

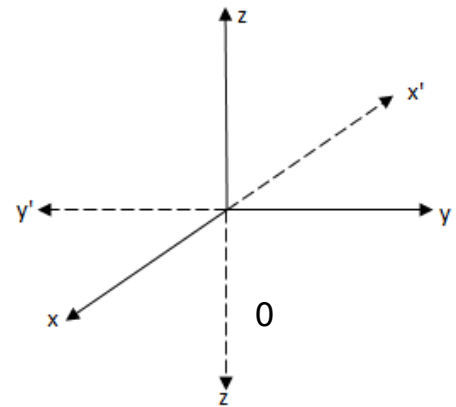


UNIT - 6

THREE DIMENSIONAL GEOMETRY AND VECTORS IN SPACE

1. Coordinate axes and coordinate planes in space

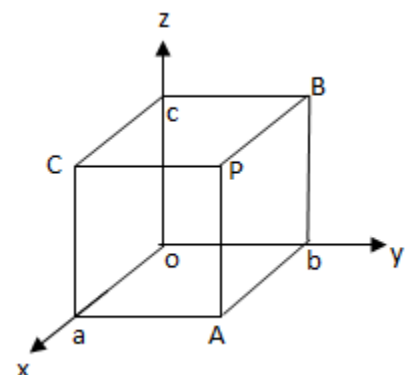
- Let $x' ox$, $y' oy$, $z' oz$ be three mutually perpendicular lines that pass through a point O such that $x' ox$ and $y' oy$ lies in the plane of the paper and line $z' oz$ is perpendicular to the plane of paper. These three lines are called rectangular axes (lines $x' ox$, $y' oy$ and $z' oz$ are called x - axis, y - axis and z - axis respectively). We call this coordinate system a three dimensional space or simply space.
- The three axes taken together in pairs determine xy , yz , zx - plane, i.e. three coordinate planes.
- The three coordinate planes (xy , yz and zx) together divide the space into eight regions (parts) called octants, namely
- $oxyz$
 - $oxyz'$
 - $oxy'z$
 - $oxy'z'$
 - $ox'yz$
 - $ox'yz'$
 - $ox'y'z$
 - $ox'y'z'$



Octants	Coordinates		
	x	y	z
I ($oxyz$)	+	+	+
II ($oxyz'$)	+	+	-
III ($oxy'z$)	+	-	+
IV ($oxy'z'$)	+	-	-
V ($ox'yz$)	-	+	+
VI ($ox'yz'$)	-	+	-
VII ($ox'y'z$)	-	-	+
VIII ($ox'y'z'$)	-	-	-

2. Coordinates of A point in space

- An arbitrary point P in three dimensional space is assigned coordinates (a, b, c) provided that :-
1. The plane through P parallel to the yz plane intersects the x - axis at $(a, 0, 0)$.
 2. The plane through P parallel to the xz plane intersects the y - axis at $(0, b, 0)$
 3. The plane through P parallel to the xy plane intersects the z - axis at $(0, 0, c)$





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- The space coordinates (a, b, c) are called the Cartesian coordinates of P or simply the rectangular coordinates of P.

A is $(a, b, 0)$

B is $(0, b, c)$

C is $(a, 0, c)$

P is (a, b, c)

Note: equations of axes and planes in coordinate space

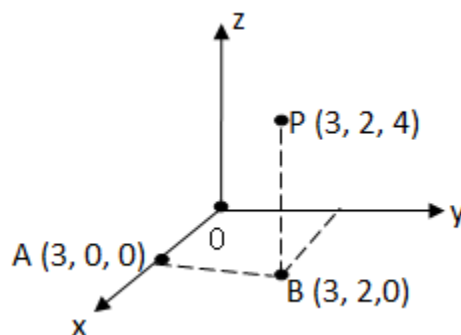
1. x - axis = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } y = z = 0\}, y = z = 0$
2. y - axis = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } x = z = 0\}, x = z = 0$
3. z - axis = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } x = y = 0\}, x = y = 0$
4. xy - plane = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } z = 0\}, z = 0$
5. yz - plane = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } x = 0\}, x = 0$
6. xz - plane = $\{(x, y, z): x, y, z \in \mathbb{R} \text{ and } y = 0\}, y = 0$

