= \frac{1}{2}\left(\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}(0 - (-1)) = \frac{1}{2}. \end{aligned}$$

9. Find the indefinite integral $\int xe^{-x^2} dx$.

Solution: Put $u = -x^2$, so that $\frac{du}{dx} = -2x$, so that $du = -2xdx$. Putting everything in terms of u , we get

$$\int xe^{-x^2} dx = -\frac{1}{2} \int e^{-x^2} (-2xdx) = -\frac{1}{2} \int e^u du = -\frac{e^u}{2} + C = -\frac{e^{-x^2}}{2} + C.$$

10. Find the indefinite integral $\int \sin^3 x \cos x \, dx$.

Solution: Put $u = \sin x$, so that $\frac{du}{dx} = \cos x$, so that $du = \cos x dx$. Putting everything in terms of u , we get

$$\int \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C.$$

11. Find the indefinite integral $\int \frac{4x-3}{x^2+1} \, dx$.

Solution: We split the integral into two pieces:

$$\int \frac{4x-3}{x^2+1} \, dx = \int \frac{4x}{x^2+1} \, dx - 3 \int \frac{dx}{x^2+1}.$$

For the first piece, put $u = x^2 + 1$, so that $\frac{du}{dx} = 2x$, so that $du = 2x dx$. Hence

$$\int \frac{4x}{x^2+1} \, dx = 2 \int \frac{1}{x^2+1} (2x dx) = 2 \int \frac{1}{u} \, du = 2 \ln |u| + C = 2 \ln(x^2+1) + C,$$

noting that $x^2 + 1$ is always positive.

Continuing, we have

$$\int \frac{4x-3}{x^2+1} \, dx = \int \frac{4x}{x^2+1} \, dx - 3 \int \frac{dx}{x^2+1} = 2 \ln(x^2+1) - 3 \tan^{-1} x + C.$$

12. Find the definite integral $\int_{-1}^0 x\sqrt{x+1} \, dx$.

Solution: Put $u = \sqrt{x+1} = (x+1)^{1/2}$. Hence

$$\frac{du}{dx} = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}} = \frac{1}{2u}.$$

This becomes $du = \frac{dx}{2u}$, which we may rewrite as $dx = 2u du$. Note also that $u^2 = x+1$, so that $x = u^2 - 1$.

Hence, expressing everything in terms of u , we get

$$\begin{aligned} \int x\sqrt{x+1} \, dx &= \int (u^2 - 1)(u)(2u du) = 2 \int (u^2 - 1)u^2 \, du = 2 \int (u^4 - u^2) \, du \\ &= 2 \left(\frac{u^5}{5} - \frac{u^3}{3} \right) + C = \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C. \end{aligned}$$

We can now evaluate the definite integral:

$$\int_{-1}^0 x\sqrt{x+1} \, dx = \left[\frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} \right]_{-1}^0 = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15}.$$

Notes for ‘Odd and even functions’

Important Ideas and Useful Facts:

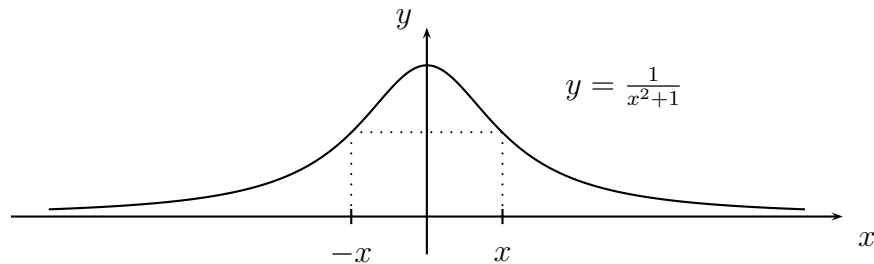
- (i) **Even functions:** An *even function* has the property that its graph has *reflectional symmetry about the y-axis*, by which we mean that if we reflect the curve in the y-axis then we get the curve back again. If $y = f(x)$ is a function, then this is equivalent to the algebraic condition that if x is in the domain of f then $-x$ is also in the domain and

$$f(-x) = f(x) .$$

For example, if $f(x) = \frac{1}{x^2 + 1}$, the graph is the familiar *witch of Maria Agnesi*, and

$$f(-x) = \frac{1}{(-x)^2 + 1} = \frac{1}{x^2 + 1} = f(x) ,$$

so the function is even.



