Bahir Dar University

STEM Incubation Center

Extra Exercises on Statistics and Probability

1) The following grouped frequency distribution shows the number of inches of rainfall received in West Gojjam zone in one year for 25 woredas.

Number of inches	frequency
5.5 – 20.5	2
20.5 – 35.5	3
35.5 – 50.5	8
50.5 – 65.5	6
65.5 – 80.5	3
80.5 – 95.5	3

Then, determine

a) Mean, median, mode, the number of woredas that get the middle 50% of the recorded rainfall in inches.

b) Variance, standard deviation, discuss how dispersed the rain fall record is in each quartile division.

2) The following grouped frequency distribution shows the number of inches of rainfall received in East Gojjam zone in one year for 25 woredas.

Number of inches	frequency
5.5 – 20.5	5
20.5 – 35.5	2
35.5 – 50.5	6
50.5 – 65.5	9
65.5 – 80.5	1
80.5 – 95.5	2

Then, discuss which zone has got more consistent rainfall per annum.

3) In a dancing soiree, there are 12 young ladies and 15 young gentlemen. In how many ways can four dancing pairs be chosen?

4) Given 15 capital letters, 10 vowel letters and 11 consonants, then how many different words can be formed using these letters, if each word begins by capital letter, and consists of 4 different vowel letters and 6 different consonants.

5) Show that

$$\frac{\binom{n}{0}}{n+1}+\frac{\binom{n}{1}}{n}+\frac{\binom{n}{2}}{n-1}+\cdots+\frac{\binom{n}{n}}{1}=\frac{2}{n+1}\left(2^n-\frac{1}{2}\right).$$

6) Determine the value(s) of k so that $\binom{k}{1}+6\binom{k}{2}+6\binom{k}{3}=9k^2-14k$.

7) Determine the value of m and n if the ratios of

$$\binom{n-1}{m}:\left[\binom{n}{m-2}+\binom{n-2}{m-2}+2\binom{n-1}{m-2}\right]:\binom{n+1}{m}=3:5:5.$$

8) Determine the number of permutations of n different things, taken all at a time, when m specified things
   a) always come together
   b) never come together

9) Show that the number of ways of n different things selecting at least one of them is 2^n-1.

10) Prove or disprove the following statements.
   a) Given A and B are independent events, then $P(A\cup B)=1-P(A')P(B')$.
   b) If A and B are independent events, then A and B' are independent events.
   c) Given any events A and B, $P(B)=P(A)P(B\backslash A)+P(A')P(B\backslash A')$.

11) Suppose k people are selected from the general population. What is the probability that at least two of those k were born in the same day?

12) Let S be the sample space and let $E_1,E_2,E_3,\ldots,E_n$ be n mutually exclusive and exhaustive events associated with random experiment. If E is an event which occurs together with either of E_i, then show that $P(E)=\sum_{i=1}^{n}P(E_i)P(E\backslash E_i)$.

$\binom{n+1}{0}$

Solns for Worksheet

Grade 11 Mathematics (Worksheet 5)

(1) & (2) are easily solvable.

(3) 12 young ladies
15 young gentlemen } → four dancing pairs?

Soln 12 young ladies can be distributed in four person groups in $^{12}C_4$ ways. (i.e. $^{12}C_4 = \frac{12!}{4!8!} = \frac{\cancel{12}^{3} \times 11 \times \cancel{10}^{5} \times 9}{\cancel{4} \times \cancel{3} \times \cancel{2}} = 55 \times 9 = 495$)

And the 15 young gentlemen in $^{15}C_4$ ways

(i.e. $^{15}C_4 = \frac{15!}{11!4!} = \frac{15 \times \cancel{14}^{7} \times 13 \times \cancel{12}^{3}}{\cancel{4} \times \cancel{3} \times \cancel{2}} = 15 \times 91 = 1365$.)

(Bc in each group of young ladies (or young gentlemen) order is an issue, each of these groups can be ordered in $^4P_4$ ways.

As a result (we apply the multiplication Rule), we will have

$^{12}C_4 \times {}^{15}C_4 \times {}^4P_4 = 16,216,200$.

(4) We choose a capital letter in 15 dif ways. Then from 10 vowel letters we choose 4 vowel letters in ($^{10}C_4$ different ways). Finally,

* If 4 gentlemen & 4 ladies must be included & only paired with fixed person, then
# = $1 \times 1 \times 1 \times 1$ = 1 //

* If 2 gentlemen & 2 ladies must be included and paired with fixed person, then
# = $^{1\times1\times}12C_2 \times {}^{1\times1\times}13C_2 \times {}^2P_2$

we choose 6 consonant letters from 11 letters in

$^{11}C_6$ diff ways.

