MATIMATICS GRADE 12
UNIT 5 SUMMARY

Introduction to integral calculus

BY :ADDIS ABABA EDUCATION BUREAU

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5.2 TECHNIQUES OF INTEGRATION

- ✓ Integration by substitution
- ✓ Integration by partial fraction
- ✓ Integration by parts

1. **INTEGRATION BY SUBSTITUTION**

- It is a method of finding integrals by changing variables, so that a new variable may be simpler to integrate.

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Examples: Evaluate the following integral

1. $\int 2x(x^2+1)^5 dx$

Examples: Evaluate the following integral

Solution: Let $u = x^2 + 1, then\ \frac{du}{dx} = 2x\ and\ du = 2xdx$

Solution: Let

$$\int 2x(x^2+1)^5 dx = \int u^5 du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{(x^2+1)^6}{6} + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

2. $\int x\sqrt{x^2+5}\,dx$

Solution: Let $u = x^2 + 5, then\ \frac{du}{dx} = 2x\ and\ \frac{du}{2} = xdx$

2.

Solution: Let

$$\int x\sqrt{x^2+5}\,dx = \int \sqrt{u}\frac{du}{2}$$

$$= \frac{1}{2}\int \sqrt{u}du$$

$$= \frac{1}{2}.\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}(x^2+5)^{\frac{3}{2}} + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

3. $\int xe^{x^2} dx$

3. Solution: Let $u = x^2, then$

Solution: Let $\frac{du}{dx} = 2x \; and \; \frac{du}{2} = xdx$

$$\int xe^{x^2} dx = \int e^u \frac{du}{2}$$

$$= \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

4. $\int (3x-2)\sqrt{x+6}\,dx$

Solution: Let $u = x+6, then$

4. $$du = dx \; and \; x = u-6$$

Solution: Let $$\int (3x-2)\sqrt{x+6}\,dx = \int [3(u-6)-2]\sqrt{u}du$$

$$= \int (3u-20)u^{\frac{1}{2}}du$$

$$= \int \left(3u^{\frac{3}{2}} - 20u^{\frac{1}{2}}\right)du$$

$$= \int 3u^{\frac{3}{2}}du - \int 20u^{\frac{1}{2}}du$$

$$= 3.\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 20.\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= 3.\frac{2}{5}u^{\frac{5}{2}} - 20.\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{6}{5}(x+6)^{\frac{5}{2}} - \frac{40}{3}(x+6)^{\frac{3}{2}} + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

5. $\int f(x)f'(x)dx$

5. Solution: Let $u = f(x)$,

Solution: Let $then \frac{du}{dx} = f'(x)\ and\ du = f'(x)dx$

$$\int f(x)f'(x)dx = \int udu$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(f(x))^2}{2} + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

6. $\int \frac{f'(x)}{f(x)}\,dx$

6.

Solution: Let $u = f(x), then \frac{du}{dx} = f'(x)\ and\ du = f'(x)dx$

Solution: Let

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|f(x)| + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

7. $\int cotx\ dx$

Solution: $cotx = \frac{cosx}{sinx}$

Solution:

$$\int cotx\ dx = \int \frac{cosx}{sinx} dx$$

Let $u = sinx, then\ \frac{du}{dx} = cosx\ and\ du = cosx\ dx$

$$\int cotx\ dx = \int \frac{cosx}{sinx} dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|sinx| + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

8. $\int \frac{1}{x \ln x} dx$

8. Solution: $\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \frac{1}{x} dx$

Solution: Let $u = \ln x$,then

Let then

$$\frac{du}{dx} = \frac{1}{x} \; and \; du = \frac{1}{x} dx$$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{\ln x} \frac{1}{x} dx$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|lnx| + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

9. $\int \sin(2x+4)dx$

Solution: Let $u = 2x + 4, then$

Solution: Let

$$\frac{du}{dx} = 2 \; and \; \frac{du}{2} = dx$$

$$\int \sin(2x+4)dx = \int \sin u \frac{du}{2}$$

$$= \frac{1}{2}(-cosu) + C$$

$$= \frac{-1}{2}\cos(2x+4) + C$$

INTEGRATION BY SUBSTITUTION(CONT'D)