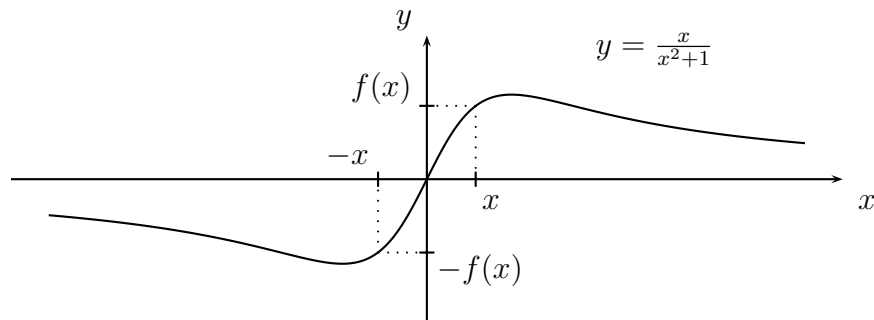
- (ii) **Odd functions:** An *odd function* has the property that its graph has 180° *rotational symmetry about the origin*, by which we mean that if we rotate the curve 180° about the origin then we get the curve back again. If $y = f(x)$ is a function, then this is equivalent to the algebraic condition that if x is in the domain of f then $-x$ is also in the domain and

$$f(-x) = -f(x) .$$

For example, if $f(x) = \frac{x}{x^2 + 1}$, the graph is *Newton's serpentine*, and

$$f(-x) = \frac{-x}{(-x)^2 + 1} = -\frac{x}{x^2 + 1} = -f(x) ,$$

so the function is odd.



- (iii) **Integrating an even function over a symmetric interval:** An interval in the real line is called *symmetric* if it has the form $[-a, a]$ for some real number a . The area under the curve of an even function $y = f(x)$ over a symmetric interval $[-a, a]$ is twice the area over the nonnegative interval $[0, a]$, that is,

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx .$$

For example,

$$\int_{-1}^1 \frac{dx}{x^2 + 1} = 2 \int_0^1 \frac{dx}{x^2 + 1} = 2 [\tan^{-1} x]_0^1 = 2 \times \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{2} .$$

- (iv) **Integrating an odd function over a symmetric interval:** The area under the curve of an odd function $y = f(x)$ over a symmetric interval $[-a, a]$ is zero, that is,

$$\int_{-a}^a f(x) dx = 0 .$$

For example, we get the following automatically:

$$\int_{-1}^1 \frac{x}{x^2 + 1} dx = 0 ,$$

without any need to antidifferentiate the integrand.

- (v) **Arithmetic of odd and even functions:** The sum, difference, product or quotient of two even functions is even.

For example, $y = x^2$ and $y = \cos x$ are both even, so the following are even also:

$$y = x^2 + \cos x , \quad y = x^2 - \cos x , \quad y = x^2 \cos x , \quad y = \frac{x^2}{\cos x} , \quad y = \frac{\cos x}{x^2} .$$

The sum and difference of two odd functions is odd, but the product and quotient of two odd functions is even.

For example, $y = \sin x$ and $y = x$ are both odd, so the following are odd also:

$$y = x + \sin x , \quad y = x - \sin x , \quad -x + \sin x ,$$

but the following are even:

$$y = x \sin x , \quad y = \frac{x}{\sin x} , \quad y = \frac{\sin x}{x} .$$

The product or quotient of an even function with an odd function is odd.

For example, $y = \cos x$ is even and $y = \sin x$ is odd, so the following are odd:

$$y = \cos x \sin x , \quad y = \frac{\sin x}{\cos x} = \tan x , \quad y = \frac{\cos x}{\sin x} .$$

The inverse f^{-1} of an invertible odd function f is odd. This follows because if x is in the domain of f^{-1} then $x = f(a)$ for some a so that

$$-x = -f(a) = f(-a) ,$$

since f is odd, so that $-x$ is in the domain of f^{-1} , and further

$$f^{-1}(-x) = -a = -f^{-1}(x) .$$

For example, the function $y = \sin x$ is odd and invertible when restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so that the inverse function $y = \sin^{-1}(x)$ is also odd.

Examples and derivations:

1. Taking the sum or difference of an even function with an odd function may produce a function that is neither odd nor even. For example, $y = x$ is odd and $y = 1 + x^2$ is even. Let f and g be functions with the rules

$$f(x) = 1 + x + x^2 \quad \text{and} \quad g(x) = 1 - x + x^2 .$$

Then f is neither even nor odd. For example, $f(1) = 3$, whilst $f(-1) = 1$, which does not equal $f(1)$ or $-f(1)$. Similarly, g is neither even nor odd. For example, $g(1) = 1$, whilst $g(-1) = 3$, which does not equal $g(1)$ or $g(-1)$.

2. Let f be the function with rule $y = f(x) = x^n$, where n is an integer. If n is even, say $n = 2k$, for some integer k , then

$$f(-x) = (-x)^n = (-1)^n x^n = (-1)^{2k} x^n = ((-1)^2)^k x^n = (1)x^n = x^n = f(x) ,$$

which shows that f is even. For example, the following functions are even:

$$y = 1 = x^0, \quad y = x^8, \quad y = x^{-2} = \frac{1}{x^2}, \quad y = x^{-8} = \frac{1}{x^8} .$$

On the other hand, if n is odd, say $n = 2k + 1$, for some integer k , then

$$f(-x) = (-x)^n = (-x)^{2k+1} = (-x)^{2k}(-x) = x^{2k}(-x) = -x^{2k+1} = -f(x) ,$$

which shows that f is odd. For example, the following functions are odd:

$$y = x = x^1, \quad y = x^9, \quad y = x^{-3} = \frac{1}{x^3}, \quad y = x^{-9} = \frac{1}{x^9} .$$

3. From the previous example, and the arithmetic of even and odd functions, it follows that any polynomial or rational function involving only even powers of x is even. For example, the following are even:

$$y = 1 + 2x^2 - 3x^4 + 7x^6 \quad \text{and} \quad y = \frac{3 - 7x^2 + 5x^4 - x^{10}}{1 - x^2 + x^6 - x^{12}} .$$

Note that each of the constants (even if odd integers) represent even functions!