Therefore, using multiplication rule, we choose the required letters to form the word in $15 \times {}^{10}C_4 * {}^{11}C_6$. But after placing the capital letter at the beginning, with $(4+6)$ letters we can form $(4+6)!$ permutations. Each such permutations yields a new word.

Therefore, in total, $15 \times ({}^{10}C_4 \times {}^{11}C_6 \times {}^{10}P_{10}) (4+6)!$ different words can be formed.

(6) Soln

$$\binom{k}{1} + 6\binom{k}{2} + 6\binom{k}{3} = 9k^2 - 14k$$

How many of these words have all vowels together

$\# = {}^{15}C_1 \times {}^{10}C_4 \times {}^{11}C_6 \times {}^{7}P_7$

$$\Leftrightarrow (k) + 6\frac{(k)(k-1)}{2\times 1} + 6\frac{(k)(k-1)(k-2)}{3\times 2\times 1} = 9k^2 - 14k$$

$$\Leftrightarrow k + 3k^2 - 3k + k(k-1)(k-2) = 9k^2 - 14k$$

$$\Leftrightarrow 3k^2 - 2k + k(k^2 - 3k + 2) = 9k^2 - 14k$$

$$\Leftrightarrow 3k^2 - 2k + k^3 - 3k^2 + 2k - 9k^2 + 14k = 0$$

$$\Leftrightarrow k^3 - 9k^2 + 14k = 0 \Leftrightarrow k(k^2 - 9k + 14) = 0$$

$$\Leftrightarrow k = 0 \text{ or } k = 2 \text{ or } k = 7$$

Then $k \neq 0$ b/c $\binom{k}{1}$ has no meaning.

$k \neq 2$ b/c $\binom{k}{3}$ has no meaning.

Therefore, $k = 7$ is the required value.

(5) Option 1

2 TS

$$\frac{\binom{n}{0}}{n+1} + \frac{\binom{n}{1}}{n} + \frac{\binom{n}{2}}{n-1} + \cdots + \frac{\binom{n}{n}}{1} = \frac{2}{n+1}\left(2^n - \frac{1}{2}\right)$$

Now,

$$\frac{\binom{n}{0}}{n+1} + \frac{\binom{n}{1}}{n} + \frac{\binom{n}{2}}{n-1} + \cdots + \frac{\binom{n}{n}}{1} = \frac{n!\binom{n}{0} + (n+1)(n-1)!\binom{n}{1} + (n+1)(n)(n-2)!\binom{n}{2} + \cdots}{(n+1)(n)(n-1)(n-2)\cdots(1)} \quad \frac{(n+1)!}{(n+1)!}$$

$$= \frac{n!\left(\frac{n!}{0!n!}\right) + (n+1)(n-1)!\left(\frac{n!}{(n-1)!1!}\right) + (n+1)(n)(n-2)!\left(\frac{n!}{(n-2)!2!}\right) + \cdots + (n+1)!\left(\frac{n!}{0!n!}\right)}{(n+1)!}$$

$$= \frac{n! + (n+1)\left(\frac{n!}{1!}\right) + (n+1)(n)\left(\frac{n!}{2!}\right) + \cdots + (n+1)!}{(n+1)!}$$

$$= \frac{n!\left[\frac{1}{0!} + \frac{n+1}{1!} + \frac{(n+1)(n)}{2!} + \cdots + (n+1)\right]}{(n+1)!}$$

$$= \frac{\overbrace{\left[\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n}\right]}^{(*)}}{(n+1)} \quad \cdots (*_1)$$

But $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1} = 2^{n+1} \quad \cdots (*_2)$

Therefore, $*_1 = \frac{*_2 - \binom{n+1}{n+1}}{n+1} = \frac{2^{n+1} - 1}{n+1} = \frac{2}{n+1}\left[2^n - \frac{1}{2}\right]$ (QED)

Option 2 (4)

$$\frac{\binom{n}{0}}{n+1} + \frac{\binom{n}{1}}{n} + \frac{\binom{n}{2}}{n-1} + \cdots + \frac{\binom{n}{n}}{1} = \frac{2}{n+1}\left(2^n - \frac{1}{2}\right)$$

