



☞ Dear students: - Copy the note and do the given homework on your exercise book!

Subject:- MATHEMATICS

Grade & section: 11 (A)

Unit: 10

Main topic: Introduction to linear programming

Sub topic/s: introduction to linear programming

Page range from: 403-412

MONDAY, MAY 25/2020 G.C

Objective

After completing this unit, you should be able to:

- ❖ Identify regions of inequality graphs.
- ❖ create real life examples of linear programming problems using Inequality and solve them

INTRODUCTION

Many real life problems involve finding the optimum (maximum or minimum) value of a function under certain conditions. In particular, linear programming is a field of mathematics that deals with the problem of finding the maximum or minimum value of a given linear function, known as the objective function, subject to certain conditions expressed as linear inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resource limitations such as market demand, labour time, production capacity, etc.

.1 REVISIONS ON LINEAR GRAPHS

Given a non horizontal line ℓ on the xy-coordinate plane, it intersects with the x-axis at exactly one point. The angle α measured from the x-axis to ℓ in the counter clockwise direction is called the inclination of the line ($0 \leq \alpha < 180^\circ$). In order to determine the equation of ℓ , we pick two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on ℓ as shown in Figure 10.1. Then we define the slope m of ℓ by $M = \text{Rise} / \text{run} = \text{Opposite side} / \text{adjacent side} = y_2 - y_1 / x_2 - x_1$

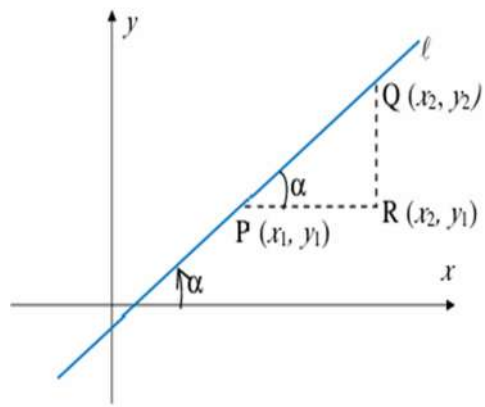


Figure 10.1

Two non-vertical lines ℓ_1 and ℓ_2 with slopes m_1 and m_2 , respectively, are parallel if and only if they have the same slope; i.e, $m_1 = m_2$.

Example : Verify that the line ℓ_1 through the points A (1, 1) and B (−2, 3) is parallel to the line ℓ_2 through the points C (3, 2) and D (−3, 6).

Solution:- the slope of the line that pass through a and b is

$$y_2 - y_1 / x_2 - x_1$$

$$3 - 1 / -2 - 1$$

$2 / -3 = -2/3$ the slope of the line that pass through c and d is

$$y_2 - y_1 / x_2 - x_1$$

$$= 6 - 2 / -3 - 3$$

$$= 4 / -6$$

$$= -2/3$$

the two lines have the same slope. hence they are parallel. The equation of a line ℓ is an equation in two variables x and y such that a point $P(x, y)$ is on ℓ if and only if x and y satisfy the equation. Recall that if a line ℓ has slope m and passes through a point $P(x_1, y_1)$, then the point-slope form of equation of ℓ is given by $y - y_1 = m(x - x_1)$ If the line passes through $(0, 0)$, its equation is $y = mx$

Example: The equation of the line passing through $P(-2, 3)$ with slope $m = 2$ is given by

$$y - 3 = 2(x - (-2))$$

$$= 2(x + 2) = 2x + 4$$

$$\text{or } y = 2x + 7.$$

If the y -intercept of a line with slope m is $(0, b)$, then its equation in the slope-intercept form is $y = mx + b$

Example 3 The equation of a line ℓ with slope 4 and y -intercept −3 is given by

$$y = mx + b$$

$$y = 4x - (-3)$$

$$y = 4x + 3$$

Note:

1. An equation of a vertical line passing through the point (h, k) is given by $x = h$. A vertical line has no slope.
2. An equation of a horizontal line passing through the point (h, k) is given by $y = k$. A horizontal line has zero slope.
3. Two lines are perpendicular, if and only if their slopes are negative reciprocals of each other. That is, if ℓ_1 has slope m_1 and ℓ_2 has slope m_2 , then ℓ_1 is perpendicular to ℓ_2 , if and only if $m_1 \cdot m_2 = -1$