Any polynomial involving only odd powers of x will be odd. For example, the following function is odd:

$$y = 2x + 3x^5 - 6x^7 + x^{99} .$$

The situation involving mixtures of odd and even, when forming rational functions, is more delicate. If $y = f(x) = \frac{p(x)}{q(x)}$ is a rational function, say, a ratio of polynomials $p(x)$ and $q(x)$, then f is even if both $p(x)$ and $q(x)$ are odd, but odd if only one of $p(x)$ and $q(x)$ is even and the other is odd. For example, the following function is even:

$$y = \frac{x - x^3 + x^7}{x + x^5 + x^9} ,$$

and the following functions are odd:

$$y = \frac{x - x^3 + x^7}{1 + x^4 + x^8} \quad \text{and} \quad \frac{1 + x^4 + x^8}{x - x^3 + x^7} .$$

4. Verify algebraically that if f and g are odd with the same domain, then h is even, where h has the rule $h(x) = f(x)g(x)$.

Solution: For x in the domain of h , we have, using the fact that both f and g are odd,

$$h(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = h(x).$$

5. Verify algebraically that if f is even and g is odd with the same domain, then h is odd, where h has the rule $h(x) = f(x)g(x)$.

Solution: For x in the domain of h , we have, using the fact that f is even and g is odd,

$$h(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -h(x).$$

6. This exercise is very difficult and challenging, and included to demonstrate the surprising power of the notions of even and odd functions. Find

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^{-x}} dx.$$

As a hint, if you graph the curve $y = \frac{1}{1+e^{-x}}$, you will discover that it has 180° rotational symmetry about the point of inflection, so can be turned into an odd function by an appropriate translation in the plane.

Solution: We begin by sketching the curve

$$y = \frac{1}{1 + e^{-x}} = (1 + e^{-x})^{-1}.$$

By the Chain Rule,

$$y' = -(1 + e^{-x})^{-2}(-e^{-x}) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

which is clearly positive for all x . Hence the curve is increasing over all of the real line $\mathbb{R}$. By the Quotient Rule,

$$\begin{aligned} y'' &= \frac{(1 + e^{-x})^2(-e^{-x}) - (e^{-x})(2)(1 + e^{-x})(-e^{-x})}{(1 + e^{-x})^4} \\ &= \frac{-e^{-x}(1 + e^{-x}) + 2(e^{-x})(e^{-x})}{(1 + e^{-x})^3} = \frac{e^{-x}(e^{-x} - 1)}{(1 + e^{-x})^3} = \frac{e^{-2x}(1 - e^x)}{(1 + e^{-x})^3}, \end{aligned}$$

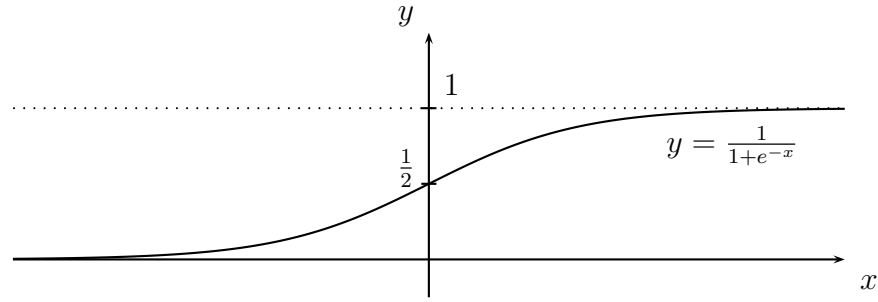
which is zero only when $1 - e^x = 0$, that is, when $e^x = 1$, giving $x = 0$. Further, the sign of y'' is the same as the sign of $1 - e^x$, so we get the following sign diagram:

$$\begin{array}{c|ccc} x & & 0 & \\ \hline y'' & & 0 & \\ & + & & - \\ & \cup & & \cap \end{array}$$

Thus the curve is increasing with an inflection at $x = 0$, that is at the y -intercept $y(0) = \frac{1}{2}$. Observe that

$$\lim_{x \rightarrow \infty} \frac{1}{1 + e^{-x}} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{1 + e^{-x}} = 0,$$