Let $A = \frac{\binom{n}{0}}{n+1} + \frac{\binom{n}{1}}{n} + \frac{\binom{n}{2}}{n-1} + \cdots + \frac{\binom{n}{n}}{1}$

$$= \frac{\binom{n}{n}}{n+1} + \frac{\binom{n}{n-1}}{n} + \frac{\binom{n}{n-2}}{n-1} + \cdots + \frac{\binom{n}{0}}{1}$$

$\Rightarrow$ (×) $\frac{(n+1)}{n+1}\binom{n}{n} + \frac{n+1}{n}\binom{n}{n-1} + \cdots + \frac{n+1}{1}\binom{n}{0} = A(n+1)$

$\Leftrightarrow \frac{(n+1)}{1}\binom{n}{0} + \left(\frac{n+1}{2}\right)\binom{n}{1} + \cdots + \frac{n+1}{n}\binom{n}{n-1} + \frac{n+1}{n+1}\binom{n}{n}$

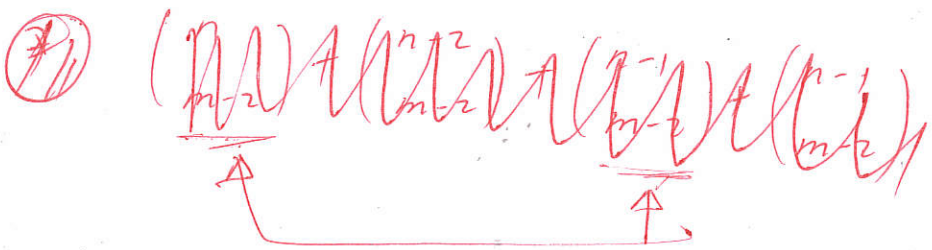
$= A(n+1)$

$\Leftrightarrow \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \cdots + \binom{n+1}{n} + \binom{n+1}{n+1} = A(n+1)$

Then $\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n+1} = \binom{n+1}{0} + A(n+1)$

$\Rightarrow A(n+1) = 2^{n+1} - \binom{n+1}{0} = 2^n \cdot 2 - 1$

$\Rightarrow A = \frac{2}{n+1}\left(2^n - \frac{1}{2}\right)$

(#1) 

(7) $\binom{m}{n-1} : \underbrace{\binom{m-2}{n} + \binom{m-2}{n-2} + 2\binom{m-2}{n-1}}_{(*)} : \binom{m}{n+1} = 3:5:5$

Recall: $\boxed{\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}}$

Let us simplify, first (*), $\binom{m-2}{n} + \binom{m-2}{n-2} + 2\binom{m-2}{n-1} =$

$$\binom{m-2}{n} + \binom{m-2}{n-2} + \binom{m-2}{n-1} + \binom{m-2}{n-1}$$

$$= \binom{m-1}{n} + \binom{m-1}{n-1} = \binom{m}{n}$$

Hence $(*) = \binom{m}{n}$

Now, $\dfrac{\binom{m}{n-1}}{\binom{m}{n}} = \dfrac{3}{5}$ & $\dfrac{\binom{m}{n}}{\binom{m}{n+1}} = \dfrac{5}{5}$

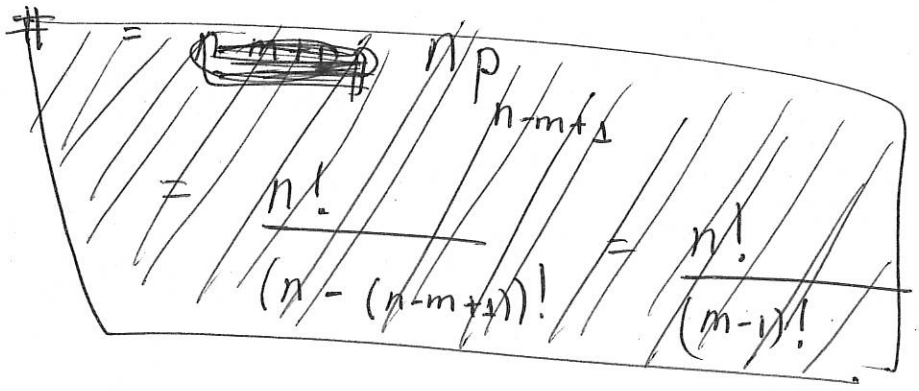
$$\Rightarrow \frac{m!}{(m-n+1)!\,(n-1)!} \cdot \frac{(m-n)!\,n!}{m!} = \frac{3}{5} \quad \& \quad \frac{m!}{(m-n)!\,n!} \cdot \frac{(m-n-1)!\,(n+1)!}{m!} = \frac{5}{5}$$

$$\Rightarrow \frac{n}{m-n+1} = \frac{3}{5} \quad \& \quad \frac{n+1}{m-n} = 1 \Rightarrow \begin{cases} 5n+3n-3m-3=0 \\ 2n-m+1=0 \end{cases}$$

$$\begin{cases} 8n - 3m = 3 \\ 2n - m = -1 \end{cases} \Rightarrow \begin{matrix} m = 7 \\ n = 3 \end{matrix}$$

(8) Permutation of n objects, taking all at a time, when m is specified things.

a) m things always come together



$m! \times (n-m+1)!$ Where A_1 = Permutation of the m objects among themselves.