10. $\int sin^2 x dx$

Solution: $\cos 2x = cos^2 x - sin^2 x \ (\cos(x+y) = \cos x \cos y - \sin x \sin y)$

$= 1 - sin^2 x - sin^2 x \ (\sin^2 x + \cos^2 x = 1)$

$= 1 - 2sin^2 x$

$$\Longrightarrow 2sin^2 x = 1 - cos2x$$

$$\Longrightarrow sin^2 x = \frac{1 - cos2x}{2}$$

$$\int sin^2 x dx = \int \frac{1 - cos2x}{2} dx$$

$$= \int \frac{1}{2} dx - \int \frac{1}{2} cos2x dx$$

$$= \frac{1}{2}x - \frac{1}{2} . \frac{1}{2} sin2x + C$$

$$= \frac{1}{2}x - \frac{1}{4} sin2x + C$$

EXERCISE
EVALUATE THE FOLLOWING INTEGRALS

1. $\int cos^2 x dx$
2. $\int tanx\ dx$
3. $\int \frac{x}{\sqrt{x+1}} dx$
4. $\int \frac{sinx}{cos^2 x} dx$
5. $\int x e^{x^2+7} dx$

2. INTEGRATION BY PARTIAL FRACTION

$$\int \left(\frac{1}{x+2}+\frac{3}{x-1}\right)dx = _____$$

Solution:

Solution: $\int \left(\frac{1}{x+2}+\frac{3}{x-1}\right)dx = \int \frac{1}{x+2}dx + \int \frac{3}{x-1}dx$

$$= \ln|x+2| + 3\ln|x-1| + C$$

And

And $\int (\frac{1}{x+2}+\frac{3}{x-1})dx =$

$$\int \frac{x-1+3(x+2)}{(x+2)(x-1)}dx = \int \frac{4x+5}{(x+2)(x-1)}dx$$

INTEGRATION BY PARTIAL FRACTION(CONT'D)

Examples:

Examples:

1. $\int \frac{1}{x(x+1)}\,dx = ____$

Solution:

Solution: $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$$= \frac{A(x+1)+Bx}{x(x+1)}$$

For $1 = A(x+1) + Bx$

And for $x = 0, 1 = A$

And for $x = -1, B = -1$

INTEGRATION BY PARTIAL FRACTION(CONT'D)

Therefore, $\frac{1}{x(x+1)} = \frac{1}{x} + \frac{-1}{x+1}$

Therefore, $\int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} + \frac{-1}{x+1} \right) dx$

$$= \int \frac{1}{x} dx + \int \frac{-1}{x+1} dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$= \ln \left| \frac{x}{x+1} \right| + C$$

INTEGRATION BY PARTIAL FRACTION(CONT'D)

$$2.\int \frac{x}{x^3-3x+2}\,dx$$

Solution: First factorize the denominator and decompose the rational expression in to partial fraction

$$\frac{x}{x^3-3x+2} = \frac{x}{(x-1)^2(x+2)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)}$$

INTEGRATION BY PARTIAL FRACTION(CONT'D)

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

For $x = 1, 1 = 3B \Longrightarrow B = \frac{1}{3}$

For

For $x = -2, -2 = C(-3)^2 \Longrightarrow C = \frac{-2}{9}$

For

For $x = 0, 0 = A(-1)(2) + \frac{1}{3}(2) - \frac{2}{9} \Longrightarrow A = \frac{2}{9}$