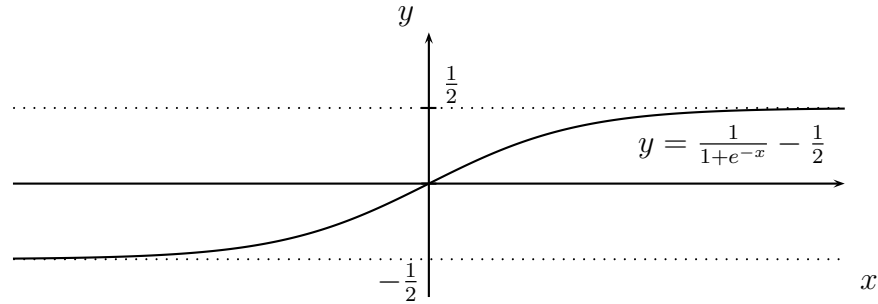
since $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$, and $e^{-x} \rightarrow \infty$ as $x \rightarrow -\infty$. Thus the x -axis and the line $y = 1$ are horizontal asymptotes, and we get the following sketch:



The graph appears to have 180° rotational symmetry about the point of inflection, so if we shift the graph uniformly $\frac{1}{2}$ vertically downwards, we should be able to get rotational symmetry about the origin. We therefore form the function f defined by the rule

$$f(x) = \frac{1}{1+e^{-x}} - \frac{1}{2} = \frac{2 - (1+e^{-x})}{2(1+e^{-x})} = \frac{1-e^{-x}}{2(1+e^{-x})},$$

whose graph should look as follows:



If we are right, then f should be an odd function. How can we check? We observe that, for any input x ,

$$f(-x) = \frac{1 - e^{-(-x)}}{2(1 + e^{-(-x)})} = \frac{1 - e^x}{2(1 + e^x)} = \frac{e^{-x} - 1}{2(e^{-x} + 1)} = -\frac{1 - e^{-x}}{2(1 + e^{-x})} = -f(x),$$

which proves, without any doubt, that f is odd. Hence the function $y = f(x) \cos x$ must be odd, since $y = \cos x$ is even. Now we can return to the problem at hand and calculate as follows, exploiting the fact that the definite integral of an odd function over a symmetric interval is zero, and we double the definite integral of an even function over the nonnegative half of the interval:

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{-x}} dx &= \int_{-\pi/2}^{\pi/2} \cos x \left(\frac{1}{1+e^{-x}} \right) dx = \int_{-\pi/2}^{\pi/2} \cos x \left(\frac{1}{1+e^{-x}} - \frac{1}{2} + \frac{1}{2} \right) dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x \left(f(x) + \frac{1}{2} \right) dx = \int_{-\pi/2}^{\pi/2} f(x) \cos x dx + \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos x dx \\ &= 0 + \frac{2}{2} \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1. \end{aligned}$$

Notes for ‘The logistic function’

Important Ideas and Useful Facts:

- (i) **The exponential growth model:** Consider a population of size $x = x(t) > 0$, which is a function of time t . The exponential growth model assumes that the growth rate of x with respect to t is proportional to x , that is,

$$\frac{dx}{dt} = kx$$

for some constant $k > 0$. This makes precise the idea that, the more there is of the population or colony, the faster it should grow. One can show that the only solutions to this *differential equation* (an equation involving a derivative) have the form

$$x = x(t) = Ae^{kt}$$

for some constant $A > 0$, which is the usual rule for exponential growth.

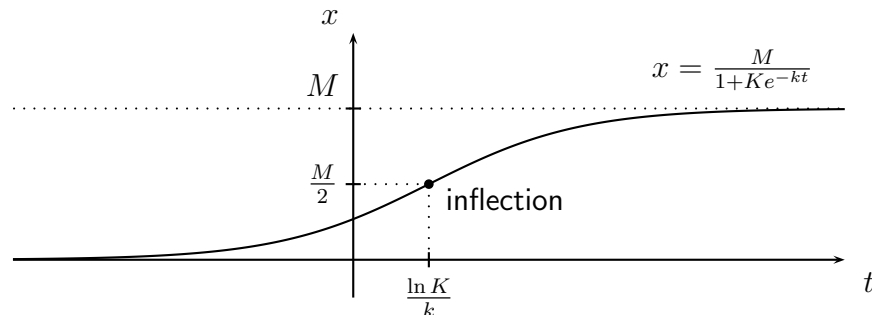
- (ii) **The logistic growth model:** The exponential growth model is modified to take into account constraints on the growth of the population caused by limited resources. A more sophisticated model presumes there is a largest sustainable population size, say $M > 0$, and adjusts the differential equation above for $x = x(t)$, where $k > 0$, by including an *inhibition factor* $(1 - \frac{x}{M})$, which is close to one when x is small, but close to zero when x is close to M :

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right).$$

This (differential equation) is called the *logistic equation*. Its solutions are called *logistic functions*, and they all have the form

$$x = x(t) = \frac{M}{1 + Ke^{-kt}}$$

for some constant $K > 0$. The graph of x (where x labels the vertical axis, whilst t labels the horizontal axis) has a *sigmoid* shape, concave up followed by concave down. The curve is increasing, with horizontal asymptotes $x = 0$ and $x = M$, and has 180° rotational symmetry about a point of inflection that occurs when $x = \frac{M}{2}$ and $t = \frac{\ln K}{k}$.