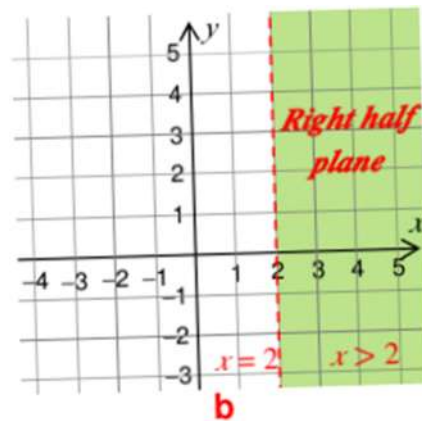
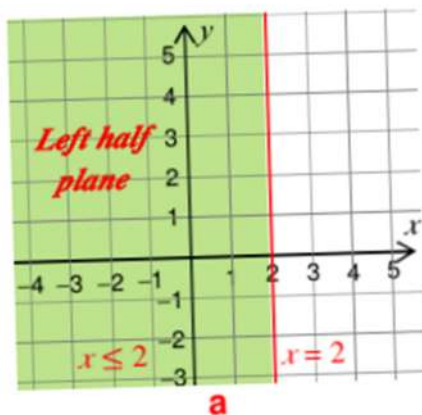
Exercise

1. Determine the equation of the line a that has slope 4 and passes through $P(-1, 3)$.
 - b. that passes through the points $P(1, 2)$ and $Q(-4, 1)$.
 - c whose slope is -2 with y -intercept $(0, 5)$.
2. Determine the value of k so that the line with equation $4x + ky = 8$ is parallel to the line with equation $x + 2y = 0$.
- 3 Draw the graphs of the following lines using the same coordinate axes.
 - a. $y = 2x - 1$
 - b . $y = 2x + 3$
 - c. $3x - 2y = 4$

GRAPHICAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

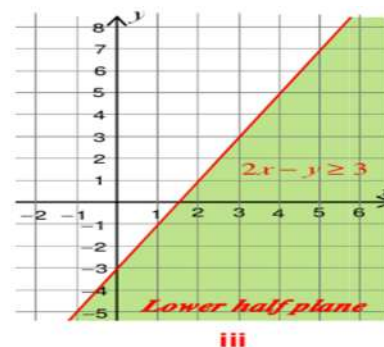
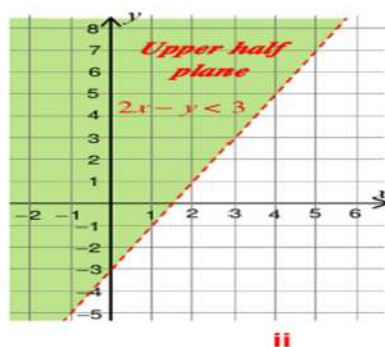
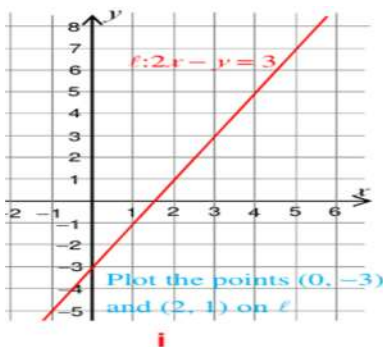
In this section, you use graphs to determine the solution set of a system of linear inequalities in two variables. Every line $\ell: ax + by = c$ in the plane divides the plane into two regions, one on each side of the line. Each of these regions is called a half plane. A vertical line $\ell: x = a$ divides the plane into left and right half planes. A point (x, y) is to the left of the line $x = a$, if and only if, $x < a$. Hence the graph of the inequality $x < a$ is the half plane lying to the left of the line $x = a$. Similarly, the graph of the inequality $x > a$ is the half plane lying to the right of the line $x = a$.

Example 1. consider the vertical line $x = 2$



Observe that in a the left half plane $x \leq 2$ contains the points on the line $x = 2$ and hence the line is a bold (unbroken) line; whereas in b the right half plane $x > 2$ does not include the points on the line $x = 2$ (broken line). A non-vertical line divides the plane into two regions which can be called upper and lower half planes.

Example 2. consider the graph of the linear equation $2x - y = 3$ and the related linear inequalities $2x - y \geq 3$ and $2x - y < 3$. First graph the line $2x - y = 3$ by plotting two points on the line. To identify which half plane belongs to which inequality, test a point that does not lie on the line (usually the origin).



Test $(0, 0)$; $2(0) - 0 = 0 < 3$

Observe the broken line for $2x - y < 3$ and solid line for $2x - y \geq 3$

.solved simultaneously. A graphical solution of a system of linear inequalities is the graph of all ordered pairs (x, y) that satisfy all the inequalities. Such a graph is called the solution region (or feasible region).

