

Unit 5 Geometry and Measurement

5.1 Regular Polygons

A. Revision on Polygons

Definition: A polygon is simple closed figure formed by the union of three or more line segments such that:

1. The line segment that have a common end points are non-collinear.
2. Each line segment intersects only at their end points.

The line segments are called the sides of polygon.

The end points of the sides of the sides are called the vertices.

Convex polygon is a polygon such that no line containing a side of the polygon contains a point in the interior of the polygon. Each interior angle of convex polygon is less than 180° .

Concave polygon is a polygon that is not convex. One or more of interior angle of concave polygon is greater than 180° .

Polygons can be classified according to the number of sides

- A polygon with three sides is called triangle.
- A polygon with four sides is called quadrilateral.
- A polygon with five sides is called pentagon.
- A polygon with six sides is called hexagon.
- A polygon with seven sides is called heptagon.
- A polygon with eight sides is called octagon.
- A polygon with nine sides is called nonagon.
- A polygon with ten sides is called decagon.
- A polygon with eleven sides is called undecagon.
- A polygon with twelve sides is called dodecagon.
- A polygon with thirteen sides is called tri decagon.
- .
- .
- .
- A polygon with nineteen sides is called innenadecagon.
- A polygon with twenty sides is called icosagon.
- .
- .
- .

B. Angles of a polygon

There are two types of angles of polygon. These are:

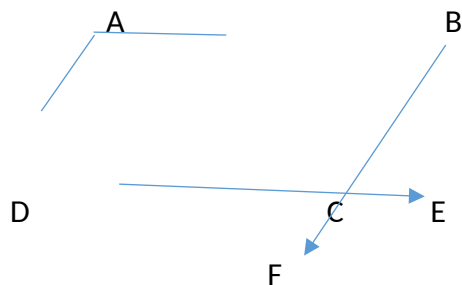
- i. Interior angles of polygon and
- ii. Exterior angles of polygon.

Definition: Interior angles of a polygon is an angle at a vertex on the inside of the polygon.

Definition: Exterior angles of a polygon is an angle that is supplementary to the interior angle. It is formed between one side of the polygon and the extended adjacent side.

Example: In the polygon ABCD in figure below

$\angle DCB$ is an interior angle; $\angle BCE \wedge \angle DCF$ are exterior angles of the polygon at the vertex C.



There are two possible exterior angles at any vertex which are equal.

The sum of the measures of the interior angles of polygon.

Theorem 5.1 Angle sum theorem

The sum of the measures of the three interior angles of any triangle is 180° .

Convex polygon of n -sides have

- i. $(n - 3)$ diagonals from one vertex
- ii. $(n - 2)$ triangles

Theorem 5.2

The sum of the measures of the interior angle of a convex polygon of n -sided is given by

$$S = (n - 2) \times 180^\circ$$

Example1: Find the sum of the measures of the interior angles of each convex polygon of given side

- a. Decagon
- b. Dodecagon
- c. 26- gon
- d. hexagon
- e. heptagon

Solution: a. $n = 10$; $S = (n - 2) \times 180^\circ$ therefore $S = (10 - 2) \times 180^\circ = 1440^\circ$

b. $n = 12$

$$S = (n - 2) \times 180^\circ \text{ therefore } S = (12 - 2) \times 180^\circ = 1800^\circ$$

c. $n = 26$

$$S = (n-2) \times 180^\circ \text{ therefore } S = (26-2) \times 180^\circ = 4320^\circ$$

d. Exercise left for you

e. Exercise left for you

Example2: Find the number of diagonal from one vertex and sides of a polygon if the sum of the measures of the interior angles of a polygon is

a. 3600°

b. 4500°

c. 5400°

Solution: We use $S = (n-2) \times 180^\circ \wedge d = n-3$

a. $3600^\circ = (n-2) \times 180^\circ \implies n-2 = \frac{3600^\circ}{180^\circ} = 20, \text{ therefore, } n = 22$

And all possible diagonal drawn from one vertex is $22 - 3 = 19$.

b. $4500^\circ = (n-2) \times 180^\circ \implies n-2 = \frac{4500^\circ}{180^\circ} = 25, \text{ therefore, } n = 27$

And all possible diagonal drawn from one vertex is $27 - 3 = 24$.

c. **Exercise left for you!**

Example3: Determine the measure of the interior angle of pentagon polygon with measure of $x^\circ, 2x^\circ, (x+30)^\circ, (x-10)^\circ \wedge (x+40)^\circ$

Solution: $n = 5, S = (5-2) \times 180^\circ = 540^\circ \leftarrow \sum \text{ of all interior angle}$

$$\implies x^\circ + 2x^\circ + (x+30)^\circ + (x-10)^\circ + (x+40)^\circ = 540^\circ$$

$$\implies 6x^\circ + 60^\circ = 540^\circ \text{ implies } x = 80^\circ$$

Measures of Angles of a regular polygon

The sum of the measures of the exterior angle of any polygon, taking one angle at each vertex is 360° .

Example1: Determine the sum of the measures of the exterior angle of the following polygon.

a. Triangle

b. quadrilateral

c. hexagon

Solution: If $e_1, e_2, e_3, \dots, e_n$ are the measures of the exterior angles of n-sided polygon, then

$$e_1 + e_2 + e_3 + \dots + e_n = 360^\circ$$

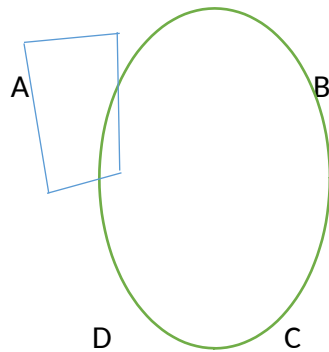
Thus a. 360°

b. 360°

c. 360°

Definition: A regular polygon is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angle are equal.

A polygon is said to be inscribed in a circle if all of its vertices lie on the circle. Any regular polygon can be inscribed in a circle. Because of this, the center and radius of a circle can be taken as the center and radius of an inscribed regular polygon.



An example of a quadrilateral polygon inscribed in a circle.