For

Therefore,

Therefore, $\frac{x}{x^3-3x+2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

$$= \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

INTEGRATION BY PARTIAL FRACTION(CONT'D)

$$\int \frac{x}{x^3-3x+2}\,dx = \int \left(\frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}\right) dx$$

$$= \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int (x-1)^{-2}\,dx - \frac{2}{9}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{9}\ln|x-1| + \frac{1}{3}.\frac{(x-1)^{-2+1}}{-2+1} - \frac{2}{9}\ln|x+2| + C$$

$$= \frac{2}{9}[\ln|x-1| - \ln|x+2|] - \frac{1}{3}.\frac{1}{x-1} + C$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Integration by partial fraction(cont'd)

3. $\int \frac{5x^2-8x+5}{(x-2)(x^2-x+1)}dx = _____$

Solution:

Solution: $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2-x+1}$

$$= \frac{A(x^2-x+1)+(x-2)(Bx+C)}{(x-2)(x^2-x+1)}$$

For

$$5x^2 - 8x + 5 = A(x^2 - x + 1) + (x - 2)(Bx + C)$$

For ,

For $x = 2, 5(2)^2 - 8(2) + 5 = A(2^2 - 2 + 1) \Longrightarrow A = 3$

For

For $x = 0, 5(0)^2 - 8(0) + 5 = 3(0^2 - 0 + 1) + (0 - 2)(C) \Longrightarrow C = -1$

Therefore,

For $x = 1, 5(1)^2 - 8(1) + 5 = 3(1^2 - 1 + 1) + (1 - 2)(B(1) - 1) \Longrightarrow B = 2$

Therefore, $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)} = \frac{3}{(x-2)} + \frac{2x-1}{x^2-x+1}$

Integration by partial fraction(cont'd)

$$hence, \int \frac{5x^2-8x+5}{(x-2)(x^2-x+1)}dx = \int \left(\frac{3}{(x-2)} + \frac{2x-1}{x^2-x+1}\right)dx$$

$$= \int \frac{3}{(x-2)}dx + \int \frac{2x-1}{x^2-x+1}dx$$

$$= 3\ln|x-2| + _____$$

Let and

$$Let\ u = x^2 - x + 1\ and \frac{du}{dx} = 2x - 1 \Longrightarrow du = (2x-1)dx$$

Therefore,

$$\int \frac{2x-1}{x^2-x+1}dx = \int \frac{du}{u} = \ln|u| = \ln|x^2-x+1|$$

Therefore, $\int \frac{5x^2-8x+5}{(x-2)(x^2-x+1)}dx = 3\ln|x-2| + \ln|x^2-x+1| + C$

Integration by partial fraction(cont'd)

4. $\int \frac{x^3}{x^2-4x+3}\,dx = _____$

Solution: the rational expression is an improper fraction, hence before factorizing the denominator we use long division to obtain

Solution: the rational expression is an improper fraction, hence before factorizing the denominator we use long division to obtain

$$\frac{x^3}{x^2-4x+3} = x + 4 + \frac{13x-12}{x^2-4x+3}$$

$$= x + 4 + \frac{13x-12}{(x-1)(x-3)}$$

Therefore,

$$= x + 4 + \frac{-1}{2(x-1)} + \frac{27}{2(x-3)}$$

Therefore, $\int \frac{x^3}{x^2-4x+3}\,dx = \int \left(x + 4 + \frac{-1}{2(x-1)} + \frac{27}{2(x-3)}\right) dx$

$$= \int x\,dx + \int 4\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{27}{2}\int \frac{1}{x-3}\,dx$$

$$= \frac{x^2}{2} + 4x - \frac{1}{2}\ln|x-1| + \frac{27}{2}\ln|x-3| + C$$

Integration by partial fraction(cont'd)

