ISI - 2015

Q. ~~Let (X,d) be a is a sequence in $[0,1)$~~

Qs Let (X,d) be a metric space. If there exist exist an uncountable set $A \subset X$ such that $\inf\{d(x,y) : x \in A, y \in A, x \neq y\} > 0$. show that X is not separable.

Soln: Let X be separable, so it would be M

First we take → According to question

$$d(x,y) = \inf\{d(x,y)\}. \text{ (given question)}$$

$$= \alpha > 0$$

Let X be separable, so it would be M.

Now take $x \in M$ and $(x_n)_{n \in \mathbb{N}} \subset M : x_n \longrightarrow x$

Standing to the hypothesis you would have

$d(x_n, x) = \alpha \; \forall \; n$.

so that is a contradiction that $d(x_n, x) = 0$

Note Here M is an uncountable discrete subsets of X

This mean that for $x \in M$. $\{x\}$ is open.

Q1 Prove that a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly to a continuous function f if and only if $f_n(x_n) \to f(x)$ whenever $x_n \to x$

Sol:-

Let $\varepsilon > 0$, since $f_n \to f$ uniformly, there is f_n so that $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in [0,1]$.

Since f_n is uniformly continuous, there is $\delta > 0$ so that $|f_n(x) - f_n(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. Thus $|f_n(x) - f_n(b)| < \varepsilon$.

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta$.

Thus f is uniformly continuous.

Q2 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose $\delta = \inf_{x \in \mathbb{R}} f'(x) > 0$. Prove that $f(a) = 0$ for some $a \in \mathbb{R}$

Sol:-

If $f(0) = 0$, there is nothing to show. Otherwise let $a \in \mathbb{R}$ be non-zero.

According to the mean value - theorem there exists c in between a and 0 satisfying

$$\frac{f(a) - f(0)}{a - 0} = f'(c) > \delta$$

Assume first that $f(0) < 0$. Then for each $a > 0$ the above observation implies that

$$\frac{f(a) - f(0)}{a} = f'(c)$$

$$\frac{f'(a) - \quad}{a} \Rightarrow \frac{f'(a) - f(0)}{a} > \delta.$$

$$f'(a) - f(0) > a\delta.$$

$$\boxed{f'(a) > f(0) + a\delta} \Rightarrow f(a) > f(0) + a\delta.$$

and by taking a sufficiently large we get $f(0) > 0$.

Now, $f(0) > 0$.
Then for each $a < 0$, the above observation implies that

$$\frac{f(a) - f(0)}{a} \cdot \frac{}{-0} = f'(c)$$

$$\Rightarrow \frac{f(a) - f(0)}{+a} < \delta. \Rightarrow f(a) - f(0) < +a\delta$$

$$\Rightarrow f(a) < f(0) + a\delta$$

$\Rightarrow$ by taking a sufficiently large

$$f(a) < 0$$

Now By using Intermediate ~~theorem~~ Value theorem, we get a $f(a) = 0$ for some $a \in \mathbb{R}$.

Q3. Show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$ where $f(x) = \left|\frac{\sin x}{x}\right|$

Sol:

$$|\sin(x)| \geq \sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

Hence $\left|\frac{\sin x}{x}\right| \geq \frac{1}{2x} - \frac{1}{2}\frac{\cos 2x}{x}$.

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$$

or

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \geq \int_\pi^\infty \left|\frac{\sin x}{x}\right|$$

$$= \lim_{n\to\infty} \int_\pi^{n\pi} \left|\frac{\sin x}{x}\right| dx$$

$$= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx$$

$$\geq \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \int_0^{(n+1)\pi} |\sin x|\,dx$$

Hence $\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx = 2$

$$= \sum_{n=0}^{\infty} \frac{2}{(n+1)\pi} = \infty$$

Indeed for all n

$$\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx = 2$$

Q1. If (a_n) is a sequence in $(0,1)$, show that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \longrightarrow 0 \text{ if and only if } \frac{1}{n}\sum_{k=1}^{n} a_k^2 \longrightarrow 0$$