Any regular polygon of **n-sides** has:

- i. Measure of each interior angle $i = \frac{(n-2)180^\circ}{n}$
- ii. Measure of each central angle $c = \frac{360^\circ}{n}$
- iii. Measure of each exterior angle $e = \frac{360^\circ}{n}$

Example2: Find

- i. The measure of each interior angle (i)
- ii. The measure of each central angle
- iii. The measure of each exterior angle of a regular polygon with
 - a. 3 sides
 - b. 8 sides
 - c. 18 sides

Solution: a. $n = 3, \Rightarrow i(\text{each interior angle}) = \frac{(3-2)180^\circ}{3} = 60^\circ$

ii. each central angle $= c \left(\frac{360^\circ}{3} \right) = 120^\circ$

iii. (e) each exterior angle $= e \left(\frac{360^\circ}{3} \right) = 120^\circ$

b. $n = 8, \Rightarrow i(\text{each interior angle}) = \frac{(8-2)180^\circ}{8} = 135^\circ$

ii. each central angle $= c \left(\frac{360^\circ}{8} \right) = 45^\circ$

iii. (e) each exterior angle $= e \left(\frac{360^\circ}{8} \right) = 45^\circ$

c. Left exercise for you!!

Example3: ABCD is a quadrilateral such that the measures of three of its interior angles are given as $(m \angle D) = 112^\circ$, $(m \angle C) = 75^\circ$ and $(m \angle B) = 51^\circ$. Find $(m \angle A)$.

Solution: Since a quadrilateral is a 4 - sided polygon and the sum of its interior angle is

$$S = (n - 2)180^\circ \implies (4 - 2)180^\circ = 360^\circ$$

$$\text{Hence, } (m \angle A) + (m \angle B) + (m \angle C) + (m \angle D) = 360^\circ$$

$$(m \angle A) + 51^\circ + 75^\circ + 112^\circ = 360^\circ$$

$$(m \angle A) + 238^\circ = 360^\circ$$

$$(m \angle A) = 360^\circ - 238^\circ$$

$$(m \angle A) = 122^\circ$$

Example4: Find the number of side of a regular polygon, if the measure of each of its interior angle is

a. 150°

b. 160°

c. 147.27°

Solution: Since, $i = \frac{(n-2)180^\circ}{n}$ this implies that $n = \frac{360^\circ}{180^\circ - i}$

a. $n = \frac{360^\circ}{180^\circ - 150^\circ} = 12$

b. $n = \frac{360^\circ}{180^\circ - 160^\circ} = 18$

c. *for you!*

Example5: If the measure of a central angle of a regular polygon is 18° , find the measure of each of its interior angles.

Solution: To find the measure of each interior angle of a regular polygon first we must calculate the number of sides from the central angle.

Since, measure of each central angle $i = \frac{360^\circ}{n}$, this implies $n = \frac{360^\circ}{\text{measure of each central angle}}$

$$n = \frac{360^\circ}{18^\circ} = 20$$

Therefore, $i = \frac{(n-2)180^\circ}{n}$ implies $i = \frac{(20-2)180^\circ}{20} = 162^\circ$

Example6: Find the number of sides of a regular polygon if each exterior angle is

a. 5°

b. 164°

d. 16°

Solution:

Since for n – sided regular polygon each exterior angle

$$e = \frac{360^\circ}{n} \iff n = \frac{360^\circ}{e}$$

- a. $n = \frac{360^\circ}{e} = \frac{360^\circ}{5^\circ} = 72 \text{ sides}$
- b. $n = \frac{360^\circ}{e} = \frac{360^\circ}{164^\circ} = 2.19$. *It is impossible because $n=2.19$ is not a natural number*
- c. Left exercise for you!

Properties of Regular Polygon

A symmetric properties of regular polygon

A plane figure can be divided exactly into two identical parts by a line (l) the line l is called **line of symmetry** also called the **line of reflection**.

A figure is called **symmetric** if it has at least one line of symmetry.

Regular polygons are symmetric.

Some figures have more than one line of symmetry. In each cases, the lines of symmetry always intersect at one point.

Note that equilateral triangle, squares and regular pentagons have as many lines of symmetry as their sides.

In general, A regular n-sided polygon has n-lines of symmetry.

Example: How many lines of symmetry can have?

- | | | |
|-----------------------|------------------|---------------------|
| a. Circle | d. Rectangle | g. square |
| b. Isosceles triangle | e. Rhombus | h. regular heptagon |
| c. Scalene triangle | f. Parallelogram | i. Regular icosagon |

Solution:

- a. Circle has infinity line of symmetry.
- b. Isosceles triangle has **one line of symmetry**.
- c. Scalene triangle has **no line of symmetry**.
- d. Rectangle has **two line of symmetry**.
- e. **Left exercise for you.**
- f. **Left exercise for you.**
- g. $n = 4$, therefore square has **4 line of symmetry** because square is regular polygon.
- h. $n = 7$, therefore regular heptagon has **7 line of symmetry** because regular heptagon is regular polygon.

i. Left exercise for you.

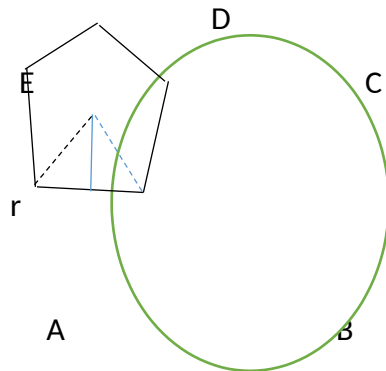
Note: A regular polygon of odd number of sides has every line of symmetry passes through the vertex

Inscribed Regular Polygon and Circumscribed Regular Polygon

- An inscribed polygon is a polygon all of whose sides are chords of a circle.
- The circle is called circumscribed about the regular polygon.
- Inscribed polygon is the same as the term circumscribed circle.
- A regular polygons have a special relation to circles.
- A regular polygon can always be inscribed in or circumscribed about a circle.

The Relation between Apothem, Perimeter, Area and sides of a Regular Polygon.

Definition: An **apothem** of a regular polygon is segment from its center perpendicular to one of sides.



- The distance **a** is an apothem.