Examples and derivations:

1. Find integers A and B such that $\frac{1}{119} = \frac{1}{7 \times 17} = \frac{A}{7} + \frac{B}{17}$.

Solution: We want $1 = 17A + 7B$. We look for multiples of 17 and 7 that differ by 1, say $34 = 2 \times 17$ and $35 = 5 \times 7$, so take $A = -2$ and $B = 5$, and then we have

$$\frac{1}{119} = \frac{-2}{7} + \frac{5}{17}.$$

2. Find constants A and B such that

$$\frac{1}{2x - x^2} = \frac{1}{x(2 - x)} = \frac{A}{x} + \frac{B}{2 - x}.$$

Solution: After rewriting the right-hand side with a common denominator, we find that we need

$$1 = A(2 - x) + Bx = 2A + (B - A)x.$$

To get constants to match, we want $1 = 2A$, so that $A = \frac{1}{2}$. It follows that $B = \frac{1}{2}$ also, so we have

$$\frac{1}{2x - x^2} = \frac{1}{x(2 - x)} = \frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{2 - x}.$$

3. Find the indefinite integral $\int \frac{dx}{2 - x}$ (assuming $0 < x < 2$).

Solution: Put $u = 2 - x$, so that $\frac{du}{dx} = -1$. Hence $du = -dx$ and $dx = -du$, and we get

$$\int \frac{dx}{2 - x} = \int \frac{-du}{u} = - \int \frac{du}{u} = -\ln u + C = -\ln(2 - x) + C,$$

noting that $u > 0$, for the third step.

4. Find the indefinite integral $\int \frac{dx}{2x - x^2}$ (assuming $0 < x < 2$).

Solution: Using the solutions to the two previous exercises, we get

$$\begin{aligned} \int \frac{dx}{2x - x^2} &= \int \left(\frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{2 - x} \right) dx = \frac{1}{2} \left(\int \frac{dx}{x} + \int \frac{dx}{2 - x} \right) \\ &= \frac{1}{2} \left(\ln x - \ln(2 - x) \right) + C = \frac{1}{2} \ln \left(\frac{x}{2 - x} \right) + C. \end{aligned}$$

5. Consider the logistic function

$$x = x(t) = \frac{M}{1 + Ke^{-kt}},$$

where $M, K, k > 0$. Use interval notation to describe the range and show that the derivative $x'(t)$ is maximised when

$$t = \frac{\ln K}{k},$$

given that this occurs when $x = \frac{M}{2}$.

Solution: The curve is sandwiched between horizontal asymptotes $x = 0$ and $x = M$, so that the range is the interval $(0, M)$. The derivative is maximised when

$$\frac{M}{2} = \frac{M}{1 + Ke^{-kt}} ,$$

so that $2 = 1 + Ke^{-kt}$, giving $Ke^{-kt} = 1$, so that $K = e^{kt}$. Taking natural logarithms and rearranging, we get

$$t = \frac{\ln K}{k} .$$

6. A rumour has been spread, amongst a population of 50,000 people, that calculus is fun. It was seeded by ten people, and after 10 days, it is estimated that 1,000 people know about the rumour. Let $x(t)$ denote the number of people that know about the rumour after t days, and assume a logistic model, so that

$$x = x(t) = \frac{50,000}{1 + Ke^{-kt}}$$

for some positive constants K and k . Find

- (a) exact expressions for K and k ,
- (b) the number of days after which the rumour is being spread most rapidly, and
- (c) the number of days after which 40,000 people know about the rumour.

Solution: (a) We know that

$$10 = x(0) = \frac{50,000}{1 + Ke^0} = \frac{50,000}{1 + K} ,$$

so that $1 + K = 5000$, so that $K = 4999$.

We also know that

$$1,000 = x(10) = \frac{50,000}{1 + Ke^{-10k}} .$$

Hence $1 + Ke^{-10k} = 50$, so that $Ke^{-10k} = 49$, which becomes $e^{10k} = \frac{K}{49}$. Taking natural logarithms and rearranging, we get

$$k = \frac{\ln(K/49)}{10} = \frac{\ln(4999/49)}{10} .$$

(b) The rumour is being spread most rapidly when

$$t = \frac{\ln K}{k} = \frac{10 \ln(4999)}{\ln(4999/49)} \approx 18.4 ,$$

which is after about 18 days (to the nearest day).

(c) We want t such that

$$40,000 = \frac{50,000}{1 + Ke^{-kt}}$$

so that $1 + Ke^{-kt} = \frac{5}{4}$, that is, $Ke^{-kt} = \frac{1}{4}$, so that $e^{kt} = 4K$. This becomes

$$t = \frac{\ln(4K)}{k} = \frac{10 \ln(4 \times 4999)}{\ln(4999/49)} \approx 21.4 ,$$

which is after about 21 days (to the nearest day).

7. Suppose a yeast colony starts off with a population of 100 cells and is growing most rapidly 10 hours later, when it reaches 500 cells. Assuming logistic growth, find an upper bound for the colony size, and how long it takes to reach 75% and then 90% of this maximum.

Solution: Let t be the number of hours and $x = x(t)$ be the size of the colony, so that

$$x = \frac{M}{1 + Ke^{-kt}} ,$$

for some constants $M, K, k > 0$.