A_2 = permutation of the remaining $n-m$ objects and the m objects as one entity.

Then, # of ways of doing A_1 and A_2 together:

$m! \times (n-m+1)!$

b) m objects never together:

$n! - (m! \times (n-m+1)!)$

(bic the total number of permutation of n objects taking all at a time: ${}_nP_n = n!$)

(9) The number of ways of selecting at least one of them from n objects : given by

$$\#\text{ of ways}_{\text{at least one}} = {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \cdots + {}^{n}C_n$$

$$= \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

But we know that, $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

[Obtained from binomial expansion.

$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$

When $x = y = 1$

$\Rightarrow 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$]

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - \binom{n}{0} = \underline{2^n - 1} \quad \underline{(\text{QED})}$$

Solns for Worksheet

(1) A & B independent, then

$$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= (1 - P(A)) - P(B)[1 - P(A)]$$

$$= (1 - P(A))(1 - P(B))$$

$$= P(A')P(B')$$

$\Rightarrow$ A' & B' are independent.

(2) A & B independent, the A' & B are independent.

A & B independent $\iff P(A \cap B) = P(A)P(B)$

But $B = (A \cap B) \cup (B \cap A')$

$$\Rightarrow P(B) = P(A \cap B) + P(B \cap A') - P(A \cap B \cap B \cap A')$$

$$= P(A \cap B) + P(B \cap A')$$

$$\Rightarrow P(B \cap A') = P(B) - P(A \cap B) = P(B)[1 - P(A)] = P(B)P(A')$$

$\Rightarrow$ A' & B are independent.

(3) If A & B independent events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

$$= 1 - P(A') + P(B)\left[1 - P(A)\right]$$

$$= 1 - P(A') + P(B)\,P(A')$$

$$= 1 - P(A')\left[1 - P(B)\right]$$

$$= 1 - P(A')\,P(B')$$

Hence $P(A \cup B) = 1 - P(A')\,P(B')$

(4) If A & B independent, then WTS A and B' are independent.

$$P(A \cap B) = P(A) \cdot P(B)$$

But $A = (A \cap B) \cup (A \setminus B) = (A \cap B) \cup (A \cap B')$

$$P(A) = P(A \cap B) + P(A \cap B') - P(A \cap B \cap A \cap B')$$

$$\Rightarrow P(A) = P(A \cap B) + P(A \cap B')$$

$$\Rightarrow P(A \cap B') = P(A) - P(A) \cdot P(B)$$

$$= P(A)\left[1 - P(B)\right]$$

$$= P(A)\,P(B')$$

Thus A & B' are independent.

Partition theorem If $B_1, B_2, \ldots, B_m$ form a partition of $\mathcal{S}$, then

$$P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(A \mid B_i)\, P(B_i)$$ for any event A.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \& \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow \boxed{P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)}$$

Note: B and B' Partition Sample space ($\mathcal{S}$), then

$$P(A) = P(A \cap B) + P(A \cap B')$$

$$= P(A \mid B) P(B) + P(A \mid B') P(B')$$ for any event A & B.

Baye's theorem: $P(B \mid A) = \dfrac{P(A \mid B)\, P(B)}{P(A)}$

More generally, If $B_1, B_2, \ldots, B_m$ form a partition of $\mathcal{S}$, then

$$P(B_j \mid A) = \frac{P(A \mid B_j)\, P(B_j)}{\sum_{i=1}^{m} P(A \mid B_i) P(B_i)}$$ for any j

Chains of events: For any events $A_1, A_2, \ldots, A_n$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2 \mid A_1)\, P(A_3 \mid (A_2 \cap A_1)) \cdots P(A_n \mid (A_{n-1} \cap A_{n-2} \cdots \cap A_1))$$

(11) Suppose the k individuals lined up in a row to form an ordered sequence. Each person might have any of the 365 birthdays.

- By multiplication Rule, the group as a whole generates a sample space of 365^k birthday sequences.