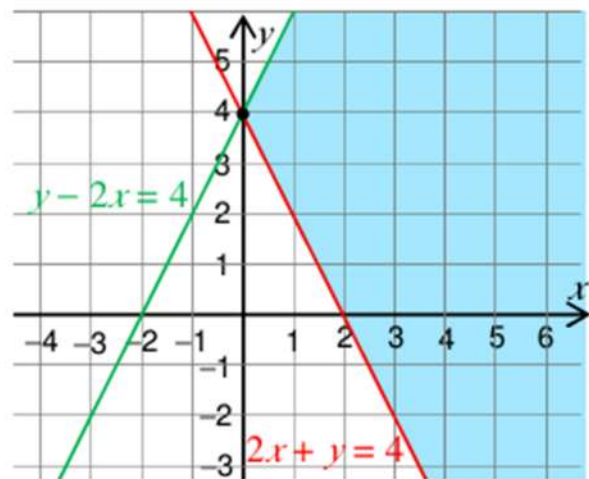
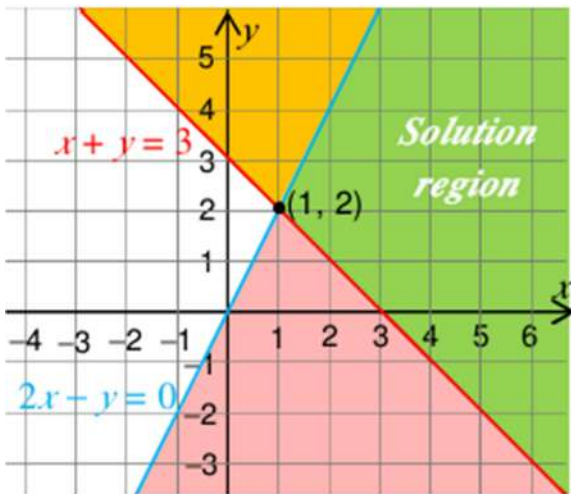
Example 3 Find a graphical solution to the system of linear inequalities.

$$\begin{cases} x + y \geq 3 \\ 2x - y \geq 0 \end{cases}$$

Solution:-first draw the lines $x + y = 3$ and $2x - y = 0$ by plotting two points for each line. Then shade the regions for the two inequalities. The solution region is the intersection of the two regions. To find the point of the intersection of the two lines

$$\begin{cases} x + y \geq 3 \\ 2x - y \geq 0 \end{cases}$$

simultaneously, to get the point $(1, 2)$.



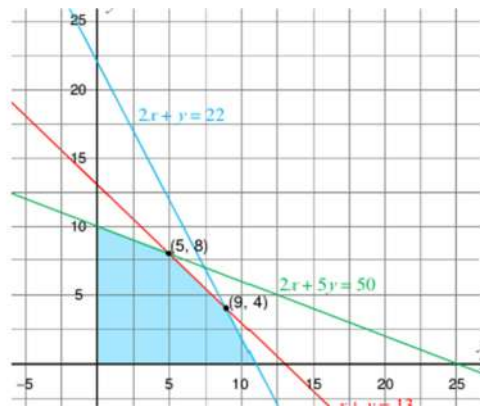
More on solving linear inequalities

Definition 10.1 A point of intersection of two or more boundary lines of a solution region is called a vertex or corner point of the region

Example 5 Solve the following system of linear inequalities.

$$\left. \begin{array}{l} 2x + y \leq 22 \\ x + y \leq 13 \\ 2x + 5y \leq 50 \\ x \geq 0 \\ y \geq 0 \end{array} \right\}$$

Solution:- The last two inequalities $x \geq 0$ and $y \geq 0$ are known as non-negative inequalities (or non-negative requirements). They indicate that the solution region is in the first quadrant of the plane. Draw the lines $\ell_1 : 2x + y = 22$, $\ell_2 : x + y = 13$ and $\ell_3 : 2x + 5y = 50$. To determine the solution region, test the point $O(0, 0)$ which is not in any of these 3 lines, and find the intersection of all half planes to get the shaded region in the figure.



This solution region has five corner points. The vertices $O(0, 0)$, $P(0, 10)$ and $Q(11, 0)$ can be easily determined. To find the other two vertices R and S , solve simultaneously the following two pairs of equations:

$$\begin{array}{l} \ell_1 : 2x + y = 22 \\ \ell_2 : x + y = 13 \end{array} \left. \vphantom{\begin{array}{l} \ell_1 : 2x + y = 22 \\ \ell_2 : x + y = 13 \end{array}} \right\} \text{ to get } S(9, 4) \quad \text{and} \quad \begin{array}{l} \ell_2 : x + y = 13 \\ \ell_3 : 2x + 5y = 50 \end{array} \left. \vphantom{\begin{array}{l} \ell_2 : x + y = 13 \\ \ell_3 : 2x + 5y = 50 \end{array}} \right\} \text{ to get } R(5, 8)$$

Observe that the point of intersection of ℓ_1 and ℓ_3 is not a corner point of the solution region.