The population is growing most rapidly after $t = 10$ hours, at which time it reaches 500. This occurs at the point of inflection of the logistic curve, when the population has reached half of its potential maximum, so that $\frac{M}{2} = 500$. Hence, an upper bound for the colony size is $M = 1,000$.

We have

$$100 = x(0) = \frac{M}{1 + Ke^0} = \frac{1000}{1 + K} ,$$

which quickly rearranges to give $K = 9$.

We also have

$$500 = x(10) = \frac{M}{1 + Ke^{-10k}} = \frac{1000}{1 + 9e^{-10k}} ,$$

which quickly rearranges to give

$$e^{-10k} = \frac{1}{9} .$$

Hence $e^{10k} = 9$, so that, taking natural logarithms and rearranging, we get

$$k = \frac{\ln 9}{10} .$$

To reach 75% of the maximum, we need $x = 750$, so that $750 = \frac{1000}{1 + 9e^{-kt}}$, which rearranges to give

$$1 + 9e^{-kt} = \frac{1000}{750} = \frac{4}{3} .$$

Thus $e^{-kt} = \frac{1}{27}$, so that $e^{kt} = 27$. Taking natural logarithms and rearranging, we get

$$t = \frac{\ln(27)}{k} = \frac{10 \ln(27)}{\ln 9} = \frac{10 \ln(3^3)}{\ln(3^2)} = \frac{10 \times 3 \ln 3}{2 \ln 3} = 15 .$$

To reach 90% of the maximum, we need $x = 900$, so that $900 = \frac{1000}{1 + 9e^{-kt}}$, which rearranges to give

$$1 + 9e^{-kt} = \frac{1000}{900} = \frac{10}{9} .$$

Thus $e^{-kt} = \frac{1}{81}$, so that $e^{kt} = 81$. Taking natural logarithms and rearranging, we get

$$t = \frac{\ln(81)}{k} = \frac{10 \ln(81)}{\ln 9} = \frac{10 \ln(9^2)}{\ln 9} = \frac{10 \times 2 \ln 9}{\ln 9} = 20 .$$

Thus we expect to reach 75% of the maximum after 15 hours, and 90% after 20 hours.

8. We explain why the exponential growth model

$$\frac{dx}{dt} = kx ,$$

for some constant $k > 0$, only has solutions of the form $x = Ae^{kt}$ for some constant $A > 0$. We assume throughout that $x = x(t)$ is positive.

Proof: We multiply through by dt to get an equation involving differentials:

$$dx = kxdt$$

and then divide both sides by x to get

$$\frac{dx}{x} = kdt .$$

We then integrate both sides, to get an equation linking antiderivatives:

$$\int \frac{dx}{x} = \int k dt = kt + C ,$$

for some constant of integration C . On the left-hand side, we integrate to get the natural logarithm function, and can suppose that the constant of integration has been moved across and absorbed in the constant on the right, and we get

$$\ln x = kt + C .$$

Raising Euler's number e to the power of each side of this equation, we get

$$x = e^{\ln x} = e^{kt+C} = e^C e^{kt} = Ae^{kt} ,$$

for some constant A , taking $A = e^C$, completing the proof.

9. We explain why the logistic growth model

$$\frac{dx}{dt} = kx \left(1 - \frac{x}{M} \right) ,$$

for some constants $k, M > 0$, only has solutions of the form

$$x = \frac{M}{1 + Ke^{-kt}}$$

for some constant $K > 0$. We assume throughout that $0 < x < M$.

Proof: We multiply through by dt to get an equation involving differentials:

$$dx = kx \left(1 - \frac{x}{M} \right) dt$$

and then divide both sides by $x(1 - \frac{x}{M})$, putting everything involving x to the left-hand side, to get

$$\frac{dx}{x(1 - \frac{x}{M})} = kdt .$$

(This is an example of the technique of *separating variables*, which is explained in greater generality in more advanced courses in calculus.) We then integrate both sides, to get an equation linking antiderivatives:

$$\int \frac{dx}{x(1 - \frac{x}{M})} = \int k dt = kt + C, \quad (1)$$

for some constant of integration C . To unravel the left-hand side, we first observe that

$$\frac{1}{x(1 - \frac{x}{M})} = \frac{1}{x} + \frac{\frac{1}{M}}{1 - \frac{x}{M}} = \frac{1}{x} + \frac{1}{M - x}.$$

It is easy to check this equation holds. It is not so clear where it came from. It arises by postulating a decomposition of the left-hand side as a sum of individual fractions with denominators made up from each of the factors of the denominator of the left-hand side, and then using a “guess and check” method. (A more formal, systematic approach uses the *method of partial fractions*, explained in more advanced courses in calculus.)