Possible birthdays: $\underset{1}{\underline{365}}\ \underset{2}{\underline{365}}\ \underset{3}{\underline{365}}\ \ldots\ \underset{k}{\underline{365}}$ (Person) $\longrightarrow 365^k$ different sequences of birthdays.

Let A = be the event having at least two people have the same birthday.

WTF $P(A) = ?$ NOW $P(A) = \dfrac{n(A)}{365^k}$

NOW, It is easy to find $n(A^c)$ than $n(A)$. Therefore, A^c means the event in which all k people have different birthdays.

Hence, $n(A) = 365^k$ - number of sequences where all k people have different birthdays.

But, the number of ways to form birthday sequences for k people subject to the restriction that all k birthdays must be different is simply the number of ways to form permutations of length k from a set

of 365 distinct objects:

$$^{365}P_k = (365)(364)\cdots(365-k+1)$$

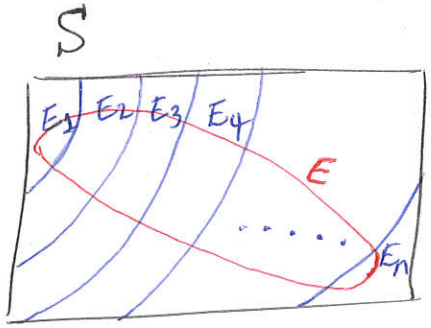
Therefore, $P(A) = P$(at least two people have the same birthday))

$$= \frac{365^k - (365)(364)\cdots(365-k+1)}{365^k}$$

(11) S = Sample space, $E_1, E_2, \ldots, E_n$ mutually exclusive & exhaustive

E is an event s.t. $E \cap E_i \neq \phi$, $\forall i = 1, 2, \ldots, n$.

WTS $P(E) = \sum_{i=1}^{n} P(E_i)\, P(E \mid E_i)$.



Now, $E = (E_1 \cap E) \cup (E_2 \cap E) \cup (E_3 \cap E) \cdots \cup (E_n \cap E)$

(But each $E_i \cap E$ & $E_j \cap E$ are mutually exclusive events for $i \neq j$ (bic E_i's are pairwise mutually exclusive).

$\Rightarrow P(E) = P(E_1 \cap E) + P(E_2 \cap E) + \cdots + P(E_n \cap E)$

$= P(E_1) \cdot P(E \mid E_1) + P(E_2) \cdot P(E \mid E_2) + \cdots + P(E_n) \cdot P(E \mid E_n)$

$\Rightarrow P(E) = \sum_{i=1}^{n} P(E_i) \cdot P(E \mid E_i)$

// (QED)

Basic Combination Identities

1) $\binom{n}{m} = \binom{n}{n-m}$

2) $\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}$

3) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1}$

(i.e. $\binom{n}{n-k} = \binom{n-1}{n-k} + \binom{n-2}{n-k-1} + \binom{n-3}{n-k-2} + \cdots + \binom{k-1}{0}$)

4) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

⊛ The $(k+1)$ term in $(x+y)^n$ is given by

$$T_{k+1} = \binom{n}{k} x^{n-k} y^k, \quad k = 0, 1, 2, \ldots, n.$$

* $\binom{n}{k} = \frac{(n-k+1)}{k} \cdot \binom{n}{k-1}$.

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{(n)(n-1)\cdots(n-k+1)(n-k)!}{(n-k)!\,k!} = \frac{(n)(n-1)\cdots(n-k+1)}{(k)(k-1)!}$

$= \frac{n-k+1}{k}\binom{n}{k-1}$

5) $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$

6) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (Generally, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ //)

Verification

(6) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Now, $\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}$

$= \frac{(n-1)!}{(n-k-1)!\,(k-1)!}\left[\frac{1}{n-k} + \frac{1}{k}\right]$

$= \frac{(n-1)!\,(k+n-k)}{(n-k-1)!\,(k-1)!\,(k)(n-k)}$

$= \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

Therefore, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

(2) $\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}$

Now, $\binom{n}{m-1} + \binom{n}{m} = \frac{n!}{(n-m+1)!\,(m-1)!} + \frac{n!}{(n-m)!\,m!}$

$= \frac{n!}{(n-m)!\,(m-1)!}\left[\frac{1}{(n-m+1)} + \frac{1}{m}\right]$

$= \frac{n!}{(n-m)!\,(m-1)!}\left[\frac{n+1}{(n-m+1)(m)}\right]$

$= \frac{(n+1)!}{(n-m+1)!\,m!} = \binom{n+1}{m}$ //