Definition. A solution region of a system of linear inequalities is said to be bounded, if it can be enclosed by a rectangle, otherwise it is unbounded.

Exercise:- Find a graphical solution for each of the following.

a. $x \geq 0$

$$y \geq 0$$

$$2x + 3y \leq 4$$

$$x \leq 4$$

b. $x - y \leq 2$

$$x + y \geq 2$$

$$x + 2y \leq 8$$

$$3x + 5y \geq 30$$

c. $x \geq 0$

$$y \geq 0$$

$$x + y \geq 8$$

$$4x + y \leq 48$$

d. $x, y \geq 0$

$$2x + 3y \leq 60$$

$$2x + y \leq 28$$

Maximum and minimum values

Definition

Suppose f is a function with domain $I = \{x \mid a \leq x \leq b\}$

- A number $M = f(c)$ for some c in I is called the maximum value of f on I , if $M \geq f(x)$, for all x in I .
- A number $m = f(d)$ for some d in I is called the minimum value of f on I , if $m \leq f(x)$, for all x in I .
- A value which is either a maximum or a minimum is called an optimum (or extremum) value of f on I .

Many optimization problems involve maximizing or minimizing a linear function (the objective function) subject to one or more linear equations or inequalities (constraints). In this section, problems with only two variables are going to be considered since such problems can easily be solved by a graphical method.

Example 1 Find the values of x and y which will maximize the value of the objective function $Z = f(x, y) = 2x + 5y$, subject to the linear constraints:

$$x \geq 0$$

$$y \geq 0$$

$$3x + 2y \leq 6$$

$$-2x + 4y \leq 8$$

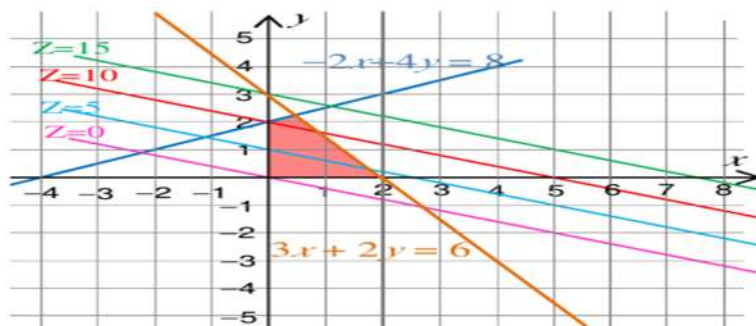
Solution: First you sketch the graphical solution S of the given constraints using the methods of Section This bounded region S is also called the region of feasible solution or feasible region. Any point in the interior or on the boundary of S satisfies all the above constraints. Next you find a point (x, y) of the feasible region that gives the maximum value of the objective function Z . Let's first draw some lines which represent the objective function for values of $Z = 0, 5, 10$ and 15 ; which are,, the lines.

$$2x + 5y = 0$$

$$2x + 5y = 10$$

$$2x + 5y = 5$$

$$2x + 5y = 15$$



From the above Figure you can observe that as the value of Z increases, the lines are moving upwards and the line for $Z = 15$ is outside the feasible region. The maximum possible value of Z will be obtained if we draw a line between $Z = 10$ and $Z = 15$ parallel to them that just "touches" the feasible region. This occurs at the vertex (corner point) P which is the point of intersection of the lines.

$$\begin{cases} 3x + 2y = 6 \\ -2x + 4y = 8 \end{cases} \Rightarrow x = \frac{1}{2} \quad \text{and} \quad y = \frac{9}{4}$$

The value of Z at this point is

$$Z = 2x + 5y = 2 \left(\frac{1}{2} \right) + 5 \left(\frac{9}{4} \right) = \frac{49}{4} = 12 \frac{1}{4}$$

Thus the maximum value of Z under the given conditions is $Z = 12.25$

Fundamental theorem of linear programming

Theorem 10.1

If the feasible region of a linear programming problem is **non-empty** and **bounded**, then the objective function attains both a maximum and a minimum value and those occur at corner points of the feasible region. If the feasible region is **unbounded**, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at corner points.