Example

Locate the points

- $(3, 2, 4)$ and
 - $(-2, -4, 3)$ is space
- i. To locate the points $(3, 2, 4)$ in space, we move 3 units from the origin 0 long the positive direction of x - axis. Let this point be A $(3, 0, 0)$. From the point A moves 2 units parallel to positive directions of y - axis. Let this point be B $(3, 2, 0)$. From the point B moves 4 units along positive direction of z - axis. Let this point be P $(3, 2, 4)$

- ii. From
along
axis.
the
parallel
axis, let

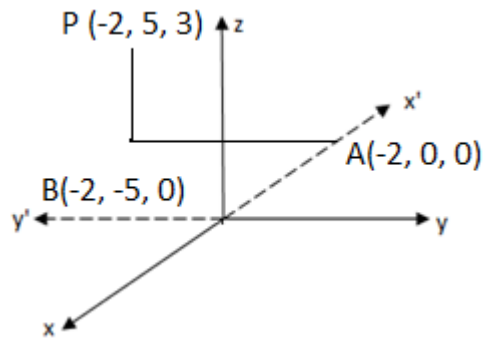


the origin moves 2 units
the negative direction of x -
Let it be A $(-2, 0, 0)$. From
point A moves 5 units
to negative direction of y -
this point be B $(-2, -5, 0)$.

From point B moves 3 units parallel to positive direction of z - axis,
let this result point P $(-2, -5, 3)$



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Do exercise 6.1 on page 275

Distance between two points in space

Definition:- let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be points in space, the distance between the points P and Q denoted by PQ is given as

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example

Find the distance between the points $P\left(-1, -\frac{1}{2}, -\frac{1}{4}\right)$ and $Q(-4, 0, 1)$

Solution:

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-4 - (-1))^2 + \left(0 - \left(-\frac{1}{2}\right)\right)^2 + \left(-1 - \left(-\frac{1}{4}\right)\right)^2} \\ &= \sqrt{(-4 + 1)^2 + \left(0 + \frac{1}{2}\right)^2 + \left(-1 + \frac{1}{4}\right)^2} \\ &= \sqrt{9 + \frac{1}{4} + \frac{9}{16}} \end{aligned}$$



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$$i \sqrt{\frac{144+4+9}{16}}$$

$$i \frac{\sqrt{157}}{4}$$

Midpoint of a line segment in space

Definition:- if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are end point of a line segment in space, given as $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

- You can see figure 6.10 on your text book

Example:-

Find the mid-point the line segment with end points

$$P\left(\frac{1}{2}, \frac{1}{3}, -1\right) \wedge Q\left(\frac{-1}{2}, 0, \frac{1}{4}\right)$$

Solution:-

The mid-point of $\overline{PQ}$ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

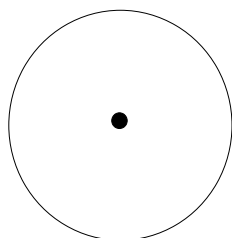
$$i \left(\frac{\frac{1}{2} + \left(\frac{1}{2}\right)}{2}, \frac{\frac{1}{2} + 0}{2}, \frac{-1 + \frac{1}{4}}{2} \right)$$

$$i \left(0, \frac{1}{6}, \frac{-3}{8} \right)$$

Do exercise 6.2 and 6.3 on page 279 and 281

Equation of sphere { {

Definition:- sphere is a set of infinite points in space equidistant from a fixed point the fixed point is called center of the sphere.





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Note:- The equation of a sphere centered at $c(a, b, c)$ whose radius r is given by

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Example:-

Write the equation of the sphere with center at $C(-2, 3, 1)$ and radius 2 units

Solution:- The equation is $(x + 2)^2 + (y - 3)^2 + (z - 1)^2 = 4$

Note:- Let $P(a, b, c)$ be any point in space and S be a sphere whole equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

- P is outside of the sphere if $d > r$
- P is on the sphere if $d = r$
- P is inside of the sphere if $d < r$, where d is the distance between the center of the sphere and the point P .

- The shortest distance between the sphere and the point p is
 - $D - r$ if P is outside of the sphere
 - 0 if P is on the sphere
 - $R - d$ if P is inside of the sphere.