Definition: A **central angle** of a regular polygon is an angle between two radii (r) given by

$$m(\angle AOB) = \frac{360^\circ}{n}, \text{ where } n \text{ is the number of sides.}$$

Theorem:

Formula for the length of side s , Apothem a , perimeter P and area A of a regular polygon with n sides and radius r are:

$$1. \quad s = 2r \sin \frac{180^\circ}{n}$$

$$2. \quad a = r \cos \frac{180^\circ}{n}$$

$$3. \quad P = 2nr \sin \frac{180^\circ}{n}$$

$$4. A = \frac{1}{2} aP$$

Prove of this theorem is Left exercise for (refer your text book page 186 - 187)

Example1:

- Find the length of the side of an equilateral triangle if its radius is $\sqrt{12} \text{ cm}$.
- Find the area of a regular hexagon whose radius is 5 cm .
- Find the apothem of a square whose radius is $\sqrt{8} \text{ cm}$

Solution:

- By the formula, the length of side is $s = 2r \sin \frac{180^\circ}{n}$

So, replacing r by $\sqrt{12}$ and n by 3, we have,

$$s = 2 \times \sqrt{12} \times \sin \frac{180^\circ}{3} = 2 \times \sqrt{12} \times \sin 60^\circ = 2 \times \sqrt{12} \times \frac{\sqrt{3}}{2} = \sqrt{36} = 6$$

Therefore, the length of the side of the equilateral triangle is 6 cm .

- To find the area of the regular hexagon, we use the formula $A = \frac{1}{2} aP$, where a is the apothem and P is the perimeter of the regular hexagon.

Therefore,

$$A = \frac{1}{2} aP = \frac{1}{2} \left(r \cos \frac{180^\circ}{n} \right) \left(2nr \sin \frac{180^\circ}{n} \right), \text{ substituting formulae for } a \wedge P$$

$$= \frac{1}{2} \left(5 \times \cos \frac{180^\circ}{6} \right) \left(2 \times 6 \times 5 \times \sin \frac{180^\circ}{6} \right)$$

$$= \frac{1}{2} \times 5 \times \frac{\sqrt{3}}{2} \times 2 \times 6 \times 5 \times \frac{1}{2}; \left(\cos 30^\circ = \frac{\sqrt{3}}{2}, \sin 30^\circ = \frac{1}{2} \right)$$

$$= \frac{75\sqrt{3}}{2} \text{ cm}^2$$

- Left exercise for you!

Example2: Show that the length of each side of a regular hexagon is equal to the length of the radius of the hexagon.

Solution: Since, $s = 2r \sin \frac{180^\circ}{n} \wedge n=6$, Thus

$$s = 2 \times r \times \sin \frac{180^\circ}{6} = 2 \times r \times \sin 30^\circ = 2 \times r \times \frac{1}{2} = r \implies s = r$$

Example3: Find

- i. The length of side
- ii. The apothem
- iii. The area, of the regular polygon inscribed in a circle of radius $r = \sqrt{3} \text{ cm}$ of the following regular polygon.
 - a. Triangle
 - b. hexagon

Solution: Since the inscribed polygon is regular, so that

Use, $s = 2r \sin \frac{180^\circ}{n}$ to find side

$a = r \cos \frac{180^\circ}{n}$, to find the apothem

$A = \frac{1}{2} aP = \frac{1}{2} n r^2 \sin \frac{360^\circ}{n}$, to find the area

a. Left exercise for you!!

b. Regular hexagon, $n = 6$, $r = \sqrt{3}$

$$\text{i. } s = 2r \sin \frac{180^\circ}{n} = 2(\sqrt{3}) \left(\sin \frac{180^\circ}{6} \right) = \sqrt{3} \text{ cm}$$

$$\text{ii. } a = r \cos \frac{180^\circ}{n} = \sqrt{3} \left(\cos \frac{180^\circ}{6} \right) = (\sqrt{3}) \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2} \text{ cm}$$

$$\text{iii. Perimeter (p), } P = ns = (6)(\sqrt{3}) = 6\sqrt{3} \text{ cm}$$

$$\text{Therefore, } A = \frac{1}{2} aP = \frac{1}{2} \times \frac{3}{2} \times 6\sqrt{3} = \frac{9\sqrt{3}}{2} \text{ cm}^2$$

Example4: The side length of **Inscribed** regular polygon is 10 unit long. Find the area of each the following regular polygon.

- a. Triangle
- b. nonagon

Solution:

a. Regular triangle, $n = 3$, and $s = 10$ unit. Since $s = 2r \sin \frac{180^\circ}{n}$

$$\implies 10 = 2r \sin \frac{180^\circ}{3} \implies 2r \left(\frac{\sqrt{3}}{2} \right) = 10 \implies r = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

$$\text{Therefore } A = \frac{1}{2} n r^2 \sin \frac{360^\circ}{n} = \frac{1}{2} (3) \left(\frac{10\sqrt{3}}{3} \right)^2 \left(\sin \frac{360^\circ}{3} \right) = 25\sqrt{3} \text{ cm}^2$$

b. Left exercise for you!

Example5: Find the length of a side and the perimeter of a regular nine-sided polygon with radius 5 units.

Solution: $s = 2(5) \sin \frac{180^\circ}{9} = 10\sqrt{20} \text{ unit}$

$$P = 2 n r \sin \frac{180^\circ}{n} = 2(9)(5) \sin \frac{180^\circ}{9} = 90\sqrt{20} \text{ unit}$$

Example6: Find the radius of a regular hexagon with perimeter 48 units.

Solution: Since, $p = 48$ units, $n = 6$, then we have

$$P = 2 n r \sin \frac{180^\circ}{n}$$

$$\Rightarrow 48 = 2(6) r \left(\sin \frac{180^\circ}{6} \right) \Rightarrow 12 r \sin 30^\circ = 48$$

$$\Rightarrow 12(r) \left(\frac{1}{2} \right) = 48 \Rightarrow r = 8 \text{ units}$$

Example7: The area of inscribed regular hexagon is $18\sqrt{3} \text{ cm}^2$. Find the

- Radius of a circle
- Side length
- Apothem
- Perimeter

Solution: To find r , Use the relation $A = \frac{1}{2} n r^2 \sin \frac{360^\circ}{n}$

a. $n = 6$, Area $A = 18\sqrt{3}$

$$\Rightarrow 18\sqrt{3} = \frac{1}{2} (6) r^2 \left(\frac{360^\circ}{6} \right) = 3 r^2 \sin 60^\circ = 3 r^2 \left(\frac{\sqrt{3}}{2} \right)$$

$$\Rightarrow 3 r^2 \sqrt{3} = 36\sqrt{3}$$

$$\Rightarrow r^2 = 12 \Rightarrow r = \sqrt{12} = 2\sqrt{3} \text{ cm}$$

b. $s = 2r \sin \frac{180^\circ}{n}$ implies $s = 2(2\sqrt{3}) \sin \frac{180^\circ}{6} \Rightarrow 4\sqrt{3} \left(\frac{1}{2} \right) = 2\sqrt{3} \text{ cm}$

c. Since, $A = \frac{1}{2}nP$ implies $2A = nP \wedge p = ns$

$$\Rightarrow a = \frac{2A}{P} = \frac{2(18)(\sqrt{3})}{(6)(4\sqrt{3})} = \frac{3}{2} \text{ cm}$$

d. Left exercise!

Further on Congruency and Similarity

- Two geometric figures are said to be congruent if they have the same shape and size.
- To line segments $\overline{AB} \wedge \overline{CD}$ are congruent if $AB = CD$.
- Two angles are congruent if and only if their measures are equal.

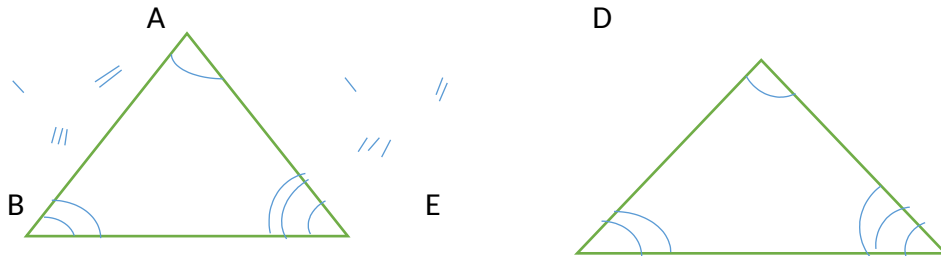
Congruency of Triangles

Triangles that fit exactly are called congruent triangle.

Congruent triangles are triangles that have the same size and shape. That is, the six parts of the triangle (the three sides and the three angles) are correspondingly congruent.

If two triangles are congruent; their corresponding sides and angles must be congruent.

Consider the two triangles given below



The congruent six part of a triangle are the three corresponding angle and the three corresponding sides (size) of $\triangle ABC \wedge \triangle DEF$ denoted by $\triangle ABC \cong \triangle DEF$.

These are:

Corresponding sides $\left\{ \begin{array}{l} \overline{AB} \cong \overline{DE} \\ \overline{BC} \cong \overline{EF} \\ \overline{AC} \cong \overline{DF} \end{array} \right.$ and

Corresponding angles $\left\{ \begin{array}{l} \angle A \cong \angle D \\ \angle B \cong \angle E \\ \angle C \cong \angle F \end{array} \right.$

Note: The order in which the vertices match determine a correspondence between the sides and angles.

Four Basic Principles of Congruency of Triangles

Two triangles are congruent if the following corresponding parts of the triangles are congruent.

1. SSS (three sides)
2. ASA (two angles and the included side)
3. SAS (two sides and the included angle)
4. RHS (Right angle hypotenuse and a side)

Example1: Given that $\triangle ABC \cong \triangle PQR$, which of the following is necessarily true?

- A. $\overline{AC} \equiv \overline{PQ}$
 B. $\overline{AB} \equiv \overline{QR}$
 C. $\angle BAC \equiv \angle QRP$
 D. $\angle BCA \equiv \angle PRQ$

Solution: $\triangle ABC \cong \triangle PQR$ Provided that

$$\begin{cases} \overline{AB} \equiv \overline{PQ} \\ \overline{BC} \equiv \overline{QR} \\ \overline{AC} \equiv \overline{PR} \end{cases} \quad \text{and} \quad \begin{cases} \angle A \equiv \angle P \\ \angle B \equiv \angle Q \\ \angle C \equiv \angle R \end{cases} \therefore \text{Answer D}$$

Example2: Given $\triangle ABC \cong \triangle RST$. Find the value y , if $m(\angle A) = 40^\circ$ and $m(\angle R) = (2y + 10)^\circ$.

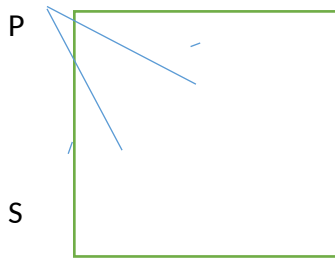
Solution: Since $\triangle ABC \cong \triangle RST$, the corresponding angle are congruent. So,

$$m(\angle A) \cong m(\angle R)$$

Therefore, $m(\angle A) = m(\angle R)$, i.e., $40^\circ = (2y + 10)^\circ$. So, $y = 15^\circ$

Example3: In figure given below, PQRS is a square. A and B are points on $\overline{QR} \wedge \overline{SR}$, such that $\overline{QA} \equiv \overline{SB}$.

Prove that: $\angle PAQ \equiv \angle PBS$.

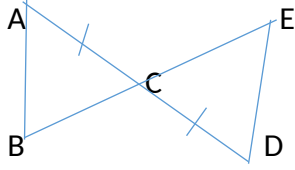


Solution: $\overline{QA} \equiv \overline{SB}$. (side of square is congruent)

$$\overline{QA} \equiv \overline{SB}. \text{ (given)}$$
 $\angle Q \equiv \angle S$ (angle of square which is right angles)

Therefore, $\angle PAQ \equiv \angle PBS$ by SAS or by corresponding angles of congruent triangles.