We then get

$$\int \frac{dx}{x(1 - \frac{x}{M})} = \int \frac{dx}{x} + \int \frac{dx}{M - x} = \ln x - \ln(M - x) + C = \ln\left(\frac{x}{M - x}\right) + C,$$

where the second antiderivative can be deduced quickly by a “guess and check” method or using integration by substitution. Putting this together with (1), by combining the constants of integration on the right, we get

$$\ln\left(\frac{1}{\frac{M}{x} - 1}\right) = \ln\left(\frac{x}{M - x}\right) = kt + C.$$

Raising Euler’s number e to the power of each side of this equation, we get

$$\frac{1}{\frac{M}{x} - 1} = e^{kt+C} = e^C e^{kt},$$

so that, after reciprocating,

$$\frac{M}{x} - 1 = e^{-C} e^{-kt} = K e^{-kt},$$

for some constant K , taking $K = e^{-C}$, which, after rearranging, produces

$$x = \frac{M}{1 + K e^{-kt}},$$

completing the proof.

10. It is worth noting, in the rule above for the logistic function, that when $K e^{-kt}$ is large, then it dominates the fraction, so that the addition of 1 becomes negligible, so that

$$x = \frac{M}{1 + K e^{-kt}} \approx \frac{M}{K e^{-kt}} = \left(\frac{M}{K}\right) e^{kt}.$$

Thus, towards the left of the inflection point of the curve, the logistic function behaves approximately as an exponential function.

This makes sense in terms of the model: for small x , the inhibition factor is close to zero, and the logistic model is approximated well by the exponential growth model. As the population increases, its growth rate is maximised when it reaches $\frac{M}{2}$, after which the population continues to increase, but the growth rate decreases, and dwindles to a trickle the closer the population gets to the maximum carrying capacity M .

Notes for ‘The escape velocity of a rocket’

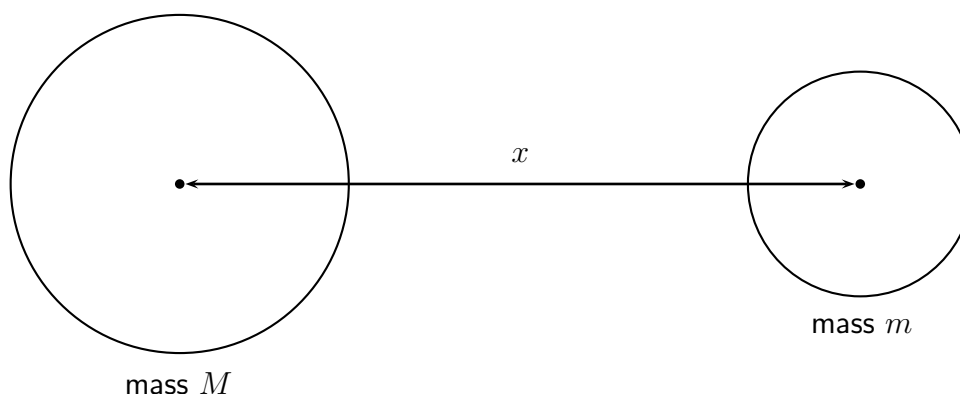
We aim to reconstruct, at least in spirit, Isaac’s Newton’s thought processes in estimating the *escape velocity* of an object, that is, the velocity one needs to impart to an object for it to escape the earth’s gravitational field. This turns out to be independent of the mass of the object.

The term ‘escape velocity’ is often associated with rockets, but of course, modern rockets have a continuous supply of fuel. Newton was in fact thinking in terms of imparting enough energy to a cannonball, so that it can, in principle, escape from the earth. In the following development, we have the benefit of hindsight, and use Leibniz notation and the Riemann integral, neither of which were available to Newton, who had his own heuristic and other devices for conceiving the underlying mathematics.

The main idea is that the gravitational force $F = F(x)$ between two objects, which is a function of the distance x units between them, is proportional to their masses, say M and m respectively, and inversely proportional to the square of the distance x . Hence, we have

$$F(x) = \frac{GMm}{x^2},$$

for some constant G (known as the *universal gravitational constant*). In the diagram below, the objects are perfectly spherical, but they could be irregular. The distance x is measured between their respective centres of gravity, which, in the case of spheres, becomes their usual centres.

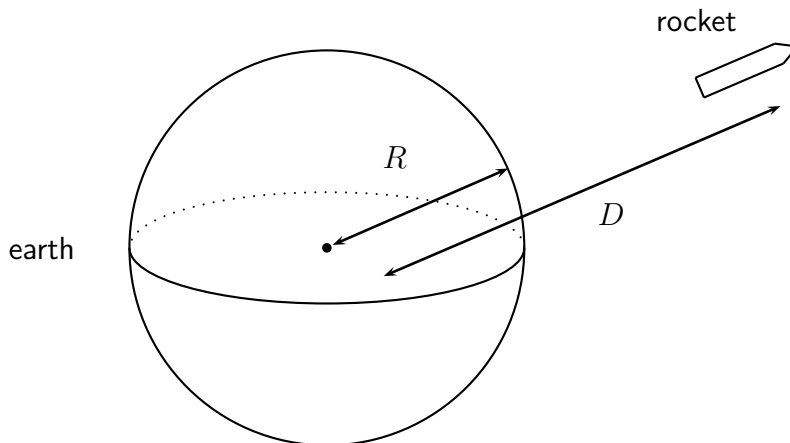


The numerator of the fraction in the expression for $F(x)$ makes sense, as the larger the masses, the greater one expects the gravitational force to be between them.