SOLUTIONS OF LINEAR PROGRAMMING

Steps to solve a linear programming problem by the graphical method

1. Draw the graph of the feasible region.
2. Compute the coordinates of the corner points.
3. Substitute the coordinates of the corner points into the objective function to see which gives the optimal value.
4. If the feasible region is unbounded, this method is misleading: optimal solutions always exist when the feasible region is bounded, but may or may not exist when it is unbounded. In the above examples, the vertices are on the points

$$s (0, 0), (2, 0), \left(\frac{1}{2}, \frac{9}{4}\right) \text{ and } (0, 2)$$

check each value as shown in the figure below

Vertex Point	Value of $Z = 2x + 5y$
(0, 0)	$Z = 2(0) + 5(0) = 0$
(2, 0)	$Z = 2(2) + 5(0) = 4$
$\left(\frac{1}{2}, \frac{9}{4}\right)$	$Z = 2\left(\frac{1}{2}\right) + 5\left(\frac{9}{4}\right) = \frac{49}{4}$
(0, 2)	$Z = 2(0) + 5(2) = 10$

Comparing the values of Z , you get the maximum value of $Z = \frac{49}{4}$ obtained at $\left(\frac{1}{2}, \frac{9}{4}\right)$.

We also have the minimum value $Z = 0$ at $(0, 0)$.

Example 2 . Solve the following linear programming problem. Find the maximum value of the objective function $Z = 3x + 2y$, subject to the following constraints:

$$x \geq 0$$

$$y \geq 0$$

$$x + 2y \leq 4$$

$$x - y \leq 1$$

Solution: from the constraints you sketch the feasible region shown in Figure 10.9. The vertices of this region are $(0, 0)$, $(1, 0)$, $(2, 1)$ and $(0, 2)$.

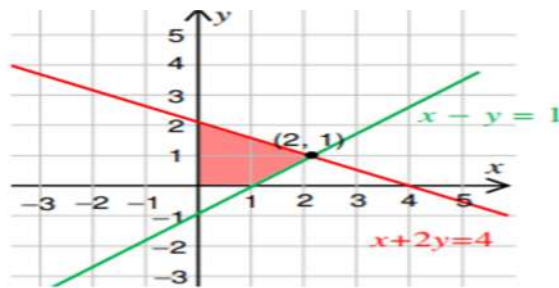


Figure 10.9

Their functional values at Z are given in the following table:

Vertex	Value of $Z = 3x + 2y$
(0, 0)	$Z = 3(0) + 2(0) = 0$
(1, 0)	$Z = 3(1) + 2(0) = 3$
(2, 1)	$Z = 3(2) + 2(1) = 8$
(0, 2)	$Z = 3(0) + 2(2) = 4$

Thus, the maximum value of Z is 8, and occurs when $x = 2$ and $y = 1$.

Example 3 Solve the following linear programming problem. Find the maximum value of

$Z = 4x + 6y$ subject to the following constraints

$$x \geq 0$$

$$y \geq 0$$

$$-x + y \leq 11$$

$$x + y \leq 27$$

$$2x + 5y \leq 90$$

Solution:-The feasible region bounded by the constraints is shown in Figure below The vertices of the feasible region are (0, 0), (27, 0), (15, 12), (5, 16) and (0, 11).

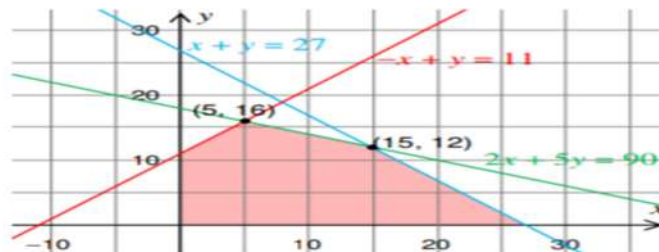


Figure 10.10

Testing the objective function at the vertices gives

Vertex	Value of $Z = 2x + 4y$
(0, 0)	$Z = 4(0) + 6(0) = 0$
(27, 0)	$Z = 4(27) + 6(0) = 108$
(15, 12)	$Z = 4(15) + 6(12) = 132$
(5, 16)	$Z = 4(5) + 6(16) = 116$
(0, 11)	$Z = 4(0) + 6(11) = 66$

Thus the maximum value of Z is 132 when $x = 15$ and $y = 12$.

EXERCISE:- Find the maximum and minimum values of $Z = 4x + 5y$

Subject to: $x > 0$

$$y > 0$$

$$2x + 2y \leq 10$$

$$x + 2y < 6$$