Do exercise 6.4 on page 284

Under in space

- The notation of vectors in space let the initial point of a vector in space be at $P(x_1, y_1, z_1)$ and let its terminal point be at $Q(x_2, y_2, z_2)$ then the vector

$$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

Where $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$ are standard unit vectors in the direction of the positive x - axis, positive y - axis and positive, z axis respectively.

(see figure 6.15 on your text book)

- The magnitude of the vector $\vec{V} = ai + bj + ck$ denoted by $|\vec{V}|$ is given as

$$|\vec{V}| = \sqrt{a^2 + b^2 + c^2}$$
- The unit vector in the direction of $\vec{V}(a, b, c)$ is $\vec{U} = \frac{ai + bj + ck}{\sqrt{a^2 + b^2 + c^2}}$



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Example:- let $\vec{V}$ be a vector in space whose initial point P(-1, 2, 1) and terminal point at Q(3, 1, 2)

$$A) \vec{V} = \overrightarrow{PQ} = (3 - (-1), 1 - 2, 2 - 1) = 4i - j + k$$

$$B) |\vec{V}| = \sqrt{4^2 + (-1)^2 + 1^2} = \sqrt{16 + 1 + 1} = \sqrt{18} = 3\sqrt{2}$$

C) The unit vector in the direction of the vector $\vec{V}$ is

$$\vec{U} = \frac{4i - j + k}{3\sqrt{2}} = \frac{4}{3\sqrt{2}}i - \frac{1}{3\sqrt{2}}j + \frac{1}{3\sqrt{2}}k = \frac{2\sqrt{2}}{3}i - \frac{2\sqrt{2}}{6}j + \frac{2\sqrt{2}}{6}k$$

Addition and subtraction of vectors

If $\vec{a}$ and $\vec{b}$ are vectors in space whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively then $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $k\vec{a}$ are vectors in space with

1. $\vec{a} + \vec{b} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$
2. $\vec{a} - \vec{b} = (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$
3. $k\vec{a} = (kx_1, ky_1, kz_1) = kx_1i + ky_1j + kz_1k$

Example:- if $\vec{a} = (1, 2, 3)$ and $\vec{b} = (-2, 1, 3)$ then find $\vec{a} + 3\vec{b}$ and $2\vec{a} - \vec{b}$

- i. $\vec{a} + \vec{b} = (1, 2, 3) + (-2, 1, 3) = (-1, 3, 6) = -i + 3j + 6k$
- ii. $2\vec{a} - \vec{b} = 2(1, 2, 3) - (-2, 1, 3) = (2, 4, 6) - (-2, 1, 3) = (4, 3, 3) = 4i + 3j + 3k$

Properties of addition of vectors

i. Vector addition is commutative

For any two vectors $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

ii. Vector addition is associative

For any three vectors $\vec{a}, \vec{b}, \vec{c}$ in space

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

- iii. For any two vectors $\vec{a}$ and $\vec{b}$ in space and any scalar k, $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Do exercise 6.5 on page 290

Scalar or dot product of vectors in space



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If $\vec{a}$ and $\vec{b}$ are two vectors in space the dot (scalar) product of $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \cdot \vec{b}$ is defined as

$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$ where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

If $\vec{a} = (x_1, y_1, z_1) = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$ and

$\vec{b} = (x_2, y_2, z_2) = x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}$ then

$$\vec{a} \cdot \vec{b} = (x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}) \cdot (x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

→ The angle between the vector $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$ is

$$\theta = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\Rightarrow \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \sqrt{x_2^2 + y_2^2 + z_2^2}}$$

Example

Let $\vec{a} = (-5, 0, 1)$ and $\vec{b} = (1, -3, -2)$

i. $\vec{a} \cdot \vec{b} = (-5, 0, 1) \cdot (1, -3, -2) = -5 \times 1 + 0 \times (-3) + 1 \times (-2) = -5 + 0 - 2 = -7$

ii. The cosine of the angle between $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned} \cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{(-5, 0, 1) \cdot (1, -3, -2)}{\sqrt{(-5)^2 + 0^2 + 1^2} \sqrt{1^2 + (-3)^2 + (-2)^2}} \\ &= \frac{-7}{\sqrt{26} \sqrt{14}} = \frac{-7}{2\sqrt{91}} = \frac{-7\sqrt{91}}{182} \end{aligned}$$

Do exercise 6.6 on page 292