It also makes sense that $F(x)$ should be inversely proportional to the square of the distance. Remember, we showed that the surface area of a sphere is 4π times the square of the radius, so one can imagine the gravitational force radiating uniformly outwards from a spherical body, distributed evenly over 4π times the square of the distance from the centre. The gravitational force on any unit of area is therefore proportional to the inverse of the total surface area of an expanding sphere, which, in turn, is proportional to the inverse of the square of the distance to the centre of the sphere.

In fact, F becomes a function of several variables, if you include the masses of the objects as extra variables, though, in what follows, we think of M and m as constants, so that $F = F(x)$ becomes a function of the single variable x .

Denote the radius of the earth by R . Consider a rocket, moving away from the earth, and located in space, say, D units from the centre of the earth. We can ask how much energy $E = E(D)$, as a function of D , was expended, getting the rocket from the surface of the earth to its present position in space.



This can be described by the following definite integral:

$$E = E(D) = \int_R^D F(x) dx ,$$

which we briefly explain. From elementary physics, if an object is moved a small distance, say Δx by a force $F(x)$, then the work done by the force, equivalently the energy expended, is calculated by multiplying the force by the distance, that is

$$F(x) \times \Delta x .$$

If we imagine the interval $[R, D]$ partitioned into tiny pieces,

$$x_0 = R < x_1 < x_2 < \dots < x_{n-1} < x_n = D$$

and put $\Delta x_i = x_i - x_{i-1}$ for $i = 1$ to n , then an approximation of the work done, to get the rocket to its present position, is the Riemann sum

$$F(x_1)\Delta x_1 + F(x_2)\Delta x_2 + \dots + F(x_n)\Delta x_n = \sum_{i=1}^n F(x_i)\Delta x_i .$$

By imagining this partition getting finer and finer, the approximation approaches the exact amount of work done, which then becomes the definite integral above. We can find this using the formula for $F = F(x)$, where m is the mass of the rocket and M is the mass of the earth, applying the Fundamental Theorem of Calculus:

$$\begin{aligned}
E &= E(D) = \int_R^D F(x) dx = \int_R^D \frac{GMm}{x^2} dx = GMm \int_R^D \frac{dx}{x^2} \\
&= GMm \int_R^D x^{-2} dx = GMm \left[-x^{-1} \right]_R^D = GMm \left(-D^{-1} - (-R^{-1}) \right) \\
&= GMm \left(\frac{1}{R} - \frac{1}{D} \right).
\end{aligned}$$

Imagine now the rocket travelling arbitrarily far way from the earth (which vanishes to a single point from the view of the rocket), that is, $D \rightarrow \infty$, so that $\frac{1}{D} \rightarrow 0$. We then get the following limit:

$$\lim_{D \rightarrow \infty} E(D) = \lim_{D \rightarrow \infty} GMm \left(\frac{1}{R} - \frac{1}{D} \right) = GMm \left(\frac{1}{R} - 0 \right) = \frac{GMm}{R},$$

which represents the amount of energy required for the rocket to escape the earth's gravitational field completely. Newton was in fact thinking about cannonballs, not rockets, and this calculation tells us that $\frac{GMm}{R}$ is the energy required to impart to a cannonball, when fired from the earth's surface, so that it manages to escape from the earth, where now m denotes the mass of the cannonball.

Newton wanted to estimate the initial velocity v that would need to be imparted to the cannonball for this to occur, and we call v the *escape velocity*.

From elementary physics, the energy of an object of mass m and velocity v is given by the formula $\frac{1}{2}mv^2$. Newton therefore wanted v such that

$$\frac{mv^2}{2} = \frac{GMm}{R}.$$

Cancelling m from both sides, and rearranging, we get $v^2 = \frac{2GM}{R}$, so that, taking the positive square root,

$$v = \sqrt{\frac{2GM}{R}}.$$

To turn this into an actual figure, observe that

$$v = \sqrt{2R} \times \sqrt{\frac{GM}{R^2}}.$$

The diameter of the earth is $2R$, which was calculated to be approximately 12.74×10^6 metres, by the Greek scholar and mathematician Eratosthenes, from about 200 BC, using clever measurements and geometry, and this was known to Newton. Further, at the surface of the earth, the gravitational force (in appropriate units) acting on an object of mass m is

$$F(R) = \frac{GMm}{R^2} = m \left(\frac{GM}{R^2} \right).$$

But force equals the mass m times the acceleration $\frac{GM}{R^2}$, for an object falling freely near the earth's surface. Newton knew this was about 9.8 m/sec^2 . Hence, we may take the numerical value of $\frac{GM}{R^2}$ to be about 9.8, and so the estimate of the escape velocity becomes

$$v = \sqrt{\frac{2GM}{R}} \approx \sqrt{12.74 \times 10^6} \times \sqrt{9.8} \approx 11,000 \text{ m/sec}.$$

Thus the escape velocity of any object is about 11 kilometres per second.