5. $\int \frac{4x-5}{5x-4}\,dx = _____$

Solution: the rational expression is an improper fraction, so we use long division to obtain

$$\frac{4x-5}{5x-4} = \frac{4}{5} + \frac{\frac{-9}{5}}{5x-4}$$

$$= \frac{4}{5} - \frac{9}{5(5x-4)}$$

Therefore,

$$\int \frac{4x-5}{5x-4}\,dx = \int \left(\frac{4}{5} - \frac{9}{5(5x-4)}\right) dx$$

$$= \int \frac{4}{5}\,dx - \frac{9}{5}\int \frac{1}{5x-4}\,dx$$

$$= \frac{4}{5}x - \frac{9}{5}.\frac{1}{5}\ln|5x-4| + C$$

$$= \frac{4}{5}x - \frac{9}{25}\ln|5x-4| + C$$

Exercise
Evaluate the following integrals

1. $\int \frac{4x+1}{x^2-3x+2}\,dx$
2. $\int \frac{3x+5}{x+2}\,dx$
3. $\int \frac{x^2+4}{x^2-1}\,dx$
4. $\int \frac{x^3}{(x+1)^2(x+2)}\,dx$
5. $\int \frac{x^2+2x-3}{x^2(x^2-5x+6)}\,dx$

3. Integration by parts

- Let u and v be functions of x , $i.e.\ u = u(x)\ and\ v = v(x)$

Then, $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

Integrating both sides:

$$\int \frac{d}{dx}(uv)dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$(uv) = \int vdu + \int udv$$

- In this method, you should be able to choose parts and

x

-

$$\int udv = uv - \int vdu$$

- In this method, you should be able to choose parts u and dv

$u =$ ____ x $dv =$ ____

$du =?$ - $v =?$

Examples: Evaluate the following integral

1. $\int xe^x dx = _____$

Solution:

Let $u = x$ $dv = e^x$

$du = 1$ $v = e^x$

Therefore, $\int xe^x dx = xe^x - \int e^x dx$

$= \boldsymbol{xe^x - e^x + C}$

If you let $u = e^x$ $dv = x$

$du = e^x$ $v = \frac{x^2}{2}$

$$\int xe^x dx = e^x . \frac{x^2}{2} - \int e^x . \frac{x^2}{2} dx$$

- The proper selection of is important to find the integral of the given function.
- The proper selection of $u \; and \; dv$ is important to find the integral of the given function.

Guidelines in selecting u and dv when you apply integration by parts

1. For integral of the form $\int x^n e^{ax}\,dx$, $\int x^n \sin ax\,dx$ or $\int x^n \cos ax\,dx$

   Choose $u = x^n$ and $dv = e^{ax}dx$, $\sin axdx, or\ \cos ax\,dx$

2. For integral of the form $\int x^n \ln x\,dx$

   Choose $u = \ln x$ and $dv = x^n dx$

3. For integral of the form $\int e^{ax} \sin bx\,dx\ or\ \int e^{ax} \cos bx\,dx$

   Choose $u = \sin bx\ \ or\ \ \cos bx\ \ and\ dv = e^{ax}dx$

Example 2

$$\int x^2 e^x dx = _____$$

Solution: Let

Solution: Let $u = x^2$ $dv = e^x$

$du = 2x$ $v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

)+C

$$= x^2 e^x - 2\int x e^x dx$$

$$= x^2 e^x - 2(\boldsymbol{x e^x - e^x})\text{+C}$$

$$= x^2 e^x - 2\boldsymbol{x e^x} + \boldsymbol{2 e^x} + \boldsymbol{C}$$

$$= e^x(x^2 - 2\boldsymbol{x} + \boldsymbol{2}) + \boldsymbol{C}$$

Alternative way of integration by part

Tabular method

This method works well for integral of the form

$$\int x^n e^{ax}\,dx\,,\ \int x^n \sin ax\,dx\ \ or\ \int x^n \cos ax\,dx$$

Example: let us do example 2,( )using tabular form

Example: let us do example 2,($\int x^2 e^x dx$)using tabular form

Solution:

Solution:

alternate signs and its derivative and its anti derivative

alternate signs	u and its derivative	dv and its anti derivative
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x
$-$	0	e^x

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

Example 3

$$\int \ln x \, dx = _____$$

Solution:

Solution: $\int \ln x \, dx = \int 1.\ln x \, dx$

Let

Let $u = \ln x$ $dv = 1$

$du = \frac{1}{x}$ $v = x$

$$\int 1.\ln x \, dx = x ln\, x - \int \frac{1}{x}.x \, dx$$

$$= x ln\, x - x + C$$

Example 4

$$\int \log x \, dx = _____$$

Solution:

Solution: $\int \log x \, dx = \int \frac{\ln x}{\ln 10} dx$

$$= \frac{1}{\ln 10} \int \ln x \, dx$$

$$= \frac{1}{\ln 10} (x ln x - x) + C$$

Example 5

$$\int \log(2x+5)dx = _____$$

Solution: Let

Solution: Let $u = 2x+5$, then $\frac{du}{dx} = 2 \Longrightarrow \frac{du}{2} = dx$

$$\int \log(2x+5)dx = \int \log u \, \frac{du}{2}$$

$$= \frac{1}{2}\int \log u \, du$$

$$= \frac{1}{2}.\frac{1}{\ln 10}(u \ln u - u) + C$$

$$= \frac{1}{2}.\frac{1}{\ln 10}[(2x+5)\ln(2x+5) - (2x-5)] + C$$

$$= \frac{2x+5}{2\ln 10}[\ln(2x+5) - 1] + C$$

Example 6

$\int x^r \ln x \, dx =$ ____ where r is real number different from -1

Solution:

Let

$$u = \ln x \qquad dv = x^r$$

$$du = \frac{1}{x} \qquad v = \frac{x^{r+1}}{r+1}$$

$$\int x^r \ln x \, dx = \frac{x^{r+1}}{r+1} \ln x - \int \frac{1}{x} . \frac{x^{r+1}}{r+1} dx$$

$$= \frac{x^{r+1}}{r+1} \ln x - \frac{1}{r+1} \int x^r dx$$

$$= \frac{x^{r+1}}{r+1} \ln x - \frac{1}{r+1} . \frac{x^{r+1}}{r+1} + C$$

$$= \frac{x^{r+1}}{r+1} \ln x - \frac{x^{r+1}}{(r+1)^2} + C$$

In the above example, if $r = -1$

$$\int x^r \ln x \, dx = \int x^{-1} \ln x \, dx$$

$$= \int \frac{1}{x} . \ln x \; dx$$

Let

Let $u = \ln x \;\; and \;\; \frac{du}{dx} = \frac{1}{x} \Longrightarrow du = \frac{1}{x} dx$

$$\int x^{-1} \ln x \, dx = \int \frac{1}{x} . \ln x \; dx$$

$$= \int u du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{ln^2 x}{2} + C$$

Example 7

$$\int e^{2x} \sin x \, dx = _____$$

Solution:

Solution:

Let

Let $u = \sin x$ $dv = e^{2x}$

$du = \cos x$ $v = \frac{1}{2} e^{2x}$

Again applying integration by part for the second part of integral

Let

Again applying integration by part for the second part of integral

Let $u = \cos x$ $dv = e^{2x}$

$du = -\sin x$ $v = \frac{1}{2} e^{2x}$

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} [\frac{1}{2} e^{2x} \cos x - \int \frac{-1}{2} e^{2x} \sin x \, dx]$$

Example 7(cont'd)

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x - \int \frac{-1}{2}e^{2x}\sin x\,dx\right]$$

$$\Rightarrow \int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$$

$$\Rightarrow \int e^{2x}\sin x\,dx + \frac{1}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\Rightarrow \frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\Rightarrow \int e^{2x}\sin x\,dx = \frac{4}{5}.\frac{1}{2}e^{2x}\sin x - \frac{4}{5}.\frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C$$

Exercise
Evaluate the following integrals

1. $\int xe^{5x+2}dx$
2. $\int(4x-3)e^{-x}dx$
3. $\int x^2 \sin x\, dx$
4. $\int x.5^x dx$
5. $\int \frac{\ln x}{x^2} dx$
6. $\int(\ln x)^2 dx$
7. $\int e^x \cos 2x\, dx$
8. $\int 3^x \cos x\, dx$