Example4: In the following figure $\overline{AD} \wedge \overline{BE}$ bisect each other at C. The appropriate theorem used to prove that $\triangle ABC \cong \triangle DEC$ is



A. AAS

B. ASA

C. SAS

D. SSS

Solution: Given that $\overline{AC} \equiv \overline{EC}$. And $\overline{BC} \equiv \overline{DC}$. (By bisection of $\overline{AD} \cap \overline{BE}$)

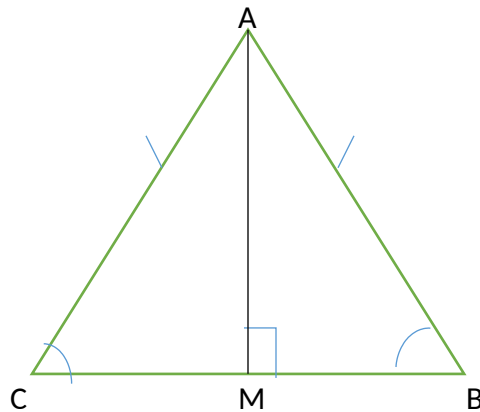
$m(\angle ACB) \equiv m(\angle DCE)$ by VOA (vertical opposite angle.)

Therefore, $\Delta ABC \cong \Delta DEC$ by SAS.

Hence, Answer C.

Example5: ABC is an isosceles triangle with $\overline{AC} \equiv \overline{AB}$, and M is mid-point of $\overline{BC}$. Prove that $\angle ABC \equiv \angle ACB$.

Solution: First draw the isosceles triangle as follows.



Statement	Reason
1. $AC = AB$	1. Given
2. $m(\angle AMC) = m(\angle AMB) = 90^\circ$	2. M is midpoint of $\overline{BC}$ and $\overline{AM} \perp \overline{BC}$
3. $\Delta AMC \cong \Delta AMB$	3. By RHS
4. $\angle ABC \equiv \angle ACB$	4. Corresponding angle of congruent triangle and by 1,2 and 3

Definition of Similar Figures

- ✓ Figures that have the same shape but that might have different sizes are called similar.
- ✓ In similar figures:
 - One is the enlargement of the other.
 - Angles in corresponding position are congruent.
 - Corresponding sides have the same ratio (proportional)

Definition: Two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

Example: Referring to figure 5.42 on your text book page 198, if ABCDE and HIJKL. Then find the length of:

- a. $\overline{IJ}$ b. $\overline{CD}$ c. $\overline{HL}$

Solution: Since ABCDE and HIJKL, we have, $\frac{AB}{HI} = \frac{BC}{IJ} = \frac{CD}{JK} = \frac{DE}{KL} = \frac{AE}{HL}$

- a. To find the length of $\overline{IJ}$, we use $\frac{AB}{HI} = \frac{BC}{IJ} \Rightarrow \frac{5}{4} = \frac{7}{x}$ ($AB=5, HI=4, BC=7, IJ=x$)

$$\text{So, } x = \frac{4 \times 7}{5} = 5.6$$

Therefore, the length of side, $\overline{IJ}$ is 5.6 m

- b. , $\frac{AB}{HI} = \frac{CD}{JK} \Rightarrow \frac{5}{4} = \frac{a}{8}$, So, $a=10$

Therefore, the length of side, $\overline{CD}$ is 10 m

- c. Left Exercise for you!

Theorems on Similarity of Triangle

Definition: Two triangles are said to be similar, if

1. Their corresponding sides are proportional (have equal ratio), and
2. Their corresponding angles are congruent.

That is, $\triangle ABC \sim \triangle DEF$, if and only if

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \wedge \angle$$

$$\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$$

SAS Similarity Theorem

If the two sides of one triangle are proportional to two sides of another triangle and their included angles are congruent, then the triangles are similar.

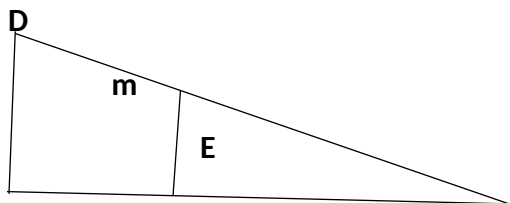
AA Similarity Theorem

If two angles of one triangle are equal in measures to two angles of another triangle, then the two triangles are similar.

SSS Similarity Theorem

If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the triangles are similar.

Example: Find x , y and m using the triangle below given $DC=9\text{cm}$, $EB=3\text{cm}$ and $AE = 5\text{cm}$.



Here you have two triangles $\triangle ABE \sim \triangle ACD$, which is similar by AA similarity theorem. Since $\angle A$ is a common angle and $m(\angle C) = m(\angle B) = 90^\circ$.

➤ Thus, $\frac{AB}{AC} = \frac{BE}{CD} = \frac{AE}{AD}$

$$\frac{x}{x+y} = \frac{3\text{ cm}}{9\text{ cm}} = \frac{5\text{ cm}}{m+5\text{ cm}}$$

$$\bullet \quad \frac{3\text{ cm}}{9\text{ cm}} = \frac{5\text{ cm}}{m+5\text{ cm}} \Rightarrow 3(m+5) = 9(5) \Rightarrow 3m+15 = 45 \Rightarrow m = 10\text{ cm}$$

- A triangle ABE is right angle triangle and we can find the measure of $AB=x$, using Pythagoras theorem:

$$(AE)^2 = (AB)^2 + (BE)^2 \Rightarrow (AB)^2 = (AE)^2 - (BE)^2 = 5^2 - 3^2 = 25 - 9 = 16$$

$$\Rightarrow AB = \sqrt{16} = 4\text{ cm}.$$

Therefore, $AB = x = 4\text{ cm}$.

$$\bullet \quad \frac{x}{x+y} = \frac{3\text{ cm}}{9\text{ cm}} \Rightarrow \frac{4\text{ cm}}{4\text{ cm}+y} = \frac{3\text{ cm}}{9\text{ cm}} \Rightarrow 3(4+y) = 9(4) \Rightarrow 12+3y = 36 \Rightarrow m = 8\text{ cm}$$

For more examples refer your text book page 200-202.

Note:

- If two triangle are congruent, then they are similar but vice-verse is not all ways true.
- All equilateral triangle and all square are always congruent also they are similar.

5.2.4 Theorems on similar plane figures

Ratio of perimeters and ratio of areas of similar plane figures.

Theorem: If the ratio of the length of any two corresponding sides of two similar figure (any similar triangle, any similar polygon, etc) is k , then

- i. The ratio of their perimeter is k , that is; $\frac{P_1}{P_2} = \frac{S_1}{S_2} = k$.
- ii. The ratio of their areas is k^2 , that is $\frac{A_1}{A_2} = k^2 \iff \frac{A_1}{A_2} = \left(\frac{P_1}{P_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^2$.

Example: The area of two similar triangle are 144 cm^2 and 81 cm^2 . Find

- i. The ratio of their perimeters and
- ii. If the side of the first is 6 cm long, what is the length of the other corresponding side of the second?

Solution:

i. Since $\frac{A_1}{A_2} = \left(\frac{P_1}{P_2}\right)^2 \implies \frac{P_1}{P_2} = \sqrt{\frac{A_1}{A_2}} = \sqrt{\frac{144}{81}} = \frac{\sqrt{144}}{\sqrt{81}} = \frac{12}{9}$.

Therefore $P_1 : P_2 = 12 : 9$

- ii. Let $S_1 = 6 \text{ cm}$, then find S_2

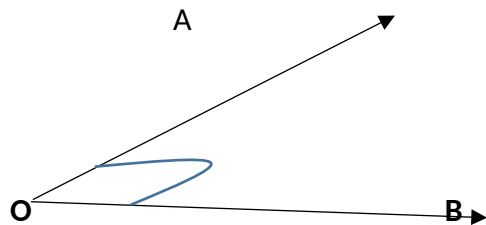
$$\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 \implies \frac{144}{81} = \frac{6^2}{S_2^2} \implies S_2^2 = \frac{81 \times 36}{144} \implies S_2 = \sqrt{\frac{81 \times 36}{144}} = \frac{9 \times 6}{12} = \frac{9}{2}$$

5.2.5 Construction of Similar Figures and 5.2.6 Real-life problems using Congruency and Similarity. It is easy so refer your text book page 206 – 209.

5.3 Further on Trigonometry

Radian Measure of an Angle

- ✓ An angle is formed by two rays with a common end point.



- i. Measuring angles in Degree

Definition: A degree denoted by ($^{\circ}$), is defined as the measure of the central angle subtended by an arc of a circle equal in length to $\frac{1}{360}$ of the circumference of the circle.

- ❖ A minute which is denoted by ($'$), is $\frac{1}{60}$ of a degree. i.e; $1' = \frac{1}{60}^{\circ}$ implies $1^{\circ} = 60'$
- ❖ A second which is denoted by ($''$), is $\frac{1}{60}$ of a minute. i.e; $1'' = \frac{1}{60}'$ implies $1' = 60''$
- ❖ A second which is denoted by ($''$), is $\frac{1}{3600}$ of a degree. i.e; $1'' = \frac{1}{3600}^{\circ}$ implies $1^{\circ} = 3600''$.

ii. Measuring angles in Radians

Definition: A radian denoted by (rad), is defined as the measure of the central angle subtended by an arc of a circle equal in length to the radius of the circle.

Note:

- a. 1 revolution = $360^{\circ} = 2\pi$ radians i.e; $180^{\circ} = \pi$ radians
- b. 1 radians = $\left(\frac{180}{\pi}\right)^{\circ} \approx 57.3^{\circ}$
- c. $1^{\circ} = \frac{\pi}{180}$ radians ≈ 0.0175 radian
- d. To convert radians to degrees, multiply by $\frac{180^{\circ}}{\pi}$.
- e. To convert degrees to radians multiply by $\frac{\pi}{180^{\circ}}$

Example:

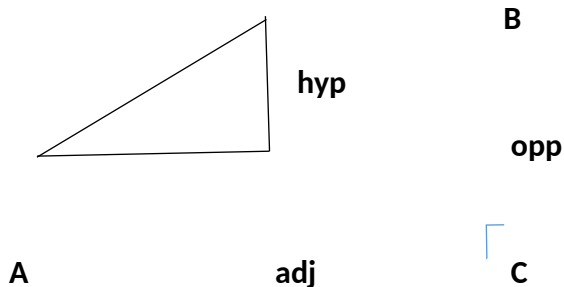
1. Convert 30° to radians
2. Convert $\frac{\pi}{4} rad$ to degree.

Solution:

1. $30^{\circ} = 30^{\circ} \times \frac{\pi}{180^{\circ}} = \frac{\pi}{6} rad$
2. $\frac{\pi}{4} rad = \frac{\pi}{4} \times \frac{180^{\circ}}{\pi} = 45^{\circ}$

Trigonometrical Ratios to Solve Right- angled Triangles

Using the figure below



Note:

- Opp is abbreviation of opposite side to angle A.
- adj is abbreviation of adjacent side to angle A.
- hyp is abbreviation of hypotenuse side to angle A.

So we can define sine, cosine and tangent function as follows

1. $\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$
2. $\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB}$
3. $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC}$

Note:

- a. The trigonometric values of an acute angle is between 0 and 1.
- b. Every angles can be expressed in terms of an acute angle.
- c. Trigonometric values of obtuse angle is given as if θ is obtuse angle

$$\sin \theta = \sin (180^\circ - \theta)$$

$$\cos \theta = -\cos (180^\circ - \theta)$$

$$\tan \theta = -\tan (180^\circ - \theta)$$

From the above relationship cosine and tangent of an obtuse angle is negative.

- d. The values of sine cosine and tangent of the angles of $0^\circ, 30^\circ, 45^\circ, 60^\circ \wedge 90^\circ$ are summerized as follows

$\angle A$	0°	30°	45°	60°	90°
$\sin A$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos A$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan A$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

5.4 Circles

Symmetrical Properties of Circles

Definition: Circle is the set of points in a given plane, each of which is at the same distance from a fixed point of the plane. The fixed point is called the center, and the distance is the radius of the circle.

A line segment through the center of a circle with end points on the circle is called a diameter.

A chord of a circle is a line segment whose end points lie on the circle.

If the figure can be folded or divided into half so that the two halves match exactly then such a figure is called symmetrical figure.

If you recall regular polygon is also symmetrical figure and line of symmetry is equal to the number of sides.

Symmetrical figure is the figure that has at least one line of symmetry. Thus a circle has infinitely many line of symmetry.

Theorem:

The line segment joining the center of a circle to the mid-point of a chord is perpendicular to the chord.

Theorem:

The line segment drawn from the center of a circle perpendicular to a chord bisects the chord

Theorem:

- 1. If two chords of a circle are equal, then they are equidistance from the center.*
- 2. If two chords of a circle are equidistance from the center, then their lengths are equal.*

Theorem:

If two tangent segments are drawn to a circle from an external point, then

- I. The tangents are equal in length, and*
- II. The line segment joining the center to the external point bisects the angle between the tangents.*

Angle Properties of Circles

Remark that:

- A part of a circle (part of its circumference) between any two points on the circle is called an **arc denoted by $\widehat{AB}$ if A and B end point of the circle.**
- **Semicircle** is half of the circumference of the circle and measure 180° .
- An arc less than semicircle is called **minor arc**.
- An arc greater than semicircle is called **major arc**.

- **A central angle** of a circle is an angle whose vertex is at the center of the circle and whose sides are radii of the circle. The measure of the central angle is equal to the angle measure of the arc it intercepts.
- **An inscribed angle in a circle** is an angle whose vertex is on the circle and whose sides are chords of the circle. The measure of the inscribed angle is one half of the angle measure of the arc it intercepts or one half of the measure of the central angle.

Theorem:

The measure of the central angle subtended by an arc is twice the measure of an inscribed angle in the circle subtended by the same arc.

Corollary1.

Angles inscribed in the same arc of the circle are equal.

Corollary2.

The angle inscribed in a semi-circle is a right angle. That is its degree measure is 90° .

Corollary3.

A quadrilateral inscribed in a circle is called **cyclic quadrilateral**. The opposite angles of a cyclic quadrilateral are supplementary. That is their sum measure 180° .

5.4.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

Remember that: (Refer figure 5.97 on your text book page 228).

- ✓ Circumference of a circle $C = 2\pi r$ or πd .
- ✓ Area of a circle $A = \pi r^2$.
- ✓ Part of the circumference of a circle is called an arc.
- ✓ A segment of a circle is a region bounded by a chord and an arc.
- ✓ A sector of a circle is a region bounded by two radii and an arc.

Arc Length

- The length l of an arc of the circle of radius r that subtends an angle of θ at the center is given by

$$l = \frac{\theta}{360^\circ} \times 2\pi r = \frac{\pi r \theta}{180^\circ}$$

The Perimeters and Areas of a Sectors

- The area A of a sector of radius r and central angle θ is given by

$$A = \frac{\theta}{360^\circ} \times \pi r^2 = \frac{\pi r^2 \theta}{360^\circ}$$

- The perimeter P of the sector is the sum of the radii and the arc that bound it.

$$P = 2r + \frac{\pi r \theta}{180^\circ}$$

The Perimeters and Areas of a Segment

- The area A and perimeter P of a segment of a circle of radius r, cut off by a chord subtending an angle θ at the center of a circle is given by

$$A = \frac{\pi r^2 \theta}{360^\circ} - \frac{1}{2} r^2 \sin \theta \quad (\text{sector area} - \text{triangle area})$$

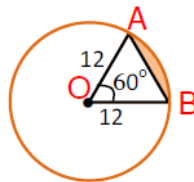
$$P = 2r \sin \frac{\theta}{2} + \frac{\pi r \theta}{180^\circ} \quad (\text{chord length} + \text{arc length})$$

Example: A segment of a circle of a radius 12 cm is cut off by a chord subtending an angle 60° at the center of the circle. Find:

a. The area of the segment

b. the perimeter of the sector

Solution:



- From the figure given above,

Area of the segment (the shaded part) = area of the sector OAB - area of triangle OAB;

$$\text{Area of the sector OAB} = \frac{\pi \times 12^2 \times 60}{360^\circ} = 24\pi \text{ cm}^2$$

$$\text{Area of triangle OAB} = \frac{1}{2} r^2 \sin \theta = \frac{1}{2} \times 12^2 \sin 60 = 36\sqrt{3} \text{ cm}^2$$

$$\text{Therefore, segment area} = 24\pi \text{ cm}^2 - 36\sqrt{3} \text{ cm}^2 = (24\pi - 36\sqrt{3}) \text{ cm}^2$$

Perimeter of the sector = $2 \times \text{radius} + \text{length of arc AB}$

$$2 \times 12 + \frac{\pi \times 12 \times 60}{180^\circ} = (24 + 4\pi) \text{ cm}$$

5.5 Measurement

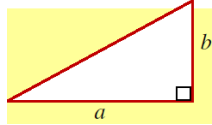
5.5.1 Areas of Triangles and Parallelograms

A. Areas of Triangles

Case i.

The area A of a right angle triangle with perpendicular sides of length a and b is given by

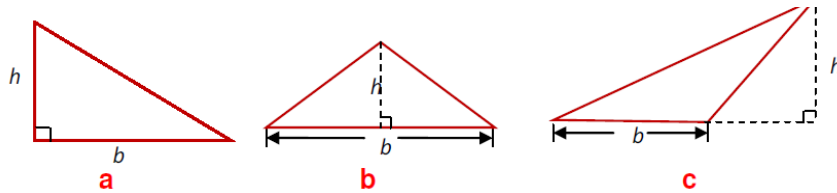
$$A = \frac{1}{2} ab$$



Case i.

The area A of any triangle with base b and the corresponding height h is given by

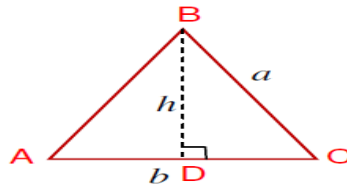
$$A = \frac{1}{2} bh$$



Case iii.

The area A of any triangle with side a and b units long and angle C ($\angle C$) included between these sides is

$$A = \frac{1}{2} ab \sin(\angle C)$$



Theorem 5.14: (Heron's Formula)

The area A of a triangle with sides a , b and c units long and semi-perimeter

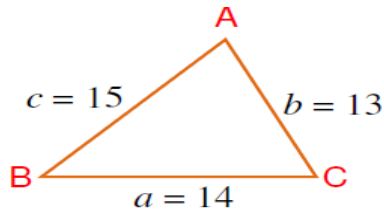
$$S = \frac{1}{2}(a+b+c) \text{ is given by}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example: Given $\triangle ABC$. If $AB=15$ units, $BC=14$ units and $AC=13$ units, find

- The area of $\triangle ABC$
- The length of the altitude from the vertex A

- c. The measure of $\angle B$.



Solution:

a. $S = \frac{1}{2}(a+b+c)$ implies $S = \frac{1}{2}(14+13+15) = 21$

$a=14$ $s-a=7$

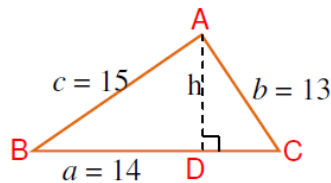
$b=13$ $s-b=8$

$c=15$ $s-c=6$

Therefore, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$\therefore \sqrt{21(7)(8)(6)} = 84 \text{ sq. cm.}$

- a. Let the altitude from the vertex A to the corresponding base BC be h, meeting BC at D as shown in figure below.



Then, area of $\Delta ABC = \frac{1}{2} BC \times h$.

Therefore, $84 = \frac{1}{2} \times 14 \times h = 7h$.

Hence, $h = \frac{84}{7} = 12 \text{ units}$

Therefore, the altitude of ΔABC from the vertex A is 12 units long.

- b. In the right angle triangle ABD shown in figure above, we see that

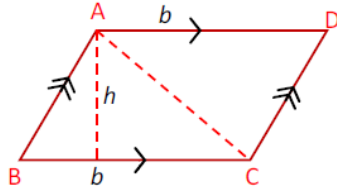
$\sin(\angle B) = \frac{AD}{AB} = \frac{h}{c} = \frac{12}{15} = \frac{4}{5} = 0.8$.

B. Areas of Parallelograms

Theorem 5.15

The area of a parallelogram with base b and perpendicular height h is

$$A = bh$$



Example: If one pair of opposite sides of a parallelogram have length 40 cm and the distance between them is 15 cm, find the area of the parallelogram.

Solution: $A = bh = 40 \text{ cm} \times 15 \text{ cm} = 600 \text{ sq. cm}$

5.5.2 Further on Surface Areas and Volumes of Cylinders and Prisms

A. Prism

A. Prism

- A prism is a solid figure formed by two congruent polygonal regions in parallel planes, along with three or more parallelograms, joining the two polygons. The polygons in parallel planes are called bases.
- A prism is named by its base. Thus, a prism is called triangular, rectangular, pentagonal, etc., if its base is a triangle, a rectangle, a pentagon, etc., respectively.

In a prism

- ✓ The lateral edges are equal and parallel.
- ✓ The lateral faces are parallelograms.
- A right prism is a prism in which the base is perpendicular to a lateral edge. Otherwise it is an oblique prism.
- In a right prism
 - ✓ All lateral edges are perpendicular to both bases.
 - ✓ The lateral faces are rectangles
 - ✓ The altitude is equal to the length of each lateral edge.
- A regular prism is a right prism whose base is a regular polygon.

Surface area and volume of prisms

- The lateral surface area of a prism is the sum of the area of its lateral faces

$A_L = Ph$, where P is perimeter $\wedge$ h is altitude

- The total surface area of a prism is the sum the lateral areas and the area of the bases.

$$A_T = 2A_B + A_L$$

- The volume of any prism is equal to the product of its bases and its altitude.

$$V = A_B h$$

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$$A_T = 2A_B + A_L$$
- o The volume of any prism is equal to the product of its bases and its altitude.

$$V = A_B h$$

Example 1. The altitude of a rectangular prism is 4 units and the width and length of its base are 3 and 2 units respectively. Find:

- the total surface area of the prism.
- the volume of the prism.

Solution:

a. To find A_T , first we have to find the base area and the lateral surface area.

$AB = 2 \times 3 = 6$ sq unit and $AL = Ph = (3 + 2 + 3 + 2) \times 4 = 40$ sq unit.

$$AT = 2AB + AL = 2 \times 6 + 40 = 52 \text{ sq. unit.}$$

So, the total surface area is 52 sq unit.

b. $V = AB h = 6 \times 4 = 24$ cubic unit

Cylinder

- A circular cylinder is a simple closed surface bounded on two ends by circular bases.
- A more general definition of a cylinder replaces the circle with any simple closed curve. Since not all cylinder is circular.

Surface area and volume of Circular Cylinders

- The lateral surface area of a right circular cylinder is the product of its height h and the circumference C of its base.

$$A_L = Ch \vee A_L = 2\pi rh$$

- The total surface area of a right circular cylinder is the sum of twice the area of the circular base and the area of the curved surface (lateral surface area).

$$A_T = 2A_L + 2A_B = 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$$

- The volume V of the right circular cylinder is equal to the product of its base area and height

$$V = A_B h = \pi r^2 h$$

Example 1. The altitude of a rectangular prism is 4 units and the width and length of its base are 3 and 2 units respectively. Find:

a. the total surface area of the prism.

B. the volume of the prism.

Solution:

a. To find A_T , first we have to find the base area and the lateral surface area.

$$AB = 2 \times 3 = 6 \text{ sq unit and } AL = Ph = (3 + 2 + 3 + 2) \times 4 = 40 \text{ sq unit.}$$

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So, the total surface area is 52 sq unit.

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- **A circular cylinder** is a simple closed surface bounded on two ends by circular bases.
- A more general definition of a cylinder replaces the circle with any simple closed curve. Since not all cylinder is circular.

Surface area and volume of Circular Cylinders

- The lateral surface area of a right circular cylinder is the product of its height h and the circumference C of its base.

$$A_L = Ch \vee A_L = 2\pi rh$$

- The total surface area of a right circular cylinder is the sum of twice the area of the circular base and the area of the curved surface (lateral surface area).

$$A_T = 2A_L + 2A_B = 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$$

- The volume V of the right circular cylinder is equal to the product of its base area and height

$$V = A_B h = \pi r^2 h$$

Example 3 If the height of a right circular cylinder is 8 cm and the radius of its base is 5 cm find the following giving your answers in terms of π .

a. its lateral surface area

b. its total surface area

c. its volume

Solution:

a. The lateral surface area of the right circular cylinder is given by

$$A_L = 2\pi rh = 2\pi \times 5 \times 8 = 80\pi \text{ cm}^2$$

$$\text{b. } A_T = 2\pi r(r+h)$$

$$= 80\pi + 50\pi = 130\pi \text{ sq. cm}$$

c. The volume of the cylinder is

$$V = \pi r^2 h = \pi \times 5^2 \times 8 = 200\pi \text{ cm}^3$$