Solution: Suppose (a_n) is a sequence in $(0,1)$ such that

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

We aim to show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Using Bryan College Station, Wikipedia, Bunyakovsky - Cauchy-Schwarz inequality:

$$0 \le \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 = \left(\left\langle \frac{1}{n}, \cdots \frac{1}{n} \right\rangle \langle a_1, \cdots a_n \rangle\right)^2$$

$$\le \left\| \left\langle \frac{1}{n}, \cdots \frac{1}{n} \right\rangle \right\|^2 \left\| \langle a_1, \cdots a_n \rangle \right\|^2$$

(Cauchy-Schwarz inequality)

$$= \left(\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^2 \right)(a_1^2 + \cdots + a_n^2)$$

$$= \frac{a_1^2 + \cdots + a_n^2}{} \times \left(\frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} \right)$$

$$= (a_1^2 + \cdots + a_n^2) \left(\frac{n}{n^2} \right)$$

$$= \frac{(a_1^2 + \cdots + a_n^2)}{n}$$

Then since $\frac{a_1^2 + \cdots + a_n^2}{n} \to 0$ as $n \to \infty$

It follows that $\left(\frac{a_1 + \cdots + a_n^2}{n} \right)^2 \to 0$ as $n \to \infty$

hence $\left(\frac{a_1 + \cdots + a_n}{n} \right) \to 0$ as $n \to \infty$

Hence proved.

Q. Section - B

Let $T : X \longrightarrow Y$ be an $\mathbb{R}$- linear map of $\mathbb{R}$ vector spaces X and Y of finite dimension.
Let $W \subseteq Y$ be an $\mathbb{R}$ vector-subspace such that $\text{Im}\,T$ and W together span Y.
Let $Z = T^{-1}(W)$. Show that

$$\dim X + \dim W = \dim Y + \dim Z.$$

Soln:

Let $r_X(T)$ for a rank as a map from X and $n_X(T)$ for its nullity

1. $r_X(T) + n_X(T) = \dim X$.
$r_Z(T) + n_Z(T) = \dim Z$

2.) As we have T that is acting on Z

$$T(Z) = T(X) \cap W$$

$$\text{So } r_Z(T) = \dim(T(X) \cap W)$$

(3) Inclusion - exclusion on an appropriate basis gives, $\dim Y + \dim(T(X) \cap W)$

$$= r_X(T) + \dim W$$

$$\Rightarrow \dim Y = r_X(T) + \dim W - \dim(T(X) \cap W)$$

(4) let $\ker T \in W$, so Z must include $\ker T$, so

$$\boxed{n_Z(T) = n_X(T)}$$

Subtracting the two equation from (1) and using (4), gives $r_x(T) - r_z(T) = \dim X - \dim Z$

i.e $r_x(T) + n_x(T) - r_z(T) - n_z(T)$

$= \dim X - \dim Z$

where $n_x(T) = n_z(T)$

$\Rightarrow$ Whole substituting into the one in (3.) from (2) gives

$\Rightarrow \dim Y = r_x(T) + \dim W - \dim(T(X) \cap W)$

$\Rightarrow \dim Y = r_x(T) + \dim W - r_z(T)$

$\Rightarrow \dim Y + r_z(T) = r_x(T) + \dim W$

$\Rightarrow \dim Y + \dim Z = \dim X + \dim W$

Q. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n > 0$. Let $p_1, p_2, \ldots, p_{n+1}$ be distinct prime number. Show that there exist a non-zero polynomial $g \in \mathbb{Q}[x]$ such that $fg = \sum_{i=1}^{n+1} c_i x^{p_i}$ with $c_i \in \mathbb{Q}$

Soln: Consider the polynomial q in $\mathbb{Q}[x]$ with degree at most n.

Now consider the remainder

$r_i(x) = x^{p_i} - f(x)h_i(x)$

$\forall i = 1, 2, \ldots, n+1$ (obtained

By division algorithm.

$$x^{p_i} = f(x)h_i(x) + r_i(x)$$

There are $n+1$, r_i's in the n dimensional vector space V. So there is a non-zero linear combination of $r_i(x)$ equal to 0 ; i.e there exist $c_1, c_2 \ldots c_{n+1} \in \mathbb{Q}$ such that

$$\sum_{i=1}^{n+1} c_i r_i(x) = 0$$

Now set $g(x) = \sum_{1}^{n+1} c_i h_i(x)$

So $f(x)g(x) = \sum_{1}^{n+1} c_i f(x) h_i(x)$

$$= \sum_{1}^{n+1} c_i x^{p_i} - \sum_{1}^{n+1} c_i r_i(x)$$

$$= \sum_{1}^{n+1} c_i x^{p_i}$$

as $\boxed{f(x)h_i(x) = x^{p_i} - r_i(x)}$

Q3. prove that the largest possible number 1's in an $n \times n$ invertible matrix with all entries are 0, or 1, is $n^2 - n + 1$

Note: Suppose there are $n^2 - n + 2$ ones and $n-2$ zero in the given matrix

Then, we have to distribute $n-2$ zeros among n rows.

By ~~pigeonhole~~ pigeonhole principles (or a Variation), we get ~~atleast~~ atleast two row must be filled with one completely. This causes the determinant to be zero, as those rows would be the same.

By the same logic follows if we had $n-k$ zeros for $n \geq k \geq 2$

Now Another proof:-

Consider the following $n \times n$ matrix

$$a_{ij} = \begin{cases} 1 & j = n \\ 0 & j \neq n,\ i = j \\ 1 & i \neq n,\ i \neq j \end{cases}$$

Take $n = 3$, we have

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Here all rows are linearly independent, hence the determinant of this Matrix is non-zero (as in all the above cases it is 1).

~~so 1 is the same ans.~~

$\det A = 1$

Q4 Let G be a group which has only finitely many subgroups. prove that G must be finite.

Soln: Suppose if G were infinite.

If G contains an element of infinite order, then G has infinite number of subgroups.

Otherwise, every element of G has a finite order and if $x_1, x_2, \ldots$ x_n are any finite sets of elements of G. Then the subgroups $\langle x_i \rangle$ cover only finitely many elements of G.

Therefore. G has finite number number of subgroup

Q6. Suppose the polynomial x^4+x+1 has multiple roots over a fields of characteristics p. Determine the possible values of p.

Soln: Let $f = x^4 + x + 1$.

Suppose the multiple roots of x is m.

So, f, f' (the formal derivative of f) have roots in the field of characteristics p.

Hence $m^4 + m + 1 = 0 \pmod p$

$4m^3 + 1 = 0 \pmod p$

So $3m + 1 = 4(m^3 + m + 1) - m(4m^3 + 1) = 0 \pmod p$

$3m+4 \equiv 0 \pmod{p}$

$3m = -4 \pmod{p}$

$27m^3 \equiv 64 \pmod{p}$

We also have $4m^3 \equiv -1 \pmod{p}$

Denote $m^3 = x$, then

$27x \equiv 64 \pmod{p}$ — (i)

$4x = -1 \pmod{p}$ — (ii)

Now multiply by 4 and (ii) by (27)

$x \cdot 27x \times 4 = 64 \times 4 \pmod{p}$

$x \cdot 4 \times 27 = -1 \times 27 \pmod{p}$

$0 = 229 \pmod{p}$

~~mod p — 22~~

$p = 229$ is required prime.

5. Let A be a commutative ring with unity.
Prove that the set

$$Z = \{a \in A : ab = 0 \text{ for some non-zero } b \in A\}$$

Contains a prime ideal of A.

Sol^n:-

Let us consider the set Ω of ideal contain in Z.

clearly $\Omega \neq \phi$ as $\{0\} \in \Omega$.

Using Zorn (lemma):-

Suppose a partially ordered set P has the property that every chain in P has an upper bound in P.
Then set P contains atleast one maximal elements.

$\Rightarrow$ Now by using Zorn's lemma one see that this set Ω contains a maximal ideal m.

we can show that m is a prime ideal
i.e Z contains a prime ideal m, i.e
an ideal consisting of any zero divisor ($\neq m \subseteq Z$)
such that $x, y \in m^c \Rightarrow xy \